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Abstract

We study the structure of bialgebras in cointeraction on noncrossing partitions appearing in the theory of free probability. Its first coproduct is given by separation of the blocks of the partitions into two parts, with respect to the nestings, while the second one is given by fusion of blocks. This structure implies the existence of a unique polynomial invariant $\phi_{\mathcal{NCP}}$ respecting the product and both coproducts: we give a combinatorial interpretation of this polynomial invariant, study its values at -1 and use it for the computation of the antipode. We also give several results on its coefficients, in the simplest case where the considered noncrossing partitions have no nesting. This leads to unexpected links with harmonic nested sums, Riordan arrays and generalized Stirling numbers. This polynomial invariant is related to other ones, counting increasing or strictly increasing maps for the nesting order on noncrossing partitions, through the action of several characters.

Keywords. Noncrossing partitions; polynomial invariants; bialgebras in cointeraction.

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Contents

Introduction

Noncrossing partitions have a rich combinatorics which has been used for example in the theory of free probability [2, 3, 13, 14]. They can be given several algebraic structures which are used to study relations between cumulants and moments in the theory of free probability [5, 4]. In the present article, we study the double bialgebraic structure on noncrossing partitions, already defined in [6] with operadic tools, and its applications to polynomial invariants. The presentation is based here on our construction of contraction-extraction coproducts [10] in the context of species.

We shall denote by $\mathcal{NCP}$ the set of noncrossing partitions:

$$\mathcal{NCP}=\left\{\begin{array}{c} |;\ \sqcup,\ |\,|;\ \sqcup\!\sqcup,\ \sqcup\,|,\ |\,\sqcup,\ \overset{|}{\sqcup},\ |\,|\,|;\\ \sqcup\!\sqcup\!\sqcup,\ \sqcup\!\sqcup\,|,\ |\,\sqcup\!\sqcup,\ \overset{|}{\sqcup}\!\sqcup,\ \sqcup\!\overset{|}{\sqcup},\ \sqcup\,\sqcup,\ \overset{\sqcup}{\sqcup},\\ \sqcup\,|\,|,\ |\,\sqcup\,|,\ |\,|\,\sqcup,\ \overset{|}{\sqcup}\,|,\ |\,\overset{|}{\sqcup},\ \overset{|\,|}{\sqcup},\ |\,|\,|\,|,\ldots \end{array}\right\}.$$

Graphically, noncrossing partitions will be represented with legs (vertical segments), united in blocks by horizontal segments. The gradation which is used here is not given by the number of legs as usual, but rather by the number of blocks, see Definition 2.1. The object which will be considered all along the text is the polynomial algebra generated by $\mathcal{NCP}$, which we denote by $\mathbb{K}[\mathcal{NCP}]$: a basis of this algebra is given by commutative monomials in noncrossing partitions. In order to avoid confusion between these monomials and non connected noncrossing partitions, the product of $\mathbb{K}[\mathcal{NCP}]$ is denoted by $\cdot$. For example, in $\mathbb{K}[\mathcal{NCP}]$, $|\cdot\sqcup$, $|\,\sqcup$ and $\sqcup\,|$ are pairwise different. The first coproduct, introduced in Proposition 2.6, is given by separation of the blocks of a noncrossing partitions into two parts respecting the nesting order relation. For example,

$$\begin{aligned}
\Delta(|)&=|\otimes 1+1\otimes |,\\
\Delta(\sqcup)&=\sqcup\otimes 1+1\otimes\sqcup,\\
\Delta(|\,|)&=|\,|\otimes 1+1\otimes |\,|+2|\otimes |,\\
\Delta(\sqcup\!\sqcup)&=\sqcup\!\sqcup\otimes 1+1\otimes\sqcup\!\sqcup,\\
\Delta(\sqcup\,|)&=\sqcup\,|\otimes 1+1\otimes\sqcup\,|+\sqcup\otimes |+|\otimes\sqcup,\\
\Delta(|\,\sqcup)&=|\,\sqcup\otimes 1+1\otimes |\,\sqcup+\sqcup\otimes |+|\otimes\sqcup,\\
\Delta(\overset{|}{\sqcup})&=\overset{|}{\sqcup}\otimes 1+1\otimes\overset{|}{\sqcup}+\sqcup\otimes |,\\
\Delta(|\,|\,|)&=|\,|\,|\otimes 1+1\otimes |\,|\,|+|\otimes(|\cdot|+2|\,|)+3|\,|\otimes |.
\end{aligned}$$

This gives a Hopf algebra, graded in two different ways, firstly by the number of legs of noncrossing partitions, secondly by the number of blocks. Using the formalism defined in [10], we give $\mathbb{K}[\mathcal{NCP}]$ a second coproduct, given by fusion of blocks (Proposition 3.8). For example,

$$\begin{aligned}
\delta(|)&=|\otimes |,\\
\delta(\sqcup)&=\sqcup\otimes\sqcup,\\
\delta(|\,|)&=|\,|\otimes |\cdot|+\sqcup\otimes |\,|,\\
\delta(\sqcup\!\sqcup)&=\sqcup\!\sqcup\otimes\sqcup\!\sqcup,\\
\delta(\sqcup\,|)&=\sqcup\,|\otimes\sqcup\cdot|+\sqcup\!\sqcup\otimes\sqcup\,|,\\
\delta(|\,\sqcup)&=|\,\sqcup\otimes\sqcup\cdot|+\sqcup\!\sqcup\otimes |\,\sqcup,\\
\delta(\overset{|}{\sqcup})&=\overset{|}{\sqcup}\otimes\sqcup\cdot|+\sqcup\!\sqcup\otimes\overset{|}{\sqcup},\\
\delta(|\,|\,|)&=|\,|\,|\otimes |\cdot|\cdot|+(\sqcup\,|+|\,\sqcup+\overset{|}{\sqcup})\otimes |\cdot|\,|+\sqcup\!\sqcup\otimes |\,|\,|.
\end{aligned}$$

These two coproducts were also studied in [6], with applications to the theory of free probability. We obtained that $(\mathbb{K}[\mathcal{NCP}],m,\Delta,\delta)$ is a bialgebra in cointeraction (we shall here call these objects double bialgebras), that is to say that the bialgebra $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ is a bialgebra in the category of right comodules over the bialgebra $(\mathbb{K}[\mathcal{NCP}], m, \delta)$, with the coaction given by δ itself. The most remarkable property that this implies is

$$(\Delta \otimes \mathrm{Id}) \circ \delta = m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta,$$

where $m_{1,3,24} : \mathbb{K}[\mathcal{NCP}]^{\otimes 4} \longrightarrow \mathbb{K}[\mathcal{NCP}]^{\otimes 3}$ send $a_1 \otimes a_2 \otimes a_3 \otimes a_4$ to $a_1 \otimes a_3 \otimes a_2 \cdot a_4$.

Let us give a few reminders on the results we obtained on double bialgebras, see [7, 8] for a detailed exposition. More detailed can be found in the first section of this text. Recall first that the polynomial algebra $\mathbb{K}[X]$ is also a double bialgebra, with its multiplicative coproducts defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \delta(X) = X \otimes X.$$

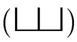
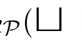
A polynomial invariant is a bialgebra morphism from $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$. Then, there exists a unique polynomial invariant $\phi_{\mathcal{NCP}}$ which is also a bialgebra morphism from $(\mathbb{K}[\mathcal{NCP}], m, \delta)$ to $(\mathbb{K}[X], m, \delta)$. If π is a noncrossing partition,

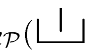
$$\phi_{\mathcal{NCP}}(\pi) = \sum_{k=0}^{\infty} \epsilon_\delta^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(\pi) \frac{X(X-1)\dots(X-n+1)}{n!},$$

where ϵ_δ is the counit of δ and $\tilde{\Delta}^{(n-1)}$ are the iterated reduced coproducts associated to Δ. We give a combinatorial interpretation of $\phi_{\mathcal{NCP}}(\pi)$ in Proposition 3.1: for any integer N, $\phi_{\mathcal{NCP}}(\pi)(N)$ counts the number of N-valid colorations of π, that is to say maps f from π to $\{1, \dots, N\}$ such that the two following conditions are satisfied:

- If b, b' are two blocks of π such that b' is nested into b, then $f(b) < f(b')$.
- If b and b'' are two different blocks of π such that $f(b) = f(b'')$ and $\max(b) < \min(b'')$, then there exists a block b' of π intersecting $]\max(b), \min(b'')[$ such that $f(b') < f(b) = f(b'')$.

The second condition will be called the b-condition. We give in Proposition 3.2 an inductive way to compute this chromatic polynomial $\phi_{\mathcal{NCP}}(\pi)$. We obtain, for example,

$\phi_{\mathcal{NCP}}($$) = X,$ $\qquad$ $\phi_{\mathcal{NCP}}($$) = X(X-1),$

$\phi_{\mathcal{NCP}}($$) = X(X-1),$ $\qquad$ $\phi_{\mathcal{NCP}}($$) = \dfrac{X(X-1)}{2},$

$\phi_{\mathcal{NCP}}($$) = X(X-1)\left(X - \dfrac{3}{2}\right).$

To this polynomial invariant is attached a character $\mu_{\mathcal{NCP}}$, defined by

$$\mu_{\mathcal{NCP}}(x) = \phi_{\mathcal{NCP}}(x)(-1),$$

for any $x \in \mathbb{K}[\mathcal{NCP}]$. The antipode of $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ is then given by

$$S = (\mu_{\mathcal{NCP}} \otimes \mathrm{Id}) \circ \delta.$$

We show that this character can be inductively computed (Lemmas **??** and **??**), and we prove that it sends any noncrossing partition to a product of Catalan numbers.

We then turn to the computation $\phi_{\mathcal{NCP}}(\pi)$, when π has no nesting. In this case, $\phi_{\mathcal{NCP}}(\pi)$ depends only on the number n of blocks of π. We decompose these polynomials, first in the basis of Hilbert polynomials $\left(\dfrac{X(X-1)\dots(X-n+1)}{n!}\right)_{n\geqslant 0}$, then in the basis of monomials $(X^n)_{n\geqslant 0}$. In the first case, their coefficients are inductively given in Proposition 3.5. Closed formulas for certain of these coefficients are given in Corollary 3.7, with the help of multiple nested harmonic sums. In the second case, Proposition 3.8 shows that these coefficients are related to the exponentiation of a particular infinite matrix, namely the Riordan array of $(1+X, X(1+X))$. We also give a combinatorial interpretation of the coefficient of X^n and X^{n-1} in $\phi_{\mathcal{NCP}}(\pi)$ when π has n blocks in Corollaries 3.14 and 3.16, using a particular invertible character $\lambda_{\mathcal{NCP}}$ of $(\mathbb{K}[\mathcal{NCP}], m, \delta)$. This character is also related to the infinitesimal generator of $X(1+X)$, see Proposition 3.17, and to generalized Stirling numbers, see Example 3.5.

We introduce two other bialgebra morphisms Λ and Λ^s from $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$, counting colorations where the b-condition is abandoned and, in the Λ case, the increasing condition is weakened (Theorem 3.20). For example,

$$\begin{aligned}
\Lambda(\sqcup\!\sqcup) &= X, & \Lambda^s(\sqcup\!\sqcup) &= X,\\
\Lambda(\sqcup\ |) &= X^2, & \Lambda^s(\sqcup\ |) &= X^2,\\
\Lambda(|\ \sqcup) &= X^2, & \Lambda^s(|\ \sqcup) &= X^2,\\
\Lambda(\sqcup\!\!|\!\!\sqcup) &= \frac{X(X+1)}{2}, & \Lambda^s(\sqcup\!\!|\!\!\sqcup) &= \frac{X(X-1)}{2},\\
\Lambda(|\ |\ |) &= X^3, & \Lambda^s(|\ |\ |) &= X^3.
\end{aligned}$$

These two morphisms are related by a duality principle: for any noncrossing partition π,

$$\Lambda(\pi)(X) = (-1)^{|\pi|}\Lambda^s(\pi)(-X).$$

We also show in Corollary 3.21 that they are related to $\phi_{\mathcal{NCP}}$ trough two simple characters λ and λ^s, such that

$$\Lambda = (\phi_{\mathcal{NCP}} \otimes \lambda) \circ \delta, \qquad \Lambda^s = (\phi_{\mathcal{NCP}} \otimes \lambda^s) \circ \delta.$$

The inverses μ and μ^s for the convolution associated to δ of these characters are also studied. Note that

$$\phi_{\mathcal{NCP}} = (\Lambda \otimes \mu) \circ \delta = (\Lambda^s \otimes \mu^s) \circ \delta.$$

We give a simple formula for μ^s in Proposition 3.24 and a more complicated way to compute μ in Proposition 3.29.

Finally, in the last section of this paper, we prove that there is no double bialgebra morphism from $\mathbb{K}[\mathcal{NCP}]$ to the double bialgebras of hypergraphs [11] or mixed graphs [12], sending a noncrossing partition to a sum of hypergraphs or to a mixed graph, showing in this way the specificity of the combinatorics of noncrossing partitions, see Proposition 4.1.

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Notations 0.1. 1. We denote by $\mathbb{K}$ a commutative field of characteristic zero. Any vector space in this text will be taken over $\mathbb{K}$.

2. For any $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$. In particular, $[0] = \emptyset$.

3. If (C, Δ) is a (coassociative but not necessarily counitary) coalgebra, we denote by $\Delta^{(n)}$ the n-th iterated coproduct of C: $\Delta^{(0)} = \mathrm{Id}_C$, $\Delta^{(1)} = \Delta$ and if $n \geqslant 2$,

$$\Delta^{(n)} = \left(\Delta \otimes \mathrm{Id}_C^{\otimes(n-1)}\right) \circ \Delta^{(n-1)} : C \longrightarrow C^{\otimes(n+1)}.$$

4. If (B, m, Δ) is a bialgebra of unit 1_B and of counit ε_B, let us denote by $B_+ = \mathrm{Ker}(\varepsilon_B)$ its augmentation ideal. We define a coproduct on B_+ by
$$\forall x \in B_+, \qquad \tilde{\Delta}(x) = \Delta(x) - x \otimes 1_B - 1_B \otimes x.$$
Then $(B_+, \tilde{\Delta})$ is a coassociative (generally not counitary) coalgebra. In consequence, we shall be able to consider the iterated reduced coproducts $\tilde{\Delta}^{(n-1)}$, with $n \geqslant 1$.

5. For any $n \geqslant 1$, we denote by $H_n(X)$ the n-th Hilbert polynomial:
$$H_n(X) = \frac{X(X-1)\ldots(X-n+1)}{n!} \in \mathbb{K}[X].$$
By convention, $H_0 = 1$.

1 Reminders

1.1 On species

We denote by **Set** the category of finite sets with bijections and by **Vect** the category of vector spaces. A (linear) species is a functor $\mathbf{P}$ from **Set** to **Vect**. Let $\mathbf{P}$ be a species. We fix the notations:

- For any finite set X, the vector space associated to X by $\mathbf{P}$ is denoted by $\mathbf{P}[X]$.
- For any bijection $\sigma : X \longrightarrow Y$ between two finite sets, the linear map associated to σ by $\mathbf{P}$ is denoted by $\mathbf{P}[\sigma] : \mathbf{P}[X] \longrightarrow \mathbf{P}[Y]$.

Note that for any finite set X, $\mathbf{P}[\mathrm{Id}_X] = \mathrm{Id}_{\mathbf{P}[X]}$ and that for any bijections $\sigma : X \longrightarrow Y$ and $\tau : Y \longrightarrow Z$ between finite sets, $\mathbf{P}(\tau \circ \sigma) = \mathbf{P}(\tau) \circ \mathbf{P}(\sigma)$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be two species. A morphism between $\mathbf{P}$ and $\mathbf{Q}$ is a natural transformation between the two functors $\mathbf{P}$ and $\mathbf{Q}$, that is to say, for any finite set X, a linear map $f_X : \mathbf{P}[X] \longrightarrow \mathbf{P}[Y]$ such that for any bijection $\sigma : X \longrightarrow Y$ between two finite sets, the following diagram commutes:
$$\begin{tikzcd} \mathbf{P}[X] \arrow[r, "\mathbf{P}[\sigma]"] \arrow[d, "f_X"'] & \mathbf{P}[Y] \arrow[d, "f_Y"] \\ \mathbf{Q}[X] \arrow[r, "\mathbf{Q}[\sigma]"'] & \mathbf{Q}[Y] \end{tikzcd}$$

Species form a symmetric monoidal category, with the Cauchy tensor product $\otimes$: if $\mathbf{P}$ and $\mathbf{Q}$ are two species, for any finite set X,
$$\mathbf{P} \otimes \mathbf{Q}[X] = \bigoplus_{X = Y \sqcup Z} \mathbf{P}[Y] \otimes \mathbf{Q}[Z],$$
and if $\sigma : X \longrightarrow Y$ is a bijection between two finite sets, then
$$\mathbf{P} \otimes \mathbf{Q}[\sigma] = \bigoplus_{X = Y \sqcup Z} \mathbf{P}[\sigma_{|Y}] \otimes \mathbf{Q}[\sigma_{|Z}].$$
For any species $\mathbf{P}$ and $\mathbf{Q}$, the flip $c_{\mathbf{P},\mathbf{Q}} : \mathbf{P} \otimes \mathbf{Q} \longrightarrow \mathbf{Q} \otimes \mathbf{P}$ is defined by the following: for any pair (X, Y) of disjoint sets,
$$c_{\mathbf{P},\mathbf{Q}} : \left\{ \begin{array}{rcl} \mathbf{P}[X] \otimes \mathbf{Q}[Y] & \longrightarrow & \mathbf{Q}[Y] \otimes \mathbf{P}[X] \\ x \otimes y & \longmapsto & y \otimes x. \end{array} \right.$$

A twisted algebra (resp. coalgebra, bialgebra) is an algebra (resp. coalgebra, bialgebra) in the symmetric monoidal category of species with the Cauchy tensor product. Let us now give more details.

A twisted algebra is a pair $(\mathbf{P}, m)$ where $\mathbf{P}$ is a species and, for any pair (X, Y) of disjoint finite sets, $m_{X,Y} : \mathbf{P}[X] \otimes \mathbf{P}[Y] \longrightarrow \mathbf{P}[X \sqcup Y]$ is a linear map, with the following properties:

- If (X, Y) and (X', Y') are two pairs of disjoint finite sets, and if $\sigma : X \longrightarrow X'$ and $\tau : Y \longrightarrow Y'$ are bijections, then the following diagram commutes:

$$\begin{tikzcd}
\mathbf{P}[X] \otimes \mathbf{P}[Y] \arrow[r, "m_{X,Y}"] \arrow[d, "\mathbf{P}[\sigma]\otimes\mathbf{P}[\tau]"'] & \mathbf{P}[X \sqcup Y] \arrow[d, "\mathbf{P}[\sigma\sqcup\tau]"] \\
\mathbf{P}[X'] \otimes \mathbf{P}[Y'] \arrow[r, "m_{X',Y'}"'] & \mathbf{P}[X' \sqcup Y']
\end{tikzcd}$$

- The product m is associative: for any triple (X, Y, Z) of pairwise disjoint sets,

$$m_{X\sqcup Y,Z} \circ (m_{X,Y} \otimes \mathrm{Id}_{\mathbf{P}[Z]}) = m_{X,Y\sqcup Z} \circ (\mathrm{Id}_{\mathbf{P}[X]} \otimes m_{Y,Z}).$$

- The product m has a unit $1_{\mathbf{P}} \in \mathbf{P}[\varnothing]$: for any finite set X, for any $x \in \mathbf{P}[X]$,

$$m_{\varnothing,X}(1_{\mathbf{P}} \otimes x) = m_{X,\varnothing}(x \otimes 1_{\mathbf{P}}) = x.$$

A twisted coalgebra is a pair (P, Δ) where $\mathbf{P}$ is a species and for any pair (X, Y) of finite sets, $\Delta_{X,Y} : \mathbf{P}[X \sqcup Y] \longrightarrow \mathbf{P}[X] \otimes \mathbf{P}[Y]$ is a linear map, with the following properties:

- If (X, Y) and (X', Y') are two pairs of disjoint finite sets, and if $\sigma : X \longrightarrow X'$ and $\tau : Y \longrightarrow Y'$ are bijections, then the following diagram commutes:

$$\begin{tikzcd}
\mathbf{P}[X \sqcup Y] \arrow[r, "\Delta_{X,Y}"] \arrow[d, "\mathbf{P}[\sigma\sqcup\tau]"'] & \mathbf{P}[X] \otimes \mathbf{P}[Y] \arrow[d, "\mathbf{P}[\sigma]\otimes\mathbf{P}[\tau]"] \\
\mathbf{P}[X' \sqcup Y'] \arrow[r, "\Delta_{X',Y'}"'] & \mathbf{P}[X'] \otimes \mathbf{P}[Y']
\end{tikzcd}$$

- The coproduct Δ is coassociative: for any triple (X, Y, Z) of pairwise disjoint finite sets,

$$(\Delta_{X,Y} \otimes \mathrm{Id}_{\mathbf{P}[Z]}) \circ \Delta_{X\sqcup Y,Z} = (\mathrm{Id}_{\mathbf{P}[Z]} \otimes \Delta_{Y,Z}) \circ \Delta_{X,Y\sqcup Z}.$$

- The coproduct Δ has a counit $\varepsilon_\Delta \in \mathbf{P}[\varnothing]^*$: for any finite set X,

$$(\varepsilon_\Delta \otimes \mathrm{Id}_{\mathbf{P}[X]}) \circ \Delta_{\varnothing,X} = (\mathrm{Id}_{\mathbf{P}[X]} \otimes \varepsilon_\Delta) \circ \Delta_{X,\varnothing} = \mathrm{Id}_{\mathbf{P}[X]}.$$

A twisted bialgebra is a triple $(\mathbf{P}, m, \Delta)$ such that:

- $(\mathbf{P}, m)$ is a twisted algebra. Its unit is denoted by $1_{\mathbf{P}}$.
- $(\mathbf{P}, \Delta)$ is a twisted coalgebra. Its counit is denoted by ε_Δ.
- For any pairs (X_1, X_2) and (Y_1, Y_2) of disjoint finite sets, such that $X_1 \sqcup X_2 = Y_1 \sqcup Y_2$,

$$\begin{aligned}
\Delta_{Y_1,Y_2} \circ m_{X_1,X_2} = {} & (m_{X_1\cap Y_1,X_2\cap Y_1} \otimes m_{X_1\cap Y_2,X_2\cap Y_2}) \\
& \circ (\mathrm{Id}_{\mathbf{P}[X_1\cap Y_1]} \otimes c_{\mathbf{P}[X_1\cap Y_2,X_2\cap Y_1]} \otimes \mathrm{Id}_{\mathbf{P}[X_2\cap Y_2]}) \\
& \circ (\Delta_{X_1\cap Y_1,X_1\cap Y_2} \otimes \Delta_{X_2\cap Y_1,X_2\cap Y_2}).
\end{aligned}$$

- For any $x,y\in \mathbf{P}[\varnothing]$, $\varepsilon_\Delta(xy)=\varepsilon_\Delta(x)\varepsilon_\Delta(y)$.
- $\Delta_{\varnothing,\varnothing}(1_\mathbf{P})=1_\mathbf{P}\otimes 1_\mathbf{P}$ and $\varepsilon_\Delta(1_\mathbf{P})=1$.

The bosonic Fock functor [1] sends any species $\mathbf{P}$ to

$$\mathcal{F}[\mathbf{P}]=\bigoplus_{n=0}^\infty \mathrm{coInv}(\mathbf{P}[n])=\bigoplus_{n=0}^\infty \frac{\mathbf{P}[n]}{\mathrm{Vect}(\mathbf{P}[\sigma](p)-p\mid \sigma\in\mathfrak{S}_n,\ p\in\mathbf{P}[n])}.$$

This is a functor of symmetric monoidal categories from species to (graded) vector spaces. Therefore, if $\mathbf{P}$ is a twisted algebra (resp. coalgebra, bialgebra), then $\mathcal{F}[\mathbf{P}]$ is a (graded) algebra (resp. coalgebra, bialgebra).

1.2 On double bialgebras

We refer to [7, 8, 9] for the details.

Definition 1.1. *A double bialgebra is a family (B,m,Δ,δ) such that:*

1. *(B,m,Δ) and (B,m,δ) are bialgebras. Their common unit is denoted by 1_B. The counits of Δ and δ are respectively denoted by ε_Δ and ϵ_δ.*
2. *(B,m,Δ) is a bialgebra in the category of right comodules over (B,m,δ), with the coaction δ, seen as a coaction over itself. This is equivalent to the two following assertions:*

$$\forall x\in B,\qquad (\varepsilon_\Delta\otimes \mathrm{Id}_B)\circ\delta(x)=\varepsilon_\Delta(x)1_B,$$
$$(\Delta\otimes \mathrm{Id}_B)\circ\delta=m_{1,3,24}\circ(\delta\otimes\delta)\circ\Delta,$$

where $m_{1,3,24}: B^{\otimes 4}\longrightarrow B^{\otimes 3}$ sends $x_1\otimes x_2\otimes x_3\otimes x_4$ to $x_1\otimes x_3\otimes x_2x_4$. Note that $m_{1,3,24}\circ(\delta\otimes\delta)$ is in fact the coaction of $B\otimes B$.

An example of double bialgebra is given by the usual polynomial algebra $\mathbb{K}[X]$, with its usual product m and the two (multiplicative) coproducts defined by

$$\Delta(X)=X\otimes 1+1\otimes X,\qquad \delta(X)=X\otimes X.$$

The counits are given by

$$\varepsilon_\Delta:\left\{\begin{array}{rcl}\mathbb{K}[X]&\longrightarrow&\mathbb{K}\\ P(X)&\longmapsto&P(0),\end{array}\right.\qquad \epsilon_\delta:\left\{\begin{array}{rcl}\mathbb{K}[X]&\longrightarrow&\mathbb{K}\\ P(X)&\longmapsto&P(1).\end{array}\right.$$

Proposition 1.2. *Let (B,m,Δ,δ) be a double bialgebra.*

1. *We denote by $\mathrm{Char}(B)$ the set of characters of B, that is to say the set of algebra morphisms from B to $\mathbb{K}$. This sets inherits two associative and unitary products defined by*

$$\forall \lambda,\mu\in\mathrm{Char}(B),\qquad \lambda*\mu=(\lambda\otimes\mu)\circ\Delta,\qquad \lambda\star\mu=(\lambda\otimes\mu)\circ\delta.$$

The units of the products $*$ and $\star$ are respectively ε_Δ and ϵ_δ.

2. *Let (A,m,Δ) be a bialgebra. We denote by $M_{B\to A}$ the set of bialgebra morphisms from (B,m,Δ) to (A,m,Δ). Then the monoid $(\mathrm{Char}(B),\star)$ acts on $M_{B\to A}$ via the following right action:*

$$\leftsquigarrow:\left\{\begin{array}{rcl}M_{B\to A}\times\mathrm{Char}(B)&\longrightarrow&M_{B\to A}\\ (\phi,\lambda)&\longmapsto&\phi\leftsquigarrow\lambda=(\phi\otimes\lambda)\circ\delta.\end{array}\right.$$

The double structure allows to find the antipode for the first structure, whenever it exists:

Theorem 1.3. *[9, Corollary 2.3] Let (B, m, Δ, δ) be a double bialgebra.*

1. *Then (B, m, Δ) is a Hopf algebra if, and only if, the character ϵ_δ has an inverse μ_B for the convolution product $*$ dual to Δ. Moreover, if this holds, the antipode of (B, m, Δ) is given by*
$$S = (\mu_B \otimes \mathrm{Id}_B) \circ \delta.$$
2. *Let $\phi_B : B \longrightarrow \mathbb{K}[X]$ be a double bialgebra morphism. Then ϵ_δ has an inverse for the convolution product $*$, given by*
$$\mu_B : \begin{cases} B & \longrightarrow & \mathbb{K} \\ x & \longmapsto & \phi_B(x)(-1). \end{cases}$$

A double bialgebra (B, m, Δ, δ) is connected if the reduced coproduct $\tilde{\Delta}$ is locally nilpotent: in other words, for any $x \in \mathrm{Ker}(\varepsilon_\Delta)$, there exists $n \geqslant 1$ such that $\tilde{\Delta}^{(n)}(x) = 0$. If so, we obtain more results:

Theorem 1.4. *Let (B, m, Δ, δ) be a connected double bialgebra.*

1. *There exists a unique double bialgebra morphism ϕ_B from B to $\mathbb{K}[X]$. Moreover,*
$$\forall x \in \mathrm{Ker}(\varepsilon_B), \qquad \phi_B(x) = \sum_{k=1}^{\infty} \epsilon_\delta^{\otimes k} \circ \tilde{\Delta}^{(k-1)}(x) H_k(X).$$
2. *The two following maps are bijective, inverse one from the other:*
$$\begin{cases} \mathrm{Char}(B) & \longrightarrow & M_{B \to \mathbb{K}[X]} \\ \lambda & \longmapsto & \phi_B \leftsquigarrow \lambda, \end{cases} \qquad \begin{cases} M_{B \to \mathbb{K}[X]} & \longrightarrow & \mathrm{Char}(B) \\ \phi & \longmapsto & \begin{cases} B & \longrightarrow & \mathbb{K} \\ x & \longmapsto & \phi(x)(1). \end{cases} \end{cases}$$
3. *For any $\lambda \in \mathrm{Char}(B)$, for any $x \in \mathrm{Ker}(\varepsilon_\Delta)$,*
$$\phi_B \leftsquigarrow \lambda(x) = \sum_{k=1}^{\infty} \lambda^{\otimes k} \circ \tilde{\Delta}^{(k-1)}(x) H_k(X).$$

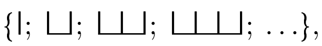
2 Bialgebraic structures on noncrossing partitions

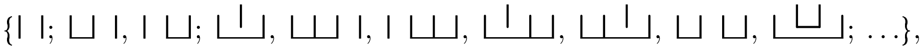
2.1 Reminders on noncrossing partitions

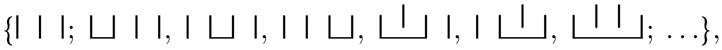
Definition 2.1. *A noncrossing partition is a partition π of a set $[n]$ with $n \in \mathbb{N}$, such that*
$$\forall b \neq b' \in \pi,\ \forall x, z \in b,\ \forall y, t \in b', \qquad x < y < z < t \text{ does not hold.}$$
The elements of π are called its blocks. The number of blocks of a noncrossing partition π is denoted by $|\pi|$. The set of noncrossing partitions with k blocks is denoted by $\mathcal{NCP}_k$.

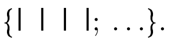
We shall represent noncrossing partitions by diagrams. Elements of $[n]$ will be represented by vertical segments, from left to right, related by horizontal segments corresponding to the blocks. For example, the noncrossing partition $\{\{1,4\},\{2\},\{3\}\}$ is represented by .

Note that for any $k \geqslant 1$, $\mathcal{NCP}_k$ is an infinite set. For example,

$\mathcal{NCP}_1 =$ 

$\mathcal{NCP}_2 =$ 

$\mathcal{NCP}_3 =$ 

$\mathcal{NCP}_4 =$ 

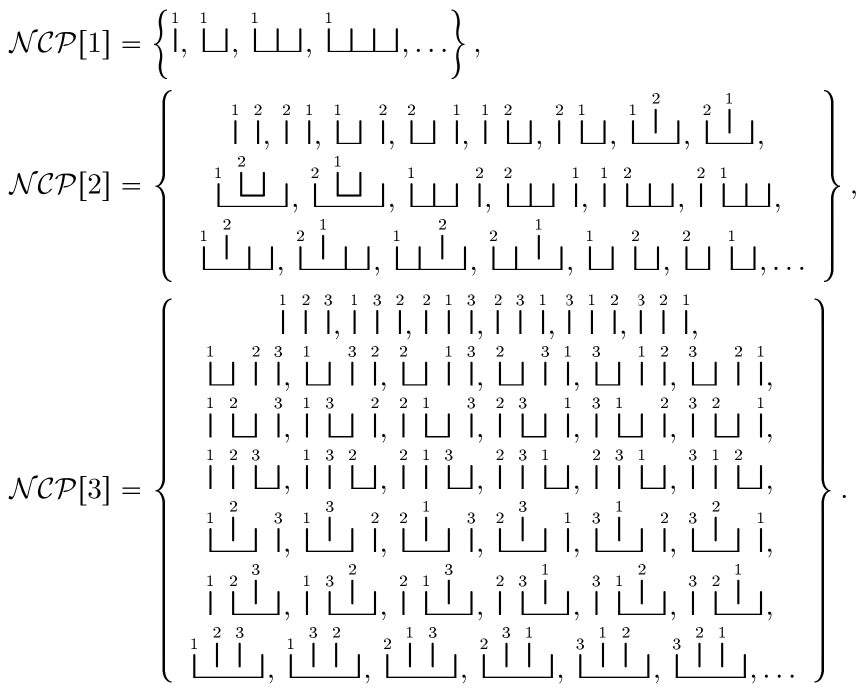
By convention, $\mathcal{NCP}_0 = \varnothing$.

Definition 2.2. *Let X be a finite set. We denote by $\mathcal{NCP}[X]$ the set of bijections f between an element π of $\mathcal{NCP}_{|X|}$ and X. If $\sigma : X \longrightarrow Y$ is a bijection between two finite sets and $f : \pi \longrightarrow X$ is an element of $\mathcal{NCP}[X]$, we put $\mathcal{NCP}[\sigma][f] = \sigma \circ f : \pi \longrightarrow Y$. This defines a species $\mathcal{NCP}$, which linearization is denoted by* **NCP***: for any finite set X,*

$$\mathbf{NCP}[X] = \mathrm{Vect}(\mathcal{NCP}[X]).$$

Notations 2.1. the elements of $\mathcal{NCP}[X]$ can be described as noncrossing partitions which blocks are indexed by the set X. Such an object$f : \pi \longrightarrow X$ will be denoted by $\pi = (\pi_x)_{x\in X}$, where for any $x \in X$, $\pi_x = f^{-1}(x)$. We shall represent graphically the elements of $\mathcal{NCP}[X]$, the index of the blocks being attached to their leftmost leg.

Example 2.1.



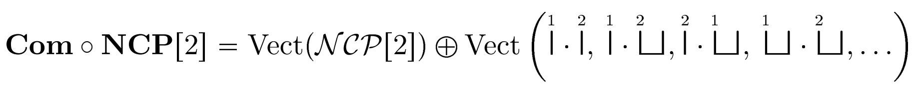
Notations 2.2. We shall consider the species $\mathbf{Com} \circ \mathbf{NCP}$: for any finite set X,

$$\mathbf{Com} \circ \mathbf{NCP}[X] = \bigoplus_{\sim \in \mathcal{E}[X]} \left(\bigotimes_{I \in X/\sim} \mathbf{NCP}[I] \right),$$

where $\mathcal{E}[X]$ is the set of all equivalence relations on X. As **Com** is a commutative twisted algebra and the endofunctor $\mathcal{F}_{\circ\mathbf{NCP}}$ of the category of species is compatible with the Cauchy tensor product, $\mathbf{Com} \circ \mathbf{NCP}$ is naturally a commutative twisted algebra. Its product is denoted by $\cdot$. In other terms, a basis of $\mathbf{Com}\circ\mathbf{NCP}[X]$ is given by commutative monomials of noncrossing partitions $\pi_1 \cdot \ldots \cdot \pi_k$, where for any i, π_i is indexed by a nonempty set I_i, such that $I_1 \sqcup \ldots \sqcup I_k = X$. The unit is the empty monomial $1 \in \mathbf{Com} \circ \mathbf{NCP}[\emptyset]$.

Example 2.2.

$$\mathbf{Com} \circ \mathbf{NCP}[1] = \mathrm{Vect}(\mathcal{NCP}[1]),$$



Note that

2.2 The first coproduct

Let us now define a coproduct on the twisted algebra $\mathbf{Com} \circ \mathbf{NCP}$.

Notations 2.3. Let $\pi = (\pi_x)_{x \in X}$ be an X-indexed noncrossing partition and let $Y \subseteq X$. We put $I_Y = \bigcup_{y \in Y} \pi_y$ and we denote by f_Y the unique increasing bijection from I_Y to $[|I_Y|]$. The Y-indexed noncrossing partition $\pi_{|Y}$ is

$$\pi_{|Y} = (f_Y(\pi_y))_{y \in Y}.$$

Graphically, π_Y is obtained by deleting the blocks of π which are not indexed by an element of Y. By convention, $\pi_{|\varnothing} = 1$. Note that $\pi_{|X} = \pi$.

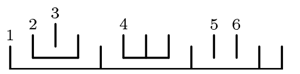
Definition 2.3. *Let $\pi = (\pi_x)_{x \in X}$ be an X-indexed noncrossing partition and $I \subseteq X$. We put $I' = \bigcup_{x \in I} \pi_x \subseteq [n]$, which we decompose into connected components $I'_1 \sqcup \ldots \sqcup I'_k$, that is to say:*

- *For any $i \in [k]$, I'_i is an interval of $[n]$.*
- *For any $i \in [k-1]$, $\max(I'_i) < \min(I'_{i+1}) - 1$.*

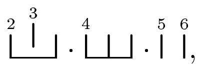
We shall say that I is an ideal of π if for any $i \in [k]$, I'_i is the union of blocks of π. If I is an ideal of π, we put

$$\pi_{|\cdot I} = \dot{\prod_{i \in [k]}} \pi_{|I'_i}.$$

Example 2.3. Let us consider

$\pi =$  $\in \mathcal{NCP}[7]$.

Then $\{2,3,4,5,6\}$ is an ideal of π. The associated connected components are $\{2,3,4\}$, $\{6,7,8\}$ and $\{10,11\}$, and

$\pi_{|\cdot\{2,3,4,5,6\}} =$ , $\qquad \pi_{|\{1,7\}} =$ .

Let us give a more usual characterization of the ideals of a noncrossing partitions, with the help of a classical order on the blocks.

Definition 2.4 (**Nesting order**). *Let π a noncrossing partition. We define a relation on π by*

$$\forall b, b' \in \pi, \qquad b \leqslant_\pi b' \Longleftrightarrow b' \subseteq [\min(b), \max(b)].$$

Then $\leqslant_\pi$ is a partial order on π.

Lemma 2.5. *Let $\pi = (\pi_x)_{x \in X}$ be an X-indexed noncrossing partition and $I \subset X$. Then I is an ideal of π if, and only if,*

$$\forall x, y \in X, \qquad (x \in I \text{ and } \pi_x \leqslant_\pi \pi_y) \Longrightarrow y \in I.$$

Proof. $\Longrightarrow$. Let $\pi_x \neq \pi_y \in \pi$, with $x \in I$ and $\pi_x \leqslant_\pi \pi_y$. As I is an ideal of π, there exists a connected component I'_i containing π_x. As $\pi_y \subseteq [\min(\pi_x), \max(\pi_y)] \subseteq I'_i$, $y \in I$.

$\Longleftarrow$. Let π_x be a block of π intersecting I'_i. Let us assume that $\min(\pi_x) \notin I'_i$. As J is the union of blocks of π, $\min(\pi_x) \in I'$, so belongs to a I'_p with $p < i$. By definition of the connected components, there exists $k \in [n] \backslash J$, such that $\max(I'_p) < k < \min(I'_i)$. This element k belongs to a block π_y of π. Then $\min(\pi_x) < k \in \pi_y < \min(I'_i) \leqslant \max(\pi_x)$. As π is noncrossing, $\pi_x \leqslant_\pi \pi_y$. As I is an ideal, $y \in I$, so $k \in I'$: this is a contradiction, so $\min(\pi_x) \in I'_i$. Similarly, $\max(\pi_y) \in I'_i$. As I'_i is an interval, $\pi_x \subseteq I'_i$. Therefore, I'_i is the union of the blocks of π it intersects. $\square$

Proposition 2.6. *Let $X = I \sqcup J$ be a finite set and $\pi \in \mathcal{NCP}[X]$. We define $\Delta_{I,J}(\pi) \in \mathbf{NCP}[I] \otimes \mathbf{Com} \circ \mathbf{NCP}[J]$ by*

$$\Delta(\pi) = \begin{cases} \pi_{|I} \otimes \pi_{|.J} \text{ if } J \text{ is an ideal of } \pi, \\ 0 \text{ otherwise.} \end{cases}$$

We extend Δ as an algebra morphism from $\mathbf{Com} \circ \mathbf{NCP}$ to $\mathbf{Com} \circ \mathbf{NCP}^{\otimes 2}$. Then the triple $(\mathbf{Com} \circ \mathbf{NCP}, m, \Delta)$ is a twisted bialgebra.

Proof. By definition, Δ is an algebra morphism. Let $\pi \in \mathcal{NCP}[X]$. As $\varnothing$ and X are obviously ideals of π,

$$\Delta_{\varnothing,X}(\pi) = 1 \otimes \pi, \qquad \Delta_{X,\varnothing}(\pi) = \pi \otimes 1,$$

Therefore, Δ has a counit ε_Δ, defined as the linear map from $\mathbf{Com} \circ \mathbf{NCP}[\varnothing]$ to $\mathbb{K}$ sending 1 to 1.

Let us assume that $X = I \sqcup J \sqcup K$. Let $\pi \in \mathcal{NCP}[X]$. We use the notations of Definition 2.3.

$$\begin{aligned}
(\mathrm{Id} \otimes \Delta_{J,K}) \circ \Delta_{I,J \sqcup K}(\pi) &= \begin{cases} \pi_{|I} \otimes \prod\limits_{i \in [p]}^{\cdot} (\pi_{|(J \sqcup K)_i})_{|J} \otimes \prod\limits_{i \in [p]}^{\cdot} (\pi_{|(J \sqcup K)_i})_{|.K} \\ \qquad \text{if } J \sqcup K \text{ ideal of } \pi \text{ and } K \cap (J \sqcup K)_p \text{ ideal of } \pi_{|(J \sqcup K)_p} \text{ for any } p, \\ 0 \text{ otherwise} \end{cases} \\
&= \begin{cases} \pi_{|I} \otimes \pi_{|.J} \otimes \pi_{|.K} \text{ if } J \sqcup K \text{ and } K \text{ ideals of } \pi, \\ 0 \text{ otherwise} \end{cases} \\
&= \begin{cases} (\pi_{|I \sqcup J})_{|I} \otimes (\pi_{|I \sqcup J})_{|.J} \otimes \pi_{|.K} \text{ if } J \sqcup K \text{ and } K \text{ ideals of } \pi, \\ 0 \text{ otherwise} \end{cases} \\
&= (\Delta_{I,J} \otimes \mathrm{Id}) \circ \Delta_{I \sqcup J,K}(\pi).
\end{aligned}$$

By multiplicativity, Δ is coassociative. $\square$

Let us now apply the Fock functor $\mathcal{F}$ on $\mathbf{Com} \circ \mathbf{NCP}$. As an algebra, $\mathcal{F}[\mathbf{Com} \circ \mathbf{NCP}]$ is described as follows.

Definition 2.7. *We denote by $\mathbb{K}[\mathcal{NCP}]$ be free commutative algebra generated by $\mathcal{NCP}$. Its product will be denoted by $\cdot$. A basis of $\mathbb{K}[\mathcal{NCP}]$ is given by commutative monomials in $\mathcal{NCP}$, and the unit is the empty monomial 1.*

Observe that in $\mathbb{K}[\mathcal{NCP}]$,

$$| \cdot \sqcup = \sqcup \cdot |, \qquad | \cdot \sqcup \neq | \sqcup, \qquad \sqcup \cdot | \neq \sqcup\ |, \qquad | \sqcup \neq \sqcup\ |.$$

Applying the Fock functor $\mathcal{F}$ on $\mathbf{Com} \circ \mathbf{NCP}$, we obtain the following coproduct on the algebra $\mathcal{F}[\mathbf{Com} \circ \mathbf{NCP}] = \mathbb{K}[\mathcal{NCP}]$: for any noncrossing partition π of $[n]$,

$$\Delta(\pi) = \sum_{J \text{ ideal of } \pi} \pi_{|[n] \setminus J} \otimes \pi_{|.J}. \tag{1}$$

Colored version can also obtained, using the colored Fock functor of [10] – we won't use them in this paper.

Example 2.4.

$$
\begin{aligned}
\Delta(|) &= | \otimes 1 + 1 \otimes |,\\
\Delta(\sqcup) &= \sqcup \otimes 1 + 1 \otimes \sqcup,\\
\Delta(|\;|) &= |\;| \otimes 1 + 1 \otimes |\;| + 2| \otimes |,\\
\Delta(\sqcup\!\sqcup) &= \sqcup\!\sqcup \otimes 1 + 1 \otimes \sqcup\!\sqcup,\\
\Delta(\sqcup\;|) &= \sqcup\;| \otimes 1 + 1 \otimes \sqcup\;| + \sqcup \otimes | + | \otimes \sqcup,\\
\Delta(|\;\sqcup) &= |\;\sqcup \otimes 1 + 1 \otimes |\;\sqcup + \sqcup \otimes | + | \otimes \sqcup,\\
\Delta(\overset{|}{\sqcup}) &= \overset{|}{\sqcup} \otimes 1 + 1 \otimes \overset{|}{\sqcup} + \sqcup \otimes |,\\
\Delta(|\;|\;|) &= |\;|\;| \otimes 1 + 1 \otimes |\;|\;| + | \otimes (| \cdot | + 2|\;|) + 3|\;| \otimes |.
\end{aligned}
$$

This coproduct is also introduced in [6], with operadic methods.

2.3 The extraction-contraction coproduct

Recall that $\mathcal{E}$ is the species which associates to any finite set X the set of equivalence relations on X.

Definition 2.8. *Let π be an X-indexed noncrossing partition and $\sim \in \mathcal{E}[X]$.*

1. *For any $\overline{x} \in X/\sim$, we put $\pi_{\overline{x}} = \displaystyle\bigsqcup_{y \in \overline{x}} \pi_y$. We put $\pi/\sim = (\pi_{\overline{x}})_{\overline{x} \in X/\sim}$. This defines an $X/\sim$-indexed partition.*
2. *We shall say that $\sim \in \mathcal{E}_c[\pi]$ if $X/\sim \in \mathcal{NCP}[X/\sim]$.*
3. *We put $\pi \mid_{.} \sim = \displaystyle\dot{\prod_{\overline{x} \in X/\sim}} \pi_{|\overline{x}}$. This defines an element of $\mathbf{Com} \circ \mathbf{NCP}[X]$.*

Proposition 2.9. *For any X-indexed noncrossing partition, for any $\sim \in \mathcal{E}[X]$, we put*

$$
\delta_\sim(\pi) = \begin{cases} \pi/\sim \otimes \pi \mid_{.} \sim & \text{if } \sim \in \mathcal{E}_c[\pi],\\ 0 \text{ otherwise.} \end{cases}
$$

We extend this by multiplicativity to $\mathbf{Com} \circ \mathbf{NCP}$. More precisely, if $X = X_1 \sqcup \ldots \sqcup X_k$ and if for any i, $\pi_i \in \mathcal{NCP}[X_i]$, then, for any $\sim \in \mathcal{E}[X]$,

$$
\delta_\sim(\pi_1 \cdot \ldots \cdot \pi_k) = \begin{cases} \delta_{\sim \cap X_1^2}(\pi_1) \cdot \ldots \cdot \delta_{\sim \cap X_1^2}(\pi_k) \text{ if } \sim = (\sim \cap X_1^2) \sqcup \ldots \sqcup (\sim \cap X_k^2),\\ 0 \text{ otherwise.} \end{cases}
$$

Then δ is a contraction-extraction coproduct on $\mathbf{Com} \circ \mathbf{NCP}$ in the sense of [10], compatible with m and Δ.

Proof. Let $\pi \in \mathcal{NCP}[X]$ and let $\sim, \sim' \in \mathcal{E}[X]$, with $\sim' \subseteq \sim$. We denote by $\overline{\sim}$ the equivalence on $X/\sim'$ induced by $\sim$.

$$
(\delta_\sim \otimes \mathrm{Id}) \circ \delta_{\sim'}(\pi) = \begin{cases} (\pi/\sim')/\overline{\sim} \otimes (\pi/\sim')\mid\overline{\sim} \otimes \pi \mid_{.} \sim' \\ \quad \text{if } \sim' \in \mathcal{E}_c[\pi] \text{ and } \overline{\sim} \in \mathcal{E}_c[\pi/\sim'],\\ 0 \text{ otherwise.} \end{cases} \tag{2}
$$

$$
(\mathrm{Id} \otimes \delta_{\sim'}) \circ \delta_\sim(\pi) = \begin{cases} \pi/\sim \otimes (\pi \mid_{.} \sim)/\sim' \otimes (\pi \mid_{.} \sim) \mid_{.} \sim' \\ \quad \text{if } \sim \in \mathcal{E}_c[\pi] \text{ and } \sim' \in \mathcal{E}_c[\pi \mid_{.} \sim],\\ 0 \text{ otherwise.} \end{cases} \tag{3}
$$

Note that, as indexed partitions or monomials of indexed partitions,

$$(\pi/\sim')/\overline{\sim}=\pi/\sim, \qquad (\pi/\sim')\mid\overline{\sim}=(\pi\mid\sim)/\sim', \qquad (\pi\mid.\sim)\mid.\sim'=\pi\mid.\sim'.$$

Let us assume that $\sim'\in\mathcal{E}_c[\pi]$ and $\overline{\sim}\in\mathcal{E}_c[\pi/\sim']$. Then $(\pi/\sim')/\overline{\sim}=\pi/\sim$ is noncrossing, so $\sim\in\mathcal{E}_c[\pi]$. Moreover, $(\pi/\sim')\mid\overline{\sim}=(\pi\mid\sim)/\sim'$ is noncrossing as $\pi/\sim'$ is noncrossing, so $\sim'\in\mathcal{E}_c[\pi\mid.\sim]$. Conversely, let us assume that $\sim\in\mathcal{E}_c[\pi]$ and $\sim'\in\mathcal{E}_c[\pi\mid.\sim]$. Then $(\pi/\sim')/\overline{\sim}=\pi/\sim$ is noncrossing, so $\overline{\sim}\in\mathcal{E}_c[\pi/\sim']$. Moreover, $(\pi/\sim')\mid\overline{\sim}=(\pi\mid\sim)/\sim'$ is noncrossing, so $\sim'\in\mathcal{E}_c[\pi\mid.\sim]$. As $\sim'\subseteq\sim$, $\sim'\in\mathcal{E}_c[\pi]$.

Finally, the two conditions in (2) and (3) are equivalent. This gives the coassociativity of δ when applied to any noncrossing partition, then, by multiplicativity, the coassociativity of δ.

Let now $\pi\in\mathcal{NCP}[X]$ and let $\sim,\sim'\in\mathcal{E}[X]$, without $\sim'\subseteq\sim$. This means that at least one class of $\sim$ is not the union of the classes of $\sim'$ that it contains. Hence, denoting by $\overline{x}_1,\ldots,\overline{x}_k$ the classes of $\sim$, $\sim'$ is not equal to the union of the parts $\sim'\cap\overline{x}_i^2$. Consequently,

$$\delta_{\sim'}(\pi\mid.\sim)=\delta_{\sim'}(\pi_{|\overline{x}_1}\cdot\ldots\cdot\pi_{|\overline{x}_k})=0.$$

We obtain that $(\mathrm{Id}\otimes\delta_{\sim'})\circ\delta_\sim=0$.

Let ε_δ be the twisted algebra morphism from $\mathbf{Com}\circ\mathbf{NCP}$ to $\mathbf{Com}$ defined by

$$\forall\pi\in\mathcal{NCP}[X], \qquad \varepsilon_\delta[X](\pi)=\begin{cases}1\text{ if }\pi\text{ has only one block},\\0\text{ otherwise.}\end{cases}$$

Let $\pi\in\mathcal{NCP}[X]$. The unique $\sim\in\mathcal{E}_c[X]$ such that $\pi\mid.\sim$ is a monomial which all factors are noncrossing partitions with only one block is the equality of X, which we denote by $\sim_1$. Moreover, $\pi/\sim_1=\pi$. Hence, ϵ_δ is a right counit for δ. The unique $\sim\in\mathcal{E}_c[X]$ such that $\pi/\sim$ has only one block is the equivalence $\sim_0$ on X with only one class. As $\pi\mid.\sim_0=\pi$, ε_δ is a left counit for δ.

Let X and Y be two disjoint sets, $\pi\in\mathcal{NCP}[X\sqcup Y]$, $\sim_X\in\mathcal{E}[X]$ and $\sim_Y\in\mathcal{E}[Y]$. We put $\sim=\sim_X\sqcup\sim_Y$.

$$(\Delta_{X/\sim_X,Y/\sim_Y}\otimes\mathrm{Id})\circ\delta_\sim(\pi)=\begin{cases}(\pi/\sim)_{|X}/\sim_X\otimes(\pi/\sim)_{|.Y}\otimes\pi\mid.\sim\\\quad\text{if }\sim\in\mathcal{E}_c[\pi]\text{ and }Y/\sim_Y\text{ is an ideal of }\pi/\sim,\\0\text{ otherwise.}\end{cases}\tag{4}$$

$$m_{1,3,24}\circ(\delta_X\otimes\delta_Y)\circ\Delta_{X,Y}(\pi)=\begin{cases}(\pi_{|X})/\sim_X\otimes(\pi_{|.Y})/\sim_Y\otimes(\pi_{|X})\mid\sim_X\cdot(\pi_{|.Y})\mid\sim_Y\\\quad\text{if }Y\text{ is an ideal of }\pi,\\\quad\sim_X\in\mathcal{E}_c[\pi_{|X}]\text{ and }\sim_Y\in\mathcal{E}_c[\pi_{|.Y}],\\0\text{ otherwise.}\end{cases}\tag{5}$$

Note that, as indexed partitions or as monomials of indexed partitions,

$$(\pi/\sim)\mid(X/\sim_X)=(\pi_{|X})/\sim_X,\quad(\pi/\sim)_{|.Y}=(\pi_{|.Y})/\sim_Y,\quad\pi\mid.\sim=((\pi_{|X})\mid\sim_X)\cdot((\pi_{|.Y})\mid\sim_Y).$$

Let us assume that $\sim\in\mathcal{E}_c[\pi]$ and $Y/\sim_Y$ is an ideal of $\pi/\sim$. Let $\overline{x},\overline{x}'\in\pi$, such that $\overline{x}\leqslant_\pi\overline{x}'$ and $\overline{x}\in Y$. We denote by $\varpi:(X\sqcup Y)\longrightarrow(X\sqcup Y)/\sim$ the canonical surjection. Then $\varpi(\overline{x})\leqslant_{\pi/\sim}\varpi(\overline{x}')$ in $\pi/\sim$. As $Y/\sim_Y$ is an ideal of $\pi/\sim$, $\varpi(\overline{x}')\in Y/\sim_Y$, so $\overline{x}'\in Y$. By restriction, $(\pi\mid X)/\sim_X=(\pi/\sim)\mid X/\sim_X$ is noncrossing as $\pi/\sim$ is noncrossing, so $\sim_X\in\mathcal{E}_c[\pi_{|X}]$. Similarly, $\sim_Y\in\mathcal{E}_c[\pi_{|.Y}]$.

Let us now assume that Y is an ideal of π, $\sim_X\in\mathcal{E}_c[\pi_{|X}]$ and $\sim_Y\in\mathcal{E}_c[\pi_{|.Y}]$. Let $\overline{x},\overline{x}'\in(X\sqcup Y)/\sim$.

- If $\overline{x} \in Y/\sim_Y$ and $\overline{x}' \in X/\sim_X$, as $\sim_Y \in \mathcal{E}_c[\pi_{|.Y}]$, $\sim_Y$ is compatible with the connected components of Y, as defined in Definition 2.3. Hence, $\pi_{\overline{x}'}$ cannot cross $\pi_{\overline{x}}$. Similarly, if $\overline{x} \in X/\sim_X$ and $\overline{x}' \in Y/\sim_Y$, then $\pi_{\overline{x}}$ cannot cross $\pi_{\overline{x}'}$.
- If both $\overline{x}, \overline{x}'$ belong to $X/\sim_X$, As $(\pi_{|X})/\sim_X = (\pi/\sim)_{|X/\sim_X}$ is noncrossing, $\pi_{\overline{x}}$ cannot cross $\pi_{\overline{x}'}$. Similarly, if both $\overline{x}, \overline{x}'$ belong to $Y/\sim_Y$, then $\pi_{\overline{x}}$ cannot cross $\pi_{\overline{x}'}$.

So $\sim \in \mathcal{E}_c[\pi]$. By contraction, $Y/\sim_Y$ is an ideal of $\pi/\sim$.

Finally, the two conditions in (4) and (5) are equivalent. This gives the compatibility of δ and Δ. $\square$

Applying the Fock functor $\mathcal{F}$ on **Com**$\circ$**NCP**, we obtain the following coproduct on $\mathbb{K}[\mathcal{NCP}]$: for any noncrossing partition π,

$$[\delta(\pi) = \sum_{\sim \in \mathcal{E}_c[\pi]} \pi/\sim \otimes \pi\mid.\sim. \tag{6}$$

Colored versions could also be obtained with colored Fock functors [10] (when the space of colors is a commutative and cocommutative, not necessarily unitary bialgebra).

Example 2.5. In $\mathbb{K}[\mathcal{NCP}]$,

$$\begin{aligned}
\delta(\mathsf{I}) &= \mathsf{I} \otimes \mathsf{I},\\
\delta(\sqcup) &= \sqcup \otimes \sqcup,\\
\delta(\mathsf{I}\ \mathsf{I}) &= \mathsf{I}\ \mathsf{I} \otimes \mathsf{I}\cdot\mathsf{I} + \sqcup \otimes \mathsf{I}\ \mathsf{I},\\
\delta(\sqcup\!\sqcup) &= \sqcup\!\sqcup \otimes \sqcup\!\sqcup,\\
\delta(\sqcup\ \mathsf{I}) &= \sqcup\ \mathsf{I} \otimes \sqcup\cdot\mathsf{I} + \sqcup\!\sqcup \otimes \sqcup\ \mathsf{I},\\
\delta(\mathsf{I}\ \sqcup) &= \mathsf{I}\ \sqcup \otimes \sqcup\cdot\mathsf{I} + \sqcup\!\sqcup \otimes \mathsf{I}\ \sqcup,\\
\delta(\overset{\mathsf{I}}{\sqcup}) &= \overset{\mathsf{I}}{\sqcup} \otimes \sqcup\cdot\mathsf{I} + \sqcup\!\sqcup \otimes \overset{\mathsf{I}}{\sqcup},\\
\delta(\mathsf{I}\ \mathsf{I}\ \mathsf{I}) &= \mathsf{I}\ \mathsf{I}\ \mathsf{I} \otimes \mathsf{I}\cdot\mathsf{I}\cdot\mathsf{I} + (\sqcup\ \mathsf{I} + \mathsf{I}\ \sqcup + \overset{\mathsf{I}}{\sqcup}) \otimes \mathsf{I}\cdot\mathsf{I}\ \mathsf{I} + \sqcup\!\sqcup \otimes \mathsf{I}\ \mathsf{I}\ \mathsf{I}.
\end{aligned}$$

This is the second coproduct of [6].

3 Polynomial invariants

3.1 Fundamental polynomial invariants

As a double bialgebra, $\mathbb{K}[\mathcal{NCP}]$ is the symmetric algebra generated by noncrossing partitions, with the two multiplicative coproducts defined by (1) and (6). We denote by $\phi_{\mathcal{NCP}}$ the unique double bialgebra morphism from $\mathbb{K}[\mathcal{NCP}]$ to $\mathbb{K}[X]$, see Theorem 1.4. For any $\pi \in \mathcal{NCP}$,

$$\begin{aligned}
\phi_{\mathcal{NCP}}(\pi) &= \sum_{k=1}^{\infty} \sum_{\substack{\pi = I_1 \sqcup \ldots \sqcup I_k,\\ \forall p \in [k], I_p \sqcup \ldots \sqcup I_k \text{ideal of } \pi}} \epsilon_\delta(\pi_{|I_1})\epsilon_\delta(\pi_{|.I_2}) \ldots \epsilon_\delta(\pi_{|.I_k}) H_k(X)\\
&= \sum_f \epsilon_\delta(\pi_{|f^{-1}(1)})\epsilon_\delta(\pi_{|.f^{-1}(2)}) \ldots \epsilon_\delta(\pi_{|.f^{-1}(\max(f))}) H_{\max(f)}(X),
\end{aligned}$$

where the sum runs over all surjective maps $f : \pi \longrightarrow [\max(f)]$ such that for any $b, b' \in \pi$,

$$b \leqslant_\pi b' \Longrightarrow f(b) \leqslant f(b').$$

Moreover, due to the definition of ϵ_δ, the term corresponding to f is not zero if, and only if, for any $i > 1$, $\pi_{|.f^{-1}(i)}$ is a monomial of noncrossing partitions with only one block each, and $\pi_{|f^{-1}(1)}$ is a noncrossing partition with only one block. If this holds, then the contribution of f is 1. Reformulating:

Proposition 3.1. *For any noncrossing partition π, for any $n \in \mathbb{N}_{>0}$, $\phi_{\mathcal{NCP}}(G)(n)$ is the number of maps $f : \pi \longrightarrow [n]$ such that, for any $b, b' \in \pi$,*

- *If $b \leqslant_\pi b'$ and $b \neq b'$, then $f(b) < f(b')$.*
- *If $f(b) = f(b')$ and $\max(b) < \min(b')$, then there exists $b'' \in \pi$ such that*

$$]\max(b), \min(b')[\cap b'' \neq \varnothing \qquad \textit{and} \qquad f(b'') < f(b) = f(b').$$

Such a map f will be called a valid n-coloration of π.

Example 3.1.

1. A valid n-coloration of $\pi = \lfloor\,|\;|\,\rfloor$ is a map $f : [3] \longrightarrow [n]$ such that

   $$f(1) < f(2), \qquad f(1) < f(3), \qquad f(2) \neq f(3).$$

   Therefore, for any $n \geqslant 1$,

   $$\phi_{\mathcal{NCP}}(\lfloor\,|\;|\,\rfloor)(n) = \frac{n(n-1)(n-2)}{3}.$$

   We obtain that

   $$\phi_{\mathcal{NCP}}(\lfloor\,|\;|\,\rfloor) = \frac{X(X-1)(X-2)}{3}.$$

2. A valid n-coloration of $|\;|\;|$ is a map $f : [3] \longrightarrow [n]$ such that

   $$f(1) \neq f(2), \qquad f(2) \neq f(3), \qquad f(1) = f(3) \Longrightarrow f(2) < f(1).$$

   Therefore, for any $n \geqslant 1$,

   $$\phi_{\mathcal{NCP}}(|\;|\;|)(n) = n(n-1)(n-2) + \frac{n(n-1)}{2} = n(n-1)\left(n - \frac{3}{2}\right).$$

   The first term correspond to the valid n-colorations with $f(1) \neq f(3)$ and the second with $f(1) = f(3)$. We obtain that

   $$\phi_{\mathcal{NCP}}(|\;|\;|) = X(X-1)\left(X - \frac{3}{2}\right).$$

3. See Table 4 at the end of the article for more examples of $\phi_{\mathcal{NCP}}(\pi)$.

Let us now give an inductive way to compute these chromatic polynomials.

Notations 3.1. Let π be a noncrossing partition. We denote by $\mathrm{Base}(\pi)$ the set of blocks of π which are minimal for the partial order $\leqslant_\pi$.

Proposition 3.2. *Let π be a noncrossing partition. Then*

$$\begin{aligned}\phi_{\mathcal{NCP}}(\pi)(X+1) &= \phi_{\mathcal{NCP}}(\pi)\\ &\quad + \sum_{b \in \mathrm{Base}(\pi)} \phi_{\mathcal{NCP}}(\pi_{|[1,\min(b)[})\phi_{\mathcal{NCP}}(\pi_{|.[\min(b),\max(b)]\setminus b})\phi_{\mathcal{NCP}}(\pi_{|]\max(b),\infty[}).\end{aligned}$$

Together with $\phi_{\mathcal{NCP}}(\pi)(0) = 0$, this characterizes $\phi_{\mathcal{NCP}}(\pi)$.

Proof. Let $n \geqslant 1$ and let $f : \pi \longrightarrow [n+1]$ be a $(n+1)$-valid coloration of π. By the increasing condition, $f^{-1}(1)$ is included in $\mathrm{Base}(\pi)$. If it is not empty, by the second condition, it is reduced to a singleton. In the first case, it corresponds, up to a lift of 1, to a n-valid coloration of π. In the second case, denoting $\{b\} = f^{-1}(1)$, it corresponds to a n-valid coloration of $\pi_{|[1,\min(b)[} \cdot \pi_{|[\min(b),\max(b)]\setminus b} \cdot \pi_{|]\max(b),\infty[}$. Summing, we obtain the announced result. $\square$

3.2 Antipode

In order to compute the antipode, we put, for any noncrossing partition π,

$$\mu_{\mathcal{NCP}}(\pi) = \phi_{\mathcal{NCP}}(\pi)(-1).$$

This defines a character of $\mathbb{K}[\mathcal{NCP}]$. Theorem 1.3 states that the antipode of $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ is

$$S = (\mu_{\mathcal{NCP}} \otimes \mathrm{Id}_{\mathbb{K}[\mathcal{NCP}]}) \circ \delta.$$

Notations 3.2. We denote by $\mathrm{cat}_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the n-th Catalan number.

n	0	1	2	3	4	5	6	7	8	9	10
cat_n	1	1	2	5	14	42	132	429	1430	4862	16796

Recall that for any $k \geqslant 1$,

$$\mathrm{cat}_k = \sum_{i=1}^{k} \mathrm{cat}_{i-1}\mathrm{cat}_{k-i}. \tag{7}$$

The number of noncrossing partitions of $[n]$ is cat_n. The formal series of Catalan numbers is

$$C = \sum_{n=0}^{\infty} \mathrm{cat}_n X^n = \frac{1-\sqrt{1-4X}}{2X}.$$

For more details, see Entry A000108 of the OEIS [15].

Proposition 3.3. *The character $\nu_{\mathcal{NCP}}$ on $\mathbb{K}[\mathcal{NCP}]$ can be computed by induction on the number of blocks, putting for any noncrossing partition π of $[n]$,*

$$\mu_{\mathcal{NCP}}(\pi) = (-1)^{|\mathrm{Base}(\pi)|}\mathrm{cat}_{|\mathrm{Base}(\pi)|}\mu_{\mathcal{NCP}}\left(\pi_{|.[n]\setminus \mathrm{Base}(\pi)}\right).$$

The antipode S of $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ is given on any noncrossing partition π by

$$S(\pi) = \sum_{\sim \in \mathcal{E}_c[\pi]} \mu_{\mathcal{NCP}}(\pi/\sim)\pi\mid_{.}\sim.$$

Proof. We evaluate the formula of Proposition 3.2 in $X = -1$. We obtain

$$\mu_{\mathcal{NCP}}(\pi) = -\sum_{b \in \mathrm{Base}(\pi)} \mu_{\mathcal{NCP}}(\pi_{|[1,\min(b)[})\mu_{\mathcal{NCP}}(\pi_{|.[\min(b),\max(b)]\setminus b})\mu_{\mathcal{NCP}}(\pi_{|]\max(b),\infty[}).$$

We prove the formula for $\mu_{\mathcal{NCP}}(\pi)$ by induction on $k = |\mathrm{Base}(\pi)|$. We put $\mathrm{Base}(\pi) = \{b_1, \ldots, b_k\}$ with $\max(b_i) + 1 = \min(b_{i+1})$ for any $i \in [k-1]$. If $k = 1$, then $\mu_{\mathcal{NCP}}(\pi_{|\mathrm{Base}(pi)}) = -1$, which immediately gives the result. If $k \geqslant 2$, then, by the induction hypothesis,

$$\begin{aligned}
\mu_{\mathcal{NCP}}(\pi) &= -\sum_{i=1}^{k} \mu_{\mathcal{NCP}}(\pi_{|[1,\max(b_{i-1})]})\mu_{\mathcal{NCP}}(\pi_{|.[\min(b_i),\max(b_i)])\setminus b_i})\mu_{\mathcal{NCP}}(\pi_{|[\min(b_{i+1},+\infty[})\\
&= -\sum_{i=1}^{k} (-1)^{i-1}\mathrm{cat}_{i-1}\mu_{\mathcal{NCP}}(\pi_{|.[1,\max(b_{i-1})]\setminus(b_1\cup\ldots\cup b_{i-1})})\mu_{\mathcal{NCP}}(\pi_{|.[\min(b_i),\max(b_i)])\setminus b_i})\\
&\qquad (-1)^{k-i}\mathrm{cat}_{k-i}\mu_{\mathcal{NCP}}(\pi_{|.[\min(b_{i+1},+\infty[\setminus(b_{i+1}\cup\ldots\cup b_k)})\\
&= (-1)^k \mu_{\mathcal{NCP}}\left(\pi_{|.[n]\setminus\mathrm{Base}(\pi)}\right)\sum_{i=1}^{k}\mathrm{cat}_{i-1}\mathrm{cat}_{k-i}\\
&= (-1)^k \mathrm{cat}_k \mu_{\mathcal{NCP}}\left(\pi_{|.[n]\setminus\mathrm{Base}(\pi)}\right).
\end{aligned}$$

We used (7) for the last equality. □

Remark 3.1. A direct induction proves that the sign of $\mu_{\mathcal{NCP}}(\pi)$ is $(-1)^{|\pi|}$.

See Table 4 at the end of the article for examples of $\mu_{\mathcal{NCP}}(\pi)$.

Example 3.2. Here are a few examples of antipodes.

$$\begin{aligned}
S(\mid\,\mid) &= -\mid\,\mid + 2 \mid \cdot \mid,\\
S(\mid\,\mid\,\mid) &= -\mid\,\mid\,\mid + (1+2+2) \mid \cdot \mid\,\mid - 5 \mid \cdot \mid \cdot \mid\\
&= -\mid\,\mid\,\mid + 5 \mid \cdot \mid\,\mid - 5 \mid \cdot \mid \cdot \mid,\\
S(\mid\,\mid\,\mid\,\mid) &= -\mid\,\mid\,\mid\,\mid + (2+1) \mid \cdot \mid\,\mid - (5+5+5+2+2+2) \mid \cdot \mid \cdot \mid\,\mid\\
&\quad + (2+1+1+2) \mid \cdot \mid\,\mid\,\mid + 14 \mid \cdot \mid \cdot \mid \cdot \mid\\
&= -\mid\,\mid\,\mid\,\mid + 3 \mid \cdot \mid\,\mid - 21 \mid \cdot \mid \cdot \mid\,\mid + 6 \mid \cdot \mid\,\mid\,\mid + 14 \mid \cdot \mid \cdot \mid \cdot \mid.
\end{aligned}$$

3.3 More results on $\phi_{\mathcal{NCP}}$ for basic noncrossing partitions

Let us now give a way to compute the polynomial $\phi_{\mathcal{NCP}}(\pi)$, when $\pi = \mathrm{Base}(\pi)$.

Proposition 3.4. *Let π be a nonempty noncrossing partition such that $\pi = \mathrm{Base}(\pi)$. Then $\phi_{\mathcal{NCP}}(\pi)$ is a polynomial $P_{|\pi|}$ which depends uniquely on $|\pi|$, characterized by $P_0(X) = 1$ and*

$$\forall k \geqslant 1, \qquad \begin{cases} P_k(X+1) = P_k(X) + \sum_{i=1}^{k} P_{i-1}(X) P_{k-i}(X), \\ P_k(0) = 0. \end{cases} \tag{8}$$

Proof. This is a direct consequence of Proposition 3.2: if b is the i-th block of π (totally ordered from left to right), then $\pi_{|[1,\min(b)[}$ is a partition π' with $i-1$ blocks such that $\mathrm{Base}(\pi') = \pi'$, $\pi_{|[\min(b),\max(b)]\setminus b} = 1$ and $\pi_{|]\max(b),\infty[}$ a partition π'' with $|\pi| - i$ blocks such that $\mathrm{Base}(\pi'') = \pi''$. $\square$

Notations 3.3. For any $n \geqslant 1$, we denote by J_n the noncrossing partition $\{\{1\}, \ldots, \{n\}\}$:

$$J_1 = \mid, \qquad J_2 = \mid\,\mid, \qquad J_3 = \mid\,\mid\,\mid, \qquad J_4 = \mid\,\mid\,\mid\,\mid, \ldots$$

By their combinatorial nature, it is natural to decompose these polynomials P_n in the basis of Hilbert polynomials $(H_i(X))_{i\geqslant 0}$. We put

$$P_n(X) = \sum_{i=1}^{\infty} a_{i,n} H_i(X). \tag{9}$$

By construction of $P_n(X) = \phi_{\mathcal{NCP}}(J_n)$, the scalar $a_{i,n}$ is the number of surjective valid n-colorings $c : [n] \longrightarrow [i]$ of J_n, that is to say the number of surjective maps $c : [n] \longrightarrow [i]$ such that

$$\forall p, r \in [n], \qquad (p < r \text{ and } c(p) = c(r)) \Longrightarrow (\exists q \in]p, r[,\ c(q) < c(p)).$$

The following proposition allows to compute inductively these coefficients $a_{i,n}$:

Proposition 3.5. *Let $1 \leqslant i \leqslant n$. The coefficients $a_{i,n}$ defined by (9) can be computed by induction on i by*

$$a_{i,n} = \begin{cases} \delta_{n,1} \text{ if } i=1, \\ \sum_{k=1}^{[\frac{n+1}{2}]} \binom{n-k+1}{k} a_{i-1,n-k} \text{ if } i \geqslant 2. \end{cases}$$

Proof. Let $b_{k,n}$ be the number of parts I of $[n]$ with k elements such that for all $i,j \in I$, $|i-j| \neq 1$. This condition will be called the b-condition. Then

$$\begin{aligned} b_{k,l} &= |\{(i_1,\ldots,i_k) \in \mathbb{N}^k, 1 \leqslant i_1 < i_1+1 < i_2 < \ldots < i_{k-1}+1 < i_k \leqslant n\}| \\ &= |\{(j_1,\ldots,j_k) \in \mathbb{N}^k, 1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n-k+1\}| \\ &= \binom{n-k+1}{k}, \end{aligned}$$

with $j_p = i_p - p + 1$ for any p. Note that this coefficient is zero if $k > \left[\frac{n+1}{2}\right]$. If $i = 1$, then $a_{i,n} = 1$ if $n = 1$, and 0 otherwise. If $i \geqslant 2$, for any surjective map $c : [n] \longrightarrow [i]$,

- $c^{-1}(i)$ is a nonempty part I of $[n]$, satisfying the b-condition.
- Identifying $(J_n)_{|[n]\setminus I}$ with J_{n-k}, where $k = |I|$, the restriction of c to this noncrossing partitions is an element of $PV(J_{n-k})$, of maximum $i-1$.

Hence, $a_{i,n} = \sum_{k=1}^{n} b_{k,n} a_{i-1,n-k} = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n-k+1}{k} a_{i-1,n-k}$. □

Proposition 3.6. *Let $i, n \geqslant 1$. If $i > n$, then $a_{i,n} = 0$. If $n \geqslant 2^i$, then $a_{i,n} = 0$.*

Proof. The combinatorial interpretation of $a_{i,n}$ (as particular surjective maps from $[n]$ to $[i]$) implies that if $i > n$, $a_{i,n} = 0$. Let us consider

$$\mathcal{P}(X,Y) = \sum_{k=0}^{\infty} P_k(X) Y^k \in \mathbb{Q}[X][[Y]].$$

Then (8) implies that

$$\mathcal{P}(X+1,Y) = \mathcal{P}(X,Y) + Y\mathcal{P}(X,Y)^2.$$

Moreover, $\mathcal{P}(0,Y) = 1$. An easy induction then proves that $\mathcal{P}(N,Y)$ is a polynomial of $\mathbb{Q}[Y]$ of degree $2^N - 1$ for any $N \in \mathbb{N}$. Hence, $P_n(N) = 0$ if $n \geqslant 2^N$. This gives, if $n \geqslant 2^N$,

$$P_n(N) = \sum_{i=1}^{\infty} a_{i,n} H_i(N) = \sum_{i=0}^{N} a_{i,n} \binom{N}{i} + 0 = 0.$$

As the coefficients $a_{i,n}$ belong to $\mathbb{N}$, we obtain that $a_{i,n} = 0$ if $i \leqslant N$ and $n \geqslant 2^N$. In other words, if $n \geqslant 2^i$, then $a_{i,n} = 0$. □

Example 3.3. The following array contains the first values of $a_{i,n}$:

$i\backslash n$	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	0	2	1	0	0	0	0	0	0	0
3	0	0	6	10	8	4	1	0	0	0
4	0	0	0	24	86	172	254	302	298	244
5	0	0	0	0	120	756	2734	7484	17164	34612
6	0	0	0	0	0	720	7092	40148	172168	621348
7	0	0	0	0	0	0	5040	71856	585108	3589360
8	0	0	0	0	0	0	0	40320	787824	8720136
9	0	0	0	0	0	0	0	0	362880	9329760
10	0	0	0	0	0	0	0	0	0	3628800

It seems that these coefficients are not so well known. For example, no entry of the OEIS [15] contains the term $172168 = a_{6,9}$. However, we can obtain explicit formulas for some of these terms, involving multiple nested harmonic sums.

Notations 3.4. if $n, k_1, \ldots, k_p \geqslant 1$, we put $\zeta_n(k_1, \ldots, k_p) = \displaystyle\sum_{1 \leqslant n_1 < \ldots < n_p \leqslant n} \frac{1}{n_1^{k_1} \ldots n_p^{k_p}}$.

Corollary 3.7.

$$\begin{aligned}
&\forall n \geqslant 1, && a_{n,n} = n!,\\
&\forall n \geqslant 2, && a_{n-1,n} = n!\left(\frac{n+1}{2} - \zeta_n(1)\right),\\
&\forall n \geqslant 3, && a_{n-2,n} = n!\left(\zeta_n(1,1) - \frac{n}{2}\zeta_n(1) + \frac{(3n-2)(n^2+n+6)}{24n}\right).
\end{aligned}$$

The sequence $(a_{n-1,n})_{n \geqslant 2}$ is Entry A31853 in the OEIS [15].

Proof. For any $n \geqslant 2$, by Proposition 3.5,

$$a_{n,n} = \binom{n}{1} a_{n-1,n-1} + 0 = n a_{n-1,n-1},$$

as, if $k \geqslant 2$, $a_{n-1,n-k} = 0$. Consequently, $a_{n,n} = n!$.

For any $n \geqslant 3$, by Proposition 3.5,

$$\begin{aligned}
a_{n-1,n} &= \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n-k+1}{k} a_{n-2,n-k}\\
&= \binom{n}{1} a_{n-2,n-1} + \binom{n-1}{2} a_{n-2,n-2} + 0\\
&= n a_{n-2,n-1} + \frac{(n-1)!(n-2)}{2}.
\end{aligned}$$

Therefore, if $n \geqslant 3$,

$$\frac{a_{n-1,n}}{n!} = \frac{a_{n-2,n-1}}{(n-1)!} + \frac{1}{2} - \frac{1}{n}.$$

As $a_{1,2} = 0$, the result follows by directly.

If $n \geqslant 4$, by Proposition 3.5,

$$\begin{aligned}
\frac{a_{n-2,n}}{n!} &= \frac{1}{n!}\left(\binom{n}{1} a_{n-3,n-1} + \binom{n-1}{2} a_{n-3,n-2} + \binom{n-2}{3} a_{n-3,n-3}\right)\\
&= \frac{a_{n-3,n-1}}{(n-1)!} + \frac{n-2}{2n}\frac{a_{n-3,n-2}}{(n-2)!} + \frac{(n-3)(n-4)}{6n(n-1)}\frac{a_{n-3,n-3}}{(n-3)!}.
\end{aligned}$$

As $a_{1,3} = 0$, and with the preceding formulas, we deduce that

$$\frac{a_{n-2,n}}{n!} = \sum_{k=4}^{n} \left(\frac{1}{2} - \frac{1}{k}\right)\left(\frac{k-1}{2} - \zeta_{k-2}(1)\right) + \sum_{k=4}^{n} \frac{(k-3)(k-4)}{6k(k-1)}.$$

The result then follows by tedious manipulations of sums. $\square$

Let us now give a way to compute the coefficients of $P_n(X)$ in the basis of monomials $(X^k)_{k \in \mathbb{N}}$.

Proposition 3.8. *Let $M = (M_{k,l})_{k,l\geqslant 1}$ be the infinite matrix defined by*

$$M_{k,l} = \begin{cases} \binom{l}{k-l} \text{ if } k \geqslant l, \\ 0 \text{ if } k < l. \end{cases}$$

Then

$$\begin{pmatrix} P_0(X) \\ P_1(X) \\ P_2(X) \\ \vdots \end{pmatrix} = e^{X\ln(M)} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Proof. For any $n \in \mathbb{N}$, we put

$$P_n(X) = \sum_{k=0}^{\infty} b_{i,n} X^i,$$

and we consider the infinite vectors $B^{(i)} = (b_{i,n-1})_{n\geqslant 1}$. Then $B^{(0)} = (\delta_{1,n})_{n\geqslant 1}$, and for any $n \geqslant 1$, From Proposition 3.5, if $i \geqslant 2$,

$$\begin{aligned} a_{i,n} &= \sum_{k=1}^{n-1} \binom{n-k+1}{k} a_{i-1,n-k} \\ &= \sum_{k=1}^{n-1} \binom{k+1}{n-k} a_{i-1,k} \\ &= \sum_{k=0}^{n} \binom{k+1}{n-k} a_{i-1,k} - a_{i-1,n}. \end{aligned}$$

If $A^{(i)}$ is the infinite vector $(a_{i,n-1})_{n\geqslant 1}$, we obtain that if $n \geqslant 2$, then $A^{(i)} = (M-I)A^{(i-1)}$, which is still true if $i = 1$. Therefore, for any $i \geqslant 0$, $A^{(i)} = (M-I)^i A^{(0)}$, where $A^{(0)} = (\delta_{1,n})_{n\geqslant 1}$.

Let us now compute the coefficient of X^i in $H_k(X)$ for any k, n. We consider the formal series

$$f(X,Y) = \sum_{k\geqslant 0} H_k(X) Y^k = \sum_{k\geqslant 0} \binom{X}{k} Y^k = (1+Y)^X \in \mathbb{Q}[[X,Y]].$$

Therefore,

$$\sum_{k\geqslant 0} \frac{d^i H_k}{dX^i}(0) Y^k = \frac{\partial^i f}{\partial X^i}(0,Y) = \left(\ln(1+Y)^i (1+Y)^X\right)_{|X=0} = \ln(1+Y)^i.$$

Let us put $\ln(1+Y)^i = \sum_k \gamma_{i,k} Y^k$. As for any $n \geqslant 1$, $P_n = \sum_{k=1}^{\infty} a_{k,n} H_k(X)$, the coefficient of X^i in P_n is

$$b_{i,n} = \sum_{k=1}^{\infty} \frac{1}{i!} \gamma_{i,k} a_{k,n}.$$

In other words,

$$B^{(i)} = \frac{1}{i!} \sum_{k=1}^{\infty} \gamma_{i,k} A^{(k)} = \frac{1}{i!} \left(\sum_{k=1}^{\infty} \gamma_{i,k} (M-I)^k \right) B^{(0)} = \frac{1}{i!} \ln(M)^i B^{(0)}.$$

We then obtain

$$\begin{pmatrix} P_0(X) \\ P_1(X) \\ \vdots \end{pmatrix} = \sum_{i=0}^{\infty} B^{(i)} X^i = \left(\sum_{i=0}^{\infty} \frac{X^i}{i!} \ln(M)^i X^i \right) B^{(0)} = e^{X\ln(M)} B^{(0)}. \qquad \square$$

Remark 3.2. M is the Riordan array of $(1+X, X(1+X))$, see Entries A005119 and A030528 of the OEIS. Consequently, the first column of $\ln(M)$, that is to say $B^{(1)}$, gives the coefficients of the infinitesimal generator of $X(1+X)$, up to signs and multiplications by factorials, giving Entry A005119 of the OEIS.

Example 3.4.

$$\begin{aligned}
P_1 &= X,\\
P_2 &= X^2 - X,\\
P_3 &= X^3 - \frac{5X^2}{2} + \frac{3X}{2},\\
P_4 &= X^4 - \frac{13X^3}{3} + 6X^2 - \frac{8X}{3},\\
P_5 &= X^5 - \frac{77X^4}{12} + \frac{89X^3}{6} - \frac{175X^2}{12} + \frac{31X}{6},\\
P_6 &= X^6 - \frac{87X^5}{10} + \frac{175X^4}{6} - \frac{281X^3}{6} + \frac{215X^2}{6} - \frac{157X}{15},\\
P_7 &= X^7 - \frac{223X^6}{20} + \frac{1501X^5}{30} - 115X^4 + \frac{851X^3}{6} - \frac{1767X^2}{20} + \frac{649X}{30}.
\end{aligned}$$

Remark 3.3. It could be conjectured seeing these examples that P_n is a polynomial with alternating signs, as are the chromatic polynomials for graphs. This is true till $n = 28$ but fails for $n = 29$:

$$P_{29} = X^{29} - \frac{6897956948587}{80313433200}X^{28} + \cdots - \frac{14277306976985617719653}{2679791554440}X^2 - \frac{37449570182565026}{37182145}X.$$

Proposition 3.9. *Let $k, n \geqslant 1$. Then $P_k(n) = 0$ if, and only if, $k > 2^n - 1$.*

Proof. $\Longrightarrow$. Let us assume that $k \leqslant 2^n - 1$ and let us prove that $P_k(n) \neq 0$. We define a sequence $D_n = (d_{n,1}, \ldots, d_{n,2^n-1})$ of length $2^n - 1$ for any $n \geqslant 1$ by the following process: $D_1 = (1)$ and for any $n \geqslant 2$,

$$D_n = (n, d_{n-1,1}, n, d_{n-1,2}, n, \ldots, n, d_{n-1,2^{n-1}-1}, n).$$

For example,

$$\begin{aligned}
D_2 &= (2,1,2),\\
D_3 &= (3,2,3,1,3,2,3),\\
D_4 &= (4,3,4,2,4,3,4,1,4,3,4,2,4,3,4).
\end{aligned}$$

In the sequence D_n, by construction, if $i < k$ and $d_{n,i} = d_{n,k}$, then there exists j such that $i < j < k$ and $d_{n,j} < d_{n,i}$. So D_n is a n-valid coloration of J_{2^n-1}. By restriction to its first k letters, it is a n-valid coloration of J_k, so $P_k(n) \neq 0$.

$\Longleftarrow$. Let us assume that $P_k(n) \neq 0$ and let us prove that $k \leqslant 2^n - 1$. By hypothesis, there exists a valid n-coloration f of J_n. Let us prove by induction that $|f^{-1}(i)| \leqslant 2^{i-1}$ by induction on i. If $i = 1$, by the b condition, at most one element of $[k]$ is sent by f to 1, so $|f^{-1}(1)| \leqslant 1$. Let us assume the result at all ranks $< i$. By the b-condition, two elements sent by f to i are separated by at least one element of $f^{-1}([i-1])$. By the induction hypothesis,

$$|f^{-1}([i-1])| \leqslant 2^0 + \ldots + 2^{i-2} = 2^{i-1} - 1.$$

So, there can be at most 2^{i-1} elements in $f^{-1}(i)$. Finally,

$$k = \sum_{i=1}^{n} |f^{-1}(i)| \leqslant \sum_{i=1}^{n} 2^{i-1} \leqslant 2^n - 1. \qquad \square$$

Remark 3.4. In other term, the chromatic number of J_k, that is to say the minimal n such that J_k has a valid n-coloration, is the smallest integer greater that or equal to $\log_2(k+1)$.

3.4 Homogeneous polynomial invariants

The number of blocks of noncrossing partitions induces a graduation of $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ (notice that δ is not homogeneous for this graduation). It is connected (that is to say $\mathbb{K}[\mathcal{NCP}]_0 = \mathbb{K}$), but its other homogeneous components are not finite dimensional. With the help of [9, Propositions 3.10 and 5.2], we can associate a homogeneous morphism $\phi_0 : (\mathbb{K}[\mathcal{NCP}], m, \Delta) \longrightarrow (\mathbb{K}[X], m, \Delta)$ to the element $\mu \in \mathbb{K}[\mathcal{NCP}]_1^*$ such that $\mu(\pi) = 1$ for any noncrossing partition π with only one block (and μ takes the value 0 on any other monomial in noncrossing partitions).

Definition 3.10. *Let π be a noncrossing partition with n blocks. We denote by* $\mathrm{le}(\pi)$ *the number of linear extensions of $(\pi, \leqslant_\pi)$, that is to say bijections $f : \pi \longrightarrow [n]$ such that*

$$\forall b, b' \in \pi, \qquad b \leqslant_\pi b' \Longrightarrow f(b) \leqslant f(b').$$

By construction of the coproduct Δ, for any noncrossing partition π with n blocks,

$$\mu^{\otimes n} \circ \Delta^{(n-1)}(\pi) = \mathrm{le}(\pi).$$

Therefore, the morphism associated to μ is given by the following Proposition:

Proposition 3.11. *Let $\phi_0 : \mathbb{K}[\mathcal{NCP}] \longrightarrow \mathbb{K}[X]$ be the algebra morphism such that*

$$\forall \pi \in \mathcal{NCP}, \qquad \phi_0(\pi) = \frac{\mathrm{le}(\pi)}{|\pi|!} X^{|\pi|}.$$

Then ϕ_0 is a bialgebra morphism from $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$. Considering the character $\lambda_0 = \epsilon_\delta \circ \phi_0$,

$$\forall \pi \in \mathcal{NCP}, \qquad \lambda_0(\pi) = \frac{\mathrm{le}(\pi)}{|\pi|!}.$$

Moreover, with the notations of Proposition 1.2 and Theorem 1.4, $\phi_0 = \phi_{\mathcal{NCP}} \leftsquigarrow \lambda_0$.

See Table 4 at the end of the article for examples of $\lambda_0(\pi)$.

3.5 Invertible characters

In order to study invertible characters of $(\mathbb{K}[\mathcal{NCP}], m, \delta)$, let us introduce a second graduation. Let $\pi = \pi_1 \cdot \ldots \cdot \pi_k$ be a monomial in noncrossing partitions. The length of π is k and is denoted by $\mathrm{lg}(\pi)$. The number of blocks of π is

$$|\pi| = |\pi_1| + \ldots + |\pi_k|.$$

Finally, the degree of π is

$$\deg(\pi) = |\pi| - \mathrm{lg}(\pi).$$

Note that this belongs to $\mathbb{N}$.

Lemma 3.12. *With this degree, $(\mathbb{K}[\mathcal{NCP}], m, \delta)$ is a graded bialgebra. In particular, $\mathbb{K}[\mathcal{NCP}]_0$ is the subalgebra generated by noncrossing partitions with only one block.*

Proof. Let π, π' be monomials in noncrossing partitions. Then

$$|\pi \cdot \pi'| = |\pi| + |\pi'|, \qquad \mathrm{lg}(\pi \cdot \pi') = \mathrm{lg}(\pi) + \mathrm{lg}(\pi').$$

we obtain that $\deg(\pi \cdot \pi') = \deg(\pi) + \deg(\pi')$: the product is homogeneous. Let $\pi = \pi_1 \cdots \pi_k$ be a monomial in noncrossing partitions, and let $\sim_i \in \mathcal{E}_c[\pi_i]$ for any i. Then, by definition of $\mathcal{E}_c[\pi_i]$,

$$\begin{aligned}
|\pi_1 \mid_{\sim_1}| + \ldots + |\pi_k \mid_{\sim_k}| &= |\pi_1| + \ldots + |\pi_k| = |\pi|,\\
\mathrm{lg}(\pi_1 \mid_{\sim_1}) + \ldots + \mathrm{lg}(\pi_k \mid_{\sim_k}) &= \mathrm{cl}(\sim_1) + \ldots + \mathrm{cl}(\sim_k),\\
|\pi_1/\sim_1| + \ldots + |\pi_k/\sim_k| &= \mathrm{cl}(\sim_1) + \ldots + \mathrm{cl}(\sim_k),\\
\mathrm{lg}(\pi_1/\sim_1) + \ldots + \mathrm{lg}(\pi_k/\sim_k) &= k,
\end{aligned}$$

where $\mathrm{cl}(\sim_i)$ is the number of classes of $\sim_i$. Combining these equalities, we obtain that

$$\deg(\pi_1/\sim_1 \cdots \pi_k/\sim_k) + \deg(\pi_1 \mid_{\cdot\sim_1} \cdots \pi_k \mid_{\cdot\sim_k}) = |\pi| - k = \deg(\pi).$$

Therefore, the coproduct δ is homogeneous. □

Proposition 3.13. *Let λ be a character of $(\mathbb{K}[\mathcal{NCP}], m)$. It is invertible for the convolution product $\star$ associated to δ if, and only if, for any noncrossing partition π with only one block, $\lambda(\pi) \neq 0$.*

Proof. We apply [12, Proposition 3.9] with the preceding graduation, with the family of group-like elements of noncrossing partitions with only one block. □

In particular, for any noncrossing partition π with only one block, $\lambda_0(\pi) = 1$, so λ_0 is invertible. Its inverse is denoted by $\lambda_{\mathcal{NCP}}$. Therefore,

$$\phi_{\mathcal{NCP}} = \phi_0 \leftsquigarrow \lambda_{\mathcal{NCP}}. \tag{10}$$

Consequently:

Corollary 3.14. *For any noncrossing partition π,*

$$\phi_{\mathcal{NCP}}(\pi) = \sum_{\sim \in \mathcal{E}_c[\pi]} \lambda_{\mathcal{NCP}}(\pi \mid_{\cdot\sim}) \frac{\mathrm{le}(\pi/\sim)}{\mathrm{cl}(\sim)!} X^{\mathrm{cl}(\sim)}.$$

In particular, the degree of $\phi_{\mathcal{NCP}}(\pi)$ is $|\pi|$, $\lambda_{\mathcal{NCP}}(\pi)$ is the coefficient of X in $\phi_{\mathcal{NCP}}(\pi)$ and $\lambda_0(\pi)$ is the coefficient of $X^{|\pi|}$ in $\phi_{\mathcal{NCP}}(\pi)$.

Proof. The formula for $\phi_{\mathcal{NCP}}$ is a direct consequence of (10). If $\sim \in \mathcal{E}_c[\pi]$, then $\mathrm{cl}(\sim) \leqslant |\pi|$, so the degree of $\phi_{\mathcal{NCP}}(\pi)$ is smaller than $|\pi|$. There is only one element $\sim_1 \in \mathcal{E}_c[\pi]$ such that $\mathrm{cl}(\sim_0) = |\pi|$: it is the equality of π. Moreover, $\pi \mid \cdot \sim_1$ is a product of noncrossing partitions with a unique block, so $\lambda_{\mathcal{NCP}}(\pi \mid_{\cdot\sim_1}) = 1$, whereas $\pi/\sim_1 = \pi$. Therefore, the coefficient of $X^{|\pi|}$ is $\phi_{\mathcal{NCP}}(\pi)$ is $\lambda_0(\pi)$, which is non zero: the degree of $\phi_{\mathcal{NCP}}(\pi)$ is $|\pi|$.

There is only one element $\sim_0 \in \mathcal{E}_c[\pi]$ such that $\mathrm{cl}(\sim_0) = 1$: for any $b, b' \in \pi$, $b \sim_0 b'$. Moreover, $\pi \mid_{\cdot\sim_0} = \pi$ and $\pi/\sim_0$ has only one block, so $\dfrac{\mathrm{le}(\pi/\sim_0)}{\mathrm{cl}(\sim_0)!} = 1$: consequently, the coefficient of X in $\phi_{\mathcal{NCP}}(\pi)$ is $\lambda_{\mathcal{NCP}}(\pi)$. □

Let us now give an interpretation of the coefficient of $X^{|\pi|-1}$ in $\phi_{\mathcal{NCP}}(\pi)$.

Definition 3.15. *Let π a noncrossing partition and let b, b' be two different blocks of π.*

- *We shall say that (b, b') is a close pair if* $\max(b) < \min(b')$ *and* $[\max(b)+1, \min(b')-1]$ *is a union of blocks of π.*
- *We shall say that (b, b') is a nested pair if $b \leqslant_\pi b'$ and if for any $b'' \in \pi$,*

$$b \leqslant_\pi b'' \leqslant_\pi b' \Longrightarrow b'' = b \text{ or } b'' = b'.$$

If (b, b') is a close or a nested pair of π, we denote by $\pi/(b, b')$ the partition which blocks are $b \sqcup b'$ and the other blocks of π. By definition of close and nested pairs, it is a noncrossing partition.

Corollary 3.16. *Let π be a noncrossing partition with n blocks. Then $\phi_{\mathcal{NCP}}(\pi)$ is a polynomial of degree n. The coefficient of X^{n-1} is*

$$-\frac{1}{(|\pi|-1)!}\left(\sum_{(b,b') \text{ close pair of } \pi} \mathrm{le}(\pi/(b,b')) + \frac{1}{2} \sum_{(b,b') \text{ nested pair of } \pi} \mathrm{le}(\pi/(b,b'))\right).$$

Proof. Let $\sim\in \mathcal{E}_c[\pi]$ such that $\mathrm{cl}(\sim) = |\pi| - 1$. There exists exactly one class of $\sim$ made of two blocks b and b', and the other classes are made of only one blocks. Two possibilities can occur:

- Up to a permutation of b and b', (b, b') is a close pair. Then $\pi/\sim = \pi/(b, b')$. Moreover, $\pi\mid_{\cdot}\sim$ is a monomial made of noncrossing partitions with only one block and a noncrossing partition with two non nested blocks. Hence, $\lambda_0(\pi\mid_{\cdot}\sim) = 1$. As $\pi\mid_{\cdot}\sim$ is a skew-primitive element for δ, $\lambda_{\mathcal{NCP}}(\pi\mid_{\cdot}\sim) = -1$.

- Up to a permutation of b and b', (b, b') is a nested pair. Then $\pi/\sim = \pi/(b, b')$. Moreover, $\pi\mid_{\cdot}\sim$ is a monomial made of noncrossing partitions with only one block and a noncrossing partition with two nested blocks. Hence, $\lambda_0(\pi\mid_{\cdot}\sim) = \frac{1}{2}$. As $\pi\mid_{\cdot}\sim$ is a skew-primitive element for δ, $\lambda_{\mathcal{NCP}}(\pi\mid_{\cdot}\sim) = -\frac{1}{2}$.

The result on the coefficient of $X^{|\pi|-1}$ immediately follows. □

Example 3.5. 1. Let us consider J_n (see Notation 3.3), with $n \geqslant 2$. Then $P_n = \phi_{\mathcal{NCP}}(J_n)$ is a monic polynomial of degree n, as $\lambda_0(J_n) = 1$. Moreover, J_n has no nested pair, any any (i, j) with $1 \leqslant i < j \leqslant n$ is a close pair of J_n. Therefore, the coefficient of X^{n-1} in J_n is

$$b_{n-1,n} = -\sum_{1\leqslant i<j\leqslant n} \frac{1}{j-i} = \sum_{1\leqslant i\leqslant n-1} \sum_{k=1}^{n-i} \frac{1}{k} = \sum_{1\leqslant j\leqslant n-1} \sum_{k=1}^{j} \frac{1}{k} = n\zeta_n(1) - n.$$

Consequently, $((n-1)!b_{n-1,n})_{n\geqslant 0}$ is the sequence of generalized Stirling numbers, see Entry A001705 in the OEIS [15].

2. See Table 4 at the end of the article for examples of $\lambda_{\mathcal{NCP}}(\pi)$.

Let us give more results about $\lambda_{\mathcal{NCP}}(J_n)$.

Proposition 3.17. *The sequence $((n-1)!\lambda_{\mathcal{NCP}}(J_n))_{n\geqslant 1}$ is (up to the signs) the infinitesimal generator of $X(1 + X)$, see Entries A005119 and A179199 of the OEIS [15]. Consequently,*

$$\forall n \geqslant 2, \qquad \lambda_{\mathcal{NCP}}(J_n) = -\frac{1}{n-1} \sum_{i=1}^{[\frac{n}{2}]} \binom{n-i+1}{i+1} \lambda_{\mathcal{NCP}}(J_{n-i}).$$

Proof. This is a direct consequence of Proposition 3.8, Remark 3.2 and the recursive formula given in Entry A005119 of the OEIS. □

Example 3.6.

n	1	2	3	4	5	6	7	8	9	10
$\lambda_{\mathcal{NCP}}(J_n)$	1	-1	$\frac{3}{2}$	$-\frac{8}{3}$	$\frac{31}{6}$	$-\frac{157}{15}$	$\frac{649}{30}$	$-\frac{9427}{210}$	$\frac{19423}{210}$	$-\frac{6576}{35}$

Remark 3.5. It could be conjectured from these values that the sign of $\lambda_{\mathcal{NCP}}(J_n)$ is $(-1)^{n+1}$ for any n. This is false. The first counterexample is

$$\lambda_{\mathcal{NCP}}(J_{29}) = -\frac{37449570182565026}{37182145}.$$

Other counterexamples can be found for $n = 30, 33, 34, 38, 39, 42, 43, 47, 48, 51, 52, 55, 56$.

3.6 Linear extensions

Let us first generalize some combinatorial notions from noncrossing partitions to monomial of noncrossing partitions.

Definition 3.18. *Let $P = \pi_1 \cdot \ldots \cdot \pi_k$ be a monomial in noncrossing partitions. The set of blocks of P is*

$$\mathrm{Bl}(P) = \pi_1 \sqcup \ldots \sqcup \pi_k.$$

We also put $|P| = |\mathrm{Bl}(P)|$. The set $\mathrm{Bl}(P)$ is partially ordered by $\leqslant_P = \leqslant_{\pi_1} \sqcup \ldots \sqcup \leqslant_{\pi_k}$. An ideal of P is a subset I of $\mathrm{Bl}(P)$ such that

$$\forall b, b' \in \mathrm{Bl}(P), \qquad (b \in I \text{ and } b \leqslant_P b') \Longrightarrow (b' \in I).$$

With this definition, we can write, for any monomial in noncrossing partitions,

$$\Delta(P) = \sum_{J \text{ ideal of } P} P_{|\mathrm{Bl}(P)\setminus J} \otimes P_{|.J}.$$

Definition 3.19. *Let P be a monomial in noncrossing partitions and let $n \geqslant 1$.*

1. *A n-linear extension of P is a map $f : \mathrm{Bl}(P) \longrightarrow [n]$ such that*

$$\forall b, b' \in \mathrm{Bl}(P), \qquad b \leqslant_P b' \Longrightarrow f(b) \leqslant f(b').$$

The set of n-linear extensions of P is denoted by $\mathrm{Lin}(P, n)$.

2. *A strict n-linear extension of P is a map $f : \mathrm{Bl}(P) \longrightarrow [n]$ such that*

$$\forall b, b' \in \mathrm{Bl}(P), \qquad (b \leqslant_P b' \text{ and } b \neq b') \Longrightarrow f(b) < f(b').$$

The set of strict n-linear extensions of P is denoted by $\mathrm{Lin}^s(P, n)$.

Remark 3.6. Obviously, $\mathrm{Lin}^s(P, n) \subseteq \mathrm{Lin}(P, n)$ for any $n \geqslant 1$.

Theorem 3.20. 1. *For any monomial P in noncrossing partitions, there exists a unique $\Lambda(P)(X) \in \mathbb{K}[X]$, such that*

$$\forall n \geqslant 1, \qquad \Lambda(P)(n) = |\mathrm{Lin}(P, n)|.$$

Moreover, the extension of Λ as a linear map from $\mathbb{K}[\mathcal{NCP}]$ to $\mathbb{K}[X]$ is a Hopf algebra map from $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$.

2. *For any monomial P in noncrossing partitions, there exists a unique $\Lambda^s(P)(X) \in \mathbb{K}[X]$, such that*

$$\forall n \geqslant 1, \qquad \Lambda^s(P)(n) = |\mathrm{Lin}^s(P, n)|.$$

Moreover, the extension of Λ^s as a linear map from $\mathbb{K}[\mathcal{NCP}]$ to $\mathbb{K}[X]$ is a Hopf algebra map from $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ to $(\mathbb{K}[X], m, \Delta)$.

Proof. We only prove the first point. The proof of the second point is very similar. Let us first consider a monomial P in noncrossing partitions. For any $k \geqslant 1$, let us denote by $l_k(P)$ the number of surjective k-linear extensions of P. Note that if $k > |P|$, $l_k(P) = 0$. Then, for any $n \geqslant 1$,

$$|\mathrm{Lin}(P, n)| = \sum_{k=1}^{|P|} l_k(P) \binom{n}{k}.$$

We therefore take

$$\Lambda(P) = \sum_{k=1}^{|P|} l_k(P) H_k(X) \in \mathbb{K}[X].$$

Then, for any $n \geqslant 1$, $\lambda(P)(n) = |\mathrm{Lin}(P,n)|$.

By convention, $\Lambda(1) = 1$. If $P \neq 1$, then

$$\Lambda(P)(0) = \sum_{k=1}^{|P|} l_k(P)\binom{0}{k} = 0,$$

so $\varepsilon_\Delta \circ \Lambda(P) = 0 = \varepsilon_\Delta(P)$. Let us now prove the compatibility of Λ with the products. Let P, Q be two monomials in noncrossing partitions. Then $\mathrm{Bl}(P \cdot Q) = \mathrm{Bl}(P) \sqcup \mathrm{Bl}(Q)$ and $\leqslant_{P \cdot Q} = \leqslant_P \sqcup \leqslant_Q$. As a consequence, for any $n \geqslant 1$,

$$\mathrm{Lin}(P \cdot Q, n) = \{f \sqcup g \mid (f,g) \in \mathrm{Lin}(P,n) \times \mathrm{Lin}(Q,n)\}.$$

Taking the cardinalities, we obtain that for any $n \geqslant 1$,

$$\Lambda(P \cdot Q)(n) = \Lambda(P)(n)\Lambda(Q)(n).$$

This gives $\Lambda(P \cdot Q) = \Lambda(P)\Lambda(Q)$.

Let us finally prove the compatibility of Λ with the coproducts Δ. We fix a monomial P in noncrossing partitions. Let $m, n \geqslant 1$. If $f \in \mathrm{Lin}(P, m+n)$, then:

- $f^{-1}(\{m+1, \ldots, m+n\})$ is an ideal of P, which we denote I_f.
- $f_{|\mathrm{Bl}(P) \setminus I_f}$ is a m-linear extension of $P_{|\mathrm{Bl}(P) \setminus I_f}$.
- $f_{|I} - m$ is a n-linear extension of $P_{|.I_f}$.

Conversely, if I is an ideal of P, $f \in \mathrm{Lin}(P_{|\mathrm{Bl}(P) \setminus I}, m)$ and $g \in \mathrm{Lin}(P_{|.I}, n)$ the the following belongs to $\mathrm{Lin}(P, m+n)$:

$$\begin{cases} \mathrm{Bl}(P) & \longrightarrow & [m+n] \\ b & \longmapsto & \begin{cases} f(b) \text{ if } b \notin I, \\ g(b) + m \text{ if } b \in I. \end{cases} \end{cases}$$

Taking the cardinalities, we obtain that

$$\Lambda(P)(m+n) = \sum_{I \text{ ideal of } P} \Lambda(P_{|\mathrm{Bl}(P) \setminus I})(m)\Lambda(P_{|.I})(n).$$

Identifying $\mathbb{K}[X,Y]$ and $\mathbb{K}[X]^{\otimes 2}$, we obtain that for any $m, n \geqslant 1$,

$$\Delta \circ \Lambda(P)(m,n) = (\Lambda \otimes \Lambda) \circ \Delta(P)(m,n).$$

Therefore, $\Delta \circ \Lambda = (\Lambda \otimes \Lambda) \circ \Delta$. $\square$

Corollary 3.21. *Let λ and λ^s be the characters of $\mathbb{K}[\mathcal{NCP}]$ defined by*

$$\lambda : \begin{cases} \mathbb{K}[\mathcal{NCP}] & \longrightarrow & \mathbb{K} \\ \pi \in \mathcal{NCP} & \longmapsto & 1, \end{cases} \qquad \lambda^s : \begin{cases} \mathbb{K}[\mathcal{NCP}] & \longrightarrow & \mathbb{K} \\ \pi \in \mathcal{NCP} & \longmapsto & \begin{cases} 1 \text{ if } \pi = \mathrm{Base}(\pi), \\ 0 \text{ otherwise.} \end{cases} \end{cases}$$

Then, with the notations of Proposition 1.2 and Theorem 1.4,

$$\Lambda = \phi_{\mathcal{NCP}} \leftsquigarrow \lambda, \qquad \Lambda^s = \phi_{\mathcal{NCP}} \leftsquigarrow \lambda^s. \tag{11}$$

Proof. We use the bijection of Theorem 1.4. We define two elements of $\mathrm{Char}(\mathbb{K}[\mathcal{NCP}])$ by

$$\forall x \in \mathbb{K}[\mathcal{NCP}], \qquad \lambda(x) = \Lambda(x)(1), \qquad \lambda^s(x) = \Lambda^s(x)(1).$$

Theorem 1.4 gives (11). It remains to prove the formulas for λ and λ^s. Let π be a noncrossing partition. As the unique map $f : \mathrm{Bl}(\pi) \longrightarrow [1]$ is indeed a linear extension of π,

$$\lambda(\pi) = \Lambda(\pi)(1) = |\mathrm{Lin}(\pi, 1)| = 1.$$

Moreover, the unique map $f : \mathrm{Bl}(\pi) \longrightarrow [1]$ is a strict linear extension of π if, and only if, $\pi = \mathrm{Base}(\pi)$, so

$$\lambda^s(\pi) = \Lambda^s(\pi)(1) = |\mathrm{Lin}^s(\pi, 1)| = \begin{cases} 1 \text{ if } \pi = \mathrm{Base}(\pi), \\ 0 \text{ otherwise.} \end{cases}$$

□

Proposition 3.22 (Duality principle). *For any $\pi \in \mathcal{NCP}$,*

$$\Lambda(\pi)(X) = (-1)^{|\pi|}\Lambda^s(\pi)(-X).$$

Proof. Let us consider the two maps

$$\phi_1 : \left\{ \begin{array}{rcl} \mathbb{K}[X] & \longrightarrow & \mathbb{K}[X] \\ P(X) & \longmapsto & P(-X), \end{array} \right. \qquad \phi_2 : \left\{ \begin{array}{rcl} \mathbb{K}[\mathcal{NCP}] & \longrightarrow & \mathbb{K}[\mathcal{NCP}] \\ \pi \in \mathcal{NCP} & \longrightarrow & (-1)^{|\pi|}\pi. \end{array} \right.$$

The map ϕ is a bialgebra isomorphism (it is in fact the antipode of $\mathbb{K}[X]$) and, as the bialgebra $(\mathbb{K}[\mathcal{NCP}], m, \Delta)$ is graded by the number of blocks, ϕ_2 is also a bialgebra map. By composition, $\phi = \phi_1 \circ \Lambda^s \circ \phi_2$ is a bialgebra map. Moreover, for any $\pi \in \mathcal{NCP}$,

$$\phi(\pi) = (-1)^{|\pi|}\Lambda^s(\pi)(-X).$$

By Theorem 1.4, there exists a (unique) $\nu \in \mathrm{Char}(\mathbb{K}[\mathcal{NCP}])$ such that $\phi = \phi_{\mathcal{NCP}} \leftsquigarrow \nu$. Is is now enough to prove that $\nu = \lambda$. For any $\pi \in \mathcal{NCP}$,

$$\nu(\pi) = \phi(\pi)(1) = (-1)^{|\pi|}\Lambda^s(\pi)(-1),$$

so it is enough to prove that for any noncrossing partition π, $\Lambda^s(\pi)(-1) = (-1)^{|\pi|}$.We proceed by induction on $|\pi|$. If $|\pi| = 1$, then $\Lambda^s(\pi) = X$ and the result holds. Let us assume that $|\pi| \geqslant 2$ and that the result holds for any noncrossing partition π' with $|\pi'| < |\pi|$. If $f \in \mathrm{Lin}^s(\pi)$, observe that $f^{-1}(1) \subseteq \mathrm{Base}(\pi)$. Conversely, if $B \subseteq \mathrm{Base}(\pi)$ and $g \in \mathrm{Lin}^s(\pi_{|.\pi\setminus B}, n)$, one defines $f \in \mathrm{Lin}^s(\pi, n+1)$ by

$$\forall b \in \pi, \qquad f(b) = \begin{cases} 1 \text{ if } b \in B, \\ g(b) + 1 \text{ otherwise.} \end{cases}$$

Consequently,

$$\Lambda^s(\pi)(X+1) = \sum_{B \subseteq \mathrm{Base}(\pi)} \Lambda^s(\pi_{|.\pi\setminus B})(X).$$

Taking $X = -1$ and using the induction hypothesis on $\pi_{|.\pi\setminus B}$ when $B \neq \varnothing$, we obtain

$$\begin{aligned} \Lambda^s(\pi)(0) &= \epsilon_\delta(\pi) = 0 \\ &= \Lambda^s(\pi)(-1) + \sum_{\varnothing \subsetneq B \subseteq \mathrm{Base}(\pi)} \Lambda^s(\pi_{|.\pi\setminus B})(-1) \\ &= \Lambda^s(\pi)(-1) + \sum_{\varnothing \subsetneq B \subseteq \mathrm{Base}(\pi)} (-1)^{|\pi|-|B|} \\ &= \Lambda^s(\pi)(-1) + (-1)^{|\pi|}\left(\sum_{B \subseteq \mathrm{Base}(\pi)} (-1)^{|B|} - 1\right) \\ &= \Lambda^s(\pi)(-1) - (-1)^{|\pi|}. \end{aligned}$$

Note that $\sum_{B \subseteq \mathrm{Base}(\pi)} (-1)^{|B|} = 0$, as $\mathrm{Base}(\pi) \neq \varnothing$. Finally, the result holds for π. □

Corollary 3.23. *For any $\pi \in \mathcal{NCP}$, $\lambda(\pi) = (-1)^{|\pi|}\lambda^s \circ S(\pi)$.*

Proof. From the proof of the duality principle, with the same notations, $\Lambda = \phi_1 \circ \Lambda^s \circ \phi_2 = S \circ \Lambda^s \circ \phi_2$. As Λ^s is a bialgebra morphism, $\Lambda = \Lambda^s \circ S \circ \phi_2$. Evaluating this in π and then taking $X = 1$ gives the result. $\square$

Notations 3.5. By Proposition 3.13, both λ and λ^s are invertible for the convolution product $\star$, dual to the coproduct δ. We denote by μ and μ^s their respective inverses. We immediately obtain

$$\phi_{\mathcal{NCP}} = \Lambda \leftsquigarrow \mu, \qquad \phi_{\mathcal{NCP}} = \Lambda^s \leftsquigarrow \mu^s.$$

Let us first compute μ^s.

Proposition 3.24. *For any $\pi \in \mathcal{NCP}$, $\mu^s(\pi) = \begin{cases} 0 \text{ if } \pi \neq \mathrm{Base}(\pi), \\ (-1)^{|\pi|-1} \text{ if } \pi = \mathrm{Base}(\pi). \end{cases}$*

Proof. For any noncrossing partition π,

$$\lambda^s \star \mu^s(\pi) = \epsilon_\delta(\pi) = \sum_{\sim \in \mathcal{E}_c[\pi]} \lambda^s(\pi/\sim)\mu^s(\pi).$$

We denote by $\pi_1, \ldots, \pi_n$ the connected components of π, totally ordered from left to right. As λ^s only charges noncrossing partitions equal to their bases, in this sum we restrict ourselves with $\sim$ which classes are unions of consecutive π_j, which gives

$$\epsilon_\delta(\pi) = \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n,\\ n_1,\ldots,n_k \geqslant 1}} \mu^s(\pi_1 \ldots \pi_{n_1}) \ldots \mu^s(\pi_{n_1+\cdots+n_{k-1}+1} \ldots \pi_{n_1+\cdots+n_k}).$$

Let us assume firstly that $\pi \neq \mathrm{Base}(\pi)$ and let us show that $\mu^s(\pi) = 0$. Observe that $\epsilon_\delta(\pi) = 0$. We proceed by induction on n. If $n = 1$, we immediately obtain that $0 = \mu^s(\pi)$. If $n \geqslant 2$, for any $n_1, \ldots, n_k \geqslant 1$ such that $n_1 + \cdots + n_k = n$, at least one of the noncrossing partitions $P_{n_1+\cdots+n_i+1} \ldots P_{n_1+\cdots+n_{i+1}}$ is not equal to its base. Therefore, if $k \geqslant 2$, by the induction hypothesis, it vanishes under μ^s. We obtain $0 = \mu^s(\pi) + 0$, which ends this proof by induction.

Let us now assume that $\pi = \mathrm{Base}(\pi)$. We define a sequence $(a_n)_{n\geqslant 1}$ by

$$a_n = \begin{cases} 1 \text{ if } n = 1, \\ -\displaystyle\sum_{k=2}^{n} \sum_{\substack{n_1+\cdots+n_k=n,\\ n_1,\ldots,n_k \geqslant 1}} a_{n_1} \ldots a_{n_k} \text{ if } n \geqslant 2. \end{cases}$$

Let us prove that $\mu^s(\pi) = a_{|\pi|}$ by induction on $|\pi|$. If $|\pi| = 1$, then $\mu^s(\pi) = 1 = a_1$. If $|\pi| \geqslant 2$, the induction hypothesis gives

$$\begin{aligned} 0 &= \mu^s(\pi) + \sum_{k=2}^{n} \sum_{\substack{n_1+\cdots+n_k=n,\\ n_1,\ldots,n_k \geqslant 1}} \mu^s(\pi_1 \ldots \pi_{n_1}) \ldots \mu^s(\pi_{n_1+\cdots+n_{k-1}+1} \ldots \pi_{n_1+\cdots+n_k}) \\ &= \mu^s(\pi) + \sum_{k=2}^{n} \sum_{\substack{n_1+\cdots+n_k=n,\\ n_1,\ldots,n_k \geqslant 1}} a_{n_1} \ldots a_{n_k} \\ &= \mu^s(\pi) - a_n, \end{aligned}$$

so $\mu^s(\pi) = a_n$. It remains to prove that $a_n = (-1)^{n-1}$ for any $n \geqslant 1$. Let us consider the formal series $A = \sum_{k=1}^{\infty} a_k X^k \in \mathbb{X}[[X]]$. By definition of the sequence $(a_n)_{n\geqslant 1}$,

$$\begin{aligned} A &= X - \sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\sum_{\substack{n_1+\dots+n_k=n,\\ n_1,\dots,n_k\geqslant 1}} a_{n_1}\dots a_{n_k} X^n \\ &= X - \sum_{k=2}^{\infty}\sum_{\substack{n_1+\dots+n_k=n,\\ n_1,\dots,n_k\geqslant 1}} a_{n_1}X^{n_1}\dots a_{n_k}X^{n_k} \\ &= X - \sum_{k=2}^{\infty} A^k \\ &= X - \frac{A^2}{1-A}. \end{aligned}$$

Hence, $\frac{A}{1-A} = X$ and $A = \frac{X}{1+X}$, which finally implies that $a_n = (-1)^{n-1}$ for any $n \geqslant 1$. $\square$

This finally gives:

Proposition 3.25. *For any noncrossing partition π, $\phi_{\mathcal{NCP}}(\pi) = \sum_{\sim\in\mathcal{E}_c[\pi]} (-1)^{\mathrm{bl}(\pi|\sim)-1}\Lambda^s(\pi/\sim)$.*

In order to describe μ, we shall introduce a family of characters:

Proposition 3.26. *For any $q \in \mathbb{K}$, we define $\gamma_q \in \mathrm{Char}(\mathbb{K}[\mathcal{NCP}])$ by*

$$\gamma_q : \left\{ \begin{array}{rcl} \mathbb{K}[\mathcal{NCP}] & \longrightarrow & \mathbb{K} \\ \pi \in \mathcal{NCP} & \longmapsto & \begin{cases} q \text{ if } |\pi| = 1, \\ 0 \text{ otherwise.} \end{cases} \end{array} \right.$$

In particular, $\gamma_1 = \epsilon_\delta$. Then:

1. *For any $\lambda \in \mathrm{Char}(\mathbb{K}[\mathcal{NCP}])$, for any $\pi \in \mathcal{NCP}$, $\lambda \star \gamma_q(\pi) = q^{|\pi|}\lambda(\pi)$.*
2. *For any $q, q' \in \mathbb{K}$, $\gamma_{q'} \star \gamma_q = \gamma_{qq'}$.*
3. *If $q \neq 0$, γ_q is invertible for $\star$, and $\gamma_q^{\star -1} = \gamma_{q^{-1}}$.*
4. *For any $\lambda \in \mathrm{Char}(\mathbb{K}[\mathcal{NCP}])$, for any $\pi \in \mathcal{NCP}$, $\gamma_q \star \lambda(\pi) = q\lambda(\pi)$.*

Proof. 1. For any $\pi \in \mathcal{NCP}$,

$$\lambda \star \gamma_q(\pi) = \sum_{\sim\in\mathcal{E}_c[\pi]} \lambda(\pi/\sim)\gamma_q(\pi\mid\sim).$$

Let us consider $\sim\in \mathcal{E}_c[\pi]$ such that $\gamma_q(\pi\mid\sim) \neq 0$: $\pi\mid\sim$ is a monomial of noncrossing partitions with a single block, which means that $\sim$ is the equality $\sim_1$ of π. Therefore,

$$\lambda \star \gamma_q(\pi) = \lambda(\pi/\sim_1)\gamma_q(\pi\mid\sim_1) = \lambda(\pi)q^{|\pi|}.$$

2. In the particular case where $\lambda = \gamma_{q'}$, for any $\pi \in \mathcal{NCP}$,

$$\gamma_{q'} \star \gamma_q(\pi) = q^{|\pi|}\gamma_{q'}(\pi) = \begin{cases} qq' \text{ if } |\pi| = 1, \\ 0 \text{ otherwise.} \end{cases}$$

So $\gamma_{q'} * \gamma_q = \gamma_{qq'}$.

3. Easy consequence of 2.

4. For any $\pi \in \mathcal{NCP}$,
$$\gamma_q \star \lambda(\pi) = \sum_{\sim \in \mathcal{E}_c[\pi]} \gamma_q(\pi/\sim)\lambda(\pi \mid \sim).$$

Let us consider $\sim \in \mathcal{E}_c[\pi]$ such that $\gamma_q(\pi/\sim) \neq 0$: $\pi/\sim$ is a noncrossing partition with a single block, which means that $\sim$ is equivalence $\sim_0$ on π with a single class. Therefore,

$$\gamma_q \star \lambda(\pi) = \gamma_q(\pi/\sim_0)\lambda(\pi \mid_\cdot \sim_0) = q\lambda(\pi).$$ □

Lemma 3.27. *For any* $\lambda \in \mathrm{Char}(\mathbb{K}[\mathcal{NCP}])$, $\lambda \circ S = \mu_{\mathcal{NCP}} \star \lambda$.

Proof. Recall that $S = (\mu_{\mathcal{NCP}} \otimes \mathrm{Id}_{\mathbb{K}[\mathcal{NCP}]}) \circ \delta$. Therefore,

$$\lambda \circ S = \lambda \circ (\mu_{\mathcal{NCP}} \otimes \mathrm{Id}_{\mathbb{K}[\mathcal{NCP}]}) \circ \delta = (\mu_{\mathcal{NCP}} \otimes \lambda) \circ \delta = \mu_{\mathcal{NCP}} \star \lambda.$$ □

Lemma 3.28. $\mu_{\mathcal{NCP}} \star \mu_{\mathcal{NCP}} = \epsilon_\delta$.

Proof. By Lemma 3.27, for $\lambda = \epsilon_\delta$,

$$\epsilon_\delta \circ S = \mu_{\mathcal{NCP}} \star \epsilon_\delta = \mu_{\mathcal{NCP}}.$$

Still by this Lemma, with $\lambda = \mu_{\mathcal{NCP}}$,

$$\mu_{\mathcal{NCP}} \star \mu_{\mathcal{NCP}} = \mu_{\mathcal{NCP}} \circ S = \epsilon_\delta \circ S^2 = \epsilon_\delta.$$

Indeed, as $\mathbb{K}[\mathcal{NCP}]$ is commutative, $S^2 = \mathrm{Id}_{\mathbb{K}[\mathcal{NCP}]}$. □

Proposition 3.29. *In* $\mathrm{Char}(\mathbb{K}[\mathcal{NCP}])$, $\lambda = \mu_{\mathcal{NCP}} \star \lambda^s \star \gamma_{-1}$ *and* $\mu = \gamma_{-1} \star \mu^s \star \mu_{\mathcal{NCP}}$. *Moreover,*

$$\forall \pi \in \mathcal{NCP}, \qquad \mu(\pi) = \sum_{\substack{\sim \in \mathcal{E}_c[\pi],\\ \pi/\sim = \mathrm{Base}(\pi/\sim)}} (-1)^{\mathrm{cl}(\sim)} \mu_{\mathcal{NCP}}(P \mid_\cdot \sim),$$

where $\mathrm{cl}(\sim)$ *is the number of classes of* $\sim$.

Proof. By Corollary 3.23, Lemma 3.27 and Proposition 3.26-1, $\lambda = \mu_{\mathcal{NCP}} \star \lambda^s \star \gamma_{-1}$. By proposition 3.26-3 and Lemma 3.28, the inverses (for $\star$) of γ_{-1} and $\mu_{\mathcal{NCP}}$ are themselves, so

$$\mu = \lambda^{\star -1} = \gamma_{-1}^{\star -1} \star (\lambda^s)^{\star -1} \star \mu_{\mathcal{NCP}}^{\star -1} = \gamma_{-1} \star \mu^s \star \mu_{\mathcal{NCP}}.$$

By Proposition 3.26-4, for any noncrossing partition π,

$$\mu(\pi) = -\mu^s \star \mu_{\mathcal{NCP}}(\pi) = -\sum_{\sim \in \mathcal{E}_c[\pi]} \mu^s(\pi/\sim)\mu_{\mathcal{NCP}}(P \mid_\cdot \sim).$$

The formula then comes from Proposition 3.24. □

Example 3.7. See Table 4 at the end of the article for examples of $\mu^s(\pi)$ and $\mu(\pi)$.

Corollary 3.30. *Let* $\pi \in \mathcal{NCP}$, *such that* $\pi = \mathrm{Base}(\pi)$. *Then* $\mu(\pi) = (-1)^{|\pi|-1}\mathrm{cat}_{|\pi|-1}$.

Proof. Let $\pi \in \mathcal{NCP}$, such that $\pi = \mathrm{Base}(\pi)$. The blocks of π are denoted by $\pi_1, \ldots, \pi_n$, totally ordered from left to right. If $\sim \in \mathcal{E}_c[\pi]$ such that $\pi/\sim = \mathrm{Base}(\pi/\sim)$, then the classes of $\sim$ are formed by consecutive π_i. This gives

$$\begin{aligned}\mu(\pi) &= \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n,\\ n_1,\ldots,n_k \geqslant 1}} (-1)^k \mu_{\mathcal{NCP}}(\pi_1 \ldots \pi_{n_1}) \ldots \mu_{\mathcal{NCP}}(\pi_{n_1+\cdots+n_{k-1}+1} \ldots \pi_{n_1+\cdots+n_k})\\ &= \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n,\\ n_1,\ldots,n_k \geqslant 1}} (-1)^{k+n_1+\cdots+n_k} \mathrm{cat}_{n_1} \ldots \mathrm{cat}_{n_k}.\end{aligned}$$

We put, for any $n \geqslant 1$,

$$a_n = \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n,\\ n_1,\ldots,n_k \geqslant 1}} (-1)^{k+n_1+\cdots+n_k} \mathrm{cat}_{n_1} \ldots \mathrm{cat}_{n_k},$$

so that $\mu(\pi) = a_{|\pi|}$. We consider the formal series

$$A = \sum_{n=1}^{\infty} a_n X^n \in \mathbb{Q}[[X]], \qquad C = \sum_{n=1}^{\infty} \mathrm{cat}_n X^n = \frac{1-\sqrt{1-4X}}{2X} - 1 \in \mathbb{Q}[[X]].$$

By definition of a_n,

$$A = \sum_{k=1}^{\infty} (-1)^k C(-X)^k = \frac{C(-X)}{1+C(-X)}.$$

Replacing the expression of C in this formula and working out, we obtain

$$A = \frac{1-\sqrt{1+4X}}{2} = X(1+C(-X)) = \sum_{n=1}^{\infty} (-1)^{n-1} \mathrm{cat}_{n-1} X^n,$$

which gives the result for $\mu(\pi)$. □

4 Combinatorial morphisms from noncrossing partitions to graphs

We now turn to the bialgebras of hypergraphs or mixed graphs defined in [11, 12]. We obtain the following negative results:

Proposition 4.1. *1. Let $\curlywedge \in \{\cap, \subset\}$. There exists no double bialgebra morphism ψ from $\mathbb{K}[\mathcal{NCP}]$ to the double bialgebra of hypergraphs $(\mathcal{F}[\mathbf{H}], m, \Delta, \delta^{(\curlywedge)})$ of [11], such that $\psi(J_3)$ is a sum of hypergraphs.*

2. There exists no double bialgebra morphism ψ from $\mathbb{K}[\mathcal{NCP}]$ to the double bialgebra of mixed graphs $\mathcal{H}_{\mathbf{G}}$ of [12], such that $\psi(J_3)$ is a mixed graph.

Proof. 1. Let ψ be such a morphism. Then $\phi_{\mathcal{NCP}} \circ \psi : \mathbb{K}[\mathcal{NCP}] \longrightarrow \mathbb{K}[X]$ is a double bialgebra morphism by composition, so is equal to ϕ_{chr}. Hence, there exists a sum of hypergraphs $G_1 + \cdots + G_k = \psi(J_3)$ such that

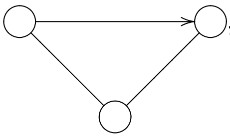
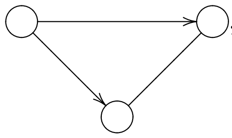
$$\phi_{chr}(G_1 + \cdots + G_k) = X^3 - \frac{5X^2}{2} + \frac{3X}{2}.$$

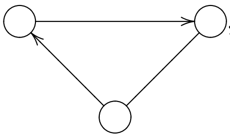
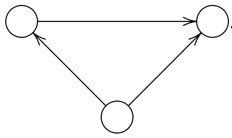
This contradicts [11, Proposition 2.5], stating that the coefficients of $\phi_{chr}(H)$ are integers for any hypergraph.

2. Let ψ be such a morphism. Then $\phi_{\mathcal{NCP}} \circ \psi : \mathbb{K}[\mathcal{NCP}] \longrightarrow \mathbb{K}[X]$ is a double bialgebra morphism by composition, so is equal to ϕ_{chr}. Hence, there exists a mixed graph $G = \psi(J_3)$ such that

$$\phi_{chr}(G) = X^3 - \frac{5X^2}{2} + \frac{3X}{2}.$$

By [12, Proposition 3.14], G has no cycle and has three vertices. It is not a graph, as for any graph G, the coefficients of $\phi_{\mathcal{NCP}}(G) = P_{chr}(G)$ are integers. As it is not divisible by X^2, G is connected. It is not one of the mixed graphs of [12, Example 3.2]. So G is one of the four following mixed graphs:

$G_1 =$ , $G_2 =$ ,

$G_3 =$ , $G_4 =$ .

But for any of these graphs, $\phi_{chr}(G_i)(2) = 0$, whereas $\phi_{chr}(G)(2) = 1$. So such a ψ does not exist. $\square$

Remark 4.1. Of course, there exist double bialgebra morphisms from $\mathbb{K}[\mathcal{NCP}]$ to $\mathcal{F}[\mathbf{H}]$ or to $\mathcal{H}_{\mathbf{G}}$. For example, observe that the two following define double bialgebra morphisms:

$$\begin{cases} \mathbb{K}[X] & \longrightarrow & \mathcal{F}[\mathbf{H}] \\ X & \longmapsto & \bullet, \end{cases} \qquad \begin{cases} \mathbb{K}[X] & \longrightarrow & \mathcal{H}_{\mathbf{G}} \\ X & \longmapsto & \bullet. \end{cases}$$

The composition with ϕ_{chr} gives two double bialgebra morphisms, which of course do not satisfy the combinatorial conditions of Proposition 4.1.

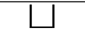
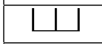
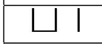
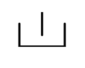
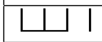
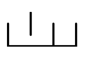
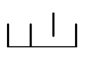
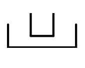
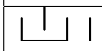
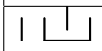
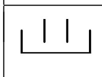
π	$\phi_{\mathcal{NCP}}(\pi)$	$\mu_{\mathcal{NCP}}(\pi)$	$\lambda_0(\pi)$	$\lambda_{\mathcal{NCP}}(\pi)$	$\mu^s(\pi)$	$\mu(\pi)$
	X	-1	1	1	1	1
	X	-1	1	1	1	1
	$X(X-1)$	2	1	-1	-1	-1
	X	-1	1	1	1	1
	$X(X-1)$	2	1	-1	-1	-1
	$X(X-1)$	2	1	-1	-1	-1
	$\frac{X(X-1)}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
	$X(X-1)\left(X-\frac{3}{2}\right)$	-5	1	$\frac{3}{2}$	1	2
	X	-1	1	1	1	1
	$X(X-1)$	2	1	-1	-1	-1
	$X(X-1)$	2	1	-1	-1	-1
	$\frac{X(X-1)}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1
	$\frac{X(X-1)}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1
	$X(X-1)$	2	1	-1	-1	-1
	$\frac{X(X-1)}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1
	$X(X-1)\left(X-\frac{3}{2}\right)$	-5	1	$\frac{3}{2}$	1	2
	$X(X-1)\left(X-\frac{3}{2}\right)$	-5	1	$\frac{3}{2}$	1	2
	$X(X-1)\left(X-\frac{3}{2}\right)$	-5	1	$\frac{3}{2}$	1	2
	$X(X-1)^2$	-2	1	1	0	3
	$X(X-1)^2$	-2	1	1	0	3
	$\frac{X(X-1)(X-2)}{3}$	-2	$\frac{1}{3}$	$\frac{2}{3}$	0	2
	$X(X-1)(X-2)\left(X-\frac{4}{3}\right)$	14	1	$-\frac{8}{3}$	-1	-5

Table 1: examples

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