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# A new approach to agglomeration problems <sup>\*</sup>

Adriana Navarro-Ramos <sup>†‡§</sup>

## Abstract

In this paper, we study a location problem with positive externalities. We define a new transferable utility game, considering there is no restriction on the transfer of benefits between firms. We prove that the core of this game is non-empty, provide an expression for it, and an axiomatic characterization. We also study several core allocations, selected by means of a certain bankruptcy problem.

**Keywords:** location problem; positive externalities; cooperative game; core; axiomatic characterization.

## 1 Introduction

Consider that a firm is planning to open a new plant in a certain territory. This territory is divided into different geographical units or regions where the plant could be installed. A finite number of firms are located in these regions, all of which obtain some benefit from this new incorporation. These positive externalities are well-studied in agglomeration economies theory Marshall (1890) and cluster theory (Porter, 1998). These theories state that firms in related fields of business come geographically together because concentration generates several advantages. For example, if the new firm is a common supplier, its insertion in the region would imply savings in the transportation of goods and new buyer-supplier relationships. Or the firm may be a new distribution network, which would generate an increase in the number of potential clients.

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In a decentralized mechanism, the newcomer firm would locate its plant in the region where it optimizes its individual benefit. Nevertheless, the aggregate utility of all firms could be maximized in a different region. Thus, it is reasonable for firms to transfer something to the new firm in order to incentivize it to open the plant in their region. This transfer is not utility *per se*, but rather some suitable medium of exchange (commercial agreements, money, etc.).

Location theory deals with two main problems. First, finding the optimal location for plants or facilities in a territory. Then, allocating the costs or benefits generated by this new incorporation over the agents involved in the problem. In the literature, diverse authors addressing the second issue by means of cooperative game theory can be found. For instance, in the facility-location cost-allocation problem on networks Granot (1987), where the aim is to locate one public-service facility in a tree network. Or in the NIMBY (Not In My BackYard) problem, where a locally undesirable but globally necessary facility must be installed Ambec and Kervinio (2016); Lejano and Davos (2001); Sakai (2012); Shapley and Shubik (1969). The problem studied in this paper differs from the above mentioned location problems in two aspects. First, the plant to be located generates exclusively positive externalities to the firms involved. Second, the newcomer firm is not just a facility but an active agent.

This problem was introduced in Bergantiños and Navarro-Ramos (2022) under the name of *agglomeration problem*. They associate a cooperative game with any agglomeration problem. This game considers that only coalitions formed by the newcomer firm and all firms in a given region can transfer their benefits. The idea behind this approach is that regional governments step in to offer subsidies to the new company. These subsidies are paid by all firms in the region through their taxes. However, if the benefits between firms are obtained through commercial agreements, government intervention is not necessary and each firm is free to negotiate or not with the new firm. So, the first aim of this paper is to define a new transferable utility game that reflects firms' freedom of individual negotiations. There are several advantages of associating different games with the same problem. For instance, the same solution could provide different allocations, probably more intuitive in certain contexts. Also, a game can give us more information about agents involved in the problem. Some examples in the literature where the same problem is approached by several games are the minimum cost spanning tree problems Bird (1976); Granot and Huberman (1981), bankruptcy problems Driessen (1995); O'Neill (1982), and hub network problems Skorin-Kapov (1998).

There is another important difference between the approach taken in this paper and the one in Bergantiños and Navarro-Ramos (2022). They propose single-valued functions for distributing the global benefit. In the case of this paper, we propose set solutions *i.e.* a set of feasible ways to allocate the total benefit among the firms. In order to do this, we study the core of the game,

describe it, and give an axiomatic characterization. Finally, we also analyze allocations in the core through bankruptcy problems.

The present paper is organized as follows. Section 2 introduces positive-externalities location problems. Section 3 formally defines the game and discusses its properties. Section 4 studies the core of the game and presents an axiomatic characterization. Section 5 discusses several allocations in the core. The proofs of the results are presented in Appendix A.

## 2 The model

Let  $\mathbb{N}$  be set of potential firms and let  $\mathcal{N}$  be the family of all finite (non-empty) subsets of  $\mathbb{N}$ . A **positive-externalities location problem** (or problem) is a triple  $A = (N_0, P, b)$  where

- $N_0 = N \cup \{0\}$ .  $N \in \mathcal{N}$  is the set of firms already located in a territory and 0 is the firm which will open a plant.
- $P = (P_k)_{k \in R}$  with  $\bigcup_{k \in R} P_k = N$  is an indexed collection of pairwise disjoint subsets of  $N$ , where  $R = \{1, \dots, r\}$  is the set of regions in the territory.  $P_k$  denotes the set of firms located in region  $k$ .
- $b = \{b_i^k : i \in N_0 \text{ and } k \in R\}$ .  $b_i^k \geq 0$  denotes the benefit obtained by firm  $i$  when 0 locates its plant in region  $k$ . It is assumed that, for all  $k \in R$  and all  $i \in N \setminus P_k$ ,  $b_i^k = 0$ .

Let  $\mathcal{A}^N$  denote the set of all problems over  $N$ . For any  $S \subseteq N_0$  and every  $k \in R$ , let  $b^k(S)$  denote the aggregate benefit of firms in  $S$  if 0 is located in region  $k$ , that is,  $b^k(S) = \sum_{i \in S} b_i^k$ . For each  $i \in N$ ,  $k(i) \in R$  denotes the region where firm  $i$  is located, *i.e.*,  $i \in P_{k(i)}$ .

Consider a problem  $A \in \mathcal{A}^N$ . The **global benefit** of  $A$  is  $g(A) = \max_{k \in R} \{b^k(N_0)\}$ . This value is the maximum aggregate benefit that all firms can obtain. A region  $k^* \in R$  is an **optimal region** if for each  $k \in R \setminus \{k^*\}$ ,  $b^{k^*}(N_0) \geq b^k(N_0)$ . The global benefit is reached when 0 locates its plant in any optimal region. Obviously,  $k^*$  may not be unique and  $g(A) = b^{k^*}(N_0) = b^{k^*}(P_{k^*} \cup \{0\})$ , for each optimal region  $k^*$ . We denote by  $R^*(A) \subseteq R$  the set of optimal regions in  $A$ .

Let  $s(A)$  denote the global benefit obtained by all firms in  $N_0$  when the plant is located in the second most profitable region (this region may not be unique). Formally, given an optimal region  $k^*$ ,  $s(A) = \max_{k \in R \setminus \{k^*\}} \{b^k(N_0)\}$ . When there are several optimal regions,  $s(A) = g(A)$ . Otherwise,  $s(A) < g(A)$ . If there is a single region,  $s(A) = 0$ .

Let  $I_0(A)$  be the maximum benefit that can be obtained without the cooperation of firms in an optimal region  $k^*$ . This is,  $I_0(A) = \max_{k \in R} \{b^k(N_0 \setminus P_{k^*})\} = \max \{b_0^{k^*}, s(A)\}$ . If there is more than one optimal region, we can see that  $I_0(A) = g(A)$ . Thus,  $I_0(A)$  does not depend on the

chosen  $k^*$ . The importance of  $I_0(A)$  is that it reflects a certain “negotiation power” of firm 0 in the problem. Consider that it has been decided to open the plant in an optimal region  $k^*$ . Firms in  $k^*$  agree to make a transfer  $x$  to firm 0. Seeking to determine which is the minimum value of  $x$  that it should accept, firm 0 starts negotiations with regions in  $R \setminus \{k^*\}$ . The region that can offer the most is the second best region, say region  $k$ . The maximum that firm 0 can obtain from firms in  $k$  is  $s(A)$ . However, if  $s(A)$  is lower than  $b_0^{k^*}$ , firm 0 has no incentives to accept the transfer from region  $k$  and would locate the plant in region  $k^*$ . In this case, firm 0 would expect to receive at least  $b_0^{k^*}$ . Therefore,  $I_0(A)$  can be viewed as the minimum firm 0 expects to receive for opening the plant in region  $k^*$ .

For an optimal region  $k^*$  in  $A$ , firm  $i \in P_{k^*}$  is called **essential** if  $b^{k^*}(N_0 \setminus \{i\}) < s(A)$ . If an essential firm does not cooperate, region  $k^*$  would not longer be an optimal region. Let  $E_{k^*}(A) \subseteq P_{k^*}$  denote the set of essential firms of region  $k^*$ . This set may be empty.

If no confusion arises, we denote  $R^*$  instead of  $R^*(A)$  and  $E_{k^*}$  instead of  $E_{k^*}(A)$ .

Given  $A \in \mathcal{A}^N$  and  $i \in N$ , the  **$i$ -reduced problem** of  $A$  is defined as  $A \setminus i = (N_0 \setminus \{i\}, P \setminus i, b \setminus i) \in \mathcal{A}^{N \setminus \{i\}}$  where  $P \setminus i = (P \setminus P_{k(i)}) \cup (P_{k(i)} \setminus \{i\})$ , and  $(b \setminus i)_j^k = b_j^k$ , for all  $j \in N_0 \setminus \{i\}$ , and every  $k \in R$ . In the  $i$ -reduced problem, firm  $i$  is removed from  $A$ , keeping the rest of the problem the same.

Once it has been established which is the most convenient location for firm 0 in a problem, the question is how the global benefit should be divided. In order to answer this question, we define solutions for any problem.

An **allocation** in  $A$  is a vector  $x \in \mathbb{R}^{n+1}$  such that it is **efficient**, *i.e.*,  $x(N_0) := \sum_{i \in N_0} x_i = g(A)$ . A **solution**  $\sigma$  on  $\mathcal{A}^N$  assigns to each problem  $A \in \mathcal{A}^N$  a set of allocations  $\sigma(A) \neq \emptyset$ . A solution gives a set of possible ways to completely allocate the global benefit among all firms. It is crucial that a solution is non-empty because the main goal is to answer the question of how  $g(A)$  should be divided up.

**Example 1.** Let  $N = \{1, 2, 3, 4\}$ ,  $P = \{\{1\}, \{2, 3\}, \{4\}\}$ , and  $b$  given in Table 1.

| Regions | Firms |   |   |    |   |
|---------|-------|---|---|----|---|
|         | 0     | 1 | 2 | 3  | 4 |
| 1       | 10    | 1 | 0 | 0  | 0 |
| 2       | 7     | 0 | 2 | 11 | 0 |
| 3       | 9     | 0 | 0 | 0  | 7 |

Table 1: Values of  $b_i^k$ , for all  $i \in N$  and  $k \in R$ .

In the problem  $A = (N_0, P, b)$ , the unique optimal region is region 2, the global benefit is  $g(A) = 20$ ,  $I_0(A) = s(A) = 16$ , and firm 3 is the only essential firm.

To end this section, we introduce some properties of the positive-externalities location problems.

**Lemma 1.** *Let  $A \in \mathcal{A}^N$  and  $k^* \in R^*$ .*

- (a) *If  $I_0(A) = b_0^{k^*}$ , then  $E_{k^*} = \emptyset$ .*
- (b) *If  $E_{k^*} \neq \emptyset$ , then  $I_0(A) = s(A)$ .*
- (c) *If  $|R^*| > 1$  and  $i \in P_{k^*}$  is such that  $b_i^{k^*} \neq 0$ , then  $i \in E_{k^*}$ .*

### 3 A transferable utility game

In this section, we associate with every  $A \in \mathcal{A}^N$  a transferable utility game. By associating a game with any positive-externalities location problem, we seek to analyze not only the maximum that could be generated by a certain subset of firms, but also firms' capability to attract the newcomer firm to their region.

A **transferable utility game** (thereafter TU game) is a pair  $(N, v)$  where  $N \subset \mathbb{N}$  is the finite set of players and  $v : 2^N \rightarrow \mathbb{R}$  with  $v(\emptyset) = 0$  is the *characteristic function*. For any  $S \subseteq N$ ,  $v(S)$  is the worth of coalition  $S$  and represents the amount that its members can obtain if they cooperate. The coalition  $N$  is referred to as the *grand coalition*. When possible, we address  $v$  as a game.

We define the worth of a coalition  $S \subseteq N_0$  according to the following reasoning: consider that a subset of firms  $S$  decide to cooperate. First, if firm 0 is not in this coalition, the new plant will not be located in the territory and therefore, the worth of  $S$  is zero. If firm 0 is in the coalition, then we must look at the aggregate benefit of the firms in the coalition at each region. The firm would locate the plant where the aggregate benefit is maximized and this value is the worth of the coalition. This approach is different from that of Bergantiños and Navarro-Ramos (2022) in the sense that here, we consider that if firm 0 locates the new plant in region  $k$ , then it can cooperate with any subset of firms already installed in  $k$ . Formally, for any  $A \in \mathcal{A}^N$  and every  $S \subseteq N_0$ , the game  $(N_0, w^A)$  is defined as

$$w^A(S) = \begin{cases} \max_{k \in R} \{b^k(S)\}, & \text{if } 0 \in S, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

**Example 2.** *Consider the problem  $A$  in Example 1. Table 2 shows the worth of coalitions  $S \subseteq N_0$ , with  $0 \in S$  according to  $w^A$ .*

| $S$        | $w^A(S)$ | $S$           | $w^A(S)$ | $S$              | $w^A(S)$ | $S$                 | $w^A(S)$ |
|------------|----------|---------------|----------|------------------|----------|---------------------|----------|
| $\{0\}$    | 10       | $\{0, 4\}$    | 16       | $\{0, 2, 3\}$    | 20       | $\{0, 1, 2, 4\}$    | 16       |
| $\{0, 1\}$ | 11       | $\{0, 1, 2\}$ | 11       | $\{0, 2, 4\}$    | 16       | $\{0, 1, 3, 4\}$    | 18       |
| $\{0, 2\}$ | 10       | $\{0, 1, 3\}$ | 18       | $\{0, 3, 4\}$    | 18       | $\{0, 2, 3, 4\}$    | 20       |
| $\{0, 3\}$ | 18       | $\{0, 1, 4\}$ | 16       | $\{0, 1, 2, 3\}$ | 20       | $\{0, 1, 2, 3, 4\}$ | 20       |

Table 2:  $w^A(S)$  for all  $S \subseteq N_0$ , with  $0 \in S$ .

Even though region 2 is the only optimal region, in several coalitions (e.g.  $\{0, 2, 4\}$ ), the aggregate benefit in the optimal region is not enough to attract the newcomer firm to region 2.

Next, we present some properties of a game. A TU game  $(N, v)$  is:

**Monotone** if  $v(S) \leq v(T)$  whenever  $S \subseteq T$ , for all  $S, T \subseteq N$ .

**Superadditive** if for  $S, T \subseteq N$  with  $S \cap T = \emptyset$ ,  $v(S \cup T) \geq v(S) + v(T)$ .

**Convex** if  $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ , for all  $S \subseteq T \subseteq N \setminus \{i\}$ .

Monotonicity states that the worth of a coalition increases as more players join it. Superadditivity says that it is more profitable for two disjointed coalitions to merge. In a convex game, the marginal contribution of a player is monotone with respect to the size of the coalition that they join. The next proposition states which of these properties are fulfilled by the game defined in (1).

**Proposition 1.** *For any problem  $A \in \mathcal{A}^N$ , the game  $w^A$  is monotone and superadditive but may not be convex.*

## 4 The core solution

In this section, we introduce the core Shapley (1955), a well-known solution for TU games. We define a solution for positive-externalities location problems as the core of  $w^A$ . We discuss another solution called the core cover Tijs and Lipperts (1982), and we prove that it also coincides with the core solution. To end the section, an axiomatic characterization of the core solution is given.

Given a game  $(N, v)$ , an **allocation** is a vector  $x \in \mathbb{R}^n$  such that  $x(N) = v(N)$ . A **solution**  $\phi$  is a correspondence that associates with each  $v$  a (possibly empty) set of allocations  $\phi(v)$ . If a solution selects a single allocation, then it is commonly referred to as a **single-valued solution**.

An allocation  $x$  is **individually rational** if, for each  $i \in N$ ,  $x_i \geq v(\{i\})$ , *i.e.*, no player gets less than what they could get by themselves. The **imputation set** of a TU game consists of all the allocations that are individually rational, *i.e.*,  $I(v) = \{x \in \mathbb{R}^n : x(N) = v(N), x_i \geq v(\{i\}), \forall i \in N\}$ .

The core of a game is defined by imposing a coalitional rationality principle to  $I(v)$ . An imputation  $x$  is **coalitionally rational** if  $x(S) \geq v(S)$ , for every  $S \subseteq N$ . This principle ensures that no coalition can improve by working on its own. The **core** of  $v$  is defined as  $C(v) = \{x \in I(v) : x(S) \geq v(S), \forall S \subseteq N\}$ .

## 4.1 Expression of the core

In the next result, we give an expression for the core of  $w^A$  and it has the following interpretation. Firm 0 receives at least its negotiation power and, at most, the global benefit. Every firm receives a non-negative payoff. Firms in optimal regions are distinguished in two cases: essential and non-essential firms. An essential firm receives at most what is left after giving  $I_0(A)$  to firm 0. Non-essential firms receive at most their individual benefit. Firms in any other region receive zero.

**Theorem 1.** *Given a problem  $A \in \mathcal{A}^N$  and  $k^* \in R^*$ , the core of  $w^A$  is non-empty and is given by*

$$C(w^A) = \left\{ x \in \mathbb{R}^{n+1} : \begin{array}{l} I_0(A) \leq x_0 \leq g(A), \\ 0 \leq x_i \leq g(A) - I_0(A), \forall i \in E_{k^*}, \\ 0 \leq x_i \leq b_i^{k^*}, \forall i \in P_{k^*} \setminus E_{k^*}, \\ x_i = 0, \forall i \in N \setminus P_{k^*} \end{array} \right\}. \quad (2)$$

Notice that if  $|R^*| > 1$  or if  $P_{k^*} = \emptyset$  for any  $k^* \in R^*$ , the core of  $w^A$  consists of a single element in which firm 0 gets the global benefit and the rest of the firms get zero. This also happens if  $b_i^{k^*} = 0$ , for all  $i \in P_{k^*}$ .

For all  $A \in \mathcal{A}^N$ , the **core solution**  $\gamma$  on  $\mathcal{A}^N$  is defined as  $\gamma(A) = C(w^A)$ . Since the core of  $w^A$  is non-empty for any problem  $A \in \mathcal{A}^N$ ,  $\gamma$  is well-defined.

## 4.2 The core cover

We now study the core cover, another interesting solution for TU games. As its name suggests, the core cover (when it is non-empty) contains the core of any game.

For any  $(N, v)$  and every  $i \in N$ , let  $M_i(v) = v(N) - v(N \setminus \{i\})$  be player  $i$ 's *marginal contribution* to the grand coalition. The vector  $M(v) = (M_i(v))_{i \in N}$  is called the **utopia vector** of  $v$  and  $M_i(v)$  is referred to as the utopia payoff of player  $i$ . The **minimum right vector** is  $m(v) = (m_i(v))_{i \in N}$  where  $m_i(v) = \max_{S \subseteq N: i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right\}$ . The minimum right payoff,  $m_i(v)$ , can be seen as the maximum payoff that player  $i$  can guarantee for herself by offering to every other player their marginal contribution to the grand coalition.



The **core cover** of  $(N, v)$  is defined as  $CC(v) = \{x \in \mathbb{R}^n : x(N) = v(N), m(v) \leq x \leq M(v)\}$ . The elements of the core cover can be interpreted as efficient compromises between the minimum right and the utopia payoffs.

In the following proposition, we describe the utopia vector and minimum right vector of the game  $w^A$ , for any problem  $A \in \mathcal{A}^N$ .

**Proposition 2.** *For any problem  $A \in \mathcal{A}^N$ , every  $k^* \in R^*$ , and all  $i \in N_0$ ,*

$$M_i(w^A) = \begin{cases} g(A), & \text{if } i = 0, \\ g(A) - I_0(A), & \text{if } i \in E_{k^*}, \\ b_i^{k^*}, & \text{if } i \in P_{k^*} \setminus E_{k^*}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$m_i(w^A) = \begin{cases} I_0(A), & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since for any problem  $A \in \mathcal{A}^N$ ,  $0 \leq I_0(A) \leq g(A)$  and  $b_i^k \geq 0$ , for all  $i \in N_0$  and each  $k \in R$ , we conclude that  $CC(w^A) \neq \emptyset$ . Moreover, looking at the expression of the core given by (2) and Proposition 2, we have that the core cover and the core of the game  $w^A$  coincide. With this result, we also have an alternative interpretation of  $\gamma(A)$ . The core solution insures that every firm gets at least their minimal right and at most their utopia payoff.

### 4.3 Axiomatic characterization

This section ends with an axiomatic characterization of  $\gamma(A)$ . We introduce three natural axioms that characterize this solution.

Let  $\sigma$  be a solution on  $\mathcal{A}^N$ . For any problem  $A \in \mathcal{A}^N$ ,  $\sigma$  satisfies:

**Positive externalities effect (PE).**  $x \geq 0$ , for all  $x \in \sigma(A)$ .

*PE* states that no firm should end up with a negative transfer. In other words, no cost should be allocated to any firm. This axiom reflects one of the main assumptions in the problem, that is, the entry of firm 0 in the territory generates exclusively positive externalities.

**Maximal aspiration (MA).**  $x_i \leq g(A) - g(A \setminus i)$ , for all  $x \in \sigma(A)$  and all  $i \in N$ .

The contribution that a firm makes to a problem can be seen as the increase in the global benefit when this firm enters the problem. This axiom states that each firm should get at most its contribution to the problem.

**Newcomer firm monotonicity (NFM).** Let  $A = (N_0, P, b)$  and  $A' = (N_0, P, b')$  such that  $b^{k^*}(N) > b'^{k^*}(N)$ , for  $k^* \in R^*(A)$  and  $b_j^{k^*} = b_j^k$ , for all  $j \in N_0$  and  $k \in R \setminus \{k^*\}$ . Then  $x_0 \geq \min_{y \in \sigma(A)} \{y_0\}$ , for all  $x \in \sigma(A')$ .

If the aggregate benefit of an optimal region increases, *NFM* says that firm 0 should get at least the minimum of what it could have received before. A solution satisfying this axiom ensures that the negotiation power of firm 0 does not decrease if the positive externalities on the optimal region increase.

The following result states that any solution satisfying *PE* and *MA* is a larger set of the core solution.

**Lemma 2.** *Consider a solution  $\sigma$  on  $\mathcal{A}^N$  satisfying *PE* and *MA*. Then,  $\gamma(A) \subseteq \sigma(A)$ , for all  $A \in \mathcal{A}^N$ .*

Now, we state that  $\gamma(A)$  is the unique solution that satisfying positive externalities effect, maximal aspiration and newcomer firm monotonicity.

**Theorem 2.** *A solution  $\sigma$  on  $\mathcal{A}^N$  satisfies *PE*, *MA* and *NFM* if and only if  $\sigma(A) = \gamma(A)$ , for any  $A \in \mathcal{A}^N$ .*

## 5 Allocations in the core

Since we have fully described and axiomatically characterized the core solution, we now study some allocations belonging to this solution. In this section, we provide a procedure to select allocations in  $\gamma(A)$  by means of a certain bankruptcy situation. We study three specific allocations and establish the link that they have with three single-valued solutions for TU games.

Consider a problem  $A \in \mathcal{A}^N$  and the expression of the core solution given by (2). To select an allocation belonging to  $\gamma(A)$ , there are four conditions that have to be met: (a) firm 0 certainly receives  $I_0(A)$ , (b) essential firms and firm 0 receive up to  $g(A) - I_0(A)$ , (c) non-essential firms in the optimal region receive up to their individual benefit, and (d) firms outside the optimal region receive zero.

We provide the following procedure to find an allocation  $x \in \gamma(A)$ . To ensure that condition (a) is met, first we give the payoff  $(I_0(A), 0, \dots, 0)$ . To allocate the surplus,  $g(A) - I_0(A)$ , fulfilling conditions (b), (c) and (d), we consider a certain bankruptcy problem.

In bankruptcy problems, a group of agents have some claims over a resource, but it is insufficient to satisfy all the demands. Let  $N$  be a finite set of agents. Each agent  $i \in N$  has a *claim*  $c_i \in \mathbb{R}^+$  over an *estate*  $E \in \mathbb{R}^+$ . A **bankruptcy problem** on  $N$  is a pair  $(E, c)$ , where  $E$  is the estate and

$c = (c_i)_{i \in N}$  is a vector of claims, with  $0 \leq E \leq \sum_{i \in N} c_i$ . The class of bankruptcy problems on  $N$  is denoted by  $BR^N$ .

A **bankruptcy rule** (or simply, a rule) is a function  $f : BR^N \rightarrow \mathbb{R}^n$  that assigns to each bankruptcy problem  $(E, c) \in BR^N$  a vector  $f(E, c) \in \mathbb{R}^n$  such that  $\sum_{i \in N} f_i(E, c) = E$  and  $0 \leq f(E, c) \leq c$ . A rule gives a distribution of the estate as a function of the agents' claims.

Now, for any  $A \in \mathcal{A}^N$ , let  $(g(A) - I_0(A), c^A) \in BR^{N_0}$  be such that, for all  $i \in N_0$ ,

$$c_i^A = \begin{cases} g(A) - I_0(A), & \text{if } i \in E_{k^*} \cup \{0\}, \\ b_i^{k^*}, & \text{if } i \in P_{k^*} \setminus E_{k^*}, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

In this bankruptcy problem, the estate to distribute is the surplus and every firm claims their maximum possible payoff. Recall that a bankruptcy rule gives to each agent an non-negative amount that is at most what they claim. Due to this definition and the claims given in (3), we can ensure that any allocation of the form  $x = (I_0(A), 0, \dots, 0) + f(g(A) - I_0(A), c^A)$  belongs to the core solution. Therefore, to divide the surplus, it is enough to solve the bankruptcy problem  $(g(A) - I_0(A), c^A) \in BR^{N_0}$ .

We propose three well-known rules for bankruptcy problems to allocate the surplus: the Talmud rule (Aumann and Maschler, 1985), the proportional rule, and the random arrival rule (O'Neill, 1982).

**Talmud rule.** For every  $(E, c) \in BR^N$  and all  $i \in N$ ,

$$TAL_i(E, c) = \begin{cases} \min \left\{ \lambda, \frac{1}{2} c_i \right\} & \text{if } 2E \leq \sum_{j \in N} c_j, \\ \max \left\{ c_i - \mu, \frac{1}{2} c_i \right\}, & \text{otherwise,} \end{cases}$$

where  $\lambda$  and  $\mu$  are chosen such that  $\sum_{i \in N} TAL_i(E, c) = E$ .

This rule establishes that no agent will get more than half of their claim if the estate is below half of the aggregate claim, and nobody will lose more than half of their claim if the estate is above half of the aggregate claim.

**Proportional rule.** For every  $(E, c) \in BR^N$  and all  $i \in N$ ,  $PR_i(E, c) = \frac{c_i}{\sum_{j \in N} c_j} E$ .

This rule divides the estate proportionally to the agents' claims.

**Random arrival rule.** For every  $(E, c) \in BR^N$  and all  $i \in N$ ,

$$RA_i(E, c) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \min \left\{ c_i, \max \left\{ E - \sum_{j \in S_\pi(i)} c_j, 0 \right\} \right\},$$

where  $\Pi^N$  is the set of all orderings of  $N$  and  $S_\pi(i) = \{j \in N | \pi(j) < \pi(i)\}$  denote the set of elements of  $N$  which come before  $i$  in the order given by  $\pi$ .

This rule considers all possible arrivals of agents and assigns them their claim or what is left over after those who arrived earlier have already received their claims.

Although there are many other rules for bankruptcy problems (for a survey, see (Thomson, 2015)), we propose these rules because of the link they have with three interesting single-valued solutions for TU games: the nucleolus (Schmeidler, 1969), the  $\tau$ -value (Tijs, 1981), and the Alexia value (Tijs et al., 2011).

Given a game  $(N, v)$ , the **excess** of  $S \subseteq N$  with respect to  $x \in I(v)$  is defined as  $e(S, x) = v(S) - x(S)$ . The excess is a measurement of the dissatisfaction that coalition  $S$  has when the imputation  $x$  is realized. For each  $x \in I(v)$ , let  $\theta(x) \in R^{2^n}$  be the vector of all excesses arranged in non-increasing order, *i.e.*,  $\theta_i(x) \geq \theta_j(x)$  if  $1 \leq i < j \leq 2^n$ . For any  $x, y \in I(v)$ , we say that  $x$  is **more acceptable** than  $y$  (and we write  $x \succ y$ ) if there is an integer  $1 \leq j \leq 2^n$  such that  $\theta_i(x) = \theta_i(y)$  if  $1 \leq i < j$  and  $\theta_j(x) < \theta_j(y)$ . As usual,  $x \succeq y$  if either  $x \succ y$  or  $x = y$ . The nucleolus consists of those imputations such that there is no other more acceptable. Formally, the **nucleolus** of  $v$  is the set  $\eta(v) = \{x \in I(v) | x \succeq y, \forall y \in I(v)\}$ . It is known that if  $I(v) \neq \emptyset$ , the nucleolus is non-empty and contains a unique allocation. Furthermore if  $C(v) \neq \emptyset$ , the nucleolus belongs to the core.

In the subclass of games where the core cover is non-empty, the  $\tau$ -value is defined as the unique efficient vector which is a convex combination of  $m(v)$  and  $M(v)$ . Let  $(N, v)$  be such that  $CC(v) \neq \emptyset$ . The  **$\tau$ -value** of  $v$  is defined as  $\tau(v) = \rho M(v) + (1 - \rho)m(v)$ , with  $\rho \in [0, 1]$  such that  $\sum_{i \in N} \tau_i(v) = v(N)$ .

Now, let  $(v, N)$  be such that  $C(v) \neq \emptyset$ . For any  $\pi \in \Pi^N$ , the vector  $\lambda^\pi \in \mathbb{R}^n$  is called the **lexinal** and it is defined as  $\lambda_{\pi(k)}^\pi(v) = \max \left\{ x_{\pi(k)} : x \in C(v) \text{ and } x_{\pi(\ell)} = \lambda_{\pi(\ell)}^\pi(v), \forall \ell \in \{1, \dots, k-1\} \right\}$ , for all  $k \in \{1, \dots, n\}$ . The lexinal is recursively defined such that every player gets the maximum they can obtain inside the core considering that players before them (in the order given by  $\pi$ ) obtain their restricted maxima. Once the lexinal for each possible order is calculated, the **Alexia value** of  $v$  is defined as the average over all of them, *i.e.*,  $\alpha(v) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \lambda^\pi(v)$ .

Notice that while the nucleolus and the Alexia value are by definition core allocations (when the core is non-empty), the  $\tau$ -value not always provides a core element. Nevertheless, the fact

that the core cover and the core are equivalent for  $w^A$  ensures that the  $\tau$ -value is a core allocation.

The next theorem shows some results in the literature establishing the links between the bankruptcy rules and the single-valued solutions for TU games we have introduced.

**Theorem 3.** *Let  $(N, v)$  be a game such that  $CC(v) = C(v) \neq \emptyset$ . Then,*

- $\eta(v) = m(v) + TAL(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v))$  (Quant et al., 2005),
- $\tau(v) = m(v) + PR(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v))$  (González-Díaz et al., 2005),
- $\alpha(v) = m(v) + RA(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v))$  (Tijjs et al., 2011).

From Proposition 2 and the definition of the claims  $c^A$  given by (3), one can easily verify that  $w^A(N_0) - \sum_{i \in N} m_i(w^A) = g(A) - I_0(A)$  and  $M(w^A) - m(w^A) = c^A$ . This implies that if we apply the Talmud rule to the bankruptcy problem in the provided procedure, the allocation selected is the nucleolus of  $w^A$ ; when we use the proportional rule, we have the  $\tau$ -value; and if the random arrival rule is used, we get the Alexia rule. We formally state this in the following corollary, which is direct result from Theorem 3 and Proposition 2.

**Corollary 1.** *For every  $A \in \mathcal{A}^N$ ,*

- $\eta(w^A) = (I_0(A), 0, \dots, 0) + TAL(g(A) - I_0(A), c^A)$ ,
- $\tau(w^A) = (I_0(A), 0, \dots, 0) + PR(g(A) - I_0(A), c^A)$ , and
- $\alpha(w^A) = (I_0(A), 0, \dots, 0) + RA(g(A) - I_0(A), c^A)$ .

Then, we have two interpretations for these allocations, one by means of bankruptcy problems and the other from TU games. Other conclusion that we can get from this result is that, in the subset of problems where  $P_{k^*} = E_{k^*}$ , the nucleolus, the  $\tau$ -value, and the Alexia value of  $w^A$  coincide with the egalitarian location rule (Bergantiños and Navarro-Ramos, 2022). This rule works as the procedure we have provided, but the surplus is divided equally among all firms in  $P_{k^*} \cup \{0\}$ .

**Example 3.** *Consider the problem  $A$  in Example 1. We calculate the allocations resulting from applying the three rules to the provided procedure. Since  $g(A) = 20$  and  $I_0(A) = 16$ , the claim of any  $i \in N_0$  is*

$$c_i^A = \begin{cases} 4, & \text{if } i \in \{0, 3\}, \\ 2, & \text{if } i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, any allocation  $x = (16, 0, 0, 0, 0) + f(4, (4, 0, 2, 4, 0))$  belongs to the core solution. Table 3 contains the allocations  $x$  resulting from applying the Talmud rule, the proportional rule and the random arrival rule to the bankrupt problem.

| Bankruptcy rule     | Solution for $w^A$ | Allocation                |
|---------------------|--------------------|---------------------------|
| Talmud rule         | Nucleolus          | (17.50, 0, 1.00, 1.50, 0) |
| Proportional rule   | $\tau$ -value      | (17.60, 0, 0.80, 1.60, 0) |
| Random arrival rule | Alexia value       | (17.67, 0, 0.67, 1.67, 0) |

Table 3: Three allocations in  $\gamma(A)$ .

*Deciding which of these allocations is more convenient depends on the criteria to consider. For example, the non-essential firm receives the most in the allocation given by applying the Talmud rule; then the proportional rule; and they receive the least when we apply the random arrival rule. In this sense, we could say that the nucleolus “treats smaller companies better”.*

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## Appendix A. Proof of the results

*Proof of Lemma 1.* Let  $A \in \mathcal{A}^N$  and  $k^* \in R^*$ .

- (a) Assume that  $I_0(A) = b_0^{k^*}$ , then  $b_0^{k^*} \geq s(A)$ . Let  $i \in P_{k^*}$ . Since  $b_j^{k^*} \geq 0$ , for all  $j \in N_0$ ,  $b^{k^*}(N_0 \setminus \{i\}) = b_0^{k^*} + b^{k^*}(N_0 \setminus \{i\}) \geq s(A)$ . Therefore,  $i$  is non-essential and  $E_{k^*} = \emptyset$ .
- (b) Assume that there exists  $i \in E_{k^*}$ . By the definition of essential firm,  $b^{k^*}(N_0 \setminus \{i\}) < s(A)$ . Then,  $b_0^{k^*} \leq b^{k^*}(N_0 \setminus \{i\}) < s(A)$ . Thus,  $I_0(A) = \max\{b_0^{k^*}, s(A)\} = s(A)$ .
- (c) Assume that  $|R| > 1$ . We prove that if  $i$  is not essential, its individual benefit is zero. Take  $\ell \in R^* \setminus \{k^*\}$ . Because  $b_i^\ell = 0$  and  $\ell$  is an optimal region,  $b^{k^*}(N_0 \setminus \{i\}) \leq b^\ell(N_0 \setminus \{i\}) = g(A) = s(A)$ . Assume that  $i \in P_{k^*} \setminus E_{k^*}$ . Therefore,  $b^{k^*}(N_0 \setminus \{i\}) \geq s(A) = g(A)$ . Then,  $b^{k^*}(N_0 \setminus \{i\}) = g(A) = b^{k^*}(N_0)$ , that is,  $b_i^{k^*} = 0$ .

□



*Proof of Proposition 1.* Let  $S \subseteq T \subseteq N_0$ . Since  $b_i^k \geq 0$  for all  $i \in N_0$  and all  $k \in R$ ,  $b^k(S) \leq b^k(T)$  for every  $k \in R$ . Then,  $w^A(S) \leq w^A(T)$  and  $w^A$  is monotone.

Now, let  $S, T \subseteq N_0$  be such that  $S \cap T = \emptyset$ . If  $0 \notin S \cup T$ ,  $w^A(S \cup T) = w^A(S) = w^A(T) = 0$ . If  $0 \in S \cup T$ , assume, without loss of generality, that  $0 \in S$ . Then  $0 \notin T$  and since  $w^A$  is monotone,  $w^A(S \cup T) \geq w^A(S) = w^A(S) + w^A(T)$ . Therefore,  $w^A$  is superadditive.

Finally, we prove that  $w^A$  may not be convex. Consider the problem  $A$  in Example 1. Take  $i = 3$ ,  $S = \{0, 1\}$  and  $T = \{0, 1, 4\}$ . Although  $S \subset T$ ,  $w^A(S \cup \{i\}) - w^A(S) = 18 - 11 = 7 > w^A(T \cup \{i\}) - w^A(T) = 18 - 16 = 2$ . Then,  $w^A$  may not be convex.  $\square$

*Proof of Theorem 1.* First, we prove “ $\subseteq$ ”, that is, we prove that every element of  $C(w^A)$  fulfills the inequalities in (2). Let  $x \in C(w^A)$ . By definition,  $x(N_0) = w^A(N_0) = g(A)$ . Let  $i \in N$ . Since  $w^A(\{i\}) = 0$ ,  $x_i \geq 0$  holds. Further since  $x(N_0 \setminus \{i\}) \geq w^A(N_0 \setminus \{i\})$ ,  $x_i = w^A(N_0) - x(N_0 \setminus \{i\}) \leq w^A(N_0) - w^A(N_0 \setminus \{i\})$ .

Let  $k^* \in R^*$ . If  $k^*$  is not unique,  $w^A(N_0 \setminus \{i\}) = w^A(N_0)$ . Therefore  $x_i = 0$ , for all  $i \in N$  and  $x_0 = w^A(N_0) = g(A)$ . This is,  $C(w^A) = \{x\} = \{(g(A), 0, \dots, 0)\}$  and then,  $x$  fulfills the inequalities in (2).

Now, assume that  $k^*$  is unique. We face two cases:

- $i \in N \setminus P_{k^*}$ . Since  $b_i^{k^*} = 0$ ,  $w^A(N_0 \setminus \{i\}) = w^A(N_0)$ . Hence,  $x_i = 0$ .
- $i \in P_{k^*}$ . There are two possibilities:
  - $i \in E_{k^*}$ . By Lemma 1-(b),  $w^A(N_0 \setminus \{i\}) = \max_{k \in R} \{b^k(N_0 \setminus \{i\})\} = s(A) = I_0(A)$ . Then,  $x_i \leq w^A(N_0) - I_0(A) = g(A) - I_0(A)$ .
  - $i \notin E_{k^*}$ . Since  $b^{k^*}(N_0 \setminus \{i\}) \geq b^k(N_0 \setminus \{i\})$ , for all  $k \in R \setminus \{k^*\}$ , then  $w^A(N_0 \setminus \{i\}) = b^{k^*}(N_0 \setminus \{i\}) \Rightarrow x_i \leq w^A(N_0) - w^A(N_0 \setminus \{i\}) = b^{k^*}(N_0) - b^{k^*}(N_0 \setminus \{i\}) = b_i^{k^*}$ .

We now prove that  $x_0 \geq I_0(A)$ . Assume first that  $I_0(A) = b_0^{k^*}$ . By the definition of  $I_0(A)$ ,  $b_0^k \leq b_0^{k^*}$ , for any  $k \in R \setminus \{k^*\}$ . Then,  $w^A(\{0\}) = \max_{k \in R} \{b_0^k\} = b_0^{k^*}$ . Thus,  $x_0 \geq w^A(\{0\}) = b_0^{k^*} = I_0(A)$ .

Assume now that  $I_0(A) = s(A)$ . Then there exists  $\ell \in R \setminus \{k^*\}$  such that  $b^\ell(N_0) = I_0(A)$ . Since  $x_i = 0$ , for all  $i \in N \setminus P_{k^*}$ ,  $x_0 = x_0 + \sum_{i \in P_\ell} x_i \geq w^A(\{0\} \cup P_\ell) = b_0^\ell + b^\ell(P_\ell) = I_0(A)$ .

Finally,  $x_0 = w^A(N_0) - \sum_{i \in N} x_i \leq g(A) - w^A(N) = g(A)$ .

To prove the reverse inclusion, it is sufficient to show that for every  $x \in \mathbb{R}^{n+1}$  in the set defined by (2),  $x(S) \geq w^A(S)$ , for all  $S \subseteq N_0$ .

Let  $S \subseteq N_0$ . If  $0 \notin S$ ,  $w^A(S) = 0$  and  $x(S) \geq 0 = w^A(S)$ . If  $0 \in S$ , there are two possibilities:

- $w^A(S) \leq I_0(A)$ . Then, since  $x_i = 0$  for all  $i \in N \setminus P_{k^*}$ ,  $x_i \geq 0$ , for all  $i \in P_{k^*}$  and  $x_0 \geq I_0(A)$ ,  $x(S) = x_0 + x(S \cap P_{k^*}) \geq I_0(A) \geq w^A(S)$ .
- $w^A(S) > I_0(A)$ . We need to prove a couple of things:
  - $w^A(S) = b^{k^*}(S)$ . Assume that there is  $k \in R \setminus \{k^*\}$  such that  $w^A(S) = b^k(S)$ . Then we have  $w^A(S) = b_0^k + b^k(S \cap P_k) \leq b_0^k + b^k(P_k) = b^k(N_0 \setminus P_{k^*}) \leq I_0(A)$ , a contradiction. Therefore,  $w^A(S) = b^{k^*}(S)$ .
  - $E_{k^*} \subset S$ . If  $E_{k^*} = \emptyset$ , the inclusion is true. If  $E_{k^*} \neq \emptyset$ , assume that there is  $i \in E_{k^*}$  such that  $i \notin S$ . Since  $S \cap P_{k^*} \subseteq P_{k^*} \setminus \{i\}$ , then  $w^A(S) = b_0^{k^*} + b^{k^*}(S \cap P_{k^*}) \leq b_0^{k^*} + b^{k^*}(P_{k^*} \setminus \{i\}) < s(A) \leq I_0(A)$ . Again, this is a contradiction. Therefore,  $E_{k^*} \subset S$ .

Then, since  $x_i \geq b_i^{k^*}$  for all  $i \in P_{k^*} \setminus S$ ,  $x(S) = x(N_0) - x(N_0 \setminus S) = g(A) - x(P_{k^*} \setminus S) \geq g(A) - b^{k^*}(P_{k^*} \setminus S) = b_0^{k^*} + b_i^{k^*}(S \cap P_{k^*}) = b^{k^*}(S) = w^A(S)$ .

□

*Proof of Proposition 2.* Let  $k^* \in R^*$ . Assume that  $k^*$  is not unique. Then,  $w^A(N_0 \setminus \{i\}) = g(A)$ , for all  $i \in N$  and  $M(w^A) = (g(A), 0, \dots, 0)$ . We know by Lemma 1-(c) that every firm in an optimal region either is essential or their individual benefit is 0. Therefore, we have the desired expression for  $M(w^A)$ . Now, let  $S \subseteq N_0$  such that  $i \in S$ , for any  $i \in N$ . If  $0 \notin S$ ,  $w^A(S) - \sum_{j \in S \setminus \{i\}} M_j(w^A) = 0$ . If  $0 \in S$ ,  $w^A(S) - \sum_{j \in S \setminus \{i\}} M_j(w^A) = w^A(S) - g(A)$ . Since  $w^A$  is monotone,  $g(A) \geq w^A(S)$ , for all  $S \subseteq N_0$ . Therefore, the maximum value that  $w^A(S) - g(A)$  can reach is zero. Hence,  $m_i(w^A) = 0$  for all  $i \in N$ . Again, since  $w^A$  is monotone and the fact that in this situation  $g(A) = I_0(A)$ ,

$$m_0(w^A) = \max_{S \subseteq N_0: 0 \in N_0} \left\{ w^A(S) - \sum_{j \in S \setminus \{i\}} M_j(w^A) \right\} = \max_{S \subseteq N_0: 0 \in N_0} \{w^A(S)\} = g(A) = I_0(A).$$

Now, assume that  $k^*$  is unique. Let  $i \in N$ . If  $i \in N \setminus P_{k^*}$ ,  $w^A(N_0 \setminus \{i\}) = g(A)$ . Then,  $M_i(w^A) = 0, \forall i \in N \setminus P_{k^*}$ . If  $i \in P_{k^*}$ , there are two cases:

- $i \in E_{k^*}$ . In the proof of Theorem 1, we showed that  $w^A(N_0 \setminus \{i\}) = I_0(A)$ . Then,  $M_i(w^A) = g(A) - I_0(A)$ .
- $i \in P_{k^*} \setminus E_{k^*}$ . Then,  $w^A(N_0 \setminus \{i\}) = b^{k^*}(N_0 \setminus \{i\})$  and  $M_i(w^A) = g(A) - b^{k^*}(N_0 \setminus \{i\}) = b_i^{k^*}$ .

Finally,  $M_0(w^A) = g(A) - 0 = g(A)$ .

We now compute  $m(w^A)$ . Let  $i \in N$  and  $S \subseteq N_0$  with  $i \in S$ . If  $0 \notin S$ , since  $M_j(w^A) \geq 0$ , for all  $j \in N_0$  and  $w^A(S) = 0$ ,  $w^A(S) - \sum_{j \in S \setminus \{i\}} M_j(w^A) \leq 0$ . If  $0 \in S$ , since  $M_0(w^A) = g(A)$  and  $w^A(S) \leq w^A(N_0) = g(A)$ ,

$$w^A(S) - \sum_{j \in S \setminus \{i\}} M_j(w^A) \leq g(A) - g(A) - \sum_{j \in S \setminus \{0, i\}} M_j(w^A) = - \sum_{j \in S \setminus \{0, i\}} M_j(w^A) \leq 0.$$

Finally, we need to find  $T \subseteq N_0$  with  $i \in T$  such that  $w^A(T) - \sum_{j \in T \setminus \{i\}} M_j(w^A) = 0$ . We can achieve this by taking  $T = \{i\}$ . Therefore  $m_i(w^A) = 0$ , for all  $i \in N$ .

Let  $S \subseteq N_0$  with  $0 \in S$ . Assume that  $w^A(S) \leq I_0(A)$ . Since  $M_j(w^A) \geq 0 \forall j \in N$ ,  $w^A(S) - \sum_{j \in S \setminus \{0\}} M_j(w^A) \leq w^A(S) \leq I_0(A)$ .

Now, assume that  $w^A(S) > I_0(A)$ . In this case (see proof of Theorem 1),  $w^A(S) = b^{k^*}(S)$  and  $E_{k^*} \subset S$ . Then,

$$\begin{aligned} w^A(S) - \sum_{j \in S \setminus \{0\}} M_j(w^A) &= b^{k^*}(S) - \sum_{j \in S \cap (P_{k^*} \setminus E_{k^*})} M_j(w^A) - \sum_{j \in S \cap E_{k^*}} M_j(w^A) \\ &= b^{k^*}((S \cap P_{k^*}) \cup \{0\}) - b^{k^*}(S \cap (P_{k^*} \setminus E_{k^*})) - |E_{k^*}|(g(A) - I_0(A)) \\ &= b^{k^*}(E_{k^*} \cup \{0\}) - |E_{k^*}|(g(A) - I_0(A)) \\ &\leq g(A) - (g(A) - I_0(A)) = I_0(A). \end{aligned}$$

As before, it only remains to find a coalition  $T \subseteq N_0$  with  $0 \in T$  such that  $w^A(T) - \sum_{j \in T \setminus \{0\}} M_j(w^A) = I_0(A)$ . We need to consider the two possibilities over  $I_0(A)$ . If  $I_0(A) = b_0^{k^*}$ , take  $T = \{0\}$ . Then  $w^A(T) - \sum_{j \in T \setminus \{0\}} M_j(w^A) = b_0^{k^*} = I_0(A)$ . Now, if  $I_0(A) = s(A)$ , consider  $T = P_k \cup \{0\}$  such that  $w^A(P_k \cup \{0\}) = s(A) = I_0(A)$ . Since  $k \neq k^*$ ,  $w^A(P_k \cup \{0\}) - \sum_{j \in P_k} M_j(w^A) = w^A(P_k \cup \{0\}) = I_0(A)$ . Therefore,  $m_0(w^A) = I_0(A)$ .  $\square$

*Proof of Lemma 2.* Let  $A \in \mathcal{A}^N$ ,  $x \in \sigma(A)$  and  $i \in N$ . By the definitions of global benefit,  $i$ -reduced problem and the game  $w^A$ ,  $g(A \setminus i) = \max_{k \in R} \{(b \setminus i)^k(N_0 \setminus \{i\})\} = \max_{k \in R} \{b^k(N_0 \setminus \{i\})\} = w^A(N_0 \setminus \{i\})$ .

Considering the proof of Proposition 2, we have that, for all  $i \in N$  and every  $k^* \in R^*$ ,

$$g(A \setminus i) = \begin{cases} I_0(A), & \text{if } i \in E_{k^*}, \\ b^{k^*}(N_0 \setminus \{i\}), & \text{if } i \in P_{k^*} \setminus E_{k^*}, \\ g(A), & \text{otherwise.} \end{cases}$$

Since  $\sigma$  satisfies  $MA$ , for every  $k^* \in R^*$ ,

$$\sigma(A) = \left\{ x \in \mathbb{R}^{n+1} : \begin{array}{l} x_i \leq g(A) - I_0(A), \forall i \in E_{k^*}, \\ x_i \leq b_i^{k^*}, \forall i \in P_{k^*} \setminus E_{k^*}, \\ x_i \leq 0, \forall i \in N \setminus P_{k^*} \end{array} \right\}. \quad (\text{A.1})$$

Now, by  $PE$ ,  $x_i \geq 0$  for all  $i \in N_0$ . Furthermore, since  $x$  is an allocation *i.e.*  $x(N_0) = g(A)$ , we conclude that  $x_0 \leq g(A)$ . Combining the above with (A.1), we have

$$\sigma(A) = \left\{ x \in \mathbb{R}^{n+1} : \begin{array}{l} 0 \leq x_0 \leq g(A), \\ 0 \leq x_i \leq g(A) - I_0(A), \forall i \in E_{k^*}, \\ 0 \leq x_i \leq b_i^{k^*}, \forall i \in P_{k^*} \setminus E_{k^*}, \\ x_i = 0, \forall i \in N \setminus P_{k^*} \end{array} \right\}. \quad (\text{A.2})$$

Clearly  $\gamma(A) \subseteq \sigma(A)$ , by the expression of the core of  $w^A$  given in (2).  $\square$

*Proof of Theorem 2.* First, we prove that  $\gamma(A)$  satisfies the three axioms. By Theorem 1, we know that  $C(w^A)$  satisfies  $PE$  and  $MC$ , therefore,  $\gamma(A)$  also satisfies  $PE$  and  $MC$ . Now, consider  $A, A' \in \mathcal{A}^N$  as the statement of  $NFM$ . Then, for all  $x \in \gamma(A)$ ,  $x_0 \geq I_0(A) = I_0(A') = \min_{y \in \gamma(A')} \{y_0\}$ . Hence,  $\gamma(A)$  satisfies  $NFM$ .

For the uniqueness, let  $\sigma$  be a solution satisfying the three axioms. Lemma 2 tells us that  $\gamma(A) \subseteq \sigma(A)$ . Now, we prove the reverse inclusion. By the definition of a solution, we can take  $x \in \sigma(A)$ .

Assume first that  $|R^*| > 1$ . In this case,  $g(A) = I_0(A)$ . By (2),  $\gamma(A) = C(w^A) = \{(g(A), 0, \dots, 0)\}$ . Let  $i \in N$  and  $k^*$  any optimal region. Because  $\sigma$  satisfies  $PE$  and  $MA$ ,  $x$  belongs to the set given by (A.2). Then,  $x_i = 0$  if  $i \in N \setminus P_{k^*}$ . Consider now that  $i \in P_{k^*}$ . By Lemma 1-(c), we know that either  $i \in E_{k^*}$  or  $b_i^{k^*} = 0$ . Considering the expression (A.2), we conclude that  $x_i = 0$ . Now, if  $i \in E_{k^*}$ ,  $0 \leq x_0 \leq g(A) - I_0(A) = 0$ . Therefore,  $x_i = 0$ , for all  $i \in N$  and  $x_0 = g(A)$  *i.e.*  $\sigma(A) = \{x\} = \gamma(A)$ .

Now, assume that  $R^* = \{k^*\}$ . By expressions (2) and (A.2), it only remains to show that  $x_0 \geq I_0(A)$ . There are two possible cases on  $I_0(A)$ .

- $I_0(A) = s(A) > b_0^{k^*}$ . Let  $A^1 = (N_0, P, b^1) \in \mathcal{A}^N$  such that  $b^{1k^*}(P_{k^*}) = s(A) - b_0^{k^*}$  and  $b_k^{1k} = b_k^k$ , otherwise. In  $A^1$  there are at least two optimal regions,  $k^*$  and any other region which aggregate benefit is  $s(A)$ . Since  $g(A^1) = s(A) = I_0(A)$ , by the expression of  $\sigma$  given in (A.2),  $\sigma(A^1) = \{(I_0(A), 0, \dots, 0)\}$ . Since  $g(A) > I_0(A) = g(A^1)$  and  $\min_{y \in \sigma(A^1)} \{y_0\} = I_0(A)$ , by  $NFM$ ,  $x_0 \geq I_0(A)$ .

- $I_0(A) = b_0^{k^*} \geq s(A)$ . Let  $A^2 = (N_0, P, b^2) \in \mathcal{A}^N$  such that  $b_i^{2k^*} = 0$ , for all  $i \in P_{k^*}$  and  $b_k^{2k} = b_k^k$ , otherwise. In this problem, all firms in  $P_{k^*}$  are non-essential because  $b^{2k^*}(N_0 \setminus \{i\}) = b_0^{2k^*} = b_0^{k^*} \geq s(A^2)$ . Since for every  $y \in \sigma(A^2)$   $0 \leq y_i \leq b_i^{k^*}$  and  $g(A^2) = b_0^{k^*}$  then,  $\sigma(A^2) = \{(I_0(A), 0, \dots, 0)\}$ . Following the same line of reasoning as before, by *NFM*,  $x_0 \geq I_0(A)$ .

By the expression of the core given by (2),  $x \in C(w^A) = \gamma(A)$ . Hence,  $\sigma(A) \subseteq \gamma(A)$ . By Lemma 2,  $\sigma(A) = \gamma(A)$ .  $\square$