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Density results for smooth functions in weighted Sobolev spaces

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1 Introduction

The study of elliptic PDEs is tightly linked with Sobolev spaces. In particular, the presence of a divergence term with a weight function leads to introduce the notion of weighted Sobolev spaces. The Navier-Stokes equations with singular viscosity represent a good example of a situation where this kind of function spaces is needed.

Generally speaking, it seems difficult to give a general definition of Sobolev spaces, given the various types of weights likely to be used in practice. Many examples and definitions are given for instance in [4, 8].

The type of weight used in each situation obviously depends on the domain. Some whole space cases are treated namely in [5, 6], but the present chapter focuses on a regular bounded domain of \mathbb{R}^N , named Ω .

The density of smooth functions in such a normed space is an interesting, but tricky issue. Some examples and counter-examples are given in [9, 11, 10], while [7] states regularity and density results, and some of their applications to non-linear PDEs.

This chapter considers a bounded open domain of regularity \mathcal{C}^1 in \mathbb{R}^N , named Ω . The weight ϱ is a \mathcal{C}^1 positive function such that

$$(1.1) \quad \lim_{\substack{x \in \Omega \\ d(x, \partial\Omega) \rightarrow 0}} \frac{\varrho(x)}{d(x, \partial\Omega)} = 1 \quad \text{and} \quad \forall \omega \subset\subset \Omega, \quad \varrho_\omega = \sup_\omega \varrho > 0.$$

This corresponds to the power weights defined in [4, 7]. When the density does not stand, it can be necessary to define a trace operator for Sobolev functions, which is often a hard task. This is made for instance in [2] in a 2D square.

Let $\eta \in [0; 1[$, we start by focusing on the following η -weighted space

$$(1.2) \quad W_\eta(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) / u \in L^1(\Omega), \varrho^{\eta/2} \nabla u \in L^2(\Omega)^N \right\},$$

equipped with the norm

$$(1.3) \quad \|u\|_{W_\eta(\Omega)} = \|u\|_{L^1(\Omega)} + \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N}.$$

More precisely, we prove that the subspace $\mathcal{D}(\Omega)$ of C^∞ compact-supported functions in Ω is not dense in $W_\eta(\Omega)$. Actually, its closure is the subspace of null-trace $W_\eta(\Omega)$ functions, which has a sense if and only if a trace can be defined on $W_\eta(\Omega)$. This is done in Section 2.

This space is used for instance in [1], in order to find weak solutions to a turbulence model derived from the Navier-Stokes equations. Unfortunately, the limit case $\eta = 1$ cannot be treated the same way, and still remains an open problem.

Nonetheless, it is possible to state similar results in the following double-weighted space

$$(1.4) \quad H_\varrho(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) / \varrho^{-1/2} u \in L^2(\Omega), \varrho^{1/2} \nabla u \in L^2(\Omega)^N \right\},$$

which is a Hilbert space when equipped with the scalar product

$$(1.5) \quad (u, v)_{H_\varrho} = \int_\Omega \frac{uv}{\varrho} + \int_\Omega \varrho \nabla u \cdot \nabla v,$$

inducing the following Hilbert norm

$$(1.6) \quad \|u\|_{H_\varrho} = \left(\|\varrho^{-1/2} u\|_{L^2(\Omega)}^2 + \|\varrho^{1/2} \nabla u\|_{L^2(\Omega)^N}^2 \right)^{1/2}.$$

This space is used for instance in [SOLUTIONS RENORMALISEES], and Section 3 is devoted to build a trace operator on $H_\varrho(\Omega)$ and characterize the closure of $\mathcal{D}(\Omega)$ in it.

2 Density in the η -weighted space

Let $\eta \in [0; 1[$, and $W_\eta(\Omega)$ be defined by (1.2). In this section, we start by showing that $W_\eta(\Omega)$ equipped with the norm (1.3) is a Banach space, and is embedded in a $W^{1,p}(\Omega)$ space. This directly implies the existence of a L^p trace on $\partial\Omega$ for $W_\eta(\Omega)$ functions. This notion of trace is necessary to characterize the closure of $\mathcal{D}(\Omega)$ in $W_\eta(\Omega)$.

Remark 2.1. Let $u \in W_\eta(\Omega)$, then $\nabla u \in L^2_{loc}(\Omega)$. Indeed, let K be any compact subset of Ω , we have

$$(2.1) \quad \int_K |\nabla u|^2 \leq \|\varrho^{-\eta}\|_{L^\infty(K)} \int_K \varrho^\eta |\nabla u|^2.$$

Hence

$$(2.2) \quad \|u\|_{L^2(K)} \leq \|\varrho^{-\eta}\|_{L^\infty(K)}^{1/2} \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N} < +\infty.$$

2.1 Analytical results on $W_\eta(\Omega)$

2.1.1 A Banach space

Theorem 2.1. *The weighted Sobolev space $W_\eta(\Omega)$ defined just above is a Banach space when equipped with the norm*

$$(2.3) \quad \|u\|_{W_\eta} = \|u\|_{L^1(\Omega)} + \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N}.$$

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $W_\eta(\Omega)$. In particular, $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$ and $(\varrho^{\eta/2} \nabla u_n)_{n \in \mathbb{N}}$ satisfies the same property in $L^2(\Omega)^N$. Since any L^p space is complete, we obtain the existence of $u \in L^1(\Omega)$ and $\mathbf{w} \in L^2(\Omega)^N$ such that

$$(2.4) \quad u_n \xrightarrow{n \rightarrow +\infty} u \text{ in } L^1(\Omega),$$

$$(2.5) \quad \varrho^{\eta/2} \nabla u_n \xrightarrow{n \rightarrow +\infty} \mathbf{w} \text{ in } L^2(\Omega)^N.$$

Since $\varrho \in \mathcal{C}^1(\Omega)$, there exists a sequence $(\varrho_m)_{m \in \mathbb{N}}$ of $\mathcal{C}^\infty(\bar{\Omega})$ functions such that $\|\varrho_m - \varrho\|_{L^\infty(\Omega)} \xrightarrow{m \rightarrow +\infty} 0$. This yields

$$(2.6) \quad \varrho_m^{\eta/2} \nabla u_n \xrightarrow{m, n \rightarrow +\infty} \mathbf{w} \text{ in } L^2(\Omega)^N.$$

Let $\varphi \in \mathcal{D}(\Omega)$, we obtain

$$(2.7) \quad \begin{aligned} \langle \mathbf{w}, \varphi \rangle &= \lim_{m, n \rightarrow +\infty} \langle \varrho_m^{\eta/2} \nabla u_n, \varphi \rangle = \lim_{m, n \rightarrow +\infty} \langle \nabla u_n, \varrho_m^{\eta/2} \varphi \rangle = - \lim_{m, n \rightarrow +\infty} \langle u_n, \nabla (\varrho_m^{\eta/2} \varphi) \rangle \\ &= - \lim_{m \rightarrow +\infty} \langle u, \nabla (\varrho_m^{\eta/2} \varphi) \rangle = \lim_{m \rightarrow +\infty} \langle \nabla u, \varrho_m^{\eta/2} \varphi \rangle = \lim_{m \rightarrow +\infty} \langle \varrho_m^{\eta/2} \nabla u, \varphi \rangle \\ &= \langle \varrho^{\eta/2} \nabla u, \varphi \rangle. \end{aligned}$$

In other words, $\mathbf{w} = \varrho^{\eta/2} \nabla u$ in $\mathcal{D}'(\Omega)^N$. Thus, $\varrho^{\eta/2} \nabla u \in L^2(\Omega)^N$, which yields $u \in W_\eta(\Omega)$ and $u_n \xrightarrow{n \rightarrow +\infty} u$ in $W_\eta(\Omega)$. \square

2.1.2 A continuous Sobolev embedding

This subsection is devoted to state a continuous embedding for $W_\eta(\Omega)$ functions. More precisely, $W_\eta(\Omega) \hookrightarrow W^{1,p}(\Omega)$ for $p \in [1, p_c[$ with $p_c < 2$ depending on η . We start by giving a general lemma, which is a generalization of Gagliardo-Nirenberg inequality.

Lemma 2.2. *Let $p \in]1; +\infty[$ and $u \in L^1(\Omega)$ such that $\nabla u \in L^p(\Omega)^N$. Then $u \in W^{1,p}(\Omega)$ and there exists a constant $c > 0$ such that*

$$(2.8) \quad \|u\|_{L^p(\Omega)} \leq c \left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N} \right).$$

Proof. We build an iteration of critical Sobolev exponents

$$(2.9) \quad \begin{cases} p_0 = 1, \\ \forall k \in \llbracket 0; N-2 \rrbracket, \quad p_{k+1} = \left(\frac{1}{p_k} - \frac{1}{N} \right)^{-1}. \end{cases}$$

Simple computations yield

$$(2.10) \quad \forall k \in \llbracket 0; N-1 \rrbracket, \quad p_k = \frac{N}{N-k}.$$

In particular, $p_{N-1} = N$.

For all $k \in \llbracket 0; N-1 \rrbracket$ such that $p_k < p$, we have

$$(2.11) \quad \|u\|_{L^{p_k}(\Omega)} \leq c_k \left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N} \right).$$

This is easily proved by induction. The case $k = 0$ is straightforward, and we assume that (2.11) is true at rank $k-1$. Then Gagliardo-Nirenberg inequality yields

$$(2.12) \quad \|u\|_{L^{p_k}(\Omega)} \leq C \|u\|_{W^{1,p_{k-1}}(\Omega)}.$$

On one hand, $\|\nabla u\|_{L^{p_{k-1}}(\Omega)^N} \leq M \|\nabla u\|_{L^p(\Omega)^N}$, with $M > 0$ independent of u .

On the other hand, the induction assumption yields

$$(2.13) \quad \|u\|_{L^{p_{k-1}}(\Omega)} \leq c_{k-1} \left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N} \right).$$

Then (2.12) becomes

$$(2.14) \quad \|u\|_{L^{p_k}(\Omega)} \leq c_{k-1} C \|u\|_{L^1(\Omega)} + C(M + c_{k-1}) \|\nabla u\|_{L^p(\Omega)^N}.$$

Inequality (2.11) is then proved.

Two cases have to be distinguished.

- If $p \leq N$, there exists $k \in \llbracket 0; N-2 \rrbracket$ such that $p_k < p \leq p_{k+1}$. Then (2.11) yields

$$(2.15) \quad \|u\|_{L^p(\Omega)} \leq M \|u\|_{L^{p_{k+1}}(\Omega)} \leq c_{k+1} \left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N} \right).$$

- If $p > N$, (2.11) for $k = N-1$ is written

$$(2.16) \quad \|u\|_{L^N(\Omega)} \leq c_{N-1} \left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N} \right).$$

We deduce that $u \in W^{1,N}(\Omega)$, and Sobolev embedding yields that $u \in L^q(\Omega)$ for any $q > N$. In particular, $u \in L^p(\Omega)$ and

$$(2.17) \quad \|u\|_{L^p(\Omega)} \leq C \|u\|_{W^{1,N}(\Omega)} \leq c_{N-1} C \|u\|_{L^1(\Omega)} + C(M + c_{N-1}) \|\nabla u\|_{L^p(\Omega)^N}.$$

In both cases, the result is proved. □

The following result is due to Kufner (see [7]), and is necessary to obtain a $W^{1,p}(\Omega)$ regularity on $W_\eta(\Omega)$ functions.

Theorem 2.3. Let $\eta \in [0, 1[$ and $u \in \mathcal{D}'(\Omega)$ such that $\varrho^{\eta/2} \nabla u \in L^2(\Omega)^N$. Let $p \in \left[1; \frac{2}{1+\eta}\right]$, then $\nabla u \in L^p(\Omega)^N$ and we have

$$(2.18) \quad \|\nabla u\|_{L^p(\Omega)^N} \leq c \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N},$$

where the constant $c > 0$ does not depend on u .

The following continuous embedding can be deduced from Lemma 2.2 and Theorem 2.3.

Corollary 2.4. *The continuous embedding*

$$(2.19) \quad W_\eta(\Omega) \hookrightarrow W^{1,p}(\Omega)$$

stands for any $p \in \left[1; \frac{2}{1+\eta}\right]$.

Proof. Let $u \in W_\eta(\Omega)$ and $p \in \left[1; \frac{2}{1+\eta}\right]$. Theorem 2.3 yields that $\nabla u \in L^p(\Omega)$ and Lemma 2.2 yields that $u \in W^{1,p}(\Omega)$.

In addition, (2.18) and (2.8) yield

$$(2.20) \quad \begin{aligned} \|u\|_{W^{1,p}(\Omega)} &= \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N} \leq c \left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N} \right) + \|\nabla u\|_{L^p(\Omega)^N} \\ &\leq c' \left(\|u\|_{L^1(\Omega)} + \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N} \right). \end{aligned}$$

□

The continuous embedding given by Corollary 2.4 directly yields the existence of a trace for $W_\eta(\Omega)$ functions, which is at least in $L^p(\partial\Omega)$ for $p \in \left[1; \frac{2}{1+\eta}\right]$. This allows to set the subspace of $W_\eta(\Omega)$ functions with a vanishing trace.

2.2 Density result in $W_\eta(\Omega)$

The end of this section is the proof of the following theorem.

Theorem 2.5. *The closure of $\mathcal{D}(\Omega)$ in $W_\eta(\Omega)$ for the topology given by (1.3) is*

$$(2.21) \quad \overline{\mathcal{D}(\Omega)}^{W_\eta(\Omega)} = \{u \in W_\eta(\Omega) / \text{tr } u = 0 \text{ a.e. on } \partial\Omega\}.$$

The existence and continuity of the trace operator given by Corollary 2.4 make the work easier. Indeed, we have the following result.

Lemma 2.6. *Let $u \in \overline{\mathcal{D}(\Omega)}^{W_\eta(\Omega)}$, then $\text{tr } u = 0$ a.e. on $\partial\Omega$.*

Proof. Let u be in this closure. Then there exists a sequence $(u_n)_{n \in \mathbb{N}}$ of $\mathcal{D}(\Omega)$ functions such that $u_n \xrightarrow[n \rightarrow +\infty]{} u$ in $W_\eta(\Omega)$.

The continuous embedding of Corollary 2.4 yields that $u_n \xrightarrow[n \rightarrow +\infty]{} u$ in $W^{1,p}(\Omega)$, which means

$$(2.22) \quad \text{tr } u_n \xrightarrow[n \rightarrow +\infty]{} \text{tr } u \text{ in } L^p(\partial\Omega),$$

for any $p \in \left[1; \frac{2}{1+\eta}\right]$, by continuity of the trace operator. Given the smoothness of $(u_n)_{n \in \mathbb{N}}$, $\text{tr } u_n = 0$ a.e. on $\partial\Omega$ for any $n \in \mathbb{N}$. This yields $\text{tr } u = 0$ a.e. on $\partial\Omega$. □

The following of this subsection aims at proving that every null-trace $W_\eta(\Omega)$ function is the limit of a sequence of smooth functions in $W_\eta(\Omega)$.

2.2.1 Vanishing trace and density result

This subsection is focused on null-trace $W_\eta(\Omega)$ functions. More precisely, we prove the density of smooth functions in this subspace, which is the main result of this section.

Let the subspace $W_{0,\eta}(\Omega)$ be defined by

$$(2.23) \quad W_{0,\eta}(\Omega) = \{u \in W_\eta(\Omega) / \operatorname{tr} u = 0 \text{ a.e. on } \partial\Omega\}.$$

We start by defining a norm on $W_{0,\eta}(\Omega)$, only involving the weighted L^2 estimate.

Theorem 2.7. *The application*

$$(2.24) \quad \left[u \mapsto \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N} \right]$$

is a norm on $W_{0,\eta}(\Omega)$, which is equivalent to $\|\cdot\|_{W_\eta}$.

Proof. This application is clearly a seminorm. Let $u \in W_{0,\eta}(\Omega)$, both Corollary 2.4 and the null-trace assumption yield that $u \in W_0^{1,p}(\Omega)$ for $p \in \left[1; \frac{2}{1+\eta}\right]$. For such p , Poincaré inequality and (2.18) yield

$$(2.25) \quad \|u\|_{L^p(\Omega)} \leq c \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N}.$$

Let $u \in W_{0,\eta}(\Omega)$ such that $\|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N} = 0$. The previous inequality yields that $u = 0$ a.e. in Ω . Therefore, (2.24) is a norm.

Then, the definition of $\|\cdot\|_{W_\eta}$ directly yields for any $u \in W_{0,\eta}(\Omega)$

$$(2.26) \quad \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N} \leq \|u\|_{W_\eta}.$$

Conversely, Poincaré inequality and (2.18) in $W_0^{1,1}(\Omega)$ yield for any $u \in W_{0,\eta}(\Omega)$

$$(2.27) \quad \|u\|_{L^1(\Omega)} \leq c_1 \|\nabla u\|_{L^1(\Omega)} \leq c_2 \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N},$$

hence $\|u\|_{W_\eta(\Omega)} \leq (1 + c_2) \|\varrho^{\eta/2} \nabla u\|_{L^2(\Omega)^N}$. The equivalence of norms is proved. \square

Now the $W_{0,\eta}$ topology is set, the result of density of smooth functions can be stated.

Theorem 2.8. *The space $\mathcal{D}(\Omega)$ is dense in $W_{0,\eta}(\Omega)$ at the sense of the norm given by (2.24).*

The proof of Theorem 2.8 is divided into several steps, corresponding to the next subsections. The problem is first reduced by showing that $\mathcal{D}(\Omega)$ is dense in the space of compact-supported $W_\eta(\Omega)$ functions. Then, in order to give a better understanding, the fact that any $W_{0,\eta}(\Omega)$ function is the limit of a sequence of compact-supported functions is proved in small dimensions, before being extended to the general case.

2.2.2 Functions with compact support

Let the space $W_\eta^c(\Omega)$ be defined by

$$(2.28) \quad W_\eta^c(\Omega) = \{u \in W_\eta(\Omega) / \text{Supp}(u) \subset\subset \Omega\}.$$

The purpose is showing that $\mathcal{D}(\Omega)$ is dense in $W_\eta^c(\Omega)$ for the $W_{0,\eta}(\Omega)$ norm defined by (2.24).

Lemma 2.9. *Let $u \in W_\eta^c(\Omega)$. There exists a sequence $(u_n)_{n \in \mathbb{N}}$ of $\mathcal{D}(\Omega)$ functions such that*

$$(2.29) \quad \|\varrho^{\eta/2}(\nabla u_n - \nabla u)\|_{L^2(\Omega)^N} \xrightarrow{n \rightarrow +\infty} 0.$$

Proof. First, we set $K = \text{Supp}(u) \subset\subset \Omega$, and write

$$(2.30) \quad \int_\Omega |\nabla u|^2 \leq \left\| \frac{1}{\varrho} \right\|_{L^\infty(K)}^\eta \int_\Omega \varrho^\eta |\nabla u|^2.$$

This yields that $u \in L^1(\Omega)$ and $\nabla u \in L^2(\Omega)^N$. Therefore, Lemma 2.2 yields that $u \in H^1(\Omega)$. And given that u is compact-supported, we obtain $u \in H_0^1(\Omega)$.

Consequently, there exists a sequence $(u_n)_{n \in \mathbb{N}}$ of $\mathcal{D}(\Omega)$ functions such that $u_n \xrightarrow{n \rightarrow +\infty} u$ in $H^1(\Omega)$. This implies in particular $\|\nabla u_n - \nabla u\|_{L^2(\Omega)^N} \xrightarrow{n \rightarrow +\infty} 0$, which yields

$$(2.31) \quad \|\varrho^{\eta/2}(\nabla u_n - \nabla u)\|_{L^2(\Omega)^N} \leq \|\varrho\|_{L^\infty(\Omega)}^{\eta/2} \|\nabla u_n - \nabla u\|_{L^2(\Omega)^N} \xrightarrow{n \rightarrow +\infty} 0.$$

The result is proved. \square

The following of this subsection aims at proving that $W_\eta^c(\Omega)$ is dense in $W_{0,\eta}(\Omega)$. This is first proved in an interval (dimension 1), and then on a bounded block (dimension $N \geq 2$). Finally, a change of variables allows to go from a bounded block to any bounded \mathcal{C}^1 domain.

2.2.3 Proof in dimension 1

In dimension 1, Theorem 2.4 and Sobolev embeddings yield

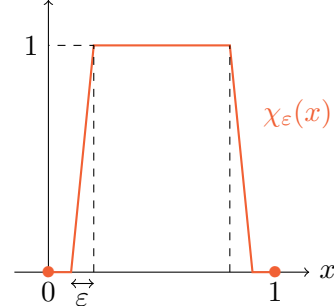
$$(2.32) \quad W_\eta(0; 1) \hookrightarrow \mathcal{C}^0([0; 1]).$$

We deduce another way to characterize $W_{0,\eta}(0; 1)$:

$$(2.33) \quad W_{0,\eta}(0; 1) = \{u \in W_\eta(0; 1) / u(0) = u(1) = 0\}.$$

Let the continuous approximation $(\chi_\varepsilon)_{\varepsilon > 0}$ be defined by

$$(2.34) \quad \chi_\varepsilon : x \mapsto \begin{cases} 0 & \text{if } 0 \leq x \leq \varepsilon, \\ \frac{x}{\varepsilon} - 1 & \text{if } \varepsilon < x < 2\varepsilon, \\ 1 & \text{if } 2\varepsilon \leq x \leq 1 - 2\varepsilon, \\ \frac{1-x}{\varepsilon} - 1 & \text{if } 1 - 2\varepsilon < x < 1 - \varepsilon, \\ 0 & \text{if } 1 - \varepsilon \leq x \leq 1. \end{cases}$$



The density of $W_\eta^c(0; 1)$ in $W_{0,\eta}(0; 1)$ corresponds to the following result.

Theorem 2.10. *Let $u \in W_{0,\eta}(0; 1)$ and $(\chi_\varepsilon)_{\varepsilon>0}$ be defined like just above. Then*

$$(2.35) \quad u\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u \quad \text{in } W_{0,\eta}(0; 1).$$

Proof. First, we clearly see that $0 \leq \chi_\varepsilon \leq 1$ and $\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 1$ a.e. in $[0; 1]$. Lebesgue theorem then yields

$$(2.36) \quad \varrho^{\eta/2} u' \chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \varrho^{\eta/2} u' \quad \text{in } L^2(0; 1)^N.$$

It remains to prove the convergence

$$(2.37) \quad \varrho^{\eta/2} u \chi_\varepsilon' \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L^2(0; 1)^N.$$

We first write

$$(2.38) \quad \int_0^1 \varrho^\eta |u \chi_\varepsilon'|^2 dx = \frac{1}{\varepsilon^2} \left(\int_\varepsilon^{2\varepsilon} \varrho(x)^\eta |u(x)|^2 dx + \int_{1-2\varepsilon}^{1-\varepsilon} \varrho(x)^\eta |u(x)|^2 dx \right).$$

We focus on the first term of (2.38) (the second may be treated a similar way). The fact that $\varrho(x)$ and x are equivalent when x goes to zero can be written

$$(2.39) \quad \forall x \in]0; \varepsilon_0], \quad M_1 x \leq \varrho(x) \leq M_2 x,$$

where $\varepsilon_0 > 0$ is sufficiently small, and $C_1, C_2 > 0$ do not depend on x . Since $u \in W_{0,\eta}(0; 1)$, (2.33) yields the following Taylor formula

$$(2.40) \quad u(x) = \int_0^x u'(t) dt \quad \text{a.e. } x \in [0; 1].$$

Hölder inequality then yields

$$(2.41) \quad |u(x)|^2 \leq \left(\int_0^x \varrho(t)^\eta |u'(t)|^2 dt \right) \left(\int_0^x \frac{dt}{\varrho(t)^\eta} \right) \quad \text{a.e. } x \in [0; 1].$$

Let $\varepsilon > 0$ such that $2\varepsilon \leq \varepsilon_0$, (2.39) yields

$$(2.42) \quad \forall x \in [\varepsilon; 2\varepsilon], \quad \int_0^x \frac{dt}{\varrho(t)^\eta} \leq c_1(\eta) \int_0^x \frac{dt}{t^\eta} = c_1(\eta) \frac{x^{1-\eta}}{1-\eta}.$$

Multiplying by $\varrho(x)^\eta$ and integrating on $[\varepsilon; 2\varepsilon]$ yield

$$(2.43) \quad \begin{aligned} \int_\varepsilon^{2\varepsilon} \varrho(x)^\eta |u(x)|^2 dx &\leq c_2(\eta) \int_\varepsilon^{2\varepsilon} x^\eta |u(x)|^2 dx \\ &\leq c_3(\eta) \left(\int_0^{2\varepsilon} \varrho(t)^\eta |u'(t)|^2 dt \right) \left(\int_\varepsilon^{2\varepsilon} x dx \right). \end{aligned}$$

We deduce

$$(2.44) \quad \frac{1}{\varepsilon^2} \int_\varepsilon^{2\varepsilon} \varrho(x)^\eta |u(x)|^2 dx \leq c_4(\eta) \int_0^{2\varepsilon} \varrho(t)^\eta |u'(t)|^2 dt.$$

Given that $u \in W_\eta(0; 1)$, Lebesgue theorem yields

$$(2.45) \quad \int_0^{2\varepsilon} \varrho(t)^\eta |u'(t)|^2 dt \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The result is proved. □

Remark 2.2. The previous result is on the interval $[0; 1]$, but a classical linear change of variables may yield the convergence on any interval $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$.

We now aim at extending Theorem 2.10 to higher dimensions.

2.2.4 Proof on the unit hypercube in dimension $N \geq 2$

All along this subsection, Q_N refers to the unit hypercube in dimension $N \geq 2$, that is to say $Q_N =]0; 1[^N$.

We start by recalling the Taylor formula.

Lemma 2.11. *Let $u \in W_{0,\eta}(Q_N)$. For any $j \in \llbracket 1; N \rrbracket$, we have*

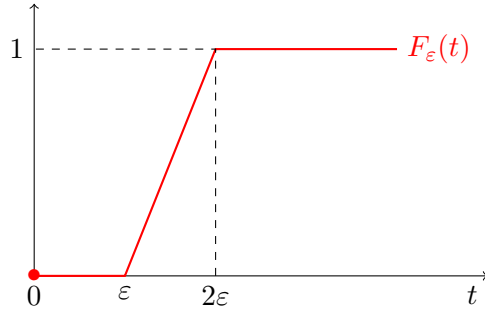
$$(2.46) \quad u(x) = \int_0^{x_j} \partial_j u(\dots, x_{j-1}, t, x_{j+1}, \dots) dt \quad \text{a.e. in } Q_N.$$

Similarly to dimension 1, the continuous approximation sequence $(\chi_\varepsilon)_{\varepsilon > 0}$ is defined as follows.

$$(2.47) \quad \chi_\varepsilon : (x; y) \mapsto F_\varepsilon(d(x, \partial Q)),$$

where $(F_\varepsilon)_{\varepsilon > 0}$ is given by

$$(2.48) \quad F_\varepsilon : t \mapsto \begin{cases} 0 & \text{if } t \leq \varepsilon, \\ \frac{t}{\varepsilon} - 1 & \text{if } \varepsilon < t < 2\varepsilon, \\ 1 & \text{if } t \geq 2\varepsilon. \end{cases}$$



We start by the $N = 2$ case. The proof is made on the unit square $Q_2 =]0, 1[^2$.

The density of $W_\eta^c(Q_2)$ in $W_{0,\eta}(Q_2)$ corresponds to the following result.

Theorem 2.12. *Let $u \in W_{0,\eta}(Q_2)$ and $(\chi_\varepsilon)_{\varepsilon > 0}$ be defined by (2.47) with $\Omega = Q_2$. Then $u\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u$ in $W_{0,\eta}(Q_2)$.*

Proof. Similarly to dimension 1, the approximation sequence satisfies for all $\varepsilon > 0$

$$(2.49) \quad 0 \leq \chi_\varepsilon \leq 1 \quad \text{and} \quad \chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 1 \quad \text{a.e. in } Q_2.$$

Thus, Lebesgue theorem yields

$$(2.50) \quad \varrho^{\eta/2} \nabla u \chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \varrho^{\eta/2} \nabla u \text{ in } L^2(Q_2)^2.$$

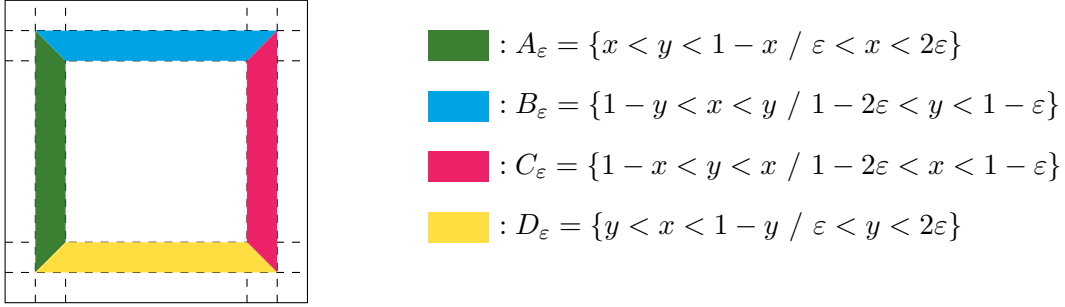
It remains to show that

$$(2.51) \quad \varrho^{\eta/2} u \nabla \chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ in } L^2(Q_2)^2.$$

First, we write

$$(2.52) \quad \int_{Q_2} \varrho^\eta |u \nabla \chi_\varepsilon|^2 = \frac{1}{\varepsilon^2} \int_{\{\varepsilon < d(\cdot, \partial Q_2) < 2\varepsilon\}} \varrho(x, y)^\eta |u(x, y)|^2 dx dy.$$

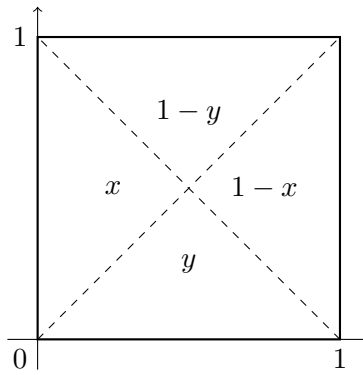
The subset $\{\varepsilon < d(\cdot, \partial Q_2) < 2\varepsilon\}$ is partitioned as on the following graph.



We obtain $\{\varepsilon < d(\cdot, \partial Q_2) < 2\varepsilon\} = A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon \cup D_\varepsilon$, which means

$$(2.53) \quad \int_{Q_2} \varrho^\eta |u \nabla \chi_\varepsilon|^2 = \frac{1}{\varepsilon^2} \left(\int_{A_\varepsilon} \varrho(x, y)^\eta |u(x, y)|^2 dx dy + \int_{B_\varepsilon} \cdots + \int_{C_\varepsilon} \cdots + \int_{D_\varepsilon} \cdots \right).$$

These zones correspond to the different expressions for the distance to the boundary.



We focus on the first term. The three others may be treated similarly.

Let $\varepsilon > 0$. In A_ε , we have the equivalence

$$(2.54) \quad \varrho(x; y) \underset{x \rightarrow 0}{\sim} x.$$

In particular, there exist two constants $M_1, M_2 > 0$ such that

$$(2.55) \quad M_1 x \leq \varrho(x; y) \leq M_2 x \quad \text{a.e. in } A_\varepsilon.$$

Taylor formula (2.46) applied on $u \in W_{0,\eta}(Q_2)$ is written

$$(2.56) \quad u(x, y) = \int_0^x \partial_1 u(t, y) dt \quad \text{a.e. } x, y \in A_\varepsilon.$$

Hölder inequality then yields

$$(2.57) \quad |u(x, y)|^2 \leq \left(\int_0^x |\partial_1 u(t, y)| dt \right)^2 \leq \left(\int_0^x \frac{dt}{t^\eta} \right) \left(\int_0^x t^\eta |\partial_1 u(t, y)|^2 dt \right).$$

On one hand,

$$(2.58) \quad \int_0^x \frac{dt}{t^\eta} = \frac{x^{1-\eta}}{1-\eta}.$$

On the other hand, (2.55) yields

$$(2.59) \quad \int_0^x t^\eta |\partial_1 u(t, y)|^2 dt \leq \frac{1}{M_1^\eta} \int_0^x \varrho(t, y)^\eta |\partial_1 u(t, y)|^2 dt.$$

We obtain

$$(2.60) \quad \begin{aligned} |u(x, y)|^2 &\leq c_1(\eta) x^{1-\eta} \int_{t=0}^x \varrho(t, y)^\eta |\partial_1 u(t, y)|^2 dt \\ &\leq c_1(\eta) x^{1-\eta} \int_{t=0}^{2\varepsilon} \varrho(t, y)^\eta |\partial_1 u(t, y)|^2 dt. \end{aligned}$$

Multiplying by $\varrho(x, y)^\eta$ and integrating (2.60) on A_ε yield

$$(2.61) \quad \begin{aligned} \iint_{A_\varepsilon} \varrho(x, y)^\eta |u(x, y)|^2 dx dy &\leq c_2(\eta) \left(\int_0^{2\varepsilon} \varrho(t, y)^\eta |\partial_1 u(t, y)|^2 dt \right) \left(\int_\varepsilon^{2\varepsilon} x dx \right) \\ &= c_3(\eta) \varepsilon^2 \int_0^{2\varepsilon} \varrho(t, y)^\eta |\partial_1 u(t, y)|^2 dt. \end{aligned}$$

We deduce

$$(2.62) \quad \frac{1}{\varepsilon^2} \iint_{A_\varepsilon} \varrho(x, y)^\eta |u(x, y)|^2 dx dy \leq c_3(\eta) \int_0^{2\varepsilon} \varrho(t, y)^\eta |\partial_1 u(t, y)|^2 dt \xrightarrow{\varepsilon \rightarrow 0} 0,$$

by Lebesgue theorem.

The convergence of the three terms in B_ε , C_ε and D_ε is obtained a similar way. \square

Now the previous proof can be adapted to the unit hypercube in dimension $N \geq 2$. This is the following result.

Multiplying by $\varrho(x)^\eta$ and integrating on A_ε yield

$$(2.71) \quad \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \varrho(x)^\eta |u(x)|^2 dx \leq c_2(\eta) \int_{x' \in Q_{N-1}} \int_{t=0}^{2\varepsilon} \varrho(x', t)^\eta |\partial_N u(x', t)|^2 dt dx' \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The conclusion is the same as Theorem 2.12. \square

Remark 2.4. The result of Theorem 2.13 can be extended to any bounded open block of the form $Q = \prod_{j=1}^N]a_j, b_j[$, where $(a_j)_{1 \leq j \leq N}$ and $(b_j)_{1 \leq j \leq N}$ are real numbers satisfying

$$(2.72) \quad \forall j \in \llbracket 1; N \rrbracket, \quad a_j < b_j.$$

Similarly to dimension 1, it uses a linear change of variables.

2.2.5 Proof in a bounded \mathcal{C}^1 domain

The extension of Theorem 2.13 to any bounded \mathcal{C}^1 domain of \mathbb{R}^N uses a change of variables by local maps and partition of unity, following the method detailed in [3, Chapter IX].

In this subsection, $N \geq 2$ and Q refers to the cube of size 2 in dimension N

$$(2.73) \quad Q =]-1; 1[^N.$$

Then, the subsets Q^+ and Q^0 are defined by

$$(2.74) \quad Q^+ = \{x \in Q / x_N > 0\} \quad \text{and} \quad Q^0 = \{x \in Q / x_N = 0\}.$$

Definition 2.1. Let Ω be a bounded open subset of \mathbb{R}^N . Since $\partial\Omega$ is compact, we can consider a family $(U_i)_{1 \leq i \leq k}$ of open subsets of \mathbb{R}^N such that

$$(2.75) \quad \partial\Omega \subset \bigcup_{i=1}^k U_i.$$

The set Ω is of class \mathcal{C}^1 if there exists a family of \mathcal{C}^1 -diffeomorphisms $(H_i)_{1 \leq i \leq k}$ satisfying for every $i \in \llbracket 1; k \rrbracket$

- $H_i : Q \rightarrow U_i$ is bijective,
- $H_i(Q^+) = U_i \cap \Omega$,
- $H_i \in \mathcal{C}^1(\bar{Q})$,
- $H_i^{-1} \in \mathcal{C}^1(\bar{U}_i)$,
- $H_i(Q^0) = U_i \cap \partial\Omega$.

In such an open set, a partition of unity can be introduced.

Theorem 2.14 (Partition of unity). *Let Ω be a bounded open set of \mathbb{R}^N , and $(U_i)_{1 \leq i \leq k}$ be defined as in the previous definition. There exists a family $(\psi_i)_{0 \leq i \leq k}$ of functions satisfying*

- $\psi_0 \in \mathcal{D}(\mathbb{R}^N \setminus \partial\Omega)$ and $\forall i \in \llbracket 1; k \rrbracket, \quad \psi_i \in \mathcal{D}(U_i)$,

- $\forall i \in \llbracket 0; k \rrbracket, \quad 0 \leq \psi_i \leq 1,$
- $\forall x \in \mathbb{R}^N, \quad \sum_{i=0}^k \psi_i(x) = 1.$

These tools are used to prove the following result.

Theorem 2.15. *Let Ω be any bounded C^1 domain of \mathbb{R}^N , and $u \in W_{0,\eta}(\Omega)$. There exists a sequence $(u_n)_{n \in \mathbb{N}}$ of $\mathcal{D}(\Omega)$ functions such that $u_n \xrightarrow[n \rightarrow +\infty]{} u$ in $W_{0,\eta}(\Omega)$.*

Proof. Lemma 2.9 ensures that $\mathcal{D}(\Omega)$ is dense in $W_{0,\eta}^c(\Omega)$ for the $W_{0,\eta}(\Omega)$ topology. It remains to prove that u is the limit of a sequence of compact-supported $W_{0,\eta}(\Omega)$ functions.

We consider a family $(U_i)_{1 \leq i \leq k}$ a family of open sets such that

$$(2.76) \quad \partial\Omega \subset \bigcup_{i=1}^k U_i,$$

and $(\psi_i)_{0 \leq i \leq k}$ the associated partition of unity.

Given that $u \in W_0^{1,p}(\Omega)$ for $p \in \left[1; \frac{2}{1+\eta}\right]$, the extension by 0 of u (named \tilde{u}) is a $W^{1,p}(\mathbb{R}^N)$ function. We write the decomposition with respect to the partition of unity

$$(2.77) \quad \tilde{u} = \sum_{i=0}^k \tilde{u} \psi_i.$$

Let $i \in \llbracket 1; k \rrbracket$. In the following, the product $\tilde{u} \psi$ is simply called $u^{(i)}$. Similarly, we define

$$(2.78) \quad v^{(i)} = u^{(i)} \circ H_i : Q^+ \rightarrow \mathbb{R}.$$

It can be easily checked that $\text{tr}(v^{(i)}) = 0$ on ∂Q^+ . From now on, let ω be any open subset of \mathbb{R}^N , ϱ_ω refers to any function satisfying the same assumptions as ϱ , with respect to ω . In particular, we can write for any $z \in Q^+$

$$(2.79) \quad M_1 \varrho_{Q^+}(z) \leq \varrho_\Omega(H_i(z)) \leq M_2 \varrho_{Q^+}(z),$$

where $M_1, M_2 > 0$ do not depend on z .

An intuitive change of variables yields

$$(2.80) \quad \begin{aligned} \int_{Q^+} \varrho_{Q^+}(z)^\eta |\partial_j v^{(i)}(z)|^2 dz &\leq \int_{Q^+} c_1 \varrho_\Omega(H_i(z))^\eta |\partial_j u^{(i)}(H_i(z))|^2 \cdot |\partial_j H_i(z)|^2 dz \\ &\leq c_2 \int_{U_i \cap \Omega} \varrho_\Omega(x)^\eta |\partial_j u^{(i)}(x)|^2 \cdot |J_{H_i^{-1}}(x)| dx \\ &\leq c_3 \int_{U_i \cap \Omega} \varrho_\Omega^\eta |\partial_j u^{(i)}|^2 < +\infty. \end{aligned}$$

We deduce that $v^{(i)} \in W_{0,\eta}(Q^+)$. Theorem 2.13 ensures the existence of a sequence $(v_n^{(i)})_{n \in \mathbb{N}}$ of compact-supported $W_{0,\eta}(Q^+)$ functions such that

$$(2.81) \quad v_n^{(i)} \xrightarrow[n \rightarrow +\infty]{} v^{(i)} \quad \text{in } W_{0,\eta}(Q^+).$$

We set for any $n \in \mathbb{N}$, $u_n^{(i)} = v_n^{(i)} \circ H_i^{-1}$. By a similar way, we prove that $u_n^{(i)} \in W_\eta^c(U_i \cap \Omega)$ for every $n \in \mathbb{N}$. The same change of variables as previously yields

$$(2.82) \quad \int_{U_i \cap \Omega} \varrho_\Omega^\eta |\partial_j u_n^{(i)} - \partial_j u^{(i)}|^2 \leq c \int_{Q^+} \varrho_{Q^+}^\eta |\partial_j v_n^{(i)} - \partial_j v^{(i)}|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Actually, we have proved that $u_n^{(i)} \xrightarrow{n \rightarrow +\infty} u^{(i)}$ in $W_{0,\eta}(U_i \cap \Omega)$.

Let the function u_n be defined by

$$(2.83) \quad u_n = u(\psi_0)|_\Omega + \sum_{i=1}^k u_n^{(i)}.$$

Each function u_n is a finite sum of $W_\eta^c(\Omega)$ functions. Then $u_n \in W_\eta^c(\Omega)$ and recalling that

$$(2.84) \quad u = u(\psi_0)|_\Omega + \sum_{i=1}^k u^{(i)} \quad \text{a.e. in } \Omega,$$

we obtain

$$(2.85) \quad \|u_n - u\|_{W_{0,\eta}(\Omega)} \leq \sum_{i=1}^k \|u_n^{(i)} - u^{(i)}\|_{W_{0,\eta}(\Omega)} \xrightarrow{n \rightarrow +\infty} 0.$$

The result is proved. □

3 Density in the double-weighted space

This section is focused on the space $H_\varrho(\Omega)$ defined by (1.4). Defining a trace is a bit more difficult than for the η -weighted space. Indeed, we are not able to state any continuous embedding in a $W^{1,p}(\Omega)$ space. However, it remains possible to build a trace operator in $H_\varrho(\Omega)$, which is detailed in the following subsection. This definition is necessary to characterize the closure of $\mathcal{D}(\Omega)$ in $H_\varrho(\Omega)$.

3.1 Trace of functions

Definition 3.1. Let the subspace $D_\varrho(\Omega)$ be defined by

$$(3.1) \quad D_\varrho(\Omega) = C^\infty(\bar{\Omega}) \cap H_\varrho(\Omega).$$

Lemma 3.1. *Let $u \in D_\varrho(\Omega)$. Then $u|_{\partial\Omega} = 0$.*

Proof. By contradiction, we assume that there exists $x_0 \in \partial\Omega$ such that $u(x_0) \neq 0$. We write

$$(3.2) \quad \ell = u(x_0) = \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} u(x) \neq 0.$$

The convergence may be written

$$(3.3) \quad \forall \varepsilon > 0, \exists \alpha > 0, \forall x \in \Omega, |x - x_0| \leq \alpha \Rightarrow |u(x) - \ell| \leq \varepsilon.$$

In particular for $\varepsilon = \frac{|\ell|}{2}$, we obtain the existence of $\alpha > 0$ such that

$$(3.4) \quad \forall x \in \Omega, \quad |x - x_0| \leq \alpha \Rightarrow |u(x)| \geq \frac{|\ell|}{2}.$$

Integrating yields

$$(3.5) \quad \int_{\Omega} \frac{|u(x)|^2}{\varrho(x)} dx \geq \frac{|\ell|}{2} \int_{\Omega \cap \bar{B}(x_0, \alpha)} \frac{dx}{\varrho(x)} = +\infty,$$

according to Kufner (REFERENCE).

The result is proved. \square

Lemma 3.2. *Let $\varphi \in D_{\varrho}(\Omega)$. There exists a constant $c > 0$ not depending on φ such that*

$$(3.6) \quad \|\varphi\|_{L^2(\partial\Omega)} \leq c \|\varphi\|_{H_{\varrho}}.$$

Proof. Let $\varphi \in D_{\varrho}(\Omega)$. On one hand, we have

$$(3.7) \quad \int_{\Omega} |\varphi^2| \leq \|\varrho\|_{L^{\infty}(\Omega)} \int_{\Omega} \frac{|\varphi|^2}{\varrho} \leq \|\varrho\|_{L^{\infty}(\Omega)} \|\varphi\|_{H_{\varrho}}^2.$$

On the other hand, Young inequality yields

$$(3.8) \quad \int_{\Omega} |\nabla(\varphi^2)| \leq \int_{\Omega} 2|\varphi| \cdot |\nabla\varphi| \leq \int_{\Omega} \frac{|\varphi|^2}{\varrho} + \int_{\Omega} \varrho |\nabla\varphi|^2 = \|\varphi\|_{H_{\varrho}}^2.$$

Adding (3.7) and (3.8) yields

$$(3.9) \quad \|\varphi^2\|_{W^{1,1}(\Omega)} \leq (1 + \|\varrho\|_{L^{\infty}(\Omega)}) \|\varphi\|_{H_{\varrho}}^2.$$

The definition of the trace operator on $W^{1,1}(\Omega)$ and the regularity of φ yield the existence of $c_1 > 0$ not depending on φ such that

$$(3.10) \quad \|\varphi^2\|_{L^1(\partial\Omega)} \leq c_1 \|\varphi^2\|_{W^{1,1}(\Omega)}.$$

By setting $c_2 = c_1 (1 + \|\varrho\|_{L^{\infty}(\Omega)})$, we obtain

$$(3.11) \quad \|\varphi^2\|_{L^1(\partial\Omega)} \leq c_2 \|\varphi\|_{H_{\varrho}}^2.$$

Finally, setting $c = \sqrt{c_2}$ yields (3.6). \square

Corollary 3.3. *Let $\overline{D_{\varrho}(\Omega)}$ be the closure of $D_{\varrho}(\Omega)$ for the H_{ϱ} topology, and $u \in \overline{D_{\varrho}(\Omega)}$. Then u admits a trace in $L^2(\partial\Omega)$ and $\text{tr } u = 0$ a.e. on $\partial\Omega$.*

Proof. Let $u \in \overline{D_{\varrho}(\Omega)}$, then there exists a sequence $(u_n)_{n \in \mathbb{N}}$ of D_{ϱ} functions such that $u_n \xrightarrow{n \rightarrow +\infty} u$ in H_{ϱ} . Lemma 3.2 then yields

$$(3.12) \quad \forall n \in \mathbb{N}, \quad \|u_n\|_{L^2(\partial\Omega)} \leq c \|u_n\|_{H_{\varrho}}.$$

Fatou Lemma yields

$$(3.13) \quad \|u\|_{L^2(\partial\Omega)} \leq \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2(\partial\Omega)} \leq c \liminf_{n \rightarrow +\infty} \|u_n\|_{H_{\varrho}} = \|u\|_{H_{\varrho}}.$$

Therefore, u admits a trace in $L^2(\partial\Omega)$. According to Lemma 3.1, for all $n \in \mathbb{N}$ we have $\text{tr } u_n = 0$ a.e. on $\partial\Omega$. The continuity of the operator directly yields that $\text{tr } u = 0$ a.e. on $\partial\Omega$. \square

3.2 Density Results

The end of this section is devoted to prove a similar result as Theorem 2.5, but for $H_\varrho(\Omega)$. More precisely, this is the following theorem.

Theorem 3.4. *The closure of $\mathcal{D}(\Omega)$ in $H_\varrho(\Omega)$ for the topology given by (1.6) is*

$$(3.14) \quad \overline{\mathcal{D}(\Omega)}^{H_\varrho(\Omega)} = \{u \in H_\varrho(\Omega) / \text{tr } u = 0 \text{ a.e. on } \partial\Omega\}.$$

Until the end of this section, $H_{0,\varrho}(\Omega)$ refers to the right hand side of (3.14).

Remark 3.1. The first inclusion is easily proved. Indeed, $\mathcal{D}(\Omega) \subset D_\varrho(\Omega)$, which implies $\overline{\mathcal{D}(\Omega)}^{H_\varrho(\Omega)} \subset \overline{D_\varrho(\Omega)}^{H_\varrho(\Omega)}$. Yet, Corollary 3.3 yields that $\overline{D_\varrho(\Omega)}^{H_\varrho(\Omega)} \subset H_{0,\varrho}(\Omega)$. This directly yields

$$(3.15) \quad \overline{\mathcal{D}(\Omega)}^{H_\varrho(\Omega)} \subset H_{0,\varrho}(\Omega).$$

The end of this section corresponds to the different steps of the proof of the other inclusion in Theorem 3.4. Similarly to what is done in section 2, we start by considering the compact-supported $H_\varrho(\Omega)$ functions.

Definition 3.2. Let $H_\varrho^c(\Omega)$ be the space of $H_\varrho(\Omega)$ functions with compact support in Ω .

Lemma 3.5. *Let E be a normed vector space, ω an open subset of E and K a compact set such that $K \subset \omega$. There exists a rank $\varepsilon > 0$ such that*

$$(3.16) \quad \bar{B}(0, \varepsilon) + K \subset \omega.$$

Dois-je refaire la preuve de ce lemme ?

Theorem 3.6. *The space $\mathcal{D}(\Omega)$ is dense in $H_\varrho^c(\Omega)$ at the sense of the H_ϱ topology.*

Proof. Let $u \in H_\varrho^c(\Omega)$. We consider its extension by 0 named \tilde{u} , and a mollifier $(\theta_\varepsilon)_{\varepsilon>0}$. A well-known analysis result ensures that

$$(3.17) \quad \theta_\varepsilon * \tilde{u} \xrightarrow{\varepsilon \rightarrow 0} u \text{ in } L^2(\Omega).$$

Let ω be an open set such that $\text{Supp}(u) \subset\subset \omega \subset\subset \Omega$. Lemma 3.5 implies the existence of $\varepsilon_0 > 0$ such that

$$(3.18) \quad \forall \varepsilon \in]0; \varepsilon_0], \quad \text{Supp}(\theta_\varepsilon * \tilde{u}) \subset \omega.$$

The definition of ϱ yields the existence of a constant $\varrho_\omega > 0$ such that

$$(3.19) \quad \sup_\omega \frac{1}{\varrho} = \varrho_\omega.$$

Thus, (3.18) yields for every $\varepsilon \in]0; \varepsilon_0]$

$$(3.20) \quad \int_\Omega \frac{|\theta_\varepsilon * \tilde{u} - u|^2}{\varrho} \leq \varrho_\omega \int_\Omega |\theta_\varepsilon * \tilde{u} - u|^2.$$

Consequently, (3.17) yields that

$$(3.21) \quad \int_\Omega \frac{|\theta_\varepsilon * \tilde{u} - u|^2}{\varrho} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Given that u is compact-supported, it is easy to show that $\nabla u \in L^2(\Omega)^N$ and its extension by 0 satisfies $\tilde{u} \in H^1(\mathbb{R}^N)$. This directly yields

$$(3.22) \quad \int_{\Omega} \varrho |\nabla(\theta_{\varepsilon} * \tilde{u}) - \nabla u|^2 \leq \|\varrho\|_{L^{\infty}(\Omega)} \int_{\Omega} |\theta_{\varepsilon} * (\nabla \tilde{u}) - \nabla u|^2 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Finally, (3.21) and (3.22) yield

$$(3.23) \quad \theta_{\varepsilon} * \tilde{u} \xrightarrow{\varepsilon \rightarrow 0} u \text{ in } H_{\varrho}(\Omega).$$

The result is proved. \square

It only remains to prove the following theorem.

Theorem 3.7. *Let $u \in H_{\varrho}(\Omega)$ such that $\text{tr } u = 0$ a.e. on $\partial\Omega$. There exists a sequence $(u_{\varepsilon})_{\varepsilon > 0}$ of $H_{\varrho}^c(\Omega)$ functions satisfying*

$$(3.24) \quad u_{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} u \text{ in } H_{\varrho}(\Omega).$$

3.3 Proof of the density theorem in $H_{\varrho}(\Omega)$

The proof does not follow the same steps as Theorem 2.8, since $H_{\varrho}(\Omega)$ is not embedded in a $W^{1,p}(\Omega)$ space. For example, Taylor formulas cannot be applied in this case, and some regularity results are not valid anymore. This subsection provides ways to solve these issues.

3.3.1 On the interval $]0; 1[$

We consider the interval $\Omega =]0; 1[$. Since we did not find any continuous embedding in a Sobolev space, we start by giving a regularity result in dimension 1.

Lemma 3.8. *Let $u \in H_{\varrho}(0; 1)$. Then $|u| \in C^0([0; 1])$.*

Proof. According to the proof of Lemma 3.2, if $u \in H_{\varrho}(0; 1)$ then $u^2 \in W^{1,1}(0; 1)$, which is continuously embedded in $C^0([0; 1])$.

The continuity of the square root function on $[0; +\infty[$ yields that $|u| = \sqrt{u^2}$ is continuous on $[0; 1]$. Hence the result. \square

Remark 3.2. The previous result does not mean that u necessarily admits an image in 0 and 1. Nonetheless, if $\lim_{x \rightarrow 0^+} |u(x)| = \lim_{x \rightarrow 1^-} |u(x)| = 0$, we can deduce that $u(0) = u(1) = 0$.

Theorem 3.9. *Let $u \in H_{\varrho}(0; 1)$ such that $u(0) = u(1) = 0$. Let the sequence $(\chi_{\varepsilon})_{\varepsilon > 0}$ be defined by (2.34). Then we have*

$$(3.25) \quad u\chi_{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} u \text{ in } H_{\varrho}(0; 1).$$

Proof. The sequence $(\chi_{\varepsilon})_{\varepsilon > 0}$ obviously satisfies

• $\forall \varepsilon > 0, 0 \leq \chi_\varepsilon \leq 1,$

• $\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 1$ a.e. in $[0; 1].$

This directly yields

$$(3.26) \quad \left. \begin{array}{l} u\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u \\ \varrho^{-1/2}|u\chi_\varepsilon| \leq \varrho^{-1/2}|u| \in L^2(0; 1) \end{array} \right\}, \text{ a.e. in } [0; 1].$$

Therefore, Lebesgue theorem yields

$$(3.27) \quad \varrho^{-1/2}u\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \varrho^{-1/2}u \text{ in } L^2(0; 1).$$

We now focus on the derivative term. First, the same arguments as just before yield

$$(3.28) \quad \varrho^{1/2}u'\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \varrho^{1/2}u' \text{ in } L^2(0; 1).$$

It remains to prove that

$$(3.29) \quad \varrho^{1/2}u\chi'_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ in } L^2(0; 1),$$

which is more intricate.

First, we write

$$(3.30) \quad \int_0^1 \varrho|u|^2|\nabla\chi_\varepsilon|^2 = \frac{1}{\varepsilon^2} \left(\int_\varepsilon^{2\varepsilon} \varrho(x)|u(x)|^2 dx + \int_{1-2\varepsilon}^{1-\varepsilon} \varrho(x)|u(x)|^2 dx \right).$$

In the following of this proof, only the convergence of the first integral in (3.30) is established, the second may be done the same way.

The initial conditions directly yield that $u^2 \in W_0^{1,1}(0; 1).$ Thus, we can write the following Taylor formula

$$(3.31) \quad u(x)^2 = \int_0^x 2u'(t)u(t) dt \text{ a.e. } x \in [0; 1].$$

This yields in particular

$$(3.32) \quad |u(x)|^2 \leq \int_0^x 2|u'(t)| \cdot |u(t)| dt \leq \int_0^{2\varepsilon} 2|u'(t)| \cdot |u(t)| dt \text{ a.e. } x \in [0; 2\varepsilon].$$

Integrating on $[\varepsilon; 2\varepsilon]$ then yields

$$(3.33) \quad \int_\varepsilon^{2\varepsilon} \varrho(x)|u(x)|^2 dx \leq \left(\int_\varepsilon^{2\varepsilon} \varrho(x) dx \right) \left(\int_0^{2\varepsilon} 2|u'(t)| \cdot |u(t)| dt \right).$$

On one hand, the boundary condition on ϱ given by (1.1) yields the existence of a constant $C > 0$ such that

$$(3.34) \quad \exists \varepsilon_0 \in]0; 1[, \forall x \in [0; \varepsilon_0], \varrho(x) \leq Cx.$$

For any $\varepsilon \leq \frac{\varepsilon_0}{2}$, we obtain

$$(3.35) \quad \int_{\varepsilon}^{2\varepsilon} \varrho(x) dx \leq C \int_{\varepsilon}^{2\varepsilon} x dx = \frac{3C}{2} \varepsilon^2.$$

On the other hand, Young inequality directly yields

$$(3.36) \quad \int_0^{2\varepsilon} 2|u'(t)| \cdot |u(t)| dt \leq \int_0^{2\varepsilon} \frac{|u(t)|^2}{\varrho(t)} dt + \int_0^{2\varepsilon} \varrho(t)|u'(t)|^2 dt \xrightarrow{\varepsilon \rightarrow 0} 0,$$

by Lebesgue theorem.

Finally, inequalities (3.33), (3.35) and (3.36) yield

$$(3.37) \quad \frac{1}{\varepsilon^2} \int_{\varepsilon}^{2\varepsilon} \varrho(x)|u(x)|^2 dx \leq \frac{3C}{2} \int_0^{2\varepsilon} 2|u'(t)| \cdot |u(t)| dt \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The result is proved. \square

3.3.2 On the unit hypercube in higher dimension

Let the unit hypercube in dimension N be defined by

$$(3.38) \quad Q_N =]0; 1[^N.$$

The characterization of the closure $\overline{\mathcal{D}(Q_N)}^{H_\varrho(Q_N)}$ corresponds to the following result.

Theorem 3.10. *Let $N \geq 2$, $u \in H_{0,\varrho}(Q_N)$, and the approximation sequence $(\chi_\varepsilon)_{\varepsilon>0}$ defined by (2.47). Then we have the following convergence*

$$(3.39) \quad u\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u \text{ in } H_\varrho(Q_N).$$

Proof. This proof is quite similar to Theorem 2.13. First, the same arguments as for Theorem 3.9 yield

$$(3.40) \quad \varrho^{-1/2}u\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \varrho^{-1/2}u \text{ in } L^2(Q_N),$$

$$(3.41) \quad \varrho^{1/2}\nabla u\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \varrho^{1/2}\nabla u \text{ in } L^2(Q_N)^N.$$

It only remains to prove the following

$$(3.42) \quad \varrho^{1/2}u\nabla\chi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ in } L^2(Q_N)^N.$$

By dividing the subset $\{\varepsilon < d(\cdot, \partial Q_N) < 2\varepsilon\}$ into $2N$ areas exactly the same way as (2.66), equality (2.65) adapted for the partition (2.67) yields

$$(3.43) \quad \int_{Q_N} \varrho|u\nabla\chi_\varepsilon|^2 = \sum_{j=1}^N \left(\frac{1}{\varepsilon^2} \int_{A_\varepsilon^{(j)}} \varrho(x)|u(x)|^2 dx + \frac{1}{\varepsilon^2} \int_{B_\varepsilon^{(j)}} \varrho(x)|u(x)|^2 dx \right).$$

Similarly to Theorem 2.13, we only focus on the integral on $A_\varepsilon^{(N)}$. The $2N - 1$ other computations may be done exactly the same way.

Given that $u^2 \in W_0^{1,1}(Q_N)$, the Taylor formula is written

$$(3.44) \quad u(x)^2 = \int_0^{x_N} \partial_N (u^2) (x', t) dt = \int_0^{x_N} 2\partial_N u(x', t) \cdot u(x', t) dt, \quad \text{a.e. } x \in Q_N.$$

Young inequality directly yields

$$(3.45) \quad |u(x)|^2 \leq \int_0^{2\varepsilon} 2|\partial_N u(x', t)| \cdot |u(x', t)| dt \leq \int_0^{2\varepsilon} \frac{|u(x', t)|^2}{\varrho(x', t)} dt + \int_0^{2\varepsilon} \varrho(x', t) |\partial_N u(x', t)|^2 dt.$$

We deduce the following inequality

$$(3.46) \quad \begin{aligned} & \int_{A_\varepsilon^{(N)}} \varrho(x) |u(x)|^2 dx \\ & \leq \int_{Q_{N-1}} \int_{x_N=\varepsilon}^{2\varepsilon} \varrho(x', x_N) \left(\int_0^{2\varepsilon} \frac{|u(x', t)|^2}{\varrho(x', t)} dt + \int_0^{2\varepsilon} \varrho(x', t) |\partial_N u(x', t)|^2 dt \right) dx' dx_N. \end{aligned}$$

Yet, there exists a constant $C > 0$ and a $\varepsilon_0 > 0$ such that

$$(3.47) \quad \forall x \in [0; \varepsilon_0], \quad \varrho(x', x_N) \leq C x_N.$$

Therefore, (3.46) becomes

$$(3.48) \quad \begin{aligned} & \frac{1}{\varepsilon^2} \int_{A_\varepsilon^{(N)}} \varrho(x) |u(x)|^2 dx \\ & \leq \frac{3C}{2} \left(\int_{Q_{N-1}} \int_{t=0}^{2\varepsilon} \frac{|u(x', t)|^2}{\varrho(x', t)} dx' dt + \int_{Q_{N-1}} \int_{t=0}^{2\varepsilon} \varrho(x', t) |\partial_N u(x', t)|^2 dx' dt \right). \end{aligned}$$

This directly yields

$$(3.49) \quad \frac{1}{\varepsilon^2} \int_{A_\varepsilon^{(N)}} \varrho(x) |u(x)|^2 dx \xrightarrow{\varepsilon \rightarrow 0} 0,$$

by Lebesgue theorem. □

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