Uniform Poincaré inequalities for the discrete de Rham complex of differential forms

Daniele Antonio Di Pietro, Jérôme Droniou, Marien-Lorenzo Hanot, Silvano Pitassi

Uniform Poincaré inequalities for the discrete de Rham complex of differential forms

Daniele A. Di Pietro[1], Jérôme Droniou[1,3], Marien-Lorenzo Hanot[2], and Silvano Pitassi[1]

[1]IMAG, Univ Montpellier, CNRS, Montpellier, France, daniele.di-pietro@umontpellier.fr, jerome.droniou@umontpellier.fr
silvano.pitassi@umontpellier.fr
[2]Univ. Lille, UMR 8524 - Laboratoire Paul Painlevé, CNRS, Inria, France, marien-lorenzo.hanot@univ-lille.fr
[3]School of Mathematics, Monash University, Melbourne, Australia.

January 27, 2025

Abstract

In this paper we prove discrete Poincaré inequalities that are uniform in the mesh size for the discrete de Rham complex of differential forms developed in [Bonaldi, Di Pietro, Droniou, and Hu, *An exterior calculus framework for polytopal methods*, J. Eur. Math. Soc., to appear, arXiv preprint 2303.11093 [math.NA]]. We unify the underlying ideas behind the Poincaré inequalities for all differential operators in the sequence, extending the known inequalities for the gradient, curl, and divergence in three-dimensions to polytopal domains of arbitrary dimension and general topology. A key step in the proof involves deriving specific Poincaré inequalities for the cochain complex supported on the polytopal mesh. These inequalities are of independent interest, as they are useful, for instance, in establishing the existence and stability, on domains of generic topology, of solutions of schemes based on Mimetic Finite Differences, Compatible Discrete Operators or Discrete Geometric Approach.

Key words. Discrete de Rham complex, polytopal methods, Poincaré inequalities

MSC2020. 65N30, 65N99, 14F40

1 Introduction

The well-posedness of important classes of partial differential equations can be related to the cohomology of the de Rham complex through Poincaré-type inequalities. Such inequalities state that the L^2-norm of forms that lie in an orthogonal complement of the kernel of the exterior derivative is controlled by the L^2-norm of the latter. Mimicking Poincaré inequalities on the discrete level is required for the stability of *compatible* numerical schemes, the design of which hinges on discrete versions of the de Rham complex.

In the context of conforming Finite Element de Rham complexes, Poincaré inequalities can be derived through bounded cochain projections; see [2, Chapter 5], [3], and also [11] for a recent generalization. Owing to the need to identify an underlying computable space of (polynomial) functions, Finite Element constructions are, however, typically limited to conforming meshes with elements of simple shape. Recent works have pointed out the possibility of constructing arbitrary-order discrete de Rham complexes on general polytopal meshes, either through the use of virtual spaces, standard differential operators, and projections [4, 5], or by replacing both the spaces and operators with fully discrete counterparts [15, 19]; see also the related works [6, 8, 9, 12, 13, 25] dealing with the lowest-order case. Both the virtual and fully discrete approaches lead to *non-conforming* schemes, resulting in additional difficulties with respect to Finite Elements.

Poincaré inequalities on domains of general topology for the three-dimensional DDR complex of vector proxies of [15] have been recently proved in [20]. Using operator-specific techniques, fully general Poincaré inequalities for the gradient and the divergence had already been obtained, respectively, in [15, Theorem 3] and [16]. A Poincaré inequality for the curl on contractible domains had been obtained in [16, Theorem 20]. We also cite here [14] as a first example of Poincaré-type inequalities for two-dimensional de Rham complexes with enhanced regularity and, possibly, boundary conditions.

Very recently, Discrete de Rham (DDR) complexes of differential forms have been proposed in [7], generalizing both the constructions of [15] and of [4]; see also [22] for an extension to manifolds. The purpose of this work is to establish discrete Poincaré inequalities for the complex studied in [7, Section 3]. The proofs generalize and broaden the ideas of [20]. The starting point is the Poincaré inequality for cochains stated in Lemma 6 below. Given a polytopal mesh admitting a conforming simplicial submesh, this inequality is proved by constructing a mapping to lift polytopal cochains onto simplicial cochains, identifying the latter with conforming piecewise polynomial forms through the Whitney and de Rham maps in order to leverage continuous Poincaré inequalities, and concluding with local norms estimates. The discrete Poincaré inequality for the arbitrary-order DDR complex is then proved in Theorem 4 using the Poincaré inequality for cochains to construct, given a discrete k-form, another discrete k-form with the same discrete exterior derivative and bounded by the latter. Finally, in Corollary 5 this result is bridged to a more standard form of the Poincaré inequality for discrete k-forms that lie in an orthogonal complement of the kernel of the exterior derivative.

The rest of the paper is organized as follows. Section 2 states the main results of the paper after briefly recalling the DDR construction. The Poincaré inequality for cochains, stated and proved in Section 3, is then used in Section 4 to prove the main result. The paper is completed by three appendices. Appendix A collects results on local polynomial spaces that generalize those proved in [15, 17] for polynomial functions. Appendix B contains estimates for Whitney forms. Finally, Appendix C describes an algorithm for the computation of a basis for a subspace of simplicial cycles used in the proof of the Poincaré inequality for cochains.

2 Main results

2.1 Setting

2.1.1 Polytopal mesh

Let $k \in \{0, \ldots, n\}$. A *(flat) k-cell* f is a subset of a k-dimensional subspace of $\mathbb{R}^n$ that is homeomorphic to a closed k-dimensional ball. A *polytopal mesh* $\mathcal{M}_h$ of a polytopal domain $\Omega \subset \mathbb{R}^n$ is a finite collection of k-cells, for all $k \in \{0, \ldots, n\}$, such that the following properties hold:

(i) The union of the n-cells cover Ω;

(ii) For each $k \in \{1, \ldots, n\}$, the boundary ∂f of each k-cell f is a union of $(k-1)$-cells in $\mathcal{M}_h$;

(iii) The intersection of two distinct k-cells is either empty or is a union of lower-dimensional cells in $\mathcal{M}_h$;

(iv) Given a k-cell f and an ℓ-cell f' with $\ell \le k$, such that $f \neq f'$ and $f \cap f' \neq \emptyset$, we have $f \cap f' \subset \partial f$.

We denote by $\Delta_k(\mathcal{M}_h)$ the set of all k-cells of $\mathcal{M}_h$, and each of these k-cell is equipped by an orientation. For any k-cell $f \in \Delta_k(\mathcal{M}_h)$, we define the *polytopal submesh induced by* $\mathcal{M}_h$ *on* f to be $\mathcal{M}_h(f) \coloneqq \{f' \in \mathcal{M}_h \,:\, f' \subset f\}$. It is easy to verify that $\mathcal{M}_h(f)$ is a polytopal mesh on f. Moreover, $\mathcal{M}_h(\partial f)$ is also well-defined and is a polytopal mesh on ∂f.

A k-simplex f is a k-cell that is obtained by applying an affine transformation (i.e, composition of a translation and a non-singular linear transformation) to the *reference* k-simplex

$$\left\{(x_0,\dots,x_k)\in\mathbb{R}^{k+1}\ :\ x_i\geq 0,\sum_{i=0}^{k}x_i=1\right\}.$$

We say that $\mathcal{M}_h$ is *simplicial* if it forms a simplicial mesh in the usual sense. Given a polytopal mesh $\mathcal{M}_h$ of Ω, we say that $\mathcal{S}_h$ is a *simplicial submesh* of $\mathcal{M}_h$ if: (i) $\mathcal{S}_h$ is a simplicial mesh of Ω; (ii) each k-cell of $\mathcal{M}_h$ is a union of k-simplices of $\mathcal{S}_h$.

The notion of mesh regularity is given by [17, Definition 1.9]. Namely, we say that a sequence of meshes $(\mathcal{M}_h)_h$ is regular if there exists $\rho\in(0,1)$, independent of h, and a sequence of simplicial submeshes $(\mathcal{S}_h)_h$ of $(\mathcal{M}_h)_h$, such that:

- For each $k\in\{1,\dots,n\}$ and any k-cell f of $\mathcal{S}_h$, it holds that $\rho h_f\leq r_f$, where h_f denotes the diameter of f and r_f its inradius.
- For any k-cell f of $\mathcal{M}_h$, and all k-simplex f' of $\mathcal{S}_h$ such that $f'\subset f$, it holds that $\rho h_f\leq h_{f'}$.

To avoid the proliferation of constants, throughout the rest of the paper we write $a\lesssim b$ in place of $a\leq Cb$, with real number $C>0$ independent of the mesh size and, for local inequalities on a k-cell f, also on f, but possibly depending on other parameters such as the domain, the mesh regularity parameter, the polynomial degree of the complex, etc. We also write $a\simeq b$ for "$a\lesssim b$ and $b\lesssim a$".

The regularity requirements on the mesh imply that the number of simplices in a cell is bounded uniformly in h. Specifically, for each $k\in\{1,\dots,n\}$ and any d-cell f of $\mathcal{M}_h$ with $d\in\{k,\dots,n\}$, it holds that $\mathrm{card}(\Delta_k(\mathcal{S}_h(f)))\lesssim 1$.

2.1.2 Discrete differential forms

We give here a brief presentation, and refer to [7, Section 2 and Appendix A] for more details. Let $\mathcal{M}_h$ be a polytopal mesh. Let $d\in\{0,\dots,n\}$, and $f\in\Delta_d(\mathcal{M}_h)$ be a d-cell of $\mathcal{M}_h$. In particular, f is a (flat) d-dimensional manifold. For $k\in\{0,\dots,d\}$, we denote by $\Lambda^k(f)$ the set of differential k-forms on f. When a specific regularity is necessary, we prepend the appropriate space. For instance, $C^0\Lambda^k(f)$ is the set of k-forms defined on f and that are continuous.

We recall that the inner product on the space of k-forms can be written using the Gram determinant as:

$$\langle\omega,\mu\rangle_f=\int_f\langle\omega_x,\mu_x\rangle\,\mathrm{d}x=\int_f\det\left(\left(\langle\omega_i(x),\mu_j(x)\rangle\right)_{i,j}\right)\mathrm{d}x\qquad\forall\omega,\mu\in L^2\Lambda^k(f),\tag{1}$$

where $(\omega_j)_{j=1,\dots,k}$ and $(\mu_j)_{j=1,\dots,k}$ are 1-forms such that $\omega_x=\omega_1(x)\wedge\cdots\wedge\omega_k(x)$ and $\mu_x=\mu_1(x)\wedge\cdots\wedge\mu_k(x)$. The corresponding L^2-norm is

$$\|\omega\|_f\coloneqq\langle\omega,\omega\rangle_f^{\frac{1}{2}}.$$

We denote by $\star_f$ the Hodge star operator on a d-cell f for $d\in\{0,\dots,n\}$; it is an isomorphism $\Lambda^k(f)\to\Lambda^{d-k}(f)$. For the sake of brevity, we shall simply denote it "$\star$" in the remaining of the paper, since its domain and codomain can be inferred from its argument. We note that, using this Hodge star operator, the inner product (1) can be written

$$\langle\omega,\mu\rangle_f\coloneqq\int_f\omega\wedge\star\mu\qquad\forall\omega,\mu\in L^2\Lambda^k(f).\tag{2}$$

For each $f\in\mathcal{M}_h$ we fix a point x_f such that f is star-shaped with respect to a ball centered at x_f and of radius $\gtrsim h_f$. We define the Koszul operator κ_f on f to be the interior product against the identity vector field shifted at x_f. Specifically, for $\omega\in\Lambda^k(f)$, $\kappa_f\omega\in\Lambda^{k-1}(f)$ satisfies

$$(\kappa_f\omega)_x(v_1,\dots,v_{k-1})=(\omega)_x(x-x_f,v_1,\dots,v_{k-1}),$$

for all $x \in f$, and all $v_1, \ldots, v_{k-1}$ tangent vectors to f. From hereon, and for the same reason that we dropped the index f on the Hodge-star operator $\star$, we drop the index f in the notation for the Koszul operator.

2.2 Local spaces of forms with polynomial coefficients

For any integer $r \geq 0$, we denote by $\mathcal{P}_r\Lambda^k(f)$ the space of k-forms on f with polynomial coefficients of total degree less than or equal to r. By convention, we set $\mathcal{P}_{-1}\Lambda^k(f) \coloneqq \{0\}$. For $k \geq 1$, the following decomposition of the polynomial space holds (see [3, Eq. (3.11)]):

$$\mathcal{P}_r\Lambda^k(f) = \mathrm{d}\mathcal{P}_{r+1}\Lambda^{k-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(f).$$

With this decomposition in mind, we define the space of *trimmed* polynomials lowering by one the polynomial degree of the first component for $k \in \{1, \ldots, d\}$:

$$\begin{aligned}
&\mathcal{P}_r^-\Lambda^0(f) \coloneqq \mathcal{P}_r\Lambda^0(f),\\
&\mathcal{P}_r^-\Lambda^k(f) \coloneqq \mathrm{d}\mathcal{P}_r\Lambda^{k-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(f) \quad \text{if } k \in \{1, \ldots, d\}.
\end{aligned}$$

Using the inner product (1), we can define an associated orthogonal projector $\pi_{r,f}^{-,k} : L^2\Lambda^k(f) \to \mathcal{P}_r^-\Lambda^k(f)$ such that, for $\omega \in L^2\Lambda^k(f)$,

$$\langle \pi_{r,f}^{-,k}\omega, \mu\rangle_f = \langle \omega, \mu\rangle_f, \quad \forall \mu \in \mathcal{P}_r^-\Lambda^k(f).$$

2.3 The Discrete De Rham construction

From this point on, we fix an integer $r \geq 0$ corresponding to the polynomial degree of the discrete de Rham complex.

2.3.1 Spaces and interpolators

For any $k \in \{0, \ldots, n\}$, the discrete counterpart $\underline{X}_{r,h}^k$ of the space $H\Lambda^k(\Omega)$ is defined as

$$\underline{X}_{r,h}^k \coloneqq \bigtimes_{d=k}^{n} \bigtimes_{f \in \Delta_d(\mathcal{M}_h)} \star^{-1}\mathcal{P}_r^-\Lambda^{d-k}(f).$$

Notice that, compared with the original paper [7], we have taken $\star^{-1}\mathcal{P}_r^-\Lambda^{d-k}(f)$ instead of $\mathcal{P}_r^-\Lambda^{d-k}(f)$ as component space. In the context considered here, based on a polytopal meshes of a domain of $\mathbb{R}^n$, $\star$ is an isomorphism between spaces of polynomial forms and these choices equivalent up to an application of the Hodge star to each component. In what follows, we will therefore directly invoke results from [7] without specifying each time the adaptation (application of suitable Hodge star to the components) to the present setting. When considering general Riemannian manifolds, however, $\star$ no longer maps polynomials into polynomials, and each choice leads to a different construction. The one adopted here has the advantage to allow the generalization of the method to manifolds (as it is done in [22]), whereas the one of [7] yields a more direct link with the DDR complex in vector calculus notation of [15].

2.3.2 Local construction

For any dimension $d \in \{0, \ldots, n\}$, any face $f \in \Delta_d(\mathcal{M}_h)$, and any form degree $k \in \{0, \ldots, d\}$, we define the local interpolator $\underline{I}_{r,f}^k : C^0\Lambda^k(f) \to \underline{X}_{r,f}^k$ such that

$$\underline{I}_{r,f}^k\omega \coloneqq \left(\star^{-1}\pi_{r,f'}^{-,d'-k} \star \mathrm{tr}_{f'}\,\omega\right)_{f' \in \Delta_{d'}(\mathcal{M}_h(f)), d' \in \{k, \ldots, d\}} \qquad \forall \omega \in C^0\Lambda^k(f).$$

We set, for the sake of brevity,

$$\langle \cdot, \cdot\rangle_{\partial f} \coloneqq \sum_{f' \in \Delta_{d-1}(\mathcal{M}_h(\partial f))} \epsilon_{ff'}\langle \cdot, \cdot\rangle_{f'}, \tag{3}$$

where $\epsilon_{ff'} \in \{-1, +1\}$ is the relative orientation of f' with respect to f.

Let a form degree $k \in \{0, \dots, n\}$ and a vector of forms $\underline{\omega}_h \in \underline{X}^k_{r,h}$ be fixed. For all $d \geq k$ and all $f \in \Delta_d(\mathcal{M}_h)$, we define a local discrete exterior derivative $\mathrm{d}^k_{r,f} : \underline{X}^k_{r,f} \to \star^{-1}\mathcal{P}_r\Lambda^{d-k-1}(f)$ and a local potential reconstruction $P^k_{r,f} : \underline{X}^k_{r,f} \to \star^{-1}\mathcal{P}_r\Lambda^{d-k}(f)$. The definition is recursive on the cell dimension d:

- If $d = k$, we set
$$P^k_{r,f}\underline{\omega}_f \coloneqq \omega_f \in \star^{-1}\mathcal{P}_r\Lambda^0(f). \tag{4}$$
- For $d \in \{k+1, \dots, n\}$:
  1. We first define the discrete exterior derivative such that
  $$\langle \star \mathrm{d}^k_{r,f}\underline{\omega}_f, \mu_f\rangle_f = (-1)^{k+1}\langle \star\omega_f, \mathrm{d}\mu_f\rangle_f + \langle \star P^k_{r,\partial f}\underline{\omega}_f, \mathrm{tr}_{\partial f}\,\mu_f\rangle_{\partial f} \quad \forall \mu_f \in \mathcal{P}_r\Lambda^{d-k-1}(f), \tag{5}$$
  where $P^k_{r,\partial f}\underline{\omega}_f$ is the piecewise polynomial form on ∂f obtained patching together the polynomials $(P^k_{r,f'}\underline{\omega}_{f'})_{f' \in \Delta_{d-1}(\mathcal{M}_h(\partial f))}$ (which are made available by the previous step).
  2. We then define the potential reconstruction that satisfies
  $$\begin{aligned}(-1)^{k+1}&\langle \star P^k_{r,f}\underline{\omega}_f, \mathrm{d}\mu_f + \nu_f\rangle_f \\ &= \langle \star \mathrm{d}^k_{r,f}\underline{\omega}_f, \mu_f\rangle_f - \langle \star P^k_{r,\partial f}\underline{\omega}_{\partial f}, \mathrm{tr}_{\partial f}\,\mu_f\rangle_{\partial f} + (-1)^{k+1}\langle \star\omega_f, \nu_f\rangle_f \\ &\qquad \forall (\mu_f, \nu_f) \in \kappa\mathcal{P}_r\Lambda^{d-k}(f) \times \kappa\mathcal{P}_{r-1}\Lambda^{d-k+1}(f).\end{aligned}$$

Remark 1 (Integral formulas). Expanding the definition (2) of the inner product on f and each $f' \in \Delta_{d-1}(\mathcal{M}_h(f))$, recalling (3), and using the relation $\sigma \wedge \star\xi = \star^{-1}\sigma \wedge \xi$ valid for any couple (σ, ξ) of ℓ-forms, we note that (5) is equivalent to the following relation, which exposes a discrete Stokes formula:

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu_f = (-1)^{k+1}\int_f \omega_f \wedge \mathrm{d}\mu_f + \sum_{f' \in \Delta_{d-1}(\mathcal{M}_h(f))} \epsilon_{ff'}\int_{f'} P^k_{r,f'}\underline{\omega}_{f'} \wedge \mathrm{tr}_{f'}\,\mu_f. \tag{6}$$

The global discrete spaces are connected by the discrete differential $\underline{\mathrm{d}}^k_{r,h} : \underline{X}^k_{r,h} \to \underline{X}^{k+1}_{r,h}$ obtained by projecting each local discrete exterior derivative on the corresponding component of $\underline{X}^{k+1}_{r,h}$:

$$\underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h \coloneqq \left(\star^{-1}\pi^{-,d-k-1}_{r,f} \star \mathrm{d}^k_{r,f}\underline{\omega}_f\right)_{f \in \Delta_d(\mathcal{M}_h), d \in \{k+1,\dots,n\}}. \tag{7}$$

2.3.3 Component norms

For any form degree $k \in \{0, \dots, n\}$ and any $\underline{\omega}_h \in \underline{X}^k_{r,h}$, we define the following L^2-like norms recursively:

- For each $f \in \Delta_k(\mathcal{M}_h)$, we set
$$|||\underline{\omega}_f|||_f \coloneqq \|\omega_f\|_f. \tag{8}$$
- For $d \in \{k+1, \dots, n\}$ and $f \in \Delta_d(\mathcal{M}_h)$,
$$|||\underline{\omega}_f|||_f \coloneqq \left(\|\omega_f\|^2_f + h_f \sum_{f' \in \Delta_{d-1}(\mathcal{M}_h(f))} |||\omega_{f'}|||^2_{f'}\right)^{\frac{1}{2}}. \tag{9}$$

The global norm is then defined as

$$|||\underline{\omega}_h|||_h := \left(\sum_{f \in \Delta_n(\mathcal{M}_h)} |||\underline{\omega}_h|||_f^2 \right)^{\frac{1}{2}}. \tag{10}$$

Remark 2 (Equivalent component norms). A slightly different choice of component norms is made in [18, Eq. A.7] (with $s = 2$ in this reference). Accounting for the mesh regularity assumption (which gives, in particular, a uniform bound on the cardinality of $\Delta_{d-1}(\mathcal{M}_h(f))$ for each $f \in \Delta_d(\mathcal{M}_h)$), these choices are equivalent in terms of scaling with h, which entitles us to use the results of [18, Appendix A]. A third equivalent choice of norm is obtained replacing (9) with

$$|||\underline{\omega}_f|||_f := \left(\sum_{i=k}^{n} h_f^{n-i} \sum_{f' \in \Delta_i(\mathcal{M}_h(f))} \|\omega_{f'}\|_{f'}^2 \right)^{\frac{1}{2}}. \tag{11}$$

Remark 3 (Properties of the potential reconstruction). Recalling [7, Eq. (3.37)], we have that, for all integers $d \in \{0, \dots, n\}$ and $k \in \{0, \dots, d\}$, all $f \in \Delta_d(\mathcal{M}_h)$, and all $\underline{\omega}_f \in \underline{X}_{r,f}^k$,

$$\pi_{r,f}^{-,d-k} \star P_{r,f}^k \underline{\omega}_f = \star \omega_f. \tag{12}$$

Moreover, following the same arguments as [18, Eq. (A.10)], we also get

$$\| \star P_{r,f}^k \underline{\omega}_f \|_f \lesssim |||\underline{\omega}_f|||_f. \tag{13}$$

2.4 Poincaré inequalities for the Discrete de Rham complex

The main result of this work is the following generic Poincaré inequality, which is valid for forms of any degree and domains of general topology.

Theorem 4 (Poincaré inequality)**.** *Let $k \in \{0, \dots, n-1\}$ and let $\underline{\omega}_h \in \underline{X}_{r,h}^k$. Then, there exists $\underline{\tau}_h \in \underline{X}_{r,h}^k$ such that $\underline{\mathrm{d}}_{r,h}^k \underline{\tau}_h = \underline{\mathrm{d}}_{r,h}^k \underline{\omega}_h$ and*

$$|||\underline{\tau}_h|||_h \lesssim |||\underline{\mathrm{d}}_{r,h}^k \underline{\omega}_h|||_h.$$

Equivalently,

$$\min_{\underline{\phi}_h \in \operatorname{Ker} \underline{\mathrm{d}}_{r,h}^k} |||\underline{\omega}_h + \underline{\phi}_h|||_h \lesssim |||\underline{\mathrm{d}}_{r,h}^k \underline{\omega}_h|||_h.$$

Proof. See Section 4. □

This formulation of the Poincaré inequalities avoids any reference to an orthogonality condition. In fact, it does not even rely on the existence of an associated inner product. However, it is fully equivalent to the usual one, which is stated in the following corollary.

Corollary 5 (Poincaré inequality on orthogonal complements)**.** *Let $\langle \cdot, \cdot \rangle_h$ be an inner product on $\underline{X}_{r,h}^k$ such that its associated norm $\|\cdot\|_h$ satisfies $\|\underline{\omega}_h\|_h \simeq |||\underline{\omega}_h|||_h$ for all $\underline{\omega}_h \in \underline{X}_{r,h}^k$, and denote by $(\operatorname{Ker} \underline{\mathrm{d}}_{r,h}^k)^\perp$ the orthogonal complement for this inner product. Then, Theorem 4 is equivalent to the following statement:*

$$\|\underline{\omega}_h\|_h \lesssim \|\underline{\mathrm{d}}_{r,h}^k \underline{\omega}_h\|_h \qquad \forall \underline{\omega}_h \in (\operatorname{Ker} \underline{\mathrm{d}}_{r,h}^k)^\perp. \tag{14}$$

Proof. First, we show that Theorem 4 implies (14). Let $\underline{\omega}_h \in (\operatorname{Ker} \underline{\mathrm{d}}^k_{r,h})^\perp$, and let $\underline{P}_h$ denote the orthogonal projector onto $(\operatorname{Ker} \underline{\mathrm{d}}^k_{r,h})^\perp$ for the scalar product $\langle \cdot, \cdot \rangle_h$. Theorem 4 ensures the existence of $\underline{\phi}_h \in \operatorname{Ker} \underline{\mathrm{d}}^k_{r,h}$ such that

$$|||\underline{\omega}_h + \underline{\phi}_h|||_h \lesssim |||\underline{\mathrm{d}}^k_{r,h} \underline{\omega}_h|||_h. \tag{15}$$

By definition of $\underline{P}_h$, we have $\underline{P}_h \underline{\omega}_h = \underline{\omega}_h$ and $\underline{P}_h \underline{\phi}_h = \underline{0}$. Using the uniform equivalence of the two norms and the fact that $\|\underline{P}_h \cdot\|_h \le \|\cdot\|_h$ since $\underline{P}_h$ is an orthogonal projector, we write

$$\|\underline{\omega}_h\|_h = \|\underline{P}_h(\underline{\omega}_h + \underline{\phi}_h)\|_h \le \|\underline{\omega}_h + \underline{\phi}_h\|_h \simeq |||\underline{\omega}_h + \underline{\phi}_h|||_h \overset{(15)}{\lesssim} |||\underline{\mathrm{d}}^k_{r,h} \underline{\omega}_h|||_h \simeq \|\underline{\mathrm{d}}^k_{r,h} \underline{\omega}_h\|_h.$$

To prove the reverse implication, let $\underline{\omega}_h \in \underline{X}^k_{r,h}$ and let $\underline{\psi}_h$ be the orthogonal projection of $\underline{\omega}_h$ into $\operatorname{Ker} \underline{\mathrm{d}}^k_{r,h}$. By construction, $\underline{\omega}_h - \underline{\psi}_h \in (\operatorname{Ker} \underline{\mathrm{d}}^k_{r,h})^\perp$ and $\underline{\psi}_h \in \operatorname{Ker} \underline{\mathrm{d}}^k_{r,h}$. Moreover, it holds:

$$\begin{aligned}\inf_{\underline{\phi}_h \in \operatorname{Ker} \underline{\mathrm{d}}^k_{r,h}} |||\underline{\omega}_h + \underline{\phi}_h|||_h &\le |||\underline{\omega}_h - \underline{\psi}_h|||_h \simeq \|\underline{\omega}_h - \underline{\psi}_h\|_h \\ &\overset{(14)}{\lesssim} \|\underline{\mathrm{d}}^k_{r,h}(\underline{\omega}_h - \underline{\psi}_h)\|_h = \|\underline{\mathrm{d}}^k_{r,h} \underline{\omega}_h\|_h \simeq |||\underline{\mathrm{d}}^k_{r,h} \underline{\omega}_h|||_h. \quad \square\end{aligned}$$

3 A Poincaré inequality for cochains

The proof of Theorem 4 relies on a Poincaré inequality for cochains on $\mathcal{M}_h$. We first recall some concepts related to this notion, before stating and proving this inequality.

3.1 Setting

3.1.1 Chain complex

A *k-chain* w of $\mathcal{M}_h$ is a formal linear combination of k-cells

$$w = \sum_{f \in \Delta_k(\mathcal{M}_h)} w_f \, f, \tag{16}$$

where the w_f are real numbers. The set of all k-chains, equipped with the natural addition and scalar multiplication, is a real vector space that we denote by $C_k(\mathcal{M}_h)$. For a k-chain $w \in C_k(\mathcal{M}_h)$, its *support* $\operatorname{supp}(w)$ is defined by

$$\operatorname{supp}(w) \coloneqq \left\{ f \in \Delta_k(\mathcal{M}_h) \,:\, w_f \neq 0 \right\}.$$

Observe that each k-cell $f \in \Delta_k(\mathcal{M}_h)$ is also a k-chain $f \in C_k(\mathcal{M}_h)$, obtained by considering $w_f = 1$ as the unique non-zero real number in the linear combination (16). Therefore, the set of all k-cells $\Delta_k(\mathcal{M}_h)$ forms a basis for $C_k(\mathcal{M}_h)$, which we call the *canonical basis* of $C_k(\mathcal{M}_h)$. We also consider the following two identifications: (i) we identify the k-cell f taken with opposite orientation with the k-chain $-f \in C_k(\mathcal{M}_h)$; (ii) we identify the (topological) boundary ∂f with the *boundary $(k-1)$-chain* $\partial_k f \in C_{k-1}(\mathcal{M}_h)$ defined by

$$\partial_k f \coloneqq \sum_{f' \in \Delta_{k-1}(\mathcal{M}_h(f))} \epsilon_{ff'} \, f', \tag{17}$$

where $\epsilon_{ff'}$ is equal to $+1$ if f' has the orientation induced by that of f and -1 otherwise. The *boundary operator* $\partial_k : C_k(\mathcal{M}_h) \to C_{k-1}(\mathcal{M}_h)$ is defined by linearity on the space of k-chains by setting, for $w \in C_k(\mathcal{M}_h)$ as in (16),

$$\partial_k w \coloneqq \sum_{f \in \Delta_k(\mathcal{M}_h)} w_f \, \partial_k f.$$

It can be checked that $\partial_k \circ \partial_{k+1} = 0$ for $k \in \{1, \dots, n-1\}$. As a consequence, the sequence

$$\cdots \xrightarrow{\partial_{k+1}} C_k(\mathcal{M}_h) \xrightarrow{\partial_k} C_{k-1}(\mathcal{M}_h) \xrightarrow{\partial_{k-1}} \cdots,$$

with $C_k(\mathcal{M}_h) = \{0\}$ if $k < 0$ or $k > n$, is a *chain complex*. Accordingly, we can also define $\partial_{n+1} \coloneqq 0$ and $\partial_0 \coloneqq 0$ as the zero maps.

We define the subspace of *k-cycles*

$$Z_k(\mathcal{M}_h) \coloneqq \ker \partial_k = \{w \in C_k(\mathcal{M}_h) \,:\, \partial_k w = 0\} \subset C_k(\mathcal{M}_h),$$

and the space of *k-boundaries*

$$B_k(\mathcal{M}_h) \coloneqq \operatorname{Im} \partial_{k+1} = \{\partial_{k+1} w \,:\, w \in C_{k+1}(\mathcal{M}_h)\} \subset C_k(\mathcal{M}_h).$$

Noticing that $\partial_k \circ \partial_{k+1} = 0$ implies $B_k(\mathcal{M}_h) \subset Z_k(\mathcal{M}_h)$, we then define the k-th *homology space* as the quotient space

$$H_k(\mathcal{M}_h) \coloneqq Z_k(\mathcal{M}_h)/B_k(\mathcal{M}_h).$$

3.1.2 Cochain complex

A k-cochain λ of $\mathcal{M}_h$ is a linear map $\lambda : C_k(\mathcal{M}_h) \to \mathbb{R}$. Therefore, the set $C^k(\mathcal{M}_h)$ of all k-cochains is a vector space corresponding to the dual space of $C_k(\mathcal{M}_h)$. Given a k-chain $w \in C_k(\mathcal{M}_h)$, we denote the value of λ at w by $\langle \lambda \,, w \rangle \coloneqq \lambda(w)$.

From basic linear algebra, for each k-cell $f \in \Delta_k(\mathcal{M}_h)$, there exist a unique cochain $\hat{f} \in C^k(\mathcal{M}_h)$ such that

$$\langle \hat{f} \,, f' \rangle = \delta_{ff'} \coloneqq \begin{cases} 1 & \text{if } f = f', \\ 0 & \text{otherwise.} \end{cases}$$

The set $\left\{\hat{f} \in C^k(\mathcal{M}_h) \,:\, f \in \Delta_k(\mathcal{M}_h)\right\}$ is the *canonical basis* of $C^k(\mathcal{M}_h)$. Accordingly, a k-cochain $\lambda \in C^k(\mathcal{M}_h)$ can be uniquely written as

$$\lambda = \sum_{f \in \Delta_k(\mathcal{M}_h)} \lambda_f \, \hat{f} \text{ where } \lambda_f \coloneqq \langle \lambda \,, f \rangle. \tag{18}$$

We also define the inner product $\langle \cdot \,, \cdot \rangle$ on $\Delta_k(\mathcal{M}_h)$ such that, for any $f, f' \in \Delta_k(\mathcal{M}_h)$, $\langle f \,, f' \rangle \coloneqq \langle \hat{f} \,, f' \rangle = \delta_{ff'}$, and we extend $\langle \cdot \,, \cdot \rangle$ by linearity on $C_k(\mathcal{M}_h)$.

On the space of k-cochains we define the *coboundary operator* $\delta^k : C^k(\mathcal{M}_h) \to C^{k+1}(\mathcal{M}_h)$ as the *adjoint of the boundary operator*, by requiring that, for every k-cochain $\lambda \in C^k(\mathcal{M}_h)$, the following identity holds:

$$\langle \delta^k \lambda \,, w \rangle = \langle \lambda \,, \partial_{k+1} w \rangle \qquad \forall w \in C_{k+1}(\mathcal{M}_h). \tag{19}$$

Using (19) and $\partial_k \circ \partial_{k+1} = 0$, it can be verified that $\delta^k \circ \delta^{k-1} = 0$ for $k \in \{1, \dots, n-1\}$. As a consequence, the sequence

$$\cdots \xrightarrow{\delta^{k-1}} C^k(\mathcal{M}_h) \xrightarrow{\delta^k} C^{k+1}(\mathcal{M}_h) \xrightarrow{\delta^{k+1}} \cdots, \tag{20}$$

with $C^k(\mathcal{M}_h) = \{0\}$ if $k < 0$ or $k > n$, is a *cochain complex*. Accordingly, we also define $\delta^n \coloneqq 0$ as the zero map. We define the space of *k-coboundaries*

$$B^k(\mathcal{M}_h) \coloneqq \operatorname{Im} \delta^{k-1} = \left\{\delta^{k-1} \lambda \,:\, \lambda \in C^{k-1}(\mathcal{M}_h)\right\} \subset C^k(\mathcal{M}_h).$$

3.2 Poincaré inequality for cochains

The Poincaré inequality for cochains actually consists in finding, for each boundary cochain, a pre-image (by the coboundary operator) with coefficients in the canonical basis that can be estimated in terms of the coefficients of the coboundary cochain. In what follows, k-chains $w \in C_k(\mathcal{M}_h)$ and k-cochains $\lambda \in C^k(\mathcal{M}_h)$ will be identified whenever needed with the corresponding vectors of coefficients $(w_f)_{f \in \Delta_k(\mathcal{M}_h)}$ and $(\lambda_f)_{f \in \Delta_k(\mathcal{M}_h)}$ in their expansions (16) and (18) on the canonical bases of $C_k(\mathcal{M}_h)$ and $C^k(\mathcal{M}_h)$, respectively.

Lemma 6 (Poincaré inequality for cochains)**.** *Let $k \in \{0, \dots, n-1\}$ and let a $(k+1)$-coboundary $\xi = (\xi_f)_{f \in \Delta_k(\mathcal{M}_h)} \in B^{k+1}(\mathcal{M}_h)$ be given. Then, there exists a k-cochain $\lambda = (\lambda_f)_{f \in \Delta_k(\mathcal{M}_h)} \in C^k(\mathcal{M}_h)$ such that $\delta^k \lambda = \xi$ and*

$$\sum_{T \in \Delta_n(\mathcal{M}_h)} h_T^{n-2k} \sum_{f \in \Delta_k(\mathcal{M}_h(T))} \lambda_f^2 \lesssim \sum_{T \in \Delta_n(\mathcal{M}_h)} h_T^{n-2k-2} \sum_{f \in \Delta_{k+1}(\mathcal{M}_h(T))} \xi_f^2. \tag{21}$$

Proof. See Section 3.4. □

Remark 7 (Interpretation of Lemma 6 in the context of low-order compatible methods). Consider, for each $k \in \{0, \dots, n\}$, the canonical basis for $C^k(\mathcal{M}_h)$, and let $\mathbb{D}_k$ be the matrix associated with the linear map $\delta^k : C^k(\mathcal{M}_h) \to C^{k+1}(\mathcal{M}_h)$ for $k \in \{0, \dots, n-1\}$. By (19), $\mathbb{D}_k$ is precisely the *incidence matrix between $(k+1)$-cells and k-cells*, which is routinely used in low-order compatible schemes like the Compatible Discrete Operator method [8, 9], the Discrete Geometric Approach [12, 13, 25], or the Mimetic Finite Difference method [6]. In these works, the focus is on dimension $n = 3$, and the coboundary operator δ^k, for $k \in \{0, 1, 2\}$, acts as a topological counterpart of the continuous differential operators of the de Rham complex; in particular, δ^0 acts as a gradient, δ^1 as a curl, and δ^2 as a divergence. For this reason, the following specific notation is also of common use: $\mathbb{G} \coloneqq \mathbb{D}_0$, $\mathbb{C} \coloneqq \mathbb{D}_1$, and $\mathbb{D} \coloneqq \mathbb{D}_2$.

Applying the same reasoning of Corollary 5 to Lemma 6, we obtain the following mimetic version of the Poincaré inequality, which involves a suitable inner product and its associated norm: For all $\lambda = (\lambda_f)_{f \in \Delta_k(\mathcal{M}_h)} \in (\operatorname{Ker} \mathbb{D}^k)^\perp$, with orthogonal taken with respect to an appropriate mimetic product, it holds

$$|||\lambda|||_h \lesssim |||\mathbb{D}_k \lambda|||_h.$$

This result generalizes the known mimetic Poincaré inequalities for the gradient and the curl (c.f. [10, Lemma 7.45 and Lemma 7.47]) to domains with general topology in arbitrary dimension n.

The proof of the Poincaré inequality on the cochains on $\mathcal{M}_h$ relies on a lifting of these cochains as cochains on a simplicial submesh, that we can then identify (through the Whitney and de Rham maps) to conforming piecewise polynomial forms.

3.3 Mapping between polytopal and simplicial cochains

Let $\mathcal{S}_h$ be a given simplicial submesh of $\mathcal{M}_h$. In what follows, for clarity of notation, we use the symbols f and F to denote a generic k-cell and k-simplex of $\mathcal{M}_h$ and $\mathcal{S}_h$, respectively; the symbols T and S will also sometimes be used instead of f and F, respectively, for the special case $k = n$.

The purpose of this section is to construct a graded map $(I^k)_{k \in \{0,\dots,n\}}$ that sends polytopal k-cochains onto simplicial k-cochains, and that is a *cochain map* (i.e., it commutes with the coboundary operator). We first introduce a *selection operator* which, to each simplex, associates the polytopal element of smaller dimension that contains it; then, we explain the principle of the construction (which requires a double induction); finally, we establish the properties of this graded map: cochain property, and norm estimates.

3.3.1 Selection map

Each simplex in $\mathcal{S}_h$ is contained in one or more polytopal cells of $\mathcal{M}_h$; in the following lemma, we show that the lowest-dimensional of such cells is unique and that the simplex is actually contained in its relative interior.

Lemma 8 (Lowest-dimensional containing polytopal cell)**.** *For each $F \in \Delta_k(\mathcal{S}_h)$, there exists a unique d-cell $f \in \Delta_d(\mathcal{M}_h)$ with $d \in \{k, \dots, n\}$ such that $F \subset f$ and $F \not\subset \partial f$.*

Proof. The set $\{\ell \in \{0, \dots, n\} : \exists f \in \Delta_\ell(\mathcal{M}_h) \text{ s.t. } F \subset f\}$ is non-empty and contained in $\mathbb{N}$, so it has a minimum d. Let $f \in \Delta_d(\mathcal{M}_h)$ that contains F; we obviously have $d \geq k$ as a d-cell cannot contain

a simplex of strictly lower dimension. If we had $F \subset \partial f$, since this boundary is the union of cells in $\Delta_{d-1}(\mathcal{M}_h)$, F would be contained in some cell of $\Delta_{d-1}(\mathcal{M}_h)$, contradicting the definition of d. Hence, $F \subset f$ and $F \not\subset \partial f$.

We now prove the uniqueness of this f. Assume that we have a d'-cell $f' \neq f$ such that $F \subset f'$ and $F \not\subset \partial f'$. Without loss of generality we can assume that $d' \geq d$. Then, the intersection $f \cap f'$ is non-empty, since it contains F. However, by Point (iv) in the definition of $\mathcal{M}_h$, this would imply $F \subset \partial f'$, contradicting the choice of f'. □

Thanks to Lemma 8, we can define a function $g_k : \Delta_k(\mathcal{S}_h) \to \mathcal{M}_h$ which maps each k-simplex F to the unique d-cell $f \coloneqq g_k(F) \in \mathcal{M}_h$ such that $F \subset f$ and $F \not\subset \partial f$.

Proposition 9 (Partitions via lowest-dimensional containing cells)**.** *For each $k \in \{0, \dots, n\}$ and $d \in \{k, \dots, n\}$, define the set of all k-simplices whose lowest-dimensional containing polytopal cell is of dimension d:*

$$P_k(d) \coloneqq \{F \in \Delta_k(\mathcal{S}_h) \,:\, g_k(F) \in \Delta_d(\mathcal{M}_h)\}\,.$$

Then, the following two properties hold:

(i) The family $\{P_k(k), \dots, P_k(n)\}$ forms a partition of $\Delta_k(\mathcal{S}_h)$, i.e., $\Delta_k(\mathcal{S}_h) = \bigsqcup_{d=k}^{n} P_k(d)$;

(ii) If a k-simplex lies on the boundary of a d-cell, then the lowest-dimensional polytopal cell that contains this simplex is of dimension $< d$: for all $F \in \Delta_k(\mathcal{S}_h)$ such that $F \subset \partial f$ for some $f \in \Delta_d(\mathcal{M}_h)$, then $F \in \bigsqcup_{d'=k}^{d-1} P_k(d')$.

Proof. (i) Is a direct consequence of the definition of $P_k(d)$ and of Lemma 8.

(ii) This is a direct consequence of the reasoning in the proof of Lemma 8: if $F \subset \partial f$ for $f \in \Delta_d(\mathcal{M}_h)$ then F is fully contained in some $f' \in \Delta_{d-1}(\mathcal{M}_h(f))$ and the minimal dimension of the cells that contains F is thus $\leq d-1$. □

3.3.2 Construction

In what follows, we will consider that, for all $k \in \{0, \dots, n\}$, every simplex in $F \in P_k(k)$ inherits the orientation of the unique k-cell $g_k(F) \in \Delta_k(\mathcal{M}_h)$ in which it is contained.

For each $k \in \{0, \dots, n\}$, we construct a map $I^k : C^k(\mathcal{M}_h) \to C^k(\mathcal{S}_h)$ which associates to each polytopal k-cochain $\lambda \in C^k(\mathcal{M}_h)$ a simplicial k-cochain $I^k(\lambda) \in C^k(\mathcal{S}_h)$. Recalling the partition $\Delta_k(\mathcal{S}_h) = \bigsqcup_{d=k}^{n} P_k(d)$ and the fact that $\Delta_k(\mathcal{S}_h)$ is the canonical basis of $C_k(\mathcal{S}_h)$, we define I^k by specifying $I^k(\lambda) : P_k(d) \to \mathbb{R}$ for all $d \in \{k, \dots, n\}$ and for a generic λ.

Principles driving the construction We have to define, for each $k \in \{0, \dots, n\}$, each $\lambda \in C^k(\mathcal{M}_h)$, each $d \in \{k, \dots, n\}$, and each $F \in P_k(d)$, the value of $\langle I^k(\lambda)\,, F\rangle$. This has to be done in a way that ensures that I^k commutes with the coboundary operator, so that commutation must be inherently embedded in the choice of this value.

The definition of $\langle I^k(\lambda)\,, F\rangle$ is carried out through a double recursion. The outer recursion proceeds on the cochain index k, decreasing from $k = n$ to $k = 0$, while the inner recursion operates on the index d, increasing from $d = k$ to $d = n$. For each pair of values k and d, consider the pair of indices (k, d), which serve to track the construction process of the map I^k by identifying the specific set $P_k(d)$. Table 1 illustrates the steps and how they are intertwined.

The diagonal steps $(k, d) = (k, k)$ in this table are easy. These steps correspond to the case where F is a k-simplex contained in a k-cell $f \in \Delta_k(\mathcal{M}_h)$; since $\lambda \in C^k(\mathcal{M}_h)$, the real number $\langle \lambda\,, f\rangle$ is fully defined and $\langle I^k(\lambda)\,, F\rangle$ can then be defined as a fraction (weighted by the relative volumes of f and F) of $\langle \lambda\,, f\rangle$. This choice corresponds to (23) (in the case $k = n$) and (24) (in the case $k < n$) below.

Let us consider the first non-trivial step $(k, d) = (n-1, n)$ for $n > 1$, which is actually representative of all the other steps in the table. In this case, F is an $(n-1)$-simplex contained in the interior of an

Dimension d such that $F \in P_k(d)$	Cochain index k of λ: n	$n-1$	$n-2$	$\cdots$
0				
$\vdots$				
$n-2$			$P_{n-2}(n-2) \longrightarrow \cdots$	
			$\downarrow$	
$n-1$		$P_{n-1}(n-1) \longrightarrow$	$P_{n-2}(n-1) \longrightarrow \cdots$	
		$\downarrow$	$\downarrow$	
n	$P_n(n) \longrightarrow$	$P_{n-1}(n) \longrightarrow$	$P_{n-2}(n) \longrightarrow \cdots$	

Table 1: Illustration of the double recursion used to define the value $\langle I^k(\lambda), F\rangle$. The recursion proceeds first across the columns from left to right. Within each column, it advances row by row, starting from the diagonal element and moving down to the bottom. The arrows indicate the sets $P_k(d)$ that have already been used to define $\langle I^k(\lambda), F\rangle$ for every $F \in P_k(d)$. These sets are then used to define new values as the double recursion progresses. By the end of each column k, the mapping $I^k(\lambda) : P_k(d) \to \mathbb{R}$ is fully defined.

n-cell f. Let $z \in Z_{n-1}(\mathcal{S}_h(f))$ be a $(n-1)$-cycle and notice that, since f is homeomorphic to a closed n-dimensional ball, z is a $(n-1)$-boundary, i.e., there exists $w \in C_n(\mathcal{S}_h(f))$ such that $z = \partial_n w$. The commutation condition requires

$$\langle I^n(\delta^{n-1}\lambda), w\rangle = \langle \delta^{n-1} I^{n-1}(\lambda), w\rangle \overset{(19)}{=} \langle I^{n-1}(\lambda), \partial_n w\rangle = \langle I^{n-1}(\lambda), z\rangle. \tag{22}$$

In this expression, the left-hand side is fully known by the step $(k,d) = (n,n)$; moreover, expanding the right-hand side according to the decomposition $z = \sum_{F' \in \Delta_{n-1}(\mathcal{S}_h(f))} z_{F'} F'$ of z, the values $\langle I^{n-1}(\lambda), F'\rangle$ are also fully known by the step $(k,d) = (n-1, n-1)$ whenever $F' \in \mathrm{supp}(z)$ is contained in ∂f (since, in this case, $F' \in P_{n-1}(n-1)$).

To exploit (22) and define new values of $I^{n-1}(\lambda)$, we build upon the theory developed in [24]. Specifically, consider cycles z that belong to a complement $Z^{\mathsf{c}}_{n-1}(\mathcal{S}_h(f))$ of the space of cycles whose support is fully contained in ∂f; see (25) below. Then, we select a particular $F \not\subset \partial f$ in the support of z, fix $\langle I^{n-1}(\lambda), \hat{F}\rangle = 0$ for all $\hat{F} \in \mathrm{supp}(z)$ different from F, and determine the only remaining undefined value, $\langle I^{n-1}(\lambda), F\rangle$, so that the relation in (22) is satisfied.

For the construction above to be valid, we must be cautious that, each time we select a z to define some values $\langle I^{n-1}(\lambda), F\rangle$ (either 0 or to ensure (22) for z), we are not re-defining a value that is also fixed when selecting another z'. This is achieved by first building (see Appendix C) a particular basis $(z_i)_i$ of the cycles $Z_{n-1}(\mathcal{S}_h(f))$ that have some support in the interior of f, and such that each z_i has some $(n-1)$-simplex F_i in its support, that is interior to f but not in the support of the other z_j for $j \neq i$ (see (26) below). The process above is then applied to z_i in order to fix $\langle I^{n-1}(\lambda), F\rangle = 0$ for all $F \in \mathrm{supp}(z)$ interior to f and different from F_i, and to set $\langle I^{n-1}(\lambda), F_i\rangle$ to ensure (22) for $z = z_i$. Doing the same for another z_j in the basis does not risk re-defining any value since, by choice of the basis, $F_i \notin \mathrm{supp}(z_j)$ and, if $F \in \mathrm{supp}(z_i) \cap \mathrm{supp}(z_j)$, both the processes for z_i and for z_j set $\langle I^{n-1}(\lambda), F\rangle = 0$.

The remaining work, carried out in Lemma 11 below, consists in checking that this local enforcement of the commutation condition (22) actually leads to a global commutation property.

Detailed construction We now turn to the detailed construction of the map which is done, as indicated above, in a recursive way on the index $k \in \{n, \dots, 0\}$ (i.e., for k ranging from n down to 0) and on the

index $d \in \{k, \dots, n\}$ as described hereafter.

- *Base case* $k = n$. For each n-simplex $F \in P_n(n)$, let $f = g_n(F) \in \Delta_n(\mathcal{M}_h)$ and set, for any $\lambda \in C^k(\mathcal{M}_h)$,
$$\langle I^k(\lambda), F\rangle \coloneqq \frac{|F|}{|f|}\langle \lambda, f\rangle. \tag{23}$$

- *Induction step* $k \in \{n-1, \dots, 0\}$. Assume that I^{k+1} has been defined and let us define $I^k(\lambda)$ for a given $\lambda \in C^k(\mathcal{M}_h)$.

  – *Base case* $d = k$. For each $F \in P_k(k)$, let $f = g_k(F) \in \Delta_k(\mathcal{M}_h)$ and set
  $$\langle I^k(\lambda), F\rangle \coloneqq \frac{|F|}{|f|}\langle \lambda, f\rangle. \tag{24}$$

  – *Induction step* $d \in \{k+1, \dots, n\}$. Assume that $I^k(\lambda) : P_k(d') \to \mathbb{R}$ has been defined for all $d' \in \{k, \dots, d-1\}$ and let us define $I^k(\lambda) : P_k(d) \to \mathbb{R}$. Let $f \in \Delta_d(\mathcal{M}_h)$, and denote by $\mathcal{S}_h(f)$ the simplicial subcomplex induced by $\mathcal{S}_h$ on f. Let $Z_k^c(\mathcal{S}_h(f))$ be a subspace of $Z_k(\mathcal{S}_h(f))$ such that
  $$Z_k(\mathcal{S}_h(f)) = Z_k(\mathcal{S}_h(\partial f)) \oplus Z_k^c(\mathcal{S}_h(f)), \tag{25}$$
  and let $\mathcal{B}_k = \{z_1, \dots, z_p\}$ with $p \in \mathbb{N}$ ($p = 0$ means $\mathcal{B}_k = \emptyset$) be the basis of $Z_k^c(\mathcal{S}_h(f))$ computed with Algorithm 1 in Appendix C. This basis has the following two properties: first, for any k-cycle $z_i = \sum_{F \in \Delta_k(\mathcal{S}_h(f))} z_{i,F}\, F \in \mathcal{B}_k$, it holds $z_{i,F} \in \{-1, 0, 1\}$; second, by Lemma 21 below, there exists a set of p k-simplices $\mathcal{F}_k = \{F_1, \dots, F_p\}$ of $\mathcal{S}_h(f)$ dual to $\mathcal{B}_k$ in the sense that:
  $$\begin{aligned} F_i &\not\subset \partial f \quad \forall i \in \{1, \dots, p\}, \\ \langle F_i, z_j\rangle &= \delta_{ij} \quad \forall i, j \in \{1, \dots, p\}, \end{aligned} \tag{26}$$
  where we have assumed, without loss of generality, each F_i oriented in such a way that $\langle F_i, z_i\rangle = 1$ (note that $\langle F_i, z_i\rangle \neq 0$ is equivalent to $F_i \in \operatorname{supp}(z_i)$).
  Let us now characterize k-cycles in terms of k-boundaries. Recalling that each d-cell f is homeomorphic to a closed d-dimensional ball,[1] we have
  $$Z_k(\mathcal{S}_h(f)) = \begin{cases} B_0(\mathcal{S}_h(f)) + F^* & \text{if } k = 0, \\ B_k(\mathcal{S}_h(f)) & \text{if } k \in \{1, \dots, d\}, \end{cases} \tag{27}$$
  where $F^* \in \Delta_0(\mathcal{S}_h(f))$ is an arbitrary 0-cell on the boundary of f (the existence of F^* is guaranteed by the fact that $d > k = 0$). The case $k = 0$ can then be subsumed into the general in the spirit of *reduced homology* [1]. Specifically, we consider the substitution
  $$Z_0(\mathcal{S}_h(f)) \leftarrow Z_0(\mathcal{S}_h(f)) - F^*, \tag{28}$$
  where $Z_0(\mathcal{S}_h(f)) - F^*$ is the set obtained by subtracting to each $z \in Z_0(\mathcal{S}_h(f))$ the specific 0-cell F^* chosen in (27). With this substitution, for all $k \in \{n-1, \dots, 0\}$, we can write $Z_k(\mathcal{S}_h(f)) = B_k(\mathcal{S}_h(f))$. Therefore, by considering also the substitution
  $$\mathcal{B}_0 \leftarrow \mathcal{B}_0 - F^* = \{z_1 - F^*, \dots, z_p - F^*\}, \tag{29}$$

[1] Letting B^d be the d-ball, we have the following well-known characterization of its homology [1]:
$$H_k(B^d) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{if } k \in \{1, \dots, d\}. \end{cases}$$

for each $z_i \in \mathcal{B}_k \subset Z_k(\mathcal{S}_h(f))$, there exists a $(k+1)$-chain $w_i = \sum_{F\in\Delta_{k+1}(\mathcal{S}_h(f))} w_{i,F}\, F \in C_{k+1}(\mathcal{S}_h(f))$ such that $w_{i,F} \in \{-1, 0, 1\}$ and

$$\partial_{k+1} w_i = z_i. \tag{30}$$

For each $z_i \in \mathcal{B}_k$, let $z_i' \in C_k(\mathcal{S}_h(\partial f))$ be the *restriction of* z_i *to* ∂f, that is, the k-chain such that $\mathrm{supp}(z_i') = \mathrm{supp}(z_i) \cap \mathcal{S}_h(\partial f)$ and $\langle F\,, z_i'\rangle = \langle F\,, z_i\rangle$ for each k-simplex $F \in \mathrm{supp}(z_i')$. For each $F \in P_k(d)$ such that $f = g_k(F)$, we then let

$$\langle I^k(\lambda)\,, F\rangle \coloneqq \begin{cases} \langle I^{k+1}(\delta^k\lambda)\,, w_i\rangle - \langle I^k(\lambda)\,, z_i'\rangle & \text{if } F = F_i \in \{F_1, \dots, F_p\}, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

Observe that, if $F \in \mathrm{supp}(z_i')$, then $F \in \Delta_k(\mathcal{S}_h)$ and $F \subset \partial f$ since $z_i' \in C_k(\mathcal{S}_h(\partial f))$. Applying Point (ii) in Proposition 9, we have $F \in \bigsqcup_{d'=k}^{d-1} P_k(d')$, hence $\langle I^k(\lambda)\,, F\rangle$ (and thus $\langle I^k(\lambda)\,, z_i'\rangle$) has already been defined by the induction hypothesis on d, while I^{k+1} has been defined owing to the recurrence assumption. Hence, the recursive formula (31) is well-defined, i.e., its right-hand side is computable by induction for each $F \in P_k(d)$.

Example 10 (Definition of I^k on a 3-cell)**.** We exemplify the definition of the map $I^k : C^k(\mathcal{M}_h) \to C^k(\mathcal{S}_h)$ for the specific case of the pyramidal 3-cell f shown in Figure 1(a).

Consider, for each $k \in \{3, \dots, 0\}$ and $d \in \{k, \dots, 3\}$, the pair of indices (k, d), which keeps track of the construction process of the map I^k by identifying the specific set $P_k(d)$. First, we notice that the cases for which $k = d$, i.e., $(k, d) \in \{(3,3), (2,2), (1,1), (0,0)\}$, are trivial, since formulas (23), (24) are directly applied. Moreover, for this specific example, the cases $(k, d) \in \{(0,1), (0,3)\}$ are also trivial, since there are no 0-cells of $\mathcal{S}_h(f)$ that are internal to any 1-cell or 3-cell of $\mathcal{S}_h(f)$. We thus focus on the remaining cases, i.e., $(k, d) \in \{(2,3), (1,2), (1,3), (0,2)\}$, for which, we show the construction of the sets $\mathcal{F}_k$ and $\mathcal{B}_k$ associated with the subspace $Z_k^c(\mathcal{S}_h(f))$ of $Z_k(\mathcal{S}_h(f))$ in Figure 1(b-d).

3.3.3 Properties

Lemma 11 (Cochain map property for $(I^k)_k$)**.** *The graded map $(I^k)_k$, with $I^k : C^k(\mathcal{M}_h) \to C^k(\mathcal{S}_h)$ for $k \in \{0, \dots, n\}$, is a* cochain map*, i.e., for all $k \in \{0, \dots, n\}$,*

$$\delta^k I^k(\lambda) = I^{k+1}(\delta^k\lambda), \qquad \forall \lambda \in C^k(\mathcal{M}_h), \tag{32}$$

with the convention that $I^{n+1} \coloneqq 0$.

Proof. We verify the formula (32) by induction on $k \in \{n, n-1, \dots, 0\}$.

1) Base case $k = n$. Recalling the cochain complex (20), we have $\delta^n = 0$ and thus, by the convention on I^{n+1}, the relation (32) holds.

2) Induction step $k \in \{n-1, \dots, 0\}$. Assume that (32) holds for $k+1$ and let us prove it for k. To alleviate the notation, in the rest of the proof, for $\lambda \in C^k(\mathcal{M}_h)$, we set

$$\tilde{\lambda} \coloneqq I^k(\lambda) \in C^k(\mathcal{S}_h) \text{ and } \tilde{\xi} \coloneqq I^{k+1}(\delta^k\lambda) \in C^{k+1}(\mathcal{S}_h). \tag{33}$$

We thus need to check that $\delta^k\tilde{\lambda} = \tilde{\xi}$, that is, recalling that $\Delta_{k+1}(\mathcal{S}_h)$ is the canonical basis of $C_{k+1}(\mathcal{S}_h)$ and the relation (19), that

$$\langle \tilde{\lambda}\,, \partial_{k+1} F\rangle = \langle \tilde{\xi}\,, F\rangle \qquad \forall F \in \Delta_{k+1}(\mathcal{S}_h).$$

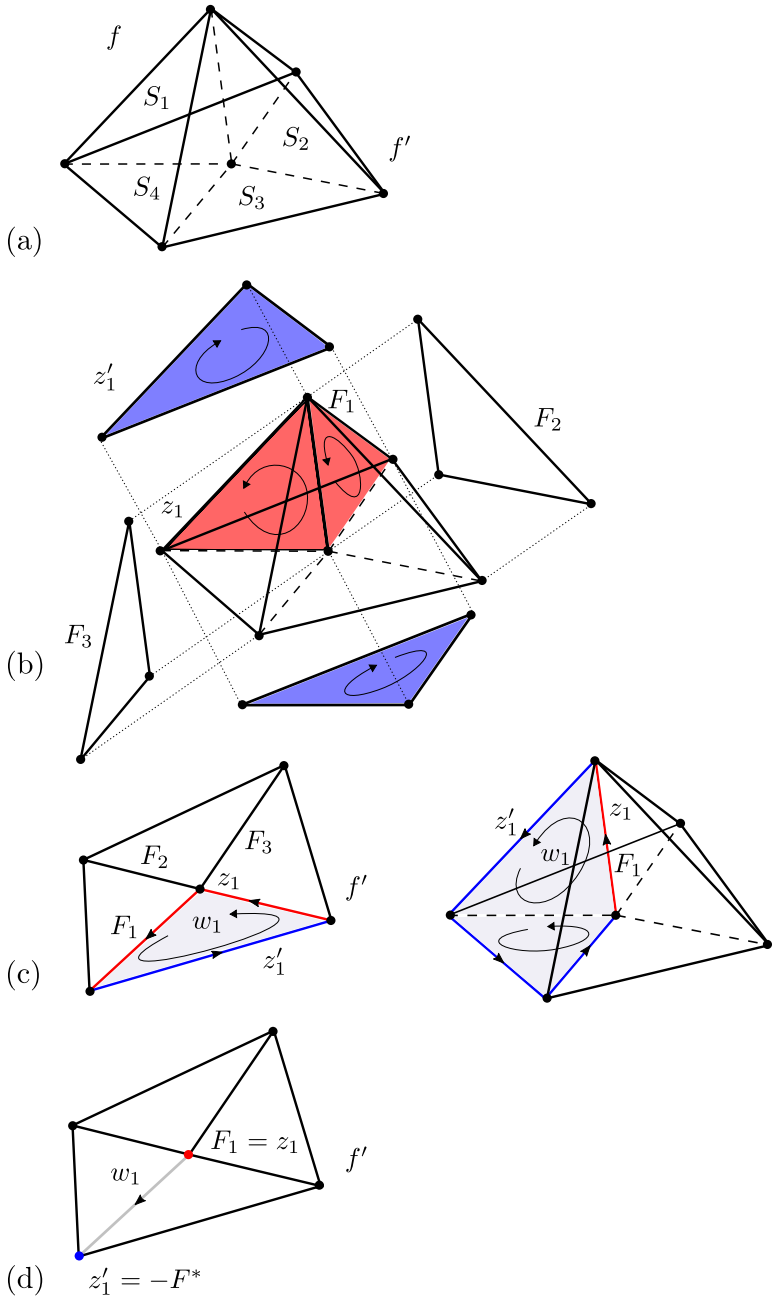

Figure 1: (a) A simplicial subdivision $\mathcal{S}_h(f)$ of a pyramidal 3-cell f with a quadrilateral base f'; $\mathcal{S}_h(f)$ is made of four 3-simplices so that $\Delta_3(\mathcal{S}_h(f)) = \{S_1, S_2, S_3, S_4\}$. (b) Case $(k,d) = (2,3)$ (2-simplices contained in the pyramid). A exploded view of the set $\mathcal{F}_2 = \{F_1, F_2, F_3\}$ of 2-simplices associated with the basis $\mathcal{B}_2$ computed with Algorithm 1 in Appendix C. In red and blue, the 2-cycle $z_1 \in \mathcal{B}_2$ satisfying $\langle F_i, z_1 \rangle = \delta_{i1}$ for $i \in \{1,2,3\}$; the 3-chain w_1 such that $\partial_3 w_1 = z_1$ is $w_1 = S_1$. In blue, the restriction $z'_1 \in C_2(\mathcal{S}_h(f))$ of z_1 to ∂f. (c) Case $(k,d) = (1,2)$ (left, corresponding to 1-simplices contained in a polygonal 2-cell) and $(k,d) = (1,3)$ (right, corresponding to 1-simplices contained in the pyramid). On the left, $d = 2$, and the set $\mathcal{F}_1 = \{F_1, F_2, F_3\}$ of 1-simplices associated with the basis $\mathcal{B}_1$ computed with Algorithm 1. In red and blue, the 1-cycle $z_1 \in \mathcal{B}_1$ satisfying $\langle F_i, z_1 \rangle = \delta_{i1}$ for $i \in \{1,2,3\}$. In blue, the restriction $z'_1 \in C_1(\mathcal{S}_h(f))$ of z_1 to ∂f, and in light grey the 2-chain w_1 such that $\partial_2 w_1 = z_1$. On the right, $d = 3$, and the set $\mathcal{F}_1 = \{F_1\}$. In red, blue, and light grey the corresponding 1-chains z_1, z'_1, and 2-chain w_1 (supported by two 2-simplices). (d) Case $(k,d) = (0,2)$. The set $\mathcal{F}_0 = \{F_1\}$ associated with the basis $\mathcal{B}_0$ computed with Algorithm 1, and the 0-cell $F^* \in \Delta_0(\mathcal{S}_h(\partial f'))$. By the substitution in (29), the 0-chain $z_1 \in \mathcal{B}_0$ is $z_1 \coloneqq F_1 - F^*$, and its restriction z'_1 to $\partial f'$ is the 0-chain $z'_1 = -F^*$. The 1-chain w_1 such that $\partial_1 w_1 = z_1$ is then simply the 1-simplex in $\mathcal{S}_h(f')$ connecting F_1 and F^*.

Since any $F \in \Delta_{k+1}(\mathcal{S}_h)$ is contained in $\mathcal{S}_h(f)$ for $f = g_k(F) \in \Delta_d(\mathcal{M}_h)$, it is sufficient to verify the following local version of the commutation property: For all $d \in \{k+1, \dots, n\}$ and all $f \in \Delta_d(\mathcal{M}_h)$,

$$\langle \tilde{\lambda}, \partial_{k+1} w \rangle = \langle \tilde{\xi}, w \rangle \qquad \forall w \in C_{k+1}(\mathcal{S}_h(f)). \tag{34}$$

Let us now focus on the boundary map ∂_{k+1} appearing in (34). By (25), we can find a basis $\{z_1, \dots, z_q\}$ of $Z_k(\mathcal{S}_h(f))$ such that $\{z_1, \dots, z_p\}$ (with $p \le q$) is the basis $\mathcal{B}_k$ of $Z_k^c(\mathcal{S}_h(f))$ obtained via Algorithm 1 in Appendix C (and used in the definition of $\tilde{\lambda}$ in the recursive step), and $\{z_{p+1}, \dots, z_q\}$ is a basis of $Z_k(\mathcal{S}_h(\partial f))$. Recall the substitution we performed in (28) and (29) for the sets $Z_0(\mathcal{S}_h(f))$ and $\mathcal{B}_0$ via the 0-cell $F^* \in \Delta_0(\mathcal{S}_h(\partial f))$ in (27), and notice also that the property (26) of the set $\mathcal{F}_0$ of 0-simplices still holds for the new set $\mathcal{B}_0$ after the substitution (29), since $F^* \subset \partial f$. All of the above substitutions ensure that $Z_k(\mathcal{S}_h(f)) = B_k(\mathcal{S}_h(f))$ whatever the value of k (including $k = 0$), and we can find $(k+1)$-chains $\{w_1, \dots, w_q\}$ such that $z_i = \partial_{k+1} w_i$ for each $i \in \{1, \dots, q\}$; we choose these chains to ensure that the first p ones are those used in the definition of I^k (see (30)). Accordingly, consider the direct decomposition

$$C_{k+1}(\mathcal{S}_h(f)) = Z_{k+1}(\mathcal{S}_h(f)) \oplus \operatorname{span}\{w_1, \dots, w_q\}, \tag{35}$$

which follows by applying the First Isomorphism Theorem to the linear map $\partial_{k+1} : C_{k+1}(\mathcal{S}_h(f)) \to B_k(\mathcal{S}_h(f))$ (cf., e.g., [26, Theorems 3.5 and 3.6]), after noticing that $\{z_1, \dots, z_q\}$ is a basis of $B_k(\mathcal{S}_h(f)) = Z_k(\mathcal{S}_h(f))$, since f is homeomorphic to a closed d-dimensional ball.

Since $\tilde{\xi} \in C^{k+1}(\mathcal{S}_h)$, by induction hypothesis it holds

$$\delta^{k+1}\tilde{\xi} \overset{(33)}{=} \delta^{k+1} I^{k+1}(\delta^k \lambda) \overset{(32)}{=} I^{k+2}\delta^{k+1}(\delta^k \lambda) = 0, \tag{36}$$

where the last equality follows from the cochain complex property $\delta^{k+1} \circ \delta^k = 0$. Let now $w \in Z_{k+1}(\mathcal{S}_h(f))$. Exploiting the fact that $Z_{k+1}(\mathcal{S}_h(f)) = B_{k+1}(\mathcal{S}_h(f))$ since the $(k+1)$-th homology group is trivial (as f is contractible), we can write $w = \partial_{k+2} t$ with $t \in C_{k+2}(\mathcal{S}_h(f))$. We then have

$$\langle \tilde{\lambda}, \partial_{k+1} w \rangle = \langle \tilde{\lambda}, 0 \rangle = 0, \tag{37}$$

$$\langle \tilde{\xi}, w \rangle = \langle \tilde{\xi}, \partial_{k+2} t \rangle \overset{(19)}{=} \langle \delta^{k+1}\tilde{\xi}, t \rangle \overset{(36)}{=} 0, \tag{38}$$

meaning that (34) is trivially verified for the elements $w \in Z_{k+1}(\mathcal{S}_h(f))$. Therefore, recalling the decomposition (35) as well as the fact that $z_i = \partial_{k+1} w_i$ for all $i \in \{1, \dots, q\}$, the relation (34) follows if we show that

$$\langle \tilde{\lambda}, z_i \rangle = \langle \tilde{\xi}, w_i \rangle \qquad \forall i \in \{1, \dots, q\}. \tag{39}$$

We check this condition by induction on the dimension $d \in \{k+1, \dots, n\}$ of f. To this end, we will treat separately the cases $i \in \{1, \dots, p\}$ (corresponding to $z_i \in Z_k^c(\mathcal{S}_h(f))$) and $i \in \{p+1, \dots, q\}$ (corresponding to $z_i \in Z_k(\mathcal{S}_h(\partial f))$).

2a) Base case $d = k+1$. Let $f \in \Delta_{k+1}(\mathcal{S}_h)$. In this case, $Z_k(\mathcal{S}_h(\partial f))$ has dimension 1 and is spanned by the k-cycle

$$z_{p+1} = \sum_{F' \in \Delta_k(\mathcal{S}_h(\partial f))} \epsilon_{ff'} F' \tag{40}$$

$$= \partial_{k+1} \underbrace{\left(\sum_{F \in \Delta_{k+1}(\mathcal{S}_h(f))} F \right)}_{=: w_{p+1}}, \tag{41}$$

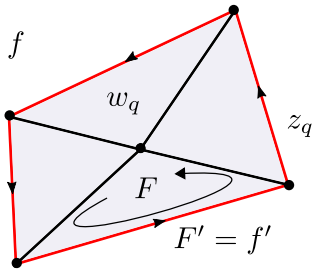

Figure 2: Illustration of the 2-chain w_{p+1} (light grey) and 1-chain z_{p+1} (red) for a simplicial subdivision $\mathcal{S}_h(f)$ of a quadrilateral 2-cell f. Note how the orientation of the 2-simplex F is the same as that of the 2-cell f, as well as how F and f induce the same orientation on the unique 1-cell $f' = F'$ (also, 1-simplex in this case) which lies in their intersection $\partial F \cap \partial f = \{f'\} = \{F'\}$ so that $\epsilon_{FF'} = \epsilon_{ff'}$.

where $f' \in \Delta_k(\mathcal{M}_h(\partial f))$ is such that $F' \subset f'$ and $\epsilon_{ff'}$ denotes its orientation relative to f (which is inherited by F'); see Figure 2. We can then write

$$\begin{aligned}
\langle \tilde{\lambda}\,, z_{p+1}\rangle &\overset{(33),(40)}{=} \sum_{F'\in\Delta_k(\mathcal{S}_h(\partial f))} \epsilon_{ff'}\,\langle I^k(\lambda)\,, F'\rangle \\
&\overset{(24)}{=} \sum_{f'\in\Delta_k(\mathcal{M}_h(\partial f))} \epsilon_{ff'}\left(\sum_{F'\in\Delta_k(\mathcal{S}_h(\partial f)),\,F'\subset f'} \frac{|F'|}{|f'|}\langle \lambda\,, f'\rangle\right) \\
&= \sum_{f'\in\Delta_k(\mathcal{M}_h(\partial f))} \epsilon_{ff'}\langle \lambda\,, f'\rangle \\
&\overset{(17),(19)}{=} \langle \delta^k\lambda\,, f\rangle,
\end{aligned} \tag{42}$$

where in the second line we have used the fact that, if $F' \in \Delta_k(\mathcal{S}_h(\partial f))$, then $F' \in P_k(k)$ by Point (ii) in Proposition 9, while, to pass to the third line, we have used the fact that $\sum_{F'\in\Delta_k(\mathcal{S}_h(\partial f)),\,F'\subset f'} \frac{|F'|}{|f'|} = 1$.

Noticing that $F \in \Delta_{k+1}(\mathcal{S}_h(f))$ implies $g_{k+1}(F) = f$ and thus $F \in P_{k+1}(k+1)$, we can apply a similar reasoning as above and use the definition of $I^{k+1}(\delta^k\lambda)$ in (24) but with $\delta^k\lambda$ in place of λ to get

$$\begin{aligned}
\langle \tilde{\xi}\,, w_{p+1}\rangle &\overset{(33),(41)}{=} \sum_{F\in\Delta_{k+1}(\mathcal{S}_h(f))} \langle I^{k+1}(\delta^k\lambda)\,, F\rangle \\
&= \sum_{F\in\Delta_{k+1}(\mathcal{S}_h(f))} \left(\frac{|F|}{|f|}\langle \delta^k\lambda\,, f\rangle\right) \\
&= \langle \delta^k\lambda\,, f\rangle.
\end{aligned} \tag{43}$$

Comparing (42) and (43), we get (39) when $i = p+1$.

Take now $i \in \{1,\dots,p\}$, i.e., z_i is an element of the basis $\mathcal{B}_k$ of $Z_k^{\mathsf{c}}(\mathcal{S}_h(f))$. Recall that $z_i' \in C_k(\mathcal{S}_h(\partial f))$ is the restriction of z_i to ∂f, so that $\mathrm{supp}(z_i - z_i') \cap \Delta_k(\mathcal{S}_h(\partial f)) = \emptyset$. The properties (26) of the set of k-simplices $\mathcal{F}_k = \{F_1,\dots,F_p\}$ imply: (i) $F_i \not\subset \partial f$ and $F_i \in \mathrm{supp}(z_i)$ for any $i \in \{1,\dots,p\}$; (ii) $F_j \notin \mathrm{supp}(z_i)$ for any $j \neq i, j \in \{1,\dots,p\}$. By Property (i), $F_i \in \mathrm{supp}(z_i - z_i')$, and so, in particular, the set $\mathrm{supp}(z_i - z_i')$ is nonempty. Thus, if $F \in \mathrm{supp}(z_i - z_i')$, then $F \subset f$ and $F \in P_k(k+1)$. By Property (ii), if $F \in \mathrm{supp}(z_i - z_i') \setminus \{F_i\}$, we have $F \notin \{F_1,\dots,F_p\}$ and thus, applying the definition of $\tilde{\lambda}$ given by (31), we infer $\langle \tilde{\lambda}\,, F\rangle = 0$. Hence, $\langle \tilde{\lambda}\,, z_i - z_i' - F_i\rangle = 0$, so that $\langle \tilde{\lambda}\,, z_i'\rangle = \langle \tilde{\lambda}\,, z_i - F_i\rangle$, and

$$\langle \tilde{\lambda}\,, F_i\rangle \overset{(31),(33)}{=} \langle \tilde{\xi}\,, w_i\rangle - \langle \tilde{\lambda}\,, z_i'\rangle = \langle \tilde{\xi}\,, w_i\rangle - \langle \tilde{\lambda}\,, z_i - F_i\rangle, \tag{44}$$

from which we infer

$$\langle \tilde{\lambda}\,, z_i\rangle = \langle \tilde{\lambda}\,, z_i - F_i\rangle + \langle \tilde{\lambda}\,, F_i\rangle \overset{(44)}{=} \langle \tilde{\xi}\,, w_i\rangle,$$

and conclude the proof of (39) for $i \in \{1,\dots,p\}$.

2b) Induction step $d \in \{k+2,\dots,n\}$. Assume that (34) holds for all d'-cells with $d' \in \{k+1,\dots,d-1\}$, and let $f \in \Delta_d(\mathcal{S}_h)$.

Suppose first $z_i \in Z_k(\mathcal{S}_h(\partial f))$. Since f is homeomorphic to a closed d-dimensional ball, its boundary ∂f is homeomorphic to a $(d-1)$-sphere[2]. Since $k < d-1$, and recalling the substitution we performed in (28) for the set $Z_0(\mathcal{S}_h(f))$ using the 0-cell $F^* \in \Delta_0(\mathcal{S}_h(\partial f))$ in (27), together with the fact that $Z_0(\mathcal{S}_h(\partial f)) \subset Z_0(\mathcal{S}_h(f))$, there exists a $(k+1)$-chain $w_i' \in C_{k+1}(\mathcal{S}_h(\partial f))$ such that $\partial_{k+1} w_i' = z_i$.

Recalling that $z_i = \partial_{k+1} w_i$ and that f is contractible, we can write $w_i = w_i' + \partial_{k+2} t$ with $t \in C_{k+2}(\mathcal{S}_h(f))$. This observation implies that verifying (39) for w_i' suffices, as (34) is trivially satisfied for $w = \partial_{k+2} t \in Z_{k+1}(\mathcal{S}_h(f))$, as shown by the relations (37) and (38).

Since every $F \in \operatorname{supp}(w_i')$ is contained in $\mathcal{S}_h(f')$, where $f' = g_{k+1}(F) \in \mathcal{M}_h(\partial f)$ is a d'-cell with $d' \in \{k+1,\dots,d-1\}$, we can apply the induction hypothesis on d to each f'. As a result, (34) is satisfied for each $w = F \in \operatorname{supp}(w_i')$. Therefore, (39) is satisfied for $w = w_i'$, and thus for $w = w_i$, as required.

Now, suppose $z_i \in Z_k^{\mathsf{c}}(\mathcal{S}_h(f))$. In this case, to verify (39) is sufficient to repeat the same argument applied for the case $d = k+1$. □

Lemma 12 (Estimate on I^k)**.** *Let $k \in \{0,\dots,n\}$ and $\lambda = (\lambda_f)_{f\in\Delta_k(\mathcal{M}_h)} \in C^k(\mathcal{M}_h)$, and set $\tilde{\lambda} = (\tilde{\lambda}_F)_{F\in\Delta_k(\mathcal{M}_h)} \coloneqq I^k(\lambda) \in C^k(\mathcal{S}_h)$. Then, for all $T \in \Delta_n(\mathcal{M}_h)$, the following uniform bound holds:*

$$\sum_{S\in\Delta_n(\mathcal{S}_h(T))}\ \sum_{F\in\Delta_k(\mathcal{S}_h(S))} \tilde{\lambda}_F^2 \lesssim \sum_{f\in\Delta_k(\mathcal{M}_h(T))} \lambda_f^2 + \sum_{f\in\Delta_{k+1}(\mathcal{M}_h(T))} (\delta^k\lambda)_f^2. \tag{45}$$

Proof. Thanks to mesh regularity of $\mathcal{M}_h$, we have $\operatorname{card}(\{S \in \Delta_n(\mathcal{S}_h(T)) \,:\, F \subset S\}) \le \operatorname{card}(\Delta_n(\mathcal{S}_h(T))) \lesssim 1$ for each $k \in \{0,\dots,n\}$, and it follows that

$$\sum_{S\in\Delta_n(\mathcal{S}_h(T))}\ \sum_{F\in\Delta_k(\mathcal{S}_h(S))} \tilde{\lambda}_F^2 \simeq \sum_{F\in\Delta_k(\mathcal{S}_h(T))} \tilde{\lambda}_F^2 = \underbrace{\sum_{F\in P_k(k),\,F\subset T} \tilde{\lambda}_F^2}_{=:A_1} + \underbrace{\sum_{d=k+1}^{n}\ \sum_{F\in P_k(d),\,F\subset T} \tilde{\lambda}_F^2}_{=:A_2}, \tag{46}$$

where, to conclude, we have used the partition $\Delta_k(\mathcal{S}_h) = P_k(k) \sqcup \big(\bigsqcup_{d=k+1}^n P_k(d)\big)$ of Proposition 9. We now focus on bounding the terms A_1 and A_2 by the right-hand side of (45). This is done by induction on $k \in \{n,\dots,0\}$.

Base case $k = n$. In this case, $A_2 = 0$ and we just have to bound A_1. Applying the definition of $\tilde{\lambda}_S$ given by (23), and observing that $\frac{|F|}{|f|} \le 1$ for each $F \in \Delta_n(\mathcal{S}_h(f))$, we have

$$\sum_{F\in\Delta_n(\mathcal{S}_h(f))} \left(\frac{|F|}{|f|}\right)^2 \le \sum_{F\in\Delta_n(\mathcal{S}_h(f))} \frac{|F|}{|f|} = 1,$$

[2] Letting S^d be the d-sphere, we have the following well-known characterization of its homology [1]:

$$H_k(S^d) = \begin{cases} \mathbb{Z} & \text{if } k \in \{0,d\} \\ 0 & \text{if } k \in \{1,\dots,d-1\}. \end{cases}$$

from which we readily infer, using $\mathrm{card}(\Delta_n(\mathcal{S}_h(f))) \lesssim 1$ (consequence of mesh regularity),

$$A_1 = \sum_{F\in\Delta_n(\mathcal{S}_h(f))} \left(\frac{|F|}{|f|}\right)^2 \lambda_f^2 \lesssim \lambda_f^2.$$

This yields (45) since $\Delta_n(\mathcal{M}_h(f)) = \{f\}$.

Induction step $k \in \{n-1,\dots,0\}$. Assume that the inequality (45) holds for $k+1$ and let us prove it for k.

Term A_1. Using the definition of $\tilde{\lambda}_F$ given by formula (24) and reasoning as above, we have

$$A_1 = \sum_{f\in\Delta_k(\mathcal{M}_h(T))} \sum_{F\in\Delta_k(\mathcal{S}_h(f))} \left(\frac{|F|}{|f|}\right)^2 \lambda_f^2 \lesssim \sum_{f\in\Delta_k(\mathcal{M}_h(T))} \lambda_f^2. \tag{47}$$

Term A_2. Let $f \in \Delta_d(\mathcal{M}_h(T))$ for $d \in \{k+1,\dots,n\}$. Let $\tilde{\zeta} = (\tilde{\zeta}_F)_{F\in\Delta_k(\mathcal{S}_h(f))} \in C^k(\mathcal{S}_h(f))$, and let $t = \sum_{F\in\Delta_k(\mathcal{S}_h(f))} t_F\, F \in C_k(\mathcal{S}_h(f))$ be such that $t_F \in \{-1,0,1\}$. Then, by mesh regularity and a Cauchy–Schwarz inequality, it holds

$$|\langle\tilde{\zeta}, t\rangle|^2 = \left|\sum_{F\in\Delta_k(\mathcal{S}_h(f))} t_F\,\tilde{\zeta}_F\right|^2 \le \mathrm{card}(\mathrm{supp}(t)) \sum_{F\in\Delta_k(\mathcal{S}_h(f))} \tilde{\zeta}_F^2 \lesssim \sum_{F\in\Delta_k(\mathcal{S}_h(f))} \tilde{\zeta}_F^2, \tag{48}$$

where we have used the fact that $t_F^2 = 1$ for each $F \in \mathrm{supp}(t)$ in the second inequality to write the squared norm in terms of the support of t, while the conclusion follows observing that $\mathrm{card}(\mathrm{supp}(t)) \le \mathrm{card}(\Delta_k(\mathcal{S}_h(f))) \lesssim 1$ by mesh regularity.

Let $F \in \Delta_k(\mathcal{S}_h(f)) \cap P_k(d)$ and recall the definition (31) of $\tilde{\lambda}_F$. Setting $\tilde{\xi} := I^{k+1}(\delta^k\lambda)$ and applying the bound (48) to both terms $\langle\tilde{\lambda}, z_i'\rangle$ and $\langle\tilde{\xi}, w_i\rangle$ leads to

$$\sum_{F\in P_k(d),\,F\subset f} \tilde{\lambda}_F^2 \lesssim \sum_{F\in\Delta_k(\mathcal{S}_h(\partial f))} \tilde{\lambda}_F^2 + \sum_{F\in\Delta_{k+1}(\mathcal{S}_h(f))} \tilde{\xi}_F^2 \qquad \forall f \in \Delta_d(\mathcal{M}_h(T)). \tag{49}$$

Observe that, by Point (ii) in Proposition 9, the first term in the right-hand side of (49) can be equivalently written as

$$\sum_{d'=k}^{d-1} \sum_{f'\in\Delta_{d'}(\mathcal{M}_h(f))} \left(\sum_{F\in P_k(d'),\,F\subset f'} \tilde{\lambda}_F^2\right),$$

and, in the case $d = k+1$, it holds for this term

$$\sum_{f'\in\Delta_k(\mathcal{M}_h(f))} \left(\sum_{F\in P_k(k),\,F\subset f'} \tilde{\lambda}_F^2\right) \overset{(24)}{\lesssim} \sum_{f'\in\Delta_k(\mathcal{M}_h(f))} \lambda_{f'}^2.$$

Hence, via a simple induction argument on the dimension d involving the bound (49) and the mesh regularity assumption (to uniformly bound the cardinalities of $\Delta_k(\mathcal{M}_h(f))$ and $\Delta_k(\mathcal{S}_h(f))$), we conclude

$$A_2 = \sum_{d=k+1}^{n} \sum_{f\in\Delta_d(\mathcal{M}_h(T))} \left(\sum_{F\in P_k(d),\,F\subset f} \tilde{\lambda}_F^2\right) \lesssim \sum_{f\in\Delta_k(\mathcal{M}_h(T))} \lambda_f^2 + \sum_{F\in\Delta_{k+1}(\mathcal{S}_h(T))} \tilde{\xi}_F^2. \tag{50}$$

Recalling that $\tilde{\xi} = I^{k+1}(\delta^k\lambda) \in C^{k+1}(\mathcal{M}_h)$ we can apply the induction hypothesis for $k+1$ with λ replaced by $\delta^k\lambda$, which states that this cochain satisfies (45). Since $\delta^{k+1}(\delta^k\lambda) = 0$ by complex property, this gives

$$\sum_{F\in\Delta_{k+1}(\mathcal{S}_h(T))} \tilde{\xi}_F^2 \lesssim \sum_{f\in\Delta_{k+1}(\mathcal{M}_h(T))} (\delta^k\lambda)_f^2. \tag{51}$$

Using (47) to estimate A_1 and plugging (51) into (50) to estimate A_2 in (46) concludes the proof of the induction step. □

3.3.4 Left inverse

For $k \in \{0, \dots, n\}$, let $(J^k)_k$ be the graded map with $J^k : C^k(\mathcal{S}_h) \to C^k(\mathcal{M}_h)$ such that, for any $\tilde{\lambda} \in C^k(\mathcal{S}_h)$, it holds

$$\langle J^k(\tilde{\lambda}), f \rangle \coloneqq \sum_{F \in \Delta_k(\mathcal{S}_h(f))} \langle \tilde{\lambda}, F \rangle \qquad \forall f \in \Delta_k(\mathcal{M}_h). \tag{52}$$

Lemma 13 (Left inverse of I^k)**.** *The graded map $(J^k)_k$ satisfies the following properties:*

(i) J^k *is a left inverse of* I^k*;*

(ii) $(J^k)_k$ *is a cochain map;*

(iii) For all $T \in \Delta_n(\mathcal{M}_h)$ *and* $\tilde{\lambda} = (\tilde{\lambda}_F)_{F \in \Delta_k(\mathcal{S}_h)} \in C^k(\mathcal{S}_h)$*, setting* $\lambda = (\lambda_f)_{f \in \Delta_k(\mathcal{M}_h)} \coloneqq J^k(\tilde{\lambda}) \in C^k(\mathcal{M}_h)$*, the following bound holds:*

$$\sum_{f \in \Delta_k(\mathcal{M}_h(T))} \lambda_f^2 \lesssim \sum_{S \in \Delta_n(\mathcal{S}_h(T))} \sum_{F \in \Delta_k(\mathcal{S}_h(S))} \tilde{\lambda}_F^2. \tag{53}$$

Proof. (i) It suffices to write, for all $f \in \Delta_k(\mathcal{M}_h)$,

$$\langle J^k(I^k(\lambda)), f \rangle \overset{(52)}{=} \sum_{F \in \Delta_k(\mathcal{S}_h(f))} \langle I^k(\lambda), F \rangle \overset{(23),(24)}{=} \langle \lambda, f \rangle,$$

where we have additionally noticed that $g_k(F) = f$ for all $F \in \Delta_k(\mathcal{S}_h(f))$ in the last step.

(ii) Let $\tilde{\lambda}$ and λ as in Point (iii) and set $\tilde{\xi} \coloneqq \delta^k \tilde{\lambda}$ and $\xi \coloneqq J^{k+1}(\tilde{\xi})$. We have to show that $\delta^k \lambda = \xi$. This is done by writing, for any $f \in \Delta_{k+1}(\mathcal{M}_h)$,

$$\begin{aligned}
\langle \delta^k \lambda, f \rangle &\overset{(19),(17),(52)}{=} \sum_{f' \in \Delta_k(\mathcal{M}_h(\partial f))} \epsilon_{ff'} \sum_{F' \in \Delta_k(\mathcal{S}_h(f'))} \langle \tilde{\lambda}, F' \rangle \\
&= \sum_{F \in \Delta_{k+1}(\mathcal{S}_h(f))} \sum_{F' \in \Delta_k(\mathcal{S}_h(\partial F))} \epsilon_{FF'} \langle \tilde{\lambda}, F' \rangle \\
&\overset{(17),(19)}{=} \sum_{F \in \Delta_{k+1}(\mathcal{S}_h(f))} \langle \tilde{\xi}, F \rangle \\
&\overset{(52)}{=} \langle \xi, f \rangle,
\end{aligned}$$

where, in the second equality, we have used the assumption that the orientation on $\mathcal{S}_h$ is the one induced by the maps $g_\bullet : \mathcal{S}_h \to \mathcal{M}_h$ (which implies $\epsilon_{FF'} = \epsilon_{ff'}$ if $f = g_{k+1}(F)$ and $f' = g_k(F')$ in these sums), together with the relation

$$\sum_{F \in \Delta_{k+1}(\mathcal{S}_h(f))} \sum_{F' \in \Delta_k(\mathcal{S}_h(\partial F)),\, F' \notin \partial f} \epsilon_{FF'} \tilde{\lambda}_{F'} = 0,$$

which translates the fact that contributions on k-simplices that do not lie on the boundary of f cancel out since $\epsilon_{F_1F'} + \epsilon_{F_2F'} = 0$ whenever F_1, F_2 are the $(k+1)$-simplices in $\mathcal{S}_h(f)$ on each side of F'.

(iii) Is a direct consequence of the definition (52) and of the regularity assumption on $\mathcal{M}_h$. □

3.4 Proof of the Poincaré inequality for cochains

We can now prove Lemma 6, which we recall here for the sake of legibility.

Lemma 6 (Poincaré inequality for cochains)**.** *Let $k \in \{0,\dots,n-1\}$ and let a $(k+1)$-coboundary $\xi = (\xi_f)_{f\in\Delta_k(\mathcal{M}_h)} \in B^{k+1}(\mathcal{M}_h)$ be given. Then, there exists a k-cochain $\lambda = (\lambda_f)_{f\in\Delta_k(\mathcal{M}_h)} \in C^k(\mathcal{M}_h)$ such that $\delta^k\lambda = \xi$ and*

$$\sum_{T\in\Delta_n(\mathcal{M}_h)} h_T^{n-2k} \sum_{f\in\Delta_k(\mathcal{M}_h(T))} \lambda_f^2 \lesssim \sum_{T\in\Delta_n(\mathcal{M}_h)} h_T^{n-2k-2} \sum_{f\in\Delta_{k+1}(\mathcal{M}_h(T))} \xi_f^2. \tag{21}$$

Proof. Let $\theta \in C^k(\mathcal{M}_h)$ be such that $\delta^k\theta = \xi$, and consider the following extensions to simplicial cochains:

$$\tilde{\xi} \coloneqq I^{k+1}(\xi) \text{ and } \tilde{\theta} \coloneqq I^k(\theta). \tag{54}$$

By Lemma 11, $(I^k)_k$ is a cochain map so we have

$$\delta^k\tilde{\theta} = \delta^k I^k(\theta) = I^{k+1}(\delta^k\theta) = \tilde{\xi}. \tag{55}$$

For $k \in \{0,\dots n\}$, denote by $R_h^k : \mathfrak{W}_k(\mathcal{S}_h) \to C^k(\mathcal{S}_h)$ the *De Rham map* restricted to the space $\mathfrak{W}^k(\mathcal{S}_h)$ of Whitney k-forms (see [2] and Appendix B), and by $W_h^k : C^k(\mathcal{S}_h) \to \mathfrak{W}_k(\mathcal{S}_h)$ the *Whitney map* on the simplicial complex $\mathcal{S}_h$ [21]. For each $\tilde{\zeta} = (\tilde{\zeta}_F)_{F\in\Delta_k(\mathcal{S}_h)} \in C^k(\mathcal{S}_h)$, $W_h^k(\tilde{\zeta})$ is explicitly written as

$$W_h^k(\tilde{\zeta}) \coloneqq \sum_{S\in\Delta_n(\mathcal{S}_h)} \sum_{F\in\Delta_k(\mathcal{S}_h(S))} \tilde{\zeta}_F \phi_{F,S}^k. \tag{56}$$

Crucially, the graded map $(W_h^k)_k$ is a cochain isomorphism between the simplicial cochain complex and the Whitney form complex supported on $\mathcal{S}_h$, and R_h^k is the inverse of W_h^k.

Let

$$\tilde{\xi}_h \coloneqq W_h^{k+1}(\tilde{\xi}) \quad \text{and} \quad \tilde{\theta}_h \coloneqq W_h^k(\tilde{\theta}). \tag{57}$$

Thanks to the cochain map property of W_h^k, we have $\mathrm{d}^k\tilde{\theta}_h = \mathrm{d}^k W_h^k(\tilde{\theta}) = W_h^{k+1}(\delta^k\tilde{\theta}) \overset{(55)}{=} \tilde{\xi}_h$. Therefore, $\tilde{\xi}_h \in \operatorname{Im}\mathrm{d}^k$. Invoking the continuous Poincaré inequality [3, Theorem 5.11], then the proof of Corollary 5 to retrieve a formulation of this inequality in the spirit of Theorem 4, we infer the existence of $\tilde{\lambda}_h \in \mathfrak{W}_k(\mathcal{S}_h)$ such that

$$\mathrm{d}^k\tilde{\lambda}_h = \tilde{\xi}_h \quad \text{and} \quad \|\tilde{\lambda}_h\|_{L^2\Lambda^k(\Omega)} \lesssim \|\tilde{\xi}_h\|_{L^2\Lambda^{k+1}(\Omega)}. \tag{58}$$

Letting $\tilde{\lambda} \coloneqq R_h^k(\tilde{\lambda}_h)$ and using the cochain map property of R_h^k, we find

$$\delta^k\tilde{\lambda} = \delta^k R_h^k(\tilde{\lambda}_h) = R_h^{k+1}(\mathrm{d}^k\tilde{\lambda}_h) \overset{(58)}{=} R_h^{k+1}\tilde{\xi}_h \overset{(57)}{=} R_h^{k+1}W_h^{k+1}(\tilde{\xi}) = \tilde{\xi}, \tag{59}$$

where the conclusion follows since R_h^{k+1} is the inverse of W_h^{k+1}. Letting $\lambda \coloneqq J^k(\tilde{\lambda})$, we then have

$$\delta^k\lambda = \delta^k J^k(\tilde{\lambda}) = J^{k+1}(\delta^k\tilde{\lambda}) \overset{(59)}{=} J^{k+1}\tilde{\xi} \overset{(54)}{=} J^{k+1}I^{k+1}(\xi) = \xi,$$

where we have used, respectively, Points (ii) and (i) in Lemma 13 to infer the second and last equalities.

It remains to prove that $\lambda = (\lambda_f)_{f\in\Delta_k(\mathcal{M}_h)}$ satisfies (21). Decomposing $\tilde{\lambda}_h = W_h^k(\tilde{\lambda})$ according to (56), noticing that $\sum_{S\in\Delta_n(\mathcal{S}_h)} \bullet = \sum_{T\in\Delta_n(\mathcal{M}_h)} \sum_{S\in\Delta_n(\mathcal{S}_h(T))} \bullet$, and observing that each $\phi_{F,S}$ is only

supported in S, we write (equations (90) and (89) are stated and proved in Appendix B)

$$\begin{aligned}
\|\tilde{\lambda}_h\|^2_{L^2\Lambda^k(\Omega)} &= \sum_{T\in\Delta_n(\mathcal{M}_h)}\sum_{S\in\Delta_n(\mathcal{S}_h(T))}\Big\|\sum_{F\in\Delta_k(\mathcal{S}_h(S))}\tilde{\lambda}_F\phi^k_{F,S}\Big\|^2_S\\
&\overset{(90)}{\simeq} \sum_{T\in\Delta_n(\mathcal{M}_h)}\sum_{S\in\Delta_n(\mathcal{S}_h(T))}\sum_{F\in\Delta_k(\mathcal{S}_h(S))}\tilde{\lambda}_F^2\|\phi^k_{F,S}\|^2_S\\
&\overset{(89)}{\simeq} \sum_{T\in\Delta_n(\mathcal{M}_h)}\sum_{S\in\Delta_n(\mathcal{S}_h(T))}\sum_{F\in\Delta_k(\mathcal{S}_h(S))}\tilde{\lambda}_F^2 h_S^{n-2k}\\
&\overset{(53)}{\lesssim} \sum_{T\in\Delta_n(\mathcal{M}_h)} h_T^{n-2k}\sum_{f\in\Delta_k(\mathcal{M}_h(T))}\lambda_f^2,
\end{aligned}\tag{60}$$

where, in the last passage, we have used the mesh regularity condition to write $h_S \simeq h_T$ for all $S \in \Delta_n(\mathcal{S}_h(T))$. Similarly, we have

$$\begin{aligned}
\|\tilde{\xi}_h\|^2_{L^2\Lambda^k(\Omega)} &\overset{(56)}{=} \sum_{T\in\Delta_n(\mathcal{M}_h)}\sum_{S\in\Delta_n(\mathcal{S}_h(T))}\Big\|\sum_{F\in\Delta_{k+1}(\mathcal{S}_h(S))}\tilde{\xi}_F\phi^{k+1}_{F,S}\Big\|^2_S\\
&\overset{(90)}{\simeq} \sum_{T\in\Delta_n(\mathcal{M}_h)}\sum_{S\in\Delta_n(\mathcal{S}_h(T))}\sum_{F\in\Delta_{k+1}(\mathcal{S}_h(S))}\tilde{\xi}_F^2\|\phi^{k+1}_{F,S}\|^2_S\\
&\overset{(89)}{\simeq} \sum_{T\in\Delta_n(\mathcal{M}_h)}\sum_{S\in\Delta_n(\mathcal{S}_h(T))}\sum_{F\in\Delta_{k+1}(\mathcal{S}_h(S))}\tilde{\xi}_F^2 h_S^{n-2k-2}\\
&\overset{(45)}{\lesssim} \sum_{T\in\Delta_n(\mathcal{M}_h)} h_T^{n-2k-2}\sum_{f\in\Delta_{k+1}(\mathcal{M}_h(T))}\xi_f^2,
\end{aligned}\tag{61}$$

where the conclusion follows from (45) applied to ξ, after noticing that $\delta^{k+1}\xi = \delta^{k+1}\delta^k\theta = 0$.

The estimate (21) follows combining (58), (60) and (61). □

4 Proof of the main result

We are now ready to prove Theorem 4.

Proof. Let $\underline{\omega}_h \in \underline{X}^k_{r,h}$ and set $\underline{\sigma}_h \coloneqq \underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h$. We will construct $\underline{\tau}_h \in \underline{X}^k_{r,h}$ such that $\underline{\mathrm{d}}^k_{r,h}\underline{\tau}_h = \underline{\sigma}_h$ and $|||\underline{\tau}_h|||_h \lesssim |||\underline{\sigma}_h|||_h$.

(i) Construction of $\underline{\tau}_h$ on k-cells. For $f \in \Delta_{k+1}(\mathcal{M}_h)$, we notice that

$$\int_f \sigma_f \overset{(7)}{=} \int_f \star^{-1}\pi^{-,0}_{r,f}\star\,\mathrm{d}^k_{r,f}\underline{\omega}_f = \int_f \mathrm{d}^k_{r,f}\underline{\omega}_f,$$

where the last equality follows from $\mathcal{P}^-_r\Lambda^0(f) = \mathcal{P}_r\Lambda^0(f)$ together with $\star\mathrm{d}^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^0(f)$. The discrete Stokes formula (6) with $\mu_f = 1 \in \mathcal{P}_r\Lambda^0(f)$ and the definition (4) then yield

$$\int_f \sigma_f = \sum_{f'\in\Delta_k(\mathcal{M}_h(f))}\epsilon_{ff'}\int_{f'}\omega_{f'}.$$

This shows that the $(k+1)$-cochain $\xi \coloneqq (\int_f \sigma_f)_{f\in\Delta_{k+1}(\mathcal{M}_h)} \in C^{k+1}(\mathcal{M}_h)$ is a coboundary (precisely, the coboundary of the k-cochain with coefficients $(\int_{f'}\omega_{f'})_{f'\in\Delta_k(\mathcal{M}_h)}$). Hence, by Lemma 6 there exists

a k-cochain $\lambda = (\lambda_{f'})_{f' \in \Delta_k(\mathcal{M}_h)}$ satisfying (21) and such that $\delta^k \lambda = \xi$, which translates into: for all $f \in \Delta_{k+1}(\mathcal{M}_h)$,

$$\int_f \sigma_f = \sum_{f' \in \Delta_k(\mathcal{M}_h(\partial f))} \epsilon_{ff'} \int_{f'} \tau_{f'}, \tag{62}$$

where we have defined

$$\tau_{f'} \coloneqq \star^{-1} \frac{\lambda_{f'}}{|f'|}. \tag{63}$$

(ii) Construction of $\underline{\tau}_h$ on d-cells for $d \in \{k+1, \dots, n\}$. The construction is done by induction on d (noticing that $d = k$ has already been done), by following the ideas in [7, Lemma 27]. For $d \in \{k+1, \dots, n\}$ and $f \in \Delta_d(\mathcal{M}_h)$, assuming that $(\underline{\tau}_{f'})_{f' \in \Delta_{d-1}(f)}$ have been constructed, we define

$$\tau_f \in \star^{-1} \mathrm{d} \mathcal{P}_r \Lambda^{d-k-1}(f) \subset \star^{-1} \mathcal{P}_r^- \Lambda^{d-k}(f) \tag{64}$$

such that, for all $\mu_f \in \kappa \mathcal{P}_{r-1} \Lambda^{d-k}(f)$,

$$(-1)^{k+1} \langle \star \tau_f, \mathrm{d} \mu_f \rangle_f = \langle \star \sigma_f, \mu_f \rangle_f - \langle \star P^k_{r,\partial f} \underline{\tau}_{\partial f}, \mathrm{tr}_{\partial f} \mu_f \rangle_{\partial f}. \tag{65}$$

The existence and uniqueness of τ_f follows from the fact that $\mathrm{d} : \kappa \mathcal{P}_{r-1} \Lambda^{d-k}(f) \to \mathrm{d} \mathcal{P}_r \Lambda^{d-k-1}(f)$ is an isomorphism.

(iii) Proof that $\underline{\mathrm{d}}^k_{r,h} \underline{\tau}_h = \underline{\sigma}_h$. Let $f \in \Delta_d(\mathcal{M}_h)$ for some $d \in \{k+1, \dots, n\}$. Plugging (65) into the definition (5) of the discrete exterior derivative for $\underline{\tau}_f$ we have, for all $\mu_f \in \kappa \mathcal{P}_{r-1} \Lambda^{d-k}(f) \subset \mathcal{P}_r \Lambda^{d-k-1}(f)$,

$$\langle \star \mathrm{d}^k_{r,f} \underline{\tau}_f, \mu_f \rangle_f = \langle \star \sigma_f, \mu_f \rangle_f. \tag{66}$$

We now prove that $\underline{\mathrm{d}}^k_{r,f} \underline{\tau}_f = \underline{\sigma}_f$ by induction on the dimension d of f. Let us assume first that $d = k+1$ and apply the definition (5) of $\mathrm{d}^k_{r,f} \underline{\tau}_f$ with a test function $\alpha_f \in \mathcal{P}_0 \Lambda^0(f)$ to get

$$\langle \star \mathrm{d}^k_{r,f} \underline{\tau}_f, \alpha_f \rangle_f = \langle \star \tau_{\partial f}, \alpha_f \rangle_{\partial f} \overset{(62)}{=} \langle \star \sigma_f, \alpha_f \rangle_f. \tag{67}$$

Adding together (66) and (67) shows that the projections of $\star \mathrm{d}^k_{r,f} \underline{\tau}_f$ and $\star \sigma_f$ on $\kappa \mathcal{P}_{r-1} \Lambda^1(f) + \mathcal{P}_0 \Lambda^0(f)$ coincide. Since this space is $\mathcal{P}_r \Lambda^0(f)$ (cf. [7, Equation (2.7a)]), which contains both $\star \mathrm{d}^k_{r,f} \underline{\tau}_f$ and $\star \sigma_f$, this shows that $\mathrm{d}^k_{r,f} \underline{\tau}_f = \sigma_f$, that is, that $\underline{\mathrm{d}}^k_{r,h} \underline{\tau}_h$ and $\underline{\sigma}_h$ have the same component on f.

We now consider the case $d \in \{k+2, \dots, n\}$. Since $\mathrm{d}^{k+1}_{r,f} \underline{\sigma}_f = \mathrm{d}^{k+1}_{r,f} \underline{\mathrm{d}}^k_{r,f} \underline{\omega}_f = 0$ (see [7, Eq. (3.32)]), we have, for all $\zeta_f \in \kappa \mathcal{P}_{r-1} \Lambda^{d-k-1}(f) \subset \mathcal{P}_r^- \Lambda^{d-k-2}(f)$,

$$\begin{aligned} 0 &\overset{(5)}{=} (-1)^{k+2} \langle \star \sigma_f, \mathrm{d} \zeta_f \rangle_f + \langle \star P^{k+1}_{r,\partial f} \underline{\sigma}_{\partial f}, \mathrm{tr}_{\partial f} \zeta_f \rangle_{\partial f} \\ &\overset{(12)}{=} (-1)^{k+2} \langle \star \sigma_f, \mathrm{d} \zeta_f \rangle_f + \langle \star \sigma_{\partial f}, \mathrm{tr}_{\partial f} \zeta_f \rangle_{\partial f}, \end{aligned} \tag{68}$$

where we have used the fact that $\mathrm{tr}_{f'} \zeta_f \in \mathcal{P}_r^- \Lambda^{d-k-2}(f')$ for all $f' \in \Delta_{d-1}(f)$ (cf. [7, Lemma 4]) to use (12) (with $(k+1, \underline{\sigma}_{f'})$, for $f' \in \Delta_{d-1}(\mathcal{M}_h(\partial f))$, instead of $(k, \underline{\omega}_f)$). Since $\zeta_f \in \mathcal{P}_{r+1}^- \Lambda^{d-k-2}(f)$, we can apply the link between discrete exterior derivatives on subcells [7, Lemma 22] to find

$$\langle \star \mathrm{d}^k_{r,f} \underline{\tau}_f, \mathrm{d} \zeta_f \rangle_f = (-1)^{k+1} \langle \underbrace{\star \, \underline{\mathrm{d}}^k_{r,\partial f} \underline{\tau}_{\partial f}}_{\sigma_{\partial f}}, \mathrm{tr}_{\partial f} \zeta_f \rangle_{\partial f} \overset{(68)}{=} \langle \star \sigma_f, \mathrm{d} \zeta_f \rangle_f, \tag{69}$$

where the equality $\mathrm{d}^k_{r,\partial f}\underline{\tau}_{\partial f} = \sigma_{\partial f}$ comes from the induction hypothesis and the link [7, Eq. (3.31)] between the potential reconstruction and the local discrete exterior derivative. Since $\mathcal{P}_r^-\Lambda^{d-k-1}(f) = \mathrm{d}(\kappa\mathcal{P}_{r-1}\Lambda^{d-k-1}(f)) + \kappa\mathcal{P}_{r-1}\Lambda^{d-k}(f)$ (see, e.g., [7, Eq. (2.16)] and recall that $d \geq k+2$ here), adding together (66) and (69) shows that $\pi^{-,d-k-1}_{r,f} \star \mathrm{d}^k_{r,f}\underline{\tau}_f = \pi^{-,d-k-1}_{r,f} \star \sigma_f$, i.e., that $\underline{\mathrm{d}}^k_{r,f}\underline{\tau}_h$ and $\underline{\sigma}_h$ have the same component on f (see (7)).

(iv) Proof that $|||\underline{\tau}_h|||_h \lesssim |||\underline{\sigma}_h|||_h$. Let $f \in \Delta_d(\mathcal{M}_h)$ with $d \in \{k+1,\dots,n\}$. By (64) we have $\star\tau_f \in \mathrm{d}\mathcal{P}_r\Lambda^{d-k-1}(f) = \mathrm{d}\kappa\mathcal{P}_{r-1}\Lambda^{d-k}(f)$ (see [7, Eq. (2.8)]). We can therefore use Lemma 16 to find $\mu_f \in \kappa\mathcal{P}_{r-1}\Lambda^{d-k}(f)$ such that $\mathrm{d}\mu_f = \star\tau_f$ and

$$\|\mu_f\|_f \lesssim h_f \|\star\tau_f\|_f. \tag{70}$$

Using this μ_f in (65) and applying Cauchy–Schwarz inequalities, we have

$$\begin{aligned}
\|\star\tau_f\|_f^2 &\lesssim \|\star\sigma_f\|_f\|\mu_f\|_f + \sum_{f'\in\Delta_{d-1}(\mathcal{M}_h(\partial f))} \|\star P^k_{r,f'}\underline{\tau}_{f'}\|_{f'}\|\mu_f\|_{f'} \\
&\overset{(84)}{\lesssim} \|\star\sigma_f\|_f\|\mu_f\|_f + h_f^{\frac12} \sum_{f'\in\Delta_{d-1}(\mathcal{M}_h(\partial f))} \|\star P^k_{r,f'}\underline{\tau}_{f'}\|_{f'}\, h_f^{-1}\|\mu_f\|_f \\
&\overset{(70)}{\lesssim} \left(h_f\|\star\sigma_f\|_f + h_f^{\frac12} \sum_{f'\in\Delta_{d-1}(\mathcal{M}_h(\partial f))} \|\star P^k_{r,f'}\underline{\tau}_{f'}\|_{f'}\right)\|\star\tau_f\|_f.
\end{aligned}$$

Using (13) (with f' instead of f) together the fact that $\star$ is an isometry, squaring the result and using $\mathrm{card}(\Delta_{d-1}(\mathcal{M}_h(\partial f))) \lesssim 1$ by mesh regularity assumption, we infer

$$\|\tau_f\|_f^2 \lesssim h_f^2\|\sigma_f\|_f^2 + h_f \sum_{f'\in\Delta_{d-1}(\mathcal{M}_h(\partial f))} |||\underline{\tau}_{f'}|||_{f'}^2. \tag{71}$$

Let us now use this relation to show by induction on $d \in \{k+1,\dots,n\}$ that, for all $f \in \Delta_d(\mathcal{M}_h)$,

$$|||\underline{\tau}_f|||_f^2 \lesssim h_f^2|||\underline{\sigma}_f|||_f^2 + h_f^{d-k} \sum_{f'\in\Delta_k(\mathcal{M}_h(f))} \|\tau_{f'}\|_{f'}^2. \tag{72}$$

For $d = k+1$, this relation is a straightforward consequence of (71), since $|||\underline{\tau}_{f'}|||_{f'} = \|\tau_{f'}\|_{f'}$ for all $f' \in \Delta_k(\mathcal{M}_h)$ (see (8)). Assuming now that (72) holds for some $d \in \{k+1,\dots,n-1\}$, we establish it for $(d+1)$-cells. By definition (9) of the local triple norm we have, for all $f \in \Delta_{d+1}(\mathcal{M}_h)$,

$$\begin{aligned}
|||\underline{\tau}_f|||_f^2 &= \|\tau_f\|_f^2 + h_f \sum_{f'\in\Delta_d(\mathcal{M}_h(\partial f))} |||\underline{\tau}_{f'}|||_{f'}^2 \\
&\overset{(71)}{\lesssim} h_f^2\|\sigma_f\|_f^2 + h_f \sum_{f'\in\Delta_d(\mathcal{M}_h(\partial f))} |||\underline{\tau}_{f'}|||_{f'}^2 \\
&\lesssim h_f^2\|\sigma_f\|_f^2 + h_f^3 \sum_{f'\in\Delta_d(\mathcal{M}_h(\partial f))} |||\underline{\sigma}_{f'}|||_{f'}^2 + h_f^{d+1-k} \sum_{f''\in\Delta_k(\mathcal{M}_h(f))} \|\tau_{f''}\|_{f''}^2 \\
&\lesssim h_f^2|||\underline{\sigma}_f|||_f^2 + h_f^{d+1-k} \sum_{f'\in\Delta_k(\mathcal{M}_h(f))} \|\tau_{f'}\|_{f'}^2,
\end{aligned}$$

where the second inequality follows from (72) applied to each $f' \in \Delta_d(\mathcal{M}_h(\partial f))$, together with $h_{f'} \leq h_f$ and $\sum_{f'\in\Delta_d(\mathcal{M}_h(\partial f))}\sum_{f''\in\Delta_k(\mathcal{M}_h(f'))} \bullet \lesssim \sum_{f''\in\Delta_k(\mathcal{M}_h(f))} \bullet$ (since each the number of occurrences of each f'' in the sum on the left is uniformly bounded by mesh regularity assumption). To conclude we have used the definition of $|||\underline{\sigma}_f|||_f$. This concludes the induction and therefore the proof of (72).

Summing (72) over n-cells, we get

$$|||\underline{\tau}_h|||_h^2 \lesssim \sum_{f\in\Delta_n(\mathcal{M}_h)} h_f^2 |||\underline{\sigma}_f|||_f^2 + \sum_{f\in\Delta_n(\mathcal{M}_h)} h_f^{n-k} \sum_{f'\in\Delta_k(\mathcal{M}_h(f))} \|\tau_{f'}\|_{f'}^2. \tag{73}$$

The definition (63) of $\underline{\tau}_h$ on k-cells and the fact that $\star$ is an isometry yields, for all $f' \in \Delta_k(\mathcal{M}_h)$,

$$\|\tau_{f'}\|_{f'}^2 = \frac{\lambda_{f'}^2}{|f'|} \lesssim h_{f'}^{-k} \lambda_{f'}^2,$$

where the inequality is obtained by invoking the mesh regularity assumption to infer $|f'| \simeq h_{f'}^k$. Plugging this relation into (73) and recalling that $(\lambda_{f'})_{f'\in\Delta_k(\mathcal{M}_h)}$ satisfies (21) with $(\xi_{f'})_{f'\in\Delta_{k+1}(\mathcal{M}_h)} = (\int_{f'} \sigma_{f'})_{f'\in\Delta_{k+1}(\mathcal{M}_h)}$, we obtain

$$\begin{aligned}
|||\underline{\tau}_h|||_h^2 &\lesssim \sum_{f\in\Delta_n(\mathcal{M}_h)} h_f^2 |||\underline{\sigma}_f|||_f^2 + \sum_{f\in\Delta_n(\mathcal{M}_h)} h_f^{n-2k} \sum_{f'\in\Delta_k(\mathcal{M}_h(f))} \lambda_{f'}^2 \\
&\lesssim \sum_{f\in\Delta_n(\mathcal{M}_h)} h_f^2 |||\underline{\sigma}_f|||_f^2 + \sum_{f\in\Delta_n(\mathcal{M}_h)} h_f^{n-2(k+1)} \sum_{f'\in\Delta_{k+1}(\mathcal{M}_h(f))} \|\sigma_{f'}\|_{f'}^2 |f'| \\
&\lesssim \sum_{f\in\Delta_n(\mathcal{M}_h)} h_f^2 |||\underline{\sigma}_f|||_f^2 + \sum_{f\in\Delta_n(\mathcal{M}_h)} h_f^{n-(k+1)} \sum_{f'\in\Delta_{k+1}(\mathcal{M}_h(f))} \|\sigma_{f'}\|_{f'}^2 \\
&\lesssim |||\underline{\sigma}_h|||_h^2
\end{aligned}$$

where we used a Cauchy–Schwarz inequality on the second line, and $|f'| \simeq h_f^{k+1}$ (by mesh regularity) for all $f' \in \Delta_{k+1}(\mathcal{M}_h(f))$ in the third line. The conclusion follows from the definition (10) of $|||\underline{\sigma}_h|||_h$ using the equivalent formulation (11) of the local norm. □

Acknowledgements

Daniele Di Pietro, Jérôme Droniou, and Marien Hanot acknowledge the funding of the European Union via the ERC Synergy, NEMESIS, project number 101115663. Silvano Pitassi acknowledges the funding of the European Union via the MSCA EffECT, project number 101146324. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

A Results on local polynomial spaces

Some of the results established in this appendix on polynomial forms have already been proved in [15, 17] for polynomial functions and in the vector calculus setting.

For $d \in \{1,\dots,n\}$, we denote by $B_d(x,r)$ the ball in $\mathbb{R}^d$ of radius r centered at x, and set $B_d \coloneqq B_d(0,1)$ for the sake of brevity. For any $f \in \Delta_d(\mathcal{M}_h)$, we define the function $\psi_f : x \to x_f + h_f x$. Denoting by ρ the mesh regularity parameter, we have $\psi_f(B_d(0,\rho)) = B_d(x_f, \rho h_f) \subset f \subset \psi_f(B_d) = B_d(x_f, h_f)$. We recall that, if g is an l-form, $\psi_f^\star g$ denotes the pullback of g by ψ_f, that is, the l-form such that $(\psi^\star g)_x(v_1,\dots,v_l) = g_{\psi_f(x)}(D\psi_f(x)v_1,\dots,D\psi_f(x)v_l)$.

Lemma 14 (Norm equivalence)**.** *For any $g \in L^2\Lambda^l(\psi_f(B_d))$, it holds*

$$\|\psi_f^\star g\|_{B_d(0,\rho)} \le h_f^{l-\frac{d}{2}} \|g\|_f \le \|\psi_f^\star g\|_{B_d}. \tag{74}$$

Proof. Let $g \in L^2\Lambda^l(\psi_f(B_d))$. By definition of the pullback, and since $D\psi_f = h_f \mathrm{Id}$, we have, by linearity of $g_{\psi_f(x)}$, $(\psi_f^\star)g_x = h_f^l g_{\psi_f(x)}$. Moreover, $|D\psi_f| = h_f^d$. Therefore, using the change of variable

formula and denoting by $\|\cdot\|_{\mathrm{Alt}^l(\mathbb{R}^d)}$ the canonical norm (induced by the canonical inner product) on the space $\mathrm{Alt}^l(\mathbb{R}^d)$ of alternating l-linear forms on $\mathbb{R}^d$, we have

$$\begin{aligned}\|\psi_f^\star g\|_{B_d}^2 &= \int_{B_d} \|(\psi_f^\star g)_x\|_{\mathrm{Alt}^l(\mathbb{R}^d)}^2 \,\mathrm{d}x = h_f^{-d} \int_{B_d} h_f^{2l} \|g_{\psi_f(x)}\|_{\mathrm{Alt}^l(\mathbb{R}^d)}^2 |D\psi_f| \,\mathrm{d}x \\ &= h_f^{2l-d} \int_{\psi_f(B_d)} \|g_x\|_{\mathrm{Alt}^l(\mathbb{R}^d)}^2 \,\mathrm{d}x = h_f^{2l-d} \|g\|_{\psi_f(B_d)}^2 = h_f^{2l-d} \|g\|_{B_d(x_f,h_f)}^2.\end{aligned}$$

Likewise, we have $\|\psi_f^\star g\|_{B_d(0,\rho)}^2 = h_f^{2l-d} \|g\|_{B_d(x_f,\rho h_f)}^2$. We conclude by using $B_d(x_f, \rho h_f) \subset f \subset B_d(x_f, h_f)$ to write

$$\|\psi_f^\star g\|_{B_d(0,\rho)}^2 = h_f^{2l-d} \|g\|_{B_d(x_f,\rho h_f)}^2 \le h_f^{2l-d} \|g\|_f^2 \le h_f^{2l-d} \|g\|_{B_d(x_f,h_f)}^2 = \|\psi_f^\star g\|_{B_d}^2. \qquad \square$$

If $P \in \mathcal{P}_r\Lambda^l(\mathbb{R}^d)$, then the equivalence of norms in finite dimension ensures that $\|P\|_{B_d(0,\rho)} \simeq \|P\|_{B_d}$, with hidden constants depending only on the space involved, that is, on r, l and d. Taking $\omega \in \mathcal{P}_r\Lambda^l(\mathbb{R}^d)$ and applying this remark to $P = \psi_f^\star \omega \in \mathcal{P}_r\Lambda^l(\mathbb{R}^d)$ (since ψ_f is linear), we see that (74) yields

$$h_f^{l-\frac{d}{2}} \|\omega\|_f \simeq \|\psi_f^\star \omega\|_{B_d} \qquad \forall \omega \in \mathcal{P}_r\Lambda^l(\mathbb{R}^d). \tag{75}$$

In the following lemmas, when A is a mapping between spaces of polynomial forms on some $f \in \Delta_d(\mathcal{M}_h)$, we denote by $|||A|||$ the norm of A induced by the $L^2(f)$-norms on its domain and co-domains.

Lemma 15 (Discrete inequalities for d and κ)**.** *For any $f \in \Delta_d(\mathcal{M}_h)$, the differential $\mathrm{d} : \mathcal{P}_r\Lambda^{d-k-1}(f) \to \mathcal{P}_{r-1}\Lambda^{d-k}(f)$ and the Koszul operator $\kappa : \mathcal{P}_r\Lambda^{d-k+1}(f) \to \mathcal{P}_{r+1}\Lambda^{d-k}(f)$ satisfy*

$$|||\mathrm{d}|||_f \lesssim h_f^{-1}, \tag{76}$$

$$|||\kappa|||_f \lesssim h_f. \tag{77}$$

Proof. The proof of (76) hinges on the fact that the exterior derivative commutes with pullbacks. The operator d is continuous on the finite dimensional space $\mathcal{P}_r\Lambda^{d-k-1}(\mathbb{R}^d)$, with a continuity constant that only depends on r, k and d. Hence,

$$\|\mathrm{d}\mu\|_{B_d} \lesssim \|\mu\|_{B_d} \qquad \forall \mu \in \mathcal{P}_r\Lambda^{d-k-1}(\mathbb{R}^d). \tag{78}$$

We then write, for all $\omega \in \mathcal{P}_r\Lambda^{d-k-1}(f)$ and since $\mathrm{d}\omega \in \mathcal{P}_{r-1}\Lambda^{d-k}(f)$,

$$\begin{aligned}\|\mathrm{d}\omega\|_f &\overset{(75)}{\simeq} h_f^{\frac{d}{2}-(d-k)} \|\psi_f^\star(\mathrm{d}\omega)\|_{B_d} = h_f^{k-\frac{d}{2}} \|\mathrm{d}(\psi_f^\star\omega)\|_{B_d} \\ &\overset{(78)}{\lesssim} h_f^{k-\frac{d}{2}} \|\psi_f^\star\omega\|_{B_d} \overset{(75)}{\simeq} h_f^{k-\frac{d}{2}} h_f^{d-k-1-\frac{d}{2}} \|\omega\|_f,\end{aligned} \tag{79}$$

which concludes the proof of (76) since $k - \frac{d}{2} + d - k - 1 - \frac{d}{2} = -1$.

The same approach can be applied to the Koszul operator, once we notice that it has a similar commutation property with the considered pullbacks. Specifically, recalling that $\kappa = i_{x-x_f}$ is the Koszul operator on f, and defining the Koszul operator at 0 by $\kappa_0 := i_x$, the relation

$$(\psi_f^\star \kappa)_x = \left(i_{h_f^{-1}(x_f+h_f x-x_f)} \psi_f^\star\right)_x = \left(\kappa_0 \psi_f^\star\right)_x \tag{80}$$

comes from the generic formula $\phi^\star (i_Y \mu) = i_{\phi_\star Y} \phi^\star \mu$, where $(\phi_\star Y)(x) := (D\phi(x))^{-1} Y(\phi(x))$. Equipped with this commutation and using the continuity of κ_0 on $\mathcal{P}_r\Lambda^{d-k+1}(\mathbb{R}^d)$, namely

$$\|\kappa_0 \mu\|_{B_d} \lesssim \|\mu\|_{B_d} \qquad \forall \mu \in \mathcal{P}_r\Lambda^{d-k+1}(\mathbb{R}^d), \tag{81}$$

we can reproduce the same arguments as in (79) to prove (77), the change of scaling from h_f^{-1} to h_f coming from the fact that, in (79), the form degree of $\mathrm{d}\omega$ is one more than that of ω, while it is one less for $\kappa\omega$. $\square$

Lemma 16 (Local discrete Poincaré inequality)**.** *For any $f \in \Delta_d(\mathcal{M}_h)$, the inverse of the differential $\mathrm{d}^{-1} : \mathrm{d}\kappa\mathcal{P}_r\Lambda^k(f) \to \kappa\mathcal{P}_r\Lambda^k(f)$ and the inverse of the Koszul operator $\kappa^{-1} : \kappa\mathrm{d}\mathcal{P}_r\Lambda^k(f) \to \mathrm{d}\mathcal{P}_r\Lambda^k(f)$ satisfy*

$$|||\mathrm{d}^{-1}|||_f \lesssim h_f, \tag{82}$$

$$|||\kappa^{-1}|||_f \lesssim h_f^{-1}. \tag{83}$$

Proof. The proof uses the same arguments as the proof of Lemma 15, using this time the continuity of $\mathrm{d}^{-1} : \mathrm{d}\kappa_0\mathcal{P}_r\Lambda^k(B_d) \to \kappa_0\mathcal{P}_r\Lambda^k(B_d)$, and of $\kappa_0^{-1} : \kappa_0\mathrm{d}\mathcal{P}_r\Lambda^k(B_d) \to \mathrm{d}\mathcal{P}_r\Lambda^k(B_f)$ (with norms only depending on r, k and d). Let $\mu \in \mathrm{d}\kappa\mathcal{P}_r\Lambda^k(f)$ and set $\omega \in \mathcal{P}_r\Lambda^k(f)$ such that $\mu = \mathrm{d}\kappa\omega$. Then $\mathrm{d}^{-1}\mu = \kappa\omega$ and

$$\begin{aligned} h_f^{k-1-\frac{d}{2}}\|\mathrm{d}^{-1}\mu\|_f &\overset{(75)}{\simeq} \|\psi_f^\star\kappa\omega\|_{B_d} \overset{(80)}{=} \|\mathrm{d}^{-1}\mathrm{d}\kappa_0\psi_f^\star\omega\|_{B_d} \\ &\lesssim \|\mathrm{d}\kappa_0\psi_f^\star\omega\|_{B_d} \overset{(78),(81)}{\lesssim} \|\psi_f^\star\omega\|_{B_d} \overset{(75)}{\simeq} h_f^{k-\frac{d}{2}}\|\omega\|_f. \end{aligned}$$

Simplifying by $h_f^{k-1-\frac{d}{2}}$ concludes the proof of (82). The proof of (83) follows from similar arguments. □

Lemma 17 (Topological decomposition)**.** *The decomposition*

$$\mathcal{P}_r\Lambda^k(f) = \mathrm{d}\mathcal{P}_{r+1}\Lambda^{k-1}(f) \oplus \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(f)$$

is topological: For all $(\mu_f, \nu_f) \in \mathrm{d}\mathcal{P}_{r+1}\Lambda^{k-1}(f) \times \kappa\mathcal{P}_{r-1}\Lambda^{k+1}(f)$,

$$\|\mu_f\|_f + \|\nu_f\|_f \simeq \|\mu_f + \nu_f\|_f.$$

Proof. The inequality $\gtrsim$ directly follows from a triangle inequality, so we focus on $\lesssim$. We have

$$\|\mu_f\|_f = \|\kappa^{-1}\kappa\mu_f\|_f \overset{(83)}{\lesssim} h_f^{-1}\|\kappa\mu_f\|_f = h_f^{-1}\|\kappa(\mu_f + \nu_f)\|_f \overset{(77)}{\lesssim} \|\mu_f + \nu_f\|_f,$$

where we have used the fact that $\kappa\nu_f \in \kappa^2\mathcal{P}_{r-1}\Lambda^{k+1}(f) = \{0\}$ in the third passage. We also have

$$\|\nu_f\|_f = \|\mathrm{d}^{-1}\mathrm{d}\nu_f\|_f \overset{(82)}{\lesssim} h_f\|\mathrm{d}\nu_f\|_f = h_f\|\mathrm{d}(\nu_f + \mu_f)\|_f \overset{(76)}{\lesssim} \|\nu_f + \mu_f\|_f,$$

where we have used $\mathrm{d}\mu_f \in \mathrm{d}^2\mathcal{P}_{r+1}\Lambda^{k-1}(f) = \{0\}$ in the third passage. Combining these two relations yields the required estimate. □

Lemma 18 (Discrete trace inequality)**.** *For all $f \in \Delta_d(\mathcal{M}_h)$ and all $\omega_f \in \mathcal{P}_r\Lambda^k(f)$ it holds*

$$\|\omega_f\|_{\partial f} \lesssim h_f^{-\frac{1}{2}}\|\omega_f\|_f. \tag{84}$$

Proof. We introduce the notations

$$\mathfrak{A}_l^d := \left\{\alpha \in \mathbb{N}^d \,:\, \sum_{i=1}^d \alpha_i = l\right\}, \quad \mathfrak{B}_k^d := \left\{\beta \in \{0,1\}^d \,:\, \sum_{i=1}^d \beta_i = k\right\},$$

for the sets of multi-indices corresponding to the set of monomials of degree l, and to a basis of k-forms. Specifically, letting $y^\alpha := y_1^{\alpha_1}\cdots y_d^{\alpha_d}$ and, if $i_1 < \ldots < i_k$ are the indices such that $\beta_i = 1$ if and only if $i \in \{i_1, \ldots, i_k\}$, $\mathrm{d}x^\beta := \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, the set of monomials $\left\{(x - x_f)^\alpha \mathrm{d}x^\beta\right\}_{l\in\{0,\ldots,r\},\alpha\in\mathfrak{A}_l^d,\beta\in\mathfrak{B}_k^d}$

forms a basis of $\mathcal{P}_r\Lambda^k(\mathbb{R}^d)$. Any $\omega_f \in \mathcal{P}_r\Lambda^k(f)$ can thus be evaluated at a point x according to the following formula:

$$(\omega_f)_x = \sum_{l=0}^{r} \sum_{\alpha\in\mathfrak{A}_l^d,\beta\in\mathfrak{B}_k^d} \lambda_{l,\alpha,\beta}(x-x_f)^\alpha \mathrm{d}x^\beta$$

and, using the equivalence of norms in finite dimension and the fact that $\left\{x^\alpha \mathrm{d}x^\beta\right\}_{l\in\{0,\dots,r\},\alpha\in\mathfrak{A}_l^d,\beta\in\mathfrak{B}_k^d}$ is a basis of $\mathcal{P}_r\Lambda^k(\mathbb{R}^d)$, we can write

$$\begin{aligned}\|\omega_f\|_f^2 &\overset{(75)}{\simeq} h_f^{-2k+d}\|\psi_f^\star\omega_f\|_{B_d}^2 = h_f^{-2k+d}\left\|\sum_{l=0}^{r}\sum_{\alpha\in\mathfrak{A}_l^d,\beta\in\mathfrak{B}_k^d}\lambda_{l,\alpha,\beta}(h_f x)^\alpha h_f^k \mathrm{d}x^\beta\right\|_{B_d}^2 \\ &\simeq h_f^{-2k+d}\sum_{l=0}^{r}\sum_{\alpha\in\mathfrak{A}_l^d,\beta\in\mathfrak{B}_k^d} h_f^{2l+2k}\lambda_{l,\alpha,\beta}^2 = h_f^d\sum_{l=0}^{r}\sum_{\alpha\in\mathfrak{A}_l^d,\beta\in\mathfrak{B}_k^d} h_f^{2l}\lambda_{l,\alpha,\beta}^2.\end{aligned} \tag{85}$$

Let $f' \in \Delta_{d-1}(\mathcal{M}_h(f))$. We define $x_{ff'}$ as the orthogonal projection of x_f onto the tangent plane to f', so that $x_f = x_{ff'} + cn_{f'}$ for some $c \in \mathbb{R}$ such that $|c| \le h_f$ and with $n_{f'}$ a unit normal vector to f' in the space spanned by f. Assuming, without loss of generality, that $n_{f'}$ is along the d-th basis vector, we notice that, for any multi-index $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$, $((x-x_f)^\alpha)_{|f'} = (x_{|f'} - x_{ff'})^{(\alpha_1,\dots,\alpha_{d-1})}c^{\alpha_d}$. Therefore, denoting by

$$(\omega_f)_{|f'} = \sum_{l=0}^{r}\sum_{\alpha'\in\mathfrak{A}_l^{d-1},\beta'\in\mathfrak{B}_k^{d-1}} \lambda'_{l,\alpha',\beta'}(x_{|f'}-x_{ff'})^{\alpha'}\mathrm{d}x^{\beta'}$$

the trace of ω_f on f', we have

$$\lambda'_{l,\alpha',\beta'} = \sum_{l'=l}^{r} c^{l'-l}\lambda_{l',(\alpha',l'-l),\beta'}.$$

Proceeding as in (85), we have

$$\begin{aligned}\|(\omega_f)_{|f'}\|_{f'}^2 &\simeq h_{f'}^{d-1}\sum_{l=0}^{r}\sum_{\alpha'\in\mathfrak{A}_l^{d-1},\beta'\in\mathfrak{B}_k^{d-1}} h_{f'}^{2l}\lambda_{l,\alpha',\beta'}'^2 \\ &\lesssim h_{f'}^{d-1}\sum_{l=0}^{r}\sum_{\alpha'\in\mathfrak{A}_l^{d-1},\beta'\in\mathfrak{B}_k^{d-1}}\sum_{l'=l}^{r} h_{f'}^{2l}c^{2(l'-l)}\lambda_{l',(\alpha',l'-l),\beta'}^2 \\ &\overset{|c|\le h_f}{\simeq} h_f^{d-1}\sum_{l=0}^{r}\sum_{\alpha'\in\mathfrak{A}_l^{d-1},\beta'\in\mathfrak{B}_k^{d-1}}\sum_{l'=l}^{r} h_f^{2l'}\lambda_{l',(\alpha',l'-l),\beta'}^2 \overset{(85)}{\lesssim} h_f^{-1}\|\omega_f\|_f^2,\end{aligned}$$

where, in the conclusion, we have used the mesh regularity assumption to write $h_{f'} \simeq h_f$. □

B Whitney forms

The proof of the Poincaré inequality in the lowest order case (Lemma 6) relies on the use of a conforming basis of polynomial forms on a simplicial mesh, given by the Whitney forms. We recall here their definition and key properties.

Any simplex T of dimension n is the convex hull of some vertices $v_0,\dots,v_n$ not contained in a hyperplane. We denote by $\lambda_0,\dots,\lambda_n$ the barycentric coordinates associated to these vertices. The

exterior derivatives $\mathrm{d}\lambda_0, \dots \mathrm{d}\lambda_n$ are constant 1-forms spanning the space of 1-forms on $\mathbb{R}^n$. They satisfy the relations

$$\sum_{i=0}^{n} \lambda_i = 1 \quad \text{and} \quad \sum_{i=0}^{n} \mathrm{d}\lambda_i = 0. \tag{86}$$

For any $k \in \{0, \dots, n\}$, we denote by $\Sigma(k)$ the set of strictly increasing maps $\{0, \dots, k\} \mapsto \{0, \dots, n\}$. For $\sigma \in \Sigma(k)$, we denote the range of σ by $[\sigma] \coloneqq \{\sigma(i)\}_{0 \le i \le k}$, and, for $p \notin [\sigma]$, we set $\epsilon(p, \sigma) \coloneqq (-1)^{\mathrm{card}(\{q \in [\sigma] \,:\, p > q\})}$. For $p \in [\sigma]$, we denote by $\sigma - p$ the unique element of $\Sigma(k-1)$ whose range is $[\sigma] \setminus p$, and conversely, for $p \notin [\sigma]$, $\sigma + p$ is the unique element of $\Sigma(k+1)$ with range $[\sigma] \cup \{p\}$. There is a one-to-one correspondence between $\Sigma(k)$ and the k-skeleton $\Delta_k(\mathcal{M}_h(T))$ of T, associating each $\sigma \in \Sigma(k)$ with the convex hull of $\{v_l\}_{l \in [\sigma]}$; so, depending on the context, the notation σ may also be used to indicate the latter. We also define the basic k-alternator $\mathrm{d}\lambda_\sigma \coloneqq \mathrm{d}\lambda_{\sigma(0)} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(k)}$.

For all $k \in \{0, \dots, n\}$, and all $\sigma \in \Sigma(k)$, we define the Whitney form $\phi^k_{T,\sigma} \in \mathcal{P}_1^- \Lambda^k(T)$ associated to σ by

$$\phi^k_{T,\sigma} \coloneqq \sum_{p \in [\sigma]} \epsilon(p, \sigma - p) \lambda_p \mathrm{d}\lambda_{\sigma - p}. \tag{87}$$

We recall the following result from [23, Eq. (III.8)]: For all $\sigma, \rho \in \Sigma(k)$,

$$\int_\sigma \phi^k_{T,\rho} = \begin{cases} \frac{1}{n!} & \text{if } \sigma = \rho, \\ 0 & \text{if } \sigma \neq \rho. \end{cases} \tag{88}$$

Lemma 19. *For all $k \in \{0, \dots, n\}$ and all $\sigma \in \Sigma(k)$, we have*

$$\|\phi^k_{T,\sigma}\|^2_T \simeq h_T^{n-2k}. \tag{89}$$

Moreover, for any family $\{\mu_\sigma\}_{\sigma \in \Sigma(k)}$ of real numbers,

$$\left\| \sum_{\sigma \in \Sigma(k)} \mu_\sigma \phi^k_{T,\sigma} \right\|^2_T \simeq \sum_{\sigma \in \Sigma(k)} \mu_\sigma^2 \|\phi^k_{T,\sigma}\|^2_T. \tag{90}$$

Proof. Let $U \in \mathbb{R}^{n \times n}$ be the matrix whose i-th column is $v_i - v_0$. The barycentric coordinates associated with T of a point x are given by the relation $x = U\lambda + v_0$. Let $\psi : \mathbb{R}^n \ni x \mapsto Ux + v_0 \in \mathbb{R}^n$, so that $\psi^{-1} : \mathbb{R}^n \ni x \mapsto U^{-1}(x - v_0) \in \mathbb{R}^n$. We notice that, for $p \neq 0$, $\lambda_p = \left(\psi^{-1}\right)^\star x_p$ and $\mathrm{d}\lambda_p = \left(\psi^{-1}\right)^\star \mathrm{d}x^p$. For $p = 0$, these relations still hold defining x_0 to satisfy the linear dependency (86). Taking the pullback of (87) by ψ and using the second relation, above, we infer

$$\psi^\star \phi^k_{T,\sigma} = \sum_{p \in [\sigma]} \epsilon(p, \sigma - p) x_p \mathrm{d}x^{\sigma - p},$$

which is the definition of the Whitney forms on the reference n-simplex S_0. The relation (88) shows that the family $\{\phi^k_{T,\sigma}\}_{\sigma \in \Sigma(k)}$ is linearly independent, since it shows that the integrals over the sub-simplices of T form a dual basis to the Whitney forms. Therefore, we infer from the equivalence of norms in finite dimension that, for all $\mu_\sigma \in \mathbb{R}$ and with a hidden constant depending only on S_0 (which, in turn, depends only on n) and k

$$\left\| \psi^\star \sum_{\sigma \in \Sigma(k)} \mu_\sigma \phi^k_{T,\sigma} \right\|^2_{S_0} \simeq \sum_{\sigma \in \Sigma(k)} \mu_\sigma^2 \|\psi^\star \phi^k_{T,\sigma}\|^2_{S_0} \quad \text{and} \quad \|\psi^\star \phi^k_{T,\sigma}\|^2_{S_0} \simeq 1. \tag{91}$$

We now want to compare, for $\mu \in \operatorname{span}\{\phi^k_{T,\sigma}\}_{\sigma\in\Sigma(k)}$, the norms of $\psi^\star\mu$ and μ. To this end, we need an estimate of the pointwise norm $\|(\psi^\star\mu)_x\|_{\mathrm{Alt}^k}$, which is equivalent to the operator norm of $(\psi^\star\mu)_x$ viewed as a multilinear form since Alt^k is a finite-dimensional space. In the following, we denote by triple bars these operator norms. For any vectors $Y_1,\dots,Y_k$, we have

$$\begin{aligned}|(\psi^\star\mu)_x(Y_1,\dots,Y_k)| = |(\phi^k_{T,\sigma})_{\psi(x)}(UY_1,\dots,UY_k)| &\le |||\mu_{\psi(x)}|||\,\|UY_1\|\dots\|UY_k\| \\ &\le |||\mu_{\psi(x)}|||\,|||U|||^k\|Y_1\|\dots\|Y_k\|.\end{aligned}$$

Applying the same argument to $U^{-1}Y_1,\dots,U^{-1}Y_k$, we infer

$$|||U^{-1}|||^{-k}|||\mu_{\psi(x)}||| \le |||(\psi^\star\mu)_x||| \le |||U|||^k|||\mu_{\psi(x)}|||.$$

The shape regularity assumption on the simplex T ensures that

$$|||U^{-1}|||^{-1} \simeq |||U||| \simeq h_T.$$

Therefore, $\|(\psi^\star\mu)_x\|^2_{\mathrm{Alt}^k} \simeq h_T^{2k}\|\mu_{\psi(x)}\|^2_{\mathrm{Alt}^k}$, and a change of variable yields

$$\begin{aligned}\|\psi^\star\mu\|^2_{S_0} &= \int_{S_0}\|(\psi^\star\mu)_x\|^2_{\mathrm{Alt}^k}\mathrm{d}x \\ &\simeq \int_{S_0} h_T^{2k}\|\mu_{\psi(x)}\|^2_{\mathrm{Alt}^l}h_T^{-n}\,|D\psi|\,\mathrm{d}x \\ &= h_T^{2k-n}\int_T\|\mu_x\|^2_{\mathrm{Alt}^k}\mathrm{d}x = h_T^{2k-n}\|\mu\|^2_T.\end{aligned} \tag{92}$$

Both relations (89) and (90) follow by combining (91) with (92). □

C Construction of a basis of $Z^{\mathsf{c}}_k(\mathcal{S}_h(f))$

We assume in this section the complement $Z^{\mathsf{c}}_k(\mathcal{S}_h(f))$ in (25) given and we show how to construct a suitable basis.

Algorithm 1 Algorithm to construct a basis of of $Z^{\mathsf{c}}_k(\mathcal{S}_h(f))$ on a given d-cell $f \in \mathcal{M}_h$.

procedure ConstructBasis(k, f)
  $\mathcal{B}_k \coloneqq \emptyset$
  $\mathcal{F}_k \coloneqq \emptyset$
  $\mathcal{V}_k \coloneqq \Delta_k(\mathcal{S}_h(\partial f))$ ▷ set of visited k-simplices
  for $F \in \Delta_k(\mathcal{S}_h(f)) \setminus \Delta_k(\mathcal{S}_h(\partial f))$ **do** ▷ loop over k-simplices that are not contained in the boundary ∂f
    if $\exists z \in Z^{\mathsf{c}}_k(\mathcal{S}_h(f)) \setminus \{0\}$ such that $\operatorname{supp}(z) \subset \mathcal{V}_k \cup \{F\}$ **then**
      $\mathcal{F}_k \leftarrow \mathcal{F}_k \cup \{F\}$
      $\mathcal{B}_k \leftarrow \mathcal{B}_k \cup \{z\}$
    else
      $\mathcal{V}_k \leftarrow \mathcal{V}_k \cup \{F\}$
    end if
  end for
  return $(\mathcal{B}_k, \mathcal{F}_k)$
end procedure

Algorithm 1 eventually terminates since the set $\mathcal{S}_h(f)$ is finite. Accordingly, let $\mathcal{B}_k$, $\mathcal{F}_k$, $\mathcal{V}_k$ be the sets obtained at the end.

Let $\mathcal{X} \subset \mathcal{V}_k$, and let $Z_k^{\mathsf{c}}(\mathcal{X}) \coloneqq \{z \in Z_k^{\mathsf{c}}(\mathcal{S}_h(f)) : \mathrm{supp}(z) \subset \mathcal{X}\}$. Observe that the "if" statement at line 6 in Algorithm 1 checks exactly whether $Z_k^{\mathsf{c}}(\mathcal{V}_k \cup \{F\}) \neq \{0\}$. Accordingly, each k-simplex $F \in \Delta_k(\mathcal{S}_h(f)) \setminus \Delta_k(\mathcal{S}_h(\partial f))$ is added to $\mathcal{F}_k$ if $Z_k^{\mathsf{c}}(\mathcal{V}_k \cup \{F\}) \neq \{0\}$ and to $\mathcal{V}_k$ otherwise and, thanks to the initialization of $\mathcal{V}_k$ at line 4, we infer the partition

$$\Delta_k(\mathcal{S}_h(f)) = \mathcal{F}_k \sqcup \mathcal{V}_k. \tag{93}$$

We now prove the correctness of Algorithm 1 via a sequence of elementary lemmas.

Lemma 20 (Acyclicity property)**.** *It holds* $Z_k^{\mathsf{c}}(\mathcal{V}_k) = \{0\}$.

Proof. In the beginning, $\mathcal{V}_k \coloneqq \Delta_k(\mathcal{S}_h(\partial f))$, and it holds $Z_k^{\mathsf{c}}(\mathcal{V}_k) = \{0\}$, otherwise there would exist a non-zero simplicial chain $z \in Z_k^{\mathsf{c}}(\mathcal{V}_k) = Z_k^{\mathsf{c}}(\Delta_k(\mathcal{S}_h(\partial f)))$ supported in $\mathcal{S}_h(\partial f)$, contradicting (25). Next, during execution, a k-simplex F is added to $\mathcal{V}_k$ precisely when $Z_k^{\mathsf{c}}(\mathcal{V}_k \cup \{F\}) = \{0\}$, therefore at termination this condition still holds true. □

Let $\mathcal{F}_k = \{F_1, \dots, F_p\}$ be an enumeration of the set $\mathcal{F}_k$ based on the order in which k-simplices are added to it during the execution of Algorithm 1, and let $\mathcal{B}_k = \{z_1, \dots, z_p\}$ be a corresponding enumeration of the set $\mathcal{B}_k$ such that $z_i \in Z_k^{\mathsf{c}}(\mathcal{V}_k \cup \{F_i\})$ for each $i \in \{1, \dots, p\}$ (observe $Z_k^{\mathsf{c}}(\mathcal{V}_k \cup \{F_i\}) \neq \{0\}$ since $F_i \in \mathcal{F}_k$).

Lemma 21 (Orthogonality property)**.** *The following orthogonality property holds:*

$$\langle F_i \, , z_j \rangle = \delta_{ij} \qquad \forall i, j \in \{1, \dots, p\}, \tag{94}$$

where we have assumed, without loss of generality, a suitable orientation for each k-simplex F_i and normalization of z_j.

Proof. If $i = j$, (94) is a direct consequence of the definition of $\mathcal{F}_k$ and $\mathcal{B}_k$, of the chosen enumeration of their elements, and of the normalization of z_j. Let us consider the case $i \neq j$. Suppose, for the sake of contradiction, that $\langle F_i \, , z_j \rangle \neq 0$ for some indices $i, j \in \{1, \dots, p\}$ with $i \neq j$, so that $F_i \in \mathrm{supp}(z_j)$. Then, $z_j \in Z_k^{\mathsf{c}}(\mathcal{V}_k \cup \{F_j\})$, and, since $F_i \neq F_j$, we infer $F_i \in \mathcal{V}_k$. This gives the desired contradiction since it would imply $F_i \in \mathcal{F}_k \cap \mathcal{V}_k \overset{(93)}{=} \emptyset$. □

Lemma 22 (Basis property)**.** *The set $\mathcal{B}_k = \{z_1, \dots, z_p\}$ is a basis of $Z_k^{\mathsf{c}}(\mathcal{S}_h(f))$.*

Proof. To start with, observe that if $z \in Z_k^{\mathsf{c}}(\mathcal{S}_h(f))$ and $z \neq 0$, then $\mathrm{supp}(z) \not\subset \mathcal{S}_h(\partial f)$, for otherwise $z \in Z_k(\mathcal{S}_h(\partial f))$ by (25). It follows easily that $\dim Z_k^{\mathsf{c}}(\mathcal{S}_h(f)) > 0$ if and only if $\mathcal{B}_k \neq \emptyset$.

Assume $\mathcal{B}_k \neq \emptyset$. The set $\mathcal{B}_k$ consists of k-cycles of $Z_k^{\mathsf{c}}(\mathcal{S}_h(f))$, hence it generates a subspace of it. Let us show that $\mathcal{B}_k$ is indeed a basis. The family $\mathcal{F}_k = \{F_1, \dots, F_p\}$ is composed by pairwise distinct k-simplices and, by Lemma 21, this implies that the elements of $\mathcal{B}_k$ are linearly independent. To show that $\mathcal{B}_k$ generates $Z_k^{\mathsf{c}}(\mathcal{S}_h(f))$, let $z \in Z_k^{\mathsf{c}}(\mathcal{S}_h(f)) \setminus \{0\}$. By Lemma 20, $\mathrm{supp}(z) \notin \mathcal{V}_k$, and thus $\mathrm{supp}(z) \cap \mathcal{F}_k \neq \emptyset$ by (93). Rearranging the indices $i \in \{0, \dots, p\}$ if necessary, let $\{F_1, \dots, F_q\} = \mathrm{supp}(z) \cap \mathcal{F}_k$ with $q \in \{1, \dots, p\}$. Let

$$z^* \coloneqq z - \sum_{i=1}^{q} \langle F_i \, , z \rangle z_i$$

recalling that $z_i \in Z_k^{\mathsf{c}}(\mathcal{V}_k \cup \{F_i\})$ for each $i \in \{1, \dots, q\}$. By (94) and (93), it follows easily that z^* is a k-cycle such that $\mathrm{supp}(z^*) \subset \mathcal{V}_k$. By Lemma 20, $z^* = 0$, and we thus conclude $z = \sum_{i=1}^{q} \langle F_i \, , z \rangle z_i$. □

References

[1] H. Allen. *Algebraic Topology,* 2002.

[2] D. Arnold. *Finite Element Exterior Calculus*. SIAM, 2018. DOI: 10.1137/1.9781611975543.

[3] D. N. Arnold, R. S. Falk, and R. Winther. "Finite element exterior calculus, homological techniques, and applications". In: *Acta Numer.* 15 (2006), pp. 1–155. DOI: 10.1017/S0962492906210018.

[4] L. Beirão da Veiga, F. Brezzi, F. Dassi, L. D. Marini, and A. Russo. "A family of three-dimensional virtual elements with applications to magnetostatics". In: *SIAM J. Numer. Anal.* 56.5 (2018), pp. 2940–2962. DOI: 10.1137/18M1169886.

[5] L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo. "$H(\mathrm{div})$ and $H(\mathrm{curl})$-conforming VEM". In: *Numer. Math.* 133 (2016), pp. 303–332. DOI: 10.1007/s00211-015-0746-1.

[6] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. *The mimetic finite difference method for elliptic problems*. Vol. 11. MS&A. Modeling, Simulation and Applications. Springer, Cham, 2014, pp. xvi+392. DOI: 10.1007/978-3-319-02663-3.

[7] F. Bonaldi, D. A. Di Pietro, J. Droniou, and K. Hu. "An exterior calculus framework for polytopal methods". In: *J. Eur. Math. Soc.* (2025). Accepted for publication. arXiv: 2303.11093 [math.NA].

[8] J. Bonelle, D. A. Di Pietro, and A. Ern. "Low-order reconstruction operators on polyhedral meshes: Application to Compatible Discrete Operator schemes". In: *Computer Aided Geometric Design* 35–36 (2015), pp. 27–41. DOI: 10.1016/j.cagd.2015.03.015.

[9] J. Bonelle and A. Ern. "Analysis of compatible discrete operator schemes for elliptic problems on polyhedral meshes". In: *ESAIM: Math. Model. Numer. Anal.* 48 (2014), pp. 553–581. DOI: 10.1051/m2an/2013104.

[10] J. Bonelle. "Compatible Discrete Operator schemes on polyhedral meshes for elliptic and Stokes equations". PhD thesis. Université Paris-Est, 2014.

[11] S. H. Christiansen and M. W. Licht. "Poincaré–Friedrichs inequalities of complexes of discrete distributional differential forms". In: *BIT Numerical Mathematics* 60.2 (2020), pp. 345–371. DOI: 10.1007/s10543-019-00784.

[12] L. Codecasa, R. Specogna, and F. Trevisan. "A new set of basis functions for the discrete geometric approach". In: *J. Comput. Phys.* 19.299 (2010), pp. 7401–7410. DOI: 10.1016/j.jcp.2010.06.023.

[13] L. Codecasa, R. Specogna, and F. Trevisan. "Base functions and discrete constitutive relations for staggered polyhedral grids". In: *Comput. Meth. Appl. Mech. Engrg.* 198.9-12 (2009), pp. 1117–1123.

[14] D. A. Di Pietro. "An arbitrary-order discrete rot-rot complex on polygonal meshes with application to a quad-rot problem". In: *IMA J. Numer. Anal.* 44.3 (2024), pp. 1699–1730. DOI: 10.1093/imanum/drad045.

[15] D. A. Di Pietro and J. Droniou. "An arbitrary-order discrete de Rham complex on polyhedral meshes: Exactness, Poincaré inequalities, and consistency". In: *Found. Comput. Math.* 23 (2023), pp. 85–164. DOI: 10.1007/s10208-021-09542-8.

[16] D. A. Di Pietro and J. Droniou. "An arbitrary-order method for magnetostatics on polyhedral meshes based on a discrete de Rham sequence". In: *J. Comput. Phys.* 429.109991 (2021). DOI: 10.1016/j.jcp.2020.109991.

[17] D. A. Di Pietro and J. Droniou. *The Hybrid High-Order method for polytopal meshes. Design, analysis, and applications*. Vol. 19. Modeling, Simulation and Application. Springer International Publishing, 2020. DOI: 10.1007/978-3-030-37203-3.

[18] D. A. Di Pietro, J. Droniou, and J. J. Qian. "A pressure-robust Discrete de Rham scheme for the Navier–Stokes equations". In: *Comput. Meth. Appl. Mech. Engrg.* 421.116765 (2024). DOI: 10.1016/j.cma.2024.116765.

[19] D. A. Di Pietro, J. Droniou, and F. Rapetti. "Fully discrete polynomial de Rham sequences of arbitrary degree on polygons and polyhedra". In: *Math. Models Methods Appl. Sci.* 30.9 (2020), pp. 1809–1855. DOI: 10.1142/S0218202520500372.

[20] D. A. Di Pietro and M.-L. Hanot. "Uniform Poincaré inequalities for the Discrete de Rham complex on general domains". In: *Results Appl. Math.* 23.100496 (2024). DOI: 10.1016/j.rinam.2024.100496.

[21] J. Dodziuk. "Finite-difference approach to the Hodge theory of harmonic forms". In: *American Journal of Mathematics* 98.1 (1976), pp. 79–104.

[22] J. Droniou, M.-L. Hanot, and T. Oliynyk. *A polytopal discrete de Rham complex on manifolds, with application to the Maxwell equations*. 2024. arXiv: 2401.16130 [math.NA].

[23] M. W. Licht. "On the A Priori and A Posteriori Error Analysis in Finite Element Exterior Calculus". PhD thesis. University of Oslo, 2017.

[24] S. Pitassi, R. Ghiloni, and R. Specogna. "Inverting the discrete curl operator: a novel graph algorithm to find a vector potential of a given vector field". In: *J. Comput. Phys.* 466.111404 (2022). DOI: 10.1016/j.jcp.2022.111404.

[25] S. Pitassi, R. Ghiloni, F. Trevisan, and R. Specogna. "The role of the dual grid in low-order compatible numerical schemes on general meshes". In: *J. Comput. Phys.* 436.110285 (2021). DOI: 10.1016/j.jcp.2021.110285.

[26] S. Roman, S. Axler, and F. Gehring. *Advanced linear algebra*. Vol. 3. Springer, 2005.