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ABSTRACT. We consider the logarithmic Schrödinger equation in a semiclassical scaling, in the presence of a smooth, at most quadratic, external potential. For initial data under the form of a single coherent state, we identify the notion of criticality as far as the nonlinear coupling constant is concerned, in the semiclassical limit. In the critical case, we prove a general error estimate, and improve it in the case of initial Gaussian profiles. In this critical case, when the initial datum is the sum of two Gaussian coherent states with different centers in phase space, we prove a nonlinear superposition principle.

1. INTRODUCTION

We consider the logarithmic Schrödinger equation in the semiclassical régime $\varepsilon \in (0,1]$,

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda\varepsilon^\alpha\psi^\varepsilon \log|\psi^\varepsilon|^2 \quad ; \quad \psi^\varepsilon_{|t=0} = \psi^\varepsilon_0, \tag{1.1}$$

where $\psi^\varepsilon = \psi^\varepsilon(t,x)$ is complex-valued, $x \in \mathbb{R}^d$ with $d \geqslant 1$, $\lambda \in \mathbb{R}\backslash\{0\}$, and $\alpha \geqslant 1$ (see the discussion below). Throughout this paper, the external potential $V = V(x)$ satisfies the following assumption:

Assumption 1.1. *The external potential V is smooth, real-valued, and at most quadratic:*

$$V \in C^\infty(\mathbb{R}^d;\mathbb{R}) \quad \textit{and} \quad \partial^\beta V \in L^\infty\left(\mathbb{R}^d\right), \quad \forall|\beta| \geqslant 2.$$

To lighten notations, we consider $t \geqslant 0$ only, which is no restriction as the equation is time reversible.

The logarithmic Schrödinger equation was introduced in [5], and has been used since in various physical fields, such as quantum mechanics [29], quantum optics [5, 21], nuclear physics [23], Bohmian mechanics [26], effective quantum gravity [30], theory of superfluidity and Bose-Einstein condensation (BEC) [4]. As proposed in [30, 31], the logarithmic nonlinearity may extend quantum mechanics thanks to a nonlinear model, likely to help understand quantum gravity. In [6], the presence of an harmonic trap was considered, in order to describe logarithmic BEC.

We mention two mathematical properties associated with the logarithmic nonlinearity in Schrödinger equations. First, and this is the main reason why this

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model was introduced in [5], it is the only nonlinearity that provides a tensorization property in the multidimensional setting. Suppose $d \geqslant 2$ and that V separates coordinates, in the sense that

$$V(x) = \sum_{j=1}^{d} V_j(x_j).$$

If the initial datum ψ_0^ε is a tensor product,

$$\psi_0^\varepsilon(x) = \prod_{j=1}^{d} \varphi_j^\varepsilon(x_j),$$

then the solution to (1.1) is given by

$$\psi^\varepsilon(t,x) = \prod_{j=1}^{d} \psi_j^\varepsilon(t,x_j),$$

where each ψ_j^ε solves a one-dimensional equation,

$$i\partial_t \psi_j^\varepsilon + \frac{1}{2}\partial_{x_j}^2 \psi_j^\varepsilon = V_j(x_j)\psi_j^\varepsilon + \lambda\varepsilon^\alpha \psi_j^\varepsilon \log|\psi_j^\varepsilon|^2 \quad ; \quad \psi_{j|t=0}^\varepsilon = \varphi_j^\varepsilon.$$

The second property, unusual in a nonlinear setting as well, is that the size of the solution is somehow irrelevant, in the sense that if, for $k \in \mathbb{C}$, Ψ_k^ε denotes the solution to (1.1) with initial datum $k\psi_0^\varepsilon$ instead of ψ_0^ε, then we have

$$\Psi_k^\varepsilon(t,x) = k\psi^\varepsilon(t,x)e^{-i\lambda\varepsilon^{\alpha-1}t\log|k|^2}.$$

We will meet this property later, when considering the semiclassically critical case for (1.1).

The semiclassical limit $\varepsilon \to 0$ for (1.1) was addressed by Ferriere [17] for ψ_0^ε a WKB state (also known as Lagrangian state), $\psi_0^\varepsilon(x) = a_0(x)e^{i\phi_0(x)/\varepsilon}$, in the case $V = 0$ with $\alpha = 0$ (which corresponds to a supercritical case as far as WKB analysis is concerned). In the case where V is a bounded potential, $\alpha = 0$, $\lambda < 0$, and ψ_0^ε is a concentrating Gausson,

$$\psi_0^\varepsilon(x) = e^{ix\cdot p_0/\varepsilon} R\left(\frac{x-q_0}{\varepsilon}\right), \quad R(x) = e^{\frac{1+d}{2}} e^{\lambda|x|^2},$$

Ardila and Squassina [2] proved that, locally uniformly in time, the solution remains concentrated on the Gausson, and the center in phase space evolves according to classical mechanics:

$$\psi^\varepsilon(t,x) = e^{ix\cdot p(t)/\varepsilon + i\theta^\varepsilon(t)} R\left(\frac{x-q(t)}{\varepsilon}\right) + w^\varepsilon(t,x), \quad \varepsilon^{-d/2}\|w^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon),$$

where $\theta^\varepsilon(t) \in \mathbb{R}$ and

$$\dot q(t) = p(t), \quad \dot p(t) = -\nabla V\left(q(t)\right) \quad ; \quad q(0) = q_0, \quad p(0) = p_0.$$

The above Hamilton system also plays a crucial role in the propagation of coherent states considered in the present paper. Indeed, we assume that the initial data ψ_0^ε is a localized wave packet of the form

$$\psi_0^\varepsilon(x) = \varepsilon^{-d/4} u_0\left(\frac{x-q_0}{\sqrt{\varepsilon}}\right) e^{i(x-q_0)\cdot p_0/\varepsilon}, \quad q_0, p_0 \in \mathbb{R}^d, \tag{1.2}$$

and so the parameter α measures the strength of the nonlinear interaction as $\varepsilon \to 0$. Our goal is to describe the behavior of ψ^ε as $\varepsilon \to 0$, according to the value of α. Note that this initial datum is such that $\|\psi_0^\varepsilon\|_{L^2} = \|u_0\|_{L^2}$ is independent of $\varepsilon > 0$. In view of the above remark, replacing the factor $\varepsilon^{-d/4}$ with any other power of ε leads to an explicit change in the expression of $\psi^\varepsilon(t,x)$ by a gauge transform.

In the linear case $\lambda = 0$, the mathematical study of the semiclassical limit $\varepsilon \to 0$ in (1.1)–(1.2) goes back to [25], then developed in [20] and many other contributions; we refer to [27] for a comprehensive presentation of the propagation of coherent states in the linear case. The propagation of coherent for nonlinear Schrödinger equations begins in [10] (power nonlinearity) and [3, 7] (Hartree type nonlinearity). In this nonlinear setting, the size of the initial data or, equivalently, the size of the parameter in front of the nonlinearity, in terms of ε, is a crucial parameter in the leading order behavior of ψ^ε as $\varepsilon \to 0$. Below, we make this discussion precise in the case of the logarithmic nonlinearity, and derive the notion of criticality for the parameter α.

1.1. **Notion of criticality.** First, we proceed formally, and we seek an approximate solution of the equation(1.1)

$$\psi^\varepsilon_{\rm app}(t,x) = \frac{1}{\varepsilon^{d/4}} u\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right) e^{i\phi(t,x)/\varepsilon},$$

where

$$\phi(t,x) = \phi_{\rm lin}(t,x) := S(t) + p(t)\cdot(x-q(t)), \tag{1.3}$$

and the rescaled space variable is

$$y = \frac{x-q(t)}{\sqrt{\varepsilon}}.$$

Classically, we consider the Taylor expansion of $V(x)$ in terms of the new space variable y (see e.g. [9]):

$$V(x) = V\left(y + q(t)\sqrt{\varepsilon}\right) = V(q(t)) + \sqrt{\varepsilon} y\cdot\nabla V(q(t)) + \frac{\varepsilon}{2}\left\langle y, \nabla^2 V(q(t)) y\right\rangle + \mathcal{O}\left(\varepsilon^{3/2}\right).$$

Plugging this expansion into (1.1), we can measure how close $\psi^\varepsilon_{\rm app}$ is to solve (1.1), by computing

$$\begin{aligned} &i\varepsilon\partial_t\psi^\varepsilon_{\rm app} + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon_{\rm app} - V(x)\psi^\varepsilon_{\rm app} - \lambda\varepsilon^\alpha\psi^\varepsilon_{\rm app}\log|\psi^\varepsilon_{\rm app}|^2 \\ &= \frac{1}{\varepsilon^{d/4}}\left(b_0 + \sqrt{\varepsilon}b_1 + \varepsilon b_2 + b_{\rm nl} + \mathcal{O}\left(\varepsilon^{3/2}\right)\right) e^{i\phi/\varepsilon}, \end{aligned} \tag{1.4}$$

where ϕ is defined in (1.3), and

$$\begin{aligned} b_0 &= -u\left(\dot S(t) - p(t)\cdot\dot q(t) + \frac{|p(t)|^2}{2} + V(q(t))\right), \\ b_1 &= -i(\dot q(t) - p(t))\cdot\nabla u - y\cdot(\dot p(t) + \nabla V(q(t)))u, \\ b_2 &= i\partial_t u + \frac{1}{2}\Delta u - \frac{1}{2}\left\langle y, \nabla^2 V(q(t)) y\right\rangle u, \\ b_{\rm nl} &= -\lambda\varepsilon^\alpha u\log|u|^2 + \lambda\frac{d}{2}u\varepsilon^\alpha\log\varepsilon. \end{aligned}$$

As indicated by the notations, the nonlinear contribution is present only in the term $b_{\rm nl}$.

When $\alpha > 1$, $b_{\rm nl}$ is incorporated into the reminder term, and we write

$$i\varepsilon\partial_t\psi^\varepsilon_{\rm app} + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon_{\rm app} - V(x)\psi^\varepsilon_{\rm app} - \lambda\varepsilon^\alpha\psi^\varepsilon_{\rm app}\log|\psi^\varepsilon_{\rm app}|^2$$
$$= \frac{1}{\varepsilon^{d/4}}\left(b_0 + \sqrt{\varepsilon}b_1 + \varepsilon b_2 + \mathcal{O}\left(\varepsilon^{\min(3/2,\alpha)}\right)\right)e^{i\phi/\varepsilon}.$$

Due to the presence of the parameter ε in front of the factor $\partial_t\psi^\varepsilon_{\rm app}$, the contribution of the remainder is expected to be $\mathcal{O}\left(\varepsilon^{\min(1/2,\alpha-1)}\right)$ in L^2, and we expect ψ^ε to be well approximated by the linear solution

$$i\varepsilon\partial_t\varphi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\varphi^\varepsilon = V(x)\varphi^\varepsilon \quad ; \quad \varphi^\varepsilon_{|t=0} = \psi^\varepsilon_0. \tag{1.5}$$

We cancel out, successively, b_1, b_0 and b_2. Setting $b_1 = 0$, we impose

$$\dot q(t) = p(t), \quad \dot p(t) = -\nabla V(q(t)).$$

To agree with the initial coherent state (1.2), we impose $q(0) = q_0$ and $p(0) = p_0$, so that (q,p) is given by the classical Hamiltonian flow,

$$\dot q = p, \quad \dot p = -\nabla V(q) \quad ; \quad q(0) = q_0, \quad p(0) = p_0, \tag{1.6}$$

Setting $b_0 = 0$ and $S(0) = 0$, we obtain the classical action (the action of classical mechanics).

$$S(t) = \int_0^t \left(\frac{|p(s)|^2}{2} - V(q(s))\right)ds, \tag{1.7}$$

Finally, setting $b_2 = 0$ and changing the notation u to v yields

$$i\partial_t v + \frac{1}{2}\Delta v = \frac{1}{2}\left\langle y, \nabla^2 V(q(t))y\right\rangle v \quad ; \quad v_{|t=0} = u_0. \tag{1.8}$$

To make notations consistent, and for future reference, we thus introduce

$$\varphi^\varepsilon_{\rm app}(t,x) = \frac{1}{\varepsilon^{d/4}}v\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right)e^{i\phi_{\rm lin}(t,x)/\varepsilon}, \tag{1.9}$$

where v solves (1.8).

In the case $\alpha = 1$, the contribution of $b_{\rm nl}$ is comparable with that of b_2. The second term in $b_{\rm nl}$ may be viewed as a constant (in (t,x)) potential, and is removed by a gauge transform. This is a manifestation of the effect of the size in logarithmic nonlinearity described above. We therefore modify the phase ϕ and now write

$$\psi_{\rm app}(t,x) = \frac{1}{\varepsilon^{d/4}}u\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right)e^{i\phi_{\rm nl}(t,x)/\varepsilon}, \tag{1.10}$$

where

$$\phi_{\rm nl}(t,x) := \phi_{\rm lin}(t,x) - \lambda\frac{d}{2}t\varepsilon\log\varepsilon = S(t) + p(t)\cdot(x-q(t)) - \lambda\frac{d}{2}t\varepsilon\log\varepsilon.$$

Since we have not modified the expression of b_0 and b_1, q, p and S are still given by (1.6) and (1.7). On the other hand, the envelope equation becomes

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\left\langle y, \nabla^2 V(q(t))y\right\rangle u + \lambda u\log|u|^2 \quad ; \quad u_{|t=0} = u_0. \tag{1.11}$$

The above potential is quadratic in y. This implies, as we will see in the sequel, that, like in the linear case (1.8) (as noticed initially in [24], see also [20, 27]), the flow of this equation preserves Gaussian structures: if a is a Gaussian, then so is $u(t,\cdot)$ for all $t \in \mathbb{R}$. In a nonlinear context, this property is a feature of the

logarithmic nonlinearity. We note that this is also a feature of the coherent states approximation, that even if V is not quadratic (in space), the potential in (1.11) is. The potential in (1.11) depends on time precisely when V is not exactly quadratic, and the explicit computation for Gaussian data can be viewed as a feature of the coherent states approximation.

The case $\alpha > 1$ will therefore be referred to as *subcritical*, and the case $\alpha = 1$ as *critical*. We note that the case $\alpha < 1$ seems to be out of reach in this setting (as well as the supercritical case $\alpha < \alpha_c$ in [10] for power-like nonlinearity; the case of Hartree type nonlinearity is different, as pointed out in [7] and [9], and some supercritical cases can be described in the case of a smooth, bounded kernel).

We emphasize that for a power-like nonlinearity $\varepsilon^\alpha |\psi^\varepsilon|^{2\sigma}\psi^\varepsilon$, $\sigma > 0$, with initial data like in (1.2), the critical case is given by $\alpha_c = 1 + \frac{d\sigma}{2}$,

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \varepsilon^{1+d\sigma/2}|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \quad ; \quad \psi^\varepsilon_{|t=0} = \psi^\varepsilon_0,$$

so the critical case $\alpha = 1$ in (1.1) may be viewed as a natural candidate for the limit $\sigma \to 0$ (see also [28, 18] for other evidences in this direction). In the case of Hartree equation with a homogeneous kernel ($\gamma > 0$), the critical case is ([7])

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda\varepsilon^{1+\gamma/2}\left(\frac{1}{|x|^\gamma} * |\psi^\varepsilon|^2\right)\psi^\varepsilon \quad ; \quad \psi^\varepsilon_{|t=0} = \psi^\varepsilon_0.$$

On the other hand, for Hartree type equations with a smooth kernel K (bounded as well as its derivatives), the critical case is ([7]), like for (1.1),

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda\varepsilon\left(K * |\psi^\varepsilon|^2\right)\psi^\varepsilon \quad ; \quad \psi^\varepsilon_{|t=0} = \psi^\varepsilon_0.$$

1.2. **Main results.** Denote

$$\Sigma^k = H^k \cap \mathcal{F}\left(H^k\right) = \left\{f \in H^k\left(\mathbb{R}^d\right), \quad x \mapsto |x|^k f(x) \in L^2\left(\mathbb{R}^d\right)\right\},$$

endowed with natural norm. When $k = 1$, we simply denote Σ^1 by Σ.

Our first result shows that indeed, the case $\alpha > 1$ is subcritical, in the sense that ψ^ε is well approximated by the linear solution φ^ε, up to some Ehrenfest time (a time of order $\log\frac{1}{\varepsilon}$). The solution φ^ε is also well approximated by $\varphi^\varepsilon_{\rm app}$ up to some Ehrenfest time.

Proposition 1.2 (Subcritical case)**.** *Suppose that $\alpha > 1$. If $u_0 \in \Sigma$, then for any $0 < \delta < \alpha - 1$, there exists $C > 0$ independent of ε such that, with φ^ε solution to the linear equation* (1.5)*, we have*

$$\|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^{\alpha-1-\delta}e^{Ct}, \quad t \geqslant 0.$$

In particular, there exists $c_0 > 0$ independent of ε such that

$$\sup_{0\leqslant t\leqslant c_0\log\frac{1}{\varepsilon}} \|\psi^\varepsilon(t) - \varphi^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

If in addition $u_0 \in \Sigma^2$, there exists $C > 0$ independent of ε such that, with $\varphi^\varepsilon_{\rm app}$ defined in (1.9)*, v given by* (1.8) *and $\phi_{\rm lin}$ by* (1.3)*, we have*

$$\|\varphi^\varepsilon(t) - \varphi^\varepsilon_{\rm app}(t)\|_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^{1/4}e^{Ct}, \quad t \geqslant 0.$$

Finally, if $u_0 \in \Sigma^3$, there exists $C > 0$ independent of ε such that

$$\|\varphi^\varepsilon(t) - \varphi^\varepsilon_{\rm app}(t)\|_{L^2(\mathbb{R}^d)} \lesssim \sqrt{\varepsilon}e^{Ct}, \quad t \geqslant 0.$$

We now come to critical phenomena, and set $\alpha = 1$. The case of Gaussian initial profiles turns out to be more convenient, even though we consider a nonlinear setting.

Theorem 1.3 (Critical case)**.** *Suppose that $\alpha = 1$.*
(1) *If $u_0 \in \Sigma^2$, then with $\psi^\varepsilon_{\rm app}$ defined in* (1.10) *(u is given by* (1.11)*), there exists $C > 0$ independent of ε such that,*

$$\|\psi^\varepsilon(t) - \psi^\varepsilon_{\rm app}(t)\|_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^{1/4} e^{e^{Ct}}, \quad t \geqslant 0.$$

In particular, there exists c independent of ε such that

$$\sup_{0\leqslant t\leqslant c\log\log\frac{1}{\varepsilon}} \|\psi^\varepsilon(t) - \psi^\varepsilon_{\rm app}(t)\|_{L^2(\mathbb{R}^d)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

(2) *Suppose that V decouples variables,*

$$V(x) = \sum_{j=1}^d V_j(x_j).$$

If u_0 is a Gaussian of the form

$$u_0(y) = b_0 \exp\left(-\frac{1}{2}\sum_{j=1}^d a_{0j} y_j^2\right), \quad a_{0j}, b_0 \in \mathbb{C},\ \operatorname{Re} a_{0j} > 0,$$

then there exists $C > 0$ independent of ε such that,

$$\|\psi^\varepsilon(t) - \psi^\varepsilon_{\rm app}(t)\|_{L^2(\mathbb{R}^d)} \lesssim \sqrt{\varepsilon} e^{Ct}, \quad t \geqslant 0.$$

We remark that in the second case, Assumption 1.1 implies that each V_j satisfies (the one-dimensional version of) Assumption 1.1. If V does not decouple variables, the computations presented in Section 4.2 become more involved, as one has to deal with time dependent matrices instead of scalars, since even if u_0 decouples variables like above, this property is lost for $t > 0$ in general. We choose to present this simplified setting to keep notations as light as possible.

Theorem 1.4 (Nonlinear superposition)**.** *Suppose that $\alpha = 1$, that V decouples variables,*

$$V(x) = \sum_{j=1}^d V_j(x_j),$$

and that the initial datum in (1.1) *is the sum of two coherent states,*

$$\psi^\varepsilon_0(x) = \varepsilon^{-d/4} \sum_{j=1}^2 u_{0j}\left(\frac{x - q_{0j}}{\sqrt{\varepsilon}}\right) e^{i(x-q_{0j})\cdot p_0/\varepsilon}, \quad q_{0j}, p_{0j} \in \mathbb{R}^d, \tag{1.12}$$

with different centers in phase space, $(q_{01}, p_{01}) \neq (q_{02}, p_{02})$. If u_{01} and u_{02} are two Gaussian functions like in the second part of Theorem 1.3, then uniformly on bounded time intervals, a nonlinear superposition holds: if $\psi^\varepsilon_{j,\rm app}$ denotes the approximate solution associated with u_{0j}, then for any $T > 0$ and any $\gamma < 1/2$,

$$\sup_{t\in[0,T]} \left\|\psi^\varepsilon(t) - \psi^\varepsilon_{1,\rm app}(t) - \psi^\varepsilon_{2,\rm app}(t)\right\|_{L^2} \lesssim \varepsilon^\gamma.$$

If in addition $d = 1$, and $E_1 \neq E_2$, where

$$E_j = \frac{p_{0j}^2}{2} + V\left(q_{0j}\right),$$

then for all $\gamma < 1/2$, there exists $C > 0$ independent of $\varepsilon \in (0,1)$ such that

$$\left\|\psi^\varepsilon(t) - \psi^\varepsilon_{1,\mathrm{app}}(t) - \psi^\varepsilon_{2,\mathrm{app}}(t)\right\|_{L^2} \lesssim \varepsilon^\gamma e^{Ct}, \quad \forall t \geqslant 0.$$

In particular, there exists $c > 0$ independent of ε such that

$$\sup_{0\leqslant t\leqslant c\log\frac{1}{\varepsilon}} \left\|\psi^\varepsilon(t) - \psi^\varepsilon_{1,\mathrm{app}}(t) - \psi^\varepsilon_{2,\mathrm{app}}(t)\right\|_{L^2} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The above result is readily generalized to the case of finitely many initial Gaussian coherent states, with pairwise distinct centers in phase space. We restrict the presentation to the case of two coherent states, to make notations as light as possible.

1.3. **Content.** In Section 2, we recall several general estimates, address the Cauchy problems for (1.1) and (1.11), proving new estimates in the case of (1.11), in Section 2.3. Proposition 1.2 is proven in Section 3, Theorem 1.3 in Section 4, and Theorem 1.4 in Section 5.

1.4. **Notations.** Throughout the paper, C denotes a constant independent of t and $\varepsilon \in (0,1]$, which may change from one line to the other.

For $f, g \geqslant 0$, we write $f \lesssim g$ whenever there exists a constant C independent of $t \geqslant 0$, $x \in \mathbb{R}^d$, and $\varepsilon \in (0,1)$, such that $f \leqslant Cg$.

2. Preliminary results

2.1. **Some technical tools.** We recall that under Assumption 1.1, Cauchy-Lipschitz Theorem implies:

Lemma 2.1. *Let $(q_0, p_0) \in \mathbb{R}^d \times \mathbb{R}^d$. Under Assumption 1.1,* (1.6) *has a unique global, smooth solution $(q,p) \in C^\infty(\mathbb{R};\mathbb{R}^d)^2$. It grows at most exponentially:*

$$\exists C_0 > 0, \quad |q(t)| + |p(t)| \lesssim e^{C_0 t}, \quad \forall t \geqslant 0.$$

The following lemma, originating in [15], is crucial in the study of Schrödinger equations with a logarithmic nonlinearity, and will be used several times in the sequel:

Lemma 2.2 (From Lemma 1.1.1 in [15])**.** *There holds*

$$\left|\mathrm{Im}\left(\left(z_2\log|z_2|^2 - z_1\log|z_1|^2\right)(\overline{z_2} - \overline{z_1})\right)\right| \leqslant 2|z_2 - z_1|^2, \quad \forall z_1, z_2 \in \mathbb{C}.$$

We recall the Gagliardo-Nirenberg inequality and a standard inequality (see e.g. (2.3) and its proof in [12]),

$$\|v\|_{L^p} \lesssim \|v\|_{L^2}^{1-\theta(p)} \|\nabla v\|_{L^2}^{\theta(p)}, \quad \text{for} \quad 2 \leqslant p < \frac{2d}{(d-2)_+}, \tag{2.1}$$

$$\|v\|_{L^p} \lesssim \|v\|_{L^2}^{1-\theta(p')} \|yv\|_{L^2}^{\theta(p')}, \quad \text{for} \quad \max\left(1, \frac{2d}{d+2}\right) < p \leqslant 2, \tag{2.2}$$

where

$$\theta(p) = d\left(\frac{1}{2} - \frac{1}{p}\right), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

2.2. **Cauchy problem.** In the case without potential, $V = 0$, the Cauchy problem for (1.1) was addressed initially in [15] in the case $\lambda < 0$, then resumed in [19] in the case $\lambda \in \mathbb{R}$, with a generalization in [12], using compactness methods. The space in which the Cauchy problem is solved was further enlarged in [13, 22], using more constructive methods. In the case where $\lambda < 0$ and V is a harmonic potential, the Cauchy problem for (1.1) was solved in Σ in [1], by adapting the approach from [15]. In the framework considered in the present paper (V satisfying Assumption 1.1 and $\lambda \in \mathbb{R}$), the Cauchy problem was solved in Σ in [11]:

Proposition 2.3 (Proposition 1.3 in [11])**.** *Let $d \geqslant 1$, $\lambda, \alpha \in \mathbb{R}$, $\varepsilon > 0$, and V satisfying Assumption 1.1. For $\psi_0^\varepsilon \in \Sigma$, (1.1) has a unique solution $\psi^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}; \Sigma) \cap C(\mathbb{R}; L^2(\mathbb{R}^d))$. Moreover, the following quantities are independent of time:*

$$\begin{aligned}
&\textit{Mass: } M^\varepsilon(t) = \|\psi^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} \equiv M^\varepsilon(0),\\
&\textit{Energy: } E^\varepsilon(t) = \frac{1}{2}\|\varepsilon\nabla\psi^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} V(x)|\psi^\varepsilon(t,x)|^2dx\\
&\qquad\qquad + \lambda\varepsilon^\alpha \int_{\mathbb{R}^d} |\psi^\varepsilon(t,x)|^2 \log|\psi^\varepsilon(t,x)|^2 dx \equiv E^\varepsilon(0).
\end{aligned}$$

We recall the main steps of the proof, to show that the above result is readily extended to the case of a time dependent potential like in (1.11). Since the role of $\varepsilon > 0$ is absent at this level of discussion, we set $\varepsilon = 1$ in order to lighten notations. In view of the singularity of the logarithm at the origin, we consider the family of regularized equations, for $\delta > 0$,

$$i\partial_t\psi^\delta + \frac{1}{2}\Delta\psi^\delta = V(x)\psi^\delta + \lambda\psi^\delta \log\left(\delta + |\psi^\delta|^2\right) \quad ; \quad \psi^\delta_{|t=0} = \psi_0. \tag{2.3}$$

For fixed $\delta > 0$, the nonlinearity is smooth, with moderate growth as $|\psi^\delta| \to \infty$ (the nonlinearity is L^2-subcritical in any dimension), so under Assumption 1.1, global existence in Σ follows for instance from [8] (where a dependence of V upon time is allowed, like in (1.11)): $\psi^\delta, \nabla\psi^\delta, x\psi^\delta \in C(\mathbb{R}; L^2(\mathbb{R}^d))$.

The sequence $(\psi^\delta)_{0<\delta\leqslant 1}$ converges, thanks to compactness arguments based on uniform a priori estimates. First, the L^2-norm of ψ^δ is independent of time, $\|\psi^\delta(t)\|_{L^2} = \|\psi_0\|_{L^2}$. For $1 \leqslant j \leqslant d$, differentiating (2.3) with respect to x_j yields

$$\begin{aligned}
i\partial_t\partial_j\psi^\delta + \frac{1}{2}\Delta\partial_j\psi^\delta &= V(x)\partial_j\psi^\delta + \partial_j V(x)\psi^\delta + \lambda\log\left(\delta + |\psi^\delta|^2\right)\partial_j\psi^\delta\\
&\quad + 2\lambda\frac{1}{\delta + |\psi^\delta|^2}\,\mathrm{Re}(\overline{\psi^\delta}\partial_j\psi^\delta)\psi^\delta.
\end{aligned}$$

By Assumption 1.1, $|\partial_j V(x)| \lesssim 1 + |x|$, so the standard L^2 estimate yields

$$\frac{d}{dt}\|\nabla\psi^\delta(t)\|^2_{L^2} \leqslant C\left(\|\psi^\delta(t)\|^2_{L^2} + \|x\psi^\delta(t)\|^2_{L^2} + \|\nabla\psi^\delta(t)\|^2_{L^2}\right),$$

where C is independent of δ. Similarly,

$$i\partial_t\left(x_j\psi^\delta\right) + \frac{1}{2}\Delta(x_j\psi^\delta) = \partial_j\psi^\delta + V(x)x_j\psi^\delta + \lambda\ln\left(\delta + |\psi^\delta|^2\right)x_j\psi^\delta,$$

hence

$$\frac{d}{dt}\|x\psi^\delta(t)\|^2_{L^2} \leqslant 2\int_{\mathbb{R}^d} |x\psi^\delta(t,x)||\nabla\psi^\delta(t,x)|dx \leqslant \|x\psi^\delta(t)\|^2_{L^2} + \|\nabla\psi^\delta(t)\|^2_{L^2}.$$

In view of the conservation of the mass, Grönwall lemma implies that there exists C independent of δ such that

$$\|x\psi^\delta(t)\|_{L^2}^2+\|\nabla\psi^\delta(t)\|_{L^2}^2\leqslant C\left(\|u_0\|_{L^2}^2+\|xu_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2\right)e^{Ct},\quad t\geqslant 0.$$

Therefore, we have compactness in space for the sequence $(\psi^\delta)_{0<\delta\leqslant 1}$. Compactness in time follows from (2.3). Arzela-Ascoli theorem yields a converging subsequence, hence the existence part of Proposition 2.3.

Uniqueness follows from the remark that any solution $\psi\in L^\infty_{\rm loc}(\mathbb{R};\Sigma)$ to (1.1) actually belongs to $C(\mathbb{R};L^2(\mathbb{R}^d))$. The standard L^2-estimate for the difference of two solutions ψ and ϕ (with a common initial datum ψ_0) yields

$$\frac{d}{dt}\|\psi-\phi\|_{L^2}^2\leqslant 2\left|\mathrm{Im}\int\left((\bar\psi-\bar\phi)\left(\psi\log|\psi|^2-\phi\log|\phi|^2\right)\right)\right|.$$

Lemma 2.2 and Grönwall lemma then imply $\psi\equiv\phi$. The conservations of mass and energy are justified by the same arguments as in [14], for instance.

Repeating the above arguments, we have a similar result in the case of (1.11). The fact that the potential may depend on time implies that the energy is no longer conserved, but, as we do not use the notion of energy here, we leave out the adapted statement regarding this aspect:

Lemma 2.4. *Let $d\geqslant 1$, V satisfying Assumption 1.1, and $\lambda\in\mathbb{R}$. For any $u_0\in\Sigma$, (1.11) has a unique solution $u\in L^\infty_{\rm loc}(\mathbb{R};\Sigma)\cap C(\mathbb{R};L^2(\mathbb{R}^d))$. In addition, $\|u(t)\|_{L^2}=\|u_0\|_{L^2}$ for all $t\in\mathbb{R}$, and there exists $C>0$ such that*

$$\|\nabla u(t)\|_{L^2}+\|yu(t)\|_{L^2}\lesssim e^{Ct},\quad\forall t\geqslant 0.$$

We note that in this statement, we say nothing about $q(t)$: this is due to the fact that Assumption 1.1 implies $\nabla^2V\in L^\infty(\mathbb{R}^d))$, which is the only relevant piece of information in the argument of the proof.

2.3. **Further estimates in the critical case.** To prove Theorem 1.3, we will need higher localization property than in Lemma 2.4, namely $|y|^2u\in L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$. Due to the presence of the quadratic potential in (1.11), the regularity is expected to be the same in space and frequency, so it is natural to work in Σ^2.

Unlike in the case without potential ([12]), it is not clear that the flow associated to (1.11) preserves the H^2 regularity. Indeed, the argument of [12] consists in adapting the proof from [14] in the case of power-like nonlinearities, where one first proves that $\partial_t u\in L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$, and then uses the equation to infer that Δu enjoys the same regularity. Here, we face two new difficulties. First, even if we know that $\partial_t u\in L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$, the possible lack of ellipticity of the operator

$$-\frac{1}{2}\Delta+\frac{1}{2}\left\langle y,\nabla^2V\left(q(t)\right)y\right\rangle$$

makes it difficult, if not impossible, to infer that $\Delta u,|y|^2u\in L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$. Typically, one can think of $V(x)=-\omega^2|x|^2/2$, a case where $\nabla^2V\left(q(t)\right)=-\omega^2I_d$ is constant. In the case where $\nabla^2V\left(q(t)\right)$ is not constant, we also have to deal with the time derivative of this term, which appears in the equation satisfied by $\partial_t u$, leading to a term behaving like $\dot q(t)|y|^2u$, and so estimating $\partial_t u$ and $|y|^2u$ are two connected questions. It is actually the presence of this term, and the possible exponential growth of $\dot q$, which explains the double exponential in the following statement.

Proposition 2.5. *Let $u_0 \in \Sigma^2$. The solution to (1.11), $u \in L^\infty_{\rm loc}(\mathbb{R};\Sigma)$, enjoys the extra regularity $u \in L^\infty_{\rm loc}(\mathbb{R};\Sigma^2)$. In addition, there exists C independent of $t \geqslant 0$ such that*

$$\int_{\mathbb{R}^d} \langle y\rangle^4 |u(t,y)|^2 dy + \int_{\mathbb{R}^d} |\Delta u(t,y)|^2 dy \leqslant Ce^{e^{Ct}}, \quad \forall t \geqslant 0.$$

Proof. We recall that u is constructed as the limit of the family $(u^\delta)_{0<\delta\leqslant 1}$ as $\delta \to 0$, where u^δ solves

$$i\partial_t u^\delta + \frac{1}{2}\Delta u^\delta = \frac{1}{2}\left\langle y, \nabla^2 V\left(q(t)\right) y\right\rangle u^\delta + \lambda u^\delta \log\left(\delta + |u^\delta|^2\right), \quad u^\delta_{|t=0} = u_0. \tag{2.4}$$

We prove that there exists C independent of $t \geqslant 0$ and $\delta \in (0,1)$ such that

$$\int_{\mathbb{R}^d} \langle y\rangle^4 |u^\delta(t,y)|^2 dy + \int_{\mathbb{R}^d} |\Delta u^\delta(t,y)|^2 dy \leqslant Ce^{e^{Ct}}. \tag{2.5}$$

The equation satisfied by $|y|^2 u^\delta$ is

$$\begin{aligned} i\partial_t(|y|^2u^\delta) + \frac{1}{2}\Delta(|y|^2u^\delta) = \frac{1}{2}\left[\Delta, |y|^2\right] u^\delta + \frac{1}{2}\left\langle y, \nabla^2 V\left(q(t)\right) y\right\rangle |y|^2 u^\delta \\ + \lambda |y|^2 u^\delta \log\left(\delta + |u^\delta|^2\right). \end{aligned}$$

By direct computations,

$$\left[\Delta, |y|^2\right] = 4y\cdot\nabla + 2d,$$

so the standard energy estimate for Schrödinger equations yields

$$\frac{d}{dt}\||y|^2u^\delta\|_{L^2} \lesssim \|y\cdot\nabla u^\delta\|_{L^2} + \|u^\delta\|_{L^2}. \tag{2.6}$$

We therefore examine the equation satisfied by $y\cdot\nabla u^\delta$. First, ∇u^δ solves

$$\begin{aligned} i\partial_t \nabla u^\delta + \frac{1}{2}\Delta\nabla u^\delta = \left(\nabla^2 V\left(q(t)\right) y\right) u^\delta + \frac{1}{2}\left\langle y, \nabla^2 V\left(q(t)\right) y\right\rangle \nabla u^\delta \\ + \lambda\left(\nabla u^\delta\right)\log\left(\delta + |u^\delta|^2\right) + 2\lambda \frac{u^\delta \,\mathrm{Re}\left(\overline{u}^\delta \nabla u^\delta\right)}{\delta + |u^\delta|^2}. \end{aligned}$$

Taking the inner product with y yields

$$\begin{aligned} i\partial_t y\cdot\nabla u^\delta + \frac{1}{2}\Delta\left(y\cdot\nabla u^\delta\right) = \frac{1}{2}\left[\Delta, y\right]\cdot\nabla u^\delta + \left\langle y, \nabla^2 V\left(q(t)\right) y\right\rangle u^\delta \\ + \frac{1}{2}\left\langle y, \nabla^2 V\left(q(t)\right) y\right\rangle y\cdot\nabla u^\delta \\ + \lambda\left(y\cdot\nabla u^\delta\right)\log\left(\delta + |u^\delta|^2\right) + 2\lambda\frac{u^\delta\,\mathrm{Re}\left(\overline{u}^\delta y\cdot\nabla u^\delta\right)}{\delta + |u^\delta|^2}, \end{aligned}$$

and we note that

$$\frac{1}{2}\left[\Delta, y\right]\cdot\nabla u^\delta = \Delta u^\delta,$$

so by energy estimate, recalling that $\nabla^2 V \in L^\infty(\mathbb{R}^d)$,

$$\frac{d}{dt}\|y\cdot\nabla u^\delta\|_{L^2} \lesssim \|\Delta u^\delta\|_{L^2} + \||y|^2u^\delta\|_{L^2} + \|y\cdot\nabla u^\delta\|_{L^2}. \tag{2.7}$$

To bound the term Δu^δ, we note that differentiating the (regularized) logarithmic nonlinearity makes it impossible to get estimates which are uniform in $\delta \in (0,1)$, so we use directly the information given by (2.4),

$$\|\Delta u^\delta\|_{L^2} \lesssim \|\partial_t u^\delta\|_{L^2} + \||y|^2u^\delta\|_{L^2} + \left\|u^\delta\log\left(\delta + |u^\delta|^2\right)\right\|_{L^2}. \tag{2.8}$$

Note the inequality

$$\begin{aligned}\left|u^\delta \log\left(\delta+|u^\delta|^2\right)\right|^2 &\lesssim |u^\delta|^2\left(\left(\delta+|u^\delta|^2\right)^{\eta/2}+\left(\delta+|u^\delta|^2\right)^{-\eta/2}\right)\\ &\lesssim |u^\delta|^{2+\eta}+|u^\delta|^{2-\eta},\end{aligned}$$

for any $\eta>0$, where the implicit multiplicative constant depends on $\eta>0$ but not on $\delta\in(0,1)$. For $\eta>0$ sufficiently small, this implies, in view of (the proof of) Lemma 2.4, since $\Sigma\hookrightarrow L^{2+\eta}\cap L^{2-\eta}$ provided that $\eta>0$ is sufficiently small (see (2.1) and (2.2)),

$$\left\|u^\delta\log\left(\delta+|u^\delta|^2\right)\right\|_{L^2}\lesssim e^{Ct},\quad t\geqslant 0. \tag{2.9}$$

To close the system of estimates (2.6)-(2.7), we estimate $\partial_t u^\delta$ in L^2. For $u_0\in\Sigma^2$, $\partial_t u^\delta$ solves an equation rather similar to the one satisfied by ∇u^δ, but we must pay attention to the initial data, given by (2.4) at $t=0$:

$$\begin{aligned}i\partial_t\left(\partial_t u^\delta\right)+\frac{1}{2}\Delta\left(\partial_t u^\delta\right) &= \frac{1}{2}\left\langle y,\nabla^2V\left(q(t)\right)y\right\rangle\partial_t u^\delta+\frac{1}{2}\left(\nabla^3V\left(q(t)\right)\cdot y\cdot y\cdot\dot q(t)\right)u^\delta\\ &\quad+\lambda\partial_t u^\delta\log\left(\delta+|u^\delta|^2\right)+2\lambda\frac{u^\delta\,\mathrm{Re}\left(\overline{u}^\delta\partial_t u^\delta\right)}{\delta+|u^\delta|^2},\\ \partial_t u^\delta_{|t=0} &= \frac{i}{2}\Delta u_0-\frac{i}{2}\left\langle y,\nabla^2V(q_0)y\right\rangle u_0-i\lambda u_0\log\left(\delta+|u_0|^2\right).\end{aligned}$$

Recalling that the third derivatives of V are bounded and that $\dot q(t)=p(t)$ grows at most exponentially in time (Lemma 2.1), energy estimate yields

$$\frac{d}{dt}\|\partial_t u^\delta\|_{L^2}\lesssim e^{Ct}\||y|^2u^\delta\|_{L^2}+\|\partial_t u^\delta\|_{L^2}. \tag{2.10}$$

We also note that in view of (1.11), $\|\partial_t u^\delta_{|t=0}\|_{L^2}\leqslant C\left(\|u_0\|_{\Sigma^2}\right)$. Summing (2.6), (2.7) and (2.10), we find, using (2.8) and (2.9),

$$\begin{aligned}\frac{d}{dt}\left(\||y|^2u^\delta\|_{L^2}+\|y\cdot\nabla u^\delta\|_{L^2}+\|\partial_t u^\delta\|_{L^2}\right) &\lesssim e^{Ct}\||y|^2u^\delta\|_{L^2}+\|\partial_t u^\delta\|_{L^2}\\ &\quad+\|y\cdot\nabla u^\delta\|_{L^2}+e^{Ct}.\end{aligned}$$

Grönwall lemma yields

$$\||y|^2u^\delta\|_{L^2}+\|y\cdot\nabla u^\delta\|_{L^2}+\|\partial_t u^\delta\|_{L^2}\leqslant C\left(\|u_0\|_{\Sigma^2}\right)e^{e^{Ct}},$$

and invoking (2.8) and (2.9), (2.5) follows. Since we already know that u^δ converges to u, the proposition stems from Fatou's lemma. □

3. Subcritical case

In this section, we prove Proposition 1.2. We start by recalling some properties of the solution φ^ε to (1.5).

3.1. **Properties of $\varphi^\varepsilon_{\rm app}$ and φ^ε.** We resume the approximate solution of the linear case,

$$\varphi^\varepsilon_{\rm app}(t,x)=\frac{1}{\varepsilon^{d/4}}v\left(t,\frac{x-q(t)}{\sqrt{\varepsilon}}\right)e^{i\phi_{\rm lin}(t,x)/\varepsilon},$$

where $\phi_{\rm lin}$ is defined in (1.3), and infer that for any $1<p<\infty$,

$$\|\varphi^\varepsilon_{\rm app}(t)\|_{L^p_x}=\varepsilon^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\|v(t)\|_{L^p_y}.$$

In the linear case, it is easy to prove (see e.g. [9]):

Lemma 3.1. *Let $k \geqslant 1$. If $u_0 \in \Sigma^k$, there exists $C > 0$ such that the solution $v \in L^\infty_{\rm loc}(\mathbb{R};\Sigma^k)$ to (1.8) satisfies $\|v(t)\|_{L^2} = \|u_0\|_{L^2}$ and, for all $\beta \in \mathbb{N}^d$ with $1 \leqslant |\beta| \leqslant k$,*

$$\|\partial_y^\beta v(t)\|_{L^2(\mathbb{R}^d)} + \|y^\beta v(t)\|_{L^2(\mathbb{R}^d)} \lesssim e^{Ct}, \quad t \geqslant 0.$$

We infer from (2.1) and (2.2) that if $u_0 \in \Sigma$, for $0 < \eta < \min\left(1, \frac{4}{d+2}\right)$, there exists $C > 0$ such that,

$$\|\varphi^\varepsilon_{\rm app}(t)\|^{2-\eta}_{L^{2-\eta}} \lesssim \varepsilon^{\frac{d\eta}{4}} e^{Ct} \quad ; \quad \|\varphi^\varepsilon_{\rm app}(t)\|^{2+\eta}_{L^{2+\eta}} \lesssim \varepsilon^{-\frac{d\eta}{4}} e^{Ct}, \quad t \geqslant 0. \tag{3.1}$$

We now show that the exact linear solution φ^ε satisfies similar estimates. Define v^ε by the same rescaling as the one relating $\varphi^\varepsilon_{\rm app}$ and v,

$$\varphi^\varepsilon(t,x) = \frac{1}{\varepsilon^{d/4}} v^\varepsilon\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right) e^{i\phi_{\rm lin}(t,x)/\varepsilon}.$$

Direct computations (see e.g. [9]) show that v^ε solves

$$i\partial_t v^\varepsilon + \frac{1}{2}\Delta v^\varepsilon = V^\varepsilon(t,y) v^\varepsilon, \tag{3.2}$$

where

$$V^\varepsilon(t,y) = \frac{1}{\varepsilon}\left(V\left(q(t) + y\sqrt{\varepsilon}\right) - V\left(q(t)\right) - \sqrt{\varepsilon} y \cdot \nabla V\left(q(t)\right)\right).$$

We classically have $\|v^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}$ for all $t \in \mathbb{R}$. Taylor formula yields

$$V^\varepsilon(t,y) = \int_0^1 (1-\theta)\left\langle y, \nabla^2 V\left(q(t) + \theta y\sqrt{\varepsilon}\right) y\right\rangle d\theta.$$

In view of Assumption 1.1, there exists $C > 0$ independent of $\varepsilon \in (0,1)$ and $t \in \mathbb{R}$ such that

$$|\nabla V^\varepsilon(t,y)| \leqslant C\left(1 + |y|\right).$$

Applying the operator ∇ to (3.2), we get:

$$\left(i\partial_t + \frac{1}{2}\Delta\right)\nabla v^\varepsilon = V^\varepsilon(t,y)\nabla v^\varepsilon + v^\varepsilon \nabla V^\varepsilon(t,y),$$

and the energy estimate yields

$$\begin{aligned}\|\nabla v^\varepsilon(t)\|_{L^2} &\lesssim \|\nabla u_0\|_{L^2} + \int_0^t \|v^\varepsilon(s)\nabla V^\varepsilon(s)\|_{L^2} ds \\ &\lesssim \|\nabla u_0\|_{L^2} + \int_0^t \|(1+|y|)v^\varepsilon(s)\|_{L^2} ds.\end{aligned}$$

Applying the multiplication operator y to (3.6), we get similarly:

$$\left(i\partial_t + \frac{1}{2}\Delta\right)(yv^\varepsilon) = V^\varepsilon y v^\varepsilon + \nabla v^\varepsilon,$$

and, by energy estimate,

$$\|yv^\varepsilon(t)\|_{L^2} \lesssim \|yu_0\|_{L^2} + \int_0^t \|\nabla v^\varepsilon(s)\|_{L^2} ds.$$

Summing these two inequalities, Grönwall lemma yields

$$\|\nabla v^\varepsilon(t)\|_{L^2} + \|yv^\varepsilon(t)\|_{L^2} \lesssim \|u_0\|_\Sigma e^{Ct}, \quad t \geqslant 0. \tag{3.3}$$

Resuming the computations presented in the case of $\varphi^\varepsilon_{\rm app}$, we infer:

Lemma 3.2. *Let $d \geqslant 1$, $0 < \eta < \min\left(1, \frac{4}{d+2}\right)$, and $u_0 \in \Sigma$. There exists $C > 0$ independent of $\varepsilon \in (0,1)$ and $t \geqslant 0$ such that*

$$\|\varphi^\varepsilon(t)\|_{L^{2-\eta}}^{2-\eta} \lesssim \varepsilon^{\frac{d\eta}{4}} e^{Ct} \quad ; \quad \|\varphi^\varepsilon(t)\|_{L^{2+\eta}}^{2+\eta} \lesssim \varepsilon^{-\frac{d\eta}{4}} e^{Ct}, \quad t \geqslant 0.$$

3.2. **Linear approximation in the subcritical case.** In this subsection, we prove the first part of Proposition 1.2. We assume $u_0 \in \Sigma$, and set $w^\varepsilon = \psi^\varepsilon - \varphi^\varepsilon$. This error satisfies

$$\begin{aligned} i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon =& V(x)w^\varepsilon + \lambda\varepsilon^\alpha\left(\psi^\varepsilon \log|\psi^\varepsilon|^2 - \varphi^\varepsilon \log|\varphi^\varepsilon|^2\right) \\ &+ \lambda\varepsilon^\alpha\varphi^\varepsilon \log|\varphi^\varepsilon|^2 \quad ; \quad w^\varepsilon_{|t=0} = 0. \end{aligned}$$

Recall the strategy for the energy estimate: multiply both sides by $\overline{w^\varepsilon}$, integrate, and take the imaginary part,

$$\begin{aligned} \frac{\varepsilon}{2}\frac{d}{dt}\|w^\varepsilon\|_{L^2}^2 &= \lambda\varepsilon^\alpha \operatorname{Im}\int \overline{\psi^\varepsilon - \varphi^\varepsilon}\left(\psi^\varepsilon \log|\psi^\varepsilon|^2 - \varphi^\varepsilon \log|\varphi^\varepsilon|^2\right) + \lambda\varepsilon^\alpha \operatorname{Im}\int \bar{w}\varphi^\varepsilon \log|\varphi^\varepsilon|^2 \\ &\leqslant 2|\lambda|\varepsilon^\alpha\|w^\varepsilon\|_{L^2}^2 + |\lambda|\varepsilon^\alpha\|w^\varepsilon\|_{L^2}\|\varphi^\varepsilon \log|\varphi^\varepsilon|^2\|_{L^2}, \end{aligned}$$

where we have used Lemma 2.2 and Cauchy-Schwarz inequality. Grönwall lemma yields

$$\|w^\varepsilon\|_{L^2} \leq \varepsilon^{\alpha-1}\int_0^t e^{C\varepsilon^{\alpha-1}(t-s)}\|\varphi^\varepsilon \log|\varphi^\varepsilon|^2(s)\|_{L^2} ds. \tag{3.4}$$

We estimate $\varphi^\varepsilon(t,\cdot)\log|\varphi^\varepsilon(t,\cdot)|^2$ in $L^2\left(\mathbb{R}^d\right)$: for $0 < \eta < \min(1, \frac{4}{d+2})$ arbitrarily small,

$$|\varphi^\varepsilon|^2\left(\log|\varphi^\varepsilon|^2\right)^2 \lesssim |\varphi^\varepsilon|^{2-\eta} + |\varphi^\varepsilon|^{2+\eta}, \tag{3.5}$$

and so, in view Lemma 3.2,

$$\|\varphi^\varepsilon(t)\log|\varphi^\varepsilon(t)|^2\|_{L^2}^2 \lesssim \varepsilon^{-\frac{d\eta}{4}} e^{Ct}, \quad t \geqslant 0.$$

Plugging this estimate into (3.4) yields the first part of Proposition 1.2.

3.3. **Error estimate in the linear case.** We prove the end of Proposition 1.2. Recall that φ^ε solves the linear equation (1.5). Its standard approximation is $\varphi^\varepsilon_{\rm app}$, see e.g. [27] or [9]. Denote the error for the envelope by $\delta_v^\varepsilon = v^\varepsilon - v$. It satisfies

$$i\partial_t\delta_v^\varepsilon + \frac{1}{2}\Delta\delta_v^\varepsilon = V^\varepsilon(t,y)\delta_v^\varepsilon + \left(V^\varepsilon(t,y) - \frac{1}{2}\left\langle y, \nabla^2 V(q(t))y\right\rangle\right)v. \tag{3.6}$$

Taylor expansion yields the uniform in time pointwise estimate

$$\left|V^\varepsilon(t,y) - \frac{1}{2}\left\langle y, \nabla^2 V(q(t))y\right\rangle\right| \lesssim \sqrt{\varepsilon}|y|^3,$$

hence, by energy estimate, since $\delta_v^\varepsilon = 0$ at $t = 0$,

$$\|\delta_v^\varepsilon(t)\|_{L^2} \lesssim \sqrt{\varepsilon}\int_0^t \||y|^3 v(s)\|_{L^2} ds \lesssim \sqrt{\varepsilon}e^{Ct}, \tag{3.7}$$

where we have invoked Lemma 3.1 with $k = 3$.

The intermediary regularity assumption $u_0 \in \Sigma^2$ will be useful in the critical case $\alpha = 1$, and consists in resuming the energy estimate: multiply (3.6) by $\overline{\delta_v^\varepsilon}$, integrate in space, and take the imaginary part: Cauchy-Schwarz inequality yields

$$\frac{1}{2}\frac{d}{dt}\|\delta_v^\varepsilon\|_{L^2}^2 \lesssim \sqrt{\varepsilon}\||y|\delta_v^\varepsilon\|_{L^2}\||y|^2 v\|_{L^2}.$$

For $u_0 \in \Sigma^2$, Lemma 3.1 guarantees that

$$\||y|^2 v(t)\|_{L^2} \lesssim e^{Ct}, \quad t \geqslant 0.$$

On the other hand, triangle inequality yields, for any $t \geqslant 0$,

$$\||y|\delta_v^\varepsilon(t)\|_{L^2} \leqslant \||y|v^\varepsilon(t)\|_{L^2} + \||y|v(t)\|_{L^2} \lesssim e^{Ct},$$

where we have used (3.3) too. We infer

$$\frac{1}{2}\frac{d}{dt}\|\delta_v^\varepsilon\|_{L^2}^2 \lesssim \sqrt{\varepsilon}e^{Ct},$$

hence the announced error estimate.

4. Critical case: single coherent state

This section is dedicated to the proof of Theorem 1.3. From now on, and in the next section as well, we assume $\alpha = 1$.

The approximate solution is

$$\psi_{\rm app}^\varepsilon(t,x) = \frac{1}{\varepsilon^{d/4}} u\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+p(t)\cdot(x-q(t)))/\varepsilon - i\lambda\frac{d}{2}t\log\varepsilon}.$$

Here, as before, (q,p) is given by the Hamiltonian flow (1.6), the classical action is given by (1.7), but now the envelope u satisfies (1.11).

4.1. General case.

In this subsection, we assume $u_0 \in \Sigma^2$. The Cauchy problem for (1.11) was addressed in Lemma 2.4, complemented by Proposition 2.5. With these preliminaries at hand, we can follow essentially the same strategy as in Section 3. We denote by $w^\varepsilon = \psi^\varepsilon - \psi_{\rm app}^\varepsilon$ the error. It solves

$$\begin{aligned} i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = V(x)w^\varepsilon + \lambda\varepsilon\left(\psi^\varepsilon\log|\psi^\varepsilon|^2 - \psi_{\rm app}^\varepsilon\log|\psi_{\rm app}^\varepsilon|^2\right) \\ + \left(V(x) - T^2_{q(t)}\right)\psi_{\rm app}^\varepsilon \quad ; \quad w^\varepsilon_{|t=0} = 0, \end{aligned}$$

where $T^2_{q(t)}$ denote the Taylor expansion of V at order two about $q(t)$,

$$T^2_{q(t)}(x) = V\left(q(t)\right) + (x-q(t))\cdot\nabla V\left(q(t)\right) + \frac{1}{2}\left\langle x-q(t), \nabla^2 V\left(x-q(t)\right)(x-q(t))\right\rangle.$$

Multiply the equation in w^ε by $\overline{w^\varepsilon}$, integrate in space, and take the imaginary part: using Lemma 2.2 to estimate the logarithmic term, we find

$$\varepsilon\frac{d}{dt}\|w^\varepsilon\|_{L^2}^2 \lesssim \varepsilon\|w^\varepsilon\|_{L^2}^2 + \int_{\mathbb{R}^d}\left|V(x) - T^2_{q(t)}\right||w^\varepsilon(t,x)||\psi_{\rm app}^\varepsilon(t,x)|dx.$$

Taylor formula yields the pointwise estimate

$$\left|V(x) - T^2_{q(t)}\right| \lesssim |x-q(t)|^3,$$

and we balance these three powers like in the previous subsection:

$$\int_{\mathbb{R}^d} \left|V(x) - T^2_{q(t)}\right| |w^\varepsilon(t,x)||\psi^\varepsilon_{\rm app}(t,x)|dx \\ \lesssim \||x-q(t)|w^\varepsilon(t)\|_{L^2} \left\||x-q(t)|^2\psi^\varepsilon_{\rm app}(t)\right\|_{L^2}.$$

Introduce the exact envelope u^ε defined by

$$\psi^\varepsilon(t,x) = \frac{1}{\varepsilon^{d/4}} u^\varepsilon\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+p(t)\cdot(x-q(t)))/\varepsilon - i\lambda\frac{d}{2}t\log\varepsilon}.$$

We first remark that $\|u^\varepsilon(t)\|_{L^2} = \|\psi^\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2}$ for all $t\in\mathbb{R}$. We readily check that the arguments presented in Sections 2 and 3 show that there exists $C>0$ independent of ε and t such that, like in Lemma 2.4,

$$\|\nabla u^\varepsilon(t)\|_{L^2} + \|y u^\varepsilon(t)\|_{L^2} \lesssim e^{Ct}, \quad t\geqslant 0.$$

In view of the triangle inequality and Lemma 2.4, this yields

$$\||x-q(t)|w^\varepsilon(t)\|_{L^2} \lesssim \sqrt{\varepsilon}e^{Ct}, \quad t\geqslant 0.$$

Invoking now Proposition 2.5, we find

$$\left\||x-q(t)|^2\psi^\varepsilon_{\rm app}(t)\right\|_{L^2} \lesssim \varepsilon e^{e^{Ct}}, \quad t\geqslant 0,$$

and thus

$$\varepsilon\frac{d}{dt}\|w^\varepsilon\|^2_{L^2} \lesssim \varepsilon\|w^\varepsilon\|^2_{L^2} + \varepsilon^{3/2}e^{e^{Ct}}.$$

The first part of Theorem 1.3 then follows from Grönwall lemma.

4.2. **Gaussian case.** The main remark is that the potential in (1.11) being quadratic in y, Gaussian initial data lead to a solution u which is Gaussian at all time. Solving the partial differential equation (1.11) thus reduces to solving ordinary differential equations that describe the time-dependent coefficients of the Gaussian $u(t,\cdot)$. We present computations in the one-dimensional case for clarity. If V decouples variables, we can use the tensorization property to address higher dimension.

4.2.1. *Approximate solution in the Gaussian case.* Suppose $d=1$, and

$$u_0(y) = b_0 e^{-a_0 y^2/2}, \quad a_0, b_0 \in \mathbb{C},\ \operatorname{Re} a_0 > 0.$$

The solution to (1.11) is sought under the form

$$u(t,y) = b(t)e^{-a(t)y^2/2}.$$

We compute

$$i\partial_t u + \frac{1}{2}\partial_y^2 u - \frac{1}{2}V''(q(t))y^2 u - \lambda u \log|u|^2\big|_{u=b(t)e^{-a(t)y^2/2}} \\ = \left(A_1(t) + A_2(t)y^2\right)e^{-a(t)y^2/2}.$$

where

$$A_1(t) = i\dot{b} - \frac{1}{2}ab - \lambda b\log|b|^2,$$
$$A_2(t) = -\frac{1}{2}i\dot{a}b + \frac{1}{2}a^2 b - \frac{1}{2}V''\left(q(t)\right)b + \lambda b \operatorname{Re} a.$$

We thus cancel A_1 and A_2,

$$i\dot{b} = \frac{1}{2}ab + \lambda b\log|b|^2, \tag{4.1}$$

$$i\dot{a} = a^2 - V''(q(t)) + 2\lambda \operatorname{Re} a. \tag{4.2}$$

Equation (4.1) is integrated like in [12]. Multiplying both sides of (4.1) by $\bar{b}$,

$$i\dot{b}\bar{b} = \frac{1}{2}a|b|^2 + \lambda|b|^2 \log|b|^2. \tag{4.3}$$

Taking the imaginary part, we find

$$\frac{d}{dt}|b|^2 = |b|^2 \operatorname{Im} a,$$

hence

$$|b(t)|^2 = |b_0|^2 \exp\left(\operatorname{Im} A(t)\right), \quad \text{where } A(t) := \int_0^t a(s)ds.$$

Plugging this expression into (4.1), we have

$$i\dot{b} = \frac{1}{2}ab + \lambda b \log|b_0|^2 + \lambda b \operatorname{Im} A(t),$$

which we directly integrate,

$$b(t) = b_0 \exp\left(-i\lambda t \log|b_0|^2 - \frac{i}{2}A(t) - i\lambda \operatorname{Im}\int_0^t A(s)ds\right). \tag{4.4}$$

Like in [12], we seek a under the form

$$a(t) = \frac{\alpha_0}{\tau(t)^2} - i\frac{\dot{\tau}(t)}{\tau(t)}, \quad \text{with } \operatorname{Re} a_0 = \alpha_0. \tag{4.5}$$

If $\beta_0 = \operatorname{Im} a_0$, we check that (4.2) becomes

$$\ddot{\tau} = \frac{\alpha_0^2}{\tau^3} + 2\lambda\frac{\alpha_0}{\tau} - V''(q(t))\,\tau \quad ; \quad \tau(0) = 1, \quad \dot{\tau}(0) = -\beta_0.$$

Since V'' is bounded, the following lemma applies to this τ:

Lemma 4.1. *Let $\lambda \in \mathbb{R}$ and $\Omega \in L^\infty(\mathbb{R};\mathbb{R})$. For any $\tau_0 > 0$ and $\tau_1 \in \mathbb{R}$, consider the ordinary differential equation*

$$\ddot{\tau} = \frac{2\lambda}{\tau} + \frac{1}{\tau^3} - \Omega(t)\tau \quad ; \quad \tau(0) = \tau_0, \quad \dot{\tau}(0) = \tau_1. \tag{4.6}$$

It has a unique, global solution $\tau \in C^1(\mathbb{R};\mathbb{R})$, and it obeys the exponential bound

$$\exists C > 0, \quad \frac{1}{\tau(t)^2} + \tau(t)^2 + \dot{\tau}(t)^2 \lesssim e^{Ct}, \quad \forall t \geqslant 0.$$

Proof. Local existence follows from the Cauchy-Lipschitz Theorem, and by continuity, $\tau > 0$ at least near $t = 0$. By symmetry, we now focus on positive time, $t > 0$. Since the right hand side of (4.6) is smooth away from the origin, either the solution is global, or there exists $T > 0$ such that

$$\frac{1}{\tau(t)} + \tau(t) + |\dot{\tau}(t)| \underset{t\to T}{\longrightarrow} +\infty.$$

(The presence of $\dot{\tau}$ follows when turning (4.6) into a first order equation, to apply the Cauchy-Lipschitz Theorem.) We first show that as long as τ is bounded, so are $1/\tau$ and $|\dot{\tau}|$. We then prove the a priori bound on τ stated in the lemma, which implies that the solution is global.

Multiply (4.6) by $\dot\tau$, and integrate between 0 and t:

$$\dot\tau(t)^2 = \underbrace{\tau_1^2 + 4\lambda\ln\tau_0 + \frac{1}{\tau_0^2}}_{=:C_0} + 4\lambda\ln\tau(t) - \frac{1}{\tau(t)^2} - 2\int_0^t \Omega(s)\tau(s)\dot\tau(s)ds. \tag{4.7}$$

For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|\ln\tau| \leqslant C_\varepsilon + \frac{\varepsilon}{\tau^2} + \varepsilon\tau^2, \quad \forall \tau > 0. \tag{4.8}$$

For t bounded and $\eta > 0$, estimate the integral by

$$\begin{aligned} 2\left|\int_0^t \Omega(s)\tau(s)\dot\tau(s)ds\right| &\leqslant 2t\|\Omega\|_{L^\infty}\sup_{s\in[0,t]}|\tau(s)||\dot\tau(s)| \\ &\leqslant t\|\Omega\|_{L^\infty}\sup_{s\in[0,t]}\left(\frac{1}{\eta}|\tau(s)|^2 + \eta|\dot\tau(s)|^2\right) \\ &\leqslant t\frac{\|\Omega\|_{L^\infty}}{\eta}\sup_{s\in[0,t]}|\tau(s)|^2 + t\eta\|\Omega\|_{L^\infty}\sup_{s\in[0,t]}|\dot\tau(s)|^2, \end{aligned}$$

where we have used Young inequality. Taking ε and η sufficiently small, we infer

$$\sup_{s\in[0,t]}\left(\dot\tau(s)^2 + \frac{1}{\tau(s)^2}\right) \leqslant C(t)\left(1 + \sup_{s\in[0,t]}|\tau(s)|^2\right). \tag{4.9}$$

In particular, if τ is bounded on $[0,T]$, so is $1/\tau$. Introduce

$$y(t) = \tau(t)^2 + \dot\tau(t)^2 \geqslant 0.$$

We compute

$$\dot y = 2\tau\dot\tau + 2\dot\tau\ddot\tau = 2\tau\dot\tau + 2\lambda\frac{\dot\tau}{\tau} + \frac{\dot\tau}{\tau^3} - \Omega\tau\dot\tau.$$

Integrating between 0 and t, and using Young inequality (for the term $\int_0^t \tau\dot\tau$),

$$y(t) \leqslant C + \int_0^t y(s)ds + 2\lambda\ln\tau(t) - \frac{1}{2\tau(t)^2} - \int_0^t \Omega(s)\tau(s)\dot\tau(s)ds.$$

Using (4.8) again, and Young inequality in the last integral,

$$y(t) \leqslant C + \int_0^t y(s)ds + 2|\lambda|\left(C_\eta + \frac{\eta}{\tau(t)^2} + \eta\tau(t)^2\right) - \frac{1}{2\tau(t)^2} + \|\Omega\|_{L^\infty}\int_0^t y(s)ds.$$

For $2|\lambda|\eta < 1/2$, the factor of $1/\tau^2$ on the right hand side is negative, and the term $2|\lambda|\eta\tau^2$ is absorbed by the left hand side, so we get

$$y(t) \lesssim 1 + \int_0^t y(s)ds.$$

In view of the above discussion, Grönwall lemma then yields global existence and the exponential bound for τ and $\dot\tau$. The exponential bound for $1/\tau$ then follows from (4.7) and (4.8). □

We infer the analogue of Lemma 3.1 in the nonlinear case for Gaussian data:

Lemma 4.2. *Let $d \geqslant 1$. Suppose that V decouples variables,*

$$V(x) = \sum_{j=1}^{d} V_j(x_j),$$

and that u_0 is a Gaussian of the form

$$u_0(y) = b_0 \exp\left(-\frac{1}{2} \sum_{j=1}^{d} a_{0j} y_j^2 \right), \quad a_{0j}, b_0 \in \mathbb{C},\ \mathrm{Re}\, a_{0j} > 0.$$

For any $k \geqslant 1$, there exists $C_k > 0$ such that the solution $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; \Sigma^k)$ to (1.11) satisfies $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and, for all $\beta \in \mathbb{N}^d$ with $1 \leqslant |\beta| \leqslant k$,

$$\|\partial_y^\beta u(t)\|_{L^2(\mathbb{R}^d)} + \|y^\beta u(t)\|_{L^2(\mathbb{R}^d)} \lesssim e^{C_k t}, \quad t \geqslant 0.$$

Proof. First, we note that the above one-dimensional computation is readily adapted, by seeking $u(t, y)$ under the form

$$u(t, y) = b(t) \exp\left(-\frac{1}{2} \sum_{j=1}^{d} a_j(t) y_j^2 \right).$$

Equation (4.1) becomes

$$i\dot{b} = \frac{1}{2} \sum_{j=1}^{d} a_j b + \lambda b \log |b|^2,$$

and (4.2) becomes a family of d uncoupled equations, since $\nabla^2 V$ is diagonal,

$$i\dot{a_j} = a_j^2 - V_j''(q_j(t)) + 2\lambda\, \mathrm{Re}\, a_j,$$

so we can use directly the one-dimensional analysis. For $\beta \in \mathbb{N}^d$, we readily compute

$$\|y^\beta u(t)\|_{L^2} = |b(t)| \prod_{j=1}^{d} \|y_j^{\beta_j} e^{-a_j(t) y_j^2/2}\|_{L^2} = |b(t)| \prod_{j=1}^{d} (\mathrm{Re}\, a_j(t))^{-\beta_j/2 - 1/4}\, \mu(\beta_j),$$

where the coefficients $\mu(\beta_j)$ stand for the momenta of the Gaussian in L^2,

$$\mu(\beta) = \left(\int_{\mathbb{R}} z^{2\beta} e^{-z^2} dz \right)^{1/2},$$

whose precise value is irrelevant here. As

$$\begin{aligned} |b(t)|^2 &= |b_0|^2 \exp\left(\sum_{j=1}^{d} \int_0^t \mathrm{Im}\, a_j(s) ds \right) \\ &= |b_0|^2 \exp\left(-\sum_{j=1}^{d} \int_0^t \frac{\dot{\tau}_j(s)}{\tau_j(s)} ds \right) = |b_0|^2 \prod_{j=1}^{d} \tau_j(t)^{-1/2}, \end{aligned}$$

with notations inherited from the one-dimensional case, we have

$$\|y^\beta u(t)\|_{L^2} \lesssim \prod_{j=1}^{d} \tau_j(t)^{-1/2} (\mathrm{Re}\, a_j(t))^{-\beta_j/2 - 1/4} \lesssim \prod_{j=1}^{d} \tau_j(t)^{-1/2 + \beta_j + 1/2}.$$

Lemma 4.1 yields $0 < \tau_j(t) \lesssim e^{c_j t}$ for some $c_j > 0$, hence the announced bound for the momenta. The bound for the derivative proceeds similarly, by considering the Fourier transform of $u(t,\cdot)$. □

4.2.2. *Error estimate in the Gaussian case.* We can now resume the notations and computations from Section 4.1. The energy estimate for w^ε implies

$$\|w^\varepsilon(t)\|_{L^2} \lesssim \frac{1}{\varepsilon}\int_0^t e^{C(t-s)}\left\||x-q(t)|^3\psi^\varepsilon_{\rm app}(s)\right\|_{L^2}ds \lesssim \sqrt{\varepsilon}\int_0^t e^{C(t-s)}\left\||y|^3u(s)\right\|_{L^2}ds,$$

so Lemma 4.2 completes the proof of Theorem 1.3.

5. Superposition in the Gaussian case

In this section, we prove Theorem 1.4. A similar result was proven in [10] in the case of a smooth power nonlinearity ($\varepsilon^{\alpha_c}|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon$, with σ an integer), and we resume the same strategy here. The major difference however lies in the estimate of interaction terms. Typically, if $\psi^\varepsilon_{1,\rm app}$ and $\psi^\varepsilon_{2,\rm app}$ have difference centers in physical space, $q_1 \neq q_2$, the error term

$$|\psi^\varepsilon_{1,\rm app}+\psi^\varepsilon_{2,\rm app}|^2\left(\psi^\varepsilon_{1,\rm app}+\psi^\varepsilon_{2,\rm app}\right) - |\psi^\varepsilon_{1,\rm app}|^2\psi^\varepsilon_{1,\rm app} - |\psi^\varepsilon_{2,\rm app}|^2\psi^\varepsilon_{2,\rm app}$$

involves only cross terms, where the factor $\psi^\varepsilon_{1,\rm app}\times\psi^\varepsilon_{2,\rm app}$ (leaving out the conjugacy to simplify the discussion) is present: since $\psi^\varepsilon_{j,\rm app}$ is centered in q_j at scale $\sqrt{\varepsilon}$, these cross terms can be shown to be small. In the case of a logarithmic nonlinearity, we cannot use the same sort of expansion: instead, we rely on a lemma proven by Ferriere in [16], which measures precisely the analogue of this decomposition, *in the case of Gaussian functions* (see Lemma 5.2 below).

We examine the superposition principle in (1.1), and distinguish two regions. When the trajectories (in physical space) of the two centers intersect, or are close to each other, we keep the nature of the nonlinear interaction as a black box, and use the fact such an event is sufficiently rare for the nonlinear interaction to be negligible. This step is an adaptation of computations from [10] to the case of a logarithmic nonlinearity. Conversely, when the centers of the two coherent states are far from each other, we apply the method of Ferriere [16], which shows the absence of interaction at leading order.

5.1. **Preparation of the proof.** As announced above, the scheme of the proof is the same as in [10], and we resume the same notations as much as possible. Under the assumptions of Theorem 1.4, denote $w^\varepsilon = \psi^\varepsilon - \psi^\varepsilon_{1,\rm app} - \psi^\varepsilon_{2,\rm app}$ the error function. It solves

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = V(x)w^\varepsilon + L^\varepsilon + \mathcal{N}^\varepsilon \quad ; \quad w^\varepsilon_{|t=0} = 0, \tag{5.1}$$

where we have now

$$L^\varepsilon(t,x) = \left(V(x) - T^2_{q_1(t)}(x)\right)\psi^\varepsilon_{1,\rm app}(t,x) + \left(V(x) - T^2_{q_2(t)}(x)\right)\psi^\varepsilon_{2,\rm app}(t,x),$$

and, like in Section 4, T^2_z stands for the Taylor second order polynomial of V about z,

$$T^2_z(x) = V(z) + (x-z)\cdot\nabla V(z) + \frac{1}{2}\left\langle x-z, \nabla^2V(x-z)x-z\right\rangle.$$

The approximate functions associated with each Gaussian are given by

$$\psi^\varepsilon_{j,\mathrm{app}}(t,x) = \frac{1}{\varepsilon^{d/4}} u_j\left(t, \frac{x-q_j(t)}{\sqrt{\varepsilon}}\right) e^{i(S_j(t)+p_j(t)\cdot(x-q_j(t)))/\varepsilon - i\lambda\frac{d}{2}t\log\varepsilon},$$

where q_j, p_j and S_j are defined by (1.6) (with initial data q_{0j} and p_{0j}) and (1.7), while u_j solves (1.11) with initial datum u_{0j}. Denote by g the function

$$g(z) = \lambda z \log|z|^2, \quad z\in\mathbb{C}.$$

The nonlinear source term is given by

$$\mathcal{N}^\varepsilon = \varepsilon\left(g\left(w^\varepsilon + \psi^\varepsilon_{1,\mathrm{app}} + \psi^\varepsilon_{2,\mathrm{app}}\right) - g\left(\psi^\varepsilon_{1,\mathrm{app}}\right) - g\left(\psi^\varepsilon_{2,\mathrm{app}}\right)\right).$$

We decompose $\mathcal{N}^\varepsilon$ as the sum of a semilinear term and an interaction source term: $\mathcal{N}^\varepsilon = \mathcal{N}^\varepsilon_S + \mathcal{N}^\varepsilon_I$, where

$$\begin{aligned}\mathcal{N}^\varepsilon_S &= \varepsilon\left(g(w^\varepsilon + \psi^\varepsilon_{1,\mathrm{app}} + \psi^\varepsilon_{2,\mathrm{app}}) - g(\psi^\varepsilon_{1,\mathrm{app}} + \psi^\varepsilon_{2,\mathrm{app}})\right),\\ \mathcal{N}^\varepsilon_I &= \varepsilon\left(g(\psi^\varepsilon_{1,\mathrm{app}} + \psi^\varepsilon_{2,\mathrm{app}}) - g\left(\psi^\varepsilon_{1,\mathrm{app}}\right) - g\left(\psi^\varepsilon_{2,\mathrm{app}}\right)\right).\end{aligned}$$

Multiply (5.1) by $\overline{w^\varepsilon}$, integrate over $\mathbb{R}^d$, and take the imaginary part: we get

$$\frac{\varepsilon}{2}\frac{d}{dt}\|w^\varepsilon\|^2_{L^2} \leqslant \|w^\varepsilon\|_{L^2}\left(\|L^\varepsilon\|_{L^2} + \|\mathcal{N}^\varepsilon_I\|_{L^2}\right) + \mathrm{Im}\int_{\mathbb{R}^d}\overline{w^\varepsilon}\mathcal{N}^\varepsilon_S.$$

The linear source term L^ε is handled like in the case of a single coherent state (Section 4):

$$\|L^\varepsilon(t)\|_{L^2} \lesssim \sum_{j=1,2}\left\||x-q_j(t)|^3\psi^\varepsilon_{j,\mathrm{app}}(t)\right\|_{L^2} = \varepsilon^{3/2}\sum_{j=1,2}\left\||y|^3 u_j(t)\right\|_{L^2}$$

In view of Lemma 4.2, we infer

$$\|L^\varepsilon(t)\|_{L^2} \lesssim \varepsilon^{3/2}e^{Ct}, \quad t\geqslant 0.$$

Also, in view of Lemma 2.2,

$$\mathrm{Im}\int_{\mathbb{R}^d}\overline{w^\varepsilon}\mathcal{N}^\varepsilon_S \leqslant 2|\lambda|\|w^\varepsilon\|^2_{L^2},$$

so this term will be handled by Grönwall lemma. Therefore the core of the proof lies in the control of N^ε_I. Like in [10], we separate the time interval that we consider into two parts: when the trajectories q_1 an q_2 are close to each other, and when they are not, each case being treated with different arguments.

5.2. **Crossing of coherent states.** The analysis of the set where trajectories are close to each other was performed in [10]:

Proposition 5.1 (Lemma 6.2 and Proposition 6.3 in [10])**.** *For $j=1,2$, let (q_j,p_j) be the solutions to (1.6) associated with initial data $(q_{01},p_{01}) \neq (q_{02},p_{02})$. Let $0<\gamma<1/2$. For $T>0$, set*

$$I^\varepsilon(T) = \left\{t\in[0,T], |q_1(t)-q_2(t)| \leqslant \varepsilon^\gamma\right\}.$$

- *If T is independent of ε, we have*

$$|I^\varepsilon(T)| = \mathcal{O}\left(\varepsilon^\gamma\right).$$

- *If in addition $d=1$ and $E_1\neq E_2$, where*

$$E_j = \frac{p_{0j}^2}{2} + V\left(q_{0j}\right),$$

then there exist $C, C_0 > 0$ *independent of* $\varepsilon \in (0,1)$ *such that*

$$|I_\varepsilon(t)| \lesssim \varepsilon^\gamma e^{C_0 t}|E_1 - E_2|^{-2}, \quad 0 \leqslant t \leqslant C \log \frac{1}{\varepsilon}.$$

In the region I^ε, nonlinear interaction may occur; however, Proposition 5.1 implies that the effects of such interaction remain negligible at leading order. Indeed, we argue like in Section 3, and write, for $0 < \eta < \min\left(1, \frac{4}{d+2}\right)$, using (3.5),

$$\|\mathcal{N}_I^\varepsilon\|_{L^2(\mathbb{R}^2)}^2 \lesssim \varepsilon^2 \sum_{j=1,2} \left(\|\psi_{j,\mathrm{app}}^\varepsilon\|_{L^{2+\eta}}^{2+\eta} + \|\psi_{j,\mathrm{app}}^\varepsilon\|_{L^{2-\eta}}^{2-\eta} \right).$$

Like in Section 3, and now referring to Lemma 4.2 in the nonlinear case for Gaussian profiles, this implies

$$\|\mathcal{N}_I^\varepsilon(t)\|_{L^2(\mathbb{R}^2)} \lesssim \varepsilon^{1-\frac{d\eta}{8}} e^{Ct}, \quad t \geqslant 0.$$

Proposition 5.1 then implies

$$\int_{I^\varepsilon(T)} \|\mathcal{N}_I^\varepsilon(t)\|_{L^2}\, dt \lesssim \int_{I^\varepsilon(T)} \varepsilon^{1-\frac{d\eta}{8}} e^{Ct} dt \lesssim \varepsilon^{\gamma+1-\frac{d\eta}{8}} e^{CT}. \tag{5.2}$$

We emphasize the fact that in view of Proposition 5.1, the above estimate is valid in general for $T > 0$ independent of $\varepsilon \in (0,1)$, and, when $d = 1$, for $T \leqslant C \log \frac{1}{\varepsilon}$ for some $C > 0$ independent of ε.

5.3. **Separation of coherent states.** The complement of the set $I^\varepsilon(T)$ is the set of times is when the trajectories of two coherent states are sufficiently far apart. We recall a crucial result from [16], which shows some orthogonality phenomenon in the case of Gaussian functions for the logarithmic nonlinearity. We state a simplified version of the result from [16], in the sense that the assumptions considered there are more general.

Lemma 5.2 (From Lemma 3.2 in [16])**.** *For any* $d \geqslant 1$*, there exists* $C_d > 0$ *such that the following holds. Let* $N \in \mathbb{N}^*$ *and consider, for* $k = 1, \dots, N$*,* $q_k \in \mathbb{R}^d$*,* $\omega_k \in \mathbb{R}$*,* $\lambda_k > 0$ *and* $\theta_k : \mathbb{R}^d \to \mathbb{R}$ *a real measurable function. Define, for all* $x \in \mathbb{R}^d$*,*

$$\varphi(x) = \sum_{k=1}^N \varphi_k(x), \quad \varphi_k(x) = e^{i\theta_k(x) + \omega_k - \lambda_k |x - q_k|^2}.$$

If

$$\rho := \left(\min_{k \neq j} |q_j - q_k| \right)^{-1} < \rho_0 := \min\left(\frac{\sqrt{\lambda_+}}{\max(\sqrt{\delta\omega + 1}, \sqrt{\ln N})}, \sqrt{\frac{\lambda_-}{d+2}} \right),$$

where $\lambda_= \max_k \lambda_k$*,* $\lambda_- = \min_k \lambda_k$*, and* $\delta\omega := \max_{j,k} |\omega_k - \omega_j|$*, then*

$$\left\| g\left(\sum_{k=1}^N \varphi_k \right) - \sum_{k=1}^N g\left(\varphi_k\right) \right\|_{L^2(\mathbb{R}^d)} \leqslant C_d N^{\frac{3}{2}} \frac{\lambda_+}{\rho^{\frac{d}{2}+1}\sqrt{\lambda_-}} e^{-\frac{\lambda_-}{4\rho^2} + \max_j \omega_j}.$$

We apply this lemma with $N = 2$,

$$\theta_k(t,x) = \arg b_k(t) + \frac{\phi_{\mathrm{nl}}(t,x)}{\varepsilon} - \operatorname{Im} a_k(t) \frac{|x - q_k(t)|^2}{2\varepsilon},$$
$$\omega_k(t) = \log |b_k(t)| - \frac{d}{4} \log \varepsilon, \quad \lambda_k(t) = \frac{\operatorname{Re} a_k(t)}{2\varepsilon}.$$

For $t \in [0,T] \setminus I^\varepsilon(T)$, we have

$$\left(e^{-Ct} \lesssim\right) \rho \leqslant \varepsilon^{-\gamma}.$$

On the other hand, in view of (4.4) and (4.5), Lemma 4.2 yields $\lambda_k(t) \gtrsim \varepsilon^{-1} e^{-Ct}$ and

$$\rho_0 \gtrsim \frac{1}{\sqrt{\varepsilon}} e^{-Ct},$$

so the condition from Lemma 5.2 is fulfilled provided that $e^{Ct} \lesssim \varepsilon^{\gamma-1/2}$, which is granted with $0 \leqslant t \leqslant c \log \frac{1}{\varepsilon}$ for some $c > 0$ independent of $\varepsilon \in (0,1)$, since $\gamma < 1/2$. Lemma 5.2 then implies, for $t \in [0,T] \setminus I^\varepsilon(T)$, and recalling Lemma 4.1 to bound $\operatorname{Re} a_k$ from above,

$$\frac{1}{\varepsilon} \|\mathcal{N}_I^\varepsilon(t)\|_{L^2} \lesssim \varepsilon^{-\frac{d}{4}-\frac{1}{2}} e^{Ct} \exp\left(-C\varepsilon^{2\gamma-1} e^{-Ct}\right), \quad t \in [0,T] \setminus I^\varepsilon(T). \tag{5.3}$$

5.4. **Conclusion.** Putting (5.2) and (5.3) together, for any $0 < \eta < \min\left(1, \frac{4}{d+2}\right)$,

$$\frac{1}{\varepsilon} \int_0^T \|\mathcal{N}_I^\varepsilon(t)\|_{L^2} dt \lesssim \varepsilon^{\gamma-\frac{d\eta}{8}} e^{CT} + \varepsilon^{-\frac{d}{4}-\frac{1}{2}} e^{CT} \exp\left(-C\varepsilon^{2\gamma-1} e^{-CT}\right).$$

In view of the preliminary estimates from Section 5.1, we get

$$\begin{aligned} \frac{d}{dt}\|w^\varepsilon(t)\|_{L^2} &\lesssim \|w^\varepsilon(t)\|_{L^2} + \frac{1}{\varepsilon}\|L^\varepsilon(t)\|_{L^2} + \frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon(t)\|_{L^2} \\ &\lesssim \|w^\varepsilon(t)\|_{L^2} + \sqrt{\varepsilon} e^{Ct} + \frac{1}{\varepsilon}\|\mathcal{N}_I^\varepsilon(t)\|_{L^2}, \end{aligned}$$

hence from Grönwall lemma and the above estimate, for any $0 < \eta < \min\left(1, \frac{4}{d+2}\right)$,

$$\begin{aligned} \sup_{t\in[0,T]} \|w^\varepsilon(t)\|_{L^2} &\lesssim \sqrt{\varepsilon} e^{CT} + \frac{1}{\varepsilon}\int_0^T e^{C(T-t)} \|\mathcal{N}_I^\varepsilon(t)\|_{L^2} dt \\ &\lesssim \sqrt{\varepsilon} e^{CT} + \varepsilon^{\gamma-\frac{d\eta}{8}} e^{CT} + \varepsilon^{-\frac{d}{4}-\frac{1}{2}} e^{CT} \exp\left(-C\varepsilon^{2\gamma-1} e^{-CT}\right). \end{aligned}$$

Recall that this estimate is always true for $T > 0$ independent of $\varepsilon \in (0,1)$, and that if $d = 1$, then it holds more generally for $0 < T \leqslant c_1 \log \frac{1}{\varepsilon}$ for some $c_1 > 0$ independent of ε. Up to decreasing c_1, the last term is always $\mathcal{O}(\varepsilon^\gamma)$ (this is obvious for T independent of ε). This yields the conclusion of Theorem 1.4, by replacing γ with $\gamma - \frac{d\eta}{8}$. But since the condition $0 < \gamma < 1/2$ is open, and $\eta > 0$ is arbitrarily small, $\gamma - \frac{d\eta}{8}$ is arbitrarily close to $1/2$, hence the result.

References

[1] A. H. Ardila, L. Cely, and M. Squassina. Logarithmic Bose-Einstein condensates with harmonic potential. *Asymptotic Anal.*, 116(1):27–40, 2020.

[2] A. H. Ardila and M. Squassina. Gausson dynamic for logarithmic Schrödinger equations. *Asymptot. Anal.*, 107(3-4):203–226, 2018.

[3] A. Athanassoulis, T. Paul, F. Pezzotti, and M. Pulvirenti. Semiclassical propagation of coherent states for the Hartree equation. *Ann. Henri Poincaré*, 12(8):1613–1634, 2011.

[4] A. V. Avdeenkov and K. G. Zloshchastiev. Quantum Bose liquids with logarithmic nonlinearity: Self-sustainability and emergence of spatial extent. *J. Phys. B: Atomic, Molecular Optical Phys.*, 44(19):195303, 2011.

[5] I. Białynicki-Birula and J. Mycielski. Nonlinear wave mechanics. *Ann. Physics*, 100(1-2):62–93, 1976.

[6] B. Bouharia. Stability of logarithmic Bose-Einstein condensate in harmonic trap. *Modern Physcis Letters B*, 29(01):1450260, 2015.

[7] P. Cao and R. Carles. Semiclassical wave packet dynamics for Hartree equations. *Rev. Math. Phys.*, 23(9):933–967, 2011.
[8] R. Carles. Nonlinear Schrödinger equation with time dependent potential. *Commun. Math. Sci.*, 9(4):937–964, 2011.
[9] R. Carles. *Semi-classical analysis for nonlinear Schrödinger equations. WKB analysis, focal points, coherent states*. Hackensack, NJ: World Scientific, 2nd edition edition, 2021.
[10] R. Carles and C. Fermanian-Kammerer. Nonlinear coherent states and Ehrenfest time for Schrödinger equation. *Commun. Math. Phys.*, 301(2):443–472, 2011.
[11] R. Carles and G. Ferriere. Logarithmic Schrödinger equation with quadratic potential. *Nonlinearity*, 34(12):8283, 2021.
[12] R. Carles and I. Gallagher. Universal dynamics for the defocusing logarithmic Schrödinger equation. *Duke Math. J.*, 167(9):1761–1801, 2018.
[13] R. Carles, M. Hayashi, and T. Ozawa. Low regularity solutions to the logarithmic Schrödinger equation. *Pure Appl. Anal.*, 6(3):859–871, 2024.
[14] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.
[15] T. Cazenave and A. Haraux. Equations d'évolution avec non linearite logarithmique. *Ann. Fac. Sci. Toulouse, Math. (5)*, 2:21–51, 1980.
[16] G. Ferriere. The focusing logarithmic Schrödinger equation: analysis of breathers and nonlinear superposition. *Discrete Contin. Dyn. Syst.*, 40(11):6247–6274, 2020.
[17] G. Ferriere. WKB analysis of the logarithmic nonlinear Schrödinger equation in an analytic framework. *Commun. Contemp. Math.*, 24(8):Paper No. 2150082, 34, 2022.
[18] M. Gallo, S. Mosconi, and M. Squassina. Power law convergence and concavity for the Logarithmic Schrödinger equation. Preprint, archived at `https://arxiv.org/abs/2411.01614`, 2024.
[19] P. Guerrero, J. López, and J. Nieto. Global solvability of the 3d logarithmic schrödinger equation. *Nonlinear Analysis: Real World Applications*, 11(1):79–87, 2010.
[20] G. A. Hagedorn. Semiclassical quantum mechanics. I. The $\hbar \to 0$ limit for coherent states. *Comm. Math. Phys.*, 71(1):77–93, 1980.
[21] T. Hansson, D. Anderson, and M. Lisak. Propagation of partially coherent solitons in saturable logarithmic media: A comparative analysis. *Phys. Rev. A*, 80(3):033819, 2009.
[22] M. Hayashi and T. Ozawa. The Cauchy problem for the logarithmic Schrödinger equation revisited. *Ann. Henri Poincaré*, 2025. To appear.
[23] E. F. Hefter. Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics. *Phys. Rev. A*, 32:1201–1204, 1985.
[24] E. J. Heller. Time dependent approach to semiclassical dynamics. *J. Chem. Phys.*, 62(1):1544–1555, 1975.
[25] K. Hepp. The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.*, 35:265–277, 1974.
[26] S. D. Martino, M. Falanga, C. Godano, and G. Lauro. Logarithmic Schrödinger-like equation as a model for magma transport. *Europhys. Lett.*, 63:472–475, 2003.
[27] D. Robert and M. Combescure. *Coherent states and applications in mathematical physics*. Springer, 2021.
[28] Z.-Q. Wang and C. Zhang. Convergence from power-law to logarithm-law in nonlinear scalar field equations. *Arch. Ration. Mech. Anal.*, 231(1):45–61, 2019.
[29] K. Yasue. Quantum mechanics of nonconservative systems. *Annals Phys.*, 114(1-2):479–496, 1978.
[30] K. G. Zloshchastiev. Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences. *Grav. Cosmol.*, 16:288–297, 2010.
[31] K. G. Zloshchastiev. Spontaneous symmetry breaking and mass generation as built-in phenomena in logarithmic nonlinear quantum theory. *Acta Phys. Polon. B*, 42(2):261–292, 2011.

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