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MATHEMATICAL ANALYSIS OF A TWO-DIMENSIONAL RADIOFREQUENCY ABLATION MODEL IN CARDIAC TISSUE WITH L^1 ENERGY DISSIPATION

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ABSTRACT. This paper deals with the existence and regularity of solutions for a nonlinear reaction-diffusion-fluid system that models radiofrequency ablation in cardiac tissue. The primary focus is on establishing the regularity of solutions in a two-dimensional setting, particularly in the presence of energy dissipation terms due to viscosity and electric fields, where these terms belong to the L^1 space. We rigorously establish the existence of global solutions for the system by employing the truncation approximation method, Schauder's fixed point theorem, and compactness arguments, under specific assumptions on the data. Finally, we present some numerical simulations to validate the theoretical findings.

1. INTRODUCTION AND CONTRIBUTIONS

We investigate a two-dimensional Electro-Thermo-Fluid model that characterizes the process of radio-frequency ablation in cardiac tissue (see Figure 1). This model consists of coupled equations representing the thermistor and the incompressible Navier-Stokes equations. The system describes the dynamics of temperature, velocity, and electric potential within cardiac tissue during the ablation procedure, formulated as follows [6]:

$$\left\{ \begin{array}{ll} \partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nabla \cdot (\nu(\theta) \mathbb{D}(\mathbf{u})) + \nabla P - \mathbf{F}(\theta) = \mathbf{0} & \text{in } \Omega_{t_f} \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega_{t_f} \\ \mathbf{u} = \mathbf{0} & \text{on } \Sigma_D \\ \mathbf{u} \cdot \boldsymbol{\eta} = 0 & \text{on } \Sigma_N \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 & \text{on } \Omega \\ \\ \partial_t \theta - \nabla \cdot (\gamma(\theta) \nabla \theta) + \mathbf{u} \cdot \nabla \theta - \nu(\theta) \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{u}) - (\lambda(\theta) \nabla \varphi) \cdot \nabla \varphi = 0 & \text{in } \Omega_{t_f} \\ (\gamma(\theta) \nabla \theta) \cdot \boldsymbol{\eta} + \alpha \theta = \alpha \theta_l & \text{on } \Sigma \\ \theta(\mathbf{x}, 0) = \theta_0 & \text{in } \Omega \\ \\ -\nabla \cdot (\lambda(\theta) \nabla \varphi) = 0 & \text{in } \Omega_{t_f} \\ (\lambda(\theta) \nabla \varphi) \cdot \boldsymbol{\eta} = g & \text{on } \Sigma_5 \\ \varphi = 0 & \text{on } \Sigma \setminus \Sigma_5 \end{array} \right. \quad (1)$$

Key words and phrases. Bio-heat equation, Navier-Stokes equation, Electro-Thermo-Fluid model, Radiofrequency ablation, Truncation method.

Here, $\Omega_{t_f} = \Omega \times (0, t_f)$, with $\Omega \subset \mathbb{R}^d$, $d = 2$, is a bounded domain with a $C^{1,1}$ boundary $\Gamma := \Gamma_D \cup \Gamma_N$, and $t_f \in (0, \infty)$ is fixed throughout the paper. Figure 1 illustrates the notations, where $\Gamma_D = \Gamma_1 \cup \Gamma_3$ and $\Gamma_N = \Gamma_2 \cup \Gamma_4 \cup \Gamma_5$. We define $\Sigma = \Gamma \times (0, t_f)$, $\Sigma_N = \Gamma_N \times (0, t_f)$, $\Sigma_D = \Gamma_D \times (0, t_f)$, and $\Sigma_i = \Gamma_i \times (0, t_f)$ for $i = 1, \dots, 5$. We assume that Γ_D and Γ_N are closed disjoint $(d-1)$ -dimensional manifolds of class $C^{1,1}$, where Γ_D represents solid surfaces and Γ_N denotes the artificial part of the boundary $\partial\Omega$. The symbol $\boldsymbol{\eta}$ denotes the unit outward normal on the boundary Γ_N .

The variables and parameters in the equations are as follows: \mathbf{u} is the flow velocity, P is the pressure scaled by the density ρ , and ν is the kinematic viscosity. The strain rate tensor is $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$, and the Cauchy stress tensor is $\mathbb{S}(\mathbf{u}, P) = \nu(\theta)\mathbb{D}(\mathbf{u}) - P\mathbf{I}$. The term $\mathbf{F}(\theta)$ represents a forcing term. The function $\gamma(\theta)$ denotes the heat conductivity, which depends on the temperature θ . The coefficient α regulates the convective heat flux through the boundary $\partial\Omega$. The functions θ_l and θ_0 represent the boundary and initial temperatures, respectively. The function $\lambda(\theta)$ represents the electric conductivity, which depends on the temperature θ , and g denotes the electric current induced via the boundary Γ_N . The potential electric is denoted by φ .

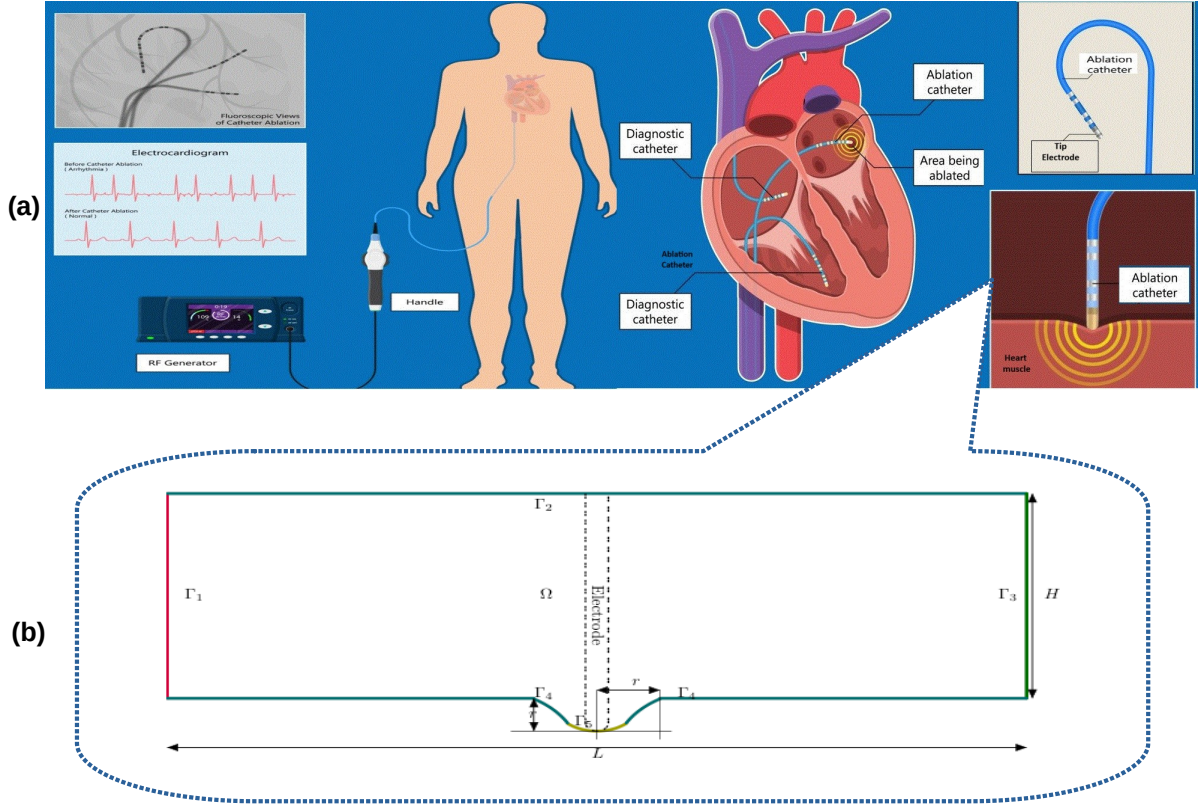


FIGURE 1. Configuration of the RFA procedure (a) and computational domain (b).

To the best of our knowledge, the mathematical analysis of models such as (1) was first introduced in [6], where the boundary condition on Σ_N is given by $\mathbb{S}(\mathbf{u}, P) \cdot \boldsymbol{\eta} := \nu(\theta)\mathbb{D}(\mathbf{u}) \cdot \boldsymbol{\eta} - P\mathbf{I} \cdot \boldsymbol{\eta} = 0$. The authors of this work focused on the existence and uniqueness of solutions in both two- and three-dimensional spaces. Further contributions to this field include the study in [7], which addresses a special case involving the elimination of dissipation viscosity ($\nu(\theta)\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{u}) = 0$) with different boundary conditions. In this study, the authors demonstrated the existence of solutions using the Faedo-Galerkin approach. In the context of the Navier-Stokes-heat coupling system, where $\varphi = 0$, there exists a significant number of theoretical works on the well-posedness of solution (existence and

uniqueness) under various data and boundary conditions. For instance, Beneš *et al.* [9] examined the impact of an external force in the heat equation that includes an energy dissipation term. They proved the existence of solutions in three-dimensional space using Galerkin’s method and Schauder’s fixed-point theorem. Additionally, Deteix *et al.* [13] explored the case with nonlinear viscosity and thermal conductivity dependent on temperature. They established the existence of solutions using Brouwer’s fixed-point theorem, without imposing restrictions on the data. Another significant contribution is found in [14], where the authors studied a variant of the coupling model, specifically the globally modified Navier-Stokes problem coupled with the heat equation in three-dimensional space. They employed Brouwer’s fixed-point theorem, the Faedo-Galerkin method, and compactness results to investigate the existence and uniqueness of solutions and assessed the stability of the time-discrete solution using the energy approach. Continuing in this direction, we turn to works related to heat-potential models, particularly those where the blood effect is neglected (i.e., $\mathbf{u} = 0$ and $P = 0$). Notable studies in this field include Allegretto and Xie [4], who demonstrated the existence of a solution using the maximum principle and a fixed-point argument. Cimatti [12] established the existence of a weak solution for an arbitrarily large time interval using the Faedo-Galerkin method. More recently, Li and Yang [23] proved the existence and uniqueness of solutions for the thermistor problem without nondegenerate assumptions.

In this study, we investigate the regularity of solutions to the system of equations (1) and establish the existence of solution, particularly in two-dimensional space. Our approach employs mathematical tools such as Schauder’s fixed-point theorem, truncated method and Sobolev’s integration theorems (see [2, 3]). A significant challenge in our analysis arises from the nonlinear terms in the equations, particularly the energy dissipation terms, which include the viscosity $\nu(\theta)\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{u})$ and the electric field $\lambda(\theta)\nabla\varphi \cdot \nabla\varphi$. These terms can belong to the L^1 space, complicating the analysis (see Remark 2.1). In contrast, the authors in [6], consider these terms to belong to the space L^2 due to the fact that the velocity \mathbf{u} and potential φ belong in the spaces $\mathbf{W}^{1,4}(\Omega)$ and $W^{1,4}(\Omega)$, respectively. In this paper, we address the challenge that the triplet $(\mathbf{u}, \theta, \varphi)$ belongs to the space $\mathbf{W}^{1,2}(\Omega) \times W^{1,q}(\Omega) \times W^{1,2}(\Omega)$, with $1 \leq q \leq \frac{d+2}{d+1}$. To address these challenges, we explore techniques from [5, 8, 10, 19], extending the results of previous studies on nonlinear PDE systems, especially when dealing with weakly regular second members and/or initial data. In particular, we applied the technique of approximation by truncated functions to these terms. Then, we construct a new system and we show that it admits a solution using the Schauder fixed point method (see Subsection 3.1). At this point of the proof, we face another challenge in analyzing the existence and uniqueness of solutions to the decoupled Navier-Stokes equations (see Remark 3.1), particularly due to the effect of nonlinear terms such as $\nabla \cdot (\mathbf{u} \otimes \mathbf{u})$. To resolve this issue, we modify the boundary condition on Σ_N to $\mathbf{u} \cdot \boldsymbol{\eta} = 0$. Subsequently, we show that the solution of the truncated problem is bounded and its blindness is independent of the truncation parameter (see Subsection 3.2). Using compactness arguments we conclude that the new system converges to the weak solution (see Subsection 3.3).

The remainder of this paper is organized as follows. In the next section, we introduce basic notations and some appropriate functional spaces, and formulate the problem within a variational framework. In Section 3, before demonstrating the existence of a solution, we first introduce assumptions on the parameters and boundary conditions. We then propose a new weak formulation with a truncated heat source. The existence of a solution to this modified problem is established using Schauder’s fixed-point theorem, along with some estimates, including the estimate for the temperature variable concerning the truncation parameter. Next, by adapting techniques intruding in [10], we derive another estimate for the temperature that is independent of the truncation parameter. With this estimate, we are able to pass to the limit as the truncation parameter approaches zero. In Section 4, we provide numerical experiments to illustrate the behavior and evolution of blood during practical ablation under weak regularity assumptions on the dissipative terms. Finally, Section 5 provides discussions and perspectives for further research on related problems.

2. MATHEMATICAL SETTINGS AND VARIATIONAL FORMULATION

We begin by introducing the main notations and results that will be used throughout this paper. The Lebesgue space $L^r(\Omega)$ is equipped with the norm $\|\cdot\|_{L^r(\Omega)}$ for an arbitrary $r \in [1, +\infty]$. We denote the Sobolev space by $W^{k,p}(\Omega)$, where $k \geq 0$ (not necessarily an integer) and $p \in [1, +\infty]$, and this space is equipped with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$. For convenience, we use the notation $\mathbf{W}^{k,p}$ to represent $W^{k,p}(\Omega)^d$ and \mathbf{L}^r to denote $L^r(\Omega)^d$. For functions depending on both space and time, given a space X of space-dependent functions, we define the spaces of functions from the interval $I := (0, t_f)$ into X as follows:

$$\begin{aligned} L^p(I; X) &:= \left\{ v : I \rightarrow X \mid v \text{ is measurable and } \int_0^{t_f} \|v(s)\|_X^p ds < \infty \right\}, \quad \text{if } p < \infty, \\ L^\infty(I; X) &:= \left\{ v : I \rightarrow X \mid \text{esssup}_{s \in (0, t_f)} \|v(s)\|_X < \infty \right\}, \quad \text{if } p = \infty. \end{aligned}$$

The corresponding norms are:

$$\begin{aligned} \|v\|_{L^p(I; X)} &:= \left(\int_0^{t_f} \|v(s)\|_X^p ds \right)^{1/p}, \quad \text{for } p \geq 1, \\ \|v\|_{L^\infty(I; X)} &:= \text{esssup}_{s \in (0, t_f)} \|v(s)\|_X. \end{aligned}$$

Let $\varphi \in W^{k,p}(\Omega)$, then we have the following embedding results (see [2] and [21] for more details):

$$\begin{aligned} W^{k,p}(\Omega) &\hookrightarrow L^q(\Omega), \quad \|\varphi\|_{L^q(\Omega)} \leq c \|\varphi\|_{W^{k,p}(\Omega)}, \quad p \leq q < \infty, \quad kp = d, \\ W^{k,p}(\Omega) &\hookrightarrow L^q(\Omega), \quad \|\varphi\|_{L^q(\Omega)} \leq c \|\varphi\|_{W^{k,p}(\Omega)}, \quad p \leq q \leq \frac{dp}{d-kp}, \quad kp < d, \\ W^{k,p}(\Omega) &\hookrightarrow L^\infty(\Omega), \quad \|\varphi\|_{L^\infty(\Omega)} \leq c \|\varphi\|_{W^{k,p}(\Omega)}, \quad kp > d. \end{aligned} \tag{2}$$

Next, for every $\varphi \in W^{k,p}(\Omega)$, there exists a continuous operator $\mathfrak{R} : W^{k,p}(\Omega) \rightarrow L^r(\partial\Omega)$ such that:

$$\|\mathfrak{R}(\varphi)\|_{L^r(\partial\Omega)} \leq c \|\varphi\|_{W^{k,p}(\Omega)} \quad \text{for } \begin{cases} 1 \leq kp < d, & r = \frac{dp-p}{d-kp}, \\ p \geq \max\{1, d/k\}, & r \in [1, \infty). \end{cases} \tag{3}$$

Additionally, for non-negative integers s, m , real numbers $1 \leq p, q, r \leq \infty$, and $\alpha \in [0, 1]$ such that:

$$\frac{1}{p} = \frac{s}{d} + \left(\frac{1}{r} - \frac{m}{d} \right) \alpha + \frac{1-\alpha}{q} \quad \text{and} \quad \frac{s}{m} \leq \alpha \leq 1,$$

then, the Gagliardo-Nirenberg interpolation inequality holds (see [25]):

$$\|\varphi\|_{W^{s,p}(\Omega)} \leq c \|\varphi\|_{W^{m,r}(\Omega)}^\alpha \|\varphi\|_{L^q(\Omega)}^{1-\alpha}, \quad \text{for all } \varphi \in L^q(\Omega) \cap W^{m,r}(\Omega). \tag{4}$$

Now, we define the following spaces:

$$\begin{aligned} \mathbf{V}_u^{k,p} &:= \overline{\left\{ \mathbf{u} \in C^\infty(\overline{\Omega}) \mid \nabla \cdot \mathbf{u} = 0, \text{supp}(\mathbf{u}) \cap \Gamma_D = \emptyset, \mathbf{u} \cdot \boldsymbol{\eta} = 0 \text{ on } \Gamma_N \right\}}^{\mathbf{W}^{k,p}(\Omega)}, \\ V_\varphi^{k,p} &:= \overline{\left\{ \varphi \in C^\infty(\overline{\Omega}) \mid \text{supp}(\varphi) \cap \Gamma_D = \emptyset \right\}}^{\mathbf{W}^{k,p}(\Omega)}, \\ V_\theta^{k,p} &:= \overline{\left\{ \theta \in C^\infty(\overline{\Omega}) \mid \text{supp}(\theta) \text{ is compact} \right\}}^{\mathbf{W}^{k,p}(\Omega)}. \end{aligned}$$

Note that $\mathbf{V}_\theta^{k,p}$, $V_\varphi^{k,p}$, and $\mathbf{V}_u^{k,p}$ are Banach spaces with the norms of $W^{k,p}(\Omega)$ and $\mathbf{W}^{k,p}(\Omega)$, respectively. Moreover, we define the followings spaces state:

$$\begin{aligned} \mathcal{H}_u &:= \left\{ \boldsymbol{\phi} \mid \boldsymbol{\phi} \in L^\infty(I; \mathbf{V}_u^{0,2}) \cap L^2(I; \mathbf{V}_u^{1,2}) \right\}, \\ \mathcal{H}_\theta &:= \left\{ \psi \mid \psi \in L^\infty(I; L^1(\Omega)) \cap L^q(I; V_\theta^{1,q}) \right\}, \quad \text{with } 1 \leq q \leq \frac{d+2}{d+1}, \\ \mathcal{H}_\varphi &:= \left\{ \chi \mid \chi \in L^2(I; V_\varphi^{1,2}) \right\}. \end{aligned}$$

The norms for these spaces are defined as follows:

$$\begin{aligned}\|\phi\|_{\mathcal{H}_u} &= \|\phi\|_{L^\infty(I; \mathbf{V}_u^{0,2})} + \|\phi\|_{L^2(I; \mathbf{V}_u^{1,2})}, \\ \|\psi\|_{\mathcal{H}_\theta} &= \|\psi\|_{L^\infty(I; L^1(\Omega))} + \|\psi\|_{L^q(I; \mathbf{V}_\theta^{1,q})}, \quad \text{with } 1 \leq q \leq \frac{d+2}{d+1}, \\ \|\chi\|_{\mathcal{H}_\varphi} &= \|\chi\|_{L^2(I; \mathbf{V}_\varphi^{1,2})}.\end{aligned}$$

To simplify mathematical formulations, we introduce the following notations, assuming that the functions \mathbf{u} , \mathbf{v} , \mathbf{w} , θ , ϕ , φ , χ , and ψ are sufficiently smooth such that the following integrals are well-defined:

$$\begin{aligned}(\mathbf{u}, \mathbf{v})_\Omega &= \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x}, & (\theta, \psi)_\Gamma &= \int_\Gamma \theta \psi \, d\mathbf{y}, \\ a_u(\theta; \mathbf{u}, \mathbf{v}) &= \int_\Omega \nu(\theta) \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, d\mathbf{x}, & a_\theta(\phi; \theta, \psi) &= \int_\Omega \gamma(\phi) \nabla \theta \cdot \nabla \psi \, d\mathbf{x}, \\ c_\varphi(\phi, \varphi, \psi) &= \int_\Omega \lambda(\phi) \nabla \varphi \cdot \nabla \varphi \psi \, d\mathbf{x}, & a_\varphi(\phi, \varphi, \chi) &= \int_\Omega \lambda(\phi) \nabla \varphi \cdot \nabla \chi \, d\mathbf{x}, \\ d(\mathbf{u}, \theta, \psi) &= \int_\Omega \mathbf{u} \cdot \nabla \theta \psi \, d\mathbf{x}, & e(\theta; \mathbf{u}, \mathbf{v}, \psi) &= \int_\Omega \nu(\theta) \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \psi \, d\mathbf{x}, \\ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_{\Gamma_N} (\mathbf{u} \otimes \mathbf{v}) : (\boldsymbol{\eta} \otimes \mathbf{w}) \, d\Gamma - \int_\Omega (\mathbf{u} \otimes \mathbf{v}) : \mathbb{D}(\mathbf{w}) \, d\mathbf{x}.\end{aligned}\tag{5}$$

By integration by parts, it is straightforward to verify that the trilinear form $b(\cdot, \cdot, \cdot)$ satisfies the skew-symmetry property, i.e., for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}_u^{1,2}$,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \text{and} \quad b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0.\tag{6}$$

The continuity of the trilinear form $b(\cdot, \cdot, \cdot)$ is given in the following lemma, where $C_b(t_f)$ is a positive constant that depends on the domain Ω and t_f , and may vary from line to line.

Lemma 2.1. *For all $(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_u^2$, $b(\mathbf{u}, \mathbf{v}, \cdot) \in L^1(I; \mathbf{V}_u^{-1,2})$, and there exists a positive constant $C_b(t_f)$ such that*

$$\|b(\mathbf{u}, \mathbf{v}, \cdot)\|_{L^1(I; \mathbf{V}_u^{-1,2})} \leq C_b(t_f) \|\mathbf{u}\|_{\mathcal{H}_u} \|\mathbf{v}\|_{\mathcal{H}_u}.\tag{7}$$

Proof. Firstly, we show that

$$\mathcal{H}_u \hookrightarrow L^{8/3}(I; \mathbf{W}^{3/4,2}).\tag{8}$$

Indeed, by the Gagliardo-Nirenberg inequality (cf. (4)), we get that

$$\|\phi(t)\|_{\mathbf{W}^{3/4,2}} \leq c \|\phi(t)\|_{\mathbf{W}^{1,2}}^{3/4} \|\phi(t)\|_{L^2}^{1/4},$$

integrating over the time interval I , and applying Young's inequality,

$$\begin{aligned}\left(\int_0^{t_f} \|\phi(t)\|_{\mathbf{W}^{3/4,2}}^{8/3} \, dt \right)^{3/8} &\leq c \left(\int_0^{t_f} \|\phi(t)\|_{\mathbf{W}^{1,2}}^2 \|\phi(t)\|_{L^2}^{2/3} \, dt \right)^{3/8} \\ &\leq c \|\phi\|_{L^2(I; \mathbf{W}^{1,2})}^{3/4} \|\phi\|_{L^\infty(I; L^2)}^{1/4} \\ &\leq c \|\phi\|_{\mathcal{H}_u},\end{aligned}$$

where $c = c(\Omega)$. Hence, we have $\mathcal{H}_u \hookrightarrow L^{8/3}(I; \mathbf{W}^{3/4,2})$. Moreover, by Hölder's inequality,

$$\begin{aligned}|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| &\leq \left| \int_{\Gamma_N} (\mathbf{u} \otimes \mathbf{v}) : (\boldsymbol{\eta} \otimes \mathbf{w}) \, d\mathbf{y} \right| + \left| \int_\Omega (\mathbf{u} \otimes \mathbf{v}) : \mathbb{D}(\mathbf{w}) \, d\mathbf{x} \right| \\ &\leq c \|\mathbf{u}\|_{L^4(\partial\Omega)} \|\mathbf{v}\|_{L^4(\partial\Omega)} \|\mathbf{w}\|_{L^2(\partial\Omega)} + c \|\mathbf{u}\|_{L^4} \|\mathbf{v}\|_{L^4} \|\mathbf{w}\|_{\mathbf{W}^{1,2}} \\ &\leq c \|\mathbf{u}\|_{\mathbf{W}^{3/4,2}} \|\mathbf{v}\|_{\mathbf{W}^{3/4,2}} \|\mathbf{w}\|_{\mathbf{W}^{1,2}},\end{aligned}$$

where we have used the embeddings (cf. (2)-(3)):

$$\mathbf{W}^{3/4,2}(\Omega) \hookrightarrow L^4(\partial\Omega), \quad \mathbf{W}^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega), \quad \mathbf{W}^{3/4,2}(\Omega) \hookrightarrow L^4(\Omega).$$

Hence, by Hölder inequality we get that

$$\begin{aligned} \|b(\mathbf{u}, \mathbf{v}, \cdot)\|_{L^1(I; \mathbf{V}_u^{-1,2})} &\leq c \left(\int_0^{t_f} \|\mathbf{u}\|_{\mathbf{W}^{3/4,2}} \|\mathbf{v}\|_{\mathbf{W}^{3/4,2}} dt \right) \\ &\leq c t_f^{1/4} \left(\int_0^{t_f} \|\mathbf{u}\|_{\mathbf{W}^{3/4,2}}^{8/3} dt \right)^{3/8} \left(\int_0^{t_f} \|\mathbf{v}\|_{\mathbf{W}^{3/4,2}}^{8/3} dt \right)^{3/8}. \end{aligned}$$

Together with (8), this yields (7). The proof is complete. \square

We denote by $C_c^1([0, t_f] \times \Omega)$ the set of all C^1 -functions with compact support in $[0, t_f] \times \Omega$. Let us recall the definition of the truncated function T_k and its primitive S_k . For $k \in \mathbb{R}^+$, define

$$T_k(z) = \begin{cases} -k & \text{if } z \leq -k, \\ z & \text{if } |z| \leq k, \\ k & \text{if } z \geq k \end{cases} \quad \text{and} \quad S_k(\zeta) = \int_0^\zeta T_k(s) ds. \quad (9)$$

Note that T_k and S_k are Lipschitz continuous functions, piecewise C^1 functions satisfying $|T_k(z)| \leq k$ and $|S_k(\zeta)| \leq k|\zeta|$. Additionally, $T_k(0) = 0$, $T_k(\theta) \in H^1(\Omega) \cap L^\infty(\Omega)$, and

$$\nabla T_k(\theta) = 1_{|\theta| \leq k} \nabla \theta = \begin{cases} \nabla \theta, & \text{if } |\theta| \leq k, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \nabla S_k(\theta) = T_k(\theta) \nabla \theta, \quad \forall \theta \in H^1(\Omega).$$

We mention that

$$0 \leq 2S_k(\zeta) = \zeta^2 1_{|\zeta| \leq k} + (2k|\zeta| - k^2) 1_{|\zeta| > k}, \quad (10)$$

where $1_{|\zeta| \leq k}$ is the characteristic function.

The existence and regularity of weak solutions for System (1) will be studied under the following assumptions:

Assumptions 2.1.

A1- The functions $\mathbf{F} = \mathbf{F}(\cdot)$, $\nu = \nu(\cdot)$, $\lambda = \lambda(\cdot)$, and $\gamma = \gamma(\cdot)$ are positive, bounded, and continuous with respect to the temperature. That is, we assume:

$$\begin{aligned} 0 &\leq F_i(\xi) \leq C_F < +\infty \quad \forall \xi \in \mathbb{R}, \quad \text{for } i = 1, \dots, d, \\ 0 &< \nu_1 \leq \nu(\xi) \leq \nu_2 < +\infty \quad \forall \xi \in \mathbb{R}, \\ 0 &< \lambda_1 \leq \lambda(\xi) \leq \lambda_2 < +\infty \quad \forall \xi \in \mathbb{R}, \\ 0 &< \gamma_1 \leq \gamma(\xi) \leq \gamma_2 < +\infty \quad \forall \xi \in \mathbb{R}, \end{aligned}$$

where C_F , ν_1 , ν_2 , λ_1 , λ_2 , γ_1 , and γ_2 are positive constants.

A2- The functions ν , λ , and γ are Lipschitz continuous. Specifically:

$$\begin{aligned} |\mathbf{F}(\zeta_1) - \mathbf{F}(\zeta_2)| &\leq L_F |\zeta_1 - \zeta_2| \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}, \\ |\nu(\zeta_1) - \nu(\zeta_2)| &\leq L_\nu |\zeta_1 - \zeta_2| \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}, \\ |\lambda(\zeta_1) - \lambda(\zeta_2)| &\leq L_\lambda |\zeta_1 - \zeta_2| \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}, \\ |\gamma(\zeta_1) - \gamma(\zeta_2)| &\leq L_\gamma |\zeta_1 - \zeta_2| \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}, \end{aligned} \quad (11)$$

where L_F , L_ν , L_λ , and L_γ are positive constants.

A3- The initial data satisfy:

$$\mathbf{u}_0 \in \mathbf{V}_u^{0,2} \quad \text{and} \quad \theta_0 \in L^1(\Omega).$$

A4- The remaining assumptions on the data are:

$$\alpha = \text{const} > 0, \quad \theta_t \in L^\infty(I; L^1(\Gamma)), \quad g \in L^2(I, L^2(\Gamma)).$$

Note that the variational formulation of (1) can be reformulated by using the test functions of the heat equation in the space $V_\theta^{1,q'} \subset W^{1,q'}(\Omega)$ with $1/q + 1/q' = 1$ and $1 \leq q \leq \frac{d+2}{d+1}$ (see Remark 2.1, below). Thus, the weak formulation reads as follows:

Definition 2.1. A triplet $(\mathbf{u}, \varphi, \theta)$ is called a variational solution of the problem (1) if:

$$\begin{aligned} \mathbf{u} &\in \mathcal{H}_{\mathbf{u}}, \quad \partial_t \mathbf{u} \in L^2(I; \mathbf{V}_{\mathbf{u}}^{-1,2}), \\ \theta &\in \mathcal{H}_{\theta}, \quad \partial_t \theta \in L^1\left(I; (W^{1,q}(\Omega))'\right) + L^1(\Omega_{t_f}), \\ \varphi &\in \mathcal{H}_{\varphi}, \end{aligned} \quad (12)$$

and satisfies the following identities for every $(\mathbf{v}, \psi, \chi) \in \mathbf{V}_{\mathbf{u}}^{1,2} \times V_{\theta}^{1,q'} \times V_{\varphi}^{1,2}$ and for almost every $t \in I$:

$$\langle \partial_t \mathbf{u}, \mathbf{v} \rangle + a_u(\theta; \mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\mathbf{F}(\theta), \mathbf{v}) = 0 \quad (13)$$

$$\langle \partial_t \theta, \psi \rangle + a_{\theta}(\theta; \theta, \psi) + d(\mathbf{u}, \theta, \psi) + \alpha(\theta - \theta_l, \psi)_{\Gamma} - e(\theta; \mathbf{u}, \mathbf{u}, \psi) - c_{\varphi}(\theta, \varphi, \psi) = 0 \quad (14)$$

$$a_{\varphi}(\theta; \varphi, \chi) - (g, \chi)_{\Gamma_N} = 0 \quad (15)$$

and

$$\begin{aligned} \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}) \quad \text{in } L^2(\Omega), \\ \theta(\mathbf{x}, 0) &= \theta_0(\mathbf{x}) \quad \text{in } L^1(\Omega). \end{aligned} \quad (16)$$

Remark 2.1. If the velocity \mathbf{u} and the potential φ belong to $\mathbf{V}_{\mathbf{u}}^{1,2} \subset H^1(\Omega)$ and $V_{\varphi}^{1,2} \subset H^1(\Omega)$, respectively, and given that the kinematic viscosity ν and the heat conductivity λ are bounded, then the quadratic terms $\lambda(\theta)|\nabla\varphi|^2$ and $\nu(\theta)\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{u})$ belong to $L^1(\Omega)$. These terms cannot be directly tested by functions in $V_{\theta}^{1,2} \subset H^1$. However, for $1 \leq q < \frac{d+2}{d+1}$, we have $q' = \frac{q}{q-1} > d$ (where q' denotes the conjugate exponent of q). By Sobolev's embedding (cf. (2)), the test function actually lies in $L^{\infty}(\Omega)$, so the terms e and c_{φ} in (14) are well-defined.

3. EXISTENCE OF THE WEAK SOLUTION

This section aims to demonstrate the existence of a weak solution to the system (1). We state the main result of this section as follows:

Theorem 3.1. Under the assumptions **A1**, **A2**, **A3** and **A4**, the problem (1) has a solution in the sense of Definition 2.1.

The proof of Theorem 3.1 is divided into several steps. First, we consider a weak problem derived from (13)-(16) by truncating the energy dissipation terms and approximating the data. We establish the existence of the solution for this truncated problem using Schauder's Fixed Point Theorem. Initial estimates are obtained for the solution, though estimates for θ depend on the truncation parameter. In the second step, we establish additional estimates for θ that are independent of the truncation parameter. Next, we prove a compactness property for the temperature and viscosity. Finally, by passing to the limit with respect to the truncation parameter, we obtain a solution to the original problem (13)-(16).

3.1. Approximation truncated problem. Let $k > 0$ be fixed and we consider the approximation of the data by

$$\left\{ \begin{array}{l} (\theta_{0,k})_{k>0} \subset L^2(\Omega), \mathbf{u}_{0,k} \in \mathbf{L}^2, (\theta_{l,k})_{k>0} \subset L^2(\Sigma_T) \text{ and } g_k \in L^2(\Sigma_T) \text{ such that:} \\ \theta_{0,k} \rightarrow \theta_0 \text{ in } L^1(\Omega), \mathbf{u}_{0,k} \rightarrow \mathbf{u}_0 \text{ in } L^2(\Omega), \theta_{l,k} \rightarrow \theta_l \text{ in } L^{\infty}(I; L^1(\Gamma)) \\ \quad \text{and } g_k \rightarrow g \text{ in } L^2(\Sigma_T) \text{ as } k \rightarrow \infty; \\ \|\theta_{0,k}\|_{L^1(\Omega)} \leq \|\theta_0\|_{L^1(\Omega)}, \|\mathbf{u}_{0,k}\|_{L^2(\Omega)} \leq \|\mathbf{u}_0\|_{L^2(\Omega)}, \|\theta_{l,k}\|_{L^1(\Sigma_T)} \leq \|\theta_l\|_{L^1(\Sigma_T)} \\ \quad \text{and } \|g_k\|_{L^2(\Sigma_T)} \leq \|g\|_{L^2(\Sigma_T)} \text{ for } k > 0. \end{array} \right. \quad (17)$$

Next, we consider the following weak truncated problem:

Find a triplet $(\mathbf{u}_k, \varphi_k, \theta_k)$ such that :

$$\begin{aligned} \mathbf{u}_k &\in L^\infty(I; \mathbf{L}^2(\Omega)) \cap L^2\left(I; \mathbf{V}_u^{1,2}\right), & \partial_t \mathbf{u}_k &\in L^2\left(I; \mathbf{V}_u^{-1,2}\right), \\ \theta_k &\in L^\infty(I; L^2(\Omega)) \cap L^2\left(I; V_\theta^{1,2}\right), & \partial_t \theta_k &\in L^2\left(I; V_\theta^{-1,2}\right), \\ \varphi_k &\in L^4\left(I; V_\varphi^{1,2}\right), \end{aligned}$$

and satisfies the following identities

$$\begin{aligned} \langle \partial_t \mathbf{u}_k, \mathbf{v} \rangle + a_u(\theta_k; \mathbf{u}_k, \mathbf{v}) + b(\mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) - (\mathbf{F}(\theta_k), \mathbf{v}) &= 0, \\ \langle \partial_t \theta_k, \psi \rangle + a_\theta(\theta_k; \theta_k, \psi) + d(\mathbf{u}_k, \theta_k, \psi) + \alpha(\theta_k, \psi)_\Gamma - \alpha(\theta_{l,k}, \psi)_\Gamma \\ - (T_k(\nu(\theta_k)\mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)), \psi)_\Omega - (T_k((\lambda(\theta_k)\nabla\varphi_k) \cdot \nabla\varphi_k), \psi)_\Omega &= 0, \\ a_\varphi(\theta_k; \varphi_k, \chi) - (g_k, \chi)_{\Gamma_N} &= 0, \end{aligned} \tag{18}$$

for every $(\mathbf{v}, \psi, \chi) \in \mathbf{V}_u^{1,2} \times V_\theta^{1,2} \times V_\varphi^{1,2}$ and for almost every $t \in I$ and

$$\begin{aligned} \mathbf{u}_k(\mathbf{x}, 0) &= \mathbf{u}_{0,k}(\mathbf{x}) & \text{in } L^2(\Omega), \\ \theta_k(\mathbf{x}, 0) &= \theta_{0,k}(\mathbf{x}) & \text{in } L^2(\Omega). \end{aligned} \tag{19}$$

Theorem 3.2. *Let assumptions **A1**, **A2**, and **A3** be satisfied, and consider the data approximations as given in (17). Then the problem (18)-(19) has a solution.*

The proof of Theorem 3.2 relies on a fixed-point argument, which is outlined in the following steps:

Step 1 - Mapping for Given Temperature: Let $\bar{\theta}_k$ be a fixed temperature in $L^2(I; L^2(\Omega))$, and the kinematic viscosity ν , right-hand side \mathbf{F} and electrical conductivity λ depend on $\bar{\theta}_k$. Then, in below subsection (cf. Subsection 3.1.1), we establish that the mapping $\mathcal{T}_1 : \bar{\theta}_k \mapsto (\mathbf{u}_k, \varphi_k)$ is well-defined from $L^2(I; L^2(\Omega))$ into $\mathcal{H}_u \times \mathcal{H}_\varphi$, where $\mathbf{u}_k \in \mathcal{H}_u$ and $\varphi_k \in \mathcal{H}_\varphi$ are the unique solutions to the decoupled Navier-Stokes equations (20) below and potential equation (21) below, respectively.

Step 2 - Mapping for Given Velocity and Potential: With $\mathbf{u}_k \in \mathcal{H}_u$ and $\varphi_k \in \mathcal{H}_\varphi$ known, we let $h_k := T_k(\nu(\bar{\theta}_k)\mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)) + T_k(\lambda(\bar{\theta}_k)\nabla\varphi_k \cdot \nabla\varphi_k) + \alpha(\theta_{l,k}, \cdot) \in L^2(\Omega_{t_f})$ and we chose the heat conductivity which depends on the fixed temperature $\bar{\theta}_k$. In Subsection 3.1.2 (below), we demonstrate that there exists a unique weak solution to the decoupled heat equation (34) (below), i.e., we show that the mapping $\mathcal{T}_2 : (\mathbf{u}_k, \varphi_k) \mapsto \theta_k$ is well-defined from $\mathcal{H}_u \times \mathcal{H}_\varphi$ into $Y := \left\{ \theta \in L^\infty(I; L^2(\Omega)) \cap L^2\left(I; V_\theta^{1,2}\right) \mid \partial_t \theta \in L^2\left(I; W^{-1,2}(\Omega)\right) \right\}$.

Step 3 - Application of Schauder's Fixed Point Theorem: Based on the preceding results, we define a fixed-point mapping $\varsigma \circ \mathcal{T}_2 \circ \mathcal{T}_1 : \bar{\theta}_k \mapsto \theta_k$, which maps $L^2(I; L^2(\Omega))$ into itself. Here, ς represents the compact injection from Y into $L^2(I; L^2(\Omega))$. As demonstrated in Subsection 3.1.3, this mapping is completely continuous and maps a ball, independent of the choice of $\bar{\theta}_k$, back into itself. By applying Schauder's fixed-point theorem, we establish the existence of a fixed point, which completes the proof.

3.1.1. Mapping for Given Temperature. In this subsection, we demonstrate that the mapping $\mathcal{T}_1 : \bar{\theta}_k \mapsto (\mathbf{u}_k, \varphi_k)$ is well-defined. To do this, we fix an arbitrary $\bar{\theta}_k \in L^2(I; L^2(\Omega))$ and seek the solutions \mathbf{u}_k and φ_k to the following systems:

$$\left\{ \begin{array}{ll} \partial_t \mathbf{u}_k - \nabla \cdot (\nu(\bar{\theta}_k)\mathbb{D}(\mathbf{u}_k)) + \nabla \cdot (\mathbf{u}_k \otimes \mathbf{u}_k) + \nabla \pi_k &= \mathbf{F}(\bar{\theta}_k) & \text{in } \Omega_{t_f}, \\ \nabla \cdot \mathbf{u}_k &= 0 & \text{in } \Omega_{t_f}, \\ \mathbf{u}_k &= \mathbf{0} & \text{on } \Sigma_D, \\ \mathbf{u}_k \cdot \boldsymbol{\eta} &= \mathbf{0} & \text{on } \Sigma_N, \\ \mathbf{u}_k(\mathbf{x}, 0) &= \mathbf{u}_{0,k}(\mathbf{x}) & \text{in } \Omega, \end{array} \right. \tag{20}$$

and

$$\begin{cases} -\nabla \cdot (\lambda(\bar{\theta}_k) \nabla \varphi_k) = 0 & \text{in } \Omega_{t_f}, \\ (\lambda(\bar{\theta}_k) \nabla \varphi_k) \cdot \boldsymbol{\eta} = g_k & \text{on } \Sigma_N, \\ \varphi_k = 0 & \text{on } \Sigma_D. \end{cases} \quad (21)$$

Observe that for any fixed $\bar{\theta}_k \in L^2(I; L^2(\Omega))$, the problem (20) reduces to a pure Navier-Stokes equation. Consequently, we have the following well-posedness result:

Theorem 3.3. *Let $\bar{\theta}_k \in L^2(I; L^2(\Omega))$. Under Assumptions 3.1 (A1 and A3), there exists a unique function $\mathbf{u}_k \in \mathcal{H}_u$ with $\partial_t \mathbf{u}_k \in L^2(I; \mathbf{V}_u^{-1,2})$ that satisfies*

$$\langle \partial_t \mathbf{u}_k, \mathbf{v} \rangle + a_u(\bar{\theta}_k; \mathbf{u}_k, \mathbf{v}) + b(\mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) = (\mathbf{F}(\bar{\theta}_k), \mathbf{v}), \quad (22)$$

for all $\mathbf{v} \in \mathbf{V}_u^{1,2}$ and almost every $t \in I$. Additionally, there exists a constant $C > 0$ independent of $\bar{\theta}_k$ and k such that

$$\|\mathbf{u}_k\|_{\mathcal{H}_u} \leq C \left(\|\mathbf{F}(\bar{\theta}_k)\|_{L^2(I; L^2)} + \|\mathbf{u}_0\|_{L^2} \right). \quad (23)$$

Proof. The proof of this result is based on the Faedo-Galerkin method. We refer the interested reader to [28] for more details. \square

Remark 3.1. *The inertia term $b(\mathbf{u}_k, \mathbf{u}_k, \cdot) = (\mathbf{u}_k \cdot \nabla) \mathbf{u}_k$ introduces significant challenges in the analysis. Specifically, in three dimensions (3D), the power of the H^1 norm in the estimation is higher than in two dimensions (2D), leading to particular difficulties with the uniqueness of weak solutions in 3D. Applying Gronwall's inequality necessitates that $\mathbf{u}_k \in L^{\frac{4}{4-d}}(I; \mathbf{H}^1)$. While this condition is met for $d = 2$, it remains an open problem for $d = 3$.*

For the decoupled potential problem (21), we establish the following result:

Theorem 3.4. *Given $\bar{\theta}_k \in L^2(I; L^2(\Omega))$ and $g_k \in L^2(I, L^2(\Gamma))$, there exists a unique function $\varphi_k \in \mathcal{H}_\varphi$ that solves the problem (21), satisfying*

$$a_{\varphi_k}(\bar{\theta}_k, \varphi_k, \chi) = (g_k, \chi)_{\Gamma_N}, \quad (24)$$

for every $\chi \in V_{\varphi_k}^{1,2}$ and almost every $t \in I$. Moreover, there exists a constant $C' > 0$ independent of $\bar{\theta}_k$ and k such that

$$\|\varphi_k\|_{\mathcal{H}_\varphi} \leq C' \|g_k\|_{L^2(I, L^2(\Gamma))}. \quad (25)$$

Proof. The proof follows from the application of the Lax-Milgram Theorem. \square

By Theorem 3.3 and Theorem 3.4, the mapping $\mathcal{T}_1 : \bar{\theta}_k \mapsto (\mathbf{u}_k, \varphi_k)$ is well-defined from $L^2(I; L^2(\Omega))$ into $\mathcal{H}_u \times \mathcal{H}_\varphi$, where $\mathbf{u}_k \in \mathcal{H}_u$ and $\varphi_k \in \mathcal{H}_\varphi$ are the unique solutions to the problems (22) and (24), respectively.

Lemma 3.1. *The mapping $\mathcal{T}_1 : \bar{\theta}_k \mapsto (\mathbf{u}_k, \varphi_k)$ is continuous from $L^2(I; L^2(\Omega))$ into $\mathcal{H}_u \times \mathcal{H}_\varphi$.*

Proof. Let $\bar{\theta}_k, \bar{\theta}_{k,n} \in L^2(I; L^2(\Omega))$ such that $\bar{\theta}_{k,n} \rightarrow \bar{\theta}_k$ in $L^2(I; L^2(\Omega))$ as $n \rightarrow \infty$. Let $\varphi_k, \varphi_{k,n} \in \mathcal{H}_\varphi$ and $\mathbf{u}_k, \mathbf{u}_{k,n} \in \mathcal{H}_u$ with $\partial_t \mathbf{u}_k, \partial_t \mathbf{u}_{k,n} \in L^1(I; \mathbf{V}_u^{-1,2})$ such that

$$\begin{aligned} \langle \partial_t \mathbf{u}_k, \mathbf{v} \rangle + a_u(\bar{\theta}_k; \mathbf{u}_k, \mathbf{v}) + b(\mathbf{u}_k, \mathbf{u}_k, \mathbf{v}) &= (\mathbf{F}(\bar{\theta}_k), \mathbf{v}), \\ \langle \partial_t \mathbf{u}_{k,n}, \mathbf{v} \rangle + a_u(\bar{\theta}_{k,n}; \mathbf{u}_{k,n}, \mathbf{v}) + b(\mathbf{u}_{k,n}, \mathbf{u}_{k,n}, \mathbf{v}) &= (\mathbf{F}(\bar{\theta}_{k,n}), \mathbf{v}), \end{aligned} \quad (26)$$

and

$$\begin{aligned} a_\varphi(\bar{\theta}_k, \varphi_k, \chi) &= \langle g_k, \chi \rangle_{\Gamma_N}, \\ a_\varphi(\bar{\theta}_{k,n}, \varphi_{k,n}, \chi) &= \langle g_k, \chi \rangle_{\Gamma_N}, \end{aligned} \quad (27)$$

for every $\mathbf{v} \in \mathbf{V}_u^{1,2}$ and $\chi \in V_\varphi^{1,2}$, and for almost every $t \in I$, with

$$\mathbf{u}_k(\mathbf{x}, 0) = \mathbf{u}_{k,0}(\mathbf{x}), \quad \mathbf{u}_{k,n}(\mathbf{x}, 0) = \mathbf{u}_{k,0}(\mathbf{x}) \quad \text{in } \Omega.$$

The continuity of the map $\mathcal{T}_1 : L^2(I; L^2(\Omega)) \rightarrow \mathcal{H}_u \times \mathcal{H}_\varphi$ is equivalent to proving that:

$$\text{If } \bar{\theta}_{k,n} \rightarrow \bar{\theta}_k \text{ in } L^2(I; L^2(\Omega)), \text{ as } n \rightarrow \infty, \text{ then} \quad (28)$$

$$\mathbf{u}_{k,n} \rightarrow \mathbf{u}_k \text{ in } \mathcal{H}_u \text{ and } \varphi_{k,n} \rightarrow \varphi_k \text{ in } L^2(I; H^1), \text{ as } n \rightarrow \infty.$$

First, we prove that $\mathbf{u}_{k,n} \rightarrow \mathbf{u}_k$, as $n \rightarrow \infty$. Denote the difference $\mathbf{z}_{k,n} = \mathbf{u}_k - \mathbf{u}_{k,n}$. Subtracting the equations in (26), we arrive at

$$\begin{aligned} & \langle \partial_t \mathbf{z}_{k,n}, \mathbf{v} \rangle + a_u(\bar{\theta}_k; \mathbf{z}_{k,n}, \mathbf{v}) \\ &= - \int_{\Omega} [\nu(\bar{\theta}_k) - \nu(\bar{\theta}_{k,n})] \mathbb{D}(\mathbf{u}_{k,n}) : \mathbb{D}(\mathbf{v}) \, d\mathbf{x} - b(\mathbf{z}_{k,n}, \mathbf{u}_{k,n}, \mathbf{v}) - b(\mathbf{u}_k, \mathbf{z}_{k,n}, \mathbf{v}) \\ & \quad + (\mathbf{F}(\bar{\theta}_k) - \mathbf{F}(\bar{\theta}_{k,n}), \mathbf{v}). \end{aligned} \quad (29)$$

Using $\mathbf{v} = \mathbf{z}_{k,n}(t)$ as a test function in (29), we obtain the following inequality:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{z}_{k,n}(t)\|_{\mathbf{L}^2}^2 + \nu_1 \|\mathbf{z}_{k,n}(t)\|_{\mathbf{V}_u^{1,2}}^2 \\ & \leq c \|\mathbf{z}_{k,n}(t)\|_{\mathbf{V}_u^{1,2}}^{2\zeta} \|\mathbf{z}_{k,n}(t)\|_{\mathbf{L}^2}^{2(1-\zeta)} \|\mathbf{u}_{k,n}(t)\|_{\mathbf{V}_u^{1,2}} + c \|\mathbf{F}(\bar{\theta}_k) - \mathbf{F}(\bar{\theta}_{k,n})\|_{\mathbf{L}^2} \|\mathbf{z}_{k,n}\|_{\mathbf{W}^{1,2}} \\ & \quad + \|(\nu(\bar{\theta}_k) - \nu(\bar{\theta}_{k,n}))\mathbb{D}(\mathbf{u}_{k,n})\|_{\mathbf{L}^2} \|\mathbb{D}(\mathbf{z}_{k,n})\|_{\mathbf{L}^2}. \end{aligned}$$

Thus,

$$\frac{d}{dt} \|\mathbf{z}_{k,n}(t)\|_{\mathbf{L}^2}^2 + \nu_1 \|\mathbf{z}_{k,n}(t)\|_{\mathbf{V}_u^{1,2}}^2 \leq R_n(t) + \beta_n(t) \|\mathbf{z}_{k,n}(t)\|_{\mathbf{L}^2}^2, \quad (30)$$

where

$$\begin{aligned} \beta_n(t) &= C_1(\nu_1) \|\mathbf{u}_{k,n}(t)\|_{\mathbf{W}^{1,2}}^{\frac{1}{1-\zeta}}, \\ R_n(t) &= C_2(\nu_1) \left[\|\mathbf{F}(\bar{\theta}_k) - \mathbf{F}(\bar{\theta}_{k,n})\|_{\mathbf{L}^2}^2 + \|(\nu(\bar{\theta}_k) - \nu(\bar{\theta}_{k,n}))\mathbb{D}(\mathbf{u}_{k,n})\|_{\mathbf{L}^2}^2 \right] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Applying Gronwall's lemma, we deduce that

$$\|\mathbf{z}_{k,n}(t)\|_{\mathbf{L}^2}^2 + \int_0^t \|\mathbf{z}_{k,n}(t)\|_{\mathbf{V}_u^{1,2}}^2 \, dt \leq C(T) \left[\int_0^T R_n(t) \, dt \right].$$

Hence, $\mathbf{u}_{k,n} \rightarrow \mathbf{u}_k$ in \mathcal{H}_u .

Next, we show that $\varphi_{k,n} \rightarrow \varphi_k$, as $n \rightarrow \infty$. Let us denote the difference by $\omega_{k,n} = \varphi_k - \varphi_{k,n}$. Subtracting the equations in (27), we obtain

$$\int_{\Omega} \lambda(\bar{\theta}_k) \nabla \omega_{k,n} \cdot \nabla \chi \, d\mathbf{x} = \int_{\Omega} [\lambda(\bar{\theta}_{k,n}) - \lambda(\bar{\theta}_k)] \nabla \varphi_{k,n} \cdot \nabla \chi \, d\mathbf{x}. \quad (31)$$

By setting $\chi = \omega_{k,n}$ in (31), we obtain the inequality

$$\lambda_1 \|\nabla \omega_{k,n}\|_{\mathbf{L}^2} \leq \|[\lambda(\bar{\theta}_{k,n}) - \lambda(\bar{\theta}_k)] \nabla \varphi_{k,n}\|_{\mathbf{L}^2}. \quad (32)$$

According to the Poincaré inequality, there exists a constant $c > 0$ such that

$$\|\omega_{k,n}\|_{H^1} \leq c \|[\lambda(\bar{\theta}_{k,n}) - \lambda(\bar{\theta}_k)] \nabla \varphi_{k,n}\|_{\mathbf{L}^2} \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (33)$$

Here, since $\bar{\theta}_{k,n} \rightarrow \bar{\theta}_k$ in $L^2(I; L^2(\Omega))$, as $n \rightarrow \infty$, the term on the right-hand side of (33) converges to zero. This follows from the application of the Lebesgue dominated convergence theorem. This completes the proof of (28). \square

3.1.2. *Mapping for Given Velocity and Potential.* In this subsection, as part of the proof of Theorem 3.2, we establish that the mapping $\mathcal{T}_2 : (\mathbf{u}_k, \varphi_k) \mapsto \theta_k$ is well-defined from $\mathcal{H}_{\mathbf{u}} \times \mathcal{H}_{\varphi}$ into Y , where the space Y is defined as

$$Y := \{\theta \in L^\infty(I; L^2(\Omega)) \cap L^2(I; V_\theta^{1,2}) \text{ with } \partial_t \theta \in L^2(I; W^{-1,2}(\Omega))\}.$$

Given $\bar{\theta}_k \in L^2(I; L^2(\Omega))$, by Subsection 3.1.1, we can define $\mathcal{T}_1(\bar{\theta}_k) = (\mathbf{u}_k, \varphi_k) \in \mathcal{H}_{\mathbf{u}} \times \mathcal{H}_{\varphi}$, where $\mathbf{u}_k \in \mathcal{H}_{\mathbf{u}}$ and $\varphi_k \in \mathcal{H}_{\varphi}$ are the unique solutions to (20) and (21), respectively. We then consider the linear heat equation:

$$\begin{cases} \partial_t \theta_k - \nabla \cdot (\gamma(\bar{\theta}_k) \nabla \theta_k) + \mathbf{u}_k \cdot \nabla \theta_k = h_k & \text{in } \Omega_{t_f}, \\ (\gamma(\bar{\theta}_k) \nabla \theta_k) \cdot \boldsymbol{\eta} + \alpha \theta_k = \alpha \theta_{l,k} & \text{on } \Sigma, \\ \theta_k(\cdot, 0) = \theta_{0,k} & \text{in } \Omega, \end{cases} \quad (34)$$

where $h_k = T_k(\nu(\bar{\theta}_k) \mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)) + T_k((\lambda(\bar{\theta}_k) \nabla \varphi_k) \cdot \nabla \varphi_k)$.

We then have the following results:

Theorem 3.5. *Let $\bar{\theta}_k \in L^2(I; L^2(\Omega))$, $\theta_{0,k} \in L^2(\Omega)$, and $\theta_{l,k} \in L^2(I; V_\theta^{0,2})$ be given functions. Then there exists a uniquely determined function $\theta_k \in L^\infty(I; L^2(\Omega)) \cap L^2(I; V_\theta^{1,2})$ with $\partial_t \theta_k \in L^2(I; W^{-1,2}(\Omega))$ such that*

$$\begin{aligned} & \langle \partial_t \theta_k, \psi \rangle + a_\theta(\bar{\theta}_k; \theta_k, \psi) + d(\mathbf{u}_k, \theta_k, \psi) + \alpha(\theta_k, \psi)_\Gamma \\ & = (T_k(\nu(\bar{\theta}_k) \mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)), \psi)_\Omega + (T_k((\lambda(\bar{\theta}_k) \nabla \varphi_k) \cdot \nabla \varphi_k), \psi)_\Omega + \alpha(\theta_{l,k}, \psi)_\Gamma, \end{aligned} \quad (35)$$

for every $\psi \in V_\theta^{1,2}$ and almost every $t \in I$,

$$\theta_k(\mathbf{x}, 0) = \theta_{0,k}(\mathbf{x}) \quad \text{in } \Omega. \quad (36)$$

Moreover, there exists a positive constant C_k , dependent on k , such that

$$\|\theta_k\|_{L^\infty(I; L^2(\Omega))} + \|\theta_k\|_{L^2(I; H^1(\Omega))} \leq C_k. \quad (37)$$

Proof. Using classical results from [22, 24], we obtain the well-posedness of the decoupled heat equation (35)-(36). By choosing $\psi = \theta_k$ in (35), and applying Assumptions 2.1, along with the estimates (23) and (25), we derive the bound given in (37). \square

Lemma 3.2. *Let ς be the injection $Y \hookrightarrow L^2(I; L^2)$. The mapping $\varsigma \circ \mathcal{T}_2 \circ \mathcal{T}_1$ is continuous and compact from $L^2(I; L^2)$ into itself for the strong topology.*

Proof. Let $\bar{\theta}_k, \bar{\theta}_{k,n} \in L^2(I; L^2(\Omega))$ such that $\bar{\theta}_{k,n} \rightarrow \bar{\theta}_k$ in $L^2(I; L^2(\Omega))$ as $n \rightarrow \infty$. By Theorem 3.3 and Theorem 3.4, there exist $\varphi_k, \varphi_{k,n} \in \mathcal{H}_{\varphi}$ and $\mathbf{u}_k, \mathbf{u}_{k,n} \in \mathcal{H}_{\mathbf{u}}$ satisfying (26) and (27), respectively. Moreover, by the continuity of \mathcal{T}_1 , we have that

$$\begin{aligned} \mathbf{u}_{k,n} & \longrightarrow \mathbf{u}_k & \text{in } \mathcal{H}_{\mathbf{u}}, \\ \varphi_{k,n} & \longrightarrow \varphi_k & \text{in } \mathcal{H}_{\varphi}, \end{aligned}$$

as $n \rightarrow \infty$. From Theorem 3.5, we conclude the existence of the unique solutions $\theta_k = \mathcal{T}_2(\mathbf{u}_k, \varphi_k)$ and $\theta_{k,n} = \mathcal{T}_2(\mathbf{u}_{k,n}, \varphi_{k,n})$ in $L^\infty(I; L^2(\Omega)) \cap L^2(I; V_\theta^{1,2})$ with $\partial_t \theta_k, \partial_t \theta_{k,n} \in L^2(I; W^{-1,2})$, such that

$$\begin{aligned} & \langle \partial_t \theta_k, \psi \rangle + a_\theta(\bar{\theta}_k; \theta_k, \psi) + d(\mathbf{u}_k, \theta_k, \psi) + \alpha(\theta_k, \psi)_\Gamma \\ & = (T_k(\nu(\bar{\theta}_k) \mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)), \psi)_\Omega + (T_k((\lambda(\bar{\theta}_k) \nabla \varphi_k) \cdot \nabla \varphi_k), \psi)_\Omega + \alpha(\theta_{l,k}, \psi)_\Gamma, \end{aligned} \quad (38)$$

$$\begin{aligned} & \langle \partial_t \theta_{k,n}, \psi \rangle + a_\theta(\bar{\theta}_{k,n}; \theta_{k,n}, \psi) + d(\mathbf{u}_{k,n}, \theta_{k,n}, \psi) + \alpha(\theta_{k,n}, \psi)_\Gamma \\ & = (T_k(\nu(\bar{\theta}_{k,n}) \mathbb{D}(\mathbf{u}_{k,n}) : \mathbb{D}(\mathbf{u}_{k,n})), \psi)_\Omega + (T_k((\lambda(\bar{\theta}_{k,n}) \nabla \varphi_{k,n}) \cdot \nabla \varphi_{k,n}), \psi)_\Omega + \alpha(\theta_{l,k}, \psi)_\Gamma, \end{aligned}$$

for every $\psi \in V_\theta^{1,2}$ and almost every $t \in I$, with initial conditions

$$\theta_k(x, 0) = \theta_0(x), \quad \theta_{k,n}(x, 0) = \theta_0^n(x) \quad \text{in } \Omega.$$

Now, we show the continuity of the map $\varsigma \circ \mathcal{T}_2 : \mathcal{H}_u \times \mathcal{H}_\varphi \rightarrow L^2(I; L^2(\Omega))$, i.e.,

$$\begin{cases} \mathbf{u}_{k,n} & \longrightarrow \mathbf{u}_k \text{ in } \mathcal{H}_u, \\ \varphi_{k,n} & \longrightarrow \varphi_k \text{ in } \mathcal{H}_\varphi, \end{cases} \implies \theta_{k,n} \longrightarrow \theta_k \text{ in } L^2(\Omega_{t_f}) \text{ as } \bar{\theta}_{k,n} \longrightarrow \bar{\theta}_k \text{ in } L^2(\Omega_{t_f}).$$

Denote the differences $\vartheta_{k,n} = \theta_k - \theta_{k,n}$ and $\mathbf{z}_{k,n} = \mathbf{u}_{k,n} - \mathbf{u}_k$. Subtracting the equations in (38), we arrive at (for all $\psi \in V_\theta^{1,2}$)

$$\begin{aligned} & \langle \partial_t \vartheta_{k,n}, \psi \rangle + a_\theta(\bar{\theta}_k; \vartheta_{k,n}, \psi) \\ &= -\alpha(\vartheta_{k,n}, \psi)_\Gamma - \int_\Omega [\gamma(\bar{\theta}_k) - \gamma(\bar{\theta}_{k,n})] \nabla \theta_{k,n} \cdot \nabla \psi \, d\mathbf{x} - d(\mathbf{z}_{k,n}, \theta_k, \psi) - d(\mathbf{u}_{k,n}, \vartheta_{k,n}, \psi) \\ &+ (T_k(\nu(\bar{\theta}_k)\mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)), \psi)_\Omega - (T_k(\nu(\bar{\theta}_{k,n})\mathbb{D}(\mathbf{u}_{k,n}) : \mathbb{D}(\mathbf{u}_{k,n})), \psi)_\Omega \\ &+ (T_k((\lambda(\bar{\theta}_k)\nabla\varphi_k) \cdot \nabla\varphi_k), \psi)_\Omega - (T_k((\lambda(\bar{\theta}_{k,n})\nabla\varphi_{k,n}) \cdot \nabla\varphi_{k,n}), \psi)_\Omega. \end{aligned} \quad (39)$$

Now, choosing $\psi = \vartheta_{k,n}$ in the previous equation, we estimate the right-hand side terms:

$$\begin{aligned} & \left| \int_\Omega [\gamma(\bar{\theta}) - \gamma(\bar{\theta}_{k,n})] \nabla \theta_{k,n} \cdot \nabla \vartheta_{k,n} \, d\mathbf{x} \right| \\ & \leq \left\| [\gamma(\bar{\theta}_k) - \gamma(\bar{\theta}_{k,n})] \nabla \theta_{k,n} \right\|_{L^2} \|\nabla \vartheta_{k,n}\|_{L^2} \\ & \leq C \|\bar{\theta}_k - \bar{\theta}_{k,n}\|_{L^2} \|\nabla \vartheta_{k,n}\|_{L^2} \\ & \leq \delta \|\nabla \vartheta_{k,n}\|_{L^2}^2 + C(\delta) \|\bar{\theta}_k - \bar{\theta}_{k,n}\|_{L^2}^2, \end{aligned} \quad (40)$$

$$\begin{aligned} |d(\mathbf{z}_{k,n}, \theta_k, \vartheta_{k,n})| &= \left| \int_\Omega \mathbf{z}_{k,n} \cdot \nabla \theta_k \vartheta_{k,n} \, d\mathbf{x} \right| \\ & \leq \|\mathbf{u}_{k,n} - \mathbf{u}_k\|_{L^4} \|\nabla \theta_k\|_{L^2} \|\vartheta_{k,n}\|_{L^4} \\ & \leq C \|\mathbf{u}_{k,n} - \mathbf{u}_k\|_{L^4} \|\vartheta_{k,n}\|_{V_\theta^{1,2}} \\ & \leq \delta \|\vartheta_{k,n}\|_{V_\theta^{1,2}}^2 + C(\delta) \|\mathbf{u}_{k,n} - \mathbf{u}_k\|_{L^4}^2, \end{aligned} \quad (41)$$

$$\begin{aligned} d(\mathbf{u}_{k,n}, \vartheta_{k,n}, \vartheta_{k,n}) &= \int_\Omega \mathbf{u}_{k,n} \cdot \nabla \vartheta_{k,n} \vartheta_{k,n} \, d\mathbf{x} \\ &= -\frac{1}{2} \int_\Omega \nabla \cdot \mathbf{u}_{k,n} \nabla (\vartheta_{k,n}^2) \, d\mathbf{x} + \frac{1}{2} \int_\Gamma \mathbf{u}_{k,n} \cdot \boldsymbol{\eta} \vartheta_{k,n}^2 \, d\mathbf{y} \\ &= 0, \end{aligned} \quad (42)$$

$$\begin{aligned} & |(T_k(\nu(\bar{\theta}_k)\mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)), \vartheta_{k,n})_\Omega - (T_k(\nu(\bar{\theta}_{k,n})\mathbb{D}(\mathbf{u}_{k,n}) : \mathbb{D}(\mathbf{u}_{k,n})), \vartheta_{k,n})_\Omega| \\ & \leq |(T_k(\nu(\bar{\theta}_k)\mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)), \vartheta_{k,n})_\Omega| + |(T_k(\nu(\bar{\theta}_{k,n})\mathbb{D}(\mathbf{u}_{k,n}) : \mathbb{D}(\mathbf{u}_{k,n})), \vartheta_{k,n})_\Omega| \\ & \leq 2k \, \text{meas}(\Omega)^{1/2} \|\vartheta_{k,n}\|_{L^2}, \end{aligned} \quad (43)$$

and

$$\begin{aligned} & |(T_k(\lambda(\bar{\theta}_k)\nabla\varphi_k) \cdot \nabla\varphi_k, \vartheta_{k,n})_\Omega - (T_k(\lambda(\bar{\theta}_{k,n})\nabla\varphi_{k,n}) \cdot \nabla\varphi_{k,n}, \vartheta_{k,n})_\Omega| \\ & \leq 2k \, \text{meas}(\Omega)^{1/2} \|\vartheta_{k,n}\|_{L^2}. \end{aligned} \quad (44)$$

Combining (40)-(44) and integrating (39) in time over $(0, t)$, for all $t \in I$, we get

$$\begin{aligned} & \|\vartheta_{k,n}(t)\|_{L^2}^2 + C_1 \int_0^t \|\vartheta_{k,n}(t)\|_{V_\theta^{1,2}}^2 \, dt \\ & \leq \|\vartheta_{k,n}(0)\|_{L^2}^2 + C_2 \int_0^t \|\vartheta_{k,n}\|_{L^2}^2 \, dt \\ & \quad + C_3 \int_0^t \left(\|\bar{\theta}_k - \bar{\theta}_{k,n}\|_{L^2}^2 + \|\mathbf{u}_{k,n} - \mathbf{u}_k\|_{\mathbf{H}^1}^2 \right) dt. \end{aligned}$$

Here, C_1 , C_2 and C_3 denote positive constants that are independent of n . Since $\bar{\theta}_{k,n} \rightarrow \bar{\theta}_k$ in $L^2(I; L^2)$ and $\mathbf{u}_{k,n} \rightarrow \mathbf{u}_k$ in \mathcal{H}_u , the right-hand side terms of the inequalities tend to zero as

$n \rightarrow \infty$. Thus, by Gronwall's inequality we get that

$$\begin{aligned} \|\vartheta_{k,n}(t)\|_{L^2}^2 &\rightarrow 0, \quad \forall t \in I \text{ as } \bar{\theta}_{k,n} \rightarrow \bar{\theta}_k \text{ in } L^2(\Omega_{t_f}), \\ \int_I \|\vartheta_{k,n}(t)\|_{H^1}^2 dt &\rightarrow 0, \text{ as } \bar{\theta}_{k,n} \rightarrow \bar{\theta}_k \text{ in } L^2(\Omega_{t_f}). \end{aligned}$$

This implies that $\vartheta_{k,n} \rightarrow 0$ in $L^2(I; L^2(\Omega))$ as $n \rightarrow \infty$, showing that $\theta_{k,n} \rightarrow \theta_k$ in $L^2(I; L^2(\Omega))$. Therefore, the mapping $\varsigma \circ \mathcal{T}_2 \circ \mathcal{T}_1$ is continuous and compact from $L^2(I; L^2)$ into itself, which concludes the proof of the lemma. \square

3.1.3. Application of Schauder's Fixed Point Theorem. To prove Theorem 3.1, we use Schauder's Fixed Point Theorem, stated as follows:

Theorem 3.6 (Schauder Fixed Point Theorem). *Let S be a closed convex set in a Banach space E , and let \mathcal{T} be a continuous mapping of S into itself such that the image $\mathcal{T}(S)$ is precompact. Then \mathcal{T} has a fixed point.*

The existence of a solution to the truncated problem (18)-(19) can be demonstrated using Theorem 3.6. We construct the fixed-point mapping \mathcal{T} based on the results established in previous sections. Specifically, we define \mathcal{T} as follows:

$$\begin{aligned} \mathcal{T} : L^2(I; L^2(\Omega)) &\rightarrow L^2(I; L^2(\Omega)) \\ \bar{\theta}_k &\mapsto \mathcal{T}(\bar{\theta}_k) = \varsigma \circ \mathcal{T}_2 \circ \mathcal{T}_1(\bar{\theta}_k). \end{aligned} \quad (45)$$

For $\bar{\theta}_k \in L^2(I; L^2(\Omega))$, Theorems 3.3 and 3.4 ensure that there exists a unique solution $(\mathbf{u}_k, \varphi_k) \in \mathcal{H}_{\mathbf{u}} \times \mathcal{H}_{\varphi}$ to the problem (20)-(21), and that this solution is bounded independently of the truncated parameter (cf. (23) and (25)). Furthermore, the solution θ_k of the problem (34) is uniquely determined by Theorem 3.5. By the a priori estimate (37), $\theta_k = \mathcal{T}(\bar{\theta}_k)$ is bounded in $L^2(I; L^2)$, independent of $\bar{\theta}_k$. Thus, we can find a fixed ball $B_k \subset L^2(I; L^2(\Omega))$ defined by

$$B_k := \{\theta \in L^2(I; L^2(\Omega)) \mid \|\theta\|_{L^2(I; L^2(\Omega))} \leq C_k\}, \quad (46)$$

where $C_k > 0$ is sufficiently large. It follows that $\mathcal{T}(B_k) \subset B_k$, and the operator $\mathcal{T} : L^2(I; L^2(\Omega)) \rightarrow L^2(I; L^2(\Omega))$ is completely continuous and compact (cf. Lemma 3.2). Since \mathcal{T} satisfies the hypotheses of Theorem 3.6, we conclude that there exists a fixed point of \mathcal{T} . Consequently, there exists a solution $(\mathbf{u}_k, \theta_k, \varphi_k)$ of the truncated problem (18)-(19).

3.2. Estimates Independent of the Truncation Parameter. First of all, let $C > 0$ be a positive constant independent of k , although its value may vary between occurrences. Next, we denote that $(\mathbf{u}_k, \varphi_k, \theta_k)$ be a solution of the truncated problem (18)-(19). From Theorem 3.3 and Theorem 3.4, we obtain the following bounds for the velocity \mathbf{u}_k and the electric potential φ_k :

$$\|\mathbf{u}_k\|_{\mathcal{H}_{\mathbf{u}}} \leq C, \quad (47)$$

$$\|\varphi_k\|_{\mathcal{H}_{\varphi}} \leq C, \quad (48)$$

where the constant C is independent of the truncation parameter k .

Although Theorem 3.5 provides a bound for the temperature θ_k , this bound depends on the truncation parameter k (cf. (37)), making it impossible to pass to the limit as $k \rightarrow \infty$.

In this section, we demonstrate that the temperature θ_k is bounded by a constant that is independent of the truncation parameter k . To derive these estimates, we apply ideas from [10, 18] to the temperature part of problem (18)-(19). Then, we have the following lemma:

Lemma 3.3. *Let $n \in \mathbb{N}$ be fixed, then there exists a positive constant $C > 0$ independent of k and n such that*

$$\max_{0 < \tau \leq t_f} \int_{\Omega} |\theta_k(\tau)| d\mathbf{x} \leq C \quad \text{and} \quad \int_{D_n} |\nabla \theta_k|^2 d\mathbf{x} dt \leq C, \quad (49)$$

where $D_n = \{(\mathbf{x}, t) \in \Omega_{t_f} : n \leq |\theta_k| \leq n+1\}$.

Proof. Let $n > 0$. Since T_n is a Lipschitz continuous function and $\theta_k \in L^2(I; H^1(\Omega))$, it follows that $T_n(\theta_k) \in L^2(I; H^1(\Omega))$. Moreover,

$$\nabla T_n(\theta_k) = \mathbf{1}_{\{|\theta_k| \leq n\}} \nabla \theta_k.$$

We choose $\psi = T_n(\theta_k)$ as a test function in (18) and integrate over $(0, t)$ with $0 < t \leq t_f$. After integration by parts and using the bound of S_k ($|S_k(\zeta)| \leq k|\zeta|$), we obtain

$$\begin{aligned} & \int_{\Omega} S_n(\theta_k)(t, x) d\mathbf{x} + \gamma_1 \int_0^t \int_{\Omega} |\nabla T_n(\theta_k)|^2 d\mathbf{x} d\tau \\ & \leq \int_{\Omega} S_n(\theta_k)(t, x) d\mathbf{x} + \int_0^t \int_{\Omega} \gamma(\bar{\theta}) |\nabla T_n(\theta_k)|^2 d\mathbf{x} d\tau \\ & \leq \int_{\Omega} |S_n(\theta_{0,k})| d\mathbf{x} + \int_0^t \int_{\Omega} |h_k T_n(\theta_k)| d\mathbf{x} d\tau + \alpha \int_0^t \int_{\partial\Omega} |\theta_k T_n(\theta_k)| d\mathbf{y} d\tau \\ & \leq n \|\theta_0\|_{L^1(\Omega)} + n \|h\|_{L^1(\Omega_{t_f})} + \alpha n \int_0^t \int_{\partial\Omega} |\theta_k| d\mathbf{y} d\tau. \end{aligned}$$

Using the definition of S_1 , we have

$$\int_{\Omega} |\theta_k| d\mathbf{x} \leq \frac{3n}{2} \text{meas}(\Omega) + \frac{1}{n} \int_{\Omega} S_1(\theta_k) d\mathbf{x}.$$

Thus,

$$\int_{\Omega} |\theta_k| d\mathbf{x} \leq \frac{3n}{2} \text{meas}(\Omega) + \|\theta_0\|_{L^1(\Omega)} + \|h\|_{L^1(\Omega_{t_f})} + \alpha C \int_0^t \int_{\Omega} |\theta_k| d\mathbf{x} d\tau.$$

Using Gronwall's inequality, we obtain

$$\max_{0 < \tau \leq t_f} \int_{\Omega} |\theta_k(\tau)| d\mathbf{x} \leq C, \quad (50)$$

for some constant $C > 0$.

New, we defined the function Φ by

$$\Phi_n(z) := \begin{cases} 0 & \text{if } z \leq n, \\ (|z| - n)\text{sign}(z) & \text{if } n \leq |z| \leq n+1, \\ \text{sign}(z) & \text{if } z \geq n+1. \end{cases} \quad (51)$$

We taking $\psi = \Phi(\theta_k)$ and integrating over $[0, t]$ we get that

$$\int_{\Omega} \theta_k^2 d\mathbf{x} + \gamma_1 \int_0^{t_f} \int_{\Omega} \Phi'(\theta_k) |\nabla \theta_k|^2 d\mathbf{x} d\tau \leq C.$$

Thus,

$$\int_{D_n} |\nabla \theta_k|^2 d\mathbf{x} d\tau \leq C, \quad (52)$$

for some constant $C > 0$. □

Therefore, we have the following estimate, based on [18, Proposition 2] for $d = 2$ or [1] for $d \geq 2$:

Corollary 3.1. *Let q satisfy $1 \leq q < \frac{d+2}{d+1}$ with $d \geq 2$. Then, there exists $C > 0$ depending on $\text{meas}(\Omega)$, T , and q such that*

$$\|\theta_k\|_{L^q(I; W^{1,q}(\Omega))} \leq C \quad \text{for } k > 0. \quad (53)$$

3.3. Passing to the Limit in the Truncation Parameter. Using the estimates from the previous section, we will demonstrate the convergence of the truncated problem (18)-(19) to the weak solution of (13)-(16) as $k \rightarrow \infty$. From Lemma 3.3, Corollary 3.1, and given that $d = 2$, it follows that the sequence $(\theta_k)_{k>0}$ is bounded in $L^q(I; W^{1,q}(\Omega))$ for $1 \leq q < \frac{d+2}{d+1}$. Consequently, since θ_k satisfies the corresponding equation, $\partial_t \theta_k$ is bounded in $L^1(I; (W^{1,q}(\Omega))') + L^1(\Omega_{t_f})$. By applying Simon's compactness result (see e.g., [26,27]), and possibly extracting subsequences (still denoted $(\theta_k)_{k>0}$), we can assume there exists a limit $\theta \in L^q(I; W^{1,q}(\Omega))$ such that as $k \rightarrow \infty$,

$$\begin{cases} \theta_k \rightarrow \theta & \text{strongly in } L^q(\Omega_{t_f}) \text{ and almost everywhere in } \Omega_{t_f}, \\ \theta_k \rightarrow \nabla \theta & \text{weakly in } L^q(I; W^{1,q}(\Omega)). \end{cases} \quad (54)$$

The compact embedding $L^q(I; W^{1,q}(\Omega)) \hookrightarrow L^q(I; L^s(\Omega))$ with $1 \leq s \leq \frac{dq}{d-q}$ (cf. (2)) implies that there exists a subsequence (still denoted $(\theta_k)_{k>0}$) converging strongly to θ in $L^q(I; L^s(\Omega))$. Similarly, using the estimates for \mathbf{u}_k in (47) and φ_k in (48), we find subsequences (still denoted $(\mathbf{u}_k)_{k>0}$ and $(\varphi_k)_{k>0}$) such that:

$$\begin{cases} \mathbf{u}_k \rightarrow \mathbf{u} & \text{weakly in } L^2(0, T; \mathbf{H}^1), \\ \partial_t \mathbf{u}_k \rightarrow \partial_t \mathbf{u} & \text{weakly in } L^1(0, T; \mathbf{H}^{-1}), \\ \nabla \mathbf{u}_k \rightarrow \nabla \mathbf{u} & \text{weakly in } L^2(0, T; \mathbf{L}^2), \\ \mathbf{u}_k \rightarrow \mathbf{u} & \text{strongly in } L^2(0, T; \mathbf{L}^2), \\ \mathbf{u}_k \rightarrow \mathbf{u} & \text{almost everywhere in } \Omega_{t_f}. \end{cases} \quad (55)$$

$$\begin{cases} \varphi_k \rightarrow \varphi & \text{weakly in } L^2(0, T; H^1(\Omega)), \\ \varphi_k \rightarrow \varphi & \text{strongly in } L^2(0, T; L^2(\Omega)), \\ \nabla \varphi_k \rightarrow \nabla \varphi & \text{weakly in } L^2(I; \mathbf{L}^2(\Omega)). \end{cases} \quad (56)$$

The truncated problem (18)-(19) can be expressed in the weak form:

$$\begin{aligned} & - \int_0^{t_f} \int_{\Omega} \mathbf{u}_k \partial_t \mathbf{v} \, d\mathbf{x} \, dt + \int_0^{t_f} \int_{\Omega} \mathbf{v}(0, x) \mathbf{u}_{0,k}(x) \, d\mathbf{x} + \int_0^{t_f} \int_{\Omega} \nu(\theta_k) \mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{v}) \, d\mathbf{x} \, dt \\ & + \int_0^{t_f} \int_{\Omega} \mathbf{u}_k \cdot \nabla \mathbf{u}_k \mathbf{v} \, d\mathbf{x} \, dt = \int_0^{t_f} \int_{\Omega} \mathbf{F}(\theta_k) \mathbf{v} \, d\mathbf{x} \, dt, \\ & - \int_0^{t_f} \int_{\Omega} \theta_k \partial_t \psi \, d\mathbf{x} \, dt + \int_0^{t_f} \int_{\Omega} \psi(0, x) \theta_{0,k}(x) \, d\mathbf{x} + \int_0^{t_f} \int_{\Omega} \gamma(\theta_k) \nabla \theta_k \cdot \nabla \psi \, d\mathbf{x} \, dt \\ & + \int_0^{t_f} \int_{\Omega} \mathbf{u}_k \cdot \nabla \theta_k \psi \, d\mathbf{x} \, dt - \int_0^{t_f} \int_{\Omega} T_k(\nu(\theta_k) \mathbb{D}(\mathbf{u}_k) : \mathbb{D}(\mathbf{u}_k)) \psi \, d\mathbf{x} \, dt \\ & - \int_0^{t_f} \int_{\Omega} T_k((\lambda(\theta_k) \nabla \varphi_k) \cdot \nabla \varphi_k) \psi \, d\mathbf{x} \, dt - \int_0^{t_f} \int_{\Sigma} \alpha(\theta_{l,k} - \theta_k) \psi \, d\mathbf{y} \, dt = 0, \\ & \int_0^{t_f} \int_{\Omega} \lambda(\theta_k) \nabla \varphi_k \cdot \nabla \chi \, d\mathbf{x} \, dt = \int_0^{t_f} \int_{\Sigma_N} g_k \chi \, d\mathbf{y} \, dt, \end{aligned}$$

where $\mathbf{v} \in \mathbf{Z}_{\mathbf{u}}$, $\psi \in \Psi_{q'}$, and $\chi \in L^2(0, T; H^1)$, with the spaces of test functions defined as:

$$\begin{aligned} \mathbf{Z}_{\mathbf{u}} &= \{ \mathbf{v} \in L^2(0, T; \mathbf{H}_0^1) : \partial_t \mathbf{v} \in \mathbf{L}^2(\Omega_{t_f}), \mathbf{v}(t_f) = 0 \}, \\ \Psi_{q'} &= \left\{ \psi \in L^{q'}(I; W_0^{1,q'}(\Omega)) \cap C^0([0, t_f]; L^\infty(\Omega)) : \partial_t \psi \in L^{q'}(\Omega_{t_f}), \psi(t_f) = 0 \right\}. \end{aligned}$$

Combining the results obtained in (54), (55), and (56), and using Assumptions 2.1, along with the Lebesgue-dominated convergence (see, for example, [11]), we can pass to the limit as $k \rightarrow \infty$ in

all terms. Thus, $(\mathbf{u}, \varphi, \theta)$ satisfies the following weak formulation:

$$\begin{aligned}
& - \int_0^{t_f} \int_{\Omega} \mathbf{u} \partial_t \mathbf{v} \, d\mathbf{x} \, dt + \int_0^{t_f} \int_{\Omega} \mathbf{v}(0, x) \mathbf{u}_0(x) \, d\mathbf{x} + \int_0^{t_f} \int_{\Omega} \nu(\theta) \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, d\mathbf{x} \, dt \\
& + \int_0^{t_f} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \, \mathbf{v} \, d\mathbf{x} \, dt = \int_0^{t_f} \int_{\Omega} \mathbf{F}(\theta) \mathbf{v} \, d\mathbf{x} \, dt, \\
& - \int_0^{t_f} \int_{\Omega} \theta \partial_t \psi \, d\mathbf{x} \, dt + \int_0^{t_f} \int_{\Omega} \psi(0, x) \theta_0(x) \, d\mathbf{x} + \int_0^{t_f} \int_{\Omega} \gamma(\theta) \nabla \theta \cdot \nabla \psi \, d\mathbf{x} \, dt \\
& + \int_0^{t_f} \int_{\Omega} \mathbf{u} \cdot \nabla \theta \, \psi \, d\mathbf{x} \, dt - \int_0^{t_f} \int_{\Omega} \nu(\theta) \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{u}) \, \psi \, d\mathbf{x} \, dt \\
& - \int_0^{t_f} \int_{\Omega} (\lambda(\theta) \nabla \varphi) \cdot \nabla \varphi \, \psi \, d\mathbf{x} \, dt - \int_0^{t_f} \int_{\Sigma} \alpha(\theta_l - \theta) \psi \, d\mathbf{y} \, dt = 0, \\
& \int_0^{t_f} \int_{\Omega} \lambda(\theta) \nabla \varphi \cdot \nabla \chi \, d\mathbf{x} \, dt = \int_0^{t_f} \int_{\Sigma_N} g \chi \, d\mathbf{y} \, dt,
\end{aligned}$$

where $\mathbf{v} \in \mathbf{Z}_{\mathbf{u}}$, $\psi \in \Psi_{q'}$, and $\chi \in L^2(0, T; H^1)$. In other words, $(\mathbf{u}, \varphi, \theta)$ is a weak solution to (13)-(16) on $(0, t_f)$. The proof of Theorem 3.1 is achieved.

4. NUMERICAL EXPERIMENTS

This section aims to validate the proposed model in a two-dimensional space domain Ω illustrated in Figure 1. We numerically explore the effects of dissipation viscosity on the dynamics of fluid flow and heat transport. The computational domain has parameters $L = 1.5$, $H = 0.5$, and $r = 0.075$. For simplicity, we assume the thickness of the electrode is negligible and omit its effect in the numerical simulations. The choice of parameters is informed by existing specifications in the literature (see [15–17]). The nonlinear functions for electrical conductivity σ , thermal conductivity η , and blood conductivity ν are modeled as temperature-dependent functions:

$$\begin{aligned}
\lambda(\theta) &= \begin{cases} \lambda_0 \exp^{0.015(\theta - \theta_b)} & \text{for } \theta \leq 99^\circ\text{C}, \\ 2.5345\lambda_0 & \text{for } 99^\circ\text{C} < \theta \leq 100^\circ\text{C}, \\ 2.5345\lambda_0 (1 - 0.198(\theta - 100^\circ\text{C})) & \text{for } 100^\circ\text{C} < \theta \leq 105^\circ\text{C}, \\ 0.025345\lambda_0 & \text{for } \theta > 105^\circ\text{C}, \end{cases} \\
\gamma(\theta) &= \begin{cases} \gamma_0 + 0.0012(\theta - \theta_b) & \text{for } \theta \leq 100^\circ\text{C}, \\ \gamma_0 + 0.0012(100^\circ\text{C} - \theta_b) & \text{for } \theta > 100^\circ\text{C}, \end{cases} \\
\nu(\theta) &= \begin{cases} 0.0021 \times 10^{-3} & \text{for } \theta \geq 30^\circ\text{C}, \\ 0.001 \times 10^{-3} & \text{for } \theta < 30^\circ\text{C}. \end{cases}
\end{aligned}$$

Here, $\lambda_0 = 0.6$ and $\gamma_0 = 0.54$ are the constant electrical conductivity and thermal conductivity, respectively, at core body temperature $\theta_b = 37^\circ\text{C}$.

4.1. Discretization and Numerical Scheme. For time discretization, we fix an integer M and define a time subdivision $t_0 = 0 < t_1 < \dots < t_M = T$, with $T = 90$ s. The time steps are given by $\tau = t_{n+1} - t_n$ for $n = 0, \dots, M - 1$. We employ a backward Euler scheme for the time semi-discretization of the problem (13)-(16). For any sequence of functions $\{f^n\}_{n=0}^N$ defined in Ω , we define the time derivative as $D_\tau f^{n+1} = \frac{f^{n+1} - f^n}{\tau}$. The spatial discretization is carried out using the finite element method. Specifically, we use the P1-Bubble finite element to approximate the velocity variable and the P1 finite element to approximate the temperature, pressure, and potential. We maintain the same notations \mathbf{u} , P , θ , and φ for the discrete versions of these variables. To handle the nonlinearities in the problem, we use an iterative scheme based on the fixed-point method. In

each iteration, the velocity and potential are computed using the temperature from the previous iteration. The proposed iterative scheme is implemented as follows:

Algorithm 1 Iterative Scheme for Numerical Solution

Step 1. : Initialize with an initial guess $(\theta^0, \mathbf{u}^0, P^0)$.

Step 2. : For $n \geq 0$, given $(\theta^n, \mathbf{u}^n, P^n)$, compute the values of $(\theta^{n+1}, \mathbf{u}^{n+1}, P^{n+1})$ as follows:

(1) Given θ^n , solve for $(\mathbf{u}^{n+1}, P^{n+1})$ from:

$$\begin{aligned} \langle D_\tau \mathbf{u}^{n+1}, \mathbf{v} \rangle + a_u(\theta^n; \mathbf{u}^{n+1}, \mathbf{v}) + b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}) - \int_\Omega P^{n+1} \nabla \cdot \mathbf{v} \, dx &= (\mathbf{F}(\theta^n), \mathbf{v}), \\ \int_\Omega q \nabla \cdot \mathbf{u}^{n+1} \, dx &= 0. \end{aligned}$$

(2) Given θ^n , solve for φ^n from:

$$a_\varphi(\theta^n; \varphi^n, \chi) - \langle g, \chi \rangle_{\Gamma_N} = 0.$$

(3) Given \mathbf{u}^{n+1} and φ^n , solve for θ^{n+1} from:

$$\begin{aligned} \langle D_\tau \theta^{n+1}, \psi \rangle + a_\theta(\theta^n; \theta^{n+1}, \psi) + d(\mathbf{u}^{n+1}, \theta^{n+1}, \psi) + \alpha \langle \theta^{n+1}, \psi \rangle_\Gamma \\ = e(\theta^n; \mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \psi) + c_\varphi(\theta^n, \varphi^n, \psi) + \alpha \langle \theta_l, \psi \rangle_\Gamma. \end{aligned}$$

(4) If the difference between $(\theta^{n+1}, \mathbf{u}^{n+1}, P^{n+1})$ and $(\theta^n, \mathbf{u}^n, P^n)$ exceeds a predefined tolerance, set $n = n + 1$ and return to Step 2. Otherwise, proceed to Step 3.

Step 3. : Conclude the iterative process when the solution converges within the tolerance.

4.2. Examples of Numerical Simulations. In this subsection, we present numerical simulations to illustrate the interactions between different variables in the model, using the finite element method with suitable parameters and initial conditions. The simulations are conducted in two phases: first, we examine the system without dissipation viscosity; second, we explore the full system, demonstrating the impact of dissipation viscosity on fluid flow dynamics and heat transport. We mention that all computations were performed using the FreeFem++ software package [20].

We begin by noting that external forces are neglected, setting $\mathbf{F} = \mathbf{0}$. A Dirichlet boundary condition is imposed on Γ_1 , where the velocity is specified as $\mathbf{u} = \begin{pmatrix} y(H - y) \\ 0 \end{pmatrix}$. For the boundaries Γ_2, Γ_4 , and Γ_5 , we set $\mathbf{u} = \mathbf{0}$. On the third boundary Γ_3 , we assume a stress-free condition, $\mathbb{S}(\mathbf{u}, P)\mathbf{n} = \mathbf{0}$. Regarding the temperature boundary conditions, on Γ_1, Γ_2 , and Γ_5 , we apply $(\gamma(\theta)\nabla\theta) \cdot \mathbf{n} + \alpha\theta = \alpha\theta_l$, where $\alpha = 1$ and $\theta_l = \theta_b = 37^\circ\text{C}$. On $\Gamma_3 \cup \Gamma_4$, homogeneous Neumann boundary conditions are imposed as an artificial boundary condition. For the potential equation, the boundary condition on Γ_5 is fixed with $g = 5$ ($\sigma(\theta)\nabla\varphi \cdot \boldsymbol{\eta} = g$ on Γ_5), while on Γ_4 , a homogeneous Neumann condition is applied ($\sigma(\theta)\nabla\varphi \cdot \boldsymbol{\eta} = 0$). The remaining boundary, $\Gamma \setminus (\Gamma_4 \cup \Gamma_5)$, is subject to a homogeneous Dirichlet condition, $\varphi = 0$.

4.2.1. Example 1: Heat Transfer and Blood Flow in the Absence of Dissipation Viscosity. This example aims to validate our model by comparing it with existing results in the literature where dissipation viscosity effects are negligible.

As shown in Figure 2, the computed potential evolves very slowly over the iterations. This is primarily because the only data in the potential equation are the source term g , which is constant, and the electrical conductivity $\sigma(\theta) = \lambda_0 \exp(0.015(\theta - \theta_b))$. The applied potential increases the temperature of the tissue near the catheter, which then diffuses into the surrounding blood vessels (Figure 3). This behavior is driven by the presence of the electric field term $\lambda(\theta)\nabla\varphi \cdot \nabla\varphi$. It is also important to note that the temperature is influenced by blood flow, specifically by the advection term $\mathbf{u} \cdot \nabla\theta$. The fluid velocity and pressure are affected by the viscosity ν , which depends on the temperature θ . Specifically, a flow recirculation zone is observed, with a localized increase in velocity upstream of the catheter, while the velocity remains nearly uniform in regions farther from the catheter at $t = \frac{T}{3}$. Over time, both velocity and pressure are increasingly influenced by the viscosity, as shown in the second column of Figure 3. In summary, this test highlights the

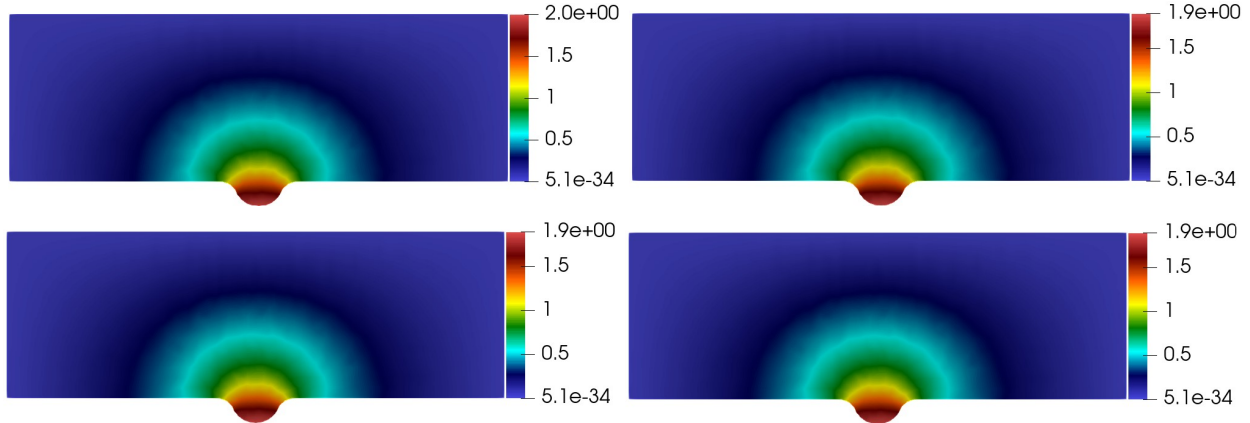


FIGURE 2. Snapshots of the potential evolution at four time points: $t = 0$, $t = \frac{T}{3}$, $t = \frac{3T}{4}$, and $t = T$.

impact of each term in the model, demonstrating the coupling effects between fluid flow, pressure, temperature, and potential. The results align well with the literature, where dissipation viscosity is neglected, as discussed in [7].

4.2.2. Example 2: Heat Transfer and Blood Flow in the Presence of Dissipation Viscosity. In this example, we illustrate the effect of dissipation viscosity on the evolution of temperature and, consequently, on the dynamics of blood flow. The theoretical challenges posed by this term are still encountered in numerical simulations, especially when using classical discretization methods. As noted in the theoretical section, the presence of this term results in solutions that are less regular, leading to significant variations and discontinuities with high jumps. To address these challenges, we have opted for methods that perform well under these conditions, while also regularizing the dissipation term through a convolution with a sequence of functions that converge to the original term.

The results of this experiment are presented in Figure 4. Comparing with the first example, we observe that the shape of the initial data is largely preserved during time iterations, indicating a slow evolution of temperature diffusion over time. Additionally, the preservation of the temperature profile's shape during iterations—despite being less regular at the initial moment—demonstrates that the initial regularity of the solution is almost maintained, implying that the non-regularizing terms of the system dominate. Moreover, the model allows for sharp temperature variations within a very small area, which could be exploited to minimize energy loss during RFA procedures, thereby reducing the risk of affecting other parts of the tissue with the injections. As a result, this approach could yield highly discontinuous solutions, with a significantly elevated temperature in the targeted area and normal body temperature elsewhere. This could shorten the duration of the RFA procedure while optimizing energy consumption. Moreover, with high-performance numerical methods such as finite volume methods or discontinuous Galerkin methods, the targeted zone can be minimized to a small-radius region. This is particularly valuable from a practical standpoint, as it addresses key challenges related to reducing the treatment area and optimizing the placement and number of injected electrodes.

5. CONCLUSION AND DISCUSSIONS

In this paper, we analyzed the existence of solutions for nonlinear Electro-Thermo-Fluid model 1 that describes radiofrequency ablation in cardiac tissue, with L^1 energy dissipation in two-dimensional space. Our results were established according to Schauder's fixed point theorem and

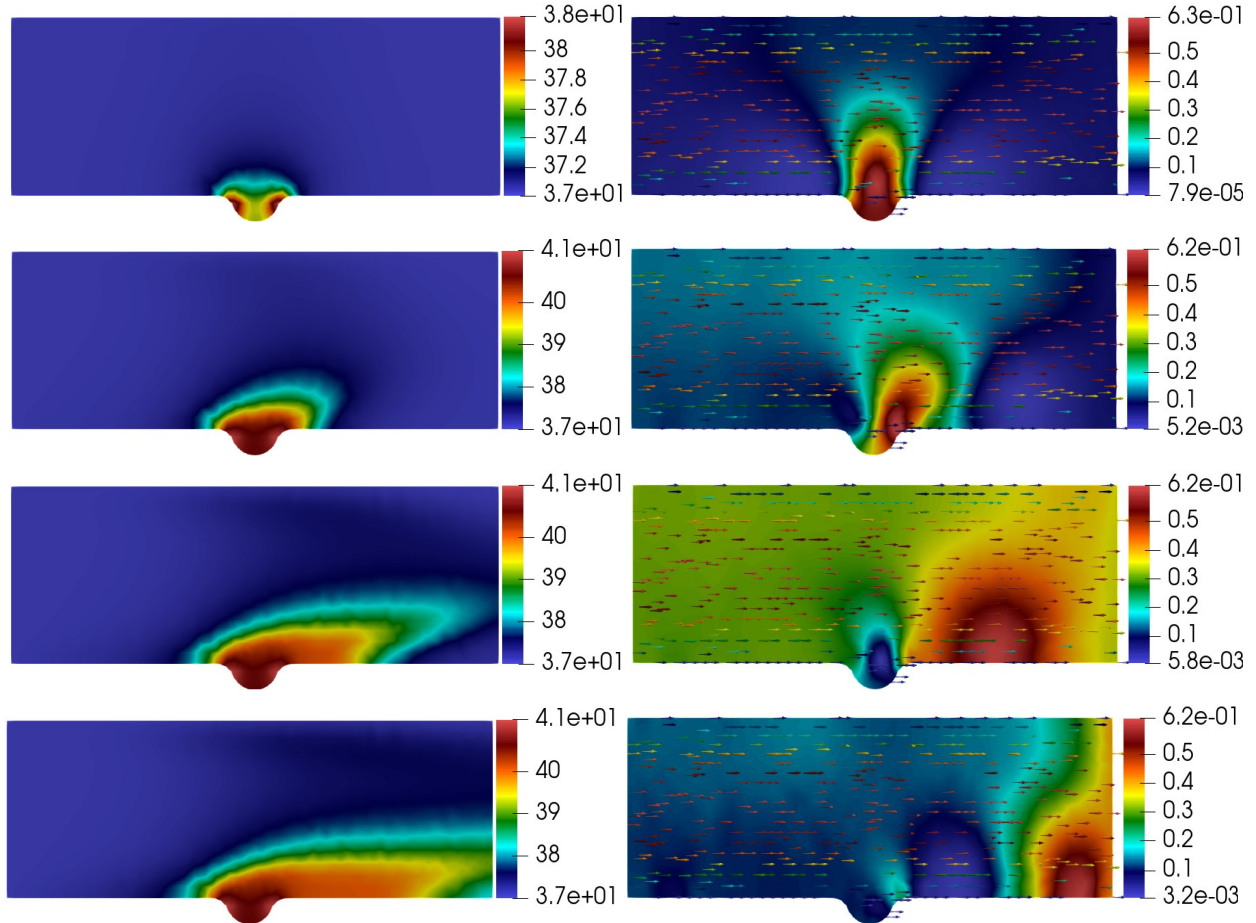


FIGURE 3. Example 1: Evolution of temperature (column 1), velocity, and pressure (column 2) at four time points: $t = 0$ (row 1), $t = \frac{T}{9}$ (row 2), $t = \frac{T}{3}$ (row 3), and $t = T$ (row 4) in the absence of dissipation viscosity.

by regularizing low-regularity terms before taking limits to prove the existence of a solution for the original problem with L^1 terms. Additionally, we provided numerical simulations to visualize the impact of source terms with lower regularities in accordance with the theoretical results. By experimenting with source terms regularized by a coefficient tending to zero, we compared the results and highlighted the advantages of analyzing the model with lower regularities.

We are convinced that our study can lead to several open questions such as:

- **Uniqueness of solution:** The possibility of achieving uniqueness of the solution in the two-dimensional space remains challenging, especially when the source terms in the heat equation have lower regularity (in L^1 -space). This is particularly difficult due to the challenges in proving the uniqueness of weak solutions for the Navier-Stokes system. Thus, this question remains open. Furthermore, the regularity of solutions for Problem (1) in the three-dimensional case is another unresolved issue.
- **Data with lower regularity:** The problem remains unresolved when data are less regular, particularly in L^1 . It is assumed that the issue arises from the lower regularity of the source terms. Consequently, the correct formulation of equation (1) in two-dimensional space remains an open problem.

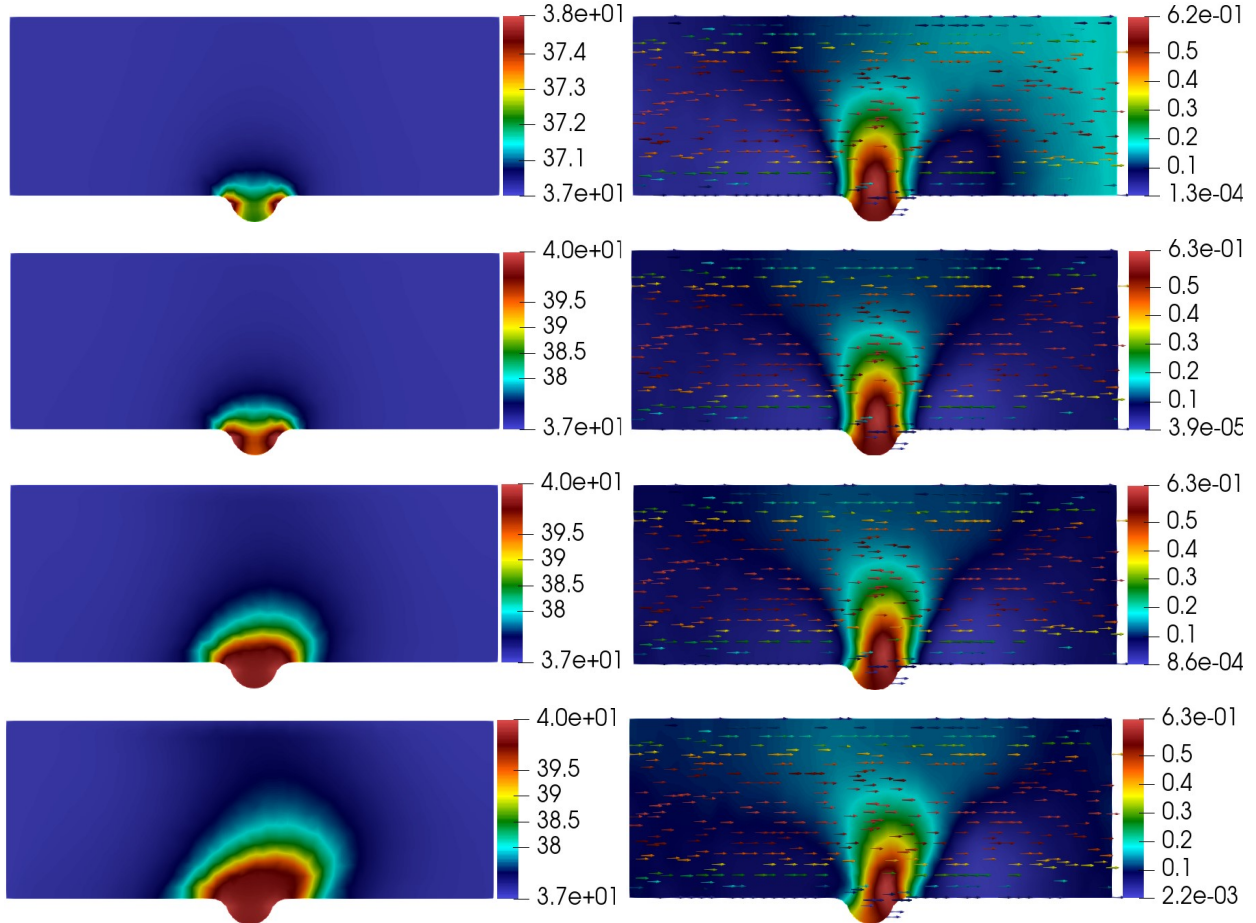


FIGURE 4. Example 2: Evolution of temperature (column 1), velocity, and pressure (column 2) at four time points: $t = 0$ (row 1), $t = \frac{T}{9}$ (row 2), $t = \frac{T}{3}$ (row 3), and $t = T$ (row 4) in the presence of dissipation viscosity.

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