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# Renormalized solutions for a simplified steady-state turbulent kinetic energy equation

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## Abstract

We consider a simplified steady-state equation for the turbulent kinetic energy used in turbulence modeling, with homogeneous boundary conditions and an eddy viscosity proportional to a regularization of the distance to the boundary, and a right hand side in  $L^1$ . After determining a suitable space function for this equation, we prove the existence and the uniqueness of a renormalized solution.

**Key words :** Fluid mechanics, Turbulence models, degenerate operators, Navier-Stokes Equations, Turbulent Kinetic Energy.

**2010 MSC:** 76D05, 35Q30, 76F65, 76D03, 35Q30.

## 1 Introduction

The aim of this paper is to prove the existence and the uniqueness of a renormalized solution to the steady-state boundary problem

$$(1.1) \quad \begin{cases} -\operatorname{div}(\ell \nabla k) + \frac{k}{\ell} = \mathbb{D} & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases}$$

where the unknown  $k$  denotes the the turbulent kinetic energy of a turbulent flow,  $\ell$  stands for the Prandtl mixing lenght, a given non negative function that degenerate near the boudary as the distance to the boundary, and  $\mathbb{D} \in L^1(\Omega)$  is the energy source term. The turbulent kinetic energy  $k$  roughly measures the deviation of the velocity field from its mean  $\bar{v}$  in a turbulent flow and is one of the main variables used in turbulence models. It is governed by a transport-diffusion equation with source and sink terms, taking the form:

$$(1.2) \quad \underbrace{\frac{Dk}{Dt}}_{\text{transport}} - \underbrace{\operatorname{div}(\mu_{\text{turb}}(k) \nabla k)}_{\text{diffusion}} = \underbrace{\mathbb{D}}_{\text{energy production}} - \underbrace{\varepsilon}_{\text{turbulent dissipation}}$$

In this equation:

1)  $\mu_{\text{turb}}(k)$  is an eddy diffusivity coefficient, expressed as

$$(1.3) \quad \mu_{\text{turb}}(k) = C\ell\sqrt{k},$$

a formula deduced from dimensional analysis, where  $\ell$  is the Prandtl mixing length,  $C$  is a dimensionless constant generally obtained from experimental data, under the assumption that  $k \geq 0$ .

2) The production term  $\mathbb{D}$  is given by

$$(1.4) \quad \mathbb{D} = \nu_{\text{turb}}(k) \left| \frac{\nabla \bar{\mathbf{v}} + \nabla \bar{\mathbf{v}}^t}{2} \right|^2,$$

where  $\nu_{\text{turb}} = \nu_{\text{turb}}(k)$  is the eddy viscosity, the form of which is similar to that of (1.3). According to the standard energy balance satisfied by  $\bar{\mathbf{v}}$ , we have  $\mathbb{D} \in L^1$  and no better.

3) The dissipation term  $\varepsilon$  is given by

$$(1.5) \quad \varepsilon = \frac{k\sqrt{k}}{\ell},$$

which is also deduced from dimensional analysis. This equation, referred to as the TKE equation, is derived from averaging the energy balance in the Navier-Stokes equations, and is central to many turbulence models, the most popular being the  $k - \varepsilon$  model, which involves an additional equation for  $\varepsilon$  [7, 10, 17].

The TKE equation is set in a bounded domain  $\Omega$  and is driven by the Prandtl mixing length  $\ell$ , representing the distance a turbulent eddy can travel before being mixed into the rest of the fluid [20]. The mixing length is somewhat equivalent to the mean free path in the kinetic theory of gases. Near the boundary of  $\Omega$ , denoted by  $\Gamma$ ,  $\ell$  is proportional to the distance to  $\Gamma$ , leading to significant mathematical challenges. For this reason, in many earlier works, Formula (1.3) is replaced by

$$(1.6) \quad \mu_{\text{turb}}(k) = \mu_0 + C\ell\tilde{\mu}(k),$$

where  $\mu_0 > 0$  and the function  $\tilde{\mu}(k) \geq 0$  is a bounded continuous approximation of  $\sqrt{k}$ , since unbounded eddy coefficients also pose significant issues (cf. [5, 8, 12, 14]).

Recently, the case with  $\mu_0 = 0$ , *i.e.*  $\mu_{\text{turb}}(k) = C\ell\tilde{\mu}(k)$ , has been studied in the steady-state case [2]. However, only relaxed cases could be addressed, namely

$$(1.7) \quad \mu_{\text{turb}}(k) = C\ell^\beta\tilde{\mu}(k), \quad \text{for } 0 < \beta < 1.$$

Depending on the value of  $\beta \in ]0, 1[$ , we were able to derive estimates for  $k$  and subsequently couple the TKE equation with Navier-Stokes equations featuring an eddy viscosity of the form  $\nu_{\text{turb}}(k) = C'\ell^\eta\tilde{\nu}(k)$ , where  $0 \leq \nu_m \leq \tilde{\nu}(k) \leq \nu_M$ , and  $0 < \eta < \eta_c < 1$  and  $0 < \beta < \beta_c < 1$ , under homogeneous boundary conditions (cf. [2, 11]).

The central idea behind these results relies on estimates in weighted spaces, embedded within regular Sobolev spaces, using a result by Kufner [9], which provides a framework for defining the trace of  $k$  at the boundary.

In this paper, we focus on the case  $\beta = 1$  in the steady-state case, for which Kufner's results are no longer applicable. A linear equation for a boundary layer with an eddy viscosity behaving as  $d(x, \Gamma)$  near the ground and a Boussinesq source term was studied in [6]. We found a solution in  $H^{1/2}$ , but no better, for which no trace can be defined.

The strategy to tackle the problem set by the TKE equation (1.2), equipped with the eddy coefficient  $\mu_{\text{turb}}(k) = \ell\tilde{\mu}(k)$  (taking  $C = 1$  for simplicity), is taking advantage of the term  $\varepsilon$ , which has the right sign. For simplicity, in this first approach, we take  $\tilde{\mu}(k)$  constant to be equal to 1, and instead of formula (1.5),

$$(1.8) \quad \varepsilon = \frac{k}{\ell},$$

and considering the steady state case, we are led to consider the linear PDE problem (1.1), assuming

$$(1.9) \quad \ell \in L^\infty(\Omega) \text{ s.t. } \ell \geq 0 \text{ a.e. } \Omega, \text{ and } \forall \omega \subset\subset \Omega, \quad \inf_{\bar{\omega}} \ell > 0,$$

$$(1.10) \quad \ell(x) \sim d(x, \Gamma) \text{ when } d(x, \Gamma) \rightarrow 0,$$

$$(1.11)$$

The key observation is that, by formally multiplying the equation (1.2) by  $T_z(k)$ ,  $T_z$  being the truncation function at height  $z > 0$  (see (3.1) below), we obtain the estimate

$$(1.12) \quad \nu_m \int_{\Omega} \ell(x) |\nabla T_z(k(x))|^2 dx + \int_{\Omega} \frac{|T_z(k(x))|^2}{\ell} \leq \int_{\Omega} \ell(x) |\nabla T_z(k(x))|^2 dx + \int_{\Omega} \frac{k(x) T_z(k(x))}{\ell(x)} \leq z \|\mathbb{D}\|_{\Omega; 0,1}.$$

In other words,  $T_z(k)$  naturally belongs to the space

$$(1.13) \quad H_\ell = \left\{ u \in \mathcal{D}'(\Omega), \quad \ell^{1/2} \nabla u \in L^2(\Omega)^N, \quad \frac{u}{\ell^{1/2}} \in L^2(\Omega) \right\}.$$

This invites us to seek a renormalized solution, in the way initially developed by Lions-Murat for elliptic equations with a right-hand side in  $L^1$  [19, 18] (see also in [13, chapter 5]). The main results of the paper is the existence of a renormalized solution and its uniqueness to Problem (1.2).

The paper is organized as follows. We first determine the space function suitable for problem (1.2), which is a subspace of  $H^{\frac{1}{2}}$ , the functions of which having a trace on  $\Gamma$  well defined and equal to 0. Then we give the definition of a renormalized solution, and prove the main results. We have gathered in an appendix technical results essential to the demonstrations of the main results.

*Acknowledgements.* The authors thank François Murat for several stimulating discussions about this problem and renormalized solutions.

## 2 Space function

Recall that we focus on the problem

$$(2.1) \quad \begin{cases} -\operatorname{div}(\ell \nabla k) + \frac{k}{\ell} = \mathbb{D} & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases}$$

### 2.1 Embeddings

We mentioned in the introduction that the natural space for solving Problem (2.1) is the space  $H_\ell$  defined by :

$$(2.2) \quad H_\ell = \left\{ u \in \mathcal{D}'(\Omega), \quad \ell^{1/2} \nabla u \in L^2(\Omega)^N, \quad \frac{u}{\ell^{1/2}} \in L^2(\Omega) \right\},$$

where the the function  $\ell$  satisfies the assumptions (1.9) and (1.10). It is an Hilbert space, equipped with the scalar product

$$(2.3) \quad (u, v)_\ell = \int_{\Omega} \ell(x) \nabla u(x) \cdot \nabla v(x) dx + \int_{\Omega} \frac{u(x)v(x)}{\ell(x)} dx.$$

We denote by  $\|u\|_\ell = \sqrt{(u, v)_\ell}$  the corresponding norm.

**Lemma 2.1.** *The following holds :*

$$(2.4) \quad H_\ell \hookrightarrow H_{00}^{\frac{1}{2}}(\Omega) = \left\{ u \in H^{\frac{1}{2}}(\Omega), \text{ s.t. } \frac{u}{\sqrt{\ell}} \in L^2(\Omega) \right\}.$$

*Proof.* From the definition of  $H_\ell$ , we get  $H_\ell \hookrightarrow L^2(\Omega; \ell^{-1}dx)$ , which combined with (1.9) yields  $H_\ell \hookrightarrow L^2(\Omega; dx) = L^2(\Omega)$ . Therefore,

$$H_\ell \hookrightarrow V_\ell = \left\{ u \in \mathcal{D}'(\Omega), \quad \ell^{1/2}\nabla u \in L^2(\Omega)^N, u \in L^2(\Omega) \right\}.$$

We know from previous works [2, 3, 4, 6], that  $V_\ell \hookrightarrow H^{\frac{1}{2}}(\Omega)$ , yielding (2.4) thanks to the definition of  $H_{00}^{\frac{1}{2}}(\Omega)$  initially given by Lions-Magenes [16].  $\square$

From the Sobolev theorem, we deduce that

$$(2.5) \quad \forall 1 \leq p < \frac{2N}{N-1}, \quad H_\ell \hookrightarrow L^p(\Omega),$$

with compact embedding.

## 2.2 Traces and homogeneous spaces

We know from [16] that any function  $u \in H_{00}^{\frac{1}{2}}(\Omega)$  can be continuously extended by zero outside  $\Omega$ , into a function  $\tilde{u} \in H^{\frac{1}{2}}(\mathbb{R}^N)$ . However, a function in  $H_{00}^{\frac{1}{2}}(\Omega)$  may lack a trace at  $\Gamma$ , which remains an open topic of discussion. Therefore, the result of the previous section is not enough to take into account the homogeneous boundary condition in the definition of a renormalized solution. This is why we introduce additional spaces. We start with

$$(2.6) \quad \mathcal{D}_\ell(\overline{\Omega}) = \mathcal{D}(\overline{\Omega}) \cap H_\ell,$$

where  $\mathcal{D}(\overline{\Omega})$  stands for the set of restrictions to  $\Omega$  of functions in  $\mathcal{D}(\mathbb{R}^N)$ . We observe that, thanks to (1.9), we have  $\mathcal{D}(\Omega) \subset \mathcal{D}_\ell(\overline{\Omega})$ . In particular,  $\mathcal{D}_\ell(\overline{\Omega}) \neq \emptyset$ . We conjecture that, in fact,  $\mathcal{D}(\Omega) = \mathcal{D}_\ell(\overline{\Omega})$ . Now we consider:

$$(2.7) \quad \begin{aligned} W_\ell &= \text{closure of } \mathcal{D}_\ell(\overline{\Omega}) \text{ in } H_\ell, \\ W_{\ell,0} &= \text{closure of } \mathcal{D}(\Omega) \text{ in } W_\ell. \end{aligned}$$

It is likely that  $H_\ell = W_\ell = W_{\ell,0}$ , but we do not have a proof of this fact yet, which is not essential for the following. We prove the following result.

**Lemma 2.2.** *For some constant  $C = C(\Omega, \|\ell\|_\infty)$ ,*

$$(2.8) \quad \forall u \in \mathcal{D}_\ell(\overline{\Omega}), \quad \|u\|_{\Gamma;0,2}^2 = \int_\Gamma |u|^2 \leq C \|u\|_\ell^2 = C \left( \int_\Omega \ell |\nabla u|^2 + \int_\Omega \frac{|u|^2}{\ell} \right).$$

*Proof.* We know from classical results about trace theory [1, 22] that  $\exists C = C(\Omega)$  such that

$$(2.9) \quad \forall w \in \mathcal{D}(\overline{\Omega}), \quad \|w\|_{\Gamma;0,1} = \int_\Gamma |w| \leq C \left( \int_\Omega |\nabla w| + \int_\Omega |w| \right) = C \|w\|_{\Omega;1,1},$$

where we still denote  $w$  the trace at  $\Gamma$  instead of  $\text{tr}(w)$ , so far no risk of confusion occurs. Let  $u \in \mathcal{D}_\ell(\overline{\Omega})$ , and take  $w = u^2 \in \mathcal{D}(\overline{\Omega})$  in (2.9). Therefore,

$$(2.10) \quad \int_\Gamma |u|^2 \leq C \left( \int_\Omega |\nabla (u^2)| + \int_\Omega |u|^2 \right).$$

On one hand,

$$(2.11) \quad \int_{\Omega} |u|^2 \leq \|\ell\|_{\infty} \int_{\Omega} \frac{|u|^2}{\ell}.$$

On the other hand, Young inequality directly yields

$$(2.12) \quad \int_{\Omega} |\nabla(u^2)| = \int_{\Omega} 2|u\nabla u| \leq \int_{\Omega} \frac{|u|^2}{\ell} + \int_{\Omega} \ell |\nabla u|^2,$$

hence (2.8) follows.  $\square$

The consequence of the inequality (2.8) is that we can extend the trace operator by density to the space  $W_{\ell}$  given by (2.7),

$$(2.13) \quad \text{tr} : \begin{cases} W_{\ell} & \rightarrow L^2(\Gamma), \\ u & \rightarrow \text{tr}(u). \end{cases}$$

Obviously,

$$W_{\ell,0} \subset \ker(\text{tr}),$$

and it is reasonable to conjecture that  $W_{\ell,0} = \ker(\text{tr})$ . However, we can claim that functions in the space  $W_{\ell,0}$  have a trace at  $\Gamma$  which is equal to zero, making  $W_{\ell,0}$  the right space to work in. Noting that  $W_{\ell,0}$  is a Hilbert space for the  $H_{\ell}$  scalar product, and that  $L^2(\Omega) \subset W'_{\ell,0}$ , the following is a direct consequence of the Lax-Milgram theorem.

**Theorem 2.1.** *Let  $\mathbb{D} \in L^2(\Omega)$ . Then there exists a unique  $k \in W_{\ell,0}$  such that*

$$(2.14) \quad \forall w \in W_{\ell,0}, \quad (k, w)_{\ell} = \int_{\Omega} \mathbb{D} w.$$

In other words, Problem (2.1) has a unique weak solution in  $W_{\ell,0}$ .

### 3 Renormalized solutions

In this section we shall use function tests of the form  $F(k)$ , for  $F$  satisfying the Stampacchia conditions defined below.

**Definition 3.1** (Stampacchia conditions). *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . We say that  $F$  satisfies the Stampacchia conditions if  $F$  is globally Lipschitz and its derivative  $F'$  has a finite number of discontinuities. We denote by  $\text{Stamp}(\mathbb{R})$  the set of all functions satisfying the Stampacchia conditions.*

We will use intensively the truncation at height  $z \in \mathbb{R}_+^*$ , denoted by  $T_z$  and given by

$$(3.1) \quad T_z : t \mapsto \begin{cases} -z & \text{if } t < -z, \\ t & \text{if } -z \leq t \leq z, \\ z & \text{if } t > z. \end{cases}$$

Note that  $\forall z \in \mathbb{R}_+^*$ ,  $T_z \in \text{Stamp}(\mathbb{R})$ , with in addition  $T_z(0) = 0$ . Therefore, by the theorem A.3 in the appendix, we deduce that  $\forall k \in W_{\ell,0}$ ,  $T_z(k) \in W_{\ell,0}$ .

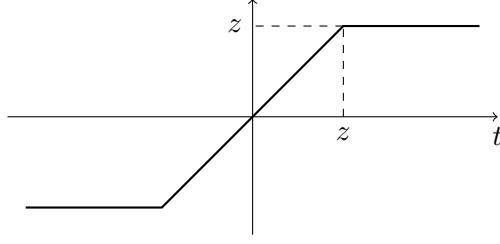


Figure 1: graph of the function  $T_z$ .

### 3.1 Convergence of the approximations

Let  $\mathbb{D} \in L^1(\Omega)$ ,  $(\mathbb{D}_n)_{n \in \mathbb{N}}$  be a sequence in  $L^2(\Omega)$  such that

$$(3.2) \quad \mathbb{D}_n \xrightarrow[n \rightarrow +\infty]{} \mathbb{D} \text{ in } L^1(\Omega),$$

$$(3.3) \quad \forall n \in \mathbb{N}, \quad |\mathbb{D}_n| \leq |\mathbb{D}| \text{ a.e. in } \Omega.$$

For example,  $\mathbb{D}_n = T_n(\mathbb{D})$  is a suitable sequence. Let  $k_n \in W_{\ell,0}$  satisfy in a weak sense as in Theorem 2.1,

$$(3.4) \quad \begin{cases} -\operatorname{div}(\ell \nabla k_n) + \frac{k_n}{\ell} = \mathbb{D}_n & \text{in } \Omega, \\ k_n = 0 & \text{at } \Gamma. \end{cases}$$

**Lemma 3.1.** *The sequence  $(k_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^1(\Omega, \ell^{-1} dx)$ . Therefore, there exists  $k \in L^1(\Omega, \ell^{-1} dx)$  such that  $k_n \xrightarrow[n \rightarrow +\infty]{} k$  in  $L^1(\Omega, \ell^{-1} dx)$ .*

*Proof.* Let  $\mathbb{H}_\eta$ ,  $\eta > 0$ , denotes the usual Lipschitz approximation of the Heaviside function,

$$(3.5) \quad \mathbb{H}_\eta : x \mapsto \begin{cases} -1 & \text{if } t < -\eta, \\ t/\eta & \text{if } -\eta \leq t \leq \eta, \\ 1 & \text{if } t > \eta. \end{cases}$$

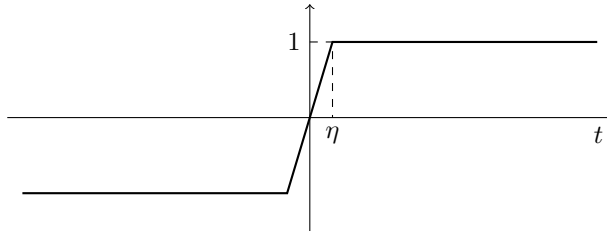


Figure 2: graph of the function  $\mathbb{H}_\eta$ .

We notice that  $\mathbb{H}_\eta \in \operatorname{Stamp}(\mathbb{R})$  and  $\mathbb{H}_\eta(0) = 0$ . By subtracting both (2.14) for  $k_p$  and  $k_q$ , we see that  $\forall v \in W_{\ell,0}$ ,

$$\int_{\Omega} \ell \nabla(k_p - k_q) \cdot \nabla v + \int_{\Omega} \frac{k_p - k_q}{\ell} v = \int_{\Omega} (\mathbb{D}_p - \mathbb{D}_q) v.$$

Taking  $v = \mathbb{H}_\eta(k_p - k_q) \in W_{\ell,0}$  (c.f. Theorem A.3) leads to

$$\int_{\Omega} \ell |\nabla(k_p - k_q)|^2 \mathbb{H}'_\eta(k_p - k_q) + \int_{\Omega} \frac{(k_p - k_q) \mathbb{H}_\eta(k_p - k_q)}{\ell} \leq \|\mathbb{D}_p - \mathbb{D}_q\|_{\Omega;0,1},$$

given that  $|\mathbb{H}_\eta(k_p - k_q)| \leq 1$ . Therefore, as  $\mathbb{H}'_\eta \geq 0$ , letting  $\eta$  go to zero, and noticing that

$$t\mathbb{H}_\eta(t) \xrightarrow{\eta \rightarrow 0} |t| \quad \text{for all } t \in \mathbb{R},$$

we deduce from Fatou's lemma

$$\|k_p - k_q\|_{L^1(\frac{dx}{\ell})} \leq \|\mathbb{D}_p - \mathbb{D}_q\|_{\Omega;0,1} \xrightarrow{p,q \rightarrow \infty} 0,$$

hence the result. As  $L^1(\Omega, \ell^{-1}dx)$  is a complete space, the sequence  $(k_n)_{n \in \mathbb{N}}$  has a limit  $k \in L^1(\Omega, \ell^{-1}dx)$ .  $\square$

**Lemma 3.2.** *Let  $k = \lim_{n \rightarrow \infty} k_n \in L^1(\Omega, \ell^{-1}dx)$  given by the statement of Lemma 3.1. Then*

$$(3.6) \quad \forall z \in \mathbb{R}_+^*, \quad T_z(k_n), T_z(k) \in W_{\ell,0},$$

and we have

$$(3.7) \quad T_z(k_n) \xrightarrow{n \rightarrow +\infty} T_z(k) \begin{cases} \text{weakly in } W_{\ell,0}, \\ \text{strongly in } L^p(\Omega), \quad 1 \leq p < \frac{2N}{N-1}, \\ \text{a.e. in } \Omega. \end{cases}$$

*Proof.* Let  $z \in \mathbb{R}_+^*$ . As already mentioned, since  $T_z \in \text{Stamp}(\mathbb{R})$  and  $T_z(0) = 0$ ,  $k_n \in W_{\ell,0}$ , then  $T_z(k_n) \in W_{\ell,0}$  by Theorem A.3. Moreover, as  $\ell \in L^\infty(\Omega)$  and  $\ell \geq 0$ , then

$$k = \lim_{n \rightarrow \infty} k_n \quad \text{in } L^1(\Omega, dx)$$

as well. Then we can extract a subsequence, still denoted by  $(k_n)_{n \in \mathbb{N}}$ , which converges to  $k$  a.e. in  $\Omega$  and by the way,

$$T_z(k_n) \xrightarrow{n \rightarrow +\infty} T_z(k)$$

in  $L^1(\Omega)$ , and a.e. in  $\Omega$ . Taking  $T_z(k_n)$  as test in (3.4) and using Stampacchia's result yields

$$(3.8) \quad \int_{\Omega} \ell |\nabla T_z(k_n)|^2 + \int_{\Omega} \frac{k_n T_z(k_n)}{\ell} = \int_{\Omega} T_z(k_n) \mathbb{D}_n \leq z \|\mathbb{D}\|_{\Omega;0,1},$$

which proves that the sequence  $(T_z(k_n))_{n \in \mathbb{N}}$  is bounded in  $W_{\ell,0}$ .

Therefore, we can extract from the sequence  $(T_z(k_n))_{n \in \mathbb{N}}$ , a subsequence, still denoted  $(T_z(k_n))_{n \in \mathbb{N}}$ , which weakly converges in  $W_{\ell,0}$  to some  $k_z \in W_{\ell,0}$ , and weakly in  $H^1(\omega)$  for all  $\omega \subset \subset \Omega$ . The convergence of  $(T_z(k_n))_{n \in \mathbb{N}}$  in  $L^1(\Omega)$  to  $T_z(k)$ , yields its convergence in  $\mathcal{D}'(\Omega)$ , hence  $k_z = T_z(k)$ . The strong convergence in  $L^p(\Omega)$  for  $1 \leq p < 2N/N - 1$ , is a consequence of Lemma 2.1 together with the Sobolev compact embedding theorem.  $\square$

### 3.2 Main result

Recall that we consider the problem

$$(3.9) \quad \begin{cases} -\text{div}(\ell \nabla k) + \frac{k}{\ell} = \mathbb{D} & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma. \end{cases}$$

**Definition 3.2.** We say that  $k$  is a renormalized solution of (3.9) if

$$(3.10) \quad k \in L^1(\Omega, \ell^{-1} dx),$$

$$(3.11) \quad \forall z \in \mathbb{R}, \quad T_z(k) \in W_{\ell,0},$$

$$(3.12) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{n \leq |k| \leq 2n\}} \ell |\nabla k|^2 = 0,$$

and  $\forall w \in W_{\ell,0} \cap L^\infty(\Omega)$  verifying

$$(3.13) \quad \exists z \in \mathbb{R}, \quad \text{s.t.} \quad \nabla w = 0 \text{ a.e. on } \{|k| \geq z\},$$

we have

$$(3.14) \quad \int_{\Omega} \ell \nabla k \cdot \nabla w + \int_{\Omega} \frac{k w}{\ell} = \int_{\Omega} \mathbb{D} w.$$

**Remark 3.1.** All the integrals in (3.14) are well defined. Indeed, because  $w \in L^\infty(\Omega)$ ,  $\mathbb{D} \in L^1(\Omega)$ ,  $k \in L^1(\Omega, \ell^{-1} dx)$ , then  $\mathbb{D} w, k w \ell^{-1} \in L^1(\Omega)$ . Moreover, because of (3.13),

$$\int_{\Omega} \ell \nabla k \cdot \nabla w = \int_{\Omega} \ell \nabla T_z(k) \cdot \nabla w,$$

which is well defined since as  $T_z(k), w \in W_{\ell,0}$ ,  $\sqrt{\ell} \nabla T_z(k), \sqrt{\ell} \nabla w \in L^2(\Omega)$ .

The main result of this section is the following.

**Theorem 3.1.** Let  $\mathbb{D} \in L^1(\Omega)$ . Then Problem (3.9) has a renormalized solution.

To this result must be added the following continuity property.

**Theorem 3.2.** Let  $(\mathbb{D}_\varepsilon)_{\varepsilon>0}$  be a sequence in  $L^1(\Omega)$  that converges to  $\mathbb{D} \in L^1(\Omega)$ ,  $(k_\varepsilon)_{\varepsilon>0}$  the corresponding sequence of renormalized solutions,  $k$  the corresponding solution with  $\mathbb{D}$  as source term. Then

$$(3.15) \quad \forall z \in \mathbb{R}_+^*, \quad T_z(k_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} T_z(k) \text{ strongly in } W_{\ell,0}.$$

*Proof of Theorem 3.1.* We argue by approximations. Let  $\mathbb{D} \in L^1(\Omega)$  and  $(\mathbb{D}_\varepsilon)_{\varepsilon>0}$  a family of  $L^\infty(\Omega)$  functions such that

$$(3.16) \quad \mathbb{D}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbb{D} \text{ in } L^1(\Omega),$$

$$(3.17) \quad \forall \varepsilon > 0, \quad |\mathbb{D}_\varepsilon| \leq |\mathbb{D}| \text{ a.e. in } \Omega.$$

Let  $k_\varepsilon \in W_{\ell,0}$  satisfy in the weak sense specified by Theorem 2.1,

$$(3.18) \quad \begin{cases} -\operatorname{div}(\ell \nabla k_\varepsilon) + \frac{k_\varepsilon}{\ell} = \mathbb{D}_\varepsilon & \text{in } \Omega, \\ k_\varepsilon = 0 & \text{at } \Gamma, \end{cases}$$

which means

$$(3.19) \quad \forall v \in W_{\ell,0}, \quad \int_{\Omega} \ell \nabla k_\varepsilon \cdot \nabla v + \int_{\Omega} \frac{k_\varepsilon v}{\ell} = \int_{\Omega} \mathbb{D}_\varepsilon v.$$

Let  $k \in L^1(\Omega, \ell^{-1} dx)$  be the limit of the family  $(k_\varepsilon)_{\varepsilon>0}$  in  $L^1(\Omega, \ell^{-1} dx)$ , the existence of which is ensured by Lemma 3.1. By the Lebesgue inverse theorem, we can extract from  $(k_\varepsilon)_{\varepsilon>0}$  a subsequence, still denoted by  $(k_\varepsilon)_{\varepsilon>0}$  such that  $k_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} k$  a.e. in  $\Omega$ .

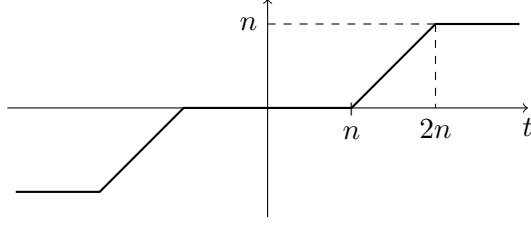


Figure 3: graph of the function  $S_n$ .

We aim at proving that  $k$  is a renormalized solution to Problem (3.9). We already know that  $k \in L^1(\Omega, \ell^{-1}dx)$ , and from Lemma 3.2, that

$$\forall z \in \mathbb{R}_+^*, \quad T_z(k) \in W_{\ell,0}.$$

It remains to prove that  $k$  satisfies (3.12) and (3.14), which we do in two steps.

**Step 1.** We start by proving that  $k$  satisfies (3.12). Let  $n \in \mathbb{N}^*$ ,  $S_n : \mathbb{R} \rightarrow \mathbb{R}$  be the odd function defined by

$$(3.20) \quad S_n : t \mapsto \begin{cases} 0 & \text{if } 0 \leq t \leq n, \\ t - n & \text{if } n < t \leq 2n, \\ n & \text{if } t > 2n. \end{cases}$$

We observe that  $\forall n \in \mathbb{N}$ ,  $S_n \in \text{Stamp}(\mathbb{R})$  and  $S_n(0) = 0$ . Therefore, by Theorem A.3,

$$\forall \varepsilon > 0, \quad \forall n \in \mathbb{N}^*, \quad S_n(k_\varepsilon) \in W_{\ell,0}.$$

Taking  $v = S_n(k_\varepsilon)$  in (3.18) yields for every  $n \in \mathbb{N}$  and  $\varepsilon > 0$ ,

$$(3.21) \quad \int_{\Omega} \ell |\nabla k_\varepsilon|^2 S_n'(k_\varepsilon) + \int_{\Omega} \frac{k_\varepsilon}{\ell} S_n(k_\varepsilon) = \int_{\Omega} \mathbb{D}_\varepsilon S_n(k_\varepsilon).$$

As  $S_n$  is an odd function and  $\ell \geq 0$ ,

$$\int_{\Omega} \frac{k_\varepsilon}{\ell} S_n(k_\varepsilon) \geq 0.$$

Moreover, from  $S_n'(t) \in \{0, 1\}$ , we deduce

$$(3.22) \quad \int_{\Omega} \ell |\nabla k_\varepsilon|^2 S_n'(k_\varepsilon) = \int_{\Omega} \ell |\nabla S_n(k_\varepsilon)|^2.$$

Moreover,

$$(3.23) \quad \int_{\Omega} \mathbb{D}_\varepsilon S_n(k_\varepsilon) \leq n \int_{\{|k_\varepsilon| \geq n\}} |\mathbb{D}_\varepsilon| \leq n \int_{\{|k_\varepsilon| \geq n\}} |\mathbb{D}|.$$

With those inequalities, (3.21) becomes

$$(3.24) \quad \frac{1}{n} \int_{\Omega} \ell |\nabla S_n(k_\varepsilon)|^2 = \frac{1}{n} \int_{\{n \leq k_\varepsilon \leq 2n\}} \ell |\nabla k_\varepsilon|^2 \leq \int_{\{|k_\varepsilon| \geq n\}} |\mathbb{D}|.$$

Using Fatou Lemma for the liminf yields

$$\frac{1}{n} \int_{\Omega} \ell |\nabla S_n(k)|^2 \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega} \ell |\nabla S_n(k_\varepsilon)|^2 \leq \liminf_{\varepsilon \rightarrow 0} \int_{\{|k_\varepsilon| \geq n\}} |\mathbb{D}| \leq \limsup_{\varepsilon \rightarrow 0} \int_{\{|k_\varepsilon| \geq n\}} |\mathbb{D}|.$$

Then, Fatou Lemma for the limsup and Lemma A.3 yield

$$\limsup_{\varepsilon \rightarrow 0} \int_{\{|k_\varepsilon| \geq n\}} |\mathbb{D}| \leq \int_{\Omega} |\mathbb{D}| \limsup_{\varepsilon \rightarrow 0} \mathbb{1}_{\{|k_\varepsilon| \geq n\}} \leq \int_{\{|k| \geq n\}} |\mathbb{D}|.$$

We finally obtain

$$(3.25) \quad \frac{1}{n} \int_{\Omega} \ell |\nabla S_n(k)|^2 = \frac{1}{n} \int_{\{n \leq |k| \leq 2n\}} \ell |\nabla k|^2 \leq \int_{\{|k| \geq n\}} |\mathbb{D}| \xrightarrow{n \rightarrow +\infty} 0.$$

Assertion (3.12) is proved.

**Step 2.** We now prove that  $k$  satisfies (3.14). Let  $w \in W_{\ell,0} \cap L^\infty(\Omega)$  that satisfies (3.13), and  $z \in \mathbb{R}_+^*$  such that  $\nabla w = 0$  a.e. over  $\{|k| \geq z\}$ . The primary intention would be to take  $w$  as test function in (3.18). Therefore, we would have to deal with the term

$$\int_{\Omega} \ell \nabla k_\varepsilon \cdot \nabla w,$$

by using that  $(\sqrt{\ell} \nabla T_r(k_\varepsilon))_{\varepsilon > 0}$  weakly converges in  $L^2(\Omega)$  to  $\sqrt{\ell} \nabla T_r(k)$  as  $\varepsilon \rightarrow 0$ , for all  $r > 0$ . Unfortunately, we only know that  $\nabla w = 0$  a.e. over  $\{|k| \geq z\}$ , and we cannot directly use this for the  $k_\varepsilon$ 's. To overcome this issue, we have to localize the equation a little bit more. To do so, we introduce for a given  $n \in \mathbb{N}$ , the even function  $L_n \in \text{Stamp}(\mathbb{R})$  given by

$$(3.26) \quad L_n : t \mapsto \begin{cases} 1 & \text{if } 0 \leq t < n, \\ \frac{1}{n}(t + 2n) & \text{if } n \leq t \leq 2n, \\ 0 & \text{if } t > 2n. \end{cases}$$

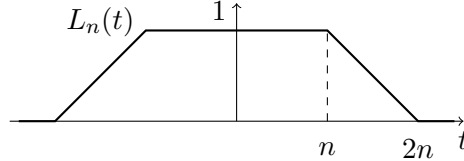


Figure 4: graph of the function  $L_n$ .

According to Theorem A.5,  $v = L_n(k_\varepsilon)w \in W_{\ell,0}$ . Therefore, it can be taken as test function in (3.18), which yields

$$(3.27) \quad \int_{\Omega} \ell \nabla k_\varepsilon \cdot \nabla w L_n(k_\varepsilon) + \int_{\Omega} \ell |\nabla k_\varepsilon|^2 w L_n(k_\varepsilon) + \int_{\Omega} \frac{k_\varepsilon w}{\ell} L_n(k_\varepsilon) = \int_{\Omega} \mathbb{D}_\varepsilon w L_n(k_\varepsilon).$$

The convergence of the two last terms is a direct consequence of Lebesgue's Theorem. On one hand, we have

$$(3.28) \quad \int_{\Omega} \mathbb{D}_\varepsilon w L_n(k_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \mathbb{D} w L_n(k).$$

And given that  $L_n \xrightarrow{n \rightarrow +\infty} 1$  a.e. in  $\mathbb{R}$ , we finally obtain

$$(3.29) \quad \int_{\Omega} \mathbb{D}_\varepsilon w L_n(k_\varepsilon) \xrightarrow[\substack{\varepsilon \rightarrow 0 \\ n \rightarrow +\infty}]{} \int_{\Omega} \mathbb{D} w.$$

On the other hand, with exactly the same method we obtain

$$(3.30) \quad \int_{\Omega} \frac{k_{\varepsilon} w}{\ell} L_n(k_{\varepsilon}) \xrightarrow[n \rightarrow +\infty]{\varepsilon \rightarrow 0} \int_{\Omega} \frac{k w}{\ell}.$$

Given the definition of  $L_n$ , the first integral of (3.27) is rewritten the following way

$$(3.31) \quad \int_{\Omega} \ell \nabla k_{\varepsilon} \cdot \nabla w L_n(k_{\varepsilon}) = \int_{\Omega} \ell \nabla T_{2n}(k_{\varepsilon}) \cdot \nabla w L_n(k_{\varepsilon}).$$

The previous results and Lebesgue Theorem yield

$$(3.32) \quad \ell^{1/2} \nabla w L_n(k_{\varepsilon}) \xrightarrow[\varepsilon \rightarrow 0]{} \ell^{1/2} \nabla w L_n(k) \text{ strong in } L^2(\Omega),$$

$$(3.33) \quad \ell^{1/2} \nabla T_{2n}(k_{\varepsilon}) \xrightarrow[\varepsilon \rightarrow 0]{} \ell^{1/2} \nabla T_{2n}(k) \text{ weak in } L^2(\Omega).$$

A classical integration result then yields, in particular when  $n \geq z$ ,

$$(3.34) \quad \int_{\Omega} \ell \nabla k_{\varepsilon} \cdot \nabla w L_n(k_{\varepsilon}) \xrightarrow[\varepsilon \rightarrow 0]{} \int_{\Omega} \ell \nabla T_{2n}(k) \cdot \nabla w L_n(k) = \int_{\Omega} \ell \nabla T_z(k) \cdot \nabla w L_n(k).$$

Finally, Lebesgue theorem yields

$$(3.35) \quad \int_{\Omega} \ell \nabla T_z(k) \cdot \nabla w L_n(k) \xrightarrow[n \rightarrow +\infty]{} \int_{\Omega} \ell T_z(k) \cdot \nabla w.$$

This corresponds to the convergence of the first term of (3.27)

$$(3.36) \quad \int_{\Omega} \ell \nabla k_{\varepsilon} \cdot \nabla w L_n(k_{\varepsilon}) \xrightarrow[n \rightarrow +\infty]{\varepsilon \rightarrow 0} \int_{\Omega} \ell T_z(k) \cdot \nabla w.$$

The last term may be bounded from above as follows

$$(3.37) \quad \left| \int_{\Omega} \ell |\nabla k_{\varepsilon}|^2 w L'_n(k_{\varepsilon}) \right| = \frac{1}{n} \left| \int_{\{n \leq |k_{\varepsilon}| \leq 2n\}} \ell |\nabla k_{\varepsilon}|^2 w \right| \leq \frac{\|w\|_{\Omega;0,\infty}}{n} \int_{\{n \leq |k_{\varepsilon}| \leq 2n\}} \ell |\nabla k_{\varepsilon}|^2,$$

which recalls what we studied in the previous step of this proof. Indeed, (3.24) yields

$$(3.38) \quad \frac{1}{n} \int_{\{n \leq |k_{\varepsilon}| \leq 2n\}} \ell |\nabla k_{\varepsilon}|^2 \leq \int_{\{|k_{\varepsilon}| \geq n\}} |\mathbb{D}|.$$

Both Fatou's Lemma for the limsup and Lemma A.3 then yield

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\{n \leq |k_{\varepsilon}| \leq 2n\}} \ell |\nabla k_{\varepsilon}|^2 \leq \int_{\{|k| \geq n\}} |\mathbb{D}|.$$

Finally, we obtain

$$(3.39) \quad \limsup_{\substack{n \rightarrow +\infty \\ \varepsilon \rightarrow 0}} \left| \int_{\Omega} \ell |\nabla k_{\varepsilon}|^2 w L'_n(k_{\varepsilon}) \right| \leq \|w\|_{\Omega;0,\infty} \limsup_{n \rightarrow +\infty} \int_{\{|k| \geq n\}} |\mathbb{D}| = 0.$$

By grouping together convergences (3.29), (3.30), (3.36) and (3.39), we obtain

$$\int_{\Omega} \ell \nabla T_z(k) \cdot \nabla w + \int_{\Omega} \frac{k w}{\ell} = \int_{\Omega} \mathbb{D} w,$$

which means, according to the remark 3.1, that  $k$  is indeed a renormalized solution of Problem (3.9).  $\square$

**Proof of Theorem 3.2.** We now prove the continuity property. Let  $(\mathbb{D}_\varepsilon)_{\varepsilon>0}$  be a  $L^\infty(\Omega)$  sequence satisfying (3.16) and (3.17), and  $(k_\varepsilon)_{\varepsilon>0}$  the corresponding sequence of renormalized solutions,  $k$  the corresponding solution with  $\mathbb{D}$  as source term. Let  $z > 0$ . We already know by Lemmas 3.1 and 3.2 that

$$(3.40) \quad k_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} k \text{ strong in } L^1(\Omega),$$

$$(3.41) \quad T_z(k_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} T_z(k) \text{ weak in } W_{\ell,0}.$$

Therefore, in order to conclude, it remains to prove the convergence of the energies, i.e.

$$(3.42) \quad \int_{\Omega} \ell |\nabla T_z(k_\varepsilon)|^2 + \int_{\Omega} \frac{|T_z(k_\varepsilon)|^2}{\ell} \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \ell |\nabla T_z(k)|^2 + \int_{\Omega} \frac{|T_z(k)|^2}{\ell}.$$

We first deduce from inequality (A.1) in the appendice that

$$(3.43) \quad \int_{\Omega} \frac{|T_z(k_\varepsilon) - T_z(k)|^2}{\ell} \leq 2z \int_{\Omega} \frac{|T_z(k_\varepsilon) - T_z(k)|}{\ell} \leq 2z^2 \int_{\Omega} \frac{|k_\varepsilon - k|}{\ell} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Consequently, it only remains to prove

$$(3.44) \quad \int_{\Omega} \ell |\nabla T_z(k_\varepsilon)|^2 \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \ell |\nabla T_z(k)|^2.$$

Taking  $v = T_z(k_\varepsilon)$  in (3.19) yields

$$(3.45) \quad \int_{\Omega} \ell |\nabla T_z(k_\varepsilon)|^2 + \int_{\Omega} \frac{k_\varepsilon T_z(k_\varepsilon)}{\ell} = \int_{\Omega} \mathbb{D}_\varepsilon T_z(k_\varepsilon).$$

First, we have

$$(3.46) \quad \int_{\Omega} \mathbb{D}_\varepsilon T_z(k_\varepsilon) = \int_{\Omega} (\mathbb{D}_\varepsilon - \mathbb{D}) T_z(k_\varepsilon) + \int_{\Omega} \mathbb{D} T_z(k_\varepsilon).$$

On one hand,

$$(3.47) \quad \left| \int_{\Omega} (\mathbb{D}_\varepsilon - \mathbb{D}) T_z(k_\varepsilon) \right| \leq z \int_{\Omega} |\mathbb{D}_\varepsilon - \mathbb{D}| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

On the other hand, it easily checked by using Lebesgue's Theorem that

$$(3.48) \quad \int_{\Omega} \mathbb{D} T_z(k_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \mathbb{D} T_z(k).$$

Similarly, we obtain

$$(3.49) \quad \int_{\Omega} \frac{k_\varepsilon T_z(k_\varepsilon)}{\ell} \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \frac{k T_z(k)}{\ell}.$$

This yields

$$(3.50) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \ell |\nabla T_z(k_\varepsilon)|^2 = \int_{\Omega} \mathbb{D} T_z(k) - \int_{\Omega} \frac{k T_z(k)}{\ell}.$$

It remains to prove this limit is equal to  $\int_{\Omega} \ell |\nabla T_z(k)|^2$ . To do so, we take  $v = T_z(k)$  in (3.14), which is allowed since  $w = T_z(k)$  satisfies (3.13). We get :

$$(3.51) \quad \int_{\Omega} \ell |\nabla T_z(k)|^2 + \int_{\Omega} \frac{u T_z(k)}{\ell} = \int_{\Omega} f T_z(k),$$

hence (3.44).

### 3.3 Uniqueness of the renormalized solution

We prove in this section the uniqueness of the renormalized solution. To do so, we need a maximum principle result, proved in the following Lemma.

**Lemma 3.3.** *Let  $g \in L^\infty(\Omega)$  and  $v \in W_{\ell,0}$  such that*

$$(3.52) \quad \forall \phi \in W_{\ell,0}, \quad \int_{\Omega} \ell \nabla v \cdot \nabla \phi + \int_{\Omega} \frac{v\phi}{\ell} = \int_{\Omega} g\phi.$$

*Then  $v \leq \|\ell g\|_{\Omega;0,\infty}$  a.e.  $\Omega$ .*

*Proof.* The proof follows the standard procedure (see in [15]). For any real valued function  $f$ , let  $f^+$  denotes its positive part, namely

$$f^+ = \sup(f, 0) = \frac{|f| + f}{2}.$$

Let  $M \in \mathbb{R}_+^*$ , and  $F_M$  be the function given by

$$(3.53) \quad F_M : \begin{cases} \mathbb{R} & \rightarrow & \mathbb{R} \\ t & \rightarrow & (t - M)^+, \end{cases}$$

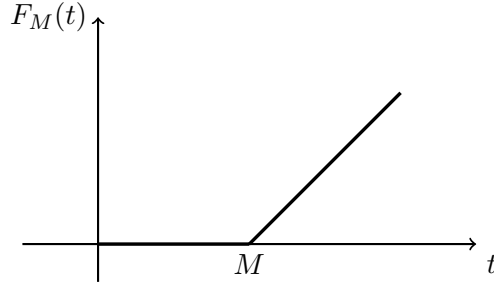


Figure 5: Graph of the function  $F_M(t) = (t - M)^+$  for  $M > 0$ .

We observe that  $F_M \in \text{Stamp}(\mathbb{R})$  and  $F(0) = 0$ . Therefore, by Theorem A.3

$$\tilde{v} = (v - M)^+ \in W_{\ell,0}.$$

It is easily checked that

$$\int_{\Omega} \frac{v\tilde{v}}{\ell} = \int_{\Omega} \frac{|\tilde{v}|^2}{\ell} + M \int_{\Omega} \frac{\tilde{v}}{\ell}, \quad \int_{\Omega} \ell \nabla v \cdot \nabla \tilde{v} = \int_{\Omega} \ell |\nabla \tilde{v}|^2.$$

Thus, taking  $\phi = \tilde{v}$  in (3.52) yields

$$(3.54) \quad \int_{\Omega} \ell |\nabla \tilde{v}|^2 + \int_{\Omega} \frac{|\tilde{v}|^2}{\ell} = \int_{\Omega} \left(g - \frac{M}{\ell}\right) \tilde{v}.$$

We take  $M = \|\ell g\|_{\Omega;0,\infty}$ , so that  $g - \frac{M}{\ell}$  is non-positive a.e.  $\Omega$ . We deduce that  $\|\tilde{v}\|_{H_\ell}^2 \leq 0$ , which means  $\tilde{v} = 0$  a.e. in  $\Omega$ . Hence the result.  $\square$

We are now in order to prove the uniqueness result, stated as follows.

**Theorem 3.3.** *Let  $\mathbb{D} \in L^1(\Omega)$ . Then the renormalized solution to Problem (3.9) is unique.*

*Proof.* Starting as in the proof of Theorem 3.1, we consider a family  $(\mathbb{D}_\varepsilon)_{\varepsilon>0}$  in  $L^\infty(\Omega)$  satisfying (3.16) and (3.17),  $k_\varepsilon \in W_{\ell,0}$  which satisfies in the weak sense specified by Theorem 2.1,

$$(3.55) \quad \begin{cases} -\operatorname{div}(\ell \nabla k_\varepsilon) + \frac{k_\varepsilon}{\ell} = \mathbb{D}_\varepsilon & \text{in } \Omega, \\ k_\varepsilon = 0 & \text{at } \Gamma, \end{cases}$$

Let  $k^*$  be any renormalized solution to Problem (3.9). We aim to prove that  $k^* = \lim_{\varepsilon \rightarrow 0} k_\varepsilon$ . Let us take

$$(3.56) \quad w_\varepsilon = \mathbb{H}_\eta(T_k(k^*) - k_\varepsilon),$$

where  $\mathbb{H}_\eta$  is defined by (3.5). As we will let  $\eta$  go to zero, we can consider  $\eta \in ]0; 1[$ . We know from Lemma 3.3 that

$$k_\varepsilon \leq \|\ell \mathbb{D}_\varepsilon\|_{\Omega;0,\infty},$$

which suggests to take

$$z_\varepsilon = \|\ell \mathbb{D}_\varepsilon\|_{\Omega;0,\infty} + 1$$

as threshold, giving

$$(3.57) \quad \mathbb{H}'_\eta(T_{z_\varepsilon}(k^*) - k_\varepsilon) = 0 \quad \text{a.e. on } \{|k^*| \geq z_\varepsilon\}.$$

We deduce in particular that  $\nabla w_\varepsilon = 0$  a.e. on  $\{|k^*| \geq z_\varepsilon\}$ . Moreover,

- $w_\varepsilon \in L^\infty$  since  $\mathbb{H}_\eta$  is bounded by 1,
- $w_\varepsilon \in W_{\ell,0}$  since  $\mathbb{H}_\eta \in \text{Stamp}(\mathbb{R})$  and  $\mathbb{H}_\eta(0) = 0$ .

we deduce that  $w_\varepsilon$  satisfies (3.13). Therefore, we can take  $w = w_\varepsilon$  in formulation (3.12) and  $v = w_\varepsilon$  in formulation (3.19), which gives

$$(3.58) \quad \int_{\Omega} \ell \nabla k^* \cdot \nabla w_\varepsilon + \int_{\Omega} \frac{k^* w_\varepsilon}{\ell} = \int_{\Omega} \mathbb{D} w_\varepsilon,$$

$$(3.59) \quad \int_{\Omega} \ell \nabla k_\varepsilon \cdot \nabla w_\varepsilon + \int_{\Omega} \frac{k_\varepsilon w_\varepsilon}{\ell} = \int_{\Omega} \mathbb{D}_\varepsilon w_\varepsilon.$$

We write the difference and replace  $w_\varepsilon$  by its expression

$$(3.60) \quad \begin{aligned} \int_{\Omega} \ell(\nabla k^* - \nabla k_\varepsilon) \cdot (\nabla T_{z_\varepsilon}(k^*) - \nabla k_\varepsilon) \mathbb{H}'_\eta(T_{z_\varepsilon}(k^*) - k_\varepsilon) + \int_{\Omega} \frac{k^* - k_\varepsilon}{\ell} \mathbb{H}_\eta(T_{z_\varepsilon}(k^*) - k_\varepsilon) \\ = \int_{\Omega} (\mathbb{D} - \mathbb{D}_\varepsilon) \mathbb{H}_\eta(T_{z_\varepsilon}(k^*) - k_\varepsilon). \end{aligned}$$

Recalling (3.57), we obtain

Therefore,

$$\begin{aligned} \int_{\Omega} \ell(\nabla k^* - \nabla k_\varepsilon) \cdot (\nabla T_{z_\varepsilon}(k^*) - \nabla k_\varepsilon) \mathbb{H}'_\eta(T_{z_\varepsilon}(k^*) - k_\varepsilon) = \\ \int_{\Omega} \ell |\nabla T_{z_\varepsilon}(k^*) - \nabla k_\varepsilon|^2 \mathbb{H}'_\eta(T_{z_\varepsilon}(k^*) - k_\varepsilon) \geq 0, \end{aligned}$$

since  $\mathbb{H}'_\eta \geq 0$ . Moreover, we deduce from Lebesgue's Theorem,

$$(3.61) \quad \int_{\Omega} \frac{k^* - k_\varepsilon}{\ell} \mathbb{H}_\eta (T_{z_\varepsilon}(k^*) - k_\varepsilon) \xrightarrow{\eta \rightarrow 0} \int_{\Omega} \frac{|k^* - k_\varepsilon|}{\ell}.$$

Finally, as  $|\mathbb{H}_\eta| \leq 1$ ,

$$(3.62) \quad \left| \int_{\Omega} (\mathbb{D} - \mathbb{D}_\varepsilon) \mathbb{H}_\eta (T_{z_\varepsilon}(k^*) - k_\varepsilon) \right| \leq \|\mathbb{D} - \mathbb{D}_\varepsilon\|_{\Omega;0,1}.$$

Consequently, when  $\eta$  goes to zero, we get from (3.60)

$$(3.63) \quad \int_{\Omega} \frac{|k^* - k_\varepsilon|}{\ell} \leq \|\mathbb{D} - \mathbb{D}_\varepsilon\|_{\Omega;0,1} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

which is what we aimed to prove. The uniqueness of the limit in  $L^1(\Omega, \ell^{-1}dx)$  directly yields the uniqueness of the renormalized solution.  $\square$

### 3.4 Equivalent definition

We give another definition of the renormalized solution to Problem (3.9). In the following, we denote

$$W = \{\beta \in \text{Stamp}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ with a compact support}\}.$$

**Definition 3.3.** *A function  $k \in H^1_{loc}(\Omega)$  is a renormalized solution to Problem (3.9) if  $u$  satisfies (3.10), (3.11) and (3.12) and*

$$\forall \beta \in W, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

$$(3.64) \quad \int_{\Omega} \ell \nabla u \cdot \nabla \varphi \beta(k) + \int_{\Omega} \ell |\nabla u|^2 \varphi \beta'(k) + \int_{\Omega} \frac{u \varphi}{\ell} \beta(k) = \int_{\Omega} \mathbb{D} \varphi \beta(k).$$

**Theorem 3.4** (Equivalence between both definitions). *The definitions 3.2 and 3.3 are equivalent.*

*Proof.* Let  $k \in H^1_{loc}(\Omega)$  that satisfies the definition 3.2,  $\beta \in W$  and  $\varphi \in \mathcal{D}(\Omega)$ . We first notice that according to Theorem A.3,

$$\beta(k) \varphi \in W_{\ell,0}, \quad \text{and} \quad |\beta(k) \varphi| \leq \|\beta\|_\infty \|\varphi\|_{0,\infty,\Omega},$$

hence  $\beta(k) \varphi \in W_{\ell,0} \cap L^\infty$ . We must show that  $w = \beta(k) \varphi$  satisfies (3.13). Let  $z \in \mathbb{R}^*$  such that  $\text{Supp}(\beta) \subset [-z, z]$ . The function  $\nabla(\beta(k) \varphi) = \beta'(k) \nabla k \varphi + \beta(k) \nabla \varphi$  vanishes a.e. in  $\{|k| \geq z\}$ , which means  $\beta(k) \varphi$  satisfies (3.13).

Now we assume that  $k \in H^1_{loc}(\Omega)$  satisfies the definition 3.3. Let  $w \in W_{\ell,0} \cap L^\infty$  that satisfies (3.13), and  $(w_m)_{m \in \mathbb{N}}$  be a sequence of smooth functions given by Theorem A.4. Taking  $\beta = L_n$  (defined by (3.26)) and  $\varphi = w_m$  in (3.64) yields

$$(3.65) \quad \int_{\Omega} \ell \nabla k \cdot \nabla w_m L_n(k) + \int_{\Omega} \ell |\nabla k|^2 w_m L'_n(k) + \int_{\Omega} \frac{w w_m}{\ell} L_n(k) = \int_{\Omega} \mathbb{D} w_m L_n(k).$$

The  $L^\infty$  bound on  $(w_m)_{m \in \mathbb{N}}$  given by Theorem A.4 and the Lebesgue theorem yield

$$(3.66) \quad \int_{\Omega} \mathbb{D}w_m L_n(k) \xrightarrow{m \rightarrow +\infty} \int_{\Omega} \mathbb{D}w L_n(k),$$

$$(3.67) \quad \int_{\Omega} \frac{k w_m}{\ell} L_n(k) \xrightarrow{m \rightarrow +\infty} \int_{\Omega} \frac{k w}{\ell} L_n(k),$$

$$(3.68) \quad \int_{\Omega} \ell |\nabla k|^2 w_m L'_n(k) \xrightarrow{m \rightarrow +\infty} \int_{\Omega} \ell |\nabla k|^2 w L'_n(k),$$

for all  $n \in \mathbb{N}$ . Moreover, the convergence  $w_m \xrightarrow{m \rightarrow +\infty} w$  in  $H_\ell$  directly yields

$$(3.69) \quad \int_{\Omega} \ell \nabla k \cdot \nabla w_m L_n(k) \xrightarrow{m \rightarrow +\infty} \int_{\Omega} \ell \nabla k \cdot \nabla w L_n(k),$$

for all  $n \in \mathbb{N}$ . Finally, Lebesgue's Theorem yields

$$(3.70) \quad \int_{\Omega} \mathbb{D}w L_n(k) \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \mathbb{D}w,$$

$$(3.71) \quad \int_{\Omega} \frac{k w}{\ell} L_n(k) \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \frac{k w}{\ell},$$

$$(3.72) \quad \int_{\Omega} \ell \nabla k \cdot \nabla w L_n(k) \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \ell \nabla k \cdot \nabla w.$$

And (3.12) yields

$$(3.73) \quad \left| \int_{\Omega} \ell |\nabla k|^2 w L'_n(k) \right| \leq \frac{1}{n} \int_{\{n \leq |k| \leq 2n\}} \ell |\nabla k|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

This yields (3.14). □

## A Technical results

### A.1 Regularity results in $W_{\ell,0}$

We begin by recalling a result of [21].

**Lemma A.1.** *Let  $u \in W^{1,1}(\Omega)$ , and  $a \in \mathbb{R}$  such that  $\{u = a\}$  has a non-zero Lebesgue measure. Then  $\nabla u = 0$  almost everywhere on  $\{u = a\}$ .*

We recall the following definition.

**Definition A.1** (Stampacchia conditions). *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . We say that  $F$  satisfies the Stampacchia conditions if  $F$  is globally Lipschitz and its derivative  $F'$  has a finite number of discontinuities.*

**Remark A.1.** *Note that if  $F$  satisfies the Stampacchia conditions, it is not necessarily bounded.*

We denote by  $\text{Stamp}(\mathbb{R})$  the set of all functions satisfying the Stampacchia conditions.

**Lemma A.2.** *Let  $F \in \text{Stamp}(\mathbb{R})$ . Then  $F' \in L^\infty(\mathbb{R})$  and we have*

$$(A.1) \quad \forall a, b \in \mathbb{R}, \quad |F(b) - F(a)| \leq \|F'\|_\infty |b - a|.$$

*Proof.* Let  $(t_j)_{1 \leq j \leq J}$  be defined as the points where  $F'$  is not continuous. Let the intervals  $(I_j)_{0 \leq j \leq J}$  be defined by

$$(A.2) \quad \begin{cases} I_0 = ] - \infty; t_1[, \\ I_j = ]t_j; t_{j+1}[ \text{ for } 1 \leq j \leq J - 1, \\ I_J = ]t_J; +\infty[. \end{cases}$$

We consider  $j \in \llbracket 0; J \rrbracket$ . The function  $F$  is continuous on  $\overline{I_j}$  and differentiable on  $I_j$ . Then for any  $x, y \in \overline{I_j}$  with  $x < y$ , there exists  $c \in ]x; y[$  such that

$$(A.3) \quad F(y) - F(x) = F'(c)(y - x).$$

We now consider  $j \in \llbracket 0; J \rrbracket$  and  $a \in I_j$  and  $(a_n)_{n \in \mathbb{N}}$  a sequence in  $I_j$  satisfying  $a_n \xrightarrow[n \rightarrow +\infty]{} a$ . We assume without losing generality that for every  $n \in \mathbb{N}$ ,  $a_n > a$ . Then, for any  $n \in \mathbb{N}$  there exists a  $c_n \in ]a; a_n[$  satisfying

$$(A.4) \quad F(a_n) - F(a) = F'(c_n)(a_n - a).$$

Given that  $F'$  is continuous on  $I_j$ , we obtain

$$(A.5) \quad |F'(a)| = \limsup_{n \rightarrow +\infty} |F'(c_n)| = \limsup_{n \rightarrow +\infty} \frac{|F(a_n) - F(a)|}{a_n - a} \leq \lambda,$$

where  $\lambda > 0$  is a Lipschitz constant for  $F$ .

Consequently,  $F'$  is bounded a.e. on  $\mathbb{R}$ , which means  $F' \in L^\infty(\mathbb{R})$ . It remains to prove (A.1).

Let  $a, b \in \mathbb{R}$ . If  $a$  and  $b$  are on the same  $\overline{I_j}$ , the mean value theorem directly yields (A.1). We now assume that there exists  $j \in \llbracket 0; J - 1 \rrbracket$  such that  $a \in \overline{I_j}$  and  $b \in \overline{I_{j+1}}$ . Then we obtain

$$(A.6) \quad \begin{aligned} |F(b) - F(a)| &\leq |F(b) - F(t_{j+1})| + |F(t_{j+1}) - F(a)| \\ &\leq \|F'\|_\infty(b - t_{j+1}) + \|F'\|_\infty(t_{j+1} - a) = \|F'\|_\infty(b - a). \end{aligned}$$

Finally, an instant induction yields (A.1). □

The following theorem is an adaptation of a Stampacchia result from [21], adapted to the space  $H_\ell$ .

**Theorem A.1.** *Let  $u \in W_{\ell,0}$  and  $F \in \text{Stamp}(\mathbb{R})$ . Then we have the following equality at the distribution sense*

$$\nabla F(u) = F'(u)\nabla u.$$

*Proof.* Let  $u \in W_{\ell,0}$ , and  $(u_n)_{n \in \mathbb{N}}$  a sequence of  $\mathcal{D}(\Omega)$  functions such that  $u_n \xrightarrow[n \rightarrow +\infty]{} u$  in  $H_\ell$ .

Let  $\theta$  be characterized by

- $\theta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$ ,
- $\text{Supp}(\theta) \subset \bar{B}(0, 1)$ ,
- $\int_{\mathbb{R}} \theta(y) dy = 1$ .

Let  $(\theta_\varepsilon)_{\varepsilon>0}$  be the mollifier given by

$$(A.7) \quad \forall \varepsilon > 0, \forall y \in \mathbb{R}, \quad \theta_\varepsilon(y) = \frac{1}{\varepsilon} \theta\left(\frac{y}{\varepsilon}\right),$$

and  $(F_\varepsilon)_{\varepsilon>0}$  be defined by  $F_\varepsilon = \theta_\varepsilon * F$ ,  $(t_j)_{1 \leq j \leq J}$  be the points where  $F'$  is not continuous. Lebesgue theorem and Young convolution inequality yield

$$(A.8) \quad F'_\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} F'(t) \quad \text{for all } t \in \mathbb{R} \setminus (t_j)_{1 \leq j \leq J},$$

$$(A.9) \quad \|F'_\varepsilon\|_\infty \leq \|F'\|_\infty \quad \text{for all } \varepsilon > 0.$$

Moreover, a linear change of variables and (A.1) yield

$$(A.10) \quad \begin{aligned} |F_\varepsilon(t) - F(t)| &\leq \int_{\mathbb{R}} \theta_\varepsilon(y) |F(t-y) - F(t)| dy = \int_{\mathbb{R}} \theta(z) |F(t-\varepsilon z) - F(t)| dz \\ &\leq \varepsilon \|F'\|_\infty \int_{\mathbb{R}} |z| \theta(z) dz \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

This yields the uniform convergence of  $(F_\varepsilon)_{\varepsilon>0}$ , i.e.

$$(A.11) \quad \|F_\varepsilon - F\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The classical results about convolution products tell that  $F_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ . Therefore, for every  $\varepsilon > 0$  and  $n \in \mathbb{N}$ ,  $F_\varepsilon(u_n) \in \mathcal{C}^\infty(\Omega)$ , giving in particular  $\nabla F_\varepsilon(u_n) = F'_\varepsilon(u_n) \nabla u_n$ . Let  $\varphi \in \mathcal{D}(\Omega)$ , we write

$$(A.12) \quad \int_{\Omega} \nabla F_\varepsilon(u_n) \varphi = - \int_{\Omega} F_\varepsilon(u_n) \nabla \varphi = - \int_{\Omega} (F_\varepsilon(u_n) - F_\varepsilon(u)) \nabla \varphi - \int_{\Omega} F_\varepsilon(u) \nabla \varphi.$$

On one hand, the regularity of  $F_\varepsilon$  and the inequality (A.9) yield

$$(A.13) \quad \begin{aligned} \left| \int_{\Omega} (F_\varepsilon(u_n) - F_\varepsilon(u)) \nabla \varphi \right| &\leq \|F'_\varepsilon\|_\infty \int_{\Omega} |u_n - u| \cdot |\nabla \varphi| \\ &\leq \|F'\|_\infty \|\ell^{-1/2}(u_n - u)\|_{\Omega;0,2} \|\ell^{1/2} \nabla \varphi\|_{\Omega;0,2} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

On the other hand, since  $(F_\varepsilon)_{\varepsilon>0}$  uniformly converges to  $F$ , we obtain

$$(A.14) \quad \left| \int_{\Omega} (F_\varepsilon(u) - F(u)) \nabla \varphi \right| \leq \|F_\varepsilon - F\|_\infty \|\nabla \varphi\|_{\Omega;0,1} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

hence  $\int_{\Omega} F_\varepsilon(u) \nabla \varphi \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} F(u) \nabla \varphi$ . This yields

$$(A.15) \quad \int_{\Omega} \nabla F_\varepsilon(u_n) \varphi = - \int_{\Omega} F_\varepsilon(u_n) \nabla \varphi \xrightarrow[n \rightarrow +\infty]{\varepsilon \rightarrow 0} - \int_{\Omega} F(u) \nabla \varphi = \int_{\Omega} \nabla F(u) \varphi.$$

We now need to show the convergence  $F'_\varepsilon(u_n) \nabla u_n \xrightarrow[n \rightarrow +\infty]{\varepsilon \rightarrow 0} F'(u) \nabla u$  at the distribution sense. Let  $\varphi \in \mathcal{D}(\Omega)$ , we write

$$(A.16) \quad \int_{\Omega} F'_\varepsilon(u_n) \nabla u_n \varphi = \int_{\Omega} F'_\varepsilon(u_n) (\nabla u_n - \nabla u) \varphi + \int_{\Omega} F'_\varepsilon(u_n) \nabla u \varphi.$$

On one hand, (A.9) yields

$$(A.17) \quad \left| \int_{\Omega} F'_{\varepsilon}(u_n)(\nabla u_n - \nabla u)\varphi \right| \leq \|F'\|_{\infty} \|\ell^{1/2}(\nabla u_n - \nabla u)\|_{0,2,\Omega} \|\ell^{-1/2}\varphi\|_{0,2,\Omega} \xrightarrow{n \rightarrow +\infty} 0.$$

On the other hand, there exists a subsequence of  $(u_n)_{n \in \mathbb{N}}$  still named the same, such that  $u_n \xrightarrow{n \rightarrow +\infty} u$  a.e. on  $\Omega$ , and the continuity of  $F'_{\varepsilon}$  yields  $F'_{\varepsilon}(u_n) \xrightarrow{n \rightarrow +\infty} F'_{\varepsilon}(u)$  a.e. on  $\Omega$ . Given that  $|\nabla u\varphi| \in L^1(\Omega)$ , we deduce from Lebesgue theorem that

$$(A.18) \quad \int_{\Omega} F'_{\varepsilon}(u_n)\nabla u\varphi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} F'_{\varepsilon}(u)\nabla u\varphi.$$

Finally, convergence (A.8) and Lemma A.1 yield that  $F'_{\varepsilon}(u)\nabla u \xrightarrow{\varepsilon \rightarrow 0} F'(u)\nabla u$  a.e. on  $\Omega$ . We obtain by Lebesgue theorem

$$(A.19) \quad \int_{\Omega} F'_{\varepsilon}(u)\nabla u\varphi \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} F'(u)\nabla u\varphi.$$

This yields

$$(A.20) \quad \int_{\Omega} F'_{\varepsilon}(u_n)\nabla u_n\varphi \xrightarrow[\varepsilon \rightarrow 0]{n \rightarrow +\infty} \int_{\Omega} F'(u)\nabla u\varphi.$$

Both convergences (A.15) and (A.20) yield the result.  $\square$

**Theorem A.2.** *Let  $F \in \text{Stamp}(\mathbb{R})$ ,  $(u_n)_{n \in \mathbb{N}}$  be a converging sequence in  $H_{\ell}$  and  $u \in H_{\ell}$  its limit.*

(i) *We assume that  $F(w) \in H_{\ell}$  for every  $w \in H_{\ell}$ . Then*

$$(A.21) \quad F(u_n) \xrightarrow{n \rightarrow +\infty} F(u) \text{ in } H_{\ell}.$$

(ii) *Let  $\varphi \in \mathcal{D}(\Omega)$ , we have*

$$(A.22) \quad F(u_n)\varphi \xrightarrow{n \rightarrow +\infty} F(u)\varphi \text{ in } H_{\ell}.$$

*Proof.* We only do the proof of (i), (ii) being proved similarly. We deduce first from (A.1) that

$$(A.23) \quad \int_{\Omega} \frac{|F(u_n) - F(u)|^2}{\ell} \leq \|F'\|_{\infty}^2 \int_{\Omega} \frac{|u_n - u|^2}{\ell} \xrightarrow{n \rightarrow +\infty} 0.$$

Then, we apply Theorem A.1 and triangular inequality on the derivative term to obtain

$$(A.24) \quad \begin{aligned} \|\ell^{1/2}(\nabla F(u_n) - \nabla F(u))\|_{\Omega;0,2} &= \|\ell^{1/2}(F'(u_n)\nabla u_n - F'(u)\nabla u)\|_{\Omega;0,2} \\ &\leq \|\ell^{1/2}F'(u_n)(\nabla u_n - \nabla u)\|_{\Omega;0,2} + \|\ell^{1/2}(F'(u_n) - F'(u))\nabla u\|_{\Omega;0,2}. \end{aligned}$$

The convergence of the first norm is straightforward. Indeed, we have

$$(A.25) \quad \|\ell^{1/2}F'(u_n)(\nabla u_n - \nabla u)\|_{\Omega;0,2} \leq \|F'\|_{\infty} \|\ell^{1/2}(\nabla u_n - \nabla u)\|_{\Omega;0,2} \xrightarrow{n \rightarrow +\infty} 0.$$

The convergence of the last norm in (A.24) is trickier. Let  $(t_j)_{1 \leq j \leq J}$  be defined as the discontinuities of  $F'$  and  $(I_j)_{0 \leq j \leq J}$  as in (A.2). We define two families of subsets of  $\Omega$

$$(A.26) \quad \begin{cases} B_j = \{u \in I_j\} & \text{for } 0 \leq j \leq J, \\ A_j = \{u = t_j\} & \text{for } 1 \leq j \leq J. \end{cases}$$

Hence the decomposition

$$(A.27) \quad \begin{aligned} \|\ell^{1/2}(F'(u_n) - F'(u))\nabla u\|_{\Omega;0,2}^2 &= \int_{\Omega} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 \\ &= \sum_{j=0}^J \int_{B_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 + \sum_{j=1}^J \int_{A_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2. \end{aligned}$$

We cannot apply immediately Lemma A.1, because  $u \notin W^{1,1}(\Omega)$ . Then we have to find another way. Let  $\varepsilon > 0$ . Since  $\ell |\nabla u|^2 \in L^1(\Omega)$ , we deduce the existence of  $\delta > 0$  such that for any borelian subset  $A$  of  $\Omega$ ,

$$(A.28) \quad |A| \leq \delta \Rightarrow \int_A \ell |\nabla u|^2 \leq \varepsilon.$$

Let  $j \in \llbracket 1; J \rrbracket$ , we write

$$(A.29) \quad \int_{A_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 \leq 2 \|F'\|_{\infty}^2 \int_{A_j} \ell |\nabla u|^2.$$

We consider  $\omega \subset\subset \Omega$  such that  $|\Omega \setminus \omega| \leq \delta$ , and write the decomposition

$$(A.30) \quad \int_{A_j} \ell |\nabla u|^2 = \int_{A_j \cap \omega} \ell |\nabla u|^2 + \int_{A_j \cap (\Omega \setminus \omega)} \ell |\nabla u|^2.$$

On one hand, (A.28) yields

$$(A.31) \quad \int_{A_j \cap (\Omega \setminus \omega)} \ell |\nabla u|^2 \leq \varepsilon.$$

On the other hand, recalling that  $u \in H^1(\omega)$ , Lemma A.1 yields

$$(A.32) \quad \nabla u = 0 \quad \text{a.e. } A_j \cap \omega.$$

We have just shown

$$(A.33) \quad \forall \varepsilon > 0, \quad \int_{A_j} \ell |\nabla u|^2 \leq \varepsilon \quad \text{hence} \quad \int_{A_j} \ell |\nabla u|^2 = 0.$$

Hence, (A.29) yields

$$(A.34) \quad \int_{A_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 = 0.$$

Let now  $j$  be an index between 0 and  $J$ , the function  $F'$  is continuous on  $B_j$ . The Lebesgue inverse theorem then yields the existence of a subsequence of  $(F'(u_n))_{n \in \mathbb{N}}$ , denoted in the same way, converging to  $F'(u)$  a.e.  $\Omega$ . We obtain for all  $j$

$$(A.35) \quad \int_{B_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 \xrightarrow{n \rightarrow +\infty} 0,$$

with the Lebesgue theorem.

We have then shown that a subsequence of  $(u_n)_{n \in \mathbb{N}}$  still named the same satisfies

$$(A.36) \quad F(u_n) \xrightarrow{n \rightarrow +\infty} F(u) \text{ in } H_\ell.$$

We recall that  $H_\ell \hookrightarrow L^1(\Omega)$  is a compact embedding. In addition, (A.1) and Hölder inequality yield

$$(A.37) \quad \begin{aligned} \int_{\Omega} |F(u_n) - F(u)| &\leq \|F'\|_{\infty} \int_{\Omega} |u_n - u| \\ &\leq \|F'\|_{\infty} \left( \int_{\Omega} \ell \right)^{1/2} \left( \int_{\Omega} \frac{|u_n - u|^2}{\ell} \right)^{1/2} \xrightarrow{n \rightarrow +\infty} 0, \end{aligned}$$

which means  $F(u_n) \xrightarrow{n \rightarrow +\infty} F(u)$  in  $L^1(\Omega)$ . We can deduce the convergence of the whole sequence

$$(A.38) \quad F(u_n) \xrightarrow{n \rightarrow +\infty} F(u) \text{ in } H_\ell.$$

□

**Theorem A.3.** *Let  $F \in \text{Stamp}(\mathbb{R})$ ,  $u \in W_{\ell,0}$ .*

(i) *Let  $\varphi \in \mathcal{D}(\Omega)$ , then  $F(u)\varphi \in W_{\ell,0}$ .*

(ii) *We assume that  $F(0) = 0$ . Then  $F(u) \in W_{\ell,0}$ .*

*Proof. We start by proving (i) :* we first show that  $F(u)\varphi \in H_\ell$ .

Let  $v \in \mathcal{D}(\Omega)$ , so that  $\|u - v\|_{H_\ell} < +\infty$ . We first apply triangular inequality to obtain

$$(A.39) \quad \|\ell^{-1/2}F(u)\varphi\|_{\Omega;0,2} \leq \|\ell^{-1/2}(F(u) - F(v))\varphi\|_{\Omega;0,2} + \|\ell^{-1/2}F(v)\varphi\|_{\Omega;0,2}.$$

On one hand, (A.1) yields

$$(A.40) \quad \int_{\Omega} \frac{|\varphi|^2}{\ell} |F(u) - F(v)|^2 \leq \|F'\|_{\infty}^2 \|\varphi\|_{\Omega;0,\infty}^2 \int_{\Omega} \frac{|u - v|^2}{\ell} < +\infty.$$

On the other hand, since  $v \in \mathcal{D}(\Omega)$  and  $F \in \mathcal{C}^0(\mathbb{R})$ , we obtain  $F(v) \in \mathcal{C}^0(\Omega)$  hence  $F(v)\varphi \in \mathcal{C}_c^0(\Omega)$ . This directly yields

$$(A.41) \quad \int_{\Omega} \frac{|F(v)\varphi|^2}{\ell} < +\infty, \text{ hence } \int_{\Omega} \frac{|F(u)\varphi|^2}{\ell} < +\infty.$$

Concerning the derivative term, the gradient of a product and triangular inequality yield

$$(A.42) \quad \|\ell^{1/2}\nabla(F(u)\varphi)\|_{\Omega;0,2} \leq \|\ell^{1/2}\nabla F(u)\varphi\|_{\Omega;0,2} + \|\ell^{1/2}F(u)\nabla\varphi\|_{\Omega;0,2}.$$

We first focus on the last term, and write

$$(A.43) \quad \|\ell^{1/2}F(u)\nabla\varphi\|_{\Omega;0,2} \leq \|\ell^{1/2}F(v)\nabla\varphi\|_{\Omega;0,2} + \|\ell^{1/2}(F(u) - F(v))\nabla\varphi\|_{\Omega;0,2}.$$

It is clear that  $\|\ell^{1/2}F(v)\nabla\varphi\|_{\Omega;0,2} < +\infty$ . Then, (A.1) yields

$$(A.44) \quad \begin{aligned} \|\ell^{1/2}(F(u) - F(v))\nabla\varphi\|_{\Omega;0,2} &\leq \|F'\|_{\infty} \|\ell^{1/2}(u - v)\nabla\varphi\|_{\Omega;0,2} \\ &\leq \|F'\|_{\infty} \|\ell\nabla\varphi\|_{\Omega;0,\infty} \|\ell^{-1/2}(u - v)\|_{\Omega;0,2} < +\infty. \end{aligned}$$

We now have to decompose the other term

$$(A.45) \quad \begin{aligned} \|\ell^{1/2}\nabla F(u)\varphi\|_{\Omega;0,2} &= \|\ell^{1/2}F'(u)(\nabla u)\varphi\|_{\Omega;0,2} \\ &\leq \|\ell^{1/2}F'(u)(\nabla u - \nabla v)\varphi\|_{\Omega;0,2} + \|\ell^{1/2}F'(u)(\nabla v)\varphi\|_{\Omega;0,2}. \end{aligned}$$

On one hand, (A.1) yields

$$(A.46) \quad \int_{\Omega} \ell|\varphi|^2|F'(u)(\nabla u - \nabla v)|^2 \leq \|F'\|_{\infty}^2 \|\varphi\|_{\Omega;0,\infty}^2 \int_{\Omega} \ell|\nabla u - \nabla v|^2 < +\infty.$$

On the other hand, since  $\nabla v \in \mathcal{D}(\Omega)$  we obtain  $\varphi\nabla v \in \mathcal{D}(\Omega)$ . Hence

$$(A.47) \quad \int_{\Omega} \ell|\varphi|^2|F'(u)\nabla v|^2 \leq \|F'\|_{\infty}^2 \int_{\Omega} \ell|\varphi\nabla v|^2 < +\infty.$$

As a result, the derivative term is finite. This means that  $F(u)\varphi \in H_{\ell}$ .

We now have to prove that  $F(u)\varphi$  is a limit of  $\mathcal{D}(\Omega)$  functions. Since  $u \in W_{\ell,0}$ , there exists a sequence  $(u_n)_{n \in \mathbb{N}}$  of smooth functions such that  $u_n \xrightarrow{n \rightarrow +\infty} u$  in  $H_{\ell}$ . We also consider the approximation  $F_{\varepsilon} = \theta_{\varepsilon} * F$  for  $\varepsilon > 0$  as in the proof of Theorem A.1.

We notice that for every  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , the function  $F_{\varepsilon}(u_n)$  is in  $\mathcal{C}^{\infty}(\Omega)$ , hence  $F_{\varepsilon}(u_n)\varphi \in \mathcal{D}(\Omega)$ . Consequently, it remains to prove the following convergence

$$(A.48) \quad F_{\varepsilon}(u_n)\varphi \xrightarrow[\varepsilon \rightarrow 0]{n \rightarrow +\infty} F(u)\varphi \text{ in } H_{\ell}.$$

Yet, we have

$$\|(F_{\varepsilon}(u_n) - F(u))\varphi\|_{H_{\ell}} \leq \|(F_{\varepsilon}(u_n) - F(u_n))\varphi\|_{H_{\ell}} + \|(F(u_n) - F(u))\varphi\|_{H_{\ell}}.$$

Theorem A.2 yields that  $\|(F(u_n) - F(u))\varphi\|_{H_{\ell}} \xrightarrow{n \rightarrow +\infty} 0$ , so that it only remains to prove

$$(A.49) \quad F_{\varepsilon}(v)\varphi \xrightarrow[\varepsilon \rightarrow 0]{H_{\ell}} F(v)\varphi, \text{ for all } v \in \mathcal{D}(\Omega).$$

First we write

$$(A.50) \quad \|\ell^{-1/2}(F_{\varepsilon}(v) - F(v))\varphi\|_{\Omega;0,2} \leq \|F_{\varepsilon} - F\|_{\infty} \|\ell^{-1/2}\varphi\|_{\Omega;0,2} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Let the two families of subsets of  $\Omega$ ,  $(A_j)_{1 \leq j \leq J}$  and  $(B_j)_{0 \leq j \leq J}$  defined similarly to (A.26), but with  $v$ . We obtain the decomposition of the derivative term

$$(A.51) \quad \begin{aligned} \int_{\Omega} \ell|\nabla F_{\varepsilon}(v) - \nabla F(v)|^2 &= \int_{\Omega} \ell|\nabla v|^2|F'_{\varepsilon}(v) - F'(v)|^2 \\ &= \sum_{j=0}^J \int_{B_j} \ell|\nabla v|^2|F'_{\varepsilon}(v) - F'(v)|^2 + \sum_{j=1}^J \int_{A_j} \ell|\nabla v|^2|F'_{\varepsilon}(v) - F'(v)|^2. \end{aligned}$$

By a similar method to the proof of Theorem A.2, the integrals on  $A_j$  vanish. Let  $j \in \llbracket 0; J \rrbracket$ , the convergence (A.8) yields that  $F'_{\varepsilon}(v) \xrightarrow{\varepsilon \rightarrow 0} F'(v)$  a.e.  $B_j$ . Moreover, by (A.9) we get

$$\ell|\nabla v|^2|F'_{\varepsilon}(v) - F'(v)|^2 \leq 2\|F'\|_{\infty}^2 \ell|\nabla v|^2 \in L^{\infty}(\Omega) \subset L^1(\Omega).$$

Consequently, the Lebesgue theorem yields the convergence

$$(A.52) \quad \int_{B_j} \ell |\nabla v|^2 |F'_\varepsilon(v) - F'(v)|^2 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

We can deduce (A.49).

**Proof of (ii) :** we first show that  $F(u) \in H_\ell$ . The method is quite similar to the one used in the previous proof. We consider  $v \in \mathcal{D}(\Omega)$ , so that  $\|u - v\|_{H_\ell} < +\infty$ . We first write

$$(A.53) \quad \|\ell^{-1/2} F(u)\|_{\Omega;0,2} \leq \|\ell^{-1/2} (F(u) - F(v))\|_{\Omega;0,2} + \|\ell^{-1/2} F(v)\|_{\Omega;0,2}.$$

By a similar argument, the first norm is finite. Then, since  $v \in \mathcal{D}(\Omega)$  and  $F(0) = 0$ , we directly obtain  $F(v) \in \mathcal{C}_c^0(\Omega)$ , which means the last norm is finite.

We now use Theorem A.1 and decompose the derivative term

$$(A.54) \quad \begin{aligned} \|\ell^{1/2} \nabla F(u)\|_{\Omega;0,2} &= \|\ell^{1/2} F'(u) \nabla u\|_{\Omega;0,2} \\ &\leq \|F'\|_\infty \left( \|\ell^{1/2} (\nabla u - \nabla v)\|_{\Omega;0,2} + \|\ell^{1/2} \nabla v\|_{\Omega;0,2} \right). \end{aligned}$$

The first norm is finite by assumption and the second is also finite given that  $v \in \mathcal{D}(\Omega)$ .

Now we show that  $F(u) \in W_{\ell,0}$ . We consider the sequence  $(G_\varepsilon)_{\varepsilon>0}$  defined by

$$(A.55) \quad G_\varepsilon = \theta_\varepsilon * F - F_\varepsilon(0).$$

This family satisfies

$$(A.56) \quad \|G_\varepsilon - F\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0,$$

$$(A.57) \quad G'_\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} F'(t) \quad \text{for } t \neq (t_j)_{1 \leq j \leq J},$$

$$(A.58) \quad \|G'_\varepsilon\|_\infty \leq \|F'\|_\infty \quad \text{for } \varepsilon > 0.$$

Let  $u \in W_{\ell,0}$  and  $(u_n)_{n \in \mathbb{N}}$  be a sequence of smooth functions converging to  $u$  in  $H_\ell$ . Let  $\varepsilon > 0$  and  $n \in \mathbb{N}$ . Given that  $G_\varepsilon(0) = 0$ , we have  $G_\varepsilon(u_n) \in \mathcal{D}(\Omega)$ . Therefore, it remains to prove

$$(A.59) \quad G_\varepsilon(u_n) \xrightarrow[\varepsilon \rightarrow 0]{n \rightarrow +\infty} F(u) \quad \text{in } H_\ell.$$

To do so, we write

$$\|G_\varepsilon(u_n) - F(u)\|_{H_\ell} \leq \|G_\varepsilon(u_n) - F(u_n)\|_{H_\ell} + \|F(u_n) - F(u)\|_{H_\ell}.$$

Theorem A.2 yields that  $\|F(u_n) - F(u)\|_{H_\ell} \xrightarrow{n \rightarrow +\infty} 0$ . Thus, we have to prove

$$(A.60) \quad G_\varepsilon(v) \xrightarrow{\varepsilon \rightarrow 0} F(v), \quad \text{for all } v \in \mathcal{D}(\Omega).$$

Let  $v \in \mathcal{D}(\Omega)$ . For every  $\varepsilon > 0$ ,  $G_\varepsilon(v) \in \mathcal{D}(\Omega)$  and  $F(v) \in \mathcal{C}_c^0(\Omega)$ . By a topological argument, there exists a compact  $K \subset \Omega$  such that for sufficiently small values of  $\varepsilon > 0$ ,  $\text{Supp}(G_\varepsilon(v)) \subset K$ , and  $A := K - \text{Supp}(F(v)) \subset \Omega$ . This yields

$$(A.61) \quad \int_\Omega \frac{|G_\varepsilon(v) - F(v)|^2}{\ell} \leq \|G_\varepsilon - F\|_\infty^2 \int_A \frac{1}{\ell} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

With a similar method to what was done to prove (i), we obtain

$$(A.62) \quad \int_\Omega \ell |\nabla G_\varepsilon(v) - \nabla F(v)|^2 \xrightarrow{\varepsilon \rightarrow 0} 0,$$

hence (A.60). □

## A.2 Additional stuff

We start by modifying the results of Corollary A.3, by focusing on  $W_{\ell,0} \cap L^\infty(\Omega)$  functions.

**Theorem A.4.** *Let  $w \in W_{\ell,0} \cap L^\infty(\Omega)$ . There exists a sequence  $(w_m)_{m \in \mathbb{N}}$  of  $\mathcal{D}(\Omega)$  functions satisfying*

$$(A.63) \quad \begin{cases} w_m \xrightarrow{m \rightarrow +\infty} w \text{ in } H_\ell, \\ \forall m \in \mathbb{N}, \quad \|w_m\|_{\Omega;0,\infty} \leq C, \end{cases}$$

where  $C > 0$  does not depend on  $m$ .

*Proof.* Given that  $w \in W_{\ell,0}$ , there exists a sequence  $(v_n)_{n \in \mathbb{N}}$  of  $\mathcal{D}(\Omega)$  functions such that  $v_n \xrightarrow{n \rightarrow +\infty} w$  in  $H_\ell$ . This sequence is not uniformly bounded in the general case. Let  $z = \|w\|_{\Omega;0,\infty}$ , and  $T_z$  the truncation at height  $z$ , as defined in (3.1). The first idea is taking  $w_m = T_z(v_m)$ , but  $T_z$  is only continuous, instead of  $\mathcal{C}^\infty$ . This is why we introduce  $(\theta_\varepsilon)_{\varepsilon > 0}$  a mollifier as in the proof of Theorem A.1, and define  $(T_z^\varepsilon)_{\varepsilon > 0}$  by

$$(A.64) \quad T_z^\varepsilon = \theta_\varepsilon * T_z - \theta_\varepsilon * T_z(0).$$

As a composition on  $\mathcal{C}^\infty$  functions,  $T_z^\varepsilon(v_n) \in \mathcal{C}^\infty(\Omega)$  for all  $\varepsilon > 0$  and  $n \in \mathbb{N}$ . Moreover,  $v_n$  is compact-supported, and  $T_z^\varepsilon(0) = 0$ , which means  $T_z^\varepsilon$  also has compact support. Finally, the same arguments as for showing (A.59) yield the convergence

$$(A.65) \quad T_z^\varepsilon(v_n) \xrightarrow[n \rightarrow +\infty]{\varepsilon \rightarrow 0} T_z(w) - T_z(0) = w,$$

given that  $|w| \leq z$  a.e. on  $\Omega$  and  $T_z(0) = 0$ .  $\square$

**Theorem A.5.** *Let  $F \in \text{Stamp}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ ,  $u \in W_{\ell,0}$  and  $w \in W_{\ell,0} \cap L^\infty(\Omega)$ . Then*

$$(A.66) \quad F(u)w \in W_{\ell,0}.$$

*Proof. Step 1 :*  $F(u)w \in H_\ell$ . First, we have

$$(A.67) \quad \|\ell^{-1/2}F(u)w\|_{\Omega;0,2} \leq \|F\|_\infty \|\ell^{-1/2}w\|_{\Omega;0,2} < +\infty.$$

Then we apply the triangular inequality on the derivative term

$$(A.68) \quad \begin{aligned} \|\ell^{1/2}\nabla(F(u)w)\|_{\Omega;0,2} &\leq \|\ell^{1/2}F'(u)(\nabla u)w\|_{\Omega;0,2} + \|\ell^{1/2}F(u)\nabla w\|_{\Omega;0,2} \\ &\leq \|F'\|_\infty \|w\|_{\Omega;0,\infty} \|\ell^{1/2}\nabla u\|_{\Omega;0,2} + \|F\|_\infty \|\ell^{1/2}\nabla w\|_{\Omega;0,2} < +\infty. \end{aligned}$$

We consider  $F_\varepsilon = \theta_\varepsilon * F$  defined as in the proof of Theorem A.1,  $(u_n)_{n \in \mathbb{N}}$  a sequence of smooth functions converging strongly to  $u$  in  $H_\ell$ , and  $(w_m)_{m \in \mathbb{N}}$  a sequence defined as in Theorem A.4. The fact that  $F_\varepsilon(u_n)w_m \in \mathcal{D}(\Omega)$  for any  $\varepsilon > 0$  and  $m, n \in \mathbb{N}$  is straightforward. We aim at showing the following

$$(A.69) \quad F_\varepsilon(u_n)w_m \xrightarrow[m, n \rightarrow +\infty]{\varepsilon \rightarrow 0} F(u)w \text{ in } H_\ell.$$

**Step 2 : the term without derivative.** The triangular inequality is written

$$(A.70) \quad \begin{aligned} \|\ell^{-1/2}(F_\varepsilon(u_n)w_m - F(u)w)\|_{\Omega;0,2} &\leq \|\ell^{-1/2}F_\varepsilon(u_n)(w_m - w)\|_{\Omega;0,2} \\ &\quad + \|\ell^{-1/2}(F_\varepsilon(u_n) - F(u_n))w\|_{\Omega;0,2} + \|\ell^{-1/2}(F(u_n) - F(u))w\|_{\Omega;0,2}. \end{aligned}$$

Respectively the bound on  $F_\varepsilon$ , the uniform convergence and the Lebesgue theorem yield

$$(A.71) \quad \|\ell^{-1/2}F_\varepsilon(u_n)(w_m - w)\|_{\Omega;0,2} \leq \|F\|_\infty \|\ell^{-1/2}(w_m - w)\|_{\Omega;0,2} \xrightarrow{m \rightarrow +\infty} 0,$$

$$(A.72) \quad \|\ell^{-1/2}(F_\varepsilon(u_n) - F(u_n))w\|_{\Omega;0,2} \leq \|F_\varepsilon - F\|_\infty \|\ell^{-1/2}w\|_{\Omega;0,2} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

$$(A.73) \quad \|\ell^{-1/2}(F(u_n) - F(u))w\|_{\Omega;0,2} \xrightarrow{n \rightarrow +\infty} 0.$$

**Step 3 : The derivative term.** The derivative term can be decomposed in two terms as follows

$$(A.74) \quad \|\ell^{1/2}(\nabla(F_\varepsilon(u_n)\nabla w_m) - \nabla(F(u)w))\|_{\Omega;0,2} \leq \|\ell^{1/2}(F_\varepsilon(u_n)\nabla w_m - F(u)\nabla w)\|_{\Omega;0,2} \\ + \|\ell^{1/2}(w_m F'_\varepsilon(u_n)\nabla u_n - w F'(u)\nabla u)\|_{\Omega;0,2}.$$

The same arguments as used in step 2 yield

$$(A.75) \quad \|\ell^{1/2}(F_\varepsilon(u_n)\nabla w_m - F(u)\nabla w)\|_{\Omega;0,2} \xrightarrow[\varepsilon \rightarrow 0]{m, n \rightarrow +\infty} 0.$$

It only remains to show the following

$$(A.76) \quad \|\ell^{1/2}(w_m F'_\varepsilon(u_n)\nabla u_n - w F'(u)\nabla u)\|_{\Omega;0,2} \xrightarrow[\varepsilon \rightarrow 0]{m, n \rightarrow +\infty} 0.$$

We write a first decomposition

$$(A.77) \quad \|\ell^{1/2}(w_m F'_\varepsilon(u_n)\nabla u_n - w F'(u)\nabla u)\|_{\Omega;0,2} \leq \|\ell^{1/2}w_m(F'_\varepsilon(u_n)\nabla u_n - F'(u)\nabla u)\|_{\Omega;0,2} \\ + \|\ell^{1/2}F'(u)\nabla u(w_m - w)\|_{\Omega;0,2}.$$

The last term is bounded from above as follows

$$(A.78) \quad \|\ell^{1/2}F'(u)\nabla u(w_m - w)\|_{\Omega;0,2} \leq \|F'\|_\infty \left( \int_\Omega \ell |\nabla u|^2 |w_m - w|^2 \right)^{1/2}.$$

First, since

$$\ell^{-1/2}w_m \xrightarrow{m \rightarrow +\infty} \ell^{-1/2}w \quad \text{in } L^2(\Omega),$$

the Lebesgue inverse yields the existence of a subsequence still named  $(w_m)_{m \in \mathbb{N}}$  such that  $w_m \xrightarrow{m \rightarrow +\infty} w$  a.e.  $\Omega$ . In addition, Theorem A.4 yields

$$(A.79) \quad \ell |\nabla u|^2 |w_m - w|^2 \leq 4C^2 \ell |\nabla u|^2 \in L^1(\Omega).$$

We can apply the Lebesgue theorem, which yields

$$(A.80) \quad \int_\Omega \ell |\nabla F(u)|^2 |w_m - w|^2 \xrightarrow{m \rightarrow +\infty} 0, \quad \text{hence } \|\ell^{1/2}F'(u)\nabla u(w_m - w)\|_{\Omega;0,2} \xrightarrow{m \rightarrow +\infty} 0.$$

Since  $(w_m)_{m \in \mathbb{N}}$  is uniformly bounded in  $L^\infty(\Omega)$ , (A.76) can be proved only by showing that

$$(A.81) \quad \|\ell^{1/2}v(F'_\varepsilon(u_n)\nabla u_n - F'(u)\nabla u)\|_{\Omega;0,2} \xrightarrow[\varepsilon \rightarrow 0]{n \rightarrow +\infty} 0,$$

for any fixed function  $v \in \mathcal{D}(\Omega)$ . This is already done in the proof of Theorem A.3.  $\square$

### A.3 An interesting result on limits

The following lemma constitutes a higher bound when  $L^1(\Omega)$  functions are integrated on a "slice", and is used for instance in the proof of Theorem 3.1.

**Lemma A.3.** *Let  $(X, \mathcal{A}, \mu)$  be a measured space,  $M > 0$  and  $(u_n)_{n \in \mathbb{N}}$  a sequence of measurable functions such that  $u_n \xrightarrow[n \rightarrow +\infty]{} u$  a.e.  $X$ . Then we have*

$$(A.82) \quad \mathbb{1}_{\limsup_{n \rightarrow +\infty} \{|u_n| \geq M\}} \leq \mathbb{1}_{\{|u| \geq M\}} \quad \text{a.e. } \Omega.$$

**Remark A.2.** *we recall that if  $(A_n)_{n \in \mathbb{N}}$  is a sequence of elements of  $\mathcal{A}$ ,*

$$\limsup_{n \rightarrow +\infty} A_n = \bigcap_{n \in \mathbb{N}} \left( \bigcup_{k > n} A_k \right).$$

*Actually,  $x \in \limsup_{n \rightarrow +\infty} A_n$  means that  $x$  is in a infinite number of  $A_n$  sets. It is well known that for any sequence  $(A_n)_{n \in \mathbb{N}}$  we have*

$$\mathbb{1}_{\limsup_{n \rightarrow +\infty} A_n} = \limsup_{n \rightarrow +\infty} \mathbb{1}_{A_n}.$$

*Proof.* We consider  $E \in \mathcal{A}$  such that for any  $x \in E$ ,  $u_n(x) \xrightarrow[n \rightarrow +\infty]{} u(x)$  and  $X \setminus E$  has null measure. Let  $x \in E \cap \limsup_{n \rightarrow +\infty} \{|u_n| \geq M\}$ . Then

$$(A.83) \quad \forall n \in \mathbb{N}, \exists k > n, \quad |u_k(x)| \geq M.$$

This means it exists a subsequence  $(u_{\phi(n)})_{n \in \mathbb{N}}$  satisfying  $|u_{\phi(n)}(x)| \geq M$  for all  $n \in \mathbb{N}$ . Passing to the limit directly yields  $|u(x)| \geq M$ . We deduce that

$$(A.84) \quad E \cap \limsup_{n \rightarrow +\infty} \{|u_n| \geq M\} \subset E \cap \{|u| \geq M\}.$$

The fact that  $X \setminus E$  has null measure yields the result. □

## References

- [1] R.A. Adams and J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [2] C. Amrouche, Leloup G., and R. Lewandowski. The model involving the distance to the wall. part 1: the relaxed case. *Preprint on Hal*, 2023.
- [3] C. Amrouche and M. Moussaoui. The dirichlet and neumann problems in lipschitz and in  $C^{1,1}$  domains. *Preprint on arXiv*, 2022.
- [4] Cherif Amrouche, Luigi C. Berselli, Roger Lewandowski, and Dinh Duong Nguyen. Turbulent flows as generalized Kelvin-Voigt materials: modeling and analysis. *Non-linear Anal.*, 196:111790, 24, 2020.
- [5] C. Bernardi, T. Chacón Rebollo, R. Lewandowski, and F. Murat. A model for two coupled turbulent fluids. I. Analysis of the system. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIV (Paris, 1997/1998)*, volume 31 of *Stud. Math. Appl.*, pages 69–102. North-Holland, Amsterdam, 2002.

- [6] Luigi C. Berselli, François Legeais, and Roger Lewandowski. Surface boundary layers through a scalar equation with an eddy viscosity vanishing at the ground. *ESAIM Math. Model. Numer. Anal.*, 58(2):489–513, 2024.
- [7] T. Chacón-Rebollo and R. Lewandowski. *Mathematical and Numerical Foundations of Turbulence Models and Applications*. Modeling and Simulation in Science, Engineering and Technology. Springer New York, 2014.
- [8] T. Gallouët, J. Lederer, R. Lewandowski, F. Murat, and L. Tartar. On a turbulent system with unbounded eddy viscosities. *Nonlinear Anal.*, 52(4):1051–1068, 2003.
- [9] A. Kufner. *Weighted Sobolev spaces*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985. Translated from the Czech.
- [10] B.E. Launder and D. Brian Spalding. *Lectures in Mathematical Models of Turbulence*. Academic Press, London, 1972.
- [11] Guillaume Leloup and Roger Lewandowski. A nonlinear elliptic equation with a degenerate diffusion and a source term in  $L^1$ . *Appl. Math. Lett.*, 153:Paper No. 109077, 4, 2024.
- [12] R. Lewandowski. Modèles de turbulence et équations paraboliques. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(9):835–840, 1993.
- [13] R. Lewandowski. *Analyse Mathématique et océanographie*. Elsevier-Masson, Paris, 1997.
- [14] R. Lewandowski. The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy viscosity. *Nonlinear Anal.*, 28(2):393–417, 1997.
- [15] Roger Lewandowski and Bijan Mohammadi. Existence and positivity results for the  $\phi$ - $\theta$  and a modified  $k$ - $\epsilon$  two-equation turbulence models. *Math. Models Methods Appl. Sci.*, 3(2), 1993.
- [16] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications*. Vol. 1, volume No. 17 of *Travaux et Recherches Mathématiques*. Dunod, Paris, 1968.
- [17] B. Mohammadi and O. Pironneau. *Analysis of the  $k$ -epsilon turbulence model*. RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.
- [18] François Murat. Soluciones renormalizadas de edp elípticas no lineales. In *Lectures notes*. University of Sevilla, 1992.
- [19] François Murat. Renormalized solutions of nonlinear elliptic equations with measure data. In *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1998)*, pages Exp. No. IX, 4. Univ. Nantes, Nantes, 1998.
- [20] L. Prandtl. Über die ausgebildeten turbulenz. *Zeitschrift für angewandte Mathematik und Mechanik*, 5:136–139, 1925.
- [21] G. Stampacchia. *Équations elliptiques du second ordre à coefficients discontinus*. Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965). Les Presses de l’Université de Montréal, Montreal, Que., 1966.

- [22] Luc Tartar. *An Introduction to Sobolev Spaces and Interpolation Spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer Berlin, Heidelberg, 2007.