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Abstract

We consider a simplified steady-state equation for the turbulent kinetic energy used in turbulence modeling, with homogeneous boundary conditions and an eddy viscosity proportional to a regularization of the distance to the boundary. After determining a suitable space function for this equation, we prove the existence and the uniqueness of a renormalized solution.

Key words : Fluid mechanics, Turbulence models, degenerate operators, Navier-Stokes Equations, Turbulent Kinetic Energy.

2010 MSC: 76D05, 35Q30, 76F65, 76D03, 35Q30.

1 Introduction

The aim of this paper is to prove the existence and the uniqueness of a renormalized solution to the steady-state boundary problem

$$\left\{\begin{array}{ll} -\mathrm{div}\,(\ell\nabla k)+\dfrac{k}{\ell}=\mathbb{D} & \text{in } \Omega,\\ k=0 & \text{at } \Gamma,\end{array}\right. \tag{1.1}$$

where the unknown k denotes the the turbulent kinetic energy of a turbulent flow, ℓ stands for the Prandtl mixing lenght, a given non negative function that degenerate near the boubdary as the distance to the boundary.
The turbulent kinetic energy k roughly measures the deviation of the velocity field from its mean $\overline{\mathbf{v}}$ in a turbulent flow and is one of the main variables used in turbulence models. It is governed by a transport-diffusion equation with source and sink terms, taking the form:

$$\underbrace{\frac{Dk}{Dt}}_{\text{transport}}-\underbrace{\mathrm{div}\,(\mu_{\text{turb}}(k)\nabla k)}_{\text{diffusion}}=\underbrace{\mathbb{D}}_{\substack{\text{energy}\\ \text{production}}}-\underbrace{\varepsilon}_{\substack{\text{turbulent}\\ \text{dissipation}}} \tag{1.2}$$

In this equation:

1) $\mu_{\text{turb}}(k)$ is an eddy diffusivity coefficient, expressed as

$$\mu_{\text{turb}}(k)=C\ell\sqrt{k}, \tag{1.3}$$

a formula deduced from dimensional analysis, where ℓ is the Prandtl mixing length, C is a dimensionless constant generally obtained from experimental data, under the assumption that $k \geq 0$.

2) The production term $\mathbb{D}$ is given by

$$\mathbb{D} = \nu_{\text{turb}}(k) \left| \frac{\nabla \overline{\mathbf{v}} + \nabla \overline{\mathbf{v}}^t}{2} \right|^2, \tag{1.4}$$

where $\nu_{\text{turb}} = \nu_{\text{turb}}(k)$ is the eddy viscosity, the form of which is similar to that of (1.3). According to the standard energy balance satisfied by $\overline{\mathbf{v}}$, we have $\mathbb{D} \in L^1$ and no better.

3) The dissipation term ε is given by

$$\varepsilon = \frac{k\sqrt{k}}{\ell}, \tag{1.5}$$

which is also deduced from dimensional analysis. This equation, referred to as the TKE equation, is derived from averaging the energy balance in the Navier-Stokes equations, and is central to many turbulence models, the most popular being the $k-\varepsilon$ model, which involves an additional equation for ε [7, 10, 17].

The TKE equation is set in a bounded domain Ω and is driven by the Prandtl mixing length ℓ, representing the distance a turbulent eddy can travel before being mixed into the rest of the fluid [20]. The mixing length is somewhat equivalent to the mean free path in the kinetic theory of gases. Near the boundary of Ω, denoted by Γ, ℓ is proportional to the distance to Γ, leading to significant mathematical challenges. For this reason, in many earlier works, Formula (1.3) is replaced by

$$\mu_{\text{turb}}(k) = \mu_0 + C\ell\widetilde{\mu}(k), \tag{1.6}$$

where $\mu_0 > 0$ and the function $\widetilde{\mu}(k) \geq 0$ is a bounded continuous approximation of $\sqrt{k}$, since unbounded eddy coefficients also pose significant issues (cf. [5, 8, 12, 14]).

Recently, the case with $\mu_0 = 0$, *i.e.* $\mu_{\text{turb}}(k) = C\ell\widetilde{\mu}(k)$, has been studied in the steady-state case [2]. However, only relaxed cases could be addressed, namely

$$\mu_{\text{turb}}(k) = C\ell^\beta\widetilde{\mu}(k), \quad \text{for } 0 < \beta < 1. \tag{1.7}$$

Depending on the value of $\beta \in]0,1[$, we were able to derive estimates for k and subsequently couple the TKE equation with Navier-Stokes equations featuring an eddy viscosity of the form $\nu_{\text{turb}}(k) = C'\ell^\eta\widetilde{\nu}(k)$, where $0 \leq \nu_m \leq \widetilde{\nu}(k) \leq \nu_M$, and $0 < \eta < \eta_c < 1$ and $0 < \beta < \beta_c < 1$, under homogeneous boundary conditions (cf. [2, 11]).

The central idea behind these results relies on estimates in weighted spaces, embedded within regular Sobolev spaces, using a result by Kufner [9], which provides a framework for defining the trace of k at the boundary.

In this paper, we focus on the case $\beta = 1$ in the steady-state case, for which Kufner's results are no longer applicable. A linear equation for a boundary layer with an eddy viscosity behaving as $d(x,\Gamma)$ near the ground and a Boussinesq source term was studied in [6]. We found a solution in $H^{1/2}$, but no better, for which no trace can be defined.

The strategy to tackle the problem set by the TKE equation (1.2), equipped with the eddy coefficient $\mu_{\text{turb}}(k) = \ell\widetilde{\mu}(k)$ (taking $C = 1$ for simplicity), is taking advantage of the term ε, which has the right sign. For simplicity, in this first approach, we take $\widetilde{\mu}(k)$ constant to be equal to 1, and instead of formula (1.5),

$$\varepsilon = \frac{k}{\ell}, \tag{1.8}$$

and considering the steady state case, we are led to consider the linear PDE problem (1.1), assuming

$$\ell \in L^\infty(\Omega) \text{ s.t. } \ell \geq 0 \text{ a.e. } \Omega, \quad \text{and} \quad \forall \omega \subset\subset \Omega, \quad \inf_{\overline{\omega}} \ell > 0, \tag{1.9}$$

$$\ell(x) \sim d(x,\Gamma) \quad \textit{when} \quad d(x,\Gamma) \to 0, \tag{1.10}$$

(1.11)

The key observation is that, by formally multiplying the equation (1.2) by $T_z(k)$, T_z being the truncation function at height $z > 0$ (see (3.1) below), we obtain the estimate

$$\begin{aligned} \nu_m \int_\Omega \ell(x) |\nabla T_z(k(x))|^2 \, dx + \int_\Omega \frac{|T_z(k(x))|^2}{\ell} \leq \\ \int_\Omega \ell(x) |\nabla T_z(k(x))|^2 \, dx + \int_\Omega \frac{k(x) T_z(k(x))}{\ell(x)} \leq z \|\mathbb{D}\|_{\Omega;0,1}. \end{aligned} \tag{1.12}$$

In other words, $T_z(k)$ naturally belongs to the space

$$H_\ell = \left\{ u \in \mathcal{D}'(\Omega), \quad \ell^{1/2} \nabla u \in L^2(\Omega)^N, \quad \frac{u}{\ell^{1/2}} \in L^2(\Omega) \right\}. \tag{1.13}$$

This invites us to seek a renormalized solution, in the way initially developed by Lions-Murat for elliptic equations with a right-hand side in L^1 [19, 18] (see also in [13, chapter 5]). The main results of the paper is the existence of a renormalized solution and its uniqueness to Problem (1.2).
The paper is organized as follows. We first determine the space function suitable for problem (1.2), which is a subspace of $H^{\frac{1}{2}}$, the functions of which having a trace on Γ well defined and equal to 0. Then we give the definition of a renormalized solution, and prove the main results. We have gathered in an appendix technical results essential to the demonstrations of the main results.

Aknowledgements. The authors thank François Murat for several stimulating discussions about this problem and renormalized solutions.

2 Space function

Recall that we focus on the problem

$$\begin{cases} -\mathrm{div}\,(\ell \nabla k) + \dfrac{k}{\ell} = \mathbb{D} & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases} \tag{2.1}$$

2.1 Embeddings

We mentionned in the introduction that the natural space for solving Problem (2.1) is the space H_ℓ defined by :

$$H_\ell = \left\{ u \in \mathcal{D}'(\Omega), \quad \ell^{1/2} \nabla u \in L^2(\Omega)^N, \quad \frac{u}{\ell^{1/2}} \in L^2(\Omega) \right\}, \tag{2.2}$$

where the the function ℓ satisfies the assumptions (1.9) and (1.10). It is an Hilbert space, equipped with the scalar product

$$(u,v)_\ell = \int_\Omega \ell(x) \nabla u(x) \cdot \nabla v(x) \, dx + \int_\Omega \frac{u(x)v(x)}{\ell(x)} dx. \tag{2.3}$$

We denote by $\|u\|_\ell = \sqrt{(u,v)_\ell}$ the corresponding norm.

Lemma 2.1. *The following holds :*

$$H_\ell \hookrightarrow H_{00}^{\frac{1}{2}}(\Omega) = \left\{ u \in H^{\frac{1}{2}}(\Omega),\ s.t.\ \frac{u}{\sqrt{\ell}} \in L^2(\Omega) \right\}. \tag{2.4}$$

Proof. From the definition of H_ℓ, we get $H_\ell \hookrightarrow L^2(\Omega; \ell^{-1}dx)$, which combined with (1.9) yields $H_\ell \hookrightarrow L^2(\Omega; dx) = L^2(\Omega)$. Therefore,

$$H_\ell \hookrightarrow V_\ell = \left\{ u \in \mathcal{D}'(\Omega), \quad \ell^{1/2}\nabla u \in L^2(\Omega)^N,\ u \in L^2(\Omega) \right\}.$$

We know from previous works [2, 3, 4, 6], that $V_\ell \hookrightarrow H^{\frac{1}{2}}(\Omega)$, yielding (2.4) thanks to the definition of $H_{00}^{\frac{1}{2}}(\Omega)$ initially given by Lions-Magenes [16]. □

From the Sobolev theorem, we deduce that

$$\forall\, 1 \le p < \frac{2N}{N-1}, \quad H_\ell \hookrightarrow L^p(\Omega), \tag{2.5}$$

with compact embedding.

2.2 Traces and homogeneous spaces

We know from [16] that any function $u \in H_{00}^{\frac{1}{2}}(\Omega)$ can be continuously extended by zero outside Ω, into a function $\widetilde{u} \in H^{\frac{1}{2}}(\mathbb{R}^N)$. However, a function in $H_{00}^{\frac{1}{2}}(\Omega)$ may lack a trace at Γ, which remains an open topic of discussion. Therefore, the result of the previous section is not enough to take into account the homogeneous boundary condition in the definition of a renormalized solution. This is why we introduce additional spaces. We start with

$$\mathcal{D}_\ell(\overline{\Omega}) = \mathcal{D}(\overline{\Omega}) \cap H_\ell, \tag{2.6}$$

where $\mathcal{D}(\overline{\Omega})$ stands for the set of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^N)$. We observe that, thanks to (1.9), we have $\mathcal{D}(\Omega) \subset \mathcal{D}_\ell(\overline{\Omega})$. In particular, $\mathcal{D}_\ell(\overline{\Omega}) \neq \emptyset$. We conjecture that, in fact, $\mathcal{D}(\Omega) = \mathcal{D}_\ell(\overline{\Omega})$. Now we consider:

$$\begin{array}{lcl} W_\ell & = & \text{closure of } \mathcal{D}_\ell(\overline{\Omega}) \text{ in } H_\ell, \\ W_{\ell,0} & = & \text{closure of } \mathcal{D}(\Omega) \text{ in } W_\ell. \end{array} \tag{2.7}$$

It is likely that $H_\ell = W_\ell = W_{\ell,0}$, but we do not have a proof of this fact yet, which is not essential for the following. We prove the following result.

Lemma 2.2. *For some constant $C = C(\Omega, \|\ell\|_\infty)$,*

$$\forall\, u \in \mathcal{D}_\ell(\overline{\Omega}), \quad \|u\|_{\Gamma;0,2}^2 = \int_\Gamma |u|^2 \le C\|u\|_\ell^2 = C\left(\int_\Omega \ell |\nabla u|^2 + \int_\Omega \frac{|u|^2}{\ell} \right). \tag{2.8}$$

Proof. We know from classical results about trace theory [1, 22] that $\exists\, C = C(\Omega)$ such that

$$\forall\, w \in \mathcal{D}(\overline{\Omega}), \quad \|w\|_{\Gamma;0,1} = \int_\Gamma |w| \le C\left(\int_\Omega |\nabla w| + \int_\Omega |w| \right) = C\|w\|_{\Omega;1,1}, \tag{2.9}$$

where we still denote w the trace at Γ instead of $\mathrm{tr}(w)$, so far no risk of confusion occurs. Let $u \in \mathcal{D}_\ell(\overline{\Omega})$, and take $w = u^2 \in \mathcal{D}(\overline{\Omega})$ in (2.9). Therefore,

$$\int_\Gamma |u|^2 \le C\left(\int_\Omega |\nabla (u^2)| + \int_\Omega |u|^2 \right). \tag{2.10}$$

On one hand,

$$\int_\Omega |u|^2 \leq \|\ell\|_\infty \int_\Omega \frac{|u|^2}{\ell}. \tag{2.11}$$

On the other hand, Young inequality directly yields

$$\int_\Omega \left|\nabla\left(u^2\right)\right| = \int_\Omega 2|u\nabla u| \leq \int_\Omega \frac{|u|^2}{\ell} + \int_\Omega \ell|\nabla u|^2, \tag{2.12}$$

hence (2.8) follows. □

The consequence of the inequality (2.8) is that we can extend the trace operator by density to the space W_ℓ given by (2.7),

$$\mathrm{tr} : \left\{ \begin{array}{ccc} W_\ell & \to & L^2(\Gamma), \\ u & \to & \mathrm{tr}(u). \end{array} \right. \tag{2.13}$$

Obviously,

$$W_{\ell,0} \subset \ker(\mathrm{tr}),$$

and it is reasonable to conjecture that $W_{\ell,0} = \ker(\mathrm{tr})$. However, we can claim that functions in the space $W_{\ell,0}$ have a trace at Γ which is equal to zero, making $W_{\ell,0}$ the right space to work in. Noting that $W_{\ell,0}$ is a Hilbert space for the H_ℓ scalar product, and that $L^2(\Omega) \subset W'_{\ell,0}$, the following is a direct consequence of the Lax-Milgram theorem.

Theorem 2.1. *Let $\mathbb{D} \in L^2(\Omega)$. Then there exists a unique $k \in W_{\ell,0}$ such that*

$$\forall\, w \in W_{\ell,0}, \quad (k, w)_\ell = \int_\Omega \mathbb{D}\, w. \tag{2.14}$$

In other words, Problem (2.1) has a unique weak solution in $W_{\ell,0}$.

3 Renormalized solutions

In this section we shall use function tests of the form $F(k)$, for F satisfying the Stampacchia conditions defined below.

Definition 3.1 (Stampacchia conditions)**.** *Let $F : \mathbb{R} \to \mathbb{R}$. We say that F satisfies the Stampacchia conditions if F is globally Lipchitz and its derivative F' has a finite number of discontinuities. We denote by* Stamp($\mathbb{R}$) *the set of all functions satisfying the Stampacchia conditions.*

We will use intensively the truncation at height $z \in \mathbb{R}_+^\star$, denoted by T_z and given by

$$T_z : t \mapsto \left\{ \begin{array}{lll} -z & \text{if} & t < -z, \\ t & \text{if} & -z \leq t \leq z, \\ z & \text{if} & t > z. \end{array} \right. \tag{3.1}$$

Note that $\forall\, z \in \mathbb{R}_+^\star$, $T_z \in$ Stamp($\mathbb{R}$), with in addition $T_z(0) = 0$. Therefore, by the theorem A.3 in the appendix, we deduce that $\forall\, k \in W_{\ell,0}$, $T_z(k) \in W_{\ell,0}$.

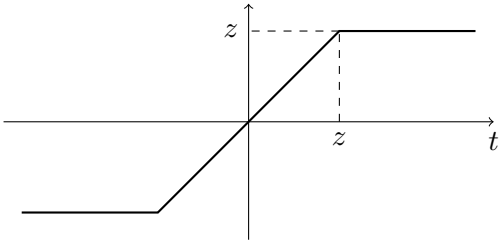

Figure 1: graph of the function T_z.

3.1 Convergence of the approximations

Let $\mathbb{D} \in L^1(\Omega)$, $(\mathbb{D}_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$ such that

$$\mathbb{D}_n \underset{n\to+\infty}{\longrightarrow} \mathbb{D} \quad \text{in } L^1(\Omega), \tag{3.2}$$

$$\forall\, n \in \mathbb{N}, \quad |\mathbb{D}_n| \leq |\mathbb{D}| \quad \text{a.e. in } \Omega. \tag{3.3}$$

For example, $\mathbb{D}_n = T_n(\mathbb{D})$ is a suitable sequence. Let $k_n \in W_{\ell,0}$ satisfy in a weak sense as in Theorem 2.1,

$$\begin{cases} -\mathrm{div}\,(\ell\nabla k_n) + \dfrac{k_n}{\ell} = \mathbb{D}_n & \text{in } \Omega, \\ k_n = 0 & \text{at } \Gamma. \end{cases} \tag{3.4}$$

Lemma 3.1. *The sequence $(k_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1\left(\Omega, \ell^{-1}dx\right)$. Therefore, there exists $k \in L^1(\Omega, \ell^{-1}dx)$ such that $k_n \underset{n\to+\infty}{\longrightarrow} k$ in $L^1(\Omega, \ell^{-1}dx)$.*

Proof. Let $\mathbb{H}_\eta$, $\eta > 0$, denotes the usual Lipchitz approximation of the Heaviside function,

$$\mathbb{H}_\eta : x \mapsto \begin{cases} -1 & \text{if} \quad t < -\eta, \\ t/\eta & \text{if} \quad -\eta \leq t \leq \eta, \\ 1 & \text{if} \quad t > \eta. \end{cases} \tag{3.5}$$

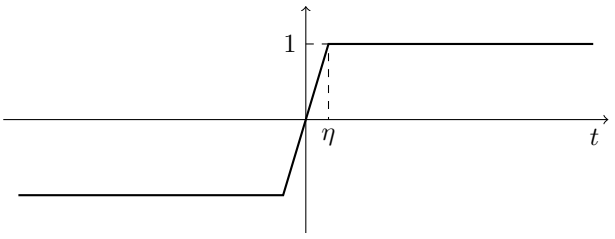

Figure 2: graph of the function $\mathbb{H}_\eta$.

We notice that $\mathbb{H}_\eta \in \mathrm{Stamp}(\mathbb{R})$ and $\mathbb{H}_\eta(0) = 0$. By substracting both (2.14) for k_p and k_q, we see that $\forall\, v \in W_{\ell,0}$,

$$\int_\Omega \ell\nabla(k_p - k_q)\cdot\nabla v + \int_\Omega \frac{k_p - k_q}{\ell} v = \int_\Omega (\mathbb{D}_p - \mathbb{D}_q)v.$$

Taking $v = \mathbb{H}_\eta(k_p - k_q) \in W_{\ell,0}$ (c.f. Theorem A.3) leads to

$$\int_\Omega \ell|\nabla(k_p - k_q)|^2 \mathbb{H}'_\eta(k_p - k_q) + \int_\Omega \frac{(k_p - k_q)\mathbb{H}_\eta(k_p - k_q)}{\ell} \leq \|\mathbb{D}_p - \mathbb{D}_q\|_{\Omega;0,1},$$

given that $|\mathbb{H}_\eta(k_p - k_q)| \leq 1$. Therefore, as $\mathbb{H}'_\eta \geq 0$, letting η go to zero, and noticing that

$$t\mathbb{H}_\eta(t) \underset{\eta\to 0}{\longrightarrow} |t| \quad \text{for all } t \in \mathbb{R},$$

we deduce from Fatou's lemma

$$\|k_p - k_q\|_{L^1\left(\frac{dx}{\ell}\right)} \leq \|\mathbb{D}_p - \mathbb{D}_q\|_{\Omega;0,1} \underset{p,q\to\infty}{\longrightarrow} 0,$$

hence the result. As $L^1\left(\Omega, \ell^{-1}dx\right)$ is a complete space, the sequence $(k_n)_{n\in\mathbb{N}}$ has a limit $k \in L^1(\Omega, \ell^{-1}dx)$. □

Lemma 3.2. *Let $k = \lim_{n\to\infty} k_n \in L^1\left(\Omega, \ell^{-1}dx\right)$ given by the statement of Lemma 3.1. Then*

$$\forall\, z \in \mathbb{R}^\star_+, \quad T_z(k_n), T_z(k) \in W_{\ell,0}, \tag{3.6}$$

and we have

$$T_z(k_n) \underset{n\to+\infty}{\longrightarrow} T_z(k) \begin{cases} \text{weakly in } W_{\ell,0}, \\ \text{strongly in } L^p(\Omega), \quad 1 \leq p < \frac{2N}{N-1}, \\ \text{a.e. in } \Omega. \end{cases} \tag{3.7}$$

Proof. Let $z \in \mathbb{R}^\star_+$. As alredy mentionned, since $T_z \in \mathrm{Stamp}(\mathbb{R})$ and $T_z(0) = 0$, $k_n \in W_{\ell,0}$, then $T_z(k_n) \in W_{\ell,0}$ by Theorem A.3. Moreover, as $\ell \in L^\infty(\Omega)$ and $\ell \geq 0$, then

$$k = \lim_{n\to\infty} k_n \quad \text{in} \quad L^1(\Omega, dx)$$

as well. Then we can extract a subsequence, still denoted by $(k_n)_{n\in\mathbb{N}}$, which converges to k *a.e.* in Ω and by the way,

$$T_z(k_n) \underset{n\to+\infty}{\longrightarrow} T_z(k)$$

in $L^1(\Omega)$, and *a.e.* in Ω. Taking $T_z(k_n)$ as test in (3.4) and using Stampacchia's result yields

$$\int_\Omega \ell|\nabla T_z(k_n)|^2 + \int_\Omega \frac{k_n T_z(k_n)}{\ell} = \int_\Omega T_z(k_n)\mathbb{D}_n \leq z\|\mathbb{D}\|_{\Omega;0,1}, \tag{3.8}$$

which proves that the sequence $(T_z(k_n))_{n\in\mathbb{N}}$ is bounded in $W_{\ell,0}$.
Therefore, we can extract from the sequence $(T_z(k_n))_{n\in\mathbb{N}}$, a subsequence, still denoted $(T_z(k_n))_{n\in\mathbb{N}}$, which weakly converges in $W_{\ell,0}$ to some $k_z \in W_{\ell,0}$, and weakly in $H^1(\omega)$ for all $\omega \subset\subset \Omega$. The convergence of $(T_z(k_n))_{n\in\mathbb{N}}$ in $L^1(\Omega)$ to $T_z(k)$, yields its convergence in $\mathcal{D}'(\Omega)$, hence $k_z = T_z(k)$. The strong convergence in $L^p(\Omega)$ for $1 \leq p < 2N/N - 1$, is a consequence of Lemma 2.1 together with the Sobolev compact embedding theorem. □

3.2 Main result

Recall that we consider the problem

$$\begin{cases} -\mathrm{div}\left(\ell\nabla k\right) + \dfrac{k}{\ell} = \mathbb{D} & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma. \end{cases} \tag{3.9}$$

Definition 3.2. *We say that k is a renormalized solution of (3.9) if*

$$k \in L^1(\Omega, \ell^{-1}dx), \tag{3.10}$$

$$\forall\, z \in \mathbb{R}, \quad T_z(k) \in W_{\ell,0}, \tag{3.11}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le|k|\le 2n\}} \ell|\nabla k|^2 = 0, \tag{3.12}$$

and $\forall\, w \in W_{\ell,0} \cap L^\infty(\Omega)$ verifying

$$\exists\, z \in \mathbb{R}, \quad s.t. \quad \nabla w = 0 \text{ a.e. on } \{|k| \ge z\}, \tag{3.13}$$

we have

$$\int_\Omega \ell\nabla k \cdot \nabla w + \int_\Omega \frac{kw}{\ell} = \int_\Omega \mathbb{D}w. \tag{3.14}$$

Remark 3.1. *All the integrals in (3.14) are well defined. Indeed, because $w \in L^\infty(\Omega)$, $\mathbb{D} \in L^1(\Omega)$, $k \in L^1(\Omega, \ell^{-1}dx)$, then $\mathbb{D}w, kw\ell^{-1} \in L^1(\Omega)$. Moreover, because of (3.13),*

$$\int_\Omega \ell\nabla k \cdot \nabla w = \int_\Omega \ell\nabla T_z(k) \cdot \nabla w,$$

which is well defined since as $T_z(k), w \in W_{\ell,0}$, $\sqrt{\ell}\nabla T_z(k), \sqrt{\ell}\nabla w \in L^2(\Omega)$.

The main result of this section is the following.

Theorem 3.1. *Let $\mathbb{D} \in L^1(\Omega)$. Then Problem (3.9) has a renormalized solution.*

To this result must be added the following continuity property.

Theorem 3.2. *Let $(\mathbb{D}_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^1(\Omega)$ that converges to $\mathbb{D} \in L^1(\Omega)$, $(k_\varepsilon)_{\varepsilon>0}$ the corresponding sequence of renormalized solutions, k the corresponding solution with $\mathbb{D}$ as source term. Then*

$$\forall\, z \in \mathbb{R}^\star_+, \quad T_z(k_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} T_z(k) \quad \text{strongly in } W_{\ell,0}. \tag{3.15}$$

Proof of Theorem 3.1. We argue by approximations. Let $\mathbb{D} \in L^1(\Omega)$ and $(\mathbb{D}_\varepsilon)_{\varepsilon>0}$ a family of $L^\infty(\Omega)$ functions such that

$$\mathbb{D}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{D} \text{ in } L^1(\Omega), \tag{3.16}$$

$$\forall\, \varepsilon > 0, \quad |\mathbb{D}_\varepsilon| \le |\mathbb{D}| \text{ a.e. in } \Omega. \tag{3.17}$$

Let $k_\varepsilon \in W_{\ell,0}$ satisfy in the weak sense specified by Theorem 2.1,

$$\begin{cases} -\mathrm{div}\,(\ell\nabla k_\varepsilon) + \dfrac{k_\varepsilon}{\ell} = \mathbb{D}_\varepsilon & \text{in } \Omega, \\ k_\varepsilon = 0 & \text{at } \Gamma, \end{cases} \tag{3.18}$$

which means

$$\forall\, v \in W_{\ell,0}, \quad \int_\Omega \ell\nabla k_\varepsilon \cdot \nabla v + \int_\Omega \frac{k_\varepsilon v}{\ell} = \int_\Omega \mathbb{D}_\varepsilon v. \tag{3.19}$$

Let $k \in L^1(\Omega, \ell^{-1}dx)$ be the limit of the family $(k_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega, \ell^{-1}dx)$, the existence of which is ensured by Lemma 3.1. By the Lebesgue inverse theorem, we can extract from $(k_\varepsilon)_{\varepsilon>0}$ a subsequence, still denoted by $(k_\varepsilon)_{\varepsilon>0}$ such that $k_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} k$ a.e. in Ω.

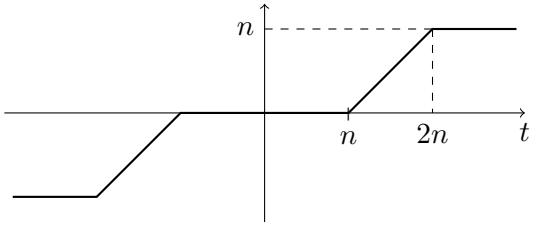

Figure 3: graph of the function S_n.

We aim at proving that k is a renormalized solution to Problem (3.9). We already know that $k \in L^1(\Omega, \ell^{-1}dx)$, and from Lemma 3.2, that

$$\forall\, z \in \mathbb{R}_+^\star, \quad T_z(k) \in W_{\ell,0}.$$

It remains to prove that k satisfies (3.12) and (3.14), which we do in two steps.

Step 1. We start by proving that k satisfies (3.12). Let $n \in \mathbb{N}^\star$, $S_n : \mathbb{R} \to \mathbb{R}$ be the odd function defined by

$$S_n : t \mapsto \begin{cases} 0 & \text{if} \quad 0 \le t \le n, \\ t-n & \text{if} \quad n < t \le 2n, \\ n & \text{if} \quad t > 2n. \end{cases} \tag{3.20}$$

We observe that $\forall\, n \in \mathbb{N}$, $S_n \in \mathrm{Stamp}(\mathbb{R})$ and $S_n(0) = 0$. Therefore, by Theorem A.3,

$$\forall\, \varepsilon > 0, \ \ \forall n \in \mathbb{N}^\star, \quad S_n(k_\varepsilon) \in W_{\ell,0}.$$

Taking $v = S_n(k_\varepsilon)$ in (3.18) yields for every $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\int_\Omega \ell |\nabla k_\varepsilon|^2 S_n'(k_\varepsilon) + \int_\Omega \frac{k_\varepsilon}{\ell} S_n(k_\varepsilon) = \int_\Omega \mathbb{D}_\varepsilon S_n(k_\varepsilon). \tag{3.21}$$

As S_n is an odd function and $\ell \ge 0$,

$$\int_\Omega \frac{k_\varepsilon}{\ell} S_n(k_\varepsilon) \ge 0.$$

Moreover, from $S_n'(t) \in \{0, 1\}$, we deduce

$$\int_\Omega \ell |\nabla k_\varepsilon|^2 S_n'(k_\varepsilon) = \int_\Omega \ell |\nabla S_n(k_\varepsilon)|^2. \tag{3.22}$$

Moreover,

$$\int_\Omega \mathbb{D}_\varepsilon S_n(k_\varepsilon) \le n \int_{\{|k_\varepsilon| \ge n\}} |\mathbb{D}_\varepsilon| \le n \int_{\{|k_\varepsilon| \ge n\}} |\mathbb{D}|. \tag{3.23}$$

With those inequalities, (3.21) becomes

$$\frac{1}{n} \int_\Omega \ell |\nabla S_n(k_\varepsilon)|^2 = \frac{1}{n} \int_{\{n \le k_\varepsilon \le 2n\}} \ell |\nabla k_\varepsilon|^2 \le \int_{\{|k_\varepsilon| \ge n\}} |\mathbb{D}|. \tag{3.24}$$

Using Fatou Lemma for the liminf yields

$$\frac{1}{n} \int_\Omega \ell |\nabla S_n(k)|^2 \le \liminf_{\varepsilon \to 0} \frac{1}{n} \int_\Omega \ell |\nabla S_n(k_\varepsilon)|^2 \le \liminf_{\varepsilon \to 0} \int_{\{|k_\varepsilon| \ge n\}} |\mathbb{D}| \le \limsup_{\varepsilon \to 0} \int_{\{|k_\varepsilon| \ge n\}} |\mathbb{D}|.$$

Then, Fatou Lemma for the limsup and Lemma A.3 yield

$$\limsup_{\varepsilon\to 0}\int_{\{|k_\varepsilon|\geq n\}}|\mathbb{D}| \leq \int_\Omega |\mathbb{D}| \limsup_{\varepsilon\to 0} \mathbb{1}_{\{|k_\varepsilon|\geq n\}} \leq \int_{\{|k|\geq n\}}|\mathbb{D}|.$$

We finally obtain

$$\frac{1}{n}\int_\Omega \ell|\nabla S_n(k)|^2 = \frac{1}{n}\int_{\{n\leq|k|\leq 2n\}}\ell|\nabla k|^2 \leq \int_{\{|k|\geq n\}}|\mathbb{D}| \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.25}$$

Assertion (3.12) is proved.

Step 2. We now prove that k satisfies (3.14). Let $w \in W_{\ell,0}\cap L^\infty(\Omega)$ that satifies (3.13), and $z \in \mathbb{R}_+^\star$ such that $\nabla w = 0$ *a.e.* over $\{|k|\geq z\}$. The primary intention would be to take w as test function in (3.18). Therefore, we would have to deal with the term

$$\int_\Omega \ell\nabla k_\varepsilon\cdot\nabla w,$$

by using that $(\sqrt{\ell}\nabla T_r(k_\varepsilon))_{\varepsilon>0}$ weakly converges in $L^2(\Omega)$ to $\sqrt{\ell}\nabla T_r(k)$ as $\varepsilon\to 0$, for all $r>0$. Unfortunately, we only know that $\nabla w = 0$ *a.e.* over $\{|k|\geq z\}$, and we cannot directly use this for the k_ε's. To overcome this issue, we have to localize the equation a little bit more. To do so, we introduce for a given $n\in\mathbb{N}$, the even function $L_n \in \mathrm{Stamp}(\mathbb{R})$ given by

$$L_n : t\mapsto \begin{cases} 1 & \text{if} \quad 0\leq t<n, \\ \frac{1}{n}(t+2n) & \text{if} \quad n\leq t\leq 2n, \\ 0 & \text{if} \quad t>2n. \end{cases} \tag{3.26}$$

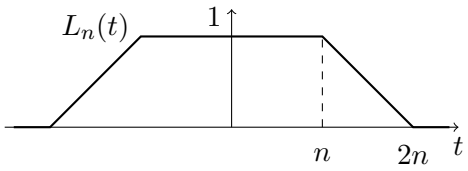

Figure 4: graph of the function L_n.

According to Theorem A.5, $v = L_n(k_\varepsilon)w \in W_{\ell,0}$. Therefore, it can be taken as test function in (3.18), which yields

$$\int_\Omega \ell\nabla k_\varepsilon\cdot\nabla w L_n(k_\varepsilon) + \int_\Omega \ell|\nabla k_\varepsilon|^2 w L_n'(k_\varepsilon) + \int_\Omega \frac{k_\varepsilon w}{\ell}L_n(k_\varepsilon) = \int_\Omega \mathbb{D}_\varepsilon w L_n(k_\varepsilon). \tag{3.27}$$

The convergence of the two last terms is a direct consequence of Lebesgue's Theorem. On one hand, we have

$$\int_\Omega \mathbb{D}_\varepsilon w L_n(k_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \mathbb{D} w L_n(k). \tag{3.28}$$

And given that $L_n \underset{n\to+\infty}{\longrightarrow} 1$ a.e. in $\mathbb{R}$, we finally obtain

$$\int_\Omega \mathbb{D}_\varepsilon w L_n(k_\varepsilon) \underset{\substack{\varepsilon\to 0\\ n\to+\infty}}{\longrightarrow} \int_\Omega \mathbb{D} w. \tag{3.29}$$

On the other hand, with exactly the same method we obtain

$$\int_\Omega \frac{k_\varepsilon w}{\ell} L_n(k_\varepsilon) \underset{\substack{\varepsilon\to 0\\ n\to+\infty}}{\longrightarrow} \int_\Omega \frac{kw}{\ell}. \tag{3.30}$$

Given the definition of L_n, the first integral of (3.27) is rewritten the following way

$$\int_\Omega \ell \nabla k_\varepsilon \cdot \nabla w L_n(k_\varepsilon) = \int_\Omega \ell \nabla T_{2n}(k_\varepsilon) \cdot \nabla w L_n(k_\varepsilon). \tag{3.31}$$

The previous results and Lebesgue Theorem yield

$$\ell^{1/2} \nabla w L_n(k_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \ell^{1/2} \nabla w L_n(k) \quad \text{strong in } L^2(\Omega), \tag{3.32}$$

$$\ell^{1/2} \nabla T_{2n}(k_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \ell^{1/2} \nabla T_{2n}(k) \quad \text{weak in } L^2(\Omega). \tag{3.33}$$

A classical integration result then yields, in particular when $n \geq z$,

$$\int_\Omega \ell \nabla k_\varepsilon \cdot \nabla w L_n(k_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \ell \nabla T_{2n}(k) \cdot \nabla w L_n(k) = \int_\Omega \ell \nabla T_z(k) \cdot \nabla w L_n(k). \tag{3.34}$$

Finally, Lebesgue theorem yields

$$\int_\Omega \ell \nabla T_z(k) \cdot \nabla w L_n(k) \underset{n\to+\infty}{\longrightarrow} \int_\Omega \ell T_z(k) \cdot \nabla w. \tag{3.35}$$

This corresponds to the convergence of the first term of (3.27)

$$\int_\Omega \ell \nabla k_\varepsilon \cdot \nabla w L_n(k_\varepsilon) \underset{\substack{\varepsilon\to 0\\ n\to+\infty}}{\longrightarrow} \int_\Omega \ell T_z(k) \cdot \nabla w. \tag{3.36}$$

The last term may be bounded from above as follows

$$\left| \int_\Omega \ell |\nabla k_\varepsilon|^2 w L_n'(k_\varepsilon) \right| = \frac{1}{n} \left| \int_{\{n\leq |k_\varepsilon| \leq 2n\}} \ell |\nabla k_\varepsilon|^2 w \right| \leq \frac{\|w\|_{\Omega;0,\infty}}{n} \int_{\{n\leq |k_\varepsilon| \leq 2n\}} \ell |\nabla k_\varepsilon|^2, \tag{3.37}$$

which recalls what we studied in the previous step of this proof. Indeed, (3.24) yields

$$\frac{1}{n} \int_{\{n\leq |k_\varepsilon| \leq 2n\}} \ell |\nabla k_\varepsilon|^2 \leq \int_{\{|k_\varepsilon| \geq n\}} |\mathbb{D}|. \tag{3.38}$$

Both Fatou's Lemma for the limsup and Lemma A.3 then yield

$$\limsup_{\varepsilon\to 0} \frac{1}{n} \int_{\{n\leq |k_\varepsilon| \leq 2n\}} \ell |\nabla k_\varepsilon|^2 \leq \int_{\{|k| \geq n\}} |\mathbb{D}|.$$

Finally, we obtain

$$\limsup_{\substack{n\to+\infty\\ \varepsilon\to 0}} \left| \int_\Omega \ell |\nabla k_\varepsilon|^2 w L_n'(k_\varepsilon) \right| \leq \|w\|_{\Omega;0,\infty} \limsup_{n\to+\infty} \int_{\{|k| \geq n\}} |\mathbb{D}| = 0. \tag{3.39}$$

By grouping together convergences (3.29), (3.30), (3.36) and (3.39), we obtain

$$\int_\Omega \ell \nabla T_z(k) \cdot \nabla w + \int_\Omega \frac{kw}{\ell} = \int_\Omega \mathbb{D} w,$$

which means, according to the remark 3.1, that k is indeed a renormalized solution of Problem (3.9). □

Proof of Theorem 3.2. We now prove the continuity property. Let $(\mathbb{D}_\varepsilon)_{\varepsilon>0}$ be a $L^\infty(\Omega)$ sequence satisfying (3.16) and (3.17), and $(k_\varepsilon)_{\varepsilon>0}$ the corresponding sequence of renormalized solutions, k the corresponding solution with $\mathbb{D}$ as source term. Let $z>0$. We already know by Lemmas 3.1 and 3.2 that

$$k_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} k \quad \text{strong in } L^1(\Omega), \tag{3.40}$$

$$T_z(k_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} T_z(k) \quad \text{weak in } W_{\ell,0}. \tag{3.41}$$

Therefore, in order to conclude, it remains to prove the convergence of the energies, *i.e.*

$$\int_\Omega \ell|\nabla T_z(k_\varepsilon)|^2 + \int_\Omega \frac{|T_z(k_\varepsilon)|^2}{\ell} \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \ell|\nabla T_z(k)|^2 + \int_\Omega \frac{|T_z(k_\varepsilon)|^2}{\ell}. \tag{3.42}$$

We first deduce from inequality (A.1) in the appendice that

$$\int_\Omega \frac{|T_z(k_\varepsilon)-T_z(k)|^2}{\ell} \le 2z\int_\Omega \frac{|T_z(k_\varepsilon)-T_z(k)|}{\ell} \le 2z^2\int_\Omega \frac{|k_\varepsilon-k|}{\ell} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.43}$$

Consequently, it only remains to prove

$$\int_\Omega \ell|\nabla T_z(k_\varepsilon)|^2 \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \ell|\nabla T_z(k)|^2. \tag{3.44}$$

Taking $v = T_z(k_\varepsilon)$ in (3.19) yields

$$\int_\Omega \ell|\nabla T_z(k_\varepsilon)|^2 + \int_\Omega \frac{k_\varepsilon T_z(k_\varepsilon)}{\ell} = \int_\Omega \mathbb{D}_\varepsilon T_z(k_\varepsilon). \tag{3.45}$$

First, we have

$$\int_\Omega \mathbb{D}_\varepsilon T_z(k_\varepsilon) = \int_\Omega (\mathbb{D}_\varepsilon - \mathbb{D})T_z(k_\varepsilon) + \int_\Omega \mathbb{D}\, T_z(k_\varepsilon). \tag{3.46}$$

On one hand,

$$\left|\int_\Omega (\mathbb{D}_\varepsilon - \mathbb{D})T_z(k_\varepsilon)\right| \le z\int_\Omega |\mathbb{D}_\varepsilon - \mathbb{D}| \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.47}$$

On the other hand, it easily checked by using Lebesgue's Theorem that

$$\int_\Omega \mathbb{D}T_z(k_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \mathbb{D}T_z(k). \tag{3.48}$$

Similarly, we obtain

$$\int_\Omega \frac{k_\varepsilon T_z(k_\varepsilon)}{\ell} \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \frac{kT_z(k)}{\ell}. \tag{3.49}$$

This yields

$$\lim_{\varepsilon\to 0}\int_\Omega \ell|\nabla T_z(k_\varepsilon)|^2 = \int_\Omega \mathbb{D}\, T_z(k) - \int_\Omega \frac{kT_z(k)}{\ell}. \tag{3.50}$$

It remains to prove this limit is equal to $\displaystyle\int_\Omega \ell|\nabla T_z(k)|^2$. To do so, we take $v = T_z(k)$ in (3.14), which is allowed since $w = T_z(k)$ satisfies (3.13). We get :

$$\int_\Omega \ell|\nabla T_z(k)|^2 + \int_\Omega \frac{uT_z(k)}{\ell} = \int_\Omega fT_z(k), \tag{3.51}$$

hence (3.44).

3.3 Uniqueness of the renormalized solution

We prove in this section the uniqueness of the renormalized solution. To do so, we need a maximum principle result, proved in the following Lemma.

Lemma 3.3. *Let $g \in L^\infty(\Omega)$ and $v \in W_{\ell,0}$ such that*

$$\forall\, \phi \in W_{\ell,0}, \quad \int_\Omega \ell \nabla v \cdot \nabla \phi + \int_\Omega \frac{v\phi}{\ell} = \int_\Omega g\phi. \tag{3.52}$$

Then $v \leq \|\ell g\|_{\Omega;0,\infty}$ a.e. Ω.

Proof. The proof follows the standard procedure (see in [15]). For any real valued function f, let f^+ denotes its positive part, namely

$$f^+ = \sup(f, 0) = \frac{|f| + f}{2}.$$

Let $M \in \mathbb{R}_+^\star$, and F_M be the function given by

$$F_M : \left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ t & \to & (t-M)^+, \end{array} \right. \tag{3.53}$$

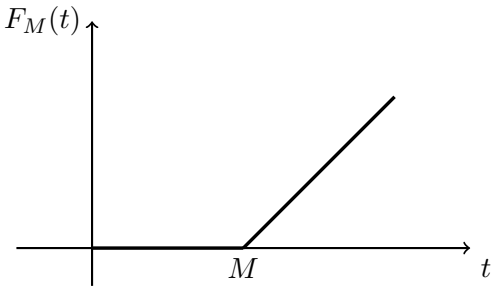

Figure 5: Graph of the function $F_M(t) = (t-M)^+$ for $M > 0$.

We observe that $F_M \in \mathrm{Stamp}(\mathbb{R})$ and $F(0) = 0$. Therefore, by Theorem A.3

$$\tilde{v} = (v - M)^+ \in W_{\ell,0}.$$

It is easily checked that

$$\int_\Omega \frac{v\tilde{v}}{\ell} = \int_\Omega \frac{|\tilde{v}|^2}{\ell} + M \int_\Omega \frac{\tilde{v}}{\ell}, \quad \int_\Omega \ell \nabla v \cdot \nabla \tilde{v} = \int_\Omega \ell |\nabla \tilde{v}|^2.$$

Thus, taking $\phi = \tilde{v}$ in (3.52) yields

$$\int_\Omega \ell |\nabla \tilde{v}|^2 + \int_\Omega \frac{|\tilde{v}|^2}{\ell} = \int_\Omega \left(g - \frac{M}{\ell} \right) \tilde{v}. \tag{3.54}$$

We take $M = \|\ell g\|_{\Omega;0,\infty}$, so that $g - \dfrac{M}{\ell}$ is non-positive a.e. Ω. We deduce that $\|\tilde{v}\|_{H_\ell}^2 \leq 0$, which means $\tilde{v} = 0$ *a.e.* in Ω. Hence the result. □

We are now in order to prove the uniqueness result, stated as follows.

Theorem 3.3. *Let $\mathbb{D} \in L^1(\Omega)$. Then the renormalized solution to Problem (3.9) is unique.*

Proof. Starting as in the proof of Theorem 3.1, we consider a familly $(\mathbb{D}_\varepsilon)_{\varepsilon>0}$ in $L^\infty(\Omega)$ satisfying (3.16) and (3.17), $k_\varepsilon \in W_{\ell,0}$ which satisfies in the weak sense specified by Theorem 2.1,

$$\left\{ \begin{array}{ll} -\mathrm{div}\,(\ell \nabla k_\varepsilon) + \dfrac{k_\varepsilon}{\ell} = \mathbb{D}_\varepsilon & \text{in } \Omega, \\ k_\varepsilon = 0 & \text{at } \Gamma, \end{array} \right. \tag{3.55}$$

Let $k^\star$ be any renormalized solution to Problem (3.9). We aim to prove that $k^\star = \lim_{\varepsilon\to 0} k_\varepsilon$. Let us take

$$w_\varepsilon = \mathbb{H}_\eta\left(T_k(k^\star) - k_\varepsilon\right), \tag{3.56}$$

where $\mathbb{H}_\eta$ is defined by (3.5). As we will let η go to zero, we can consider $\eta \in]0;1[$. We know from Lemma 3.3 that

$$k_\varepsilon \le \|\ell \mathbb{D}_\varepsilon\|_{\Omega;0,\infty},$$

which suggests to take

$$z_\varepsilon = \|\ell\, \mathbb{D}_\varepsilon\|_{\Omega;0,\infty} + 1$$

as threshold, giving

$$\mathbb{H}'_\eta\left(T_{z_\varepsilon}(k^\star) - k_\varepsilon\right) = 0 \quad \text{a.e. on } \{|k^\star| \ge z_\varepsilon\}. \tag{3.57}$$

We deduce in particular that $\nabla w_\varepsilon = 0$ a.e. on $\{|k^\star| \ge z_\varepsilon\}$. Moreover,

- $w_\varepsilon \in L^\infty$ since $\mathbb{H}_\eta$ is bounded by 1,
- $w_\varepsilon \in W_{\ell,0}$ since $\mathbb{H}_\eta \in \mathrm{Stamp}(\mathbb{R})$ and $\mathbb{H}_\eta(0) = 0$.

we deduce that w_ε satisfies (3.13). Therefore, we can take $w = w_\varepsilon$ in formulation (3.12) and $v = w_\varepsilon$ in formulation (3.19), which gives

$$\int_\Omega \ell \nabla k^\star \cdot \nabla w_\varepsilon + \int_\Omega \frac{k^\star w_\varepsilon}{\ell} = \int_\Omega \mathbb{D}\, w_\varepsilon, \tag{3.58}$$

$$\int_\Omega \ell \nabla k_\varepsilon \cdot \nabla w_\varepsilon + \int_\Omega \frac{k_\varepsilon w_\varepsilon}{\ell} = \int_\Omega \mathbb{D}_\varepsilon w_\varepsilon. \tag{3.59}$$

We write the difference and replace w_ε by its expression

$$\begin{aligned} \int_\Omega \ell(\nabla k^\star - \nabla k_\varepsilon)\cdot(\nabla T_{z_\varepsilon}(k^\star) - \nabla k_\varepsilon)\mathbb{H}'_\eta\left(T_{z_\varepsilon}(k^\star) - k_\varepsilon\right) + \int_\Omega \frac{k^\star - k_\varepsilon}{\ell}\mathbb{H}_\eta\left(T_{z_\varepsilon}(k^\star) - k_\varepsilon\right) \\ = \int_\Omega (\mathbb{D} - \mathbb{D}_\varepsilon)\mathbb{H}_\eta\left(T_{z_\varepsilon}(k^\star) - k_\varepsilon\right). \end{aligned} \tag{3.60}$$

Recalling (3.57), we obtain
Therefore,

$$\begin{aligned} \int_\Omega \ell(\nabla k^\star - \nabla k_\varepsilon)\cdot(\nabla T_{z_\varepsilon}(k^\star) - \nabla k_\varepsilon)\mathbb{H}'_\eta\left(T_{z_\varepsilon}(k^\star) - k_\varepsilon\right) = \\ \int_\Omega \ell|\nabla T_{z_\varepsilon}(k^\star) - \nabla k_\varepsilon|^2 \mathbb{H}'_\eta\left(T_{z_\varepsilon}(k^\star) - k_\varepsilon\right) \ge 0, \end{aligned}$$

since $\mathbb{H}'_\eta \geq 0$. Moreover, we deduce from Lebesgue's Theorem,

$$\int_\Omega \frac{k^\star - k_\varepsilon}{\ell} \mathbb{H}_\eta \left(T_{z_\varepsilon}(k^\star) - k_\varepsilon\right) \underset{\eta \to 0}{\longrightarrow} \int_\Omega \frac{|k^\star - k_\varepsilon|}{\ell}. \tag{3.61}$$

Finally, as $|\mathbb{H}_\eta| \leq 1$,

$$\left| \int_\Omega (\mathbb{D} - \mathbb{D}_\varepsilon) \mathbb{H}_\eta \left(T_{z_\varepsilon}(k^\star) - k_\varepsilon\right) \right| \leq \|\mathbb{D} - \mathbb{D}_\varepsilon\|_{\Omega;0,1}. \tag{3.62}$$

Consequently, when η goes to zero, we get from (3.60)

$$\int_\Omega \frac{|k^\star - k_\varepsilon|}{\ell} \leq \|\mathbb{D} - \mathbb{D}_\varepsilon\|_{\Omega;0,1} \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{3.63}$$

which is what we aimed to prove. The uniqueness of the limit in $L^1(\Omega, \ell^{-1}dx)$ directly yields the uniqueness of the renormalized solution.

□

3.4 Equivalent definition

We give another definition of the renormalized solution to Problem (3.9). In the following, we denote

$$W = \{\beta \in \mathrm{Stamp}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ with a compact support}\}.$$

Definition 3.3. *A function $k \in H^1_{loc}(\Omega)$ is a renormalized solution to Problem (3.9) if u satisfies (3.10), (3.11) and (3.12) and*

$$\begin{gathered} \forall\, \beta \in W, \quad \forall\, \varphi \in \mathscr{D}(\Omega), \\ \int_\Omega \ell \nabla u \cdot \nabla \varphi \beta(k) + \int_\Omega \ell |\nabla u|^2 \varphi \beta'(k) + \int_\Omega \frac{u\varphi}{\ell} \beta(k) = \int_\Omega \mathbb{D} \varphi \beta(k). \end{gathered} \tag{3.64}$$

Theorem 3.4 (Equivalence between both definitions)**.** *The definitions 3.2 and 3.3 are equivalent.*

Proof. Let $k \in H^1_{loc}(\Omega)$ that satisfies the definition 3.2, $\beta \in W$ and $\varphi \in \mathscr{D}(\Omega)$. We first notice that according to Theorem A.3,

$$\beta(k)\varphi \in W_{\ell,0}, \quad \text{and} \quad |\beta(k)\varphi| \leq \|\beta\|_\infty \|\varphi\|_{0,\infty,\Omega},$$

hence $\beta(k) \in W_{\ell,0} \cap L^\infty$. We must show that $w = \beta(k)\varphi$ satisfies (3.13). Let $z \in \mathbb{R}^*$ such that $\mathrm{Supp}(\beta) \subset [-z; z]$. The function $\nabla(\beta(k)\varphi) = \beta'(k)\nabla k \varphi + \beta(k)\nabla\varphi$ vanishes *a.e.* in $\{|k| \geq z\}$, which means $\beta(k)\varphi$ satisfies (3.13).

Now we assume that $k \in H^1_{loc}(\Omega)$ satisfies the definition 3.3. Let $w \in W_{\ell,0} \cap L^\infty$ that satisfies (3.13), and $(w_m)_{m\in\mathbb{N}}$ be a sequence of smooth functions given by Theorem A.4. Taking $\beta = L_n$ (defined by (3.26)) and $\varphi = w_m$ in (3.64) yields

$$\int_\Omega \ell \nabla k \cdot \nabla w_m L_n(k) + \int_\Omega \ell |\nabla k|^2 w_m L'_n(k) + \int_\Omega \frac{u w_m}{\ell} L_n(k) = \int_\Omega \mathbb{D} w_m L_n(k). \tag{3.65}$$

The L^∞ bound on $(w_m)_{m\in\mathbb{N}}$ given by Theorem A.4 and the Lebesgue theorem yield

$$\int_\Omega \mathbb{D}w_m L_n(k) \underset{m\to+\infty}{\longrightarrow} \int_\Omega \mathbb{D}wL_n(k), \tag{3.66}$$

$$\int_\Omega \frac{kw_m}{\ell} L_n(k) \underset{m\to+\infty}{\longrightarrow} \int_\Omega \frac{kw}{\ell} L_n(k), \tag{3.67}$$

$$\int_\Omega \ell|\nabla k|^2 w_m L_n'(k) \underset{m\to+\infty}{\longrightarrow} \int_\Omega \ell|\nabla k|^2 w L_n'(k), \tag{3.68}$$

for all $n\in\mathbb{N}$. Moreover, the convergence $w_m \underset{m\to+\infty}{\longrightarrow} w$ in H_ℓ directly yields

$$\int_\Omega \ell\nabla k\cdot\nabla w_m L_n(k) \underset{m\to+\infty}{\longrightarrow} \int_\Omega \ell\nabla k\cdot\nabla w L_n(k), \tag{3.69}$$

for all $n\in\mathbb{N}$.Finally, Lebesgue's Theorem yields

$$\int_\Omega \mathbb{D}wL_n(k) \underset{n\to+\infty}{\longrightarrow} \int_\Omega \mathbb{D}w, \tag{3.70}$$

$$\int_\Omega \frac{kw}{\ell} L_n(k) \underset{n\to+\infty}{\longrightarrow} \int_\Omega \frac{kw}{\ell}, \tag{3.71}$$

$$\int_\Omega \ell\nabla k\cdot\nabla w L_n(k) \underset{n\to+\infty}{\longrightarrow} \int_\Omega \ell\nabla k\cdot\nabla w. \tag{3.72}$$

And (3.12) yields

$$\left|\int_\Omega \ell|\nabla k|^2 w L_n'(k)\right| \le \frac{1}{n}\int_{\{n\le|k|\le 2n\}} \ell|\nabla k|^2 \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.73}$$

This yields (3.14).

□

A Technical results

A.1 Regularity results in $W_{\ell,0}$

We begin by recalling a result of [21].

Lemma A.1. *Let $u\in W^{1,1}(\Omega)$, and $a\in\mathbb{R}$ such that $\{u=a\}$ has a non-zero Lebesgue measure. Then $\nabla u = 0$ almost everywhere on $\{u=a\}$.*

We recall the following definition.

Definition A.1 (Stampacchia conditions)**.** *Let $F:\mathbb{R}\to\mathbb{R}$. We say that F satisfies the Stampacchia conditions if F is globally Lipchitz and its derivative F' has a finite number of discontinuities.*

Remark A.1. *Note that if F satisfies the Stampacchia conditions, it is not necessarily bounded.*

We denote by Stamp($\mathbb{R}$) the set of all functions satisfying the Stampacchia conditions.

Lemma A.2. *Let $F\in$ Stamp($\mathbb{R}$). Then $F'\in L^\infty(\mathbb{R})$ and we have*

$$\forall a,b\in\mathbb{R},\quad |F(b)-F(a)|\le\|F'\|_\infty|b-a|. \tag{A.1}$$

Proof. Let $(t_j)_{1\leq j\leq J}$ be defined as the points where F' is not continuous. Let the intervals $(I_j)_{0\leq j\leq J}$ be defined by

$$\text{(A.2)}\qquad \begin{cases} I_0 =]-\infty; t_1[, \\ I_j =]t_j; t_{j+1}[\text{ for } 1 \leq j \leq J-1, \\ I_J =]t_J; +\infty[. \end{cases}$$

We consider $j \in [\![0; J]\!]$. The function F is continuous on $\overline{I_j}$ and differentiable on I_j. Then for any $x, y \in \overline{I_j}$ with $x < y$, there exists $c \in]x; y[$ such that

$$\text{(A.3)}\qquad F(y) - F(x) = F'(c)(y - x).$$

We now consider$j \in [\![0; J]\!]$ and $a \in I_j$ and $(a_n)_{n\in\mathbb{N}}$ a sequence in I_j satisfying $a_n \underset{n\to+\infty}{\longrightarrow} a$. We assume without losing generality that for every $n \in \mathbb{N}$, $a_n > a$. Then, for any $n \in \mathbb{N}$ there exists a $c_n \in]a; a_n[$ satisfying

$$\text{(A.4)}\qquad F(a_n) - F(a) = F'(c_n)(a_n - a).$$

Given that F' is continuous on I_j, we obtain

$$\text{(A.5)}\qquad |F'(a)| = \limsup_{n\to+\infty} |F'(c_n)| = \limsup_{n\to+\infty} \frac{|F(a_n) - F(a)|}{a_n - a} \leq \lambda,$$

where $\lambda > 0$ is a Lipschitz constant for F.
Consequently, F' is bounded a.e. on $\mathbb{R}$, which means $F' \in L^\infty(\mathbb{R})$. It remains to prove (A.1).

Let $a, b \in \mathbb{R}$. If a and b are on the same $\overline{I_j}$, the mean value theorem directly yields (A.1). We now assume that there exists $j \in [\![0; J-1]\!]$ such that $a \in \overline{I_j}$ and $b \in \overline{I_{j+1}}$. Then we obtain

$$\text{(A.6)}\qquad \begin{aligned} |F(b) - F(a)| &\leq |F(b) - F(t_{j+1})| + |F(t_{j+1}) - F(a)| \\ &\leq \|F'\|_\infty (b - t_{j+1}) + \|F'\|_\infty (t_{j+1} - a) = \|F'\|_\infty (b - a). \end{aligned}$$

Finally, an instant induction yields (A.1). □

The following theorem is an adaptation of a Stampacchia result from [21], adapted to the space H_ℓ.

Theorem A.1. *Let $u \in W_{\ell,0}$ and $F \in \mathrm{Stamp}(\mathbb{R})$. Then we have the following equality at the distribution sense*

$$\nabla F(u) = F'(u)\nabla u.$$

Proof. Let $u \in W_{\ell,0}$, and $(u_n)_{n\in\mathbb{N}}$ a sequence of $\mathscr{D}(\Omega)$ functions such that $u_n \underset{n\to+\infty}{\longrightarrow} u$ in H_ℓ.
Let θ be characterized by

- $\theta \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}_+)$,
- $\mathrm{Supp}(\theta) \subset \bar{B}(0, 1)$,
- $\displaystyle\int_{\mathbb{R}} \theta(y)\, dy = 1.$

Let $(\theta_\varepsilon)_{\varepsilon>0}$ be the mollifier given by

$$\forall \varepsilon > 0,\ \forall y \in \mathbb{R},\quad \theta_\varepsilon(y) = \frac{1}{\varepsilon}\theta\left(\frac{y}{\varepsilon}\right), \tag{A.7}$$

and $(F_\varepsilon)_{\varepsilon>0}$ be defined by $F_\varepsilon = \theta_\varepsilon * F$, $(t_j)_{1\le j\le J}$ be the points where F' is not continuous. Lebesgue theorem and Young convolution inequality yield

$$F'_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} F'(t) \quad \text{for all } t \in \mathbb{R}\setminus(t_j)_{1\le j\le J}, \tag{A.8}$$

$$\|F'_\varepsilon\|_\infty \le \|F'\|_\infty \quad \text{for all } \varepsilon > 0. \tag{A.9}$$

Moreover, a linear change of variables and (A.1) yield

$$\begin{aligned} |F_\varepsilon(t) - F(t)| &\le \int_{\mathbb{R}} \theta_\varepsilon(y)|F(t-y) - F(t)|\,dy = \int_{\mathbb{R}} \theta(z)|F(t-\varepsilon z) - F(t)|\,dz \\ &\le \varepsilon\|F'\|_\infty \int_{\mathbb{R}} |z|\theta(z)\,dz \underset{\varepsilon\to 0}{\longrightarrow} 0. \end{aligned} \tag{A.10}$$

This yields the uniform convergence of $(F_\varepsilon)_{\varepsilon>0}$, *i.e.*

$$\|F_\varepsilon - F\|_\infty \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{A.11}$$

The classical results about convolution products tell that $F_\varepsilon \in \mathscr{C}^\infty(\mathbb{R})$. Therefore, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, $F_\varepsilon(u_n) \in \mathscr{C}^\infty(\Omega)$, giving in particular $\nabla F_\varepsilon(u_n) = F'_\varepsilon(u_n)\nabla u_n$. Let $\varphi \in \mathscr{D}(\Omega)$, we write

$$\int_\Omega \nabla F_\varepsilon(u_n)\varphi = -\int_\Omega F_\varepsilon(u_n)\nabla\varphi = -\int_\Omega (F_\varepsilon(u_n) - F_\varepsilon(u))\nabla\varphi - \int_\Omega F_\varepsilon(u)\nabla\varphi. \tag{A.12}$$

On one hand, the regularity of F_ε and the inequality (A.9) yield

$$\begin{aligned} \left|\int_\Omega (F_\varepsilon(u_n) - F_\varepsilon(u))\nabla\varphi\right| &\le \|F'_\varepsilon\|_\infty \int_\Omega |u_n - u|\cdot|\nabla\varphi| \\ &\le \|F'\|_\infty \|\ell^{-1/2}(u_n - u)\|_{\Omega;0,2}\|\ell^{1/2}\nabla\varphi\|_{\Omega;0,2} \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned} \tag{A.13}$$

On the other hand, since $(F_\varepsilon)_{\varepsilon>0}$ uniformly converges to F, we obtain

$$\left|\int_\Omega (F_\varepsilon(u) - F(u))\nabla\varphi\right| \le \|F_\varepsilon - F\|_\infty \|\nabla\varphi\|_{\Omega;0,1} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{A.14}$$

hence $\displaystyle\int_\Omega F_\varepsilon(u)\nabla\varphi \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega F(u)\nabla\varphi$. This yields

$$\int_\Omega \nabla F_\varepsilon(u_n)\varphi = -\int_\Omega F_\varepsilon(u_n)\nabla\varphi \underset{\substack{n\to+\infty\\ \varepsilon\to 0}}{\longrightarrow} -\int_\Omega F(u)\nabla\varphi = \int_\Omega \nabla F(u)\varphi. \tag{A.15}$$

We now need to show the convergence $F'_\varepsilon(u_n)\nabla u_n \underset{\substack{n\to+\infty\\ \varepsilon\to 0}}{\longrightarrow} F'(u)\nabla u$ at the distribution sense. Let $\varphi \in \mathscr{D}(\Omega)$, we write

$$\int_\Omega F'_\varepsilon(u_n)\nabla u_n\varphi = \int_\Omega F'_\varepsilon(u_n)(\nabla u_n - \nabla u)\varphi + \int_\Omega F'_\varepsilon(u_n)\nabla u\varphi. \tag{A.16}$$

On one hand, (A.9) yields

$$\left|\int_\Omega F_\varepsilon'(u_n)(\nabla u_n - \nabla u)\varphi\right| \leq \|F'\|_\infty \|\ell^{1/2}(\nabla u_n - \nabla u)\|_{0,2,\Omega}\|\ell^{-1/2}\varphi\|_{0,2,\Omega} \underset{n\to+\infty}{\longrightarrow} 0. \tag{A.17}$$

On the other hand, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ still named the same, such that $u_n \underset{n\to+\infty}{\longrightarrow} u$ *a.e.* on Ω, and the continuity of F_ε' yields $F_\varepsilon'(u_n) \underset{n\to+\infty}{\longrightarrow} F_\varepsilon'(u)$ *a.e.* on Ω. Given that $|\nabla u\varphi| \in L^1(\Omega)$, we deduce from Lebesgue theorem that

$$\int_\Omega F_\varepsilon'(u_n)\nabla u\varphi \underset{n\to+\infty}{\longrightarrow} \int_\Omega F_\varepsilon'(u)\nabla u\varphi. \tag{A.18}$$

Finally, convergence (A.8) and Lemma A.1 yield that $F_\varepsilon'(u)\nabla u \underset{\varepsilon\to 0}{\longrightarrow} F'(u)\nabla u$ a.e. on Ω. We obtain by Lebesgue theorem

$$\int_\Omega F_\varepsilon'(u)\nabla u\varphi \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega F'(u)\nabla u\varphi. \tag{A.19}$$

This yields

$$\int_\Omega F_\varepsilon'(u_n)\nabla u_n\varphi \underset{\substack{n\to+\infty\\ \varepsilon\to 0}}{\longrightarrow} \int_\Omega F'(u)\nabla u\varphi. \tag{A.20}$$

Both convergences (A.15) and (A.20) yield the result. □

Theorem A.2. *Let $F \in \mathrm{Stamp}(\mathbb{R})$, $(u_n)_{n\in\mathbb{N}}$ be a converging sequence in H_ℓ and $u \in H_\ell$ its limit.*

(i) We assume that $F(w) \in H_\ell$ for every $w \in H_\ell$. Then

$$F(u_n) \underset{n\to+\infty}{\longrightarrow} F(u) \quad \text{in } H_\ell. \tag{A.21}$$

(ii) Let $\varphi \in \mathscr{D}(\Omega)$, we have

$$F(u_n)\varphi \underset{n\to+\infty}{\longrightarrow} F(u)\varphi \quad \text{in } H_\ell. \tag{A.22}$$

Proof. We only do the proof of (i), (ii) being proved similarly. We deduce first from (A.1) that

$$\int_\Omega \frac{|F(u_n) - F(u)|^2}{\ell} \leq \|F'\|_\infty^2 \int_\Omega \frac{|u_n - u|^2}{\ell} \underset{n\to+\infty}{\longrightarrow} 0. \tag{A.23}$$

Then, we apply Theorem A.1 and triangular inequality on the derivative term to obtain

$$\begin{aligned}\|\ell^{1/2}(\nabla F(u_n) - \nabla F(u))\|_{\Omega;0,2} &= \|\ell^{1/2}(F'(u_n)\nabla u_n - F'(u)\nabla u)\|_{\Omega;0,2}\\ &\leq \|\ell^{1/2}F'(u_n)(\nabla u_n - \nabla u)\|_{\Omega;0,2} + \|\ell^{1/2}(F'(u_n) - F'(u))\nabla u\|_{\Omega;0,2}.\end{aligned} \tag{A.24}$$

The convergence of the first norm is straightforward. Indeed, we have

$$\|\ell^{1/2}F'(u_n)(\nabla u_n - \nabla u)\|_{\Omega;0,2} \leq \|F'\|_\infty \|\ell^{1/2}(\nabla u_n - \nabla u)\|_{\Omega;0,2} \underset{n\to+\infty}{\longrightarrow} 0. \tag{A.25}$$

The convergence of the last norm in (A.24) is trickier. Let $(t_j)_{1\leq j\leq J}$ be defined as the discontinuities of F' and $(I_j)_{0,\leq j\leq J}$ as in (A.2). We define two families of subsets of Ω

$$\left\{\begin{array}{ll} B_j = \{u \in I_j\} & \text{for} \quad 0 \leq j \leq J, \\ A_j = \{u = t_j\} & \text{for} \quad 1 \leq j \leq J. \end{array}\right. \tag{A.26}$$

Hence the decomposition

$$\begin{aligned} \|\ell^{1/2}(F'(u_n) - F'(u))\nabla u\|^2_{\Omega;0,2} &= \int_\Omega \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 \\ &= \sum_{j=0}^{J} \int_{B_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 + \sum_{j=1}^{J} \int_{A_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2. \end{aligned} \tag{A.27}$$

We cannot apply immediately Lemma A.1, because $u \notin W^{1,1}(\Omega)$. Then we have to find another way. Let $\varepsilon > 0$. Since $\ell|\nabla u|^2 \in L^1(\Omega)$, we deduce the existence of $\delta > 0$ such that for any borelian subset A of Ω,

$$|A| \leq \delta \Rightarrow \int_A \ell |\nabla u|^2 \leq \varepsilon. \tag{A.28}$$

Let $j \in [\![1; J]\!]$, we write

$$\int_{A_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 \leq 2\|F'\|^2_\infty \int_{A_j} \ell |\nabla u|^2. \tag{A.29}$$

We consider $\omega \subset\subset \Omega$ such that $|\Omega \setminus \omega| \leq \delta$, and write the decomposition

$$\int_{A_j} \ell |\nabla u|^2 = \int_{A_j \cap \omega} \ell |\nabla u|^2 + \int_{A_j \cap (\Omega \setminus \omega)} \ell |\nabla u|^2. \tag{A.30}$$

On one hand, (A.28) yields

$$\int_{A_j \cap (\Omega \setminus \omega)} \ell |\nabla u|^2 \leq \varepsilon. \tag{A.31}$$

On the other hand, recalling that $u \in H^1(\omega)$, Lemma A.1 yields

$$\nabla u = 0 \quad \text{a.e. } A_j \cap \omega. \tag{A.32}$$

We have just shown

$$\forall \varepsilon > 0, \quad \int_{A_j} \ell |\nabla u|^2 \leq \varepsilon \quad \text{hence} \quad \int_{A_j} \ell |\nabla u|^2 = 0. \tag{A.33}$$

Hence, (A.29) yields

$$\int_{A_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 = 0. \tag{A.34}$$

Let now j be an index between 0 and J, the function F' is continuous on B_j. The Lebesgue inverse theorem then yields the existence of a subsequence of $(F'(u_n))_{n\in\mathbb{N}}$, denoted in the same way, converging to $F'(u)$ a.e. Ω. We obtain for all j

$$\int_{B_j} \ell |F'(u_n) - F'(u)|^2 |\nabla u|^2 \underset{n\to+\infty}{\longrightarrow} 0, \tag{A.35}$$

with the Lebesgue theorem.

We have then shown that a subsequence of $(u_n)_{n\in\mathbb{N}}$ still named the same satisfies

$$F(u_n) \underset{n\to+\infty}{\longrightarrow} F(u) \quad \text{in } H_\ell. \tag{A.36}$$

We recall that $H_\ell \hookrightarrow L^1(\Omega)$ is a compact embedding. In addition, (A.1) and Hölder inequality yield

$$\begin{aligned}\int_\Omega |F(u_n)-F(u)| &\le \|F'\|_\infty \int_\Omega |u_n-u| \\ &\le \|F'\|_\infty \left(\int_\Omega \ell\right)^{1/2} \left(\int_\Omega \frac{|u_n-u|^2}{\ell}\right)^{1/2} \underset{n\to+\infty}{\longrightarrow} 0,\end{aligned} \tag{A.37}$$

which means $F(u_n) \underset{n\to+\infty}{\longrightarrow} F(u)$ in $L^1(\Omega)$. We can deduce the convergence of the whole sequence

$$F(u_n) \underset{n\to+\infty}{\longrightarrow} F(u) \quad \text{in } H_\ell. \tag{A.38}$$

□

Theorem A.3. *Let $F \in \mathrm{Stamp}(\mathbb{R})$, $u \in W_{\ell,0}$.*

(i) Let $\varphi \in \mathscr{D}(\Omega)$, then $F(u)\varphi \in W_{\ell,0}$.

(ii) We assume that $F(0)=0$. Then $F(u) \in W_{\ell,0}$.

Proof. **We start by proving (i) :** we first show that $F(u)\varphi \in H_\ell$.
Let $v \in \mathscr{D}(\Omega)$, so that $\|u-v\|_{H_\ell} < +\infty$. We first apply triangular inequality to obtain

$$\|\ell^{-1/2}F(u)\varphi\|_{\Omega;0,2} \le \|\ell^{-1/2}(F(u)-F(v))\varphi\|_{\Omega;0,2} + \|\ell^{-1/2}F(v)\varphi\|_{\Omega;0,2}. \tag{A.39}$$

On one hand, (A.1) yields

$$\int_\Omega \frac{|\varphi|^2}{\ell}|F(u)-F(v)|^2 \le \|F'\|_\infty^2 \|\varphi\|_{\Omega;0,\infty}^2 \int_\Omega \frac{|u-v|^2}{\ell} < +\infty. \tag{A.40}$$

On the other hand, since $v \in \mathscr{D}(\Omega)$ and $F \in \mathscr{C}^0(\mathbb{R})$, we obtain $F(v) \in \mathscr{C}^0(\Omega)$ hence $F(v)\varphi \in \mathscr{C}_c^0(\Omega)$. This directly yields

$$\int_\Omega \frac{|F(v)\varphi|^2}{\ell} < +\infty, \quad \text{hence} \quad \int_\Omega \frac{|F(u)\varphi|^2}{\ell} < +\infty. \tag{A.41}$$

Concerning the derivative term, the gradient of a product and triangular inequality yield

$$\|\ell^{1/2}\nabla(F(u)\varphi)\|_{\Omega;0,2} \le \|\ell^{1/2}\nabla F(u)\varphi\|_{\Omega;0,2} + \|\ell^{1/2}F(u)\nabla\varphi\|_{\Omega;0,2}. \tag{A.42}$$

We first focus on the last term, and write

$$\|\ell^{1/2}F(u)\nabla\varphi\|_{\Omega;0,2} \le \|\ell^{1/2}F(v)\nabla\varphi\|_{\Omega;0,2} + \|\ell^{1/2}(F(u)-F(v))\nabla\varphi\|_{\Omega;0,2}. \tag{A.43}$$

It is clear that $\|\ell^{1/2}F(v)\nabla\varphi\|_{\Omega;0,2} < +\infty$. Then, (A.1) yields

$$\begin{aligned}\|\ell^{1/2}(F(u)-F(v))\nabla\varphi\|_{\Omega;0,2} &\le \|F'\|_\infty \|\ell^{1/2}(u-v)\nabla\varphi\|_{\Omega;0,2} \\ &\le \|F'\|_\infty \|\ell\nabla\varphi\|_{\Omega;0,\infty} \|\ell^{-1/2}(u-v)\|_{\Omega;0,2} < +\infty.\end{aligned} \tag{A.44}$$

We now have to decompose the other term

$$\begin{aligned}\|\ell^{1/2}\nabla F(u)\varphi\|_{\Omega;0,2} &= \|\ell^{1/2}F'(u)(\nabla u)\varphi\|_{\Omega;0,2}\\ &\leq \|\ell^{1/2}F'(u)(\nabla u-\nabla v)\varphi\|_{\Omega;0,2}+\|\ell^{1/2}F'(u)(\nabla v)\varphi\|_{\Omega;0,2}.\end{aligned}\tag{A.45}$$

On one hand, (A.1) yields

$$\int_\Omega \ell|\varphi|^2|F'(u)(\nabla u-\nabla v)|^2 \leq \|F'\|_\infty^2\|\varphi\|_{\Omega;0,\infty}^2\int_\Omega \ell|\nabla u-\nabla v|^2<+\infty.\tag{A.46}$$

On the other hand, since $\nabla v\in\mathscr{D}(\Omega)$ we obtain $\varphi\nabla v\in\mathscr{D}(\Omega)$. Hence

$$\int_\Omega \ell|\varphi|^2|F'(u)\nabla v|^2 \leq \|F'\|_\infty^2\int_\Omega \ell|\varphi\nabla v|^2<+\infty.\tag{A.47}$$

As a result, the derivative term is finite. This means that $F(u)\varphi\in H_\ell$.

We now have to prove that $F(u)\varphi$ is a limit of $\mathscr{D}(\Omega)$ functions. Since $u\in W_{\ell,0}$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of smooth functions such that $u_n \underset{n\to+\infty}{\longrightarrow} u$ in H_ℓ. We also consider the approximation $F_\varepsilon=\theta_\varepsilon * F$ for $\varepsilon>0$ as in the proof of Theorem A.1.

We notice that for every $\varepsilon>0$ and $n\in\mathbb{N}$, the function $F_\varepsilon(u_n)$ is in $\mathscr{C}^\infty(\Omega)$, hence $F_\varepsilon(u_n)\varphi\in\mathscr{D}(\Omega)$. Consequently, it remains to prove the following convergence

$$F_\varepsilon(u_n)\varphi \underset{\substack{n\to+\infty\\ \varepsilon\to0}}{\longrightarrow} F(u)\varphi \quad\text{in } H_\ell.\tag{A.48}$$

Yet, we have

$$\|(F_\varepsilon(u_n)-F(u))\varphi\|_{H_\ell} \leq \|(F_\varepsilon(u_n)-F(u_n))\varphi\|_{H_\ell}+\|(F(u_n)-F(u))\varphi\|_{H_\ell}.$$

Theorem A.2 yields that $\|(F(u_n)-F(u))\varphi\|_{H_\ell} \underset{n\to+\infty}{\longrightarrow} 0$, so that it only remains to prove

$$F_\varepsilon(v)\varphi \xrightarrow[\varepsilon\to0]{H_\ell} F(v)\varphi, \quad\text{for all } v\in\mathscr{D}(\Omega).\tag{A.49}$$

First we write

$$\|\ell^{-1/2}(F_\varepsilon(v)-F(v))\varphi\|_{\Omega;0,2} \leq \|F_\varepsilon-F\|_\infty\|\ell^{-1/2}\varphi\|_{\Omega;0,2} \underset{\varepsilon\to0}{\longrightarrow} 0.\tag{A.50}$$

Let the two families of subsets of Ω, $(A_j)_{1\leq j\leq J}$ and $(B_j)_{0\leq j\leq J}$ defined similarly to (A.26), but with v. We obtain the decomposition of the derivative term

$$\begin{aligned}\int_\Omega \ell|\nabla F_\varepsilon(v)-\nabla F(v)|^2 &= \int_\Omega \ell|\nabla v|^2|F'_\varepsilon(v)-F'(v)|^2\\ &= \sum_{j=0}^J\int_{B_j}\ell|\nabla v|^2|F'_\varepsilon(v)-F'(v)|^2+\sum_{j=1}^J\int_{A_j}\ell|\nabla v|^2|F'_\varepsilon(v)-F'(v)|^2.\end{aligned}\tag{A.51}$$

By a similar method to the proof of Theorem A.2, the integrals on A_j vanish. Let $j\in[\![0;J]\!]$, the convergence (A.8) yields that $F'_\varepsilon(v) \underset{\varepsilon\to0}{\longrightarrow} F'(v)$ a.e. B_j. Moreover, by (A.9) we get

$$\ell|\nabla v|^2|F'_\varepsilon(v)-F'(v)|^2 \leq 2\|F'\|_\infty^2\ell|\nabla v|^2\in L^\infty(\Omega)\subset L^1(\Omega).$$

Consequently, the Lebesgue theorem yields the convergence

$$\int_{B_j} \ell |\nabla v|^2 |F'_\varepsilon(v) - F'(v)|^2 \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{A.52}$$

We can deduce (A.49).

Proof of (ii) : we first show that $F(u) \in H_\ell$. The method is quite similar to the one used in the previous proof. We consider $v \in \mathscr{D}(\Omega)$, so that $\|u - v\|_{H_\ell} < +\infty$. We first write

$$\|\ell^{-1/2} F(u)\|_{\Omega;0,2} \leq \|\ell^{-1/2}(F(u) - F(v))\|_{\Omega;0,2} + \|\ell^{-1/2} F(v)\|_{\Omega;0,2}. \tag{A.53}$$

By a similar argument, the first norm is finite. Then, since $v \in \mathscr{D}(\Omega)$ and $F(0) = 0$, we directly obtain $F(v) \in \mathscr{C}^0_c(\Omega)$, which means the last norm is finite.

We now use Theorem A.1 and decompose the derivative term

$$\begin{aligned} \|\ell^{1/2} \nabla F(u)\|_{\Omega;0,2} &= \|\ell^{1/2} F'(u) \nabla u\|_{\Omega;0,2} \\ &\leq \|F'\|_\infty \left(\|\ell^{1/2}(\nabla u - \nabla v)\|_{\Omega;0,2} + \|\ell^{1/2} \nabla v\|_{\Omega;0,2} \right). \end{aligned} \tag{A.54}$$

The first norm is finite by assumption and the second is also finite given that $v \in \mathscr{D}(\Omega)$.

Now we show that $F(u) \in W_{\ell,0}$. We consider the sequence $(G_\varepsilon)_{\varepsilon > 0}$ defined by

$$G_\varepsilon = \theta_\varepsilon * F - F_\varepsilon(0). \tag{A.55}$$

This family satisfies

$$\|G_\varepsilon - F\|_\infty \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{A.56}$$

$$G'_\varepsilon(t) \underset{\varepsilon \to 0}{\longrightarrow} F'(t) \quad \text{for } t \neq (t_j)_{1 \leq j \leq J}, \tag{A.57}$$

$$\|G'_\varepsilon\|_\infty \leq \|F'\|_\infty \quad \text{for } \varepsilon > 0. \tag{A.58}$$

Let $u \in W_{\ell,0}$ and $(u_n)_{n \in \mathbb{N}}$ be a sequence of smooth functions converging to u in H_ℓ. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Given that $G_\varepsilon(0) = 0$, we have $G_\varepsilon(u_n) \in \mathscr{D}(\Omega)$. Therefore, it remains to prove

$$G_\varepsilon(u_n) \underset{\substack{n \to +\infty \\ \varepsilon \to 0}}{\longrightarrow} F(u) \quad \text{in } H_\ell. \tag{A.59}$$

To do so, we write

$$\|G_\varepsilon(u_n) - F(u)\|_{H_\ell} \leq \|G_\varepsilon(u_n) - F(u_n)\|_{H_\ell} + \|F(u_n) - F(u)\|_{H_\ell}.$$

Theorem A.2 yields that $\|F(u_n) - F(u)\|_{H_\ell} \underset{n \to +\infty}{\longrightarrow} 0$. Thus, we have to prove

$$G_\varepsilon(v) \underset{\varepsilon \to 0}{\longrightarrow} F(v), \quad \text{for all } v \in \mathscr{D}(\Omega). \tag{A.60}$$

Let $v \in \mathscr{D}(\Omega)$. For every $\varepsilon > 0$, $G_\varepsilon(v) \in \mathscr{D}(\Omega)$ and $F(v) \in \mathscr{C}^0_c(\Omega)$. By a topological argument, there exists a compact $K \subset \Omega$ such that for sufficiently small values of $\varepsilon > 0$, $\mathrm{Supp}(G_\varepsilon(v)) \subset K$, and $A := K - \mathrm{Supp}(F(v)) \subset \Omega$. This yields

$$\int_\Omega \frac{|G_\varepsilon(v) - F(v)|^2}{\ell} \leq \|G_\varepsilon - F\|^2_\infty \int_A \frac{1}{\ell} \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{A.61}$$

With a similar method to what was done to prove (i), we obtain

$$\int_\Omega \ell |\nabla G_\varepsilon(v) - \nabla F(v)|^2 \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{A.62}$$

hence (A.60). □

A.2 Additional stuff

We wtart by modifying the results of Corollary A.3, by focusing on $W_{\ell,0}\cap L^\infty(\Omega)$ functions.

Theorem A.4. *Let $w \in W_{\ell,0} \cap L^\infty(\Omega)$. There exists a sequence $(w_m)_{m\in\mathbb{N}}$ of $\mathscr{D}(\Omega)$ functions satisfying*

$$\left\{ \begin{array}{l} w_m \underset{m\to+\infty}{\longrightarrow} w \text{ in } H_\ell, \\ \forall\, m \in \mathbb{N}, \quad \|w_m\|_{\Omega;0,\infty} \leq C, \end{array} \right. \tag{A.63}$$

where $C > 0$ does not depend on m.

Proof. Given that $w \in W_{\ell,0}$, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathscr{D}(\Omega)$ functions such that $v_n \underset{n\to+\infty}{\longrightarrow} w$ in H_ℓ. This sequence is not uniformly bounded in the general case. Let $z = \|w\|_{\Omega;0,\infty}$, and T_z the truncation at height z, as defined in (3.1). The first idea is taking $w_m = T_z(v_m)$, but T_z is only continuous, instead of $\mathscr{C}^\infty$. This is why we introduce $(\theta_\varepsilon)_{\varepsilon>0}$ a mollifier as in the proof of Theorem A.1, and define $(T_z^\varepsilon)_{\varepsilon>0}$ by

$$T_z^\varepsilon = \theta_\varepsilon * T_z - \theta_\varepsilon * T_z(0). \tag{A.64}$$

As a composition on $\mathscr{C}^\infty$ functions, $T_z^\varepsilon(v_n) \in \mathscr{C}^\infty(\Omega)$ for all $\varepsilon > 0$ and $n \in \mathbb{N}$. Moreover, v_n is compact-supported, and $T_z^\varepsilon(0) = 0$, which means T_z^ε also has compact support. Finally, the same arguments as for showing (A.59) yield the convergence

$$T_z^\varepsilon(v_n) \underset{\substack{n\to+\infty \\ \varepsilon\to0}}{\longrightarrow} T_z(w) - T_z(0) = w, \tag{A.65}$$

given that $|w| \leq z$ a.e. on Ω and $T_z(0) = 0$. □

Theorem A.5. *Let $F \in \mathrm{Stamp}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $u \in W_{\ell,0}$ and $w \in W_{\ell,0} \cap L^\infty(\Omega)$. Then*

$$F(u)w \in W_{\ell,0}. \tag{A.66}$$

Proof. **Step 1 : $F(u)w \in H_\ell$.** First, we have

$$\|\ell^{-1/2}F(u)w\|_{\Omega;0,2} \leq \|F\|_\infty \|\ell^{-1/2}w\|_{\Omega;0,2} < +\infty. \tag{A.67}$$

Then we apply the triangular inequality on the derivative term

$$\begin{aligned} \|\ell^{1/2}\nabla(F(u)w)\|_{\Omega;0,2} &\leq \|\ell^{1/2}F'(u)(\nabla u)w\|_{\Omega;0,2} + \|\ell^{1/2}F(u)\nabla w\|_{\Omega;0,2} \\ &\leq \|F'\|_\infty \|w\|_{\Omega;0,\infty}\|\ell^{1/2}\nabla u\|_{\Omega;0,2} + \|F\|_\infty \|\ell^{1/2}\nabla w\|_{\Omega;0,2} < +\infty. \end{aligned} \tag{A.68}$$

We consider $F_\varepsilon = \theta_\varepsilon * F$ defined as in the proof of Theorem A.1, $(u_n)_{n\in\mathbb{N}}$ a sequence of smooth functions converging strongly to u in H_ℓ, and $(w_m)_{m\in\mathbb{N}}$ a sequence defined as in Theorem A.4. The fact that $F_\varepsilon(u_n)w_m \in \mathscr{D}(\Omega)$ for any $\varepsilon > 0$ and $m, n \in \mathbb{N}$ is straightforward. We aim at showing the following

$$F_\varepsilon(u_n)w_m \underset{\substack{m,n\to+\infty \\ \varepsilon\to0}}{\longrightarrow} F(u)w \quad \text{in } H_\ell. \tag{A.69}$$

Step 2 : the term without derivative. The triangular inequality is written

$$\begin{aligned} \|\ell^{-1/2}(F_\varepsilon(u_n)w_m - F(u)w)\|_{\Omega;0,2} &\leq \|\ell^{-1/2}F_\varepsilon(u_n)(w_m - w)\|_{\Omega;0,2} \\ &+ \|\ell^{-1/2}(F_\varepsilon(u_n) - F(u_n))w\|_{\Omega;0,2} + \|\ell^{-1/2}(F(u_n) - F(u))w\|_{\Omega;0,2}. \end{aligned} \tag{A.70}$$

Respectively the bound on F_ε, the uniform convergence and the Lebesgue theorem yield

$$\|\ell^{-1/2}F_\varepsilon(u_n)(w_m-w)\|_{\Omega;0,2}\leq\|F\|_\infty\|\ell^{-1/2}(w_m-w)\|_{\Omega;0,2}\underset{m\to+\infty}{\longrightarrow}0, \tag{A.71}$$

$$\|\ell^{-1/2}(F_\varepsilon(u_n)-F(u_n))w\|_{\Omega;0,2}\leq\|F_\varepsilon-F\|_\infty\|\ell^{-1/2}w\|_{\Omega;0,2}\underset{\varepsilon\to0}{\longrightarrow}0, \tag{A.72}$$

$$\|\ell^{-1/2}(F(u_n)-F(u))w\|_{\Omega;0,2}\underset{n\to+\infty}{\longrightarrow}0. \tag{A.73}$$

Step 3 : The derivative term. The derivative term can be decomposed in two terms as follows

$$\begin{aligned}\|\ell^{1/2}(\nabla(F_\varepsilon(u_n)\nabla w_m)-\nabla(F(u)w))\|_{\Omega;0,2}\leq{}&\|\ell^{1/2}(F_\varepsilon(u_n)\nabla w_m-F(u)\nabla w)\|_{\Omega;0,2}\\&+\|\ell^{1/2}(w_mF'_\varepsilon(u_n)\nabla u_n-wF'(u)\nabla u)\|_{\Omega;0,2}.\end{aligned} \tag{A.74}$$

The same arguments as used in step 2 yield

$$\|\ell^{1/2}(F_\varepsilon(u_n)\nabla w_m-F(u)\nabla w)\|_{\Omega;0,2}\underset{\substack{m,n\to+\infty\\ \varepsilon\to0}}{\longrightarrow}0. \tag{A.75}$$

It only remains to show the following

$$\|\ell^{1/2}(w_mF'_\varepsilon(u_n)\nabla u_n-wF'(u)\nabla u)\|_{\Omega;0,2}\underset{\substack{m,n\to+\infty\\ \varepsilon\to0}}{\longrightarrow}0. \tag{A.76}$$

We write a first decomposition

$$\begin{aligned}\|\ell^{1/2}(w_mF'_\varepsilon(u_n)\nabla u_n-wF'(u)\nabla u)\|_{\Omega;0,2}\leq{}&\|\ell^{1/2}w_m(F'_\varepsilon(u_n)\nabla u_n-F'(u)\nabla u)\|_{\Omega;0,2}\\&+\|\ell^{1/2}F'(u)\nabla u(w_m-w)\|_{\Omega;0,2}.\end{aligned} \tag{A.77}$$

The last term is bounded from above as follows

$$\|\ell^{1/2}F'(u)\nabla u(w_m-w)\|_{\Omega;0,2}\leq\|F'\|_\infty\left(\int_\Omega\ell|\nabla u|^2|w_m-w|^2\right)^{1/2}. \tag{A.78}$$

First, since

$$\ell^{-1/2}w_m\underset{m\to+\infty}{\longrightarrow}\ell^{-1/2}w\quad\text{ in }\ L^2(\Omega),$$

the Lebesgue inverse yields the existence of a subsequence still named $(w_m)_{m\in\mathbb{N}}$ such that $w_m\underset{m\to+\infty}{\longrightarrow}w$ *a.e.* Ω. In addition, Theorem A.4 yields

$$\ell|\nabla u|^2|w_m-w|^2\leq4C^2\ell|\nabla u|^2\in L^1(\Omega). \tag{A.79}$$

We can apply the Lebesgue theorem, which yields

$$\int_\Omega\ell|\nabla F(u)|^2|w_m-w|^2\underset{m\to+\infty}{\longrightarrow}0,\quad\text{hence}\quad\|\ell^{1/2}F'(u)\nabla u(w_m-w)\|_{\Omega;0,2}\underset{m\to+\infty}{\longrightarrow}0. \tag{A.80}$$

Since $(w_m)_{m\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$, (A.76) can be proved only by showing that

$$\|\ell^{1/2}v(F'_\varepsilon(u_n)\nabla u_n-F'(u)\nabla u)\|_{\Omega;0,2}\underset{\substack{n\to+\infty\\ \varepsilon\to0}}{\longrightarrow}0, \tag{A.81}$$

for any fixed function $v\in\mathscr{D}(\Omega)$. This is already done in the proof of Theorem A.3. □

A.3 An interesting result on limits

The following lemma constitues a higher bound when $L^1(\Omega)$ functions are integrated on a "slice", and is used for instance in the proof of Theorem 3.1.

Lemma A.3. *Let $(X, \mathcal{A}, \mu)$ be a measured space, $M > 0$ and $(u_n)_{n\in\mathbb{N}}$ a sequence of measurable functions such that $u_n \underset{n\to+\infty}{\longrightarrow} u$ a.e. X. Then we have*

$$\mathbb{1}_{\limsup_{n\to+\infty}\{|u_n|\geq M\}} \leq \mathbb{1}_{\{|u|\geq M\}} \quad a.e.\ \Omega. \tag{A.82}$$

Remark A.2. *we recall that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{A}$,*

$$\limsup_{n\to+\infty} A_n = \bigcap_{n\in\mathbb{N}} \left(\bigcup_{k>n} A_k \right).$$

Actually, $x \in \limsup_{n\to+\infty} A_n$ means that x is in a infinite number of A_n sets. It is well known that for any sequence $(A_n)_{n\in\mathbb{N}}$ we have

$$\mathbb{1}_{\limsup_{n\to+\infty} A_n} = \limsup_{n\to+\infty} \mathbb{1}_{A_n}.$$

Proof. We consider $E \in \mathcal{A}$ such that for any $x \in E$, $u_n(x) \underset{n\to+\infty}{\longrightarrow} u(x)$ and $X \setminus E$ has null measure. Let $x \in E \cap \limsup_{n\to+\infty}\{|u_n| \geq M\}$. Then

$$\forall\, n \in \mathbb{N}, \exists k > n, \quad |u_k(x)| \geq M. \tag{A.83}$$

This means it exists a subsequence $(u_{\phi(n)})_{n\in\mathbb{N}}$ satisfying $|u_{\phi(n)}(x)| \geq M$ for all $n \in \mathbb{N}$. Passing to the limit directly yields $|u(x)| \geq M$. We deduce that

$$E \cap \limsup_{n\to+\infty}\{|u_n| \geq M\} \subset E \cap \{|u| \geq M\}. \tag{A.84}$$

The fact that $X \setminus E$ has null measure yields the result. □

References

[1] R.A. Adams and J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] C. Amrouche, Leloup G., and R. Lewandowski. Tke model involving the distance to the wall. part 1: the relaxed case. *Preprint on Hal*, 2023.

[3] C. Amrouche and M. Moussaoui. The dirichlet and neumann problems in lipschitz and in $C^{1,1}$ domains. *Preprint on arXiv*, 2022.

[4] Cherif Amrouche, Luigi C. Berselli, Roger Lewandowski, and Dinh Duong Nguyen. Turbulent flows as generalized Kelvin-Voigt materials: modeling and analysis. *Nonlinear Anal.*, 196:111790, 24, 2020.

[5] C. Bernardi, T. Chacón Rebollo, R. Lewandowski, and F. Murat. A model for two coupled turbulent fluids. I. Analysis of the system. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIV (Paris, 1997/1998)*, volume 31 of *Stud. Math. Appl.*, pages 69–102. North-Holland, Amsterdam, 2002.

[6] Luigi C. Berselli, François Legeais, and Roger Lewandowski. Surface boundary layers through a scalar equation with an eddy viscosity vanishing at the ground. *ESAIM Math. Model. Numer. Anal.*, 58(2):489–513, 2024.

[7] T. Chacón-Rebollo and R. Lewandowski. *Mathematical and Numerical Foundations of Turbulence Models and Applications*. Modeling and Simulation in Science, Engineering and Technology. Springer New York, 2014.

[8] T. Gallouët, J. Lederer, R. Lewandowski, F. Murat, and L. Tartar. On a turbulent system with unbounded eddy viscosities. *Nonlinear Anal.*, 52(4):1051–1068, 2003.

[9] A. Kufner. *Weighted Sobolev spaces*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985. Translated from the Czech.

[10] B.E. Launder and D. Brian Spalding. *Lectures in Mathematical Models of Turbulence*. Academic Press, London, 1972.

[11] Guillaume Leloup and Roger Lewandowski. A nonlinear elliptic equation with a degenerate diffusion and a source term in L^1. *Appl. Math. Lett.*, 153:Paper No. 109077, 4, 2024.

[12] R. Lewandowski. Modèles de turbulence et équations paraboliques. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(9):835–840, 1993.

[13] R. Lewandowski. *Analyse Mathématique et océanographie*. Elsevier-Masson, Paris, 1997.

[14] R. Lewandowski. The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy viscosity. *Nonlinear Anal.*, 28(2):393–417, 1997.

[15] Roger Lewandowski and Bijan Mohammadi. Existence and positivity results for the ϕ-θ and a modified k-ϵ two-equation turbulence models. *Math. Models Methods Appl. Sci.*, 3(2), 1993.

[16] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1*, volume No. 17 of *Travaux et Recherches Mathématiques*. Dunod, Paris, 1968.

[17] B. Mohammadi and O. Pironneau. *Analysis of the k-epsilon turbulence model*. RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.

[18] François Murat. Soluciones renormalizadas de edp elípticas no lineales. In *Lectures notes*. University of Sevilla, 1992.

[19] François Murat. Renormalized solutions of nonlinear elliptic equations with measure data. In *Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1998)*, pages Exp. No. IX, 4. Univ. Nantes, Nantes, 1998.

[20] L. Prandtl. Über die ausgebildeten turbulenz. *Zeitschrift für angewandte Mathematik und Mechanik*, 5:136–139, 1925.

[21] G. Stampacchia. *Équations elliptiques du second ordre à coefficients discontinus*. Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965). Les Presses de l'Université de Montréal, Montreal, Que., 1966.

[22] Luc Tartar. *An Introduction to Sobolev Spaces and Interpolation Spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer Berlin, Heidelberg, 2007.