On resolving the tension between visiting mathematical works and questioning the quantum world

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RESEARCH ARTICLE

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ABSTRACT
For the past forty years, researchers in didactics of mathematics have developed and utilised didactic engineering as a methodology for design-based research. In this article we discuss its second step, the so-called *a priori* analysis. In the framework of the Anthropological Theory of the Didactic, engineering is in the form of the implementation of a study and research path (SRP), an inquiry-based course. SRPs may be either finalised, that is aimed at the teaching of a particular piece of knowledge, or rather open, that is solely aimed at answering a given question. This tension between sometimes competing goals especially shows in the *a priori* analysis of SRPs, a phenomenon we address in this paper. Our study is based on pieces of evidence collected by setting up an SRP two years in a row at the interface between mathematics and quantum mechanics, in third year of bachelor's degree. We take a particular care in explicating the rationale behind the *a priori* analysis and its connection with the preliminary analysis. We bring out the significance of spelling out one's epistemological stance, as didactic engineer, emphasising the value of the notion of reference epistemological models to do so.

KEYWORDS
Didactic engineering, Anthropological Theory of the Didactic, paradigm of questioning the world, interface between mathematics and physics, quantum computing

1. Introduction

In the field of design research in mathematics education, didactic (or didactical) engineering plays an important role (Artigue, 2020a). This is especially the case in the French School of didactics of mathematics, where 'as a research methodology, didactical engineering has been strongly influenced by TDS [the Theory of Didactical Situations (Brousseau, 1997)], the dominant theory when it emerged. [...] however, [it] has continuously developed since the early eighties.' (Blum et al., 2019, p. 19).

Didactic Engineering 'mainly denotes today a research methodology based on the controlled design and experimentation of teaching sequences' (Artigue, 2020a, p. 203). As a research methodology, and from its early history, its main raison d'être was that of probing the functioning of didactic systems (Chevallard, 1982). Didactic engineering features four phases: preliminary and *a priori* analyses, a phase of observation, and a *a posteriori* analyses. Among them, the *a priori* analysis is especially crucial, as the validation of didactic engineering is internal and does not put at play any control group: it consists in accounting for the

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discrepancies between the *a posteriori* and the *a priori* analyses (see the presentation of our theoretical framework).

This study is carried out in the framework of the Anthropological Theory of the Didactic (ATD), which 'has developed its own design perspective' (Blum et al., 2019, p. 5). Indeed, the aim of the ATD 'is the elucidation of human societies' relation to "the didactic," that is to say, to all the possible factors of learning. By adopting an anthropological point of view, it purports to embrace the didactic wherever it may show itself around us, paying special attention to the institutional constructions of knowledge and the conditions established to disseminate it.' (Chevallard & Bosch, 2020a, p. 53).

So, for instance, ATD allows for a renewal of didactic engineering by taking a better account of the transpositive phenomena any such endeavour molds. Also, and quite importantly for our research, within the research program of the ATD falls the identification of 'study paradigms', chiefly involving the 'paradigm of visiting works' and the 'paradigm of questioning the world' (Chevallard & Bosch, 2020a, p. 59).

This results in an approach to didactic engineering which is at the same time genuine and novel (Chevallard, 2011). It was conceived by the time of the development of study and research paths (SRPs), which largely reflect didactic engineering as ATD fathoms it. Indeed, the goal of an SRP is to implement a pedagogy more aligned with the paradigm of questioning the world, in order to investigate the conditions and constraints weighing on its establishment (Chevallard, 2015b). As pieces of didactic engineering, SRPs should thus be analysed *a priori*, observed, and then analysed *a posteriori* (Barquero & Bosch, 2015).

The implementation of SRPs is always affected by the tensions between the prevailing paradigm of visiting works and the new paradigm of questioning the world it intends to develop. These tensions already appear in the a priori analysis and, more specifically, in the epistemological models supporting the SRPs. The aim of the present article is to illustrate this phenomenon and show what consequences it has in the didactic engineering process.

In order to perform our study, we resorted to the case study of an SRP set at the interface between mathematics and physics, at the University of Montpellier. It was set in the context of quantum computing, and started with the following generating question Q_0 : *In what respect are quantum computers indeed quantum?* It was designed for students of third year of bachelor's degree (sixth semester) and consisted of nine two- to three-hour sessions, distributed into three lab sessions and six classroom sessions.

Our research questions are the following (they partly draw on the next section):

- How does the underlying tension between paradigms affect the operation of the *a priori* analysis of an SRP?
- What part does the dominant epistemological model of the institution hosting the SRP play in the conception of the reference epistemological model and, on a broader level, in the design of the SRP?

In the following, we will start by elaborating theoretically on didactic engineering and its relation to study and research paths through the paradigm of questioning the world. Then, as our SRP about quantum computers plays the role of a case study to adress our research questions, we will explain in details how its preliminary and *a priori* analyses were conceived. Lastly, we will discuss our data in order to provided responses to our research questions.

2. Theoretical framework and design methodology

2.1. Didactic engineering as a traditional design-based research methodology

As Blum et al. (2019, p. 5) put it: 'Without design, no education is possible. It is through designed instructional material and processes [...] that learning environments for students can be created.' Among numerous kinds of design-based research, we focus in this paper on didactic engineering. Describing the French setting in didactics of mathematics, the previous authors explain:

> the design of mathematical tasks, situations and sequences of situations is essential to didactic research and is controlled by the theoretical frameworks underlying this research. This is clearly reflected in the methodology of didactical engineering within the theory of didactical situations that emerged in the early 1980s. Designs are grounded in epistemological analyses, and situations are sought that capture the epistemological essence of the mathematics to be learned. (Blum et al., 2019, p. 5)

As Artigue (2020b, p. 31) recalls, the main ingredients of didactic engineering did not fundamentally evolve throughout its forty-year long history, 'even if the formulations have evolved a little since 1990'. She elaborates on those fundamental characteristics in her entry to the *Encyclopædia of Mathematics Education* (Artigue, 2020a). In particular, didactic engineering 'is classically structured into four different phases: preliminary analyses; design and *a priori* analysis; realization, observation, and data collection; *a posteriori* analysis and validation'. Regarding the existence of *a priori* and *a posteriori* analyses, Artigue (2020b, p. 34) insists:

> What is crucial is that the validation is internal. It is not thought in terms of an external comparison between an experimental and a control group.

Thanks to this crucial aspect of didactic engineering, 'didactics has freed itself [...] from a conception of scientificity grounded on the use of experimental methods based on statistical comparisons between pre-tests and post-tests for control and experimental groups.' (Artigue, 2020b, p. 29). In this context, the *a priori* analysis is of the utmost importance, as recall for instance Bessot (2011, p. 39):

> This mathematical work on the sought knowledge in a didactic engineering is made necessary by the didactic finality of any teaching situation, which is to transform the causes of knowledge into reasons for knowledge.

That is, for didactic engineering to actually play its role of analyser of functioning didactic systems, its *a priori* analysis should be carefully executed. Indeed, the many aspects of the pieces of knowledge at play should be investigated, in order to restore their raison d'être in the classroom. Yet, in the wake of the search for suitable situations that characterize didactic engineering according to the TDS, the ATD recently put forth another approach to it:

> In the last decade, the anthropological theory of the didactic has developed its own design perspective that gives particular importance to identifying issues that question the world and have strong mathematical potential. (Blum et al., 2019, p. 5)

In the framework of ATD, not only is didactic engineering a way to probe into the functioning of a didactic system in normal pedagogical conditions. It has also become a way to investigate didactic regimes which may not be commonly found in a school setting. In particular, by this way, ATD examines the didactic regime of inquiry, which leads to the development of a new pedagogy, that of study and research paths.

2.2. From Questioning the World to study and research paths

In the framework of the ATD, a work denotes 'any human production enabling to provide answers to one or several kinds of questions, either "practical" or "theoretical" – the latter questions being the raisons d'être of the work' (Chevallard, 1996, p. 14). For instance, mathematical theorems are works. In this context, knowing means having developed a relation to an object (working with it every day, simply having heard of it, etc.). Then, there is a subtle dialectics between works and pieces of knowledge. A work can be studied to produce knowledge about it, but it can also be used to produce knowledge about other works.

To put this stance in perspective, the notion of didactic paradigm turned out to be a powerful analytical tool.

> To cut a longer story short, I define a didactic paradigm as a set of rules prescribing, however implicitly, what is to be studied — what the didactic stakes *w* [which are works] can be — and what the forms of studying them are. (Chevallard, 2015a, p. 174)

For instance, the relation to the works which is maintained at school was formalized as the 'Paradigm of Visiting Works':

> A (mathematical) work is 'visited' by a class under the supervision of the teacher as if it were a monument, even a masterpiece, that, however impudently, we are expected to revere and bow to. This leads to what I have called the 'monumentalization' of the curriculum. (Chevallard, 2019, p. 99)

As a matter of fact, this paradigm (P_1) results from an evolution (Chevallard, 2015a) and has a plurality of manifestations, which Chevallard and Strømskag (2022) describe more precisely. In this article, we are interested in its variant P_1^{+++}, which 'has been particularly studied by Guy Brousseau in his *theory of didactic situations* in mathematics'(Chevallard & Strømskag, 2022, p. 32). It consists of

> put[ting] forward the study of a question q, of a certain type Q, to which the answer A elaborated by the class under determined conditions and constraints relies essentially on the use of the work w – which will show that w "serves" to answer questions of type Q. (Chevallard & Strømskag, 2022, p. 31)

This leads to important modifications in the role of the teacher, which, 'during an "activity", consists in managing the activity, not its content, on which [she] does not intervene, but in its progress' (Chevallard & Strømskag, 2022, p. 32). Moreover, a new relationship between works and questions is thus established, through the situation. The latter, together with its generating question q, now plays the role of an epistemological model of the piece of knowledge w.

In the 21st century, the ATD developed the notion and inquired about the so-called 'Paradigm of Questioning the World' (P_2 and its variants, see below). On the one hand, and in line with Brousseau's breakthrough, it consists in restoring the dynamics between works and questions in the classroom. By putting questioning at the heart of learning, the raison d'être of the works come alive again. On the other hand, the previous focus on works to be studied is more fully amended. Instead of considering them in the first place, the paradigm of questioning the world proposes to genuinely focalise on vivid questions of our time and our world (Chevallard, 2006). They, and only they, should influence the design of the engineered sequence. As a matter of fact, the world is already questioned in many institutions, notably in research institutions. In this regard, the questioning of the world may be simply seen as a didactic characterization of study processes as they unfold outside school (Chevallard, 2007).

In order to examine this didactic regime, still quite rare in the school context, researchers in ATD have developed a pedagogical apparatus, the so-called study and research paths (SRPs).

Namely, SRPs are devices designed to install in the classroom a dissemination of praxeologies that is more in line with the one that can be observed 'in nature': 'In a certain way, SRPs represent the materialization of what the ATD considers as teaching processes based on a "functional" teaching of mathematics.' (Barquero, 2009, p. IX). More precisely, an SRP starts with the study of a question Q_0, generating of a whole inquiry:

> The study of [the generating question] Q_0 evolves and opens many other *derived questions* Q_k that appear as the starting point of new SRPs or new branches of the initial one. One needs to constantly ask whether these derived questions are relevant in the sense of being capable of leading *temporary answers* A_k that can be helpful in elaborating a *final answer* $A^\heartsuit$ for Q_0. As a result, the study of Q_0 and its derived questions Q_k leads to successive temporary answers A_k tracing out the *possible routes* to be followed in the effective experimentation of the SRP. The work of producing $A^\heartsuit$ can thus be described as a tree of questions Q_k and temporary answers A_k. (Barquero & Bosch, 2015, p. 261)

Rather unsurprisingly, this far-reaching departure from classical pedagogy also had to cause a revision of didactic engineering, as the latter was invented in the first place to probe the former. More precisely, didactic engineering in the spirit of the TDS defined pieces of knowledge by various situations and related questions, however it did not focus on the questions per se. The main objective was to remake works in the classroom and to study didactic phenomena related to these works. Questions were then a means, not an end in itself.

This especially concerns the *a priori* analysis. Indeed, it now implies envisioning those "possible routes", both to assess the viability of the SRP and the "studiableness" of its generating question. Concretely:

> A mathematical and a didactic level may be distinguished here, to first "define" or "characterize" the content (mathematical analysis), and then to propose how to make it emerge from problematic questions within a sequence of concrete situations (didactic analysis). (Barquero & Bosch, 2015, p. 251)

In the following, we will elaborate on these two levels of *a priori* analysis. First, by explaining in what respect quantum computing is a good "terrain" for students' paths to expand. Secondly, by showing how the chosen generating questions can indeed be studied given the conditions and constraints of our context, with well chosen pieces of content. However, to better understand what is at stake when doing an *a priori* analysis, we will now develop further the relationship between SRPs and paradigms of study.

2.3. Didactic engineering of study and research paths

Didactic engineering is a 'research methodology based on the controlled design and experimentation of teaching sequences' (Artigue, 2020a, p. 203). As such, it is strongly influenced by the paradigm of study prevailing in the teaching institution which is probed. This is why, as Artigue (2011, p. 21) explains, the principles of the ATD change the way didactic engineering is carried out in practice. However, throughout their twenty year-long history, SRPs did not immediately lay on the most radical versions of $P2$. Indeed:

> The emergence of Paradigm 2 first comes at a very high price. The formal requirement to investigate a question q leads to propose questions that may have only an improbable relationship with the work w, whose continued presence attests to the more or less surreptitious survival of Paradigm 1. The question q then tends to be the foil of w, which is still the main character. (Chevallard & Strømskag, 2022, p. 33)

For instance, Bosch et al. (2004) or García and Ruiz Higueras (2006) present inquiry processes which are still strongly influenced by the paradigm P_1^{+++} from the theory of didac-

tic situations. It may even happen that, at first, the dialetics between questions and works through their raison d'être is actually weakened in the paradigm P_2. This is what Chevallard and Strømskag (2022, p. 32) call the paradigm P_2^0: 'behind the question q, there is a work w that is the real learning stake and, when the didactic "set-up" is correctly designed, is, for the class, the missing key element to answer question q'. A subtle play of light and shade, between questions and works, marks the difference between P_1^{+++} and P_2^0. At any rate, 'from an epistemological point of view, P_2^0 is a regression with respect to [...] P_1^{+++}' (Chevallard & Strømskag, 2022, p. 32).

On the other side of the spectrum, the focus on the generating question of the inquiry process may be more and more literal. More instance, in the paradigm P_2^{++}, the study of the generating question leads to a set Q of intermediary questions q, overall matching a corresponding set of works. 'There then tends to be, for the questions $q \in Q$ involved, a certain definalization of their study.' (Chevallard & Strømskag, 2022, p. 36). Eventually, when the process is solely developed for the sake of answering its generating question, whatever the works encountered on the way, we reach the more radical paradigm P_2^{+++}.

This genetic tension which lasts through the history of the development of SRPs is still apparent nowadays in the distinction which is made between finalised and open SRPs. Finalised SRPs 'must lead students to encounter a complex of praxeological entities [that is, works w] fixed in advance. [...] The reason for this constraint [...] may seem clear: the teacher's contract with the school, and beyond that with society, is to have the students meet a whole set of declared praxeological entities' (Chevallard, 2011, p. 97). On the other hand. in the case of open SRPs, 'the tools used will not have been fixed in advance' (Chevallard, 2011, p. 99). So, finalised and open SRPs actually demand quite different approaches, especially at both levels of the *a priori* analysis. At the physical-mathematical level, a more open SRP compels to quite enlarge its epistemological unit of analysis. At the didactical level, one has to devise a sequence of concrete situations, even though the open character of the inquiry may slightly randomize it. Last, but not least, recalling that finalised vs open SRPs derive themselves from different paradigms: considering both at the same time can give us a quite direct view on the tensions between them.

2.4. Methodology of this study

In order to provide answers to our research questions, we will give a detailed analysis of the way we carried out the *a priori* analysis of a study and research path set up at the interface between mathematics and quantum mechanics. This SRP was generated by the question Q_0: *In what respect are quantum computers indeed quantum?* It was designed for twelve students from third year of bachelor's degree (sixth semester), six of them coming from a mathematics curriculum, whereas the six others came from a physics curriculum. It was implemented two years in a row. It consisted of nine two- to three-hour sessions, distributed into three lab sessions and six classroom sessions. Further details will be given later, as elements of our *a priori* analysis.

Before its first implementation, several analyses were conducted: in order to study the epistemological aspects put at play by this SRP, in order to study the institutional setting in which it would be implemented, as well as its ecological context. The data collected while doing these analyses was carefully logged. They make the major part of the facts we rely on in the present paper. This material is indeed the empirical basis on which we base the preliminary and *a priori* analyses we will develop shortly. Then, performing this SRP twice, two years in a row, allowed us to use components of the *a posteriori* analysis of the first implementation to enrich the *a priori* analysis of the second, as is customary in the methodology of didactic

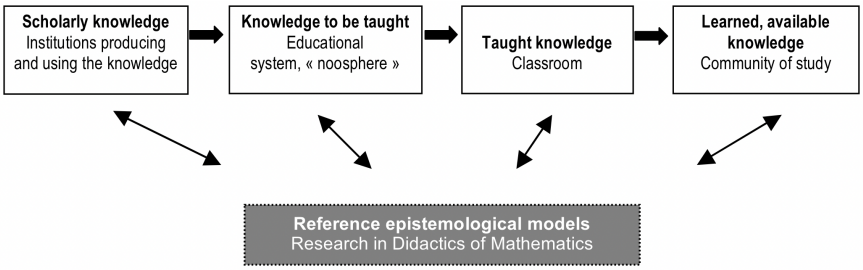


Figure 1. An REM as an outlook on the transposition phenomena effectuated by a piece of didactic engineering (Bosch & Gascón, 2006, p. 53). In the present case we deal with a 'double' didactic transposition, within mathematics and physics and between them, at both scholarly and teaching levels.

engineering. We will not give the details of our *a posteriori* analyses here, but the interested reader may consult Lombard (2023).

So, our study relies on extensive epistemological and praxeological analyses. However, in the following we simply give an operational synthesis adapted to the very context of our SRP. Then, the *a priori* analysis consists of a dialectical study of the range of possible students' paths given the generating questions considered for the SRP. The aim is to anticipate possible obstacles for students and to make sure they could be provided tools to overcome them, in the context considered where the inquiry unfolds. This is why chronogenesis, topogenesis and mesogenesis (Barquero & Bosch, 2015) should be carefully considered *a priori*, together with the writing of the *a priori* question-answer maps, the organization of study or research activities, and the choice of pieces of media that could be used. Finally, the role of some of the dialectics of learning (Chevallard, 2001, p. 7) can be conjectured (for the later-introduced but crucial media-milieu dialectics, see for instance Chevallard (2006, p. 9)).

For this stage, which includes analyses of transposition phenomena, it is required to spell out one's reference epistemological model (REM, fig. 1):

> The reference epistemological model of a body of knowledge is an alternative description of that body of knowledge elaborated by researchers [...]. [It] prevents [them] to take for granted how this body of knowledge is conceived in the institution considered. (Florensa et al., 2015, p. 2637)

The RME allows one to assess the various strategies students may adopt to answer the generating question. RMEs should not be confused with dominant epistemological models (DMEs), which are customised epistemological models used within a given institution, but which remain largely implicit. For instance, the dominant epistemological model prevailing within the schooling institution hosting the SRP entails a particular vision of the pieces of knowledge the latter will put at play. The goal of an RME is precisely to overcome this vision, both by expliciting it and then by contrasting it thanks to the study of the DEMs of other institutions.

Our RME was built upon studies in historical epistemology (see the next section), as well as praxeological analyses of the dominant epistemological model prevailing at the University of Montpellier (Lombard & Hausberger, in press). Then, we could extend our model of reference based on the literature at the interface between mathematics and quantum mechanics (see for instance Hall (2013)). Overall, we thus investigated a rather intricate transposition scheme between scholarly and schooling institutions in both mathematics and physics (see fig. 1). This combined study allowed us to gain perspective on the epistemological issues at play when mathematical aspects of quantum mechanics are taught as well as on their didactic consequence in the classroom. This would feed our preliminary and *a priori* analyses.

3. Findings of our analyses

3.1. Preliminary analysis

3.1.1. Epistemological aspects

From a study of historical epistemology (in the sense of 'Histories of epistemic things' (Feest & Sturm, 2011, p. 288)) dedicated to the interplay between quantum theory and functional analysis during their respective early developments (1900-1930), we could extract several aspects that seem particularly relevant for the design of this SRP.

First, mathematics and physics exerted a mutual influence throughout the development of quantum mechanics. In particular, from 1925-1927, mathematicians could draw results from questionings arising from physics (Lacki, 2011). This culminated in von Neumann's introduction of Hilbert spaces as an abstract structure encompassing the variety of theories of quantum mechanics known at that time. So, a physical context may be fruitful to introduce higher level mathematics via the dialectics of questions and answers.

Then, models and formulations were abundant at the interface between quantum mechanics and mathematics, with successive attempts at unifications and simplifications, which the structuralist stance in mathematics finally helps to achieve (Von Neumann, 1955, p. 28). We should thus expect the dialectics of objects and structures (Hausberger, 2017) will play a role.

Finally, interviews we performed with professors of physics and mathematical physics at the University of Montpellier lead us to consider that such phenomena still occur in their day-to-day activity. Thanks to the use of techniques not validated by the mathematical institution, and which give rise to developments at the level of the 'scholarly knowledge' (see fig. 1), physical practice can open fertile perspectives for mathematical practice. More precisely, though there seems to exist a vivid practice at the interface between mathematics and physics when considering 'scholarly knowledge', this does not seem to be the case anymore regarding 'taught knowledge', showing a lack in the 'didactic transposition' at the interface (Chevallard & Bosch, 2020b). As this is one of the issues this SRP attempts to address, we focused the study of the epistemological aspects of our preliminary analyses on this point. We now present some further results of this study.

The notion of spectrum appeared to be crucial. For instance, the word 'spectrum' is meaningful in both physics and mathematics. The Time-independant Schrödinger Equation, $\hat{H}\psi = E\psi$, establishes a link between these two objects: on the one hand, it is an eigenvalue problem, on the other hand, its resolution gives the energy levels of the physical system considered. In the case of atoms, the latter means accounting for the structure of their light spectra. However, in quantum mechanics as it is taught at the University of Montpellier, this fundamental polysemy only lives in the finite-dimensional case (two-level systems). As a consequence, the infinite well[1] is not even treated as a fully quantum system. For instance, the 'logos' of the corresponding 'praxeologies' is not based upon the postulates of the theory (Lombard & Hausberger, in press). In section 3.1.3, we explain how we may tackle this issue while respecting the ecology[2] of the institution hosting our SRP.

Quantum computing puts at play all these features of the interface between mathematics and physics. Indeed, their functioning is based on qubits, which are sets of possible states of a physical system, such as a superconductor or trapped ions. Mathematically, a qubit may me modelled as a Hilbert space. More precisely, various kinds of quantum computers, as well as various models of a given such computer (as a two-level system, as a well (see below for

[1] In physics, a well is a region of space where the graph of the potential energy has a convex shape. In quantum mechanics such a well may be idealized as a rectangular hole, sometimes having 'infinite' walls (see fig. 6).

[2] In the didactical meaning of the word. See for instance Chevallard (1989, p. 8).

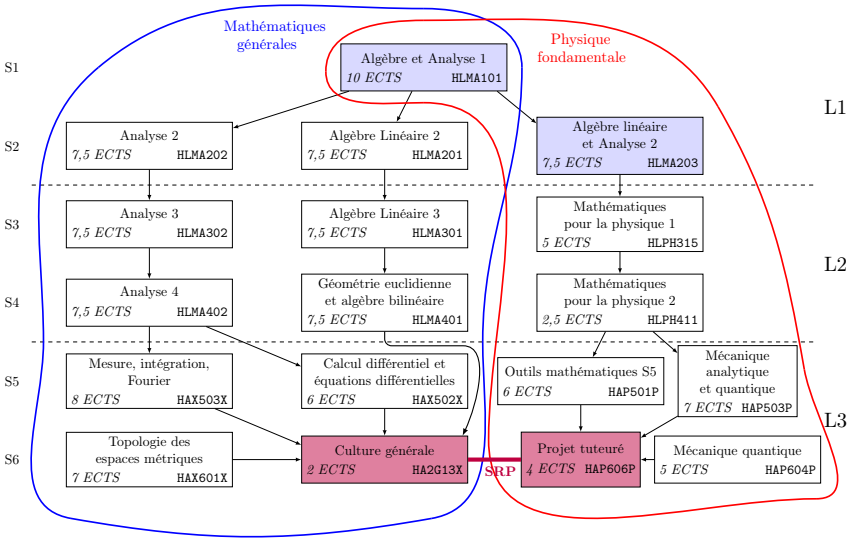


Figure 2. Overview of some TU from the 'General Mathematics' and 'Fundamental Physics' programs that are taken by students participating in the experimentation.

further explanations)), may be modelled as equally many instances of the Hilbert space structure, such as $\mathbb{C}^2$ or $L^2(\mathbb{R}^3)$. Quantum computers crucially rely on the fact that their physical states is a linear combination of pure states, which are spectral elements (eigenstates) of the Time-Independant Schrödinger Equation. So: the mixture between mathematical and physical models is constant and reversible, Hilbert spaces play a key role in understanding the functioning as well as the descriptions of quantum computers, the notion of spectrum is central in this mixed practice at the interface between mathematics and physics.

3.1.2. Institutional conditions

In Montpellier, physics and mathematics curricula are quite detached (see fig. 2). The bachelor (Licence) lasts three years (from the L1 to the L3), and it is divided into six semesters (from the S1 to the S6). Some Teaching Units (TU) taken by physics students are nevertheless taught by mathematicians: we filled them in blue. Lastly, our experimentation takes place during the sixth semester (S6), as a mixed TU (see below). It is shown in purple in the figure.

Several constraints weighted on the institutional implementation of the SRP, as it was to develop at the interface between mathematics and quantum mechanics. Firstly, physics students take their first quantum mechanics class at the fifth semester (S5, Mécanique analytique et quantique). Then, during the sixth semester, on the one hand, mathematics students have to take a 'common knowledge' class (Culture générale), whereas, on the other hand, physicists take a TU devoted to doing an experimental research project supervised by a professor (*Projet tuteuré*, French for tutored project). So, we could set up the SRP as a mixed teaching unit registered with both math and physics departments. It is the only such TU at this level of studies at the University of Montpellier. This way we could project nine two- to three-hour time-slots.

This being said, quantum computing belatedly came forth as an adaptation to an additional institutional constraint coming from physics: an experimental dimension was needed so that we could actually implement our SRP as a tutored project. In 2019, the technology company IBM initiated a partnership with the University of Montpellier, in which several members of

both the mathematics and physics departments are still involved. This is why, at that moment, quantum computing emerged as a potentially workable setting for the upcoming SRP, especially given it thoroughly puts at play the aforementioned epistemological aspects. Lastly, as the setting of an SRP should 'be regarded – by the students, by their teachers (...) – as crucial to a better understanding and mastery of their lived world' (Chevallard, 2006, p. 7-8), quantum computing seemed to fit all the more.

With this organisation came further conditions and constraints, mainly from the physics department. For instance, evaluation should include a peer-reviewing process among students taking this TU. This was actually a favorable condition. Indeed, defining the recipients of the answer to be given to the generating question is a crucial step when implementing SRPs. So, we extended the physics instructions to all students: to write a report and make a presentation that their third-year colleagues could read and understand.

3.1.3. Ecological context

Could a practice having the espistemological characteristics of the interface between mathematics and physics actually live at the third level of bachelor studies at the University of Montpellier?

Physics students had already had a quantum mechanics course during the first semester, whereas students from mathematics never did (at least at the University of Montpellier). The content of the latter quantum mechanics course corresponded to parts of the first two chapters of *Quantum Mechanics*, vol. 1 by Cohen-Tannoudji et al. (1991) which is often used in introductory quantum mechanics courses in France. We could caracterize its content in a study which provides additional inputs into the preliminary analysis (Lombard & Hausberger, in press). For instance, a tension exists between the abstractness of the Hilbert space formalism and the necessity for students to develop operative skills in order to study actual physical systems or models. In particular, the raison d'être of some elements of the formalism is never specified: why an infinite number of dimensions? Why Hilbert spaces and not Banach spaces or even pre-Hilbertian spaces, which are studied in second year by math students? This fact is reminiscent of the paradigm of study adopted in this course, akin to visiting works.

This tension particularly transpires when it comes to the passage from finite- to infinite-dimensional Hilbert spaces, as was actually acknowledged by the professor teaching the course:

> overall, there is the passage from the discrete to the continuous, from finite dimension to infinite dimension, where I am quite discreet, because I try to justify things but without really explaining. [...] Because I think it's already hard at the abstract level and that it would perhaps lose them, to try too hard to put details

As a matter of fact, the course we studied began with the infinite-dimensional case, even though the students never encountered vector spaces of this kind (all the more to solve eigenvalue problems, as is customary in quantum mechanics). In this SRP we chose to go the other way, starting with what students may already be familiar with and going from finite- to infinite-dimensional models (for such a treatment in the traditional paradigm of study, see for instance Le Bellac (2013)).

Coming back to the ecological context, we can caracterise it more precisely by describing the Dominant Praxeological Model of the institution that would host our experiment. As is customary in French universities, the Time-independant Schrödinger Equation is introduced as an eigenvalue problem, but most of the times it is solved as an ordinary differential equation

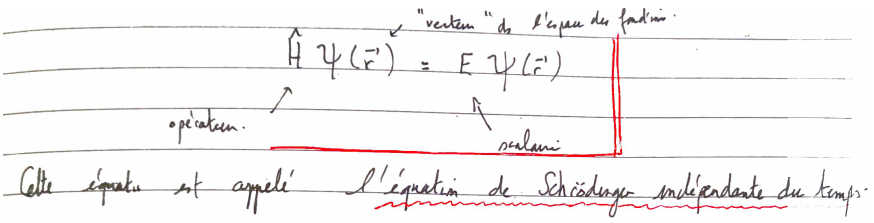


Figure 3. The Time-independant Schrödinger Equations as it appears in the notebook of the professor teaching the third-year level course *Quantum Mechanics I* at the University of Montpellier.

(the equation is taken from Basdevant (2016, p. 85)):

$$-\frac{\hbar^2}{2m}\psi_n''(x) + V(x)\psi_n(x) = E_n\psi_n(x)$$

where, as a matter of fact, $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$, and so depends on the physical setting through the function V (called the potential). The goal is then to solve this eigenvalue problem for several potential functions: a step, a barrier, an infinite well... Taking a step back thanks to our model of reference, we notice that this indeed amounts mathematically to solving an eigenvalue problem, either in finite dimension (case of the double-well) or in infinite dimension (case of the step or the infinite well). What is at stake, however, is the realization that the process is always the same, as only the underlying Hilbert space changes. So, it connects at least partially to the mathematical practice taking place at the university of Montpellier, especially considering the bachelor curriculum of Linear Algebra.

Hence the great significance that quantum computers may be modelled by potential wells, as the scholarly litterature shows. As two-levels systems, they may be considered as double-wells; as basic cavities, the infinite well fits as a first approximation (Nielsen & Chuang, 2000, p. 280). So, the dominant epistemological model prevailing within both physics and mathematics at the bachelor level at the University of Montpellier was already quite favorable to conduct a study of eigenvalue problems there, in both finite and infinite dimensions, in the context of quantum computing.

3.2. A priori analysis

3.2.1. Resources and assessment

In light of the previous considerations we undertook the a priori analysis of an SRP at the interface between mathematics and quantum mechanics, with quantum computing as its background. It would include both classroom sessions (six) and computing sessions (three), the latter being taught by an IBM representative already working with the University of Montpellier (denoted by P_{qc}). P_{qc} was used to animating workshops in the Montpellier area in order to introduce clients to quantum computing. He had already animated such workshops at the University of Montpellier. They consisted in learning how to manipulate a remote IBM quantum computer (the machine is in Zurich) by using the Qskit platform, which uses the Python programming language.

Besides P_{qc} and in order to enforce the conveyance of mathematics and physics content, we chose to let one professor from each field (P_φ and P_μ for physics and mathematics, respectively) teach once during the SRP. Finally, in order to comply with the constraints imposed on this TU by each department, we decided students would work in groups of three, accordind

to their specialy. So, as a future SRP, we considered the didactic system $S(X, Y, Q_0)$, where Q_0 is our generating question (see next section). The set of students may then be specified as $X = \{g^3_{\mu 1}\ ;\ g^3_{\mu 2}\ ;\ g^3_{\varphi 1}\ ;\ g^3_{\varphi 2}\}$ (where g^3_μ and g^3_φ are math and physics trios, respectively). Finally, the team of teachers, Y, was composed of P_{qc} and I, denoted N in the following, as a study aid: $Y = \{P_{qc}, N\}$.

Then, we planned on collecting question-answer maps per group in order to monitor the evolution of students' questioning, actually letting students draft them themselves. Complying with the institutional constraints coming from physics, we opted for a final answer in the form of a written report and an oral presentation. In particular, this implied to let a significant part of the investigation on the 'other stage' (Chevallard, 1998, p. 17). This is a questionable choice, as our research would gain in monitoring as closely as possible students' questioning. On the other hand, it enforces the customisation of the final answer $A^\heartsuit$ produced by students, making it indeed close to their hearts (Chevallard, 2019, p. 100).

3.2.2. Generating question

From this point onwards, discrepancies between paradigms begin to emerge, and so we point them out already. They will be thoroughly discussed in the last section of this paper.

For instance, our objectives were the following regarding the generating question. First and foremost, we wished to reinstall the raison d'être of the use of Hilbert spaces in quantum mechanics. In the more particular context of a PhD devoted to the didactics of algebraic structures, we wished to see them play the unifying and simplifying role for which abstract structures were conceptualized. Thus investigating the raison d'être of a mathematical notion, we were in the framework of the paradigm of didactic situations, P_1^{+++}.

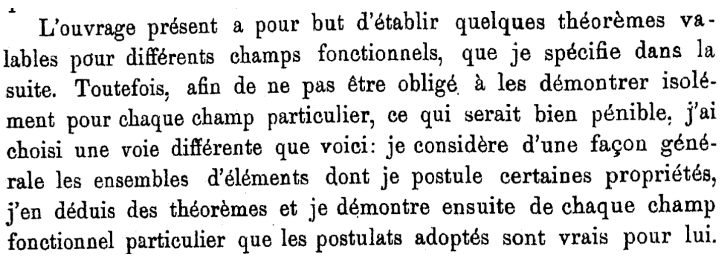
However, this endeavour was quite polarized by our preliminary analyses, which strongly suggested both a particular raison d'être for Hilbert spaces in quantum mechanics (as a unifying model for various quantum mechanical contexts) and a way to have it replayed (by questioning the quantum aspects of the functioning of quantum computers). So, by approaching Hilbert spaces (the work w) through a question while having a precise pre-determined idea of the role this notion should play in the investigation, we were rather in the framework of paradigm P_2^0.

This being said, the generating question was supposed to fuel a nine-session-long inquiry in an institutional context favorable to studying computer computers, a vivid topic. Consequently, the framework was in this respect rather that of the paradigm P_2^{++}, which is the result of a 'definalisation process' (Chevallard & Strømskag, 2022, p. 35) by which the generating question takes precedence over the works to be studied, and the SRP gets more open.

So, the goal of our *a priori* analysis of the generating question was to design one which could match this many-sided role. Actually, this clearly highlighted underlying tensions between finalised and open SRPs.

We worked primarily on the basis of our epistemological analyses to comply to this objective. In fact, the history of quantum mechanics shows Hilbert spaces were introduced precisely in order to unify physically equivalent approaches, and so avoid arduous adaptations from one formalism to another (fig. 4). For instance, Max Born, although he was among the founders of Matrix Mechanics, worked in practice with the rival frameworks Wave Mechanics because he considered it to be more operative. The generative question must thus lead to the intervention of a plurality of different models of quantum systems in the course of the investigation, so that one should find advantageous, at some point, to be able to treat them with a unified formalism, beyond the physical and (apparently diverse) mathematical contexts.

Now, as we have already mentioned, the passage from finite to infinite dimension in quantum mechanics is a crucial step in which the structure of Hilbert space comes into play. Indeed,



L'ouvrage présent a pour but d'établir quelques théorèmes valables pour différents champs fonctionnels, que je spécifie dans la suite. Toutefois, afin de ne pas être obligé à les démontrer isolément pour chaque champ particulier, ce qui serait bien pénible, j'ai choisi une voie différente que voici: je considère d'une façon générale les ensembles d'éléments dont je postule certaines propriétés, j'en déduis des théorèmes et je démontre ensuite de chaque champ fonctionnel particulier que les postulats adoptés sont vrais pour lui.

Figure 4. An excerpt from the introduction of Stefan Banach's PhD thesis, which proposes a rationale for his approach via abstract structure: 'so as not to be obliged to demonstrate them in isolation, which would be rather tedious [...], I consider in a general way the sets of elements [...], I deduce theorems from them and I then demonstrate for each particular functional field that the postulates adopted are true for it' (Banach, 1920, p. 134).

it is the very framework where the practice acquired in low dimension can relatively easily be transposed to infinite dimension (which is well shown by the conceptual development of this notion, of which ℓ^2 is the archetype). It is common in introductions to quantum mechanics to deal with several situations where finite and infinite dimensional frameworks come into play. However, in spite of this, it appears from our reference model that the particularities of the infinite dimensional case are often overlooked. Yet it is with them that the practical necessity of resorting to a more general framework emerges, necessity which is thus obscured.

As our reference model also shows, this phenomenon is particularly evident when it comes to quantization (i.e. solving eigenvalue problems), as most treatments of the infinite well illustrate: the argument for the emergence of eigenstates is often exactly the same as the one used to justify the eigenmodes of a vibrating string attached to its ends (Melde's experiment), which is a *classical* phenomenon. However, in the case of the infinite well, quantization can only be explained, in compliance with the practice at the interface, on the basis of truly quantum mathematics, or at least a mixed discourse of physics and mathematics resembling it (see Hall (2013, chap. 5)).

All things considered, we opted for the following generating question:

Q_0 – In what respect are quantum computers indeed quantum?

In order to account for the phenomena that explain the particular properties of quantum computation (which in their turn explain the interest those machines arouse), it is indeed necessary to call upon many models, in particular in order to describe the quantum computer itself: its internal functioning on the one side, the formulas used to predict the results of the algorithms on the other side. These models include for example two-level systems or anharmonic oscillators (see the next section). In each of these contexts, the question of quantization is crucial, as qubits are quantum states superposed between two spectral levels (usually marked $|0\rangle$ and $|1\rangle$). Working on models of quantum computer should thus bring forth many aspects the previous analyses have highlighted. Lastly, as the proposed questioning is rather open, we have chosen to add three questions to frame the investigation, as was already done for instance by Barquero (2009, p. 198):

Q_{0a} – What are qubits and can we calculate with them?

Q_{0b} – What phenomena allow a quantum computer to operate?

Q_{0c} – What caracteristics of quantum systems are shared by quantum computers?

This last move illustrates, again, the lasting tension between a rather finalised or a more open version of our SRP. It also appears clearly in our *a priori* analyses of both the dialectics

Date	Topic/model	Pedagogical and epistemic context
January 20th, 2022	First encounter and opening of the questioning	Video presentation of quantum mechanics and quantum computers
January 27th	First lab session	Taught by P_{qc}: first manipulations with Qiskit
February 3rd	Questions following the first lab session	
February 10th	Double quantum well	Intervention of P_{φ}: two-level systems in two-dimensional complex vector space
February 17th	Second lab session	Taught by P_{qc}
February 24th	Infinite well	Discrete infinite spectrum of eigenfunctions
March 3rd	Winter break	
March 10th	Third lab session	Taught by P_{qc}
March 17th	Finite Well	Intervention of P_{μ}; spectrum of eigenfunctions, both discrete and continuous
March 24th	Conclusion	Hilbert space as a common framework
April 15th	Peer-reviewing report	
April 18th	Written report	
April 21st	Oral exam	

Figure 5. Unfolding of the sessions of our SRP, as it was envisioned *a priori*. Notice the succession of quantum computer models (mainly quantum wells).

of media and milieus, and systems and models.

3.2.3. Media and models

The choice of media for our SRP reveals a strategy to monitor the level of openness of the inquiry, showing the enduring influence of the paradigm P_1. Indeed, we envisioned that, in the case students actually enrich their milieu with them, media could channel the questioning towards given works, be they visited or not. So, media were both supposed to increase the numbers of models of quantum computers under study (to make the use of the Hilbert space structure increasingly profitable) and to decrease the openness of the generating question (in order to balance its scope). For this reason, we have considered *a priori* a succession of models of quantum computers to be addressed in the classroom (see the table in fig. 5). Such models are the double, the infinite or the finite wells (see fig. 6 and Le Bellac (2013) for some details).

During lab sessions, students would manipulate the software Qiskit. During class sessions, book excerpts and videos would enunciate facts about physical or mathematical models of quantum computers. The goal was to provide students with answers already available in the culture, and, specifically, to put mixed objects at play in the classroom, that is notions having meanings in both mathematics and physics (though sometimes different). Such mixed objects were a spectrum, a space or an eigenvalue problem. We wished to see how students from the two disciplines would approach them, maybe enriching each other's questioning. In addition, the professors taking part in the experiment should play an important role in the media environment (see fig. 5). Lastly, we planned on producing tailor-made pieces of media, for instance to encourage the process of questioning about the links between the various models so introduced (again with the hope of favouring collective, that is interdisciplinary, inquiry).

From the generating question and the foreseen media supply, we were able to establish the *a priori* question-answer map of our SRP (figure 6). Each group of questions corresponds to

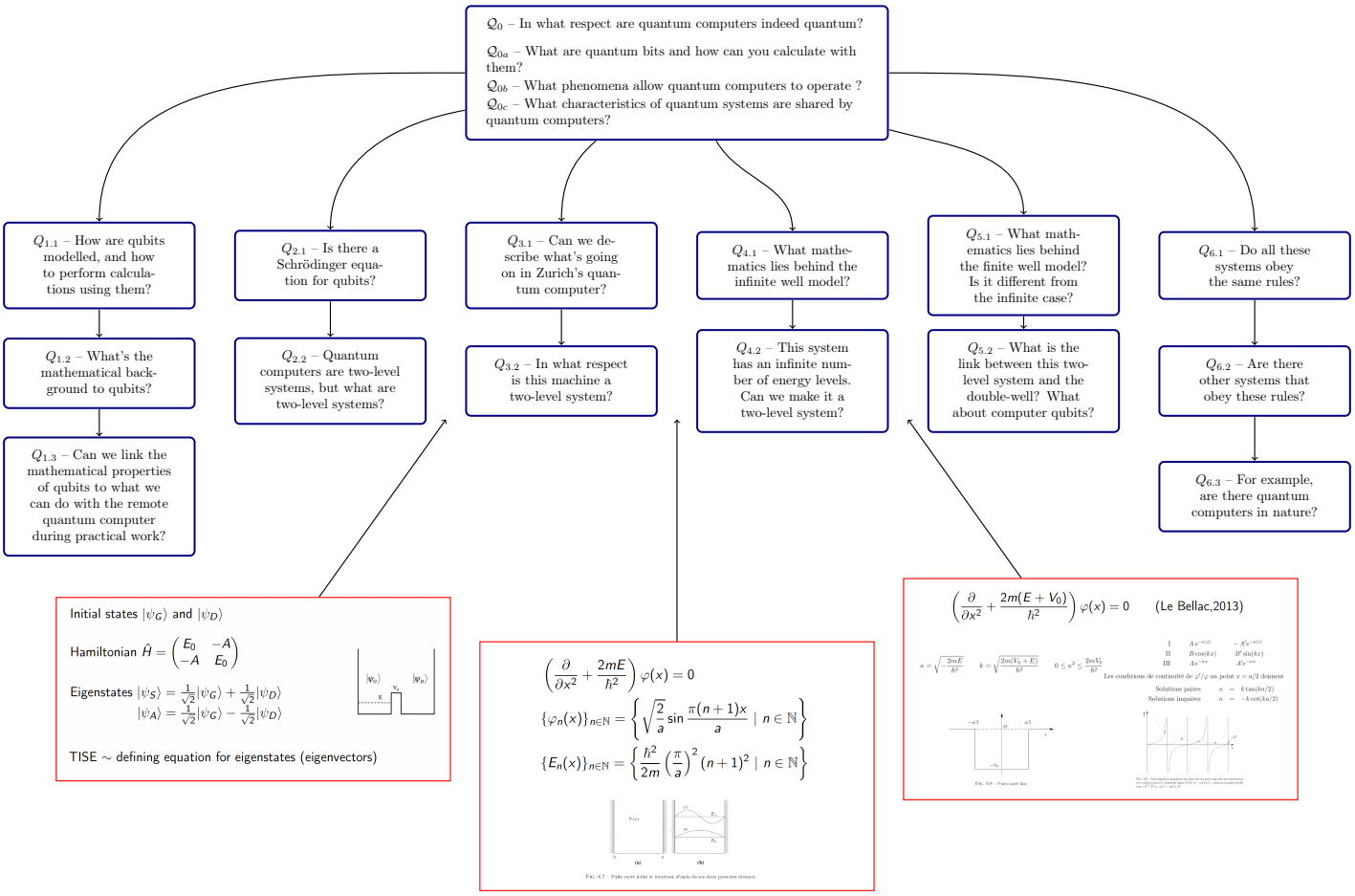


Figure 6. The *a priori* question-answer map of our SRP. This depicts our forecast regarding the intermingling of students' questioning and their study of the models introduced in the classroom. From left to right, below: the double quantum well, the infinite well, the finite well.

a possible class session (we did not consider lab sessions in this diagram), and we envisioned the encounter with three wells qua models of quantum computers as significant steps in the inquiry process. Particular videos and books excerpts were thought in correspondance with the sessions, however we will not go into further details here (see Lombard (2023)).

Based on our model of reference, we wished students to develop the view of Hilbert spaces as a general framework encompassing all models of quantum computers they would have studied: when the states of a system may be described as elements of a Hilbert space, and its observables as operators acting on the latter, it allows one to account for many special features that make it a quantum system; quantum computers are of this kind. Here again, the tension between paradigms sticks out. The latter provisional answer shows a hope to restore the raison d'être of the Hilbert setting, pointing at a tendency to adhere to the paradigm P_1^{+++}, whereas the generating question also planted the seeds for a much more open inquiry process, more in the spirit of the paradigm P_2^{++}. We will now analyse further how this tensions relates to the very decisions we took, as both research and teacher, at the broader level of the whole experimentation.

4. Discussion

Since we focus in this paper on the preliminary and *a priori* analyses of the SRP we conducted, we will not spell out the findings of our *a posteriori* analyses. Indeed, as we have shown, the *a priori* analysis itself raises numerous methodological issues, especially in the specific context of the interface between mathematics and quantum mechanics. However, in order to take a step back, we may occasionally refer to conclusions we drew from our *a posteriori* analyses.

Overall, our *a priori* analysis proved extremely valuable to monitor students' questioning throughout the SRP, while leaving them enough latitude so they could steer their inquiry towards final answers mirroring their personal relation to quantum computing. Indeed, our careful planning of several sessions to the detail of the media supply, the models at play, but also the activities students could engage in, had us explore a vast epistemological territory 'around' the envisioned questioning by students, which we then formalised as our reference epistemological model. We could then rely on it when monitoring the inquiry from inside the classroom. Thus, we helped students carrying out their own inquiry, both by expliciting yet-unaddressed questions or by putting the emphasis on answers brought by each other to the classroom.

However, in the following, we choose rather to consider problematic issues the whole process highlighted, as they seem more relevant to address our research questions.

4.1. The topos of study aid and the media supply

Firstly, the unfolding of the sessions was fairly consistent with its version *a priori* (fig. 5). However, students' questioning did not develop as much as what we contemplated, for instance when compared to fig. 6. So, at some point, there was a discrepancy between the mediatic supply and the students' inquiry. For instance, during the eighth session, as planned, P_μ presented a short theory of Hilbert spaces and the spectral issues they pose from a mathematical point of view. However, students did not rely on this piece of media in the last session, nor in their final reports.

This example shows several constraints which come from our mixed position of didactic engineer, either as a student or as a researcher. The relative inadequacy of P_μ's intervention indeed stems from the fact that, not only it had been decided by us prior to the opening of the questioning, but also we chose not to reconsider it, despite the possible tension between such content and the status of the student's paths. In fact, we wanted to propose a professorial piece of media that would shed light on the purely mathematical aspects of the inquiry. Thus, we could analyze the impact of such a piece of media on the development of students' questioning. More specifically, the goal was to introduce mathematical concepts into the milieu that we thought would enrich our data if discussed by the students. However, this very objective stems from our mixed personal position emerging from several institutional positions, each bringing forth its set of conditions and constraints.

With regard to the present analysis, our person (in ATD's sense) emerged from our dual position as study aid and PhD student in didactics. Now, our stance as a PhD student in didactics clearly affected our experimental attangement, as it was designed to answer the research questions stemming from the set of questions generating our study (our thesis subject). However, on several occasions, the students' questioning seemed to stray from the epistemological terrain in which we hoped it would take place. In fact, in order to answer our research questions as effectively as possible, it seemed important to us that specified objects be discussed, specified situations worked on by the study group, in order to bring out what would constitute

important data for highlighting the didactic phenomena that our thesis subject was focusing on (and which involve the interplay between structuralist algebra and quantum mechanics). This ambition was in fact expressed in our *a priori* analysis, the result of which came as much from our preliminary and epistemological analyses as from the conditions imposed by our subject and the limited time we had to approach it. Consequently, and on a more micro-didactic scale, one could sometimes perceive during the experimentation a tension resulting from a certain discrepancy between the student's procognition (Chevallard, 2015a, p. 68) and our experimental planning. That is, the research praxeologies we brought into play in order to generate valuable data for our future analyses go with considering students should do a precisely framed activity, of the kind of a closed SRP. Put in terms of paradigms, our process of investigation needed that the study taking place in the classroom be of the kind characterised by the paradigm P_1^{+++}, or even P_2^0.

On the other hand, as a study aid, we wished the students' inquiry would develop rather freely, as in a truly open SRP. The institutional conditions allowed this, as the SRP depended on two Teaching Units presenting rather loose constraints. Then, this very objective was pursued both as a teacher and as a researcher, as our SRP was at the same time a teaching unit and a piece of didactic engineering. As a researcher, we examined both the diffusion of given praxeologies and the conditions and constraints on the implementation of a new paradigm of study. As a teacher, we wished to favour student initiative, even if it meant moving away from the experimental ground prepared *a priori*. Depending on these stances and what triggered them in the classroom, we have adopted a more or less monumentalistic approach to pieces of knowledge, which has certainly influenced the quality of our experimentation. That is, the didactic praxeologies we brought into play in order to guide the inquiry go with considering students should study and research rather freely, in the spirit of an open SRP. Put in terms of paradigms, our way of teaching went along with the consideration that the study taking place in the classroom be of the kind characterised by the paradigm P_2^{++}, or even P_2^{+++}.

Another constraint on the topogenesis may be analysed, in conjunction with the media supply. A bit trivially, one may pick only pieces of media one has encountered herself. However, this is yet another constraint arising from the permanence of the traditional paradigm, as in the paradigm of questioning the world students should develop a capacity to supply themselves mediatically, leaving to the study aid the mere role of methodology assistant with respect to the use of sources or bibliography loopholes filler. So, regarding this aspect as well, our *a priori* analysis and the way it was used to monitor the activity in the classroom show a tension between paradigms, which is tangibly reflected by a tension between students' inquiry being finalised or open.

4.2. The dominant epistemological model of the teaching institution as a bias

From January to April 2023, we could implement our SRP for a second time. To do so, we relied on the first iteration of our *a posteriori* analysis as an input for a renewed *a priori* analysis, thus bringing into play the so-called internal validation of didactic engineering (see our theoretical framework). Naturally, in this article, we will simply mention the aspects related to the media supply and the development of students' questioning. The goal of this section is to shed light on another aspect of the tension between finalised and open SRPs, which shows the relative importance taken by the dominant epistemological model of the host institution with respect to the reference epistemological model.

As we stated, the initial goal was to have students discuss models of quantum computers, focusing on objects of the interface, such as energy levels, spectra (in both physical and mathematical senses), eigenvalue problems and quantum eigenstates. However, the models

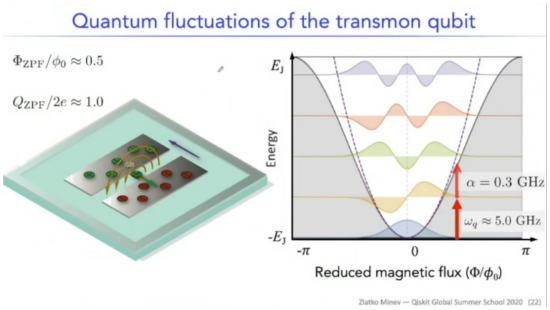


Figure 7. This image is taken from the study material of the 2020 Qiskit Summer School by the technology company IBM, which may be found online. On the left, a representation of a transmon qubit, which is used by IBM. On the right, the energy levels and their associated wave functions, of a harmonic potential (dotted), and the shape of a anharmonic potential (contiguous line). The latter is a good approximation of the transmon potential.

we have chosen in 2022 were the ones already studied at the University of Montpellier, such as the double-well and the infinite well. In order to tackle the ecological issues we raised in section 3.1.3, we conformed to the practice prevailing in the host institution. Of course, we elaborated a distinct model of reference. However it was not sufficient to 'escape the attraction' of the dominant epistemological model. So, we did not emancipate enough from it in order to accompany students' questioning, that is, to really escape a mere visit of works. Specifically, given the models (quantum wells) taught in Montpellier, we could develop our own model of reference, which allowed us to step back with respect to their technical treatment and theoretical justification, shedding light on the practice as it is taught at the interface. However, the influence of the dominant epistemological model of the host institution was still clearly visible in the very choice of these models to fuel our engineering, causing, as we have seen, a discrepancy between the questions of students and the answers these works comprise. And this relative importance of works with respect to questioning is a clear mark of a finalised SRP.

On the contrary, in 2023, we could still rely on our preliminary analyses, and most parts of our *a priori* analysis of the previous year (including of course the generating question), while reflecting more on the bias corresponding to the proximity of the dominant epistemological model of the teaching institution. In particular, the tension the latter caused with respect to the generating question could partly be released by adapting better the media supply to the questioning of students. For instance, in the second version of our SRP, the harmonic oscillators played a significant role, more in conformity with the professional practice in quantum computing (see fig. 7). Indeed, in addition to having played a very important role in the historical development of quantum mechanics, the quantum harmonic oscillator is also the best non-trivial model for a quantum computer according to the litterature. So, seing it in the classroom represents an emancipation with respect to the dominant model of the teaching institution. All the more so, since we used its explicit resolution, which amounts to solving a differential equations, instead of the usual one with ladder operators, which is less evident but taught more often (see for instance Basdevant (2016, p. 85)). Then, this model fueled the questioning on the actual functioning of IBM's transmon qubits through the model of anharmonic oscillators, two objects that are rarely visited in an introductory courses on quantum mechanics.

So, the mediatic supply is but a reflection of the researcher's or the teacher's epistemological model of reference. This observation now allows us to return to our initial questioning. Indeed, the *a priori* analysis of SRPs is especially challenging in so far as the so-called paths actually taken by students seem to change at each occurence of the experimentation:

> The openness inscribed in the Herbartian scheme, with its answers $A^\diamond$ and works W, coming from an in principle unlimited and changing universe, renders illusory the hope of a return to the same, phenomenologically, in the course of successive inquiries on the same question Q: variations are inevitable, not only because of the X [the students] and the y [the study aid], but also because of the sought media and milieus. In this respect, a better clinical knowledge of a given repertoire of questions remains to be fully developed. (Chevallard, 2011, p. 102)

However, the epistemological model of reference may appear as a stable element beyond the uncertainties pertaining to the openness of a generating question, and this is so even though it may evolve slightly after each occurence of the *a priori* analysis - *a posteriori analysis* loop. Moreover, such a model of reference allows the didactic engineer, not only to go beyond the dominant model of the host institution, but also to situate it, thus playing its original role of epistemological frame of reference.

5. Conclusion and perspectives

In this article, we have investigated the challenges posed by the *a priori* analysis of a piece of didactic engineering when it takes the form of a study and research path.

We based our study on pieces of evidence collected by setting up an SRP two years in a row at the interface between mathematics and quantum mechanics, in third year of bachelor's degree. In this specific codisciplinary context, we have exemplified the first two stages of a didactic engineering involving a questioning of the world: the preliminary analysis of its epistemological, institutional and ecological aspects, and the *a priori* analysis strictly speaking, including the design of the generating question and the conception of the *a priori* question-answer map. We have taken a particular care to explicate the rationale behind the latter analysis, as well as its connection with the preliminary analysis.

Firstly, we have shown how the *a priori* analysis of an SRP may embody the resulting tension between the paradigm of visiting works and the paradigm of questioning the world, through the paradigm of didactic situations. Indeed, a tension could clearly be analysed between, on the one hand, designing a study process in order to put at play a given work or set of works (be it by actually restoring their raison d'être or not even), and, on the other hand, designing a study process solely in order to truly answer a given question in specified conditions. This tension amounts to the one between finalised and open SRPs. In this article, it was both identified and charaterized at several steps of the process: the conception of the generating question, the choice *a priori* of media and models, but also the many aspects of the monitoring of the SRP *in vivo* which actually orginate from its *a priori* analysis.

Secondly, in order to deal with this tension, and maybe resolve it, we have brought out the importance of analysing one's epistemological stance, as didactic engineer. Indeed, the installation and monitoring of a questioning of the world, be it finalised or open, is highly dependent on one's personal relation to the objects of knowledge involved. As is customary in the ATD, this personal relation is emerging from the various institutional positions one has occupied, and from which a specific relation to objects of knowledge is developed. To make this explicit, we have relied on the notion of reference epistemological model (REM). In particular, it allows one to characterise and surpass the biases the dominant epistemological model (DEM) of the host teaching institution may represent when it comes to accompanying the students on their study and research paths. Importantly for our study, a lot of the remaining tension between paradigms shows in the (lack of) distance between the didactic engineer's REM and the host institution DEM. Lastly, we have highlighted the relatively persistent character of the epistemological model of reference throughout the iterations that are characteristic of didactic engineering, with respect to the apparently changing features of the

a priori analysis.

This immediately raises the question of the conditions and constraints weighing on the production of epistemological models of reference. If the production of such models is indeed a methodological tool, for both research and teaching, what professional gestures does it encompass? By whom and at what moment is it realised nowadays? What would be the practice of reference? This also leads to the question of the format of such models of reference, be they praxeological, made of situations (Bosch & Gascón, 2006, p. 54) or consisting of a question-answer map (Winsløw et al., 2013).

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