Distance-preserving maps between bounded symmetric domains

Bas Lemmens, Cormac Walsh

DISTANCE-PRESERVING MAPS BETWEEN BOUNDED SYMMETRIC DOMAINS

BAS LEMMENS AND CORMAC WALSH

Abstract. We study the rigidity of maps between bounded symmetric domains that preserve the Carathéodory/Kobayashi metric. We show that such maps are only possible when the rank of the codomain is at least as great as that of the domain. When the ranks are equal, and the domain is irreducible, we show that the map is either holomorphic or antiholomorphic. In the holomorphic case, we show that the map is in fact a triple homomorphism, under the additional assumption that the origin is mapped to the origin.

1. Introduction

An interesting problem in theory of several complex variables is to find conditions under which every map between two complex manifolds preserving the Carathéodory or Kobayashi distance is either holomorphic or anti-holomorphic. This is known to be the case for self maps of strictly pseudo-convex domains of $\mathbb{C}^n$. It is conjectured to hold as long as the domain is not biholomorphic to a Cartesian product of domains.

Here, we study the case of *Hermitian symmetric spaces*, or equivalently *bounded symmetric domains*. In these spaces, the Carathéodory and Kobayashi metrics coincide. Distance preserving maps between such spaces have been considered by Kim and Seo [13], who showed that if the domain is irreducible, the map is C^1-smooth, and the rank of the codomain is no greater than the rank of the domain, then the map is either holomorphic or anti-holomorphic. There are examples where the irreducibly or rank conditions do not hold and the conclusion fails.

On the other hand, it turns out that the smoothness assumption can be relaxed. This assumption was necessary in [13] to apply the techniques of differential geometry. Here, we instead consider the large scale geometry of bounded symmetric domains with their Carathéodory metric. Our main tool is the horofunction boundary of this space, which was described in detail in Chu–Cueto-Avellaneda–Lemmens [6].

Our first result is an extension to the non-smooth case of a theorem of [13], showing that there is no distance preserving map from a bounded symmetric domain to another one of lower rank.

Theorem 1.1. *Let $\phi\colon D \to D'$ be a Carathéodory distance preserving map between two finite-dimensional bounded symmetric domains. Then,* $\operatorname{rank} D \leq \operatorname{rank} D'$.

When the rank of the domain and image are the same, one can say more. Our next theorem states that, in this case, a distance preserving map respects factors, that is, each irreducible factor of the domain only influences one irreducible factor of the image.

Date: January 24, 2025.

The second author was partially funded by ANR-23-CE40-0012-03.

If $D_1, \dots, D_m$ are sets and $I := \{i_1, \dots, i_k\}$ is a subset of $\{1, \dots, m\}$, we use the notation D_I to denote the Cartesian product $D_{i_1} \times \cdots \times D_{i_k}$, where the i_j are taken in increasing order. Also, if $x := (x_1, \dots, x_m) \in D_1 \times \cdots \times D_m$, then $x_I := (x_{i_1}, \dots, x_{i_k}) \in D_I$. If J is a map, then $J^{-1}(x)$ denotes the preimage of a point x.

Theorem 1.2. *Let $\phi\colon D \to D'$ be a Carathéodory distance preserving map between two finite-dimensional bounded symmetric domains of equal rank. Let these domains be expressed in terms of their irreducible factors as follows:*

$$D = D_1 \times \cdots \times D_m$$
$$\text{and} \qquad D' = D'_1 \times \cdots \times D'_n.$$

Then, there exists a surjective map J from $\{1, \dots, m\}$ to $\{1, \dots, n\}$, and distance preserving maps ϕ_k; $k \in \{1, \dots, n\}$, with $\phi_k\colon D_{J^{-1}(k)} \to D'_k$ such that

$$\phi(x_1, \dots, x_m) = \big(\phi_1(x_{J^{-1}(1)}), \dots, \phi_n(x_{J^{-1}(n)})\big).$$

Note that the conclusion implies that D has at least as many irreducible factors as D'.

Theorem 1.3. *Let $\phi\colon D \to D'$ be a Carathéodory distance preserving map between two finite-dimensional bounded symmetric domains of equal rank. Assume that D is irreducible. Then, ϕ is either holomorphic or antiholomorphic.*

In the case where D and D' have the same number of irreducible factors, the component maps ϕ_k of Theorem 1.2 map irreducible factors to irreducible factors, and so Theorem 1.3 can be applied to obtain that each of them is either holomorphic or antiholomorphic. In particular, this is true when D' is identical to D.

Recalling that every bounded symmetric domain can be realised as the unit ball of a JB*-triple, we can strengthen the conclusion of this theorem. A *triple homomorphism* between JB*-triples is a complex-linear map that preserves the triple product.

Theorem 1.4. *Let $\phi\colon D \to D'$ be a Carathéodory distance preserving map between two finite-dimensional bounded symmetric domains of equal rank. Assume that ϕ is holomorphic, and that $\phi(0) = 0$. Then ϕ is the restriction to the unit ball of a triple homomorphism.*

2. Bounded symmetric domains and JB*-triples

A *Hermitian symmetric space* is a Riemannian symmetric space with a compatible complex structure. It was shown by Harish-Chandra that every Hermitian symmetric space can be realised as a *bounded symmetric domain*. These are the bounded domains in $\mathbb{C}^n$ such that every point in the domain is an isolated fixed point of a biholomorphic involution from the domain to itself. It was later shown that every such domain arises as the open unit ball of a *JB*-triple*. These triples provide a powerful tool for studying bounded symmetric domains, one we will use extensively in this paper. See [4] and [5] for an introduction.

A JB*-triple is a complex Banach space V endowed with a triple product

$$\{\cdot,\cdot,\cdot\}\colon V \times V \times V \to V,$$

satisfying the following axioms, for $a, b, x, y, z \in V$:

(i) $\{\cdot,\cdot,\cdot\}$ is linear and symmetric in the outer variables, and conjugate linear in the middle variable;

(ii) $\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}$;
(iii) The operator $a \,\square\, a := \{a,a,\cdot\}$ from V to V is Hermitian, and has non-negative spectrum;
(iv) $\|a \,\square\, a\| = \|a\|^2$.

The *box operator* $a \,\square\, b\colon V \to V$ is defined by

$$a \,\square\, b(x) := \{a,b,x\}, \qquad \text{for all } x \in V.$$

A *tripotent* of a JB*-triple V is an element e such that $\{e,e,e\} = e$. Each tripotent e induces a decomposition of V into eigenspaces of the box operator $e \,\square\, e$. The eigenvalues of this operator lie in the set $\{0, 1/2, 1\}$. Let

$$V_k(e) := \left\{ x \in V \,\middle|\, e \,\square\, e(x) = \frac{k}{2}x \right\}, \qquad \text{for } k \in \{0,1,2\},$$

be the corresponding eigenspaces, which are known as the *Peirce k-spaces* of e. We have the algebraic direct sum

$$V = V_0(e) \oplus V_1(e) \oplus V_2(e).$$

This is the *Peirce decomposition* associated to e. We have the following *Peirce calculus*:

$$\{V_i(e), V_j(e), V_k(e)\} \subseteq V_{i-j+k}(e), \qquad \text{if } i-j+k \in \{0,1,2\},$$

and

$$\{V_i(e), V_j(e), V_k(e)\} = \{0\}, \qquad \text{otherwise.}$$

Moreover,

$$\{V_2(e), V_0(e), V\} = \{V_0(e), V_2(e), V\} = \{0\}.$$

Each Peirce k-space $V_k(e)$ is the range of the *Peirce k-projection* $P_k(e)\colon V \to V$, defined by

$$P_2(e) := Q_e^2, \qquad P_1(e) := 2(e \,\square\, e - Q_e^2), \qquad P_0(e) := B(e,e).$$

Here, $Q_a\colon V \to V$ is, for $a \in V$, the *quadratic operator*

$$Q_a(x) = \{a,x,a\}, \qquad \text{for all } x \in V,$$

and $B(a,b)\colon V \to V$, with $a, b \in V$, is the *Bergman operator*

$$B(a,b)(x) := x - 2(a \,\square\, b)(x) + \{a,\{b,x,b\},a\}, \qquad \text{for all } x \in V. \tag{1}$$

It can be shown that, for two elements a and b of V, we have $a \,\square\, b = 0$ if and only if $b \,\square\, a = 0$. In this case the two elements are said to be *orthogonal*. Another equivalent condition is that $\{a,a,b\} = 0$. For orthogonal elements a and b,

$$\|a+b\| = \max\{\|a\|, \|b\|\};$$

see [4, Corollary 3.1.21].

The following definitions and facts are contained in [14]

One can define an ordering on the set of tripotents by writing $c \le e$ if $e = c + c'$, where c' is a tripotent orthogonal to c. A tripotent is *minimal* if it is non-zero and minimal with respect to this ordering. This is the case precisely when the tripotent e satisfies $V_2(e) = \mathbb{C}e$.

An orthogonal set of non-zero tripotents is linearly independent, and every tripotent can be written as a sum of orthogonal minimal tripotents. A *frame* is a maximal orthogonal system of minimal tripotents. The *rank* of a tripotent e is the rank of $V_2(e)$. If $e = e_1 + \cdots + e_s$ is a decomposition of e into a sum of orthogonal minimal tripotents, then the rank of e is s.

Each family $e_1, \dots, e_n$ of orthogonal tripotents defines a decomposition of V as follows. For $i, j \in \{0, 1, \dots, n\}$, define the *joint Peirce space*

$$V_{ij}(e_1, \dots, e_n) := \Big\{ z \in V \mid \{e_k, e_k, z\} = \frac{1}{2}(\delta_{ik} + \delta_{jk})z, \text{ for all } k \Big\}.$$

Here δ_{ij} is the Kronecker delta, which equals 1 if $i = j$ and is zero otherwise. The *joint Peirce decomposition* of V is

$$V = \bigoplus_{0 \le i \le j \le n} V_{ij}.$$

The joint Peirce spaces satisfy the multiplication rules

$$\begin{aligned} \{V_{ij}, V_{jk}, V_{kl}\} &\subset V_{il}, \qquad \text{for all } i, j, k, l, \quad \text{and} \\ V_{ij} \,\square\, V_{kl} &= \{0\}, \qquad \text{for } i, j \notin \{k, l\}. \end{aligned}$$

For each i and j, there is a contractive projection $P_{ij}(e_1, \dots, e_n)$ from V to $V_{ij}(e_1, \dots, e_n)$ called the *joint Peirce projection*. We will occasionally denote these simply by P_{ij} if it is clear which tripotents $e_1, \dots, e_n$ are involved.

3. The horofunction boundary

Let (X, d) be a metric space. Associate to each point $z \in X$ the function $\psi_z \colon X \to \mathbb{R}$,

$$\psi_z(x) := d(x, z) - d(b, z),$$

where $b \in X$ is some base-point. The map $\psi \colon X \to \mathcal{C}(X)$, $z \mapsto \psi_z$ is injective and continuous. Here, $\mathcal{C}(X)$ denotes the space of continuous real-valued functions on X with the topology of pointwise convergence. The closure $\operatorname{cl}\psi(X)$ is compact. If X is geodesic, meaning that every pair of points can be connected by a geodesic arc, and proper, meaning that closed balls are compact, then ψ is a homeomorphism between X and $\psi(X)$, and hence $\operatorname{cl}\psi(X)$ is a compactification of X. We call it the *horofunction compactification*. We define the *horofunction boundary* of (X, d) to be

$$X(\infty) := \big(\operatorname{cl}\psi(X)\big) \backslash \psi(X).$$

The elements of this set are the *horofunctions* of (X, d). They may be thought of as "points at infinity" of the metric space. The definition of the horofunction boundary is essentially due to Gromov [9], although he used a different topology.

Although the definition appears to depend on the choice of base-point, one can check that horofunction boundaries coming from different base-points are homeomorphic, and that corresponding horofunctions differ only by an additive constant.

3.1. Busemann points and the detour cost.

A sequence z_n in a metric space is called an *almost geodesic* if, for all $\epsilon > 0$,

$$d(b, z_j) \ge d(b, z_i) + d(z_i, z_j) - \epsilon,$$

for i and j large enough, with $i \le j$. This definition is similar to Rieffel's [15], except that here the almost geodesics are unparameterised. Note that any subsequence of an almost geodesic is also an almost geodesic.

Rieffel [15] showed that every almost-geodesic converges to a limit in the horofunction boundary. A horofunction is said to be a *Busemann point* if there is an almost-geodesic converging to it.

We define the *detour cost* for any two horofunctions ξ and η in $X(\infty)$ to be

$$H(\xi, \eta) := \liminf_{x \to \xi} \big(d(b, x) + \eta(x)\big).$$

This concept appeared first in [1]. Intuitively, it is an extension to the boundary of the excess of the triangle inequality $d(b,x)+d(x,y)-d(b,y)$, where y tends to η, and x tends to ξ. Thus, it measures the cost of taking a detour close to ξ on the way from b to η.

In the case where ξ is a Busemann point, it suffices to calculate the limit along any almost-geodesic z_n converging to it, that is,

$$H(\xi,\eta) = \lim_{n\to\infty} \big(d(b,z_n) + \eta(z_n)\big),$$

for any horofunction η,

The Busemann points can be characterised as follows: a horofunction ξ is Busemann if and only if $H(\xi,\xi) = 0$.

The detour cost is non-negative and satisfies the triangle inequality, but it is not necessarily symmetric and may take the value zero between two distinct points. We obtain better properties, however, when we symmetrise. For Busemann points ξ and η, define the *detour metric*

$$\delta(\xi,\eta) := H(\xi,\eta) + H(\eta,\xi).$$

This function is a (possibly ∞-valued) metric on the set of Busemann points. It is independent of the choice of basepoint.

We may consider a pair of Busemann points to be related if the distance between them in the detour metric is finite. This is an equivalence relation, and so partitions the set of Busemann points into what we call *parts*; these are the maximal subsets on which the detour metric is a genuine metric. When a part consists of a single Busemann point, we call that point a *singleton*. These are of particular interest because they tend to be the simplest and most tractable horofunctions.

4. The Gromov product on the horofunction boundary

We define the Gromov product of a pair of points x and y with respect to a basepoint b as follows:

$$(x,y)_b := d(x,b) + d(b,y) - d(x,y).$$

This product may be extended to the horofunction boundary:

$$(\xi,\eta)_b := \liminf_{x\to\xi,\, y\to\eta} (x,y)_b,$$

for all ξ and η in $X(\infty)$.

It is similar to the detour cost in that it is an extension to the boundary of the excess of the triangle inequality; however, the choice of the two points that go to infinity is different. Like the detour cost, is invariant under isometries, provided the basepoint is mapped to the basepoint. The two quantities provide complementary information. The Gromov product can distinguish for example hyperbolic space from Euclidean space: in the former, it is finite for all pairs of distinct points, whereas in the latter it is infinite for all pairs that are not opposite one another. In contrast, in both spaces, the detour cost is always infinite for pairs of distinct points.

When the horofunctions are Busemann points, we have the following alternative expressions for the Gromov product. Recall that a sequence of real-valued functions f_n is *almost non-increasing* if, for every $\epsilon > 0$, there exists N such that $f_j \le f_i + \epsilon$, for all i and j greater than N, with $i \le j$. Such sequences are closely related to almost geodesics, as the following proposition shows.

Proposition 4.1 ([16])**.** *Let z_n be a sequence in a metric space. Then, z_n is an almost geodesic if and only if $\phi_{z_n} := d(\cdot, z_n) - d(b, z_n)$ is an almost non-increasing sequence.*

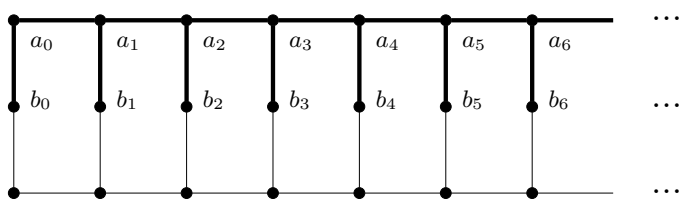

FIGURE 1. A metric space (bold) embedded in another (bold and non-bold). Two sequences, along the top and middle respectively, converge to the same horofunction in the smaller space, but not in the larger one.

Proposition 4.2. *Let x_i and y_j be two almost-geodesic sequences in a metric space (X, d), converging respectively to Busemann points ξ and η. Then,*

$$(\xi, \eta)_b = -\inf_{z \in X} \big[\xi(z) + \eta(z)\big] = \lim_{i,\,j} (x_i, y_j)_b.$$

Proof. For all x and y in X, we have $d(x, y) = \inf_z [d(x, z) + d(z, y)]$, and hence

$$(x, y)_b = -\inf_z \big[d(x, z) + d(z, y) - d(x, b) - d(b, y)\big]. \tag{2}$$

But $d(x_i, \cdot) - d(x_i, b)$ and $d(\cdot, y_j) - d(b, y_j)$ are both almost non-increasing sequences of functions, and the infimum of such functions converges to the infimum of the limiting function. We conclude that the Gromov product $(x_i, y_j)_b$ converges jointly in i and j to $-\inf[\xi(\cdot) + \eta(\cdot)]$. This establishes the second equality.

Taking the limit infimum of (2) as x tends to ξ and y tends to η, we have

$$(\xi, \eta)_b \geq -\inf_z \big[\xi(z) + \eta(y)\big].$$

Here we have used that the limit of an infimum is less than or equal to the infimum of the limit. To establish the opposite inequality, observe that

$$(\xi, \eta)_b := \liminf_{x \to \xi,\, y \to \eta} (x, y)_b \leq \lim_{i,\,j} (x_i, y_j)_b. \qquad \square$$

5. Isometric embeddings and the horofunction boundary

When there is a surjective isometry between two metric spaces, the isometry extends to a homeomorphism between their horofunction compactifications. But when one metric space is isometrically embedded in another, the situation is more complicated—the map does not necessarily extend continuously to a map between the boundaries.

Consider for example the metric spaces depicted in Figure 1. The space in bold is isometrically embedded in a larger space. The metric in both cases is the *path length metric*, where the distance between two points is the Euclidean length of the shortest path joining the two points that remains within the metric space. The smaller space has only one horofunction, and both sequences a_n, along the top, and b_n, along the middle, converge to it. On the contrary, the same sequences in the larger space converge to different points. Note that a_n is an almost-geodesic (in fact a geodesic) in both spaces, while b_n is not an almost-geodesic in either space.

When we restrict our attention to almost-geodesic sequences, however, the situation is more satisfying. Of course, an almost-geodesic is mapped by an isometric embedding to an almost-geodesic. We shall see in addition that, if two almost-geodesics converge to the same Busemann point in the domain, then their images converge to the same point.

Definition 5.1. A map $\phi\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is *distance preserving* if $d_Y(\phi(x), \phi(y)) = d_X(x, y)$, for all $x, y \in X$.

Recall the following [10, Lemma 3.18].

Proposition 5.2. *Two almost-geodesic sequences x_n and y_n in a metric space converge to the same Busemann point if and only if there exists an almost-geodesic sequence z_n that has infinitely many points in common with both x_n and y_n.*

Corollary 5.3. *Let $\phi\colon X \to Y$ be a distance-preserving map between metric spaces. Then, there exists an injective map, which we also denote by ϕ, from the set of Busemann points of X to those of Y, with the property that every almost-geodesic x_n converging to a Busemann point ξ gets mapped to an almost-geodesic $\phi(x_n)$ converging to the Busemann point $\phi\xi$.*

Proof. Given any Busemann point ξ of X, take an almost-geodesic x_n converging to it, and define $\phi\xi$ to be the limit of $\phi(x_n)$. That this is independent of the almost-geodesic chosen follows from Proposition 5.2. For the injectivity, we take two almost-geodesics y_n and z_n in $\phi(X)$ converging to the same Busemann point. We then apply Proposition 5.2 to get an almost-geodesic that has infinitely many points in common with both; this sequence may furthermore be chosen to consist entirely of points of y_n and z_n. As such, it has a preimage, which is also an almost-geodesic and so converges to a Busemann point of X. This shows that the preimages of y_n and z_n have the same limit. □

As well as this, distance-preserving maps preserve the detour cost of every pair of Busemann points and their Gromov product, assuming that the basepoint is mapped to the basepoint. The latter statement follows from Proposition 4.2.

6. The horofunction boundary of bounded symmetric domains

Chu–Cueto-Avellaneda–Lemmens [6] have determined the horofunction boundary of a bounded symmetric domain. They found that all boundary points are Busemann points and take the following form.

Theorem 6.1 (Chu–Cueto-Avellaneda–Lemmens)**.** *Let D be a bounded symmetric domain represented as the unit ball of a finite-dimensional JB*-triple of rank r. Every horofunction ξ is a Busemann point and is of the form*

$$\xi(z) = \frac{1}{2}\log\Big\| \sum_{1\le i\le j\le p} \lambda_i\lambda_j B(z,z)^{-1/2} B(z,e) P_{ij} \Big\|, \qquad \text{for all } z \in D,$$

where

$$\begin{aligned} & p \in \{1, \dots, r\}; \\ & e_1, \dots, e_p \text{ are mutually orthogonal minimal tripotents;} \\ \text{and} \quad & \lambda_1, \dots, \lambda_p \text{ are in } (0,1], \text{ with } \max_i \lambda_i = 1. \end{aligned}$$

Here, $e := e_1 + \cdots + e_p \in \partial D$, and the $P_{ij}\colon V \to V$ are the joint Peirce projections induced by $e_1, \dots, e_p$.

The following lemma concerning finiteness of the detour cost is a refinement of a result in [6].

Lemma 6.2. *Let ξ and η be Busemann points of a bounded symmetric domain, with associated tripotents e and c, respectively. Then, $H(\xi, \eta) < \infty$ if and only if $c \le e$.*

Proof. The forward implication appears in the proof of [6, Proposition 8.4]. The backward implication requires some modification of the proof [6, Proposition 8.5], where only the equality case $e = c$ was treated.

Assume that $c \leq e$. Let λ_i; $1 \leq i \leq p$, be the coefficients appearing in the definition of ξ, as in Theorem 6.1. As in the proof of [6, Proposition 8.5], consider the path

$$\gamma(t) := \sum_{i=1}^{p} \tanh(t - \alpha_i) e_i,$$

where each $\alpha_i := -\log \lambda_i$. This is a geodesic converging to ξ; see [6, Lemma 8.3].

In the proof of [6, Proposition 8.5], it is shown that, for all $w \in V_2(e)$,

$$\lim_{t\to\infty} e^{2t} B\big(\gamma(t), \gamma(t)\big)^{-1/2} B\big(\gamma(t), e\big) w = Q_{a^{-1}} Q_e w, \tag{3}$$

where $a^{-1} = \sum_{i=1}^{p} \lambda_i^{-1} e_i$.

Recall that every element of $V_2(e-c)$ is orthogonal to every element of $V_2(c)$; see [3, Lemma 1(e)]. So, for $w \in V_2(c)$,

$$\begin{aligned}(\gamma(t) \,\square\, c)w &= \{\gamma(t), c, w\} \\ &= \{\gamma(t), e, w\} - \{\gamma(t), e-c, w\} \\ &= (\gamma(t) \,\square\, e)w.\end{aligned}$$

Similarly, $\{c, w, c\} = \{e, w, e\}$. It follows, using the definition (1) of the Bergman operator, that $B(\gamma(t), c)w = B(\gamma(t), e)w$. Note also that $V_2(c) \subset V_2(e)$. So, we may replace e in the left-hand-side of equation (3) with c, when $w \in V_2(c)$.

The rest of the proof follows that of [6, Proposition 8.5]. □

We can deduce from this lemma that there is a one-to-one correspondence between the tripotents and the parts of the horofunction boundary. In fact, this result already appeared in [6].

The singleton Busemann points, that is, the ones having no other point in their part, have a particularly simple form.

Proposition 6.3. *The singletons of a bounded symmetric domain D are the functions of the form*

$$\xi(z) = \frac{1}{2} \log \big\| B(z,z)^{-1/2} B(z,e) e \big\|, \qquad \text{for all } z \in D,$$

where e is a minimal tripotent.

Proof. From Theorem 6.1 and Lemma 6.2, we see that the singletons are exactly the functions of the form

$$\xi(z) = \frac{1}{2} \log \|B(z,z)^{-1/2} B(z,e) P_2(e)\|, \qquad \text{for all } z \in D,$$

where e is a minimal tripotent. Here we are using that $P_{11}(e) = P_2(e)$. Recall that $P_2(e)$ is a contractive mapping, that is, $\|P_2(e)x\| \leq \|x\|$ for all $x \in V$. Hence the supremum of

$$\frac{\|B(z,z)^{-1/2} B(z,e) P_2(e) x\|}{\|x\|}$$

as x ranges over $V \backslash \{0\}$ is the same as its supremum as x ranges over $V_2(e) \backslash \{0\}$. But each $x \in V_2(e) \backslash \{0\}$ can be written $x = \lambda e$ for some $\lambda \in \mathbb{C} \backslash \{0\}$, and so the supremum is equal to $\|B(z,z)^{-1/2} B(z,e) e\|$. □

So, there is a one-to-one correspondence between the minimal tripotents and the singleton Busemann points. For each minimal tripotent e, we denote by Ξ_e the associated Busemann point.

7. Parts of the boundary, and tripotents

In this section, we prove Theorem 1.1, that is, that a bounded symmetric domain cannot be isometrically embedded in another of lower rank. We also show that, if the ranks are equal, then an isometric embedding takes singleton Busemann points to singleton Busemann points. Throughout the section, D is a bounded symmetric domain represented as the unit ball of a finite-dimensional JB*-triple.

The following lemma is stated in [2, Result 2.4], although no proof or reference is given.

Lemma 7.1. *Let e_1 and e_2 be orthogonal tripotents of D, and let $u \in D$ be orthogonal to $e = e_1 + e_2$. Then, u is orthogonal to both e_1 and e_2.*

Proof. The orthogonality of u and e is equivalent to $u \in V_0(e)$. Observe that

$$\{e, e, e_1\} = \{e_1, e, e\} = \{e_1, e_1, e_1\} + \{e_1, e_1, e_2\} + \{e_1, e_2, e\}.$$

The first term on the right-hand-side equals e_1; the other two terms are zero because e_1 and e_2 are orthogonal. So, we see that e_1 is in $V_2(e)$. Therefore, by the Peirce calculus, $\{u, e_1, e_1\} = 0$, which implies that u and e_1 are orthogonal. That the same holds for u and e_2 can be proved similarly. □

Recall that a *chain* in a partially ordered set is a subset that is totally ordered in the sense that every pair x, y of its elements satisfies either $x \leq y$ or $y \leq x$. A chain is *maximal* if it is not a subset of any other chain.

Lemma 7.2. *Let $e_1, \ldots, e_s$ be the elements of a chain of non-zero tripotents, written in ascending order, and take $e_0 := 0$. Then, the differences $e_i - e_{i-1}$; $i \in \{1, \ldots, s\}$, form an orthogonal family of tripotents.*

Proof. Since $e_1 \leq e_2$, we have that $e_2 - e_1$ is a tripotent orthogonal to e_1. Similarly, $e_3 - e_2$ is a tripotent orthogonal to e_2, and hence to both $e_2 - e_1$ and e_1, by Lemma 7.1. We continue inductively to get the result. □

Lemma 7.3. *Let $e_1, \ldots, e_s$ be the elements of a maximal chain, written in ascending order. Then, the rank of e_i is i.*

Proof. Fix $j \in \{1, \ldots, i-1\}$. We can write $e_{j+1} = e_j + c$, where c is a non-zero tripotent orthogonal to e_j. If c were not minimal, then we could decompose it as $c = c_1 + c_2$, into the sum of two non-zero orthogonal tripotents. By Lemma 7.1, e_j would then be orthogonal to both c_1 and c_2, and it would follow that $e_j \leq e_j + c_1 \leq e_j + c_1 + c_2 = e_{j+1}$. Thus $e_j + c_1$ could be added to the chain, contradicting its maximality. So, c must be minimal.

We conclude that each e_i can be written as the sum of the minimal tripotents $e_j - e_{j-1}$; $j \in \{1, \ldots, i\}$. Here, we are taking $e_0 := 0$. These tripotents are orthogonal, using Lemma 7.2. The conclusion follows. □

Every set of orthogonal tripotents is linearly independent. So, for any finite-dimensional bounded symmetric domain, there is an upper bound on the number of mutually orthogonal tripotents. By Lemma 7.2 then, there is the same bound on the number of elements of a chain. Therefore every chain is a subset of a maximal chain.

Lemma 7.4. *Let V be a JB*-triple of rank r. A chain of non-zero tripotents is maximal if and only if it contains exactly r elements.*

Proof. Let the elements of a maximal chain be $e_1, \dots, e_s$, written in increasing order. By Lemma 7.3, the rank of e_s is s. But e_s is a maximal tripotent, for otherwise we could add another tripotent to the end of the sequence. We conclude that $s = r$.

Now suppose that a chain $e_1, \dots, e_r$ has r elements. If it were contained in a larger chain, then the larger chain would itself be contained in a maximal chain having strictly more than r elements, which is impossible by the first part. □

Proof of Theorem 1.1. We have seen in section 5 that the map ϕ induces a map, which we again denote by ϕ, from the set of Busemann points of D to those of D' in such a way that the image under ϕ of every almost geodesic converging to a Busemann point ξ of D converges to $\phi\xi$. This map preserves the detour cost in the sense that $H(\phi\xi, \phi\eta) = H(\xi, \eta)$ for all Busemann points ξ and η of D. Therefore, two Busemann points lie in the same part if and only if their images lie in the same part. Since there is a one-to-one correspondence between parts of the horofunction boundary and tripotents, we get a map, again denoted by ϕ, from the tripotents of D to those of D'. By Lemma 6.2, the order on the set of tripotents is preserved; in fact, $e \le c$ for two tripotents if and only if $\phi e \le \phi c$, for any two tripotents e and c of D. In particular, the map ϕ on the set of tripotents is injective.

Let $e_1, \dots, e_r$ be a maximal chain of tripotents in D. Its image $\phi(e_1), \dots, \phi(e_r)$ is contained in a maximal chain of tripotents of D'. So, by Lemma 7.4, $\operatorname{rank} D = r \le \operatorname{rank} D'$. □

The next lemma will be crucial to studying the equal rank case.

Lemma 7.5. *Assume that we have an isometric embedding $\phi\colon D \to D'$ of one bounded symmetric domain in another, such that* $\operatorname{rank} D = \operatorname{rank} D'$. *Then, the induced map on the set of Busemann points (see section 5) takes singletons to singletons.*

Proof. As in the proof of Theorem 1.1, the map ϕ induces an injective map between the tripotents of D and those of D' that preserves the order.

Let e be a tripotent of D of rank s. Then, e is contained in a maximal chain $e_1, \dots, e_r$, and by Lemma 7.3, $e = e_s$. By Lemma 7.4, the chain $\phi e_1, \dots, \phi e_r$ is also maximal, and hence the rank of ϕe is s. We have shown that ϕ preserves the rank of every tripotent.

The conclusion now follows on observing that the singleton parts are precisely those corresponding to minimal tripotents, that is, tripotents of rank 1. □

8. The Gromov product in a bounded symmetric domain

We have the following expression for the Gromov product of two singleton Busemann points of a bounded symmetric domain. We take the origin 0 to be the base point.

Theorem 8.1. *Let u and v be minimal tripotents in a JB*-triple V. Then,*

$$(\Xi_u, \Xi_v)_0 = \frac{1}{2} \log \frac{4}{\|P_2(u)B(u,v)v\|}.$$

Proof. The path $t\colon (0,1) \to D$ defined by $t \mapsto tu$ is a geodesic in D and converges to Ξ_u in the horofunction compactification. The Carathéodory distance between 0 and tu is

$d(0,tu) = \tanh^{-1} t$. So, by Proposition 4.2 and Proposition 6.3,

$$\begin{aligned}(\Xi_u, \Xi_v)_0 &= \lim_{t\to 1} \big(d(0,tu) - \Xi_v(tu)\big) \\ &= \lim_{t\to 1} \frac{1}{2} \log\Big(\frac{1+t}{1-t}\Big) \big\|B(tu,tu)^{-1/2} B(tu,v)v\big\|^{-1}.\end{aligned}$$

Now we use that

$$\lim_{t\to 1} (1-t^2) B(tu,tu)^{-1/2} = P_2(u),$$

and that $B(tu,v)$ converges to $B(u,v)$; see the proof of Theorem 4.4 of [6]. The result follows. □

Lemma 8.2. *Let u be a minimal tripotent in a bounded symmetric domain, and let $b \in V_1(u)$ be such that $\{b,b,u\} = 0$. Then, $b = 0$.*

Proof. The assumption implies that b is orthogonal to u. But the orthogonality relation is symmetrical, and so $b = 2\{u,u,b\} = 0$. □

When we split one of the tripotents into its Peirce components with respect to the other, we obtain the following useful formula. We use $\mathfrak{Re}\,\mu$ to denote the real part of a complex number μ, and $\overline{\mu}$ to denote its conjugate.

Lemma 8.3. *Let u and v be minimal tripotents in a bounded symmetric domain. We decompose $v = a + b + c$ into its Peirce components $a := P_2(u)v$, $b := P_1(u)v$, and $c := P_0(u)v$, and let μ and λ in $\mathbb{C}$ be such that $a = \mu u$ and $\{b,b,u\} = \lambda u$. Then,*

$$(\Xi_u, \Xi_v)_0 = \frac{1}{2} \log \frac{2}{|\,\mathfrak{Re}\,\mu - |\mu|^2 - \lambda|}.$$

Proof. By the Peirce calculus, and the linearity and conjugate-linearity of the triple product,

$$\begin{aligned}P_2(u)B(u,v)v &= P_2(u)\big(v - 2\{u,v,v\} + \{u,v,u\}\big) \\ &= a - 2\{u,a,a\} - 2\{u,b,b\} + \{u,a,u\} \\ &= \big(\mu - 2|\mu|^2 - 2\lambda + \overline{\mu}\big)u.\end{aligned}$$

The result now follows upon applying Theorem 8.1. □

It is easy to calculate the Gromov product when one of the minimal tripotents is a complex multiple of the other.

Lemma 8.4. *Let u and v be minimal tripotents in a bounded symmetric domain, such that $v = \mu u$, for some $\mu \in \mathbb{C}$ with $|\mu| = 1$. Then,*

$$(\Xi_u, \Xi_v)_0 = \frac{1}{2} \log \frac{2}{1 - \mathfrak{Re}\,\mu}.$$

Proof. This is a simple calculation using Lemma 8.3. □

Our strategy will be to relate algebraic properties of minimal tripotents to the Gromov product of the associated Busemann points. Since the Gromov product is preserved by distance-preserving maps, the properties of the tripotents will be as well.

First, we characterise when two minimal tripotents are opposite one another.

Proposition 8.5. *Let u and v be minimal tripotents in a JB*-triple. Then, $v = -u$ if and only if $(\Xi_u, \Xi_v)_0 = 0$.*

Proof. When $v = -u$, it is easy to see from Lemma 8.4 that $(\Xi_u, \Xi_v)_0 = 0$.

To prove the converse, assume that the latter equation is true. So, by Theorem 8.1, $\|P_2(u)B(u,v)v\| = 4$. We have

$$P_2(u)B(u,v)v = P_2(u)\big(v - 2\{u,v,v\} + \{u,v,u\}\big).$$

Observe that the projections to the Peirce space $V_2(u)$ of each of the terms v, $-\{u,v,v\}$, and $\{u,v,u\}$ is a complex multiple of u. Moreover, each of these projections has norm at most 1. It follows that each of them must have norm exactly 1, and in fact must all be equal. Thus, we have $P_2(u)v = -P_2(u)\{u,v,v\} = P_2(u)\{u,v,u\} = \mu u$, for some $\mu \in \mathbb{C}$ with $|\mu| = 1$.

Write $a := P_2(u)v$, $b := P_1(u)v$, and $c := P_0(u)v$. By the Peirce calculus,

$$\mu u = -P_2(u)\{u,v,v\} = -\{u,a,a\} - \{u,b,b\}. \tag{4}$$

Since u is an eigenvector of both $a \,\square\, a$ and $b \,\square\, b$, and these operators are Hermitian with non-negative spectrum, we deduce that μ is negative, and hence equals -1. So, $a = -u$. From (4) again, we get $\{u,b,b\} = 0$, and hence by Lemma 8.2, that $b = 0$.

Observe that, since c is in $V_0(u)$, it is orthogonal to u. Using the Peirce calculus again,

$$a + c = v = \{v,v,v\} = \{a,a,a\} + \{c,c,c\}.$$

By comparing components in the various Peirce spaces, we see that both a and c are tripotents. Since v is minimal, only one of them can be non-zero, in the present case necessarily a. We have shown that $v = -u$. □

Next we characterise orthogonality of minimal tripotents.

Proposition 8.6. *Let u and v be minimal tripotents in a JB*-triple. Then, u and v are orthogonal if and only if $(\Xi_u, \Xi_v)_0 = (\Xi_u, \Xi_{-v})_0 = \infty$.*

Proof. First assume that u and v are orthogonal. In this case $B(u,v)$ is the identity map, and so we get $P_2(u)B(u,v)v = 0$. Using the formula for the Gromov product in Theorem 8.1, we see that $(\Xi_u, \Xi_v)_0 = \infty$. The same conclusion also holds when v is replaced by $-v$.

Now assume that u and v are minimal tripotents such that the two equations in the statement of the proposition hold. We use the same notation as in Lemma 8.3. Since $\{u,b,b\}$ is in $V_2(u)$, it is equal to λu for some λ in $\mathbb{C}$. So, u is an eigenvector of $b \,\square\, b$. Since this operator is Hermitian and has non-negative spectrum, λ must be non-negative. We also have that $a = \mu u$, for some $\mu \in \mathbb{C}$. So, by Lemma 8.3, the two equations can be expressed in the form

$$\mathfrak{Re}\,\mu - |\mu|^2 - \lambda = 0 = -\,\mathfrak{Re}\,\mu - |\mu|^2 - \lambda.$$

Taking their difference, we get that $\mathfrak{Re}\,\mu$ is zero, and hence that $|\mu|^2$ and λ are also zero, since both are non-negative. So, we have that $a = 0$ and, using Lemma 8.2, that $b = 0$. We have shown that $v = c$, and since c is in $V_0(u)$, it is orthogonal to u. □

Finally, we characterise when two minimal tripotents are related by a multiple $\pm i$. To do this, the following lemma will be useful.

Lemma 8.7. *Let u and v be minimal tripotents in a JB*-triple. Assume there exists a frame with u as one of its elements, such that v is orthogonal to each element of the frame apart from u. Then, $P_0(u)v = 0$.*

Proof. Let $e_1, \dots, e_r$ be the frame, with $e_1 := u$. For each $i \in \{2, \dots, r\}$, we have the following chain of equivalences:

$$\begin{aligned} v \text{ is orthogonal to } e_i &\iff P_1(e_i)v = P_2(e_i)v = 0 \\ &\iff \sum_{k \neq i} P_{ki}(e_1, \dots, e_r)v = P_{ii}(e_1, \dots, e_r)v = 0 \\ &\iff P_{ki}(e_1, \dots, e_r)v = 0, \quad \text{for all } k \in \{0, \dots, r\}. \end{aligned}$$

Also, since $e_1, \dots, e_r$ is a frame, $P_{00}(e_1, \dots, e_r)v = 0$.

Now observe that

$$P_0(u)v = P_{00}(u)v = \sum_{\substack{i \leq j \\ i \neq 1, j \neq 1}} P_{ij}(e_1, \dots, e_r)v.$$

Each term of this expression is zero because, if both i and j differ from 1, then either they are both 0 or at least one of them lies in $\{2, \dots, r\}$. In both cases, we have seen that the projection of v is zero. □

Lemma 8.8. *Let u and v be minimal tripotents in a JB*-triple. Then, $v = \pm iu$ if and only if both the following conditions hold:*

- *there exists a frame with u as one of its elements, such that v is orthogonal to each element of the frame apart from u.*
- $(\Xi_u, \Xi_v)_0 + (\Xi_u, \Xi_{-v})_0 = \log 2$.

Proof. Assume that $v = \pm iu$. Take any frame $u_1, \dots, u_r$ with $u = u_1$. Since orthogonality of two elements is preserved when one of them is multiplied by a complex scalar, v is orthogonal to each of $v_2, \dots, v_r$. This establishes the first of the two conditions. The second follows from Lemma 8.4.

Now assume that u and v satisfy the two conditions in the statement of the lemma. We decompose $v = a + b + c$, where $a := P_2(u)v$, $b := P_1(u)v$, and $c := P_0(u)v$. By Lemma 8.7, the first condition implies that $c = 0$.

Let $\mu \in \mathbb{C}$ be such that $a = \mu u$. Observe that $\{b, b, u\} = \lambda u$, for some $\lambda \in \mathbb{C}$, and since $b \,\square\, b$ is Hermitian and has non-negative spectrum, λ must be non-negative. Since v is a tripotent and $c = 0$, we have by the Peirce calculus

$$a = \{a, a, a\} + 2\{b, b, a\} = |\mu|^2 \mu u + 2\lambda \mu u = |\mu|^2 a + 2\lambda a.$$

Thus, $|\mu|^2 + 2\lambda = 1$. This implies that $|\mu|^2 + \lambda \leq 1$, with equality only when $|\mu| = 1$ and $\lambda = 0$.

So, we see that

$$\Big(\mathfrak{Re}\, \mu - |\mu|^2 - \lambda \Big) \Big(- \mathfrak{Re}\, \mu - |\mu|^2 - \lambda \Big) = \big(|\mu|^2 + \lambda \big)^2 - \big(\mathfrak{Re}\, \mu \big)^2$$

takes values in $[-1, 1]$. In fact, as can be seen from Lemma 8.3, the second condition we have imposed on u and v is equivalent to the absolute value of this expression being equal to 1. For the expression to take the value -1, one would need that $|\mu|^2 + \lambda = 0$ and $\mathfrak{Re}\, \mu = 1$, which is clearly impossible. On the other hand, for the value 1 to be obtained, we need that $|\mu|^2 + \lambda = 1$ and $\mathfrak{Re}\, \mu = 0$, which together with what we have seen above, imply that μ is either i or $-i$, and $\lambda = 0$. The conclusion now follows on applying Lemma 8.2. □

A flat in a bounded symmetric domain D is a maximal embedded Euclidean space, when one takes the Bergman metric on D. Every flat has the same dimension: the rank

r of D. If one takes the Kobayashi/Carathéodory metric instead of the Bergman metric, each flat is isometric to $\mathbb{R}^r$ with the ℓ_∞ norm $\|\cdot\|_\infty$. Let $e_1, \dots, e_r$ be a frame in D. The set

$$F := \big\{\lambda_1 e_1 + \cdots + \lambda_r e_r \mid \lambda_i \in (-1,1), \text{ for all } i\big\}$$

is a flat. In fact, every flat containing the origin is of this form.

For each $a \in D$, the *Möbius transformation* $g_a \colon D \to D$ is defined to be

$$g_a(x) := a + B(a,a)^{1/2}(\mathrm{Id} + x \,\square\, a)^{-1}(x), \qquad \text{for all } x \in D.$$

Here Id denotes the identity operator on V. The inverse operator in this definition exists because $\|x \,\square\, a\| \le \|x\|\|a\| < 1$. Observe that, for each $a \in D$, the Möbius transformation g_a maps 0 to a. Moreover, g_a is a bijection from D to itself, and its inverse is g_{-a}. It is also biholomorphic, and hence an isometry of the Kobayashi/Carathéodory metric and of the Bergman metric. The derivative of g_a at any point $b \in D$ has the following expression in terms of the Bergman operator: $\nabla g_a(b) = B(a,a)^{1/2} B(b,-a)^{-1}$; see [4, equation (3.2)]. We will see that if a and b are points lying in a common flat that contains 0, then g_a and g_b commute, and both maps leave the flat invariant.

Lemma 8.9. *Let $e_1, \dots, e_r$ be a frame, and let $a := a_1 e_1 + \cdots + a_r e_r$ and $b := b_1 e_1 + \cdots + b_r e_r$ be two elements of the associated flat, with all coefficients a_i and b_i in $\mathbb{R}$. Set $a_0 := b_0 := 0$. Then,*

$$B(a,b) = \sum_{0 \le i \le j \le r} \big(1 - a_i b_i\big)\big(1 - a_j b_j\big) P_{ij}(e_1, \dots, e_r).$$

Proof. Let $z \in P_{ij}$, with $i, j \in \{0, \dots, r\}$. We use the convention that $e_0 := 0$. Using the orthogonality of the e_k and that z is an eigenvector of each $e_k \,\square\, e_k$, we get

$$a \,\square\, b(z) = \sum_k a_k b_k \{e_k, e_k, z\} = \sum_k a_k b_k (\delta_{ik} + \delta_{jk}) z\} = \frac{1}{2}\big(a_i b_i + a_j b_j\big) z.$$

From the Jordan identity, we have, for $k, l, m, n \in \{1, \dots, r\}$,

$$\begin{aligned}\big\{e_k, \{e_m, z, e_n\}, e_l\big\} = \big\{\{z, e_m, e_k\}, e_n, e_l\big\} \\ + \big\{e_k, e_n, \{z, e_m, e_l\}\big\} - \big\{z, e_m, \{e_k, e_n, e_l\}\big\}.\end{aligned}$$

So,

$$\begin{aligned} Q_a Q_b(z) &= \sum_{m,n,k,l} a_k a_l b_m b_n \big\{e_k, \{e_m, z, e_n\}, e_l\big\} \\ &= \frac{1}{4} \sum_{m,n,k,l} a_k a_l b_m b_n \Big[\delta_{mk}\delta_{nl}(\delta_{im} + \delta_{jm})(\delta_{in} + \delta_{jn}) \\ &\qquad + \delta_{ml}\delta_{nk}(\delta_{im} + \delta_{jm})(\delta_{in} + \delta_{jn}) - 2\delta_{kn}\delta_{nl}\delta_{lm}(\delta_{im} + \delta_{jm})\Big] z \\ &= \frac{1}{4}\big[2(a_i b_i + a_j b_j)^2 - 2(a_i^2 b_i^2 + a_j^2 b_j^2)\big] z \\ &= a_i a_j b_i b_j z. \end{aligned}$$

Combining this with the result at the start, we have

$$B(a,b)z = \big(1 - a_i b_i - a_j b_j + a_i a_j b_i b_j\big) z = \big(1 - a_i b_i\big)\big(1 - a_j b_j\big) z,$$

and the conclusion follows. □

Lemma 8.10. *Let $e_1, \dots, e_r$ be a frame, and let*

$$a := \sum_i \tanh(\alpha_i) e_i, \qquad b := \sum_i \tanh(\beta_i) e_i, \qquad \text{and} \quad c = \sum_i \tanh(\alpha_i + \beta_i) e_i$$

be elements of the associated flat, with each α_i and β_i in $\mathbb{R}$. Then, $g_a \circ g_b = g_c$.

Proof. For each i, write $a_i := \tanh \alpha_i$ and $b_i := \tanh \beta_i$. Let $y = \sum_i y_i e_i$, with each $y_i := b_i/(1 + b_i a_i)$. Observe that $(\mathrm{Id} + b \,\square\, a)y = b$. We conclude that $(\mathrm{Id} + b \,\square\, a)^{-1} b = y$.

For any $z := \sum_i z_i e_i$, with each $z_i \in \mathbb{R}$, we have

$$B(a,a)z = \sum_i (1 - a_i^2)^2 z_i e_i.$$

It follows that

$$B(a,a)^{1/2} z = \sum_i (1 - a_i^2) z_i e_i.$$

Applying this to y, we get

$$\begin{aligned} g_a(b) &= a + B(a,a)^{1/2} (\mathrm{Id} + b \,\square\, a)^{-1} b \\ &= \sum_i \Big(a_i + \frac{(1 - a_i^2) b_i}{1 + b_i a_i} \Big) e_i \\ &= \sum_i \frac{a_i + b_i}{1 + b_i a_i} e_i \\ &= \sum_i \tanh(\alpha_i + \beta_i) e_i \\ &= c. \end{aligned}$$

So, the maps $g_a \circ g_b$ and g_c agree at 0.

Let z be in the joint Peirce space V_{ij}, with $i, j \in \{0, \dots, r\}$. The derivative of g_c at 0 applied to z is

$$\begin{aligned} \nabla g_c(0) z &= B(c,c)^{1/2} z \\ &= \Big[\Big(1 - \Big(\frac{a_i + b_i}{1 + a_i b_i} \Big)^2 \Big) \Big(1 - \Big(\frac{a_j + b_j}{1 + a_j b_j} \Big)^2 \Big) \Big]^{1/2} z, \end{aligned}$$

using Lemma 8.9. Similarly, the derivative of $g_a \circ g_b$ at 0 applied to z is

$$\begin{aligned} \nabla (g_a \circ g_b)(0) z &= \nabla g_a(b) . \nabla g_b(0) z \\ &= B(a,a)^{1/2} B(b, -a)^{-1} B(b,b)^{1/2} z \\ &= \left[\frac{(1 - a_i^2)(1 - a_j^2)(1 - b_i^2)(1 - b_j^2)}{(1 + a_i b_i)^2 (1 + a_j b_j)^2} \right]^{1/2} z. \end{aligned}$$

Elementary algebra shows that these two expressions are equal. Using that V decomposes as the sum of the Peirce spaces, we conclude that $g_a \circ g_b$ and g_c have the same derivative at 0.

It now follows by Cartan's Uniqueness Theorem that $g_a \circ g_b = g_c$. □

We have seen that every frame gives rise to a flat. It also defines a collection of minimal tripotents and their associated Busemann points. We need to study how these objects are related. Indeed, we will characterise when a point is in the flat in terms of values of the Busemann points there.

According to [8, Lemma 1.6], if e is a tripotent in a JB*-triple V, and x is an element with $\|x\| = 1$ and $P_2(e)x = e$, then $P_1(e)x = 0$. We will also need that if x and y in V are orthogonal, then

$$\|x + y\| = \max\big(\|x\|, \|y\|\big). \tag{5}$$

Proposition 8.11. *Let e be a minimal tripotent, and let $x \in D$. Then, $\Xi_e(x) = \Xi_{-e}(x) = 0$ if and only if $x \in V_0(e)$.*

Proof. First, assume that x is in $V_0(e)$. Using the Peirce calculus, we get that

$$B(x, e)e = e - 2\{x, e, e\} + \{x, \{e, e, e\}, x\} = e.$$

Similarly, $B(x, x)e = e$, which implies that $B(x, x)^{-1}e = e$. Since $B(x, x)^{-1}$ has positive spectrum, we deduce that $B(x, x)^{-1/2}e = e$. Therefore,

$$\Xi_e(x) = \frac{1}{2}\log\big\|B(x, x)^{-1/2}B(x, e)e\big\| = 0.$$

That $\Xi_{-e}(x)$ is also zero is proved similarly.

Now let $x \in D$ be such that $\Xi_e(x) = \Xi_{-e}(x) = 0$. So, in particular,

$$\big\|B(x, x)^{-1/2}B(x, e)e\big\| = 1.$$

Recall that $\|B(x, x)^{1/2}\| \leq 1$; see the discussion before Proposition 3.2.13 of [4]. We deduce that $\|B(x, e)e\| \leq 1$.

Write $x = a + b + c$, with $a \in V_2(e)$, $b \in V_1(e)$, and $c \in V_0(e)$. Since e is minimal, we have $a = \mu e$, for some $\mu \in \mathbb{C}$. By the Peirce calculus,

$$\begin{aligned} B(x, e)e &= e - 2\mu e - 2\{b, e, e\} + \mu^2 e + 2\mu\{b, e, e\} + \{b, e, b\} \\ &= (1 - \mu)^2 e - 2(1 - \mu)\{b, e, e\} + \{b, e, b\}. \end{aligned}$$

So, the projection of $B(x, e)e$ onto the Peirce 2-space $V_2(e)$ of e is $(1 - \mu)^2 e$. Since this projection does not increase the norm, we have $|1 - \mu| \leq 1$.

Using similar reasoning, we also get that $\|B(x, -e)e\| \leq 1$, with

$$B(x, -e)e = (1 + \mu)^2 e + 2(1 + \mu)\{b, e, e\} + \{b, e, b\},$$

and so $|1 + \mu| \leq 1$. We conclude that μ is zero, and hence so also is a. So,

$$B(x, e)e = e - 2\{b, e, e\} + \{b, e, b\}.$$

The projection of this vector onto $V_2(e)$ is e, and hence its norm is at least 1. Combining this with what we had before, its norm is actually equal to 1. Applying [8, Lemma 1.6], we get that $0 = P_1(e)B(x, e)e = -2\{b, e, e\} = -b$. We have shown that $x \in V_0(e)$. □

Recall that the inverse hyperbolic tangent function is given by

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1 + x}{1 - x}.$$

Lemma 8.12. *Let $e_1, \ldots, e_r$ be a frame of a bounded symmetric domain, and let $x := \lambda_1 e_1 + \cdots + \lambda_r e_r$ be in the associated flat, with each λ_i in $(-1, 1)$. Then $\Xi_{e_i}(x) = -\tanh^{-1}\lambda_i$, for all i.*

Proof. Since the e_k are mutually orthogonal, from the definition of the Bergman operator we have

$$B(x, e_i)e_i = e_i - 2\lambda_i e_i + \lambda_i^2 e_i = (1 - \lambda_i)^2 e_i, \qquad \text{for all } i.$$

The joint Peirce projection $P_{jk}(e_1,\dots,e_r)e_i$ equals e_i when $j=k=i$, and equals zero otherwise. So, using Lemma 8.9, we get $B(x,x)^{-1/2}e_i = (1-\lambda_i^2)^{-1}e_i$. Combining these formulae with Proposition 6.3, we see that

$$\Xi_{e_i}(x) = \frac{1}{2}\log\frac{(1-\lambda_i)^2}{1-\lambda_i^2} = \frac{1}{2}\log\frac{1-\lambda_i}{1+\lambda_i} = -\tanh^{-1}\lambda_i. \qquad \square$$

Lemma 8.13. *Let $e_1,\dots,e_r$ be a frame of a bounded symmetric domain D, and let $x := \lambda_1 e_1 + \cdots + \lambda_r e_r$ be in the associated flat, with each λ_i in $(-1,1)$. Then,*

$$\Xi_{e_i}\big(g_x(y)\big) = \Xi_{e_i}(y) - \tanh^{-1}\lambda_i, \qquad \text{for all } y\in D \text{ and all } i.$$

Proof. The flat F associated to the frame $e_1,\dots,e_r$ is isometric to the normed space $\mathbb{R}^r$ with the supremum norm $||\cdot||_\infty$. The isometry takes the form

$$\phi\colon \mathbb{R}^r \to F, \qquad (p_1,\dots,p_r)\mapsto \tanh(p_1)e_1+\cdots+\tanh(p_r)e_r.$$

For each i, let $\mu_i := \tanh^{-1}\lambda_i$, and define $M := \max_i|\mu_i|$. The restriction of g_x, when carried over to $\mathbb{R}^d$, is a translation by $(\mu_1,\dots,\mu_r)$.

For simplicity, we consider the case where $i=1$; the other cases are similar. The sequence $z_n := \phi(2nM,0,\dots,0)$ is an almost-geodesic converging to Ξ_{e_1} in the horofunction boundary. Its image $y_n := g_x(z_n)$ under the map g_x is also an almost-geodesic.

Let w_n be the sequence obtained by taking alternate terms of the sequences z_n and y_n, that is, $w_n := z_n$ for n even, and $w_n := y_n$ for n odd. Let m and n be elements of $\mathbb{N}$ such that $m\le n$. If m and n are either both even or both odd, then $d(w_m,w_n) = 2(n-m)M$. If m is odd and n is even, then $d(w_m,w_n) = 2(n-m)M+\mu_1$, while if m is even and n is odd, then $d(w_m,w_n) = 2(n-m)M-\mu_1$, So, we see that w_n is also an almost-geodesic.

It follows that the three sequences converge to the same horofunction. Therefore, using that g_x preserves the distance,

$$\begin{aligned}\Xi_{e_1}\big(g_x(y)\big) &= \lim_{n\to\infty}\Big(d\big(g_x(y),y_n\big) - d(0,y_n)\Big)\\ &= \lim_{n\to\infty}\big(d(y,z_n)-d(0,z_n)\big) - \lim_{n\to\infty}\big(d(g_{-x}(0),z_n)-d(0,z_n)\big)\\ &= \Xi_{e_1}(y) - \Xi_{e_1}(-x).\end{aligned}$$

The result now follows upon applying Lemma 8.12. $\square$

Proposition 8.14. *Let $e_1,\dots,e_r$ be a frame of a bounded symmetric domain D, and let $x\in D$. Then $\Xi_{e_i}(x)+\Xi_{-e_i}(x)=0$, for all i, if and only if x is in the flat defined by $e_1,\dots,e_r$.*

Proof. Let $x := \lambda_1 e_1+\cdots+\lambda_r e_r$ be in the flat, with each λ_i in $(-1,1)$, and take $j\in\{1,\dots,r\}$. By Lemma 8.12, we have $\Xi_{e_j}(x) = -\tanh^{-1}\lambda_j$. But $-e_1,\dots,-e_r$ is also a frame, and it gives rise to the same flat. With respect to this frame, the coordinates of x are $(-\lambda_1,\dots,-\lambda_r)$. Using the same lemma again, we get $\Xi_{-e_j}(x) = -\tanh^{-1}(-\lambda_j)$. We now use that the inverse hyperbolic tangent is an odd function to get that the sum of $\Xi_{e_j}(x)$ and $\Xi_{-e_j}(x)$ is zero.

To prove the converse, let $x\in D$ and assume that the sum of $\Xi_{e_j}(x)$ and $\Xi_{-e_j}(x)$ is zero, for each j. For each j, let $\mu_j := \Xi_{e_j}(x) = -\Xi_{-e_j}(x)$, and define $\lambda_j := \tanh\mu_j$. The maps $\{g_{\lambda_j e_j}\}$; j commute, and by Lemma 8.13 we have that, for $x\in D$ and $j,k\in$

$\{1, \dots, r\}$,

$$\Xi_{e_k}\big(g_{\lambda_j e_j}(x)\big) = \begin{cases} \Xi_{e_k}(x), & \text{if } j \neq k; \\ \Xi_{e_k}(x) - \mu_k, & \text{if } j = k \end{cases}$$

and $$\Xi_{-e_k}\big(g_{\lambda_j e_j}(x)\big) = \begin{cases} \Xi_{e_k}(x), & \text{if } j \neq k; \\ \Xi_{e_k}(x) + \mu_k, & \text{if } j = k. \end{cases}$$

So, the point

$$y := g_{\lambda_r e_r} \circ \cdots \circ g_{\lambda_1 e_1}(x)$$

satisfies $\Xi_{e_k}(y) = \Xi_{-e_k}(y) = 0$, for all k. From Proposition 8.11, we get that y is in $V_0(e_k)$, for each k. Since the e_k form a frame, it follows that $y = 0$. But 0 is in the flat defined by the frame, and therefore so also is $x = g_{-\lambda_1 e_1} \circ \cdots \circ g_{-\lambda_r e_r}(0)$. □

9. Carathéodory distance preserving maps

In this section, we use the Gromov product to study Carathéodory distance preserving maps between bounded symmetric domains of equal rank. We show that flats are mapped to flats, the Bergman metric is preserved, and the map is of differentiability class C^∞.

Recall that each Carathéodory distance preserving map ϕ between bounded symmetric domains of equal rank takes singleton Busemann points to other such points, and thus induces a map, which we have also denoted by ϕ, between their associated minimal tripotents.

Lemma 9.1. *Let $\phi\colon D \to D'$ be a Carathéodory distance preserving map between two finite-dimensional bounded symmetric domains of equal rank r. Assume that $\phi(0) = 0$. Let $e_1, \dots, e_r$ be a frame in D, and let $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with each $\lambda_i \in (-1, 1)$. Then,*

$$\phi(x) = \lambda_1 \phi(e_1) + \cdots + \lambda_r \phi(e_r). \tag{6}$$

Proof. Proposition 8.5 characterises when two minimal tripotents are opposite one another in terms of the Gromov product of their associated Busemann points. The latter is preserved by ϕ, and so $\phi(-e_i) = -\phi(e_i)$, for each i.

Orthogonality was also characterised in terms of the Gromov product, in Proposition 8.6, and thus $\phi(e_i)$ is orthogonal to $\phi(e_j)$, whenever $i \neq j$. This means that the $\phi(e_i)$ form a frame of D'.

For each i, we have $\Xi_{\phi(e_i)}(\phi x) = \Xi_{e_i}(x) = -\tanh^{-1} \lambda_i$, by Lemma 8.12, and a similar equation holds for the opposite tripotents $-e_i$. In particular,

$$\Xi_{\phi(e_i)}(\phi x) + \Xi_{-\phi(e_i)}(\phi x) = 0, \qquad \text{for all } i.$$

So, by Proposition 8.14, $\phi(x)$ lies in the flat defined by the $\phi(e_i)$. This means that $\phi(x)$ can be expressed as a real linear combination of the $\phi(e_i)$. Indeed, (6) must hold, using Lemma 8.12 again. □

This lemma has the following consequences.

Lemma 9.2. *Let $\phi\colon D \to D'$ be a Carathéodory distance-preserving map between two finite-dimensional bounded symmetric domains of equal rank. If F is a flat in D, then $\phi(F)$ is a flat in D'.*

Proof. Let x be a point in F. The map $\psi := g'_{-\phi(x)} \circ \phi \circ g_x$ preserves the Carathéodory metric and satisfies $\psi(0) = 0$. Moreover, $g_{-x}F$ is a flat containing 0. By Lemma 9.1, this flat is mapped by ψ to another flat, namely $g'_{-\phi(x)} \circ \phi(F)$. We deduce that $\phi(F)$ is a flat. □

Recall that, if $e_1, \dots, e_r$ is a frame of D and if $x \in D$ is given by $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with each $\lambda_i \in (-1, 1)$, then the Bergman distance between 0 and x is

$$d_B(0, x) = \big((\tanh^{-1} \lambda_1)^2 + \cdots + (\tanh^{-1} \lambda_r)^2\big)^{1/2}.$$

Lemma 9.3. *Let $\phi\colon D \to D'$ be a Carathéodory distance-preserving map between two finite-dimensional bounded symmetric domains of equal rank. Then, ϕ is also distance-preserving for the Bergman metric.*

Proof. Let x and y be points in D. The map $\psi := g'_{-\phi(x)} \circ \phi \circ g_x$ preserves the Carathéodory metric and satisfies $\psi(0) = 0$. Let $e_1, \dots, e_r$ be a frame of D such that $z := g_{-x}(y) = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with each $\lambda_i \in (-1, 1)$. By Lemma 9.1, we have $d'_B(0, \psi(z)) = d_B(0, z)$. Using that $g'_{\phi(x)}$ preserves the Bergman metric on D' and g_x preserves the Bergman metric on D, we get that $d'_B(\phi(x), \phi(y)) = d_B(x, y)$. □

In Theorem 11.1 of [11], it is shown that a surjective distance-preserving map from a Riemannian manifold onto itself it automatically a diffeomorphism that preserves the Riemannian structure. Actually, the same proof gives that a distance-preserving map from one C^∞ Riemannian manifold into another is C^∞ and preserves the Riemannian structure, even if it is not surjective.

Lemma 9.4. *Let $\phi\colon D \to D'$ be a Carathéodory distance-preserving map between two finite-dimensional bounded symmetric domains of equal rank. Then, ϕ is of differentiability class C^∞.*

Proof. By Lemma 9.3, the map ϕ is distance preserving for the Bergman metric. Under this metric, the bounded symmetric domains are C^∞ Riemannian manifolds, and distance preserving maps between such manifolds are C^∞. □

10. Proof of the the main results

We define the following relation on the set of minimal tripotents of a JB*-triple. We say that $u \sim v$ if there is a finite sequence $e_1, \dots, e_n$ of minimal tripotents such that $e_1 = u$ and $e_n = v$, and such that no consecutive two elements are orthogonal. This is an equivalence relation. Let $M_1, \dots, M_n$ be the equivalence classes of minimal tripotent, and let $V_1, \dots, V_n$, respectively, be the real linear spans of these sets. Observe that, if i and j are distinct, then every element of V_i is orthogonal to every element of V_j.

Recall that the set of minimal tripotents of a product of JB*-triples is the union of the sets of minimal tripotents of the factors. That is, each minimal tripotent is of the form $(e_1, 0)$ or $(0, e_2)$, where e_1 and e_2 are minimal tripotents of the respective factors. Minimal tripotents coming from different factors are of course orthogonal.

A *subtriple* of a JB*-triple is a closed subspace that is also closed with respect to the triple product.

Lemma 10.1. *Let V be a finite-dimensional JB*-triple, and let V_i be the subspaces defined above. Then, each V_i is an irreducible subtriple of V, and $V = V_1 \oplus \cdots \oplus V_n$.*

Proof. If u is a minimal tripotent and $\lambda \in \mathbb{C}$, with $|\lambda| = 1$, then $u \sim \lambda u$. It follows that each V_i is a complex linear subspace of V.

Since every element of V can be written as a linear combination of minimal tripotents, it can also be written as a sum of elements, one from each of the V_i.

To see that each V_i is closed under triple products, fix i and let $u, v, w \in V_i$. We can write $\{u, v, w\}$ as a real linear combination of elements of the form $\{e_1, e_2, e_3\}$, where each of the e_j are minimal tripotents in M_i. Consider such an element, and let c be in M_j, with $j \neq i$. So, c is orthogonal to each of e_1, e_2, and e_3. Therefore, by the Jordan identity,

$$\big\{c, c, \{e_1, e_2, e_3\}\big\} = \big\{\{c, c, e_1\}, e_2, e_3\big\} - \big\{e_1, \{c, c, e_2\}, e_3\big\} + \big\{e_1, e_2, \{c, c, e_3\}\big\} = 0.$$

So, c is orthogonal to $\{e_1, e_2, e_3\}$. We deduce that $\{u, v, w\}$ is orthogonal to every minimal tripotent apart from those in M_i, and it follows that $\{u, v, w\}$ lies in V_i.

If V_i were reducible, for some i, then we could partition its minimal tripotents into two subsets in such a way that every element of one subset was orthogonal to every element of the other; however this is clearly impossible. □

Recall that if $D = D_1 \times \cdots \times D_n$ is a product of bounded symmetric domains, and each D_i has rank r_i, then D has rank $r_1 + \cdots + r_n$, and every frame of D has exactly r_i minimal tripotents coming from D_i. Let $x := (x_1, \ldots, x_n)$ be in D, with each x_i in D_i. The Möbius transformation g_x decomposes as $g_x y = (g_{x_1} y_1, \ldots, g_{x_n} y_n)$, for all $y = (y_1, \ldots, y_n)$ in D.

Proof of Theorem 1.2. We first establish the result for the map $\psi := g'_{-\phi(0)} \circ \phi$. Here g'_x stands for the Möbius transformation of D' taking the origin to x. The statement for ϕ then follows immediately, because Möbius transformations act on each component separately. Observe that $\psi(0) = 0$.

Let e_1 and e_2 be minimal tripotents belonging to the same factor D_i of D. By Lemma 10.1, we have $e_1 \sim e_2$. From Propositions 8.5 and 8.6, and the fact that ψ preserves the Gromov product, we get that ψ, considered as a bijection between D and $\psi(D)$, preserves orthogonality in both directions. So, $\psi(e_1) \sim \psi(e_2)$. This implies that $\psi(e_1)$ and $\psi(e_2)$ are in the same factor of D', again using Lemma 10.1.

Define the map J as follows. For each factor D_i of D, choose a minimal tripotent e in D_i, and set $J(i) = k$, where D'_k is the factor of D' containing $\psi(e)$. From what we have seen in the previous paragraph, this map is well-defined.

To show that J is surjective, let D'_k be a factor of D' and take any frame $e_1, \ldots, e_r$ of D. So, $\psi(e_1), \ldots, \psi(e_r)$ is a frame of D', and hence contains a minimal tripotent of D'_k, say $\psi(e_j)$. We then have $J(l) = k$, where D_l is the factor of D containing e_j.

Fix a factor D'_i of D', and denote by P_i the projection onto this factor. For $x \in D_{J^{-1}(i)}$, define $\psi_i(x) := P_i \psi(y)$, where y is any element of D such that $y_{J^{-1}(i)} = x$. To show that this is well-defined, take any such y, and let $e_1, \ldots, e_r$ be a frame such that $y = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with each $\lambda_i \in (-1, 1)$. We order the elements of the frame so that $e_1, \ldots, e_s$ are the ones contained in $D_{J^{-1}(i)}$, with $s \leq r$. By the definition of the map J, the minimal tripotents $e_1, \ldots, e_s$ are exactly the elements of the frame that are mapped to tripotents of $D_{J^{-1}(i)}$. Applying Lemma 9.1, we have that $P_i \psi(y) = \lambda_1 \psi(e_1) + \cdots + \lambda_s \psi(e_s)$. So we see that $\psi_i(y)$ only depends on the components in $J^{-1}(i)$. □

Lemma 10.2. *Let $\phi \colon D \to D'$ be a Carathéodory distance-preserving map between two finite-dimensional bounded symmetric domains of equal rank r, contained in JB*-triple*

V and V', respectively. Assume that $\phi(0) = 0$. Then, ϕ is the restriction to D of a real linear map from V to V'.

Proof. By Lemma 9.4, the map ϕ is differentiable at the origin. Denote by $d_0\phi\colon V \to V'$ its differential there. This map is of course real linear. Let $x \in D$. So, there exists a frame $e_1, \dots, e_r$ such that $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with each $\lambda \in (-1, 1)$. For each $t \in [-1, 1]$, let

$$\gamma(t) := \sum_i t\lambda_i e_i.$$

So, γ is a smooth curve such that $\gamma(0) = 0$ and $\gamma(1) = x$. Its tangent vector at 0 is x. By Lemma 9.1,

$$\phi\big(\gamma(t)\big) := \sum_i t\lambda_i \phi(e_i), \qquad \text{for all } t \in (-1, 1).$$

The tangent vector of this curve at 0 is $\phi(x)$. Therefore, d_0 and ϕ agree at x. We conclude that ϕ is the restriction of d_0 to D. □

Lemma 10.3. *Let $\phi\colon D \to D'$ be a Carathéodory distance preserving map between two bounded symmetric domains of equal rank. Assume that $\phi(0) = 0$. Then, the induced map on the set of minimal tripotents is continuous.*

Proof. Let e be a minimal tripotent of the JB*-triple V of which D is the open unit ball, and let $\lambda \in (-1, 1)$. By considering any frame containing e and using Lemma 9.1, we get that $\phi(\lambda e) = \lambda\phi(e)$.

Now fix $\lambda \in (0, 1)$, and let e_n be a sequence of minimal tripotents of V converging to e in the norm topology of V. So, λe_n converges to λe. Hence, $\phi(\lambda e_n)$ converges to $\phi(\lambda e)$, since ϕ is continuous on D. Therefore, by what we have seen above, $\lambda\phi(e_n)$ converges to $\lambda\phi(e)$, and the conclusion follows upon dividing by λ. □

Proof of Theorem 1.3. By composing ϕ with a Möbius transformation if necessary, we can assume that $\phi(0) = 0$.

Recall that ϕ induces a map, also denoted by ϕ, from the set of Busemann points of D to those of D' with the property that every almost geodesic converging to a Busemann point ξ of D is mapped to an almost-geodesic converging to $\phi(\xi)$; see section 5. Moreover, by Lemma 7.5, singletons are mapped to singletons. Since the singletons are in one-to-one correspondence with the minimal tripotents, we get an induced map from the minimal tripotents of D to those of D'. By Lemma 10.3, this map, still denoted by ϕ, is continuous.

Combining Lemma 8.8 with Propositions 8.5 and 8.6, and using that ϕ preserves the Gromov product, we see that if u and v are minimal tripotents of D satisfying $u = \pm iv$, then $\phi(u) = \pm i\phi(v)$. In other words, for each minimal tripotent e of D, we have that either $\phi(ie) = i\phi(e)$ or $\phi(ie) = -i\phi(e)$. Define $c(e)$ to be i in the former case, and $-i$ in the latter. Thus, we obtain a function c from the set E of minimal tripotents of D to $\{i, -i\}$, This map is continuous since ϕ is continuous on E.

We have assumed that D is irreducible, and therefore E is connected. We deduce that the map c is constant, that is, takes either only the value i or only the value $-i$ on the whole of E. We will treat the former case; the latter case is similar.

Let $x \in D$. So, there exists a frame $e_1, \dots, e_r$ such that $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$, with each $\lambda_i \in (-1, 1)$. Hence, $ix = \lambda_1 ie_1 + \cdots + \lambda_r ie_r$. Applying Lemma 9.1, and using that

c is identically equal to i, we get

$$\begin{aligned}\phi(ix) &= \lambda_1\phi(ie_1) + \cdots + \lambda_r\phi(ie_r)\\ &= \lambda_1 i\phi(e_1) + \cdots + \lambda_r i\phi(e_r)\\ &= i\phi(x).\end{aligned}$$

Lemma 10.2 states that ϕ on D is the restriction of a real linear map. We have proved that it is actually the restriction of a complex linear map, and hence it is holomorphic. □

Kaup [12] showed that every *surjective* complex linear map between JB*-triples that preserves the norm is a triple homomorphism. This is not necessarily true however for maps that are not surjective. Nevertheless, Chu and Mackey [7] have shown the following.

Theorem 10.4 (Chu—Mackey)**.** *Let $\phi\colon V \to V'$ be a complex linear map between JB*-triples that preserves the norm. Denote by D and D' the open unit balls of V and V', respectively. Then, ϕ is a triple homomorphism if and only if $\phi(D)$ is invariant under the Möbius transformation $g_{\phi(x)}$, for all $x \in D$.*

For each $x \in D$, denote by S_x the geodesic symmetry in x. This means that, for each $y \in D$, the points y, x, and $S_x y$ lie on a Bergman geodesic, and $d_B(y,x) = d_B(x, S_x y)$, where d_B is the Bergman distance.

Lemma 10.5. *If w and z in D are such that $z = S_w 0$, then $g_z = S_w \circ S_0$.*

Proof. Let $w = w_1e_1 + \cdots + w_re_r$ be written in terms of some frame $e_1, \ldots, e_r$, with each $w_i \in (-1,1)$. So, $z = z_1e_1 + \cdots + z_re_r$, where $z_i = 2w_i/(1+w_i^2)$ for all i. By Lemma 8.10, $g_z = g_w \circ g_w$. We deduce that g_z maps $-w$ to w. Observe that the same is also true for $S_w \circ S_0$.

Now we compare the derivatives at $-w$. Let x be in the joint Peirce space V_{ij}, for some $i, j \in \{0, \ldots, r\}$. From Lemma 8.9,

$$\begin{aligned}\nabla g_z(-w)x &= B(z,z)^{1/2}B(-w,-z)^{-1}x\\ &= \frac{\left(1-z_i^2\right)^{1/2}\left(1-z_j^2\right)^{1/2}}{(1-w_iz_i)(1-w_jz_j)}x\\ &= x.\end{aligned}$$

We deduce that $\nabla g_z(-w)$ is the identity map. The same is clearly true for $\nabla(g_w \circ g_w)(-w)$. The result now follows from Cartan's uniqueness theorem. □

Lemma 10.6. *Let $\phi\colon D \to D'$ be a Bergman distance-preserving map between two finite-dimensional bounded symmetric domains. Then, $\phi(S_x y) = S_{\phi(x)}\phi(y)$, for all x and y in D.*

Proof. The points y, x, and $S_x y$ lie equally spaced along a Bergman geodesic in D. Therefore their images $\phi(y)$, $\phi(x)$, and $\phi(S_x y)$ lie equally spaced along a Bergman geodesic in D'. The conclusion follows. □

Proof of Theorem 1.4. By Lemma 10.2, the map ϕ is the restriction of a real linear map, which we also denote by ϕ, between V and V'. So it agrees with its derivative at the origin, which is a complex linear map since ϕ is assumed to be holomorphic.

Let $x = \lambda_1e_1 + \cdots + \lambda_re_r$ be in D, with $e_1, \ldots, e_r$ a frame of D, and each $\lambda_i \in (-1,1)$. The norm of x is $||x|| = \max(|\lambda_1|, \ldots, |\lambda_r|)$, and a similar expression holds for the norm in V'. From Lemma 9.1, we get that the norm is preserved by ϕ for elements of D. The same is true for all elements of V, by linearity.

Let z be in D. By Lemma 10.6, we have $S_{\phi(z)}\phi(w) = \phi(S_z w)$, for all $w \in D$. This shows that the set $\phi(D)$ is invariant under the point symmetry $S_{\phi(z)}$, for any $z \in D$.

Let x be in D, and let y be the midpoint of 0 and x along the unique Bergman geodesic between these two points. So, $S_y 0 = x$. From Lemma 10.6, we get that $S_{\phi(y)}0 = \phi(x)$. By Lemma 10.6, this implies that $g_{\phi(x)} = S_{\phi(y)} \circ S_0$. Since it is a composition of maps that each leave $\phi(D)$ invariant, $g_{\phi(x)}$ also leaves $\phi(D)$ invariant.

The conclusion now follows upon applying Theorem 10.4. □

References

[1] Marianne Akian, Stéphane Gaubert, and Cormac Walsh. The max-plus Martin boundary. *Doc. Math.*, 14:195–240, 2009.

[2] Gautam Bharali and Jaikrishnan Janardhanan. Proper holomorphic maps between bounded symmetric domains revisited. *Pac. J. Math.*, 271(1):1–24, 2014.

[3] María Burgos, Francisco J. Fernández-Polo, Jorge J. Garcés, Juan Martínez Moreno, and Antonio M. Peralta. Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples. *J. Math. Anal. Appl.*, 348(1):220–233, 2008.

[4] Cho-Ho Chu. *Jordan structures in geometry and analysis*, volume 190 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, 2012.

[5] Cho-Ho Chu. *Bounded symmetric domains in Banach spaces*. World Scientific, 2021.

[6] Cho-Ho Chu, María Cueto-Avellaneda, and Bas Lemmens. Horofunctions and metric compactification of noncompact Hermitian symmetric spaces, 2022. Preprint.

[7] Cho-Ho Chu and Michael Mackey. Isometries between JB*-triples. *Math. Z.*, 251(3):615–633, 2005.

[8] Y. Friedman and B. Russo. Structure of the predual of a JBW*-triple. *J. Reine Angew. Math.*, 356:67–89, 1985.

[9] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 183–213, Princeton, N.J., 1981. Princeton Univ. Press.

[10] Thomas Haettel, Anna-Sofie Schilling, Cormac Walsh, and Anna Wienhard. Horofunction compactifications of symmetric spaces, 2018.

[11] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure Appl. Math., Academic Press*. Academic Press, New York, NY, 1978.

[12] Wilhelm Kaup. A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. *Math. Z.*, 183:503–529, 1983.

[13] Sung-Yeon Kim and Aeryeong Seo. Holomorphicity of totally geodesic Kobayashi isometry between bounded symmetric domains, 2022.

[14] Ottmar Loos. *Bounded symmetric domains and Jordan pairs*. Mathematical Lectures. University of California, Irvine, 1977.

[15] Marc A. Rieffel. Group C^*-algebras as compact quantum metric spaces. *Doc. Math.*, 7:605–651 (electronic), 2002.

[16] Cormac Walsh. Hilbert and Thompson geometries isometric to infinite-dimensional Banach spaces. *Ann. Inst. Fourier*, 68(5):1831–1877, 2018.

School of Mathematics, Statistics & Actuarial Science, University of Kent, Canterbury CT2 7NX, UK

Email address: B.Lemmens@kent.ac.uk

Inria and CMAP, Ecole Polytechnique, CNRS, 91128, Palaiseau, France

Email address: cormac.walsh@inria.fr