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# ERGODICITY OF COCYLES OVER 2-DIMENSIONAL ROTATIONS

NICOLAS CHEVALLIER AND JEAN-PIERRE CONZE

ABSTRACT. We study recurrence and ergodicity of cocycles with values in  $\mathbb{R}^d$ ,  $d \geq 1$ , over rotations by badly approximable irrational numbers on  $\mathbb{T}^\rho$ ,  $\rho > 1$ . The discontinuities of the functions generating the cocycles also satisfy a Diophantine condition. For simplicity of notation we mainly consider the cases  $\rho = 2$ ,  $d = 1$  and  $2$ .

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## Introduction

Let  $T : X \rightarrow X$  be an ergodic measure preserving transformation on a probability space  $(X, \mathcal{B}, \mu)$ . Let  $\varphi$  be a measurable function on  $X$  with values in  $\mathbb{R}^d$ ,  $d \geq 1$ .

The ergodic sums of  $\varphi$  under the iteration of  $T$ , denoted by  $\varphi_n$  or  $S_n\varphi$ , are defined as

$$\varphi_n(x) := \sum_{j=0}^{n-1} \varphi(T^j x), n \geq 1, \varphi_0(x) = 0.$$

The sequence  $(\varphi_n, n \geq 0)$  will be called a ‘‘cocycle’’ (over the dynamical system  $(X, \mu, T)$ ) and denoted by  $(T, \varphi)$ , or  $(\varphi_n)$ , or  $(\varphi_{n,\alpha})$ , when  $T = T_\alpha$  is a rotation by  $\alpha$  on a torus.

We denote by  $\tilde{T}_\varphi$  the skew-product map (also called ‘‘cylinder map’’)

$$\tilde{T}_\varphi : (x, z) \rightarrow (Tx, z + \varphi(x)),$$

acting on  $X \times \mathbb{R}^d$  endowed with the infinite invariant measure  $\tilde{\mu}$  product of  $\mu$  by the Lebesgue measure  $\lambda$  (also denoted  $dz$ ) on  $\mathbb{R}^d$ .

The cocycle  $(T, \varphi)$  is said to be ergodic if the dynamical system  $(X \times \mathbb{R}^d, \tilde{\mu}, \tilde{T}_\varphi)$  is ergodic.

In what follows, after general reminders, we take for  $T$  a rotation  $T_\alpha$  on the torus  $X = \mathbb{T}^\rho = \mathbb{R}^\rho / \mathbb{Z}^\rho$ ,  $\rho \geq 1$ , with its Haar measure (denoted  $\mu$  or  $dx$ ):

$$T = T_\alpha : x = (x_1, \dots, x_\rho) \rightarrow (x_1 + \alpha_1, \dots, x_\rho + \alpha_\rho),$$

where  $\alpha = (\alpha_1, \dots, \alpha_\rho) \in \mathbb{T}^\rho$  is totally irrational (i.e.,  $1, \alpha_1, \dots, \alpha_\rho$  are linearly independent over  $\mathbb{Q}$ ).

Given a cocycle  $(T, \varphi)$ , the main questions are: is it recurrent, is it ergodic?

A first remark is that, under a mild Diophantine condition on  $\alpha$ , too much regularity for  $\varphi$  is an obstruction to ergodicity. Consequently, the presence of discontinuities plays a role in the construction of explicit ergodic cocycles.

Nevertheless, let us mention that three examples of ergodic cocycles over a 1-dimensional rotation are given in [14]: an analytic, a ‘‘smooth’’ and a continuous cocycle. The latter example is constructed over an arbitrary irrational rotation.

For  $\rho = 1$ , in particular in the class of step functions, many examples of ergodic cocycles have been given from the late Seventies and later (cf. for instance [17, 12, 7, 9]). For a one-dimensional rotation, Koksma’s inequality gives a uniform bound along the denominators of the rotation for ergodic sums of centered functions with bounded variation. It provides a way to prove the existence of non-trivial essential values and then ergodicity.

For  $\rho > 1$ , there are fewer results due to the lack of such an inequality (cf. [24]). An alternative approach is based on Lebesgue density theorem, existence of recurrence times of the cocycle and control of its discontinuities which introduces ‘‘bad Diophantine approximation’’ conditions.

The main aim here is to give examples of ergodic cocycles over rotations on  $\mathbb{T}^\rho$ ,  $\rho > 1$ . The method of proof will be based on Lebesgue density theorem and recurrence times as

mentioned above. For simplicity of notation we will present mainly examples on  $\mathbb{T}^2$ . In addition, we also review and extend some results on recurrence of cocycles over rotations.

The main recurrence result of Section 2, Theorem 2.5, applies to all rotations outside a small exceptional set, while the ergodicity results of Section 3, Theorems 3.5, 3.9 and 4.3, require badly approximable rotations and discontinuities.

## 1. Preliminaries

### 1.1. Recurrence and essential values.

For the sake of completeness, we start with reminders summarizing some results, in particular on recurrence. A basic reference on cocycles is K. Schmidt's seminal work [19].

Let  $(\varphi_n)$  be a cocycle generated by a measurable function  $\varphi$  with values in  $\mathbb{R}^d$  over an ergodic dynamical system  $(X, \mathcal{B}, \mu, T)$ .<sup>1</sup>

Recall that  $(\varphi_n)$  is recurrent if, for every neighborhood  $V$  of the origin in  $\mathbb{R}^d$ , for  $\mu$ -a.e.  $x \in X$ , there is a strictly increasing sequence  $(n_k(x))$  in  $\mathbb{N}$  such that  $\varphi_{n_k(x)}(x) \in V$ . It is recurrent if and only if  $\tilde{T}_\varphi$  is conservative. It is transient if  $\lim_n |\varphi_n(x)| = +\infty$  for  $\mu$ -a.e.  $x$ . A cocycle  $(\varphi_n)$  over an ergodic dynamical system is either recurrent or transient

Recall also that  $a \in \mathbb{R}^d \cup \{\infty\}$  is called *an essential value* of  $(\varphi_n)$  if, for every neighborhood  $V$  of  $a$ , for every measurable subset  $B$  of positive measure in  $X$ , there is  $n \in \mathbb{N}$  such that

$$(1) \quad \mu(B \cap T^{-n}B \cap \{x : \varphi_n(x) \in V\}) > 0.$$

We denote by  $\overline{\mathcal{E}}(\varphi)$  the set of essential values and by  $\mathcal{E}(\varphi)$  the set of finite essential values. Observe that  $\mathcal{E}(\varphi)$  contains always 0 and that  $\overline{\mathcal{E}}(\varphi) = \{0, \infty\}$  if the cocycle is transient.

Suppose that  $(\varphi_n)$  is recurrent. The group  $\mathcal{P}(\varphi)$  of periods of the measurable  $\tilde{T}_\varphi$ -invariant functions is a closed subgroup of  $\mathbb{R}^d$  which coincides with  $\mathcal{E}(\varphi)$ . (See [19] or [1]).

*Therefore, proving ergodicity of a cocycle  $(\varphi_n)$  with values in  $\mathbb{R}^d$  amounts to showing that  $\mathcal{E}(\varphi)$  contains elements generating a dense subgroup of  $\mathbb{R}^d$ .*

#### *Induced map and induced cocycle*

Let  $B \subset X$  be a set of positive measure. On  $B$  equipped with the measure  $\mu_B = \mu(B)^{-1}\mu|_B$ , the induced transformation is  $T_B(x) = T^{R(x)}(x)$ , with  $R(x) = R_B(x) := \inf\{j \geq 1 : T^j x \in B\}$ . It is well defined for a.e.  $x \in B$  if the system is conservative, in particular (by Poincaré recurrence property) if it has a finite measure. Clearly if  $T$  is conservative, then  $T_B$  is conservative for every  $B$  of positive measure. The successive return times of a point  $x$  in  $B$  are  $R_1(x) = R(x), R_2(x) = R(x) + R(T^{R(x)}x), \dots, R_n(x) = R(x) + R_{n-1}(T^{R(x)}x), \dots$

---

<sup>1</sup>In this general subsection, in view of Lemma 1.1,  $(X, \mathcal{B}, \mu)$  is a Radon measure space. The values of  $\varphi$  could be in a locally compact group  $G$ , but we restrict ourselves to  $G = \mathbb{R}^d$  provided with a norm  $|\cdot|$ .

Let  $\varphi$  be a measurable function on  $X$ . The "induced" cocycle (for the induced map  $T_B$  on  $B$ ) is, for  $n \geq 1$ ,

$$\varphi_n^B(x) := \varphi^B(x) + \varphi^B(T_B x) \cdots + \varphi^B(T_B^{n-1} x) = \varphi_{R_n(x)}(x), x \in B,$$

$$\text{with } \varphi^B(x) := \sum_{j=0}^{R(x)-1} \varphi(T^j x) = \varphi_{R(x)}(x).$$

We see that  $a \in \mathbb{R}^d \cup \{\infty\}$  is an essential value if, for every neighborhood  $V$  of  $a$ , and every measurable subset  $B$  of positive measure in  $X$ , there is  $n \in \mathbb{N}$  such that  $\mu(\{x \in B : \varphi_n^B(x) \in V\}) > 0$ .

**Remarks 1.** a) A cocycle  $(\varphi_n)$  is recurrent, if and only if, for each neighborhood  $V$  of the origin and each  $B \subset X$  of positive measure, there exists  $n \geq 1$  such that

$$(2) \quad \mu(B \cap T^{-n} B \cap (\varphi_n \in V)) > 0.$$

b)  $\overline{\mathcal{E}}(\varphi) = \{0\}$  if and only if  $\varphi$  is a coboundary (cf. [19]), meaning that there exists a measurable function  $\psi : X \rightarrow \mathbb{R}^d$  such that  $\varphi = \psi - \psi \circ T$ .

c) Two cocycles which differ by a coboundary have the same set of essential values.

d) A transient cocycle is never ergodic when  $(X, \mu, T)$  is aperiodic (i.e. such that the set of periodic points is  $\mu$ -negligible). Indeed, let  $a \in \mathbb{R}^d$  be different from the origin and  $V$  a neighborhood of  $a$ . By transience there is a set  $A$  of positive measure such that, for some  $N \geq 1$ ,  $\varphi_n(x) \notin V$  for  $n \geq N$  and all  $x \in A$ . By Rohklin's lemma for aperiodic dynamical systems, there is a set  $B \subset A$  such that the return time in  $B$  is  $> N$ . This shows that (1) is not satisfied and  $a \notin \mathcal{E}(\varphi)$ . Hence  $(\varphi_n)$  is not ergodic.

The following lemma will be useful in the proof of ergodicity.

**Lemma 1.1.** [19, Proposition 3.8] *If  $K \subset \mathbb{R}^d$  is a compact set such that  $K \cap \mathcal{E}(\varphi) = \emptyset$ , there exists a set  $B$  of positive measure such that  $\mu(B \cap T^{-n} B \cap (\varphi_n \in K)) = 0, \forall n \in \mathbb{Z}$ .*

*Proof.* For the sake of completeness, we give a proof. It uses induced cocycles.

The hypothesis implies the existence, for every  $z \in K$ , of a subset  $B_z$  of positive measure in  $X$  and of a neighborhood  $U_z$  of the origin such that  $\varphi_n^{B_z}(x) \notin U_z + z, \forall n \geq 0, \forall x \in B_z$ .

Let  $V_z$  be a neighborhood of the origin such that  $V_z + V_z \subset U_z$ . By compactness of  $K$ , there is a finite number of points  $z_1, \dots, z_r$  such that  $K \subset \cup_{i=1}^r (V_{z_i} + z_i)$ .

We proceed by induction on  $r$ , denoting simply  $B_i, U_i, V_i$  subsets and neighborhoods.

Suppose we have constructed a subset  $D = D_{r-1}$  of positive measure such that the values of the cocycle  $(\varphi_D^n)$  never belong to  $V_i + z_i$ , for  $i = 1, \dots, r-1$ . We are going to construct a subset  $D_r$  of positive measure of  $D$  such that the values of  $(\varphi_{D_r}^n)$  never belong to  $V_r + z_r$ .

Since  $D_r$  is a subset of  $D$ , the values of the induced cocycle  $(\varphi_{D_r}^n)$  are contained in those of  $(\varphi_D^n)$ , so they still never belong to  $V_i + z_i$ , for  $i = 1, \dots, r-1$ .

The set  $D_r$  is the set  $B$  of the statement, since, for  $x \in D_r$ , we have  $\varphi_{D_r}^n(x) \notin \cup_{i=1}^r (V_i + z_i)$ , hence  $\varphi_{D_r}^n(x) \notin K$  and we will be done.

It remains to construct  $D_r$ . By ergodicity of  $T$ , there is  $k \geq 1$  and  $D' \subset D$  of positive measure such that  $T^k D' \subset B_r$ . By Lusin's theorem, there is a subset  $D_r$  of  $D'$  of positive measure such that  $\varphi_k(x) - \varphi_k(y) \in V_r, \forall x, y \in D_r$ .

For  $k, n \geq 0$ , we have  $\varphi_n(x) = \varphi_k(x) + \varphi_n(T^k x) - \varphi_k(T^n x)$ . Therefore, if  $x, T^n x \in D_r$ , then  $\varphi_k(T^n x) - \varphi_k(x) \in V_r$  and therefore  $\varphi_n(T^k x) \in V_r + \varphi_n(x)$ .

For  $x \in D_r$  and  $n$  such that  $T^n x \in D_r$ , as  $T^k x, T^{k+n} x \in B_r$ ,  $\varphi_n(T^k x)$  is a value of the induced cocycle  $\varphi_{B_r}^n$  and therefore  $\notin U_r + z_r$ . It follows that  $\varphi_n(x) \notin V_r + z_r$ , because, otherwise,  $\varphi_n(T^k x) \in V_r + V_r + z_r \subset U_r + z_r$ , a contradiction.

We conclude that  $\varphi_{D_r}^n$  never takes its values in  $V_r + z_r$  and  $D_r$  has the desired property.  $\square$

### Regularity of a cocycle

Let  $(\varphi_n)$  be a recurrent cocycle. The function  $x \rightarrow \varphi(x) \bmod \mathcal{E}(\varphi)$  on  $X$  defines a cocycle with values in  $\mathbb{R}^d / \mathcal{E}(\varphi)$  whose finite essential values, as a quotient of  $\varphi$ , belong to the class of  $\mathcal{E}(\varphi)$ , hence are trivial. By Remark 1.a) either  $\varphi \bmod \mathcal{E}(\varphi)$  is a coboundary, or  $\mathbb{R}^d / \mathcal{E}(\varphi)$  is not compact and the set of essential values of  $\varphi \bmod \mathcal{E}(\varphi)$  is  $\{0, \infty\}$ .

In the first case, the cocycle defined by  $\varphi$  with values in  $\mathbb{R}^d$  is said to be *regular*. There exists then a measurable map  $\eta : X \rightarrow \mathbb{R}^d$  such that the cocycle  $\psi := \varphi + \eta - T\eta$  takes a.e. its values in  $\mathcal{E}(\varphi)$ . Moreover, since the group of periods of the  $\tilde{T}_\psi$ -invariant functions is still  $\mathcal{E}(\varphi)$ , it follows that  $\tilde{T}_\psi : (x, z) \rightarrow (Tx, z + \psi(x))$  is ergodic for its action on  $X \times \mathcal{E}(\varphi)$ . Therefore regularity for a cocycle  $\varphi$  means that, if it is not ergodic, it can be reduced up to a coboundary to an ergodic cocycle with values in a closed subgroup.

### 1.2. Diophantine conditions, $\text{Bad}_{\mathbb{Z}}(\alpha)$ .

**Notation.** The Hausdorff dimension of a set  $E \subset \mathbb{R}^d$  is denoted by  $\dim_H E$ .

For  $u \in \mathbb{R}$ , let  $\{u\} = u - k$  if  $u \in [k, k + 1[, k \in \mathbb{Z}$ , denote its fractional part and let  $\|u\| := \inf(\{u\}, 1 - \{u\}) = \inf_{n \in \mathbb{Z}} |u - n|$  denote its distance to  $\mathbb{Z}$ . The set  $\mathbb{Z}^2 \setminus \{(0, 0)\}$  is denoted by  $\mathbb{Z}_*^2$ .

For  $h = (h_1, h_2)$  and  $x = (x_1, x_2)$  in  $\mathbb{R}^2$ , we denote by  $\langle h, x \rangle$  or  $h.x$  the scalar product  $h_1 x_1 + h_2 x_2$ .

If  $\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$ , with partial quotients  $(a_n)_{n \geq 1}$ , is an irrational number, its denominators are  $q_n$ :  $q_0 = 1, q_1 = a_1$  and  $q_{n+1} = a_{n+1} q_n + q_{n-1}$  for  $n \geq 1$ .

We recall now some facts about Diophantine properties of irrational numbers.

For  $s \geq 0$ ,  $D(s)$  denotes the set of irrational numbers  $\alpha$  such that, for a finite constant  $A = A(\alpha, s)$ , the partial quotients of  $\alpha$  satisfy

$$(3) \quad a_n \leq A n^s, \forall n \geq 1.$$

By a theorem of Borel-Bernstein, a.e.  $\alpha$  is in  $D(s)$  for every  $s > 1$ . Moreover, thanks to the inequality  $q_{n+1} \|q_n \alpha\| \geq \frac{1}{2}$ , we see that for all  $n \geq 0$  and all  $q_n \leq k < q_{n+1}$ , we have  $\|k\alpha\| \geq \|q_n \alpha\| \geq \frac{1}{2q_{n+1}} \geq \frac{1}{4a_{n+1}k}$ . As the sequence  $(q_n)$  grows at least exponentially,

it follows that, if  $\alpha$  satisfies (3), there is a constant  $c > 0$  such that

$$(4) \quad \|k\alpha\| \geq \frac{c}{k(\log k)^s}, \quad \forall k > 1.$$

Recall also that the type (or Diophantine exponent) of an irrational number  $\alpha$  is the real  $\eta \geq 1$  such that

$$(5) \quad \inf_k [k^{\eta-\varepsilon} \|k\alpha\|] = 0, \quad \inf_k [k^{\eta+\varepsilon} \|k\alpha\|] > 0, \quad \forall \varepsilon > 0.$$

The type of  $\alpha$  satisfying (4) for some  $s > 1$  is 1 and therefore by what precedes the type of a.e.  $\alpha$  is 1.

More directly, it can be observed that, if  $\alpha$  is not of type 1, there is an integer  $r \geq 1$  such that  $k^{1+1/r} \|k\alpha\| \leq 1$  for infinitely many  $k$ . For each  $n$ , the set of  $\alpha$ 's satisfying the latter property is negligible by the Borel-Cantelli lemma.

### Badly approximable numbers

Recall that a number  $\theta$  is *badly approximable* ( $\theta \in \text{Bad}$ ), if

$$(6) \quad \exists c > 0 : \|q\theta\| \geq \frac{c}{|q|}, \quad \forall q \in \mathbb{Z} \setminus \{0\}.$$

This is equivalent for  $\theta$  to have bounded partial quotients (bpq). Clearly the type of numbers in  $\text{Bad}$  is 1. The set  $\text{Bad}$  has Lebesgue measure 0 and Hausdorff dimension 1, see [13].

The set  $\text{Bad}_{\mathbb{Z}}(\theta)$  of *badly approximable numbers with respect to an irrational  $\theta$*  is

$$\text{Bad}_{\mathbb{Z}}(\theta) = \{x \in [0, 1] : \exists c(x) > 0 : \|q\theta - x\| \geq \frac{c(x)}{|q|}, \forall q \in \mathbb{Z} \setminus \{0\}\}.$$

Observe that  $\text{Bad}_{\mathbb{Z}}(\theta) = -\text{Bad}_{\mathbb{Z}}(\theta)$  and that  $0 \in \text{Bad}_{\mathbb{Z}}(\theta)$  is equivalent to  $\theta \in \text{Bad}$ .

The set  $\text{Bad}_{\mathbb{Z}}(\theta)$  has measure 0, but its Hausdorff dimension  $\dim_H \text{Bad}_{\mathbb{Z}}(\theta)$  is 1. Actually (cf. Proposition 5.1 in appendix), the set of  $n$ -tuples  $(\beta_1, \dots, \beta_n)$  which are in  $\text{Bad}_{\mathbb{Z}}(\theta)$  as well as the differences  $\beta_j - \beta_i$  for all  $i, j, i \neq j$ , is big in the sense that its Hausdorff dimension is  $n$ .

### Role of Diophantine conditions in the question of ergodicity

Let  $(\varphi_n)$  be a recurrent cocycle with values in  $\mathbb{R}^d$  over a rotation  $x \rightarrow x + \alpha$  on  $\mathbb{T}^\rho$ . It is easily seen that first possible obstruction to ergodicity is when some component  $\varphi^i$  of the function  $\varphi$  generating the cocycle is a coboundary (meaning that there is a measurable function  $\psi : \mathbb{T}^\rho \rightarrow \mathbb{R}$  such that  $\varphi^i = \psi - T_\alpha \psi$  a.e.).

The reduction of a component to a coboundary is related to the decay of its Fourier coefficients. For an example, when  $\rho = 1$ , we can use (see [7, Lemma 2.2] for a proof):

**Lemma 1.2.** *If  $\alpha$  is an irrational of type  $\eta$ , then  $\sum_{k \geq 1} \frac{1}{k^{\eta+\delta}} \frac{1}{\|k\alpha\|} < \infty$  for every  $\delta > 0$ .*

**Proposition 1.3.** *If  $\alpha$  is of type  $\eta$  and if  $\varphi(x) = \sum_{n \neq 0} c_n(\varphi) e^{2\pi i n x}$  is such that  $c_n(\varphi) = O(n^{-(\eta+\delta)})$  for some  $\delta > 0$ , then  $\varphi$  is a coboundary:  $\varphi = T_\alpha \psi - \psi$ , with  $\psi$  continuous.*

*Proof.* The Fourier coefficients of an integrable solution  $\psi$  of the coboundary equation  $\varphi = T_\alpha \psi - \psi$ , are given by  $c_n(\psi) = \frac{c_n(\varphi)}{e^{2\pi i n \alpha} - 1}$ . By Lemma 1.2 we have

$$\sum_{n \neq 0} |c_n(\psi)| \leq \sum_{n \neq 0} \frac{|c_n(\varphi)|}{\|n\alpha\|} \leq C \sum_{n \neq 0} \frac{1}{n^{\eta+\delta}} \frac{1}{\|n\alpha\|} < +\infty.$$

Therefore the coboundary equation has a solution which is continuous.  $\square$

For example,  $\varphi : x \rightarrow x(1-x) - \frac{1}{6}$  coincides on  $[0, 1]$  with the continuous, 1-periodic and 1-lipschitz function on  $\mathbb{R}$  given by the Fourier expansion  $\frac{-1}{\pi^2} \sum_{n \geq 1} \frac{\cos(2\pi n x)}{n^2}$ .

If  $\alpha$  is of type  $< 2$ , then this function  $\varphi$  is a coboundary for the rotation by  $\alpha$  and the cocycle  $(\varphi_{n,\alpha})$  is not ergodic.

*A non-regular BV cocycle*

As an illustration of the role of Diophantine properties, let us also mention an example of a non regular (hence non ergodic) cocycle (cf. [7]).

If  $\alpha$  is an irrational  $\notin \text{Bad}$ , it can be shown that there are  $\beta, r$  in  $]0, 1[$  such that

$$\varphi : x \rightarrow \varphi(x) = 1_{[0,\beta]}(x \bmod 1) - 1_{[0,\beta]}(x + r \bmod 1)$$

satisfies  $\bar{\mathcal{E}}(\varphi) = \{0, \infty\}$ . This implies that the cocycle  $(\varphi_{n,\alpha})$  is not regular and that  $\tilde{T}_{\alpha,\varphi}$  is not ergodic on  $\mathbb{T}^1 \times \mathbb{Z}$  endowed with the product of the Lebesgue measure on  $\mathbb{T}^1$  by the counting measure on  $\mathbb{Z}$ .

## 2. About recurrence, examples

### A sufficient condition for recurrence

The question of recurrence for a cocycle with values in  $\mathbb{R}^d$ ,  $d \geq 1$  is natural and plays a key role in the proof of ergodicity. Let us first consider the general case of a cocycle  $(\varphi_n)$  generated by a function  $\varphi : X \rightarrow \mathbb{R}^d$ ,  $d \geq 1$ , over an ergodic dynamical systems  $(X, \mu, T)$ .

If  $d = 1$  and  $\varphi$  is integrable, a necessary and sufficient condition for the recurrence of  $(\varphi_n)$  is  $\int_X \varphi d\mu = 0$ . (cf. [3])

For  $d \geq 2$ , the question of recurrence is more difficult, but sometimes recurrence can be deduced from the growth rate of the cocycle  $(\varphi_n)$ . The following general lemma gives a simple sufficient condition for recurrence (cf. [6]).

For  $\varphi : X \rightarrow \mathbb{R}^d$  in  $L^2(\mu)$ , denote by  $\|\varphi_n\|_2 := (\int_X |\varphi_n(x)|^2 d\mu(x))^{1/2}$  its  $L^2$ -norm.

**Lemma 2.1.** *Let  $(\varphi_n)$  be a cocycle over a dynamical system  $(X, \mathcal{B}, \mu, T)$  with values in  $\mathbb{R}^d$ . If there exist a strictly increasing sequence of integers  $(k_n)$  and a sequence of real numbers  $(\delta_n > 0)$  such that:  $\lim_{n \rightarrow \infty} \mu(x : |\varphi_{k_n}(x)| \geq \delta_n) = 0$  and  $\delta_n = o(n^{1/d})$ , then the cocycle is recurrent.*

*In particular, if  $\varphi \in L^2(\mu)$  and  $\|\varphi_n\|_2 = o(n^{1/d})$ , the cocycle is recurrent.*



### Cocycles over 1-dimensional rotations

Let  $\mathcal{C}$  be the class of centered real valued functions with bounded variation (BV) on  $\mathbb{T}$ . It contains the centered step functions with a finite number of discontinuities. If  $\varphi$  belongs to  $\mathcal{C}$ , its Fourier coefficients  $c_r(\varphi)$  satisfy:

$$(7) \quad c_r(\varphi) = \frac{\gamma_r(\varphi)}{r}, \quad \forall r \neq 0, \quad \text{with } K(\varphi) := \sup_{r \neq 0} |\gamma_r(\varphi)| < +\infty.$$

For  $\varphi \in \mathcal{C}$  with variation  $V(\varphi)$ , for a rotation  $T_\alpha$  on  $\mathbb{T}$  and any denominator  $q$  of  $\alpha$ , the ergodic sum  $\varphi_q$  over  $T_\alpha$  satisfies:

$$(8) \quad \|\varphi_q\|_\infty = \sup_x \left| \sum_{j=0}^{q-1} f(x + j\alpha) \right| \leq V(\varphi) \quad (\text{Koksma's inequality}).$$

If  $\varphi$  centered in  $L^2(\mathbb{T})$  satisfies (7), then  $\|\varphi_q\|_2 \leq 2\pi K(\varphi)$ . Indeed, for  $\psi(x) = \{x\} - \frac{1}{2}$ , we have  $\|\psi_q\|_2 \leq \|\psi_q\|_\infty \leq V(\psi) = 1$ , by (8). Hence for  $\varphi$ , it holds:

$$\|\varphi_q\|_2 = \left( \sum_{r \neq 0} \frac{|\gamma_r(\varphi)|^2}{r^2} \left| \frac{\sin \pi q r \alpha}{\sin \pi r \alpha} \right|^2 \right)^{\frac{1}{2}} \leq K(\varphi) \left( \sum_{r \neq 0} \frac{1}{r^2} \left| \frac{\sin \pi q r \alpha}{\sin \pi r \alpha} \right|^2 \right)^{\frac{1}{2}} = 2\pi K(\varphi) \|\psi_q\|_2 \leq 2\pi K(\varphi).$$

For  $\varphi \in \mathcal{C}$ , by (8) we get a bound on the growth of the ergodic sum  $\varphi_n$  for a.e  $\alpha$ :

**Proposition 2.2.** *Let  $\varphi$  be a centered BV function on  $\mathbb{T}$ . If (3) is satisfied for some  $s \geq 0$  (a condition which holds for a.e.  $\alpha$ ), there is a constant  $K_s > 0$  such that:*

$$(9) \quad \|\varphi_n\|_\infty \leq K_s (\log n)^{1+s}, \quad \forall n \geq 2.$$

When  $\alpha$  is of type 1, for every  $\varepsilon > 0$  there is a constant  $K(\varepsilon) > 0$  such that:

$$(10) \quad \|\varphi_n\|_\infty \leq K(\varepsilon) n^\varepsilon, \quad \forall n \geq 1.$$

*Proof.* Let  $(q_k)$  be the denominators of  $\alpha$ . For  $n \geq 1$ , let  $m = m(n)$  be the integer such that  $n \in [q_{m(n)}, q_{m(n)+1}[$ . As the growth of the sequence  $(q_n)$  is at least exponential, we have  $m(n) = O(\log n)$ .

Now (9) and (10) follow easily from Koksma's inequality and the  $\alpha$ -Ostrowski's representation of the integers (cf [18]) which reads:

$$\text{if } n < q_{m+1}, \quad n = \sum_{k=0}^m b_k q_k, \quad \text{with } 0 \leq b_0 \leq a_1 - 1, \quad 0 \leq b_k \leq a_{k+1} \text{ for } 1 \leq k \leq m. \quad \square$$

If  $\varphi$  is a centered function satisfying (7), the previous proposition is valid with the  $L^2$ -norm instead of the uniform norm.

#### 2.1. Examples of recurrent cocycles over rotations in dimension $\geq 1$ .

Now we give examples where recurrence can be proved or disproved in dimension  $\geq 1$ .

2.1.1. **Examples and counterexamples.**

For cocycles over 1-dimensional rotations, (8) can be used for cocycles with values in  $\mathbb{R}^d$ ,  $d \geq 1$ . When the dimension  $\rho$  is  $> 1$ , an estimation of the rate of growth of  $(\varphi_n)$  can be obtained in some cases by Fourier analysis methods under an hypothesis on  $\alpha$ .

**Example 1.** (Case  $\rho = 1$ ) If the cocycle  $(\varphi_n)$  with values in  $\mathbb{R}^d$  is generated over a one dimensional rotation by  $\varphi$  centered with BV components (or more generally satisfying (7)), then it is recurrent for any ergodic rotation and any  $d \geq 1$  by Koksma's inequality.

**Example 2.** (Case  $d = \rho > 1$ ) Let  $\varphi = (\varphi^1, \dots, \varphi^d)$  be such that  $\varphi^j$ , for each  $j$ , is a centered BV function of the one-dimensional variable  $x_j \in \mathbb{T}$ .

In this example, the cylinder map on  $\mathbb{T}^\rho \times \mathbb{R}^\rho : (x, z) \rightarrow (x + \alpha, z + \varphi(x))$  is the product of the cylinder maps on  $\mathbb{T} \times \mathbb{R} : (x_i, z_i) \rightarrow (x_i + \alpha_i, z_i + \varphi^i(x_i))$ . Recurrence follows then immediately for a large class of rotations from Lemma 2.1 and Proposition 2.2:

**Proposition 2.3.** *If the components  $\varphi^j$  are in the class  $\mathcal{C}$  (or more generally satisfy (7)), for  $\alpha = (\alpha_1, \dots, \alpha_\rho)$ , with each  $\alpha_i$  of type 1, the cocycle is recurrent.*

As a.e.  $\alpha$  is of type 1, the conclusion holds for a.e.  $\alpha$ . However, as recalled below, for  $\varphi$  with component in  $\mathcal{C}$ , recurrence can fail in dimension  $\geq 2$  for special choices of  $\alpha$ .

**Example 3.** *Transient cocycles over a 2-dimensional rotation*

In [24] J.-C. Yoccoz constructed a centered transient cocycle given by an analytical function over a particular 2-dimensional rotation.

The following centered cocycle is another example of transient cocycle (cf. [6, Theorem 4.1]): Let  $\varphi : \mathbb{T}^2 \rightarrow \mathbb{R}^2$  be the function  $\varphi(x) = (\varphi^1(x_1), \varphi^2(x_2))$ , with  $\varphi^1 = \varphi^2 = 1_{[0, \frac{1}{3}]}(\{\cdot\}) - \frac{1}{3}$ . There exists an ergodic rotation in  $\mathbb{T}^2$ ,  $x \rightarrow x + \alpha$ ,  $\alpha = (\alpha_1, \alpha_2)$ , such that

$$\left| \sum_{k=0}^{n-1} \varphi^1(x_1 + k\alpha_1) \right| + \left| \sum_{k=0}^{n-1} \varphi^2(x_2 + k\alpha_2) \right| \xrightarrow{n \rightarrow +\infty} +\infty, \text{ for a.e. } (x_1, x_2) \in \mathbb{T}^2.$$

*A question:* Recall that a pair  $\alpha \in \mathbb{R}^2$  is singular if  $\forall \varepsilon > 0, \exists Q > 1, \forall q > Q, \exists n \leq q, d(n\alpha, \mathbb{Z}^2) \leq \varepsilon q^{-1/2}$ .

Singular pairs and more generally singular vectors were defined by Khintchine who showed that the set of singular vectors is of zero Lebesgue measure. Recently Y. Cheung [5] showed that the set of singular pairs has Hausdorff dimension  $4/3$ , hence is rather small.

In the previous examples of transient cocycles, the rotation  $T_\alpha$  on  $\mathbb{T}^2$  turns out to be associated with a singular pair  $\alpha$ . Both constructions were designed to exhibit at least one  $\alpha$  for which the cocycle is transient. So the fact that  $\alpha$  is singular might just be a technical convenience, and a natural question is to construct a transient cocycle over a rotation defined by a non singular pair  $\alpha \in \mathbb{R}^2$ .

## 2.2. Recurrence for a special class of functions.

*Notation.* Recall the notation  $|u|_+ = \max(|u|, 1)$ , for  $u \in \mathbb{R}$ . If  $\ell_1, \ell_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  are two independent linear forms, for  $h \in \mathbb{R}^2$  we put  $R(h) = R_{\ell_1, \ell_2}(h) = |\ell_1(h)|_+ |\ell_2(h)|_+$  and define for  $s > 1$ :

$$W(\ell_1, \ell_2, s) := \{\alpha \in \mathbb{R}^2 : R(h)^s \|h \cdot \alpha\| \leq 1 \text{ for infinitely many } h \in \mathbb{Z}^2\}.$$

**Definition 2.4.** We denote by  $\mathcal{G}$  the class of centered functions  $f : \mathbb{T}^2 \rightarrow \mathbb{R}$  such that there exists a finite partition of  $[0, 1]^2$  into triangles  $\Delta_j$  such that  $f$  has bounded continuous partial derivatives  $f'_x, f'_y, f''_{xy}, f''_{yx}$  on the interior of each  $\Delta_j$ .

**Theorem 2.5.** *If  $\Phi = (\varphi^1, \dots, \varphi^d) : \mathbb{T}^2 \rightarrow \mathbb{R}^d$  is such that each component  $\varphi^i \in \mathcal{G}$ , then*

- 1)  $\dim_H \{\alpha \in \mathbb{R}^2 : (\Phi_{n, \alpha})_n \text{ is not recurrent}\} \leq 2 - \frac{1}{2d-1}$ ;
- 2)  $(\Phi_{n, \alpha})_n$  is recurrent if  $\alpha$  is a totally irrational algebraic pair.

*Remark:* The class  $\mathcal{G}$  contains in particular the functions  $\varphi^\Delta := 1_\Delta - \mu(\Delta)$ , where  $1_\Delta$  is the indicator of a subset  $\Delta$  of the 2-torus whose boundary is a finite union of segments.

In this case Theorem 2.5 is related to the following result: It is shown in [6, Theorem 3.1] that the cocycle  $(\varphi_{n, \alpha}^\Delta)$  generated by  $\varphi^\Delta$  over a two dimensional rotation by  $\alpha$  satisfies, for every  $\gamma > 0$ , for almost every  $\alpha \in \mathbb{T}^2$  the bound  $\|\varphi_{n, \alpha}^\Delta\|_2 = O(n^\gamma)$ .

For a finite family  $(\varphi^{\Delta_i}, i = 1, \dots, d)$  with sets  $\Delta_i$  as  $\Delta$  above, it follows from this bound and Lemma 2.1 that the  $d$ -dimensional cocycle  $(\Phi_{n, \alpha})$  generated over the rotation by  $\alpha$  on  $\mathbb{T}^2$  by  $\Phi = (\varphi^{\Delta_i})_{i=1, \dots, d}$  is recurrent for a.e.  $\alpha$ .

Theorem 2.5 improves this result. Its proof follows the same guideline. It will be used to show that the assumptions of Theorem 3.9 below about ergodicity of some cocycles are satisfied by a set of rotations in  $\mathbb{T}^2$  of Hausdorff dimension 2. It is based on the following two propositions whose proof is postponed to the next subsection.

**Proposition 2.6.** *Let  $\ell_1, \ell_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  be two independent linear forms and let  $R(h) = R_{\ell_1, \ell_2}(h) = |\ell_1(h)|_+ |\ell_2(h)|_+$  for  $h \in \mathbb{R}^2$ . Let  $1 < t < 2$ . Then*

$$\dim_H \{\alpha \in \mathbb{R}^2 : \sum_{h \in \mathbb{Z}_*^2} \frac{1}{R(h)^2 \|h \cdot \alpha\|^t} = \infty\} \leq \frac{3 + 2/t}{1 + 2/t}.$$

**Proposition 2.7.** *If  $\varphi \in \mathcal{G}$ , there exist a constant  $C$  and  $2m$  linear forms  $\ell_1^\varphi, \dots, \ell_{2m}^\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that for  $k = 1, \dots, m$ ,  $\ell_{2k-1}^\varphi, \ell_{2k}^\varphi$  are linearly independent and the Fourier coefficients of  $\varphi$  satisfy*

$$(11) \quad |c_n(\varphi)| \leq C \sum_{k=1}^m \frac{1}{|\ell_{2k-1}^\varphi(n)|_+ |\ell_{2k}^\varphi(n)|_+}, \forall n \in \mathbb{Z}_*^2.$$

**Proof of Theorem 2.5.** 1) Let  $\Phi = (\varphi^1, \dots, \varphi^d)$  be such that each  $\varphi^i$  is in  $\mathcal{G}$  and centered. With  $\varphi = \varphi_i$ , let  $\ell_1^\varphi, \dots, \ell_{2m}^\varphi$  be the linear forms given by Proposition 2.7.

Let  $0 < t < 2$ . If  $\alpha$  is not in the set

$$E_\varphi^t = \left\{ \alpha \in \mathbb{R}^2 : \sum_{k=1}^m \sum_{h \in \mathbb{Z}_*^2} \frac{1}{(|\ell_{2k-1}^\varphi(h)|_+ |\ell_{2k}^\varphi(h)|_+)^2 \|h \cdot \alpha\|^t} = \infty \right\},$$

then by (11)

$$\begin{aligned} \sum_{h \in \mathbb{Z}_*^2} \frac{|c_h(\varphi)|^2}{\|h \cdot \alpha\|^t} &\leq C^2 \sum_{h \in \mathbb{Z}_*^2} \left( \sum_{k=1}^m \frac{1}{(|\ell_{2k-1}^\varphi(h)|_+ |\ell_{2k}^\varphi(h)|_+)^2} \right)^2 \frac{1}{\|h \cdot \alpha\|^t} \\ &\leq C^2 m \sum_{k=1}^m \sum_{h \in \mathbb{Z}_*^2} \frac{1}{(|\ell_{2k-1}^\varphi(h)|_+ |\ell_{2k}^\varphi(h)|_+)^2 \|h \cdot \alpha\|^t} < \infty. \end{aligned}$$

Since  $\varphi$  is centered, we have

$$\begin{aligned} \left\| \sum_{k=0}^{N-1} \varphi(\cdot + k\alpha) \right\|_2^2 &= \left\| \sum_{k=0}^{N-1} \sum_{h \in \mathbb{Z}_*^2} c_h(\varphi) e^{2i\pi \langle h, \cdot + k\alpha \rangle} \right\|_2^2 = \sum_{h \in \mathbb{Z}_*^2} |c_h(\varphi)|^2 \sum_{k=0}^{N-1} |e^{2i\pi k \langle h, \alpha \rangle}|^2 \\ &\leq \sum_{h \in \mathbb{Z}_*^2} |c_h(\varphi)|^2 \inf(N, \frac{1}{\|h \cdot \alpha\|})^2 \leq \sum_{h \in \mathbb{Z}_*^2} |c_h(\varphi)|^2 (N^{1-t/2} \frac{1}{\|h \cdot \alpha\|^{t/2}})^2 \leq N^{2-t} \sum_{h \in \mathbb{Z}_*^2} \frac{|c_h(\varphi)|^2}{\|h \cdot \alpha\|^t}. \end{aligned}$$

It follows that, if  $2 - \frac{2}{d} < t < 2$  and  $\alpha \notin \cup_{i=1}^d E_{\varphi_i}^t$ , then  $\left\| \sum_{k=0}^{N-1} \Phi(\cdot + k\alpha) \right\|_2^2 = O(N^{2-t}) = o(N^{\frac{2}{d}})$ , which implies that the cocycle  $(\Phi_{n,\alpha})_n$  is recurrent by Lemma 2.1.

Therefore, if  $\alpha \notin \bigcap_{2-\frac{2}{d} < t < 2} \cup_{i=1}^d E_{\varphi_i}^t$ ,  $(\Phi_n)_n$  is recurrent. Finally, by Proposition 2.6,

$$\dim_H \left( \bigcap_{2-\frac{2}{d} < t < 2} \bigcup_{i=1}^d E_{\varphi_i}^t \right) \leq \inf_{2-\frac{2}{d} < t < 2} \frac{3+2/t}{1+2/t} = \frac{3+2/(2-\frac{2}{d})}{1+2/(2-\frac{2}{d})} = 2 - \frac{1}{2d-1}.$$

2) If  $\alpha$  is algebraic, by W. Schmidt's theorem [21, Theorem 2] on simultaneous approximation to irrational numbers by rationals,  $\alpha \notin W(\ell_1, \ell_2, s)$  for  $s = 1 + \varepsilon$ , for every  $\varepsilon > 0$ . By Lemma 2.9 below it follows  $\sum_{h \in \mathbb{Z}_*^2} \frac{1}{R(h)^2 \|h \cdot \alpha\|^t} < \infty$ ,  $\forall t \in ]1, \frac{2}{1+\varepsilon}[$  and as above, taking  $\varepsilon$  small enough we obtain recurrence in any dimension  $d$  by Lemma 2.1.  $\square$

### 2.2.1. Proof of Propositions 2.6 and 2.7.

For Proposition 2.6 we need two lemmas. The first one is a simple consequence of the Hausdorff-Cantelli lemma (see [4]). The second lemma is adapted from Niederreiter [16].

**Lemma 2.8.** *Let  $\ell_1, \ell_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  be two independent linear forms and let  $s > 1$ . Then,  $\dim_H W(\ell_1, \ell_2, s) \leq \frac{3+s}{1+s}$ .*

*Proof.* Given  $h = (h_1, h_2) \in \mathbb{Z}_*^2$ , the set  $L(h) = \{x \in \mathbb{T}^2 : h \cdot x \in \mathbb{Z}\}$  is a union of  $\gcd(h_1, h_2)$  one dimensional tori. The total length of  $L(h)$  is the Euclidean norm  $|h|$  of  $h$ .

The set  $V(s, h) := \{x \in \mathbb{T}^2 : R(h)^s \|h.x\| \leq 1\}$  is included in a strip around  $L(h)$  of width  $\frac{1}{|h|R(h)^s}$  and can be covered with  $n(h) = |h|^2 R(h)^s$  balls of radius  $r(h) = \frac{4}{|h|R(h)^s}$ .

By the Hausdorff-Cantelli lemma, for  $t > 0$ , if  $\sum_{h \in \mathbb{Z}_*^2} n(h)r(h)^t = 4^t \sum_{h \in \mathbb{Z}_*^2} |h|^{2-t} R(h)^{s-st} < \infty$ , then the Hausdorff dimension of  $W(\ell_1, \ell_2, s)$  is  $\leq t$ .

For  $0 < t < 2$ , using the equivalence of norms, we obtain

$$\begin{aligned} \sum_{h \in \mathbb{Z}_*^2} |h|^{2-t} R(h)^{s-st} &\leq C \sum_{h \in \mathbb{Z}_*^2} (|\ell_1(h)| + |\ell_2(h)|)^{2-t} R(h)^{s-st} \\ &= C \sum_{h \in \mathbb{Z}_*^2} (|\ell_1(h)| + |\ell_2(h)|)^{2-t} (|\ell_1(h)|_+ + |\ell_2(h)|_+)^{s(1-t)}. \end{aligned}$$

So it suffices to bound from above the two series  $\sum_{h \in \mathbb{Z}_*^2} |\ell_i(h)|^{2-t} (|\ell_1(h)|_+ + |\ell_2(h)|_+)^{s(1-t)}$ ,  $i = 1, 2$ . The set  $\Lambda = \{(\ell_1(h), \ell_2(h)) : h \in \mathbb{Z}^2\}$  is a lattice in  $\mathbb{R}^2$ .

Let  $P = \{x \in \mathbb{R}^2 : -\frac{1}{2} \leq x_i < \frac{1}{2}, i = 1, 2\}$ . Since for all  $x \in \mathbb{R}^2$ ,  $\text{Card } \Lambda \cap (x + P) \leq C'$  for some constant  $C'$ , we have for  $1 < t < 2$ ,

$$\begin{aligned} \sum_{h \in \mathbb{Z}_*^2} |\ell_1(h)|^{2-t} (|\ell_1(h)|_+ + |\ell_2(h)|_+)^{s(1-t)} &= \sum_{(x_1, x_2) \in \Lambda \setminus \{0\}} |x_1|^{2-t} (|x_1|_+ + |x_2|_+)^{s(1-t)} \\ &= \sum_{(n_1, n_2) \in \mathbb{Z}^2} \sum_{(x_1, x_2) \in (\Lambda \setminus \{0\}) \cap ((n_1, n_2) + P)} |x_1|^{2-t} (|x_1|_+ + |x_2|_+)^{s(1-t)} \\ &\leq C' \sum_{(n_1, n_2) \in \mathbb{Z}^2} (|n_1|_+)^{2-t} (|n_1|_+ + |n_2|_+)^{s(1-t)} = C' \sum_{n_1 \in \mathbb{Z}} |n_1|_+^{2-t+s(1-t)} \sum_{n_2 \in \mathbb{Z}} |n_2|_+^{s(1-t)}. \end{aligned}$$

The product of the two series is finite when  $s(1-t) < -1$  and  $2-t+s(1-t) < -1$ . Since  $s > 1$ , the product of the series converges when  $2 > t > \max(\frac{1+s}{s}, \frac{3+s}{1+s}) = \frac{3+s}{1+s}$ .

The conclusion is the same for  $i = 2$ .  $\square$

**Lemma 2.9.** *Let  $\ell_1, \ell_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  be two independent linear forms. Let  $s > 1$  and let  $\alpha \notin W(\ell_1, \ell_2, s)$  be totally irrational. Then,*

$$\sum_{h \in \mathbb{Z}_*^2} \frac{1}{R(h)^2 \|h.\alpha\|^t} < \infty, \forall t \in ]1, 2/s[.$$

*Proof.* Since  $\alpha$  is totally irrational and since  $R_{\ell_1, \ell_2}(h)^s \|h.\alpha\| \leq 1$  has only finitely many solutions  $h \in \mathbb{Z}^2$  for  $\alpha \notin W(\ell_1, \ell_2, s)$ , there exists a constant  $c > 0$  such that for all  $h \in \mathbb{Z}_*^2$ ,  $R(h)^s \|h.\alpha\| \geq c$ .

Let us estimate the sum  $\sum_{h \in \mathbb{Z}_*^2 : |\ell_i(h)|_+ \leq n_i, i=1,2} \frac{1}{\|h.\alpha\|^t}$ , for  $n = (n_1, n_2) \in \mathbb{Z}^2$  with  $n_1, n_2 \geq 1$ .

Observe that  $\|x\| - \|y\| = \min(\|x-y\|, \|x+y\|)$  for any  $x, y \in \mathbb{R}$  (where  $\|u\| = \inf |u_n|$ , for  $u \in \mathbb{R}$ ).

For every pair  $(h, h')$  with  $h \neq h'$  and  $h, h'$  both in the domain of summation, we have  $|\ell_i(h \pm h')|_+ \leq |\ell_i(h)|_+ + |\ell_i(h')|_+ \leq 2n_i$ . It follows

$$\begin{aligned} \left| \|h.\alpha\| - \|h'.\alpha\| \right| &= \min(\|(h - h').\alpha\|, \|(h + h').\alpha\|) \\ &\geq c \min(R(h - h')^{-s}, R(h + h')^{-s}) \geq \frac{c}{(2n_1)^s (2n_2)^s}. \end{aligned}$$

Therefore, putting  $\delta = \frac{c}{4^s n_1^s n_2^s}$ , each interval  $[k\delta, (k+1)\delta[$ ,  $k = 1, \dots, \lceil 1/\delta \rceil$  contains at most one point  $\|h.\alpha\|$  with  $h$  in the domain of summation. We also have  $\|h.\alpha\| \geq \frac{c}{R(h)^s} \geq \frac{c}{n_1^s n_2^s}$  for  $h$  in the domain of summation. Since  $t > 1$ , it follows

$$\sum_{h \in \mathbb{Z}_*^2, |\ell_i(h)|_+ \leq n_i, i=1,2} \frac{1}{\|h.\alpha\|^t} \leq \sum_{k \geq 1} \frac{1}{(k\delta)^t} \leq C \frac{1}{\delta^t}.$$

Using the inequality

$$\sum_{(n_1, n_2): n_i \geq |\ell_i(h)|_+, i=1,2} \frac{1}{n_1^3 n_2^3} \geq \frac{c'}{|\ell_1(h)|_+^2 |\ell_2(h)|_+^2} = \frac{c'}{R(h)^2},$$

satisfied for some constant  $c' > 0$ , we obtain

$$\sum_{h \in \mathbb{Z}_*^2} \frac{1}{R(h)^2 \|h.\alpha\|^t} \leq C \sum_{h \in \mathbb{Z}_*^2} \frac{1}{\|h.\alpha\|^t} \sum_{(n_1, n_2): n_i \geq |\ell_i(h)|_+, i=1,2} \frac{1}{n_1^3 n_2^3}.$$

Then, by permuting the order of summation, we obtain

$$\begin{aligned} \sum_{h \in \mathbb{Z}_*^2} \frac{1}{R(h)^2 \|h.\alpha\|^t} &\leq \frac{C}{c'} \sum_{(n_1, n_2) \in \mathbb{N}_*^2} \frac{1}{n_1^3 n_2^3} \sum_{h \in \mathbb{Z}_*^2: |\ell_i(h)|_+ \leq n_i, i=1,2} \frac{1}{\|h.\alpha\|^t} \\ &\leq C' \sum_{(n_1, n_2) \in \mathbb{N}_*^2} \frac{1}{n_1^3 n_2^3} \frac{1}{\delta^t} \leq C' \sum_{(n_1, n_2) \in \mathbb{N}_*^2} \frac{1}{n_1^{3-ts} n_2^{3-ts}}. \end{aligned}$$

The last series converges if  $3 - ts > 1$ , i.e., if  $t < 2/s$ .  $\square$

**Proof of Proposition 2.6.** Let  $1 < t < 2$  and  $s \in (1, 2/t)$ . By Lemma 2.8, the Hausdorff dimension of the set  $W(\ell_1, \ell_2, s)$  is  $\leq \frac{3+s}{1+s}$ . By Lemma 2.9, if  $\sum_{h \in \mathbb{Z}_*^2} \frac{1}{R(h)^2 \|h.\alpha\|^t} = \infty$ ,

with  $1 < t < 2/s$ , then  $\alpha \in W(\ell_1, \ell_2, s)$ .

Therefore,  $\dim_H \{ \alpha \in \mathbb{R}^2 : \sum_{h \in \mathbb{Z}_*^2} \frac{1}{R(h)^2 \|h.\alpha\|^t} = \infty \} \leq \inf_{1 < s < 2/t} \frac{3+s}{1+s} = \frac{3+2/t}{1+2/t}$ .  $\square$

**Proof of Proposition 2.7.** (Bound on the Fourier coefficients of  $f$  in  $\mathcal{G}$ )

From the definition of  $\mathcal{G}$ , it suffices to prove (11) for  $f$  supported by a triangle  $\Delta$  such that  $f$  has bounded continuous partial derivatives  $f'_x, f'_y, f''_{xx}, f''_{xy}, f''_{yy}$  on the interior of  $\Delta$ . Using translations, vertical axis symmetries and by cutting the triangle along a vertical line through one of its vertices, we can reduce to the triangles  $\Delta = \Delta(a, b, c)$  where

$(0, 0)$ ,  $(a, b)$ ,  $(0, c)$  are the vertices of  $\Delta$  and  $a, b, c$  are real numbers such that  $0 < a, c \leq 1$  and  $c - 1 \leq b \leq 1$ . So we are reduced to the proposition:

**Proposition 2.10.** *Let  $f$  be a function supported in  $\Delta(a, b, c)$  with bounded continuous partial derivatives  $f'_x, f'_y, f''_{xx}, f''_{xy}, f''_{yx}$  on the interior of  $\Delta(a, b, c)$ . Then its Fourier coefficients  $c_f(s, t) = \int_0^1 \int_0^1 f(x, y) e^{-2\pi i(sx+ty)} dx dy$  satisfy, for a finite constant  $K$ ,*

$$(12) \quad |c_f(s, t)| \leq K \left( \frac{1}{|t|_+ |s|_+} + \frac{1}{|t|_+ |bt + as|_+} + \frac{1}{|t|_+ |(b-c)t + as|_+} \right), \forall s, t \in \mathbb{Z}.$$

*Proof.* We have  $c_f(s, t) = \int_0^a I_t(x) e^{-2\pi isx} dx$ , with  $I_t(x) = \int_{\frac{b}{a}x}^{\frac{b-c}{a}x+c} f(x, y) e^{-2\pi ity} dy$ .

For  $t \neq 0$ , using integration by parts we get  $I_t(x) = \frac{1}{-2\pi it} [A_t(x) - B_t(x) - C_t(x)]$ , with

$$\begin{aligned} A_t(x) &= f\left(x, \frac{b-c}{a}x + c\right) e^{-2\pi it\left(\frac{b-c}{a}x + c\right)}, \\ B_t(x) &= f\left(x, \frac{b}{a}x\right) e^{-2\pi it\frac{b}{a}x}, \quad C_t(x) = \int_{\frac{b}{a}x}^{\frac{b-c}{a}x+c} f'_y(x, y) e^{-2\pi ity} dy. \end{aligned}$$

If  $t(b-c) + sa \neq 0$ , then

$$\begin{aligned} & \int_0^a A_t(x) e^{-2\pi isx} dx = e^{-2\pi itc} \int_0^a f\left(x, \frac{b-c}{a}x + c\right) e^{-2\pi i\left(t\left(\frac{b-c}{a}\right) + s\right)x} dx \\ &= \frac{a}{-2\pi i\left(t\left(\frac{b-c}{a}\right) + sa\right)} \left[ f(a, b) e^{-2\pi i\left(t\left(\frac{b-c}{a}\right) + s\right)a} - f\left(0, c\right) e^{-2\pi itc} \right. \\ & \quad \left. - e^{-2\pi itc} \int_0^a \left( f'_x\left(x, \frac{b-c}{a}x + c\right) + \frac{b-c}{a} f'_y\left(x, \frac{b-c}{a}x + c\right) \right) e^{-2\pi i\left(t\left(\frac{b-c}{a}\right) + s\right)x} dx \right]. \end{aligned}$$

$$\begin{aligned} & \text{If } tb + sa \neq 0, \text{ then } \int_0^a B_t(x) e^{-2\pi isx} dx = \int_0^a f\left(x, \frac{b}{a}x\right) e^{-2\pi i\left(t\frac{b}{a} + s\right)x} dx \\ &= \frac{a}{-2\pi i\left(tb + sa\right)} \left[ f(a, b) e^{-2\pi i\left(t\frac{b}{a} + s\right)a} - f\left(0, 0\right) - \int_0^a \left( f'_x\left(x, \frac{b}{a}x\right) + \frac{b}{a} f'_y\left(x, \frac{b}{a}x\right) \right) e^{-2\pi i\left(t\frac{b}{a} + s\right)x} dx \right]. \end{aligned}$$

$$\begin{aligned} & \text{If } s \neq 0, \text{ then } \int_0^a C_t(x) e^{-2\pi isx} dx = \int_0^a \left( \int_{\frac{b}{a}x}^{\frac{b-c}{a}x+c} f'_y(x, y) e^{-2\pi ity} dy \right) e^{-2\pi isx} dx \\ &= \frac{1}{-2\pi is} \left[ - \int_0^c f'_y(0, y) e^{-2\pi ity} dy - \int_0^a \frac{d}{dx} \left( \int_{\frac{b}{a}x}^{\frac{b-c}{a}x+c} f'_y(x, y) e^{-2\pi ity} dy \right) e^{-2\pi isx} dx \right]. \end{aligned}$$

The last integrand above is uniformly bounded with respect to  $t, s$ , as shown by

$$\begin{aligned} & \frac{d}{dx} \left( \int_{\frac{b}{a}x}^{\frac{b-c}{a}x+c} f'_y(x, y) e^{-2\pi ity} dy \right) = \frac{b-c}{a} f'_y\left(x, \frac{b-c}{a}x + c\right) e^{-2\pi it\left(\frac{b-c}{a}x + c\right)} \\ & \quad - \frac{b}{a} f'_y\left(x, \frac{b}{a}x\right) e^{-2\pi it\frac{b}{a}x} + \int_{\frac{b}{a}x}^{\frac{b-c}{a}x+c} f''_{yx}(x, y) e^{-2\pi ity} dy. \end{aligned}$$

The previous computation shows that, for a finite constant  $K$ , if  $|t| \geq 1$ ,  $|t(b-c) + sa| \geq 1$ ,  $|tb + sa| \geq 1$  and  $|s| \geq 1$ , then

$$|c_f(s, t)| \leq \frac{K}{|t|_+ |t(b-c) + sa|_+} + \frac{K}{|t|_+ |tb + sa|_+} + \frac{K}{|t|_+ |s|_+}.$$

If  $t \neq 0$ , since the integrals  $|\int_0^a A_t(x) e^{-2\pi i s x} dx|$ ,  $|\int_0^a B_t(x) e^{-2\pi i s x} dx|$ ,  $|\int_0^a C_t(x) e^{-2\pi i s x} dx|$  are bounded by some constant depending only on  $f$ , the above inequality holds even when  $|t(b-c) + sa| \leq 1$  or  $|tb + sa| \leq 1$  or  $|s| \leq 1$ .

Likewise, if  $t = 0$  and  $s \neq 0$ , then, for a constant  $K$ ,  $|c_f(s, 0)| \leq \frac{K}{|s|_+}$ .  $\square$

**2.3. The triangle**  $\Delta_0 = \{(x, y) \in [0, 1]^2 : x < y\}$ .

Since  $\{x\} = x + 1$  for  $x \in ]-1, 0[$ , the characteristic function of  $\Delta_0 = \Delta(1, 1, 1)$  reads

$$(13) \quad 1_{\Delta_0}(x, y) = \{x - y\} + \{y\} - \{x\}, \quad (x, y) \in [0, 1]^2.$$

For this special triangle a bound for the ergodic sums generated by  $\varphi := 1_{\Delta_0} - \frac{1}{2}$  can be obtained as in the proof of Theorem 2.5 or by a simple method based on (13):

**Proposition 2.11.** *1) If the partial quotients of  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_1 - \alpha_2$  satisfy (3) for some  $s > 0$  (a condition which holds for a.e.  $\alpha = (\alpha_1, \alpha_2)$ ), there is a constant  $C_s$  such that*

$$(14) \quad \left\| \sum_{k=0}^{n-1} \varphi(\cdot + k\alpha_1, \cdot + k\alpha_2) \right\|_{\infty} \leq C_s (\log n)^{1+s}, \quad \forall n \geq 2.$$

*2) If  $\alpha_1, \alpha_2$  are algebraic, then for every  $\varepsilon > 0$ , there is a constant  $C(\varepsilon)$  such that*

$$(15) \quad \left\| \sum_{k=0}^{n-1} \varphi(\cdot + k\alpha_1, \cdot + k\alpha_2) \right\|_{\infty} \leq C(\varepsilon) n^{\varepsilon}, \quad \forall n \geq 2.$$

*Proof.* 1) Putting  $\psi(x) = \{x\} - \frac{1}{2}$ , the ergodic sums of  $1_{\Delta_0} - \frac{1}{2}$  are

$$\sum_{k=0}^{n-1} \psi(x - y + k(\alpha_1 - \alpha_2)) + \sum_{k=0}^{n-1} \psi(y + k\alpha_2) - \sum_{k=0}^{n-1} \psi(x + k\alpha_1).$$

If  $\alpha_1, \alpha_2$  and  $\alpha_1 - \alpha_2$  are in the set  $D_s$  of irrational numbers satisfying (3) for some  $s \geq 0$ , then by Proposition 2.2 there is a constant  $C > 0$  such that,  $\forall n \geq 2$ ,

$$\left\| \sum_{k=0}^{n-1} \psi(\cdot + k(\alpha_1 - \alpha_2)) \right\|_{\infty} \leq C (\log n)^{1+s}, \quad \left\| \sum_{k=0}^{n-1} \psi(\cdot + k\alpha_i) \right\|_{\infty} \leq C (\log n)^{1+s}, \quad i = 1, 2.$$

By (13) of  $1_{\Delta_0}$ , we obtain the same bound for  $\varphi$ : there is  $C_1$  such that (14) is satisfied.

The set  $D_s$  has full measure. The set  $D_{2,s}$  of pairs  $(\alpha_1, \alpha_2)$  such that  $\alpha_1, \alpha_2$  and  $\alpha_1 - \alpha_2$  are in  $D_s$  is a set of full measure in  $\mathbb{R}^2$ . This is because  $D_{2,s} = (D_s \times D_s) \cap \{(\alpha_1, \alpha_2) : \alpha_1 \in D_s + \alpha_2\}$  and by Fubini the second set in the intersection has full measure.



2) If  $\alpha_1, \alpha_2$  are algebraic, since  $\alpha_1 - \alpha_2$  is also algebraic, by Roth's theorem,  $\alpha_1, \alpha_2, \alpha_1 - \alpha_2$  are of type 1. By Proposition 2.2, for every  $\varepsilon > 0$  there is a constant  $C(\varepsilon) > 0$  such that (15) is satisfied.  $\square$

The triangle  $\Delta_0$  will be considered again in Section 4.

### 3. Examples of ergodic cocycles over rotations on $\mathbb{T}^2$

There are relatively few known examples of ergodic cocycles over a 2-dimensional rotation. Let us mention some of them:

- In [15] it is shown that for  $\varphi : (x, y) \in \mathbb{T}^2 \rightarrow \{x\} \sin 2\pi y \in \mathbb{R}$ , for uncountably many  $\alpha_1$ , there are uncountably many  $\alpha_2$  such that the cocycle  $(\varphi_n)$  over the rotation by  $(\alpha_1, \alpha_2)$  is ergodic.

- Let  $T_\alpha$ ,  $\alpha = (\alpha_1, \alpha_2)$ , be an ergodic rotation on  $\mathbb{T}^2$  with  $\alpha_1, \alpha_2 \in \text{Bad}$ . Let  $\varphi$  be a function on  $\mathbb{T}^2$  of the form  $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$  with  $\varphi_i : \mathbb{T} \rightarrow \mathbb{Z}$ ,  $i = 1, 2$ , centered step functions with rational discontinuities. In [8] it is shown that the  $\mathbb{Z}^2$ -cocycle  $(\varphi_n)$  over  $T_\alpha$  is ergodic if the jumps of  $(\varphi_1, 0)$  and  $(0, \varphi_2)$  generate  $\mathbb{Z}^2$ .

- In [2] the ergodicity of some cocycles over rotations on  $\mathbb{T}^2$  is shown for a class of examples quite different from those we consider here. The results are for rotations by  $\alpha = (\alpha_1, \alpha_2)$  of Liouville type rather than badly approximable and for cocycles generated by indicators of some rectangles.

In this section, under Diophantine conditions, we prove ergodicity for two families of 1-dimensional and 2-dimensional cocycles generated over some rotations  $T_\alpha$  on  $\mathbb{T}^2$  by functions with (locally) non zero derivatives.

*Once for all, we suppose  $\alpha = (\alpha_1, \alpha_2)$  totally irrational (i.e.,  $1, \alpha_1, \alpha_2$  linearly independent over  $\mathbb{Q}$ ), a necessary and sufficient condition for the ergodicity of the rotation  $T_\alpha$  on  $\mathbb{T}^2$ .*

#### 3.1. Class $\mathcal{F}_1$ .

First we define on  $\mathbb{T}^2$  a class  $\mathcal{F}_1$  of  $\mathbb{R}$ -valued functions with discontinuities, but with local regularity. Then, after preliminary results, we prove a result of ergodicity (Theorem 3.5).

**Definition 3.1.**  $\mathcal{F}_1$  is the class of centered functions  $\varphi$  on  $\mathbb{T}^2$  such that, for two finite sets depending on  $\varphi$ :  $J_i = \{\beta_0^i = 0 \leq \beta_1^i < \dots < \beta_{r_i-1}^i \leq \beta_{r_i}^i = 1\}$ , with  $r_i \geq 1$ ,  $i = 1, 2$ , the partial derivative  $\frac{\partial \varphi}{\partial x_1}$  exists on the open rectangles  $P_{j,j'} = ]\beta_j^1, \beta_{j+1}^1[ \times ]\beta_{j'}^2, \beta_{j'+1}^2[$ ,  $j = 0, \dots, r_1 - 1, j' = 0, \dots, r_2 - 1$ . Moreover, we suppose that  $\varphi$  and  $\frac{\partial \varphi}{\partial x_1}$  are continuous on each  $P_{j,j'}$  and can be continuously extended to its closure.

$\mathcal{P}_\varphi$  will denote the partition of the unit square <sup>2</sup> into the rectangles  $P_{j,j'}$ .

Observe that, if  $\varphi \in \mathcal{F}_1$ , the limits <sup>3</sup>

$$\varphi(\beta_-, x_2) := \lim_{x_1 \rightarrow \beta, x_1 < \beta} \varphi(x_1, x_2), \quad \varphi(\beta_+, x_2) := \lim_{x_1 \rightarrow \beta, x_1 > \beta} \varphi(x_1, x_2)$$

<sup>2</sup>In what follows, we will call ‘‘partition of the unit square’’ any finite collection of disjoint subsets of the unit square which covers it up to a Lebesgue negligible set.

<sup>3</sup>It is understood that  $\varphi(0_-, x_2) := \lim_{x_1 \rightarrow 1, x_1 < 1} \varphi(x_1, x_2)$  and  $\varphi(1_+, x_2) := \lim_{x_1 \rightarrow 0, x_1 > 0} \varphi(x_1, x_2)$ .

exist and are finite for every  $\beta \in J_1$  and every  $x_2 \notin J_2$ .

For every  $n \geq 1$ , if we write each set of numbers  $(\{\beta_j^i - \ell\alpha_i\}, j \in J_i, 0 \leq \ell < n)$ ,  $i = 1, 2$ , as an ordered set of distinct points  $(\gamma_{n,\ell}^i)_{\ell=1,\dots,p_{i,n}}$ , with  $p_{i,n} = r_i n$ , then the atoms of

$$\mathcal{P}_\varphi^n := \mathcal{P}_\varphi \wedge T_\alpha^{-1}\mathcal{P}_\varphi \wedge \dots \wedge T_\alpha^{-(n-1)}\mathcal{P}_\varphi$$

are the rectangles

$$(16) \quad R_{\ell,\ell'}^n = ]\gamma_{n,\ell}^1, \gamma_{n,\ell+1}^1[ \times ]\gamma_{n,\ell'}^2, \gamma_{n,\ell'+1}^2[.$$

On each atom of  $\mathcal{P}_\varphi^n$ ,  $\varphi_n$  is continuous, the partial derivative  $\frac{\partial \varphi_n}{\partial x_1}$  exists and can be extended to its closure.

**Examples 3.2.** a) Let  $\{0 \leq \beta_1^i < \dots < \beta_{r_1-1}^i \leq 1\}$ ,  $i = 1, 2$ , be two finite sequences in  $[0, 1]$ ,  $v_j$  continuous functions,  $\gamma_{j,j'}$  coefficients. Then the sum  $\varphi(x_1, x_2) = \sum_{j,j'} \gamma_{j,j'} [\{x_1 - \beta_j^1\} v_{j'}(\{x_2 - \beta_{j'}^2\}) - \frac{1}{2} \int_0^1 v_j dx_2]$  is in  $\mathcal{F}$ .

b) Let us taking a finite partition of the unit square into open rectangles  $P_{j,j'}$  and a family  $\varphi_{j,j'}$  such that each  $\varphi_{j,j'}$  is defined and  $C^1$  on an open set containing the closure of  $P_{j,j'}$ . The function  $\varphi$  defined by  $\varphi|_{P_{j,j'}} = \varphi_{j,j'}|_{P_{j,j'}}$  (and arbitrarily on the negligible complement  $[0, 1] \times [0, 1] \setminus \cup_{j,j'} P_{j,j'}$ ) is then in  $\mathcal{F}_1$ .

**Hypothesis  $H_1$  on  $\alpha$  and the discontinuities  $\beta_j^1$ :**

$$\beta_j^1 - \beta_{j'}^1 \in \text{Bad}_{\mathbb{Z}}(\alpha_1), \forall j, j' \in \{1, \dots, r_1\}.$$

**Remark 2.** a) In particular,  $0 \in \text{Bad}_{\mathbb{Z}}(\alpha_1)$ , meaning that  $\alpha_1 \in \text{Bad}$ . Recall that once for all  $\alpha = (\alpha_1, \alpha_2)$  is assumed to be totally irrational.

b) There is no condition on  $\beta_j^2$ . It will be shown in Subsection 5.1 that

- given  $\alpha_1 \in \text{Bad}$ , the set of  $(\beta_j^1, j = 1, \dots, r_1)$  satisfying condition  $H_1$  has Hausdorff dimension  $r_1$  in  $\mathbb{R}^{r_1}$ ;
- given  $\beta_j^1, j = 1, \dots, r_1$ , the set of  $\alpha$  such that  $H_1$  holds has Hausdorff dimension 2.

**Lemma 3.3.** *Under hypothesis  $H_1$ , there exist two constants  $0 < c \leq c'$  such that*

$$(17) \quad \frac{c}{n} \leq \gamma_{n,\ell+1}^1 - \gamma_{n,\ell}^1 \leq \frac{c'}{n}, \forall n \geq 1, \ell = 1, \dots, p_{1,n}.$$

*Proof.* By  $H_1$ , there is a constant  $c > 0$  such that for all  $j, j' \in \{1, \dots, r_1\}$  and all  $k \in \{1, \dots, n\}$ ,  $\|k\alpha_1 - (\beta_j^1 - \beta_{j'}^1)\| \geq \frac{c}{n}$ . Therefore, for all  $1 \leq j, j' \leq r_1$  and all  $0 \leq k, k' \leq n$ , if  $\beta_j + k\alpha_1 \neq \beta_{j'} + k'\alpha_1 \pmod{\mathbb{Z}}$ , then  $\|\beta_j + k\alpha_1 - (\beta_{j'} + k'\alpha_1)\| \geq \min(\frac{c}{n}, \delta) \geq \frac{\min(c,\delta)}{n}$ , with  $\delta = \min_{j \neq j'} \|\beta_j - \beta_{j'}\|$ .

For the right hand side of (17), it is enough to show that, for all positive integers  $q$ , the largest gap in  $\mathbb{T}^1 \setminus \{0, \dots, q\alpha_1\}$  is at most  $\frac{c'}{q}$  for some constant  $c'$ . First with  $j = j'$ , we obtain that  $\|n\alpha_1\| \geq \frac{c}{n}$ , for all  $n \geq 1$ . Next, let  $(\frac{p_n}{q_n})_{n \geq 0}$  be the sequence of convergents of  $\alpha_1$ . Since for all  $n \in \mathbb{N}$ ,  $q_{n+1}|q_n\alpha_1 - p_n| = q_{n+1}\|q_n\alpha_1\| \leq 1$  and  $q_n\|q_n\alpha_1\| \geq c$ , we have  $\frac{q_{n+1}}{q_n} \leq \frac{1}{c}$  and  $|k\frac{p_n}{q_n} - k\alpha_1| \leq \frac{1}{q_{n+1}}$  for all  $0 \leq k \leq q_n$ . This implies that the largest gap in

$\mathbb{T}^1 \setminus \{0, \dots, q_n \alpha_1\}$  is at most  $\frac{1}{q_n} + \frac{2}{q_{n+1}} \leq \frac{3}{q_n}$ . Hence, for all  $q_n \leq q \leq q_{n+1}$ , the largest gap in  $\mathbb{T}^1 \setminus \{0, \dots, q \alpha_1\}$  is at most  $\frac{3}{q_n} \leq \frac{3}{cq}$ .  $\square$

**Variation of the ergodic sums of  $\varphi \in \mathcal{F}_1$ .**

We associate with  $\varphi \in \mathcal{F}_1$  the following quantity  $\lambda_1(\varphi) := \int_{\mathbb{T}^2} \frac{\partial \varphi}{\partial x_1} dx$ , which reads

$$\begin{aligned} \sum_{j,j'} \int_{P_{j,j'}} \frac{\partial \varphi}{\partial x_1} dx &= \sum_{j=0, \dots, r_1-1, j'=0, \dots, r_2-1} \int_{\beta_{j'}^2}^{\beta_{j'+1}^2} [\varphi((\beta_{j+1}^1)_-, x_2) - \varphi((\beta_j^1)_+, x_2)] dx_2 \\ &= \sum_{j=0, \dots, r_1-1} \int_0^1 [\varphi((\beta_{j+1}^1)_-, x_2) - \varphi((\beta_j^1)_+, x_2)] dx_2. \end{aligned}$$

For instance, if  $\varphi^1(x_1, x_2) = \{x_1\} \{x_2\} - \frac{1}{4}$ , we have  $r_1 = r_2 = 1$ ,  $\varphi^1(1_-, x_2) = \{x_2\} - \frac{1}{4}$ ,  $\varphi^1(0_+, x_2) = -\frac{1}{4}$ , and  $\lambda_1(\varphi^1) = \int_0^1 [\varphi^1(1_-, x_2) - \varphi^1(0_+, x_2)] dx_2 = \frac{1}{2}$ .

**Lemma 3.4.** *Let  $\varphi$  be in  $\mathcal{F}_1$ . Suppose that  $\lambda_1(\varphi) \neq 0$ . Then, there exists  $N \in \mathbb{N}$  such that for all integers  $n \geq N$ ,*

- if  $x$  is not in the boundary of a rectangle of the partition  $\mathcal{P}_n$ , then  $\frac{\partial \varphi_n(x)}{\partial x_1}$  and  $\lambda_1(\varphi)$  have the same sign,
- if  $(x_1, x_2)$  and  $(x_1 + u_1, x_2)$  belong to interior of the same element of  $\mathcal{P}_n$ , then

$$\frac{1}{2}n|\lambda_1(\varphi)u_1| \leq |\varphi_n(x_1 + u_1, x_2) - \varphi_n(x_1, x_2)| \leq 2n|\lambda_1(\varphi)u_1|.$$

*Proof.* We can assume  $\lambda_1(\varphi) > 0$  w.l.g. Since the rotation  $T_\alpha$  is uniquely ergodic and since  $\frac{\partial \varphi}{\partial x_1}$  is Riemann integrable, the sequence of ergodic sums  $(\frac{1}{n} \sum_{k=0}^{n-1} \frac{\partial \varphi}{\partial x_1} \circ T_\alpha^k, n \geq 1)$

converges uniformly to  $\int_{\mathbb{T}^2} \frac{\partial \varphi}{\partial x_1} dx = \lambda_1(\varphi) > 0$ . It follows that there exists an integer  $N$  such that for every  $n \geq N$  and every  $x \in \mathbb{T}^2$  not in the boundary of a rectangle in  $\mathcal{P}_n$ ,

$$(18) \quad \frac{1}{2}\lambda_1(\varphi) \leq \frac{1}{n} \sum_{k=0}^{n-1} \frac{\partial \varphi}{\partial x_1}(T_\alpha^k(x)) \leq 2\lambda_1(\varphi).$$

It follows that  $\frac{\partial \varphi_n(x)}{\partial x_1}$  and  $\lambda_1(\varphi)$  have the same sign.

To prove the second item, we can assume  $u_1 > 0$ . By the hypothesis on  $(x_1, x_2)$  and  $(x_1 + u_1, x_2)$ , for each  $0 \leq k < n$ , their images  $T_\alpha^k(x_1, x_2), T_\alpha^k(x_1 + u_1, x_2)$ , belong to the interior of the same rectangle  $P_{j,j'}$  for some  $j, j'$ . By definition of the class  $\mathcal{F}$ , the derivative  $\frac{\partial \varphi}{\partial x_1}$  exists on each segment  $[T_\alpha^k(x_1, x_2), T_\alpha^k(x_1 + u_1, x_2)]$ , hence

$$\varphi_n(x_1 + u_1, x_2) = \varphi_n(x_1, x_2) + \int_{x_1}^{x_1+u_1} \sum_{k=0}^{n-1} \frac{\partial \varphi}{\partial x_1}(T_\alpha^k(x_1 + t, x_2)) dt.$$

Now the second item of the lemma follows, since by (18),

$$\frac{1}{2}u_1\lambda_1(\varphi) \leq \int_{x_1}^{x_1+u_1} \frac{1}{n} \sum_{k=0}^{n-1} \frac{\partial\varphi}{\partial x_1}(T_\alpha^k(x_1+t, x_2)) dt \leq 2u_1\lambda_1(\varphi). \quad \square$$

### 3.2. Ergodicity of $(\varphi_n)$ for $\varphi \in \mathcal{F}_1$ .

**Theorem 3.5.** *Let  $\varphi$  be a function in  $\mathcal{F}_1$  such that  $\lambda_1(\varphi) \neq 0$ . Let  $\alpha = (\alpha_1, \alpha_2)$  be totally irrational. Suppose that the hypothesis  $H_1$  for the discontinuities of  $\varphi$  is satisfied. Then the  $\mathbb{R}$ -valued cocycle  $(\varphi_{n,\alpha})$  over the rotation  $T_\alpha$  on  $\mathbb{T}^2$  is ergodic.*

*Proof.* We can suppose  $\lambda_1 = \lambda_1(\varphi) > 0$  w.l.g. Let  $\theta_1, \theta_2$  be positive real numbers such that, with  $c$  defined in Lemma 3.3,

$$(19) \quad 0 < \theta_1 < \theta_2 < \frac{c}{100} \lambda_1,$$

Let  $B \subset \mathbb{T}^2$  be any measurable set of positive measure. We are going to show that there are infinitely many integers  $n \in \mathbb{N}$  such that

$$(20) \quad \mu(B \cap T_\alpha^{-n}B \cap \{|\varphi_n| \in [\theta_1, \theta_2]\}) > 0.$$

As  $\theta_1, \theta_2$  are arbitrary in  $]0, \frac{c}{100} \lambda_1[$ , this will imply that  $\theta$  or  $-\theta$  is an essential value of the cocycle  $(\varphi_n)$  for every  $\theta \in ]0, \frac{c}{100} \lambda_1[$ . Since  $\mathcal{E}(\varphi)$  is a closed subgroup of  $\mathbb{R}$ , this shows ergodicity. It remains to prove (20).

*Proof of (20).*

The proof is divided into two steps. The first step aims to Inequality (21) below on density of subsets with respect to partitions  $\mathcal{U}_n$  associated with  $(\varphi_n)$ . The proof of (21) relies on a version of the Lebesgue density theorem adapted to the partitions  $\mathcal{U}_n$ . Thanks to recurrence, the second step combines Lemma 3.4 and (21).

#### 1a) Definition of the partitions $\mathcal{U}_n$

For  $\varepsilon > 0$ , let  $\omega(\varepsilon)$  be a modulus of (local) continuity for the function  $\varphi$ , i.e., if two points  $(x_1, x_2)$  and  $(y_1, y_2)$  of the torus  $\mathbb{T}^2$  are in the same rectangle of the partition  $\mathcal{P}_\varphi$  associated with  $\varphi$  and if  $\max_i |y_i - x_i| \leq \omega(\varepsilon)$ , then  $|\varphi(x_1, x_2) - \varphi(y_1, y_2)| \leq \varepsilon$ .

Let  $n \geq 1$ . The partition  $\mathcal{U}_n$  will be a refinement of the partition  $\mathcal{P}_\varphi^n$ . First, each of the sets  $J^i = \{\beta_j^i - k\alpha_i : 1 \leq j \leq r_i, 0 \leq k < n\}$ ,  $i = 1, 2$ , cuts  $\mathbb{T}^1$  into a set  $\mathcal{I}_n^i$  of half-open intervals open on the right. Next, let  $\delta_n = \min\{|I| : I \in \mathcal{I}_n^2\}$  and let  $e_n = \min(\delta_n, \omega(\frac{1}{2n^2}))$ . These quantities are non-increasing with  $n$ .

All the intervals in  $\mathcal{I}_n^2$  have a length  $\geq e_n$ . We divide each  $I \in \mathcal{I}_n^2$  into sub-intervals of length  $\frac{1}{2}e_n$ , except the last sub-interval with a length between  $\frac{1}{2}e_n$  and  $e_n$ .

We obtain a new set of non-overlapping intervals  $\mathcal{J}_n^2$  such that

- every interval in  $\mathcal{J}_n^2$  is included in an interval of  $\mathcal{I}_n^2$ ,
- every interval in  $\mathcal{I}_n^2$  is a union of intervals in  $\mathcal{J}_n^2$ ,
- the length of every interval of  $\mathcal{J}_n^2$  is in  $[\frac{e_n}{2}, e_n[$ .

In the sequel of the proof, we write simply  $T$  for the rotation  $T_\alpha$  on  $\mathbb{T}^2$ . We define

$$\mathcal{U}_n = \{R := I \times J : I \in \mathcal{I}_n^1, J \in \mathcal{J}_n^2\}.$$

We will also use the partition  $\tilde{\mathcal{U}}_n = \{\tilde{R} := T^n R : R \in \mathcal{U}_n\}$ . We denote  $R_n(x)$  (resp.  $\tilde{R}_n(x)$ ) the rectangle of  $\mathcal{U}_n$  (resp.  $\tilde{\mathcal{U}}_n$ ) that contains  $x \in \mathbb{T}^2$  (well defined for  $x$  outside a set of 0 measure).

*1b) Application of Lebesgue density theorem*

We want to use a Lebesgue density theorem (see appendix, Theorem 5.6) twice, once with the partitions  $\mathcal{U}_n$ ,  $n \geq 1$ , and once with the partitions  $\tilde{\mathcal{U}}_n$ ,  $n \geq 1$ . We need to check that Conditions (37) and (38) of Theorem 5.6 hold for both sequences of partitions. Condition (38) about the diameters is clearly satisfied.

Next, let  $L_n := \max\{|I| : I \in \mathcal{I}_n^1\}$ . The sequence  $(L_n)$  is non-increasing and, by Lemma 3.3,  $L_n \leq C\ell_n$ , where  $\ell_n = \min\{|I| : I \in \mathcal{I}_n^1\}$  and  $C = \frac{c'}{c}$ . Let  $I \times J \in \mathcal{U}_n$  or  $\tilde{\mathcal{U}}_n$ . On the one hand,  $\mu(I \times J) \geq \ell_n e_n / 2$ . On the other hand, if a product of intervals,  $I' \times J'$ , with lengths respectively  $\leq L_k$  and  $\leq e_k$  for some  $k \geq n$ , intersects  $I \times J$ , then

$$I' \times J' \subset I'' \times J'' = (I + [-L_n, L_n]) \times (J + [-e_n, e_n]).$$

Since  $\mu(I'' \times J'') \leq 3L_n \times 3e_n \leq 9C\mu(I \times J)$ , (37) holds.

Thanks to the Lebesgue density theorem for the set  $B$  with respect to the two families of rectangles  $\mathcal{U}_n$  and  $\tilde{\mathcal{U}}_n$ , we have:

$$\lim_{n \rightarrow \infty} \frac{\mu(R_n(x) \cap B)}{\mu(R_n(x))} = \lim_{n \rightarrow \infty} \frac{\mu(\tilde{R}_n(x) \cap B)}{\mu(\tilde{R}_n(x))} = 1, \text{ for a.e. } x \in B.$$

Therefore, for all  $\varepsilon > 0$ , there exist an integer  $n_\varepsilon$  and a subset  $B_\varepsilon$  of  $B$  of positive measure such that, for all  $u \in B_\varepsilon$  and all  $n \geq n_\varepsilon$ ,

$$\frac{\mu(R_n(u) \cap B)}{\mu(R_n(u))} \geq 1 - \varepsilon, \quad \frac{\mu(\tilde{R}_n(u) \cap B)}{\mu(\tilde{R}_n(u))} \geq 1 - \varepsilon.$$

Let  $n \geq n_\varepsilon$  and let  $u$  be in  $B_\varepsilon \cap T^{-n}B_\varepsilon$ .

Since  $u \in B_\varepsilon$ , the first previous inequality implies

$$\mu(B \cap R_n(u)) \geq (1 - \varepsilon)\mu(R_n(u)).$$

Since  $T^n u \in B_\varepsilon$ , the second inequality implies

$$\mu(B \cap \tilde{R}_n(T^n u)) \geq (1 - \varepsilon)\mu(\tilde{R}_n(T^n u)).$$

Now,  $T^{-n}\tilde{R}_n(T^n u) = R_n(u)$  and  $T^{-n}(B \cap \tilde{R}_n(T^n u)) = T^{-n}B \cap R_n(u)$ . Hence

$$\mu(T^{-n}B \cap R_n(u)) \geq (1 - \varepsilon)\mu(R_n(u)).$$

In brief, for every  $\varepsilon > 0$ , there are  $n_\varepsilon \geq 1$  and a set of positive measure  $B_\varepsilon \subset B$  such that

$$(21) \quad \mu(B \cap T^{-n}B \cap R_n(u)) \geq (1 - 2\varepsilon)\mu(R_n(u)),$$

for all  $n \geq n_\varepsilon$  and all  $u \in B_\varepsilon \cap T^{-n}B_\varepsilon$ . It follows, for  $n \geq n_\varepsilon$ ,

$$(22) \quad \mu((B \cap T^{-n}B)^c \cap R_n(u)) \leq 2\varepsilon\mu(R_n(u)), \forall u \in B_\varepsilon \cap T^{-n}B_\varepsilon.$$

With  $c'$  defined in Lemma 3.3, we take  $\varepsilon$  such that

$$(23) \quad 0 < \varepsilon < \frac{\theta_2 - \theta_1}{32c'\lambda_1}.$$

2) *Application of recurrence*

Let  $N$  be the integer defined in Lemma 3.4. As the function  $\varphi$  is centered, the cocycle  $(\varphi_n)$  is recurrent. Therefore, by Remark 1.a), there exists  $n \geq \max(n_\varepsilon, N)$  and  $a = (a_1, a_2) \in B_\varepsilon \cap T^{-n}B_\varepsilon$  such that  $|\varphi_n(a)| < \frac{1}{2}\theta_1$ .

Let  $R_n(a) = [s_1, t_1] \times [s_2, t_2] \in \mathcal{U}_n$  be the rectangle of  $\mathcal{U}_n$  that contains  $a$ . By Lemma 3.3,  $\frac{c}{n} \leq t_1 - s_1 \leq \frac{c'}{n}$ .

The real number  $a_1$  is either in the first half of the interval  $[s_1, t_1]$ , or in the second half of this interval. Suppose that  $a_1$  in the first half of this interval (if it is in the second half, just move in the negative direction instead of the positive direction).

Let  $x_2 \in [s_2, t_2]$  and consider the function  $f_{x_2} : t \in [0, t_1 - a_1[ \rightarrow \varphi_n(a_1 + t, x_2)$ . We want to bound from below the length of the set of  $t$  such that  $f_{x_2}(t) \in [\theta_1, \theta_2]$ . By Lemma 3.4, since  $n \geq N$ , for all  $t \in [0, t_1 - a_1]$  the derivative  $f'_{x_2}(t)$  is positive and

$$\frac{1}{2}n\lambda_1 t \leq f_{x_2}(t) - f_{x_2}(0) \leq 2n\lambda_1 t.$$

By the definitions of  $\mathcal{J}_n^2$ , of  $\omega(\frac{1}{2n^2})$  and  $e_n$ , we have  $|f_{x_2}(0)| \leq |\varphi_n(a_1, a_2)| + n \times \frac{1}{n^2} \leq \frac{1}{2}\theta_1 + \frac{1}{n}$ , so that  $|f_{x_2}(0)| \leq \theta_1$  provided that  $n \geq \frac{2}{\theta_1}$ . Using (19), we also have

$$f_{x_2}(\frac{1}{2}(t_1 - s_1)) \geq f_{x_2}(0) + \frac{1}{2}n\lambda_1 \times \frac{1}{2}(t_1 - s_1) \geq -\theta_1 + \frac{1}{4}n\lambda_1 \frac{c}{n} \geq -\theta_1 + 25\theta_2 \geq \theta_2.$$

It follows that  $[\theta_1, \theta_2] \subset f_{x_2}([0, \frac{1}{2}(t_1 - s_1)])$  which in turn implies that

$$|\{t \in [0, \frac{1}{2}(t_1 - s_1)] : \varphi_n(a_1 + t, x_2) \in [\theta_1, \theta_2]\}| \geq \frac{\theta_2 - \theta_1}{\max\{|f'_{x_2}(t)| : t \in [0, (t_1 - a_1)]\}} \geq \frac{\theta_2 - \theta_1}{2n\lambda_1}.$$

By Fubini's theorem, the set  $A := \{(x_1, x_2) \in R_n(a) : \varphi_n(x_1, x_2) \in [\theta_1, \theta_2]\}$  has a measure  $\mu(A) \geq \frac{\theta_2 - \theta_1}{2n\lambda_1} \times \frac{e_n}{2}$ . This implies by (23):

$$(24) \quad \frac{\mu(A)}{\mu(R_n(a))} \geq \frac{\frac{\theta_2 - \theta_1}{2n\lambda_1} \times \frac{e_n}{2}}{\frac{c'}{n} \times e_n} = \frac{\theta_2 - \theta_1}{4c'\lambda_1} \geq 4\varepsilon.$$

By definition of  $A$ ,  $\varphi_n(A) \subset [\theta_1, \theta_2]$ . As  $A \subset R_n(a)$ , it follows, using (22) and (24):

$$\begin{aligned} \mu(B \cap T^{-n}B \cap \varphi_n^{-1}[\theta_1, \theta_2]) &\geq \mu(B \cap T^{-n}B \cap A) \\ &= \mu(A) - \mu((B \cap T^{-n}B)^c \cap A) \geq \mu(A) - \mu((B \cap T^{-n}B)^c \cap R_n(a)) \\ &\geq \mu(A) - 2\varepsilon\mu(R_n(a)) \geq 4\varepsilon\mu(R_n(a)) - 2\varepsilon\mu(R_n(a)) = 2\varepsilon\mu(R_n(a)) > 0. \end{aligned}$$

This shows (20). □

### 3.3. Class $\mathcal{F}_2$ .

Now we consider a class  $\mathcal{F}_2$  of functions from  $\mathbb{T}^2$  to  $\mathbb{R}^2$  whose components belong to the class  $\mathcal{F}_1$ , but with a stronger regularity condition in both variables  $x_1, x_2$ .

**Definition 3.6.**  $\mathcal{F}_2$  is the class of centered functions  $\varphi = (\varphi^1, \varphi^2)$  on  $\mathbb{T}^2$  such that, for two finite sets  $J_1, J_2$  depending on  $\varphi$ :  $J_i = \{\beta_0^i = 0 \leq \beta_1^i < \dots < \beta_{r_i-1}^i \leq \beta_{r_i}^i = 1\}$ ,  $i = 1, 2$ ,  $\varphi$  is  $C^1$  on the elements of the partition  $\mathcal{P} = \mathcal{P}_\varphi$  of  $[0, 1] \times [0, 1]$  into the open rectangles  $P_{j,j'} = ]\beta_j^1, \beta_{j+1}^1[ \times ]\beta_{j'}^2, \beta_{j'+1}^2[$ ,  $j = 0, \dots, r_1 - 1, j' = 0, \dots, r_2 - 1$ .

Moreover, we assume that the partial derivatives of  $\varphi^i$ ,  $i = 1, 2$ , can be extended to continuous functions on the closure of the elements of  $\mathcal{P}_\varphi$ .

The following quantities are associated to a function  $\varphi = (\varphi^1, \varphi^2) \in \mathcal{F}_2$ :

$$\lambda_1(\varphi^i) := \int_{\mathbb{T}^2} \frac{\partial \varphi^i}{\partial x_1} dx, \quad \lambda_2(\varphi^i) := \int_{\mathbb{T}^2} \frac{\partial \varphi^i}{\partial x_2} dx, \quad i = 1, 2.$$

As in Lemma 3.4, by unique ergodicity of the rotation and since  $\frac{\partial \varphi^i}{\partial x_1}$  and  $\frac{\partial \varphi^i}{\partial x_2}$  are Riemann integrable, we have uniformly for  $x = (x_1, x_2) \in \mathbb{T}^2$ :

$$(25) \quad \lambda_j(\varphi^i) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \frac{\partial \varphi^i}{\partial x_j}(T^k(x_1, x_2)), \quad i, j = 1, 2.$$

**Hypothesis  $H_2$  on  $\alpha$  and the discontinuities  $\beta_j^i$ :**

$$\beta_j^i - \beta_{j'}^i \in \text{Bad}_{\mathbb{Z}}(\alpha_i), \quad \forall j, j' \in \{1, \dots, r_i\}, \quad \text{for } i = 1, 2.$$

**Remark 3.** In particular,  $\alpha_1, \alpha_2$  are in  $\text{Bad}$  (cf. Remark 2 a)). It will be shown in Subsection 5.1 that

- given  $\alpha$ , the set of  $(\beta_j^i, j = 1, \dots, r_i, i = 1, 2)$  satisfying  $(H_2)$  has Hausdorff dimension  $r_1 + r_2$  in  $\mathbb{R}^{r_1+r_2}$ ;
- given  $(\beta_j^i, j = 1, \dots, r_i, i = 1, 2)$ , the set of  $\alpha$  such that  $(H_2)$  hold has Hausdorff dimension 2.

**Example 3.7.** Let  $\gamma_1 < \gamma_2 \in [1, 2[$ . Consider the function  $\varphi_{\gamma_1, \gamma_2}(x_1, x_2) = \{\gamma_1 x_1\} \{\gamma_2 x_2\}$  restricted to  $[0, 1[ \times [0, 1[$ . By an elementary computation, we obtain

$$\int_0^1 \int_0^1 \varphi_{\gamma_1, \gamma_2} dx_1 dx_2 = \mu(\varphi_{\gamma_1, \gamma_2}) = \left(\frac{1}{2}\gamma_1 - 1 + \gamma_1^{-1}\right) \left(\frac{1}{2}\gamma_2 - 1 + \gamma_2^{-1}\right),$$

$$\lambda_1(\varphi_{\gamma_1, \gamma_2}) = \gamma_1 \left(\frac{1}{2}\gamma_2 - 1 + \gamma_2^{-1}\right), \quad \lambda_2(\varphi_{\gamma_1, \gamma_2}) = \gamma_2 \left(\frac{1}{2}\gamma_1 - 1 + \gamma_1^{-1}\right).$$

Putting  $\varphi^1 = \varphi_{\gamma_1, \gamma_2} - \mu(\varphi_{\gamma_1, \gamma_2})$ ,  $\varphi^2 = \varphi_{\gamma_1, \gamma_2} - \frac{1}{4}$ , the function  $\Phi := (\varphi^1, \varphi^2)$  is in  $\mathcal{F}_2$ .

The sets  $J_1, J_2$  of the definition of  $\mathcal{F}_2$  in 3.6 are:

$$J_1 = \{\beta_0^1 = 0 < \beta_1^1 = \gamma_1^{-1} < \beta_2^1 = 1\}, \quad J_2 = \{\beta_0^2 = 0 < \beta_1^2 = \gamma_2^{-1} < \beta_2^2 = 1\}.$$

If  $\alpha_1$  and  $\alpha_2$  are in  $\text{Bad}$ , the condition  $H_2$  is satisfied if  $\gamma_i$  is rational or more generally if  $\gamma_i^{-1} \in \text{Bad}_{\mathbb{Z}}(\alpha_i)$ ,  $i = 1, 2$ . Moreover, if  $\gamma_1 \neq \gamma_2$  and  $\frac{\gamma_1 + \gamma_2}{\gamma_1 \gamma_2} \neq 1$ , it holds

$$\lambda_1(\varphi_{\gamma_1, \gamma_2})\lambda_2(\varphi_{1,1}) - \lambda_2(\varphi_{\gamma_1, \gamma_2})\lambda_1(\varphi_{1,1}) = \frac{1}{2}(\gamma_2 - \gamma_1)\left(1 - \frac{\gamma_1 + \gamma_2}{\gamma_1 \gamma_2}\right) \neq 0.$$

We use the notation of the previous subsection (cf. (16)). For every  $n \geq 1$ , the ergodic sums  $\varphi_n$  are  $C^1$  on the atoms  $R_{\ell, \ell'}^n = ]\gamma_{n, \ell}^1, \gamma_{n, \ell+1}^1[ \times ]\gamma_{n, \ell'}^2, \gamma_{n, \ell'+1}^2[$  of the partition  $\mathcal{P}_\varphi^n := \mathcal{P} \wedge T_\alpha^{-1}\mathcal{P} \wedge \dots \wedge T_\alpha^{n-1}\mathcal{P}$ . We consider also the partition  $\tilde{\mathcal{P}}_n = T_\alpha^n \mathcal{P}_n$ .

We will use the following variant of Lemma 3.4:

**Lemma 3.8.** *Let  $\varphi = (\varphi^1, \varphi^2)$  be in  $\mathcal{F}_2$ . If  $x = (x_1, x_2)$  and  $x + u = (x_1 + u_1, x_2 + u_2)$  belong to the same element of the partition  $\mathcal{P}_\varphi^n$ , we have,*

$$(26) \quad \varphi_n^i(x + u) = \varphi_n^i(x) + (\lambda_1(\varphi^i)u_1 + \lambda_2(\varphi^i)u_2)n + o(n)|u| + \varepsilon(u)|u|n,$$

with  $\varepsilon(t)$ , defined for  $|t|$  small, depending only on  $\varphi$  and such that  $\lim_{t \rightarrow 0} \varepsilon(t) = 0$ .

*Proof.* By the hypothesis on  $x$  and  $x + u$ , their images  $T_\alpha^k x$ , and  $T_\alpha^k(x + u)$ , for  $0 \leq k < n$ , belong to the same rectangle  $P_{j, j'}$  for some  $j, j'$ . For  $i = 1, 2$ , as the partial derivatives of  $\varphi^i$  can be extended to continuous functions on the closure of  $P_{j, j'}$ , by the mean value theorem, we have, with  $\varepsilon$  as in the statement:

$$|\varphi^i(T_\alpha^k(x + u)) - [\varphi^i(T_\alpha^k x) + u_1 \frac{\partial \varphi^i}{\partial x_1}(T_\alpha^k x) + u_2 \frac{\partial \varphi^i}{\partial x_2}(T_\alpha^k x)]| \leq \varepsilon(u)|u|.$$

It follows:

$$|\varphi_n^i(x + u) - [\varphi_n^i(x) + u_1 \sum_{k=0}^{n-1} \frac{\partial \varphi^i}{\partial x_1}(T_\alpha^k x) + u_2 \sum_{k=0}^{n-1} \frac{\partial \varphi^i}{\partial x_2}(T_\alpha^k x)]| \leq \varepsilon(u)|u|n.$$

Using (25), we get (26). □

### 3.4. Ergodicity of $(\varphi_n)$ for $\varphi = (\varphi^1, \varphi^2) \in \mathcal{F}_2$ .

**Theorem 3.9.** *Let  $\varphi = (\varphi^1, \varphi^2) \in \mathcal{F}_2$  be such that  $\lambda_1(\varphi^1)\lambda_2(\varphi^2) - \lambda_2(\varphi^1)\lambda_1(\varphi^2) \neq 0$ .*

A) *Let  $\alpha = (\alpha_1, \alpha_2)$  be totally irrational. Suppose that the hypothesis  $H_2$  for the discontinuities of  $\varphi^1, \varphi^2$  is satisfied. If the  $\mathbb{R}^2$ -valued cocycle  $(\Phi_{n, \alpha})$  generated by  $\Phi = (\varphi^1, \varphi^2)$  over the rotation  $T_\alpha$  on  $\mathbb{T}^2$  is recurrent, then it is ergodic.*

B) *Suppose that  $\varphi^1, \varphi^2$  have bounded partial derivatives of first and second order on the interior of their continuity domain. Given the discontinuities  $\beta_j$ , the set of  $\alpha \in \mathbb{R}^2$  such that the cocycle  $(\Phi_{n, \alpha})$  is ergodic is of Hausdorff dimension 2.*

*Proof.* A) We keep the notations of Section 3.1. By Lemma 3.3, Hypothesis  $H_2'$  implies the existence of two positive constants  $c, c'$  such that

$$(27) \quad \frac{c}{n} \leq \gamma_{n, \ell+1}^i - \gamma_{n, \ell}^i \leq \frac{c'}{n}, \quad \ell = 1, \dots, |J^i|n, \quad i = 1, 2.$$



For  $x \in [0, 1] \times [0, 1] \setminus \bigcup_k \partial \mathcal{P}_\varphi^k$ , let  $R_n(x)$  be the rectangle element of the partition  $\mathcal{P}_\varphi^n$  containing  $x$ . Likewise let  $\tilde{R}_n(x)$  be the element of the partition  $\tilde{\mathcal{P}}_\varphi^n$  containing  $x$ .

Let  $x = (x_1, x_2)$  and  $u = (u_1, u_2)$  be such that  $x$  and  $x + u$  belong to the same element of the partition  $\mathcal{P}_\varphi^n$ . It follows from Lemma 3.8 for  $n$  big and  $|u|$  small:

$$\varphi_n^i(x + u) = \varphi_n^i(x) + (\lambda_1(\varphi^i)u_1 + \lambda_2(\varphi^i)u_2)n + o(n)|u| + o(|u|)n, \quad i = 1, 2.$$

We set  $M = \begin{pmatrix} \lambda_1(\varphi^1) & \lambda_2(\varphi^1) \\ \lambda_1(\varphi^2) & \lambda_2(\varphi^2) \end{pmatrix}$  and  $u^t = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ . Since  $|u| = O(\frac{1}{n})$  by (27), there is for every  $\varepsilon > 0$  an integer  $N_\varepsilon^1$  depending only on  $\varphi$  and  $\varepsilon$  such that

$$(28) \quad |\varphi_n(x + u) - [\varphi_n(x) + nMu^t]| \leq \varepsilon, \quad \text{for } n > n_\varepsilon.$$

Therefore, for  $n > N_\varepsilon^1$ ,

$$(29) \quad |\varphi_n(x + u) - nMu^t| \leq \varepsilon + |\varphi_n(x)|.$$

Recall that  $c$  and  $c'$  are defined in (27). The matrix  $M$  is invertible by the hypothesis of the theorem. For  $c_0 = \frac{1}{2}c$ , the image by  $M$  of a square  $Q = [0, c_0] \times [0, c_0]$  is a parallelogram  $\mathcal{L}_0$  with a vertex at the origin.

Suppose that  $\mathcal{E}(\varphi) \neq \mathbb{R}^2$ . According to the form of the closed subgroups of  $\mathbb{R}^2$ , there is an open ball  $U$  of radius  $r > 0$  in  $\mathbb{R}^2$ , such that its closure, the compact set  $K = \bar{U}$ , is contained in  $\mathcal{L}_0$  and is disjoint from  $\mathcal{E}(\varphi)$ . For a constant  $\nu > 0$ , we have  $\lambda(U) \geq \nu\lambda(\mathcal{L}_0)$ . Calling  $U_0 \subset U$  the ball of radius  $r - 2\varepsilon$  with the same center as  $U$ , we take  $\varepsilon$  small enough so that  $\lambda(U_0) \geq \frac{1}{2}\nu\lambda(\mathcal{L}_0)$  and  $\frac{1}{2}\nu(\frac{c_0}{c'})^2 > 2\varepsilon$ .

The theorem will be proved by contradiction if we show:

*Claim:  $K$  contains an essential value of  $\varphi$ .*

*Proof of the claim.*

Suppose that  $K \cap \mathcal{E}(\varphi) = \emptyset$ . To get a contradiction, we use Lemma 1.1 which implies that there exists  $B \in \mathcal{B}$  such that  $\mu(B) > 0$  and

$$(30) \quad \mu(B \cap T^{-n}B \cap (\varphi_n \in K)) = 0, \quad \forall n \in \mathbb{Z}.$$

As in Theorem 3.5, using the Lebesgue density theorem (Theorem 5.6) for  $B$ , we obtain:

For every  $\varepsilon > 0$ , there are  $N_\varepsilon^2 \geq 1$  and a set of positive measure  $B_\varepsilon \subset B$  such that

$$(31) \quad \mu(B \cap T^{-n}B \cap R_n(u)) \geq (1 - 2\varepsilon)\mu(R_n(u)), \quad \forall u \in B_\varepsilon \cap T^{-n}B_\varepsilon, \quad \text{for } n \geq N_\varepsilon^2.$$

The recurrence of  $(\varphi_n)$  implies (cf. Remark 1.a) that there exist infinitely many integers  $n \geq N_\varepsilon^2$  such that

$$(32) \quad \mu(B_\varepsilon \cap T_\alpha^{-n}B_\varepsilon \cap (|\varphi_n| < \varepsilon)) > 0.$$

Therefore, we can choose  $n \geq \sup(N_\varepsilon^1, N_\varepsilon^2)$  and  $x^0 \in \mathbb{T}^2$  such that  $|\varphi_n(x^0)| < \varepsilon$  and  $x^0 \in B_\varepsilon \cap T^{-n}B_\varepsilon$ , which implies by (31):

$$(33) \quad \mu(B \cap T^{-n}B \cap R_n(x^0)) \geq (1 - 2\varepsilon)\mu(R_n(x^0)).$$

We can assume that  $x^0$ , which belongs to  $R_n(x^0)$ , is one of the corners of a square  $Q_n \subset R_n(x^0)$  of size  $\frac{c_0}{n} \times \frac{c_0}{n}$ . Up to a change of the signs of  $u_1, u_2$ , we can also assume that  $x^0$  is the lower left corner.

According to the definition of  $\mathcal{L}_0$  at the beginning of the proof,  $nM(Q_n - x^0) = \mathcal{L}_0$  and the measure of the open set  $W = (nM)^{-1}U_0 + x^0$  satisfies  $\mu(W) = \mu(Q_n) \frac{\lambda(U_0)}{\lambda(\mathcal{L}_0)} \geq \frac{1}{2} \nu(\frac{c_0}{c'})^2 \mu(R_n(x^0))$  by (27). Recall that  $\varepsilon$  is such that  $\frac{1}{2} \nu(\frac{c_0}{c'})^2 - 2\varepsilon > 0$ .

Clearly,  $W \subset Q_n \subset R_n(x^0)$ . Moreover, since  $|\varphi_n(x^0)| < \varepsilon$ , by (29) we have

$$W = \{x : nM(x - x^0) \in U_0\} \subset \{x : d(\varphi_n(x), U_0) \leq 2\varepsilon\} \subset \{x : \varphi_n(x) \in U\} \subset \{x : \varphi_n(x) \in K\}.$$

Observe that for any three sets  $E_1, E_2, E_3$ ,

$$\mu(E_1 \cap E_2 \cap E_3) \geq \mu(E_1 \cap E_2) - \mu(E_2) + \mu(E_2 \cap E_3).$$

Using (33), it follows, with  $E_1 = B \cap T^{-n}B$ ,  $E_2 = R_n(x^0)$ ,  $E_3 = W$ :

$$\begin{aligned} \mu(B \cap T^{-n}B \cap R_n(x^0) \cap \{x : \varphi_n(x) \in K\}) &\geq \mu(B \cap T^{-n}B \cap R_n(x^0) \cap W) \\ &\geq (1 - 2\varepsilon) \mu(R_n(x^0)) - \mu(R_n(x^0)) + \mu(R_n(x^0) \cap W) \geq (\frac{1}{2} \nu(\frac{c_0}{c'})^2 - 2\varepsilon) \mu(R_n(x^0)) > 0. \end{aligned}$$

This gives a contradiction with (30) and concludes the proof of A).

B) If  $(\varphi^1, \varphi^2)$  is in  $\mathcal{F}$  and also satisfy the hypothesis of B), then  $\varphi^1$  and  $\varphi^2$  are in  $\mathcal{G}$  (cf. Definition 2.4).

By Theorem 2.5, the set of  $\alpha \in \mathbb{R}^2$  such that the cocycle  $(\varphi_n)$  is not recurrent has a Hausdorff dimension  $\leq 2 - 1/3$ . Let  $B$  be its complement (the set of  $\alpha$  such that  $(\Phi_n)$  is recurrent). The set  $A$  of  $\alpha$  such that the condition  $H_2$  is satisfied has a Hausdorff dimension 2 by Corollary 5.3. Therefore the Hausdorff dimension<sup>4</sup> of  $A \cap B$  is 2.

By A) it follows that the set of  $\alpha \in \mathbb{R}^2$  such that the cocycle  $(\varphi_n)$  is ergodic, is of Hausdorff dimension 2.  $\square$

*An algebraic example:* If  $\alpha_1, \alpha_2$  are quadratic, then recurrence follows from Theorem 2.5 2). As  $\alpha_1, \alpha_2$  are in Bad, we obtain ergodicity for the cocycle defined in Example 3.7. This gives an explicit example of an ergodic cocycle.

#### 4. Ergodicity of compact extensions for the triangle $\Delta_0$

In Theorems 3.5 and 3.9, the discontinuities of the function  $\Phi : \mathbb{T}^2 \rightarrow \mathbb{R}^d$  generating an ergodic cocycle lie along lines parallel to the coordinate axes. It is not easy to adapt our method to construct cocycles generated by a function with more general discontinuities.

When the set of discontinuities is the boundary of the triangle  $\Delta_0 = \{(x_1, x_2) \in [0, 1]^2 : x_1 > x_2\}$ , the diameters of the connected components of the continuity set of the ergodic sum  $\Phi_n$ , vary at least from  $1/n^\gamma$  to  $1/n$  for arbitrarily large values of  $n$  where  $\gamma = \frac{1+\sqrt{5}}{2}$  is the golden ratio.

<sup>4</sup>For the sets A and B, we have  $2 = \dim_H A \leq \max(\dim_H(A \cap B), \dim_H(A \cap B^c)) \leq \max(\dim_H(A \cap B), \dim_H(B^c)) \leq \max(\dim_H(A \cap B), 2 - 1/3)$ ; hence  $\dim_H(A \cap B) = 2$ .

Specifically, the distance from the discontinuity line  $x_1 = x_2 \pmod{\mathbb{Z}}$  to  $(-n_1\alpha_1, -n_2\alpha_2)$ , intersection of the vertical discontinuity line  $x_1 = -n_1\alpha_1 \pmod{\mathbb{Z}}$  and of the horizontal discontinuity line  $x_2 = -n_2\alpha_2 \pmod{\mathbb{Z}}$ , is  $\frac{1}{\sqrt{2}}\|n_1\alpha_1 - n_2\alpha_2\|$ . Moreover, according to a result by W.M. Schmidt [21], if  $(\alpha_1, \alpha_2)$  is totally irrational, there exist infinitely many integers  $n$  such that there are integers  $0 < n_1, n_2 < n$  with  $\|n_1\alpha_1 - n_2\alpha_2\| \leq n^{-\gamma}$ .

This large ratio between small and large diameters implies that Lebesgue's density theorem cannot be used directly. Furthermore, the diameter of the image by  $\Phi_n$  of a small component is  $\leq n \times 1/n^\gamma$ , which is insufficient to establish the existence of non-zero essential values.

Nevertheless, some partial results for  $\Delta_0$  can be shown, notably the ergodicity of compact extensions.

**Ergodicity of compact extensions for  $1_{\Delta_0}$ ,  $\Delta_0 = \{(x, y) \in [0, 1]^2 : x < y\}$ .**

For a general compact extension  $\tilde{T}_\varphi : (x, y) \rightarrow (Tx, y + \varphi(x))$ , where  $T$  is an ergodic measure preserving transformation on a probability space  $(X, \mathcal{B}, \mu)$  and  $\varphi$  a measurable function on  $X$  with values in  $\mathbb{T}^d$ ,  $d \geq 1$ , one obtains easily a criterium of ergodicity:

**Lemma 4.1.**  *$\tilde{T}_\varphi$  is ergodic on  $X \times \mathbb{T}^d$  if and only if the functional equation*

$$(34) \quad H(Tx) = e^{2\pi i \langle \underline{k}, \varphi(x) \rangle} H(x)$$

*has no measurable solution  $H$  of modulus 1 for  $\underline{k} \in \mathbb{Z}^d \setminus \{\underline{0}\}$ .*

*Proof.* Let  $F : X \times \mathbb{T}^d \rightarrow \mathbb{R}$  be a measurable  $\tilde{T}_\varphi$ -invariant function. By truncation, we can suppose  $F$  bounded. For  $\underline{k} \in \mathbb{Z}^d \setminus \{\underline{0}\}$ , its Fourier coefficient with respect to  $y$  satisfies

$$F_{\underline{k}}(x) = \int_{\mathbb{T}^d} F(x, y) e^{-2\pi i \langle \underline{k}, y \rangle} dy = \int_{\mathbb{T}^d} F(\tilde{T}_\varphi(x, y)) e^{-2\pi i \langle \underline{k}, y \rangle} dy = e^{2\pi i \langle \underline{k}, \varphi(x) \rangle} F_{\underline{k}}(Tx).$$

Therefore,  $|F_{\underline{k}}|$  is a.e.-constant and must be a.e. zero by the sufficient condition of the lemma.  $\square$

Now we take  $\underline{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$ , put  $\Phi_{\underline{a}} = 1_{\Delta_0} \underline{a}$  and consider the skew-product on  $\mathbb{T}^2 \times \mathbb{T}^d$  defined for  $\alpha = (\alpha_1, \alpha_2)$  by

$$\tilde{T}_{\alpha, \Phi_{\underline{a}}} : (x, y) \rightarrow (x + \alpha \pmod{1}, y + \Phi_{\underline{a}}(x) \pmod{1}).$$

Our aim in this section is to prove that  $\tilde{T}_{\alpha, \Phi_{\underline{a}}}$  is ergodic, when  $\alpha$  and  $\underline{a}$  are totally irrational and  $\alpha$  satisfies a Diophantine condition. We start by a definition and an auxiliary proposition.

*Coding of a map:* Given a space  $X$ , a map  $T : X \rightarrow X$  and a partition  $\mathcal{Q}$  of  $X$ , for an integer  $n \geq 1$  the  $(\mathcal{Q}, n)$ -coding of  $x \in X$  associated with  $T$  and  $\mathcal{Q}$ , is the sequence  $(w_i, i = 0, \dots, n-1) \in \mathcal{Q}^n$  such that  $T^i(x) \in w_i$  for  $i = 0, \dots, n-1$ .

**Proposition 4.2.** *Let  $(X, \mathcal{B}, \mu, T)$  be a dynamical system, where  $X$  is a compact metric space with a distance  $d_X$ ,  $\mathcal{B}$  is borelian and  $T$  is an isometry which preserves  $\mu$ . Let  $G$  be a group endowed with a bi-invariant distance  $\delta$  and  $\varphi$  a measurable function from  $X$  to  $G$  taking a finite number of values and non constant a.e. Let  $\delta_0 = \delta_0(\varphi) := \min\{\delta(g, g') : g, g' \text{ distinct values of } \varphi\} > 0$ .*

Let  $\mathcal{Q}$  be the partition of  $X$  into the sets on which  $\varphi$  is constant. Suppose that for each integer  $\ell \geq 1$ , there exists a finite partition<sup>5</sup>  $\mathcal{P}_\ell$  of  $X$  such that for each  $P \in \mathcal{P}_\ell$ , there exists  $m_\ell(P) \in \mathcal{Q}^\ell$  which is the  $(\mathcal{Q}, \ell)$ -coding of all  $x \in P$ .

Assume also that there is a sequence  $(\ell_k)_{k \geq 1}$  such that

- 1)  $\lim_{k \rightarrow \infty} \max_{P \in \mathcal{P}_{\ell_k}} \text{diam}(P) = 0$ ;
- 2) for some  $\rho \geq 1$ , for all  $k \geq 1$  and all  $P \in \mathcal{P}_{\ell_k}$ , there are at most  $\rho$  elements  $P' \in \mathcal{P}_{\ell_k}$  such that the closures satisfy  $\overline{P} \cap \overline{P'} \neq \emptyset$ ;
- 3) there is a family  $\mathcal{C}_{c, \ell_k} \subset \mathcal{P}_{\ell_k}$  and positive constants  $c$  and  $\lambda$  such that
  - 3a) for all  $P \in \mathcal{C}_{c, \ell_k}$ ,  $\mu(P) \geq \frac{c}{\text{Card}(\mathcal{P}_{\ell_k})}$ ;
  - 3b)  $N_k := \text{Card}(\mathcal{C}_{c, \ell_k}) \geq \lambda \text{Card}(\mathcal{P}_{\ell_k})$ ;
  - 3c) for all  $P \in \mathcal{C}_{c, \ell_k}$ , there is at least one element  $P'$  in  $\mathcal{C}_{c, \ell_k}$  such that  $\overline{P} \cap \overline{P'} \neq \emptyset$  and the codings  $m_{\ell_k}(P)$  and  $m_{\ell_k}(P')$  have exactly one different component.

Then there is no measurable solution  $f : X \rightarrow G$  of the functional equation

$$(35) \quad \varphi(x) = (f(x))^{-1} f(Tx), \mu\text{-a.e.}$$

*Proof.* We act by contradiction and suppose that there is a measurable function  $f : X \rightarrow G$  such that  $\varphi(x) = (f(x))^{-1} f(Tx), \mu - \text{a.e.}$

Let  $\varepsilon$  be such that  $0 < 2\varepsilon < \frac{c\lambda}{1+\rho}$ .

There exists a closed set  $F$  in  $X$  of measure  $> 1 - \varepsilon$  such that the restriction  $f|_F$  (hence also  $f|_F^{-1}$ ) is uniformly continuous. Let  $\eta > 0$  be such that the conditions  $x, y \in F$ ,  $d_X(x, y) < \eta$  imply  $\delta(f(x), f(y))$  and  $\delta((f(x))^{-1}, (f(y))^{-1}) < \delta_0/2$ .

By 1), there exists  $k_0$  such that, for all  $k \geq k_0$  and all  $P \in \mathcal{P}_{\ell_k}$ ,  $\text{diam}(P) < \eta/2$ . For  $k \geq k_0$ , if  $P$  and  $P'$  are in  $\mathcal{P}_{\ell_k}$  and such that  $\overline{P} \cap \overline{P'} \neq \emptyset$ , we have then  $d(x, y) \leq \eta$  for any  $x \in P$  and  $y \in P'$ .

Let  $H := F \cap T^{-\ell_k} F$  and  $\mathcal{B}_{c, \ell_k} := \{P \in \mathcal{C}_{c, \ell_k} : P \cap H \neq \emptyset\}$ . Let  $N_{1, k} := \text{Card}(\mathcal{B}_{c, \ell_k})$ .

Let  $P \neq P' \in \mathcal{P}_{\ell_k}$  be such that  $\overline{P} \cap \overline{P'} \neq \emptyset$  and  $m_{\ell_k}(P) = (u_0, \dots, u_{\ell_k-1})$  and  $m_{\ell_k}(P') = (u'_0, \dots, u'_{\ell_k-1})$  have exactly one different component. We claim that  $P$  and  $P'$  cannot both intersect  $H$ .

For each  $Q \in \mathcal{Q}$ , denote  $g_Q$  the constant value of  $\varphi$  on  $Q$ . Since  $\varphi_{\ell_k}(x)$  depends only on the coding of  $x$ ,  $\varphi_{\ell_k}$  is constant on  $P$  and on  $P'$ , and these constants are the products  $\pi_P = g_{u_0} \dots g_{u_{\ell_k-1}}$  and  $\pi_{P'} = g_{u'_0} \dots g_{u'_{\ell_k-1}}$ . Since  $u_i = u'_i$  for all  $i$  except for one  $i$ , say  $i_0$ , the bi-invariance of the distance  $\delta$  implies that  $\delta(\pi_P, \pi_{P'}) = \delta(g_{u_{i_0}}, g_{u'_{i_0}}) \geq \delta_0$ .

Now by assumption we have  $\varphi = f^{-1} f \circ T$ , hence  $\varphi_{\ell_k} = f^{-1} f \circ T^{\ell_k}$ . If  $x \in P$  and  $y \in P'$ , we have  $\delta(x, y) \leq \eta$  because  $\overline{P} \cap \overline{P'} \neq \emptyset$ . Since the distance  $\delta$  is bi-invariant and since  $T$

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<sup>5</sup>up to a set of 0 measure

is an isometry, if there exist  $x \in P \cap H$  and  $y \in P' \cap H$ , we have

$$\begin{aligned} \delta_0 &\leq \delta(\pi_P, \pi_{P'}) = \delta(\varphi_{\ell_k}(x), \varphi_{\ell_k}(y)) = \delta(f^{-1}(x)f(T^{\ell_k}(x)), f^{-1}(y)f(T^{\ell_k}(y))) \\ &\leq \delta((f(x))^{-1}f(T^{\ell_k}(x)), (f(y))^{-1}f(T^{\ell_k}(x))) + \delta((f(y))^{-1}f(T^{\ell_k}(x)), (f(y))^{-1}f(T^{\ell_k}(y))) \\ &< \delta_0/2 + \delta_0/2, \end{aligned}$$

hence either  $P \cap H = \emptyset$  or  $P' \cap H = \emptyset$ , which shows the claim.

Since  $\mathcal{B}_{c,\ell_k} \subset \mathcal{C}_{c,\ell_k}$ , by 3c), for each  $P \in \mathcal{B}_{c,\ell_k}$ , there exists  $P'$  in  $\mathcal{C}_{c,\ell_k}$  such that  $\overline{P} \cap \overline{P'} \neq \emptyset$  and  $m_{\ell_k}(P)$  and  $m_{\ell_k}(P')$  have exactly one different component. By the previous claim,  $P' \cap H = \emptyset$ .

On the one hand, according to 2), the number of such distinct  $P' \in \mathcal{C}_{c,\ell_k}$  when  $P$  ranges in  $\mathcal{B}_{c,\ell_k}$  is  $\geq \rho^{-1}N_{1,k}$ . Hence, by 3a)  $\mu(X \setminus H) \geq \rho^{-1}N_{1,k}c/\text{Card}(\mathcal{P}_{\ell_k})$ .

On the other hand, we have  $P \subset X \setminus H$  for all  $P \in \mathcal{C}_{c,\ell_k} \setminus \mathcal{B}_{c,\ell_k}$ , hence  $\mu(X \setminus H) \geq (N_k - N_{1,k})\frac{c}{\text{Card}(\mathcal{P}_{\ell_k})}$ . Using 3b) and  $\mu(F) \geq 1 - \varepsilon$ , we obtain

$$\begin{aligned} 2\varepsilon &\geq \mu(X \setminus H) \geq \max\left((N_k - N_{1,k})\frac{c}{\text{Card}(\mathcal{P}_{\ell_k})}, \rho^{-1}N_{1,k}\frac{c}{\text{Card}(\mathcal{P}_{\ell_k})}\right) \\ &= \frac{c}{\text{Card}(\mathcal{P}_{\ell_k})} \max\left(N_k - N_{1,k}, \frac{N_{1,k}}{\rho}\right) \geq \frac{c}{\text{Card}(\mathcal{P}_{\ell_k})} \frac{N_k}{1 + \rho} \geq \frac{c\lambda}{1 + \rho}. \end{aligned}$$

This leads to a contradiction by the choice of  $\varepsilon$ . □

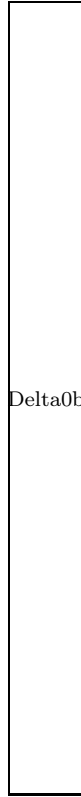


FIGURE 1. Partition  $\mathcal{P}_\ell, \ell = 20$  ( $\alpha_1 = \sqrt{2}, \alpha_2 = e$ )

**Theorem 4.3.** *Let  $(\alpha_1, \alpha_2)$  be totally irrational. If  $\alpha_1$ ,  $\alpha_2$  or  $\alpha_1 - \alpha_2$  is in  $\text{Bad}$ , then the compact extension  $\tilde{T}_{\alpha, \Phi_{\underline{a}}}$  is ergodic on  $\mathbb{T}^2 \times \mathbb{T}$  if  $\underline{a}$  is totally irrational.*

*Proof.* There are three cases, which can be reduced to the case  $\alpha_1 \in \text{Bad}$  as follows:

If  $\alpha_2 \in \text{Bad}$ , simply invert the roles of the first and the second component in the proof.

If  $\alpha_2 - \alpha_1 \in \text{Bad}$ , we use that  $\tilde{T}_{\alpha, \Phi_{\underline{a}}}$  is conjugate to

$$\tilde{T}_{\beta, \Psi_{\underline{a}}} : (x_1, x_2, y) \in \mathbb{T}^2 \times \mathbb{T}^d \rightarrow (x_1 + \beta_1, x_2 + \beta_2, y + 1_{\Delta_1} \underline{a} \bmod 1),$$

where  $\beta = (\beta_1, \beta_2) = (\alpha_1, \alpha_2 - \alpha_1)$ ,  $\Delta_1 = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\}$  and  $\Psi_{\underline{a}} = 1_{\Delta_1} \underline{a}$ .

Indeed, with  $S : (x_1, x_2, y) \in \mathbb{T}^2 \times \mathbb{T}^1 \rightarrow (x_1, x_2 - x_1, y) \in \mathbb{T}^2 \times \mathbb{T}^d$ , if  $(x_1, x_2)$  is not in the boundary of  $\Delta_1$ , we have

$$\begin{aligned} (S \circ T_{\alpha, s} \circ S^{-1})(x_1, x_2, y) &= S(x_1 + \alpha_1, x_2 + x_1 + \alpha_2, y + 1_{\Delta_0}(x_1, \{x_2 + x_1\}) \underline{a} \bmod 1) \\ &= S(x_1 + \alpha_1, x_2 + x_1 + \alpha_2, y + 1_{\Delta_1}(x, y) \underline{a} \bmod 1) \\ &= (x_1 + \alpha_1, x_2 + \alpha_2 - \alpha_1, y + 1_{\Delta_1}(x, y) \underline{a} \bmod 1) = \tilde{T}_{\beta, \Psi_{\underline{a}}}(x_1, x_2, y). \end{aligned}$$

Then we can prove the ergodicity of  $\tilde{T}_{\beta, \Psi_{\underline{a}}}$  like that of  $\tilde{T}_{\alpha, \Phi_{\underline{a}}}$ .

We suppose now that  $\alpha_1$  is in  $\text{Bad}$  and we use Proposition 4.2 with  $X = \mathbb{T}^2$ ,  $T = T_{\alpha}$ ,  $G$  the group of complex numbers of modulus 1,  $\mathcal{Q} = \{\Delta_0, \mathbb{T}^2 \setminus \Delta_0\}$  and  $\varphi = \varphi_{\underline{k}, \underline{a}} = \exp(2\pi i \langle \underline{k}, \underline{a} 1_{\Delta_0} \rangle)$  to conclude that the functional equation

$$(36) \quad H(Tx) = e^{2\pi i \langle \underline{k}, 1_{\Delta_0}(x) \underline{a} \rangle} H(x)$$

has no measurable solution  $H : X \rightarrow \mathbb{T}^d$  of modulus 1 for  $\underline{k} \in \mathbb{Z}^d \setminus \{0\}$ .

Observe that (with the notation of the proposition)  $\delta_0(\varphi_{\underline{k}, \underline{a}})$  is  $> 0$  since  $\underline{a}$  is totally irrational. It remains to check the hypotheses of Proposition 4.2.

As  $\alpha_1$  is in  $\text{Bad}$ , by Lemma 3.3 there exists a constant  $c_1 > 0$  such that for all  $n \geq 1$  the lengths of the  $n$  intervals of  $\mathbb{T}^1 \setminus \{0, \alpha_1, \dots, (n-1)\alpha_1\}$  are  $\geq \frac{c_1}{n}$ .

Let  $V_0 = \{0\} \times \mathbb{T}^1$ ,  $H_0 = \mathbb{T}^1 \times \{0\}$  and  $D_0 = \{(x, x) \bmod \mathbb{Z}^2 : x \in [0, 1]\}$ . The boundary of  $\Delta_0$  is  $V_0 \cup H_0 \cup D_0$ . For each  $\ell > 0$  consider the three sets of lines in  $\mathbb{T}^2$ ,

$$\mathcal{V}_{\ell} = \{T_{\alpha}^{-k}(V_0) : 0 \leq k < \ell\}, \mathcal{H}_{\ell} = \{T_{\alpha}^{-k}(H_0) : 0 \leq k < \ell\}, \mathcal{D}_{\ell} = \{T_{\alpha}^{-k}(D_0) : 0 \leq k < \ell\}.$$

Let  $\mathcal{P}_{\ell}$  (resp.  $\mathcal{R}_{\ell}$ ) be the set of connected components of  $\mathbb{T}^2 \setminus (\cup_{L \in \mathcal{V}_{\ell} \cup \mathcal{H}_{\ell} \cup \mathcal{D}_{\ell}} L) = \mathbb{T}^2 \setminus \cup_{0 \leq i < \ell} T^{-i}(\partial \Delta_0)$  (resp. of  $\mathbb{T}^2 \setminus (\cup_{L \in \mathcal{V}_{\ell} \cup \mathcal{H}_{\ell}} L)$ ). (See figure 1)

We will show that  $\mathcal{P}_{\ell}$  satisfies the assumption of the previous proposition. Observe first, that two points  $x$  and  $y$  in a same  $P \in \mathcal{P}_{\ell}$  have the same  $(\mathcal{Q}, \ell)$  coding because the translates  $T_{\alpha}^k([x, y])$ ,  $0 \leq k < \ell$ , of the segment  $[x, y]$  never cross a boundary of  $\Delta_0$ . Next, since  $\alpha$  is totally irrational, we have  $\lim_{\ell \rightarrow \infty} \max_{P \in \mathcal{P}_{\ell}} \text{diam}(P) = 0$ , hence 1) holds.

Let  $P \in \mathcal{P}_{\ell}$ . It is an open convex polygon with at most 6 edges and there are at most three lines through each vertex of  $P$ . It follows that there exist at most  $6 + 5 \times 6$  polygons  $P' \in \mathcal{P}_{\ell}$  such that  $\overline{P} \cap \overline{P'} \neq \emptyset$ , hence 2) holds with  $\rho = 36$ .

It remains to find the subsequence  $(\ell_k)$  and to prove that 3a), 3b) and 3c) hold.

We take  $\ell_k = q_k$ , where  $(q_k)_{k \geq 1}$  is the sequence of denominators of  $\alpha_2$ . Observe that for each  $R \in \mathcal{R}_{\ell_k}$ , the length of the vertical edge is  $\geq \frac{1}{2\ell_k}$ , while the length of the horizontal edge is  $\geq \frac{c_1}{\ell_k}$ .

Let  $\mathcal{C}_{c, \ell_k}$  be the family of polygons  $P \in \mathcal{P}_{\ell_k}$  with a vertical edge of length  $\geq \frac{1}{10\ell_k}$ .

Since the lengths of the horizontal edges of the rectangle in  $\mathcal{R}_{\ell_k}$  are  $\geq \frac{c_1}{\ell_k}$ , the measure of any  $P \in \mathcal{P}_{\ell_k}$  with a vertical edge of length  $\geq \frac{1}{10\ell_k}$  is at least  $\frac{c_2}{\ell_k^2}$  for some positive constant  $c_2$  not depending on  $\ell_k$ .

One can show by induction that  $\text{Card}(\mathcal{P}_\ell) = 3\ell^2 - \ell$  (actually the bounds  $c\ell^2 \leq \text{Card}(\mathcal{P}_\ell) \leq C\ell^2$  are sufficient for the proof). Therefore  $\mu(P) \geq \frac{c_2}{3\text{Card}(\mathcal{P}_{\ell_k})}, \forall P \in \mathcal{C}_{c, \ell_k}$ ; hence 3a).

For 3b), let  $\mathcal{R}_{5, \ell_k}$  be the family of rectangles  $R \in \mathcal{R}_{\ell_k}$  that contain at most 5 elements  $P \in \mathcal{P}_{\ell_k}$ . In each of these rectangles there exists  $P \in \mathcal{P}_{\ell_k}$  with a vertical edge of length  $\geq \frac{1}{10\ell_k}$ . Therefore  $\text{Card}(\mathcal{C}_{c, \ell_k}) \geq \text{Card}(\mathcal{R}_{5, \ell_k})$ . Now  $\text{Card}(\mathcal{R}_{5, \ell_k}) \geq \ell_k^2/2$ , because  $\text{Card}(\mathcal{R}_\ell) = \ell^2$  and  $\text{Card}(\mathcal{P}_\ell) \leq 3\ell^2$ . It follows that  $\text{Card}(\mathcal{C}_{c, \ell_k}) \geq \frac{1}{6}\text{Card}(\mathcal{P}_{\ell_k})$ ; hence 3b).

At last, let  $P_0 \in \mathcal{C}_{c, \ell_k}$ . It has a vertical edge  $e$  of length  $\geq \frac{1}{10\ell_k}$ . This edge is shared with another  $P_1 \in \mathcal{P}_{\ell_k}$ . By definition of  $\mathcal{C}_{c, \ell_k}$ , we have  $P_1 \in \mathcal{C}_{c, \ell_k}$ . Also this edge is included in  $T_\alpha^{-j}(\partial\Delta_0)$  for some  $j \in \{0, \dots, \ell_k - 1\}$ . It follows that the arc-wise connected set  $P_0 \cup e \cup P_1$  is included in  $\mathbb{T}^2 \setminus \bigcup_{0 \leq i < \ell_k, i \neq j} T_\alpha^{-i}(\partial\Delta_0)$ . Therefore, for all  $x \in P_0$ , all  $y \in P_1$  and all  $i \neq j$  we have  $T_\alpha^i(x)$  and  $T_\alpha^i(y)$  both in  $\Delta_0$  or both not in  $\Delta_0$ , while one exactly of the points  $T_\alpha^j(x)$  and  $T_\alpha^j(y)$  is in  $\Delta_0$ ; hence 3c).  $\square$

*Remarks:* 1) Under the assumptions of Theorem 4.3, the cocycle with values in  $\mathbb{R}^d$  generated by  $(1_{\Delta_0} - \frac{1}{2})\underline{a}$  is not a  $T_\alpha$ -coboundary.

2) Using irreducible representations, Proposition 4.2 provides a method for the extension of Theorem 4.3 to skew-products by topological compact groups instead of the torus  $\mathbb{T}^d$ .

## 5. Appendix

### 5.1. Badly approximable numbers and W. M. Schmidt's games.

In this section, we explain how results of W. M. Schmidt [20] combined with those of J. Tseng [23] or M. Einsiedler and J. Tseng [10] give an information about "badly approximable" numbers.

Notice that the terminology and the notation used in this section is that of the "Schmidt's games":  $\alpha$  is a number in  $]0, 1[$  and  $\theta$  is an irrational number.

**Proposition 5.1.** *Let  $\theta \in \mathbb{R}$  be an irrational number. For  $n \geq 1$ , let  $\mathcal{B}_n \subset \mathbb{R}^n$  be the set  $\{\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n \text{ such that } \beta_i \text{ and } \beta_j - \beta_i \in \text{Bad}_{\mathbb{Z}}(\theta), \text{ for all } 1 \leq i < j \leq n\}$ . Then  $\dim_H \mathcal{B}_n = n$ .*

*Proof.* We use the following results of W. M. Schmidt [20] about  $\alpha$ -winning subsets in  $\mathbb{R}$ :

- i) If  $X \subset \mathbb{R}$  is  $\alpha$ -winning for some  $\alpha \in ]0, 1[$ , then  $\dim_H X = 1$ .
- ii) A bi-Lipschitz image of an  $\alpha$ -winning subset is  $\alpha$ -winning.
- iii) Any finite or countable intersection of  $\alpha$ -winning subsets is  $\alpha$ -winning.

We proceed by induction to prove the proposition. We have  $\mathcal{B}_1 = \text{Bad}_{\mathbb{Z}}(\theta)$  which is  $\frac{1}{8}$ -winning for any irrational number  $\theta$ , by a result of J. Tseng [23]; hence  $\dim_H \mathcal{B}_1 = 1$ .

Then suppose that  $\mathcal{B}_n$  has Hausdorff dimension  $n$ . Let  $\beta = (\beta_1, \dots, \beta_n) \in \mathcal{B}_n$ . Consider the set  $E_\beta = \text{Bad}_{\mathbb{Z}}(\theta) \cap (\text{Bad}_{\mathbb{Z}}(\theta) + \beta_1) \cdots \cap (\text{Bad}_{\mathbb{Z}}(\theta) + \beta_n)$ .

If  $\beta_{n+1} \in E_\beta$ , then  $\beta_{n+1} \in \text{Bad}_{\mathbb{Z}}(\theta)$  and for all  $1 \leq i \leq n$ ,  $\beta_{n+1} - \beta_i \in \text{Bad}_{\mathbb{Z}}(\theta)$ , so that  $(\beta_1, \dots, \beta_n, \beta_{n+1}) \in \mathcal{B}_{n+1}$ .

By ii) and iii),  $E_\beta$  is  $\frac{1}{8}$ -winning, which in turn implies that  $\dim_H E_\beta = 1$ .

By Corollary 7.12 in [11], it follows that  $\dim_H \mathcal{B}_{n+1} = \dim_H \mathcal{B}_n + 1 = n + 1$ . □

*Remark:* If  $\theta \in \text{Bad}$ , since  $0 \in \text{Bad}_{\mathbb{Z}}(\theta)$ , the same conclusion holds when the condition  $1 \leq i < j \leq n$  above is replaced by  $1 \leq i, j \leq n$ .

**Proposition 5.2.** *Let  $\beta_1, \dots, \beta_r \in \mathbb{R}$ . The set of  $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$  such that*

- a)  $1, \theta_1, \theta_2$  are linearly independent over  $\mathbb{Q}$ ,
  - b)  $\theta_1$  has bounded partial quotients (i.e.,  $\theta_1 \in \text{Bad}$ ),
  - c) The differences  $\beta_j - \beta_{j'}$ ,  $j, j' \in \{1, \dots, r\}$ , are in  $\text{Bad}_{\mathbb{Z}}(\theta_1)$ ,
- is winning and therefore has Hausdorff dimension 2.*

*Proof.* By a result of M. Einsiedler and J. Tseng ([10, theorem 1.1]), given  $\beta \in \mathbb{R}$ , the set of  $\theta_1 \in \mathbb{R}$  such that  $\beta \in \text{Bad}_{\mathbb{Z}}(\theta_1)$  is  $\alpha$ -winning, for some winning parameter  $\alpha > 0$  independent of  $\beta$ .

By iii), it follows that the set

$$E(\beta_1, \dots, \beta_r) = \{\theta_1 \in \mathbb{R} : \beta_j - \beta_{j'} \in \text{Bad}_{\mathbb{Z}}(\theta_1), j, j' \in \{1, \dots, r\}\}$$

is winning which implies that  $E(\beta_1, \dots, \beta_r) \times \mathbb{R}$  is winning as a subset of  $\mathbb{R}^2$ . Since the sets of  $(\theta_1, \theta_2)$  such that a) and b) hold are winning, we are done. □

**Corollary 5.3.** *Let  $\beta_1^1, \dots, \beta_{r_1}^1, \beta_1^2, \dots, \beta_{r_2}^2 \in \mathbb{R}$ . The set of  $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$  such that*

- a)  $1, \theta_1, \theta_2$  are linearly independent over  $\mathbb{Q}$ ,
  - b)  $\theta_i, \theta_2$  have bounded partial quotients (i.e.,  $\theta_i \in \text{Bad}$ ),
  - c) the differences  $\beta_j^i - \beta_{j'}^i$ ,  $j, j' \in \{1, \dots, r_i\}$ , are in  $\text{Bad}_{\mathbb{Z}}(\theta_i)$ ,  $i = 1, 2$ ,
- is winning and therefore has Hausdorff dimension 2.*

## 5.2. A version of the Lebesgue density theorem.

In this section we recall a version of the Lebesgue density theorem used in Section 3.

Let  $(X, d)$  be a locally compact metric space equipped with a positive measure  $\mu$  on the  $\sigma$ -algebra of its Borelian sets.

For every  $n \geq 1$ , let  $\mathcal{U}_n$  be a covering (up to a set of  $\mu$ -measure 0) of  $X$  by measurable sets of positive measure. We denote by  $U_n(x)$  an element of  $\mathcal{U}_n$  containing  $x \in X$ . Assume that the following conditions are satisfied:



There is a constant  $C$  such that,

$$(37) \quad \forall n \geq 1, \forall U \in \mathcal{U}_n, \mu\left(\bigcup_{k \geq n} \bigcup_{V \in \mathcal{U}_k: \mu(U \cap V) > 0} V\right) \leq C\mu(U)$$

$$(38) \quad \lim_n \text{diam } U_n(x) = 0, \forall x \in X.$$

For a non negative integrable function  $f$  on  $(X, \mu)$ , we set

$$M(f)(x) = \sup_{n \geq 1} \frac{1}{\mu(U_n(x))} \int_{U_n(x)} f d\mu.$$

We will use the following ‘‘Vitali covering lemma’’:

**Lemma 5.4.** *Suppose that all the coverings  $\mathcal{U}_n$  are finite and that (37) holds. Let  $\mathcal{V} \subset \bigcup_{n \geq 1} \mathcal{U}_n$ . Then there exists  $\mathcal{W} \subset \mathcal{V}$  such that*

- i) for all  $U, U' \in \mathcal{W}$ ,  $U \neq U' \implies \mu(U \cap U') = 0$ ,
- ii)  $\mu(\bigcup_{U \in \mathcal{V}} U) \leq C\mu(\bigcup_{U \in \mathcal{W}} U)$ .

*Proof.* Let us define inductively a sequence of subsets  $\mathcal{W}_n \subset \mathcal{V} \cap \mathcal{U}_n$ . Let  $\mathcal{W}_1$  be a maximal subset of  $\mathcal{V} \cap \mathcal{U}_1$  such that for all  $U \neq U' \in \mathcal{W}_1$ ,  $\mu(U \cap U') = 0$ . Let  $\mathcal{W}_{n+1}$  be a maximal subset of  $\mathcal{V} \cap \mathcal{U}_{n+1}$  such that for all  $U \in \mathcal{W}_{n+1}$ ,  $\mu(U \cap U') = 0$  whenever  $U' \in \mathcal{W}_{n+1}$  and  $U' \neq U$ , or  $U' \in \bigcup_{1 \leq k \leq n} \mathcal{W}_k$ . Let  $\mathcal{W} = \bigcup_{n \geq 1} \mathcal{W}_n$ .

Clearly i) holds. Next, if  $V \in \mathcal{V} \cap \mathcal{U}_n$ , then  $V$  is either in  $\mathcal{W}$  or  $V$  cannot be add to  $\mathcal{W}_n$ , so that there exists  $U \in \bigcup_{k \leq n} \mathcal{W}_k$  such that  $\mu(U \cap V) > 0$ . It follows that any  $V \in \mathcal{V} \cap \mathcal{U}_n$  is included in

$$\bigcup_{1 \leq k \leq n} \bigcup_{U \in \mathcal{W}_k} \bigcup_{m \geq k} \bigcup_{W \in \mathcal{U}_m: \mu(W \cap U) > 0} W.$$

Therefore, by (37),

$$\mu(\bigcup_{V \in \mathcal{V}} V) \leq \sum_{k \geq 1} \sum_{U \in \mathcal{W}_k} \mu\left(\bigcup_{m \geq k} \bigcup_{W \in \mathcal{U}_m: \mu(W \cap U) > 0} W\right) \leq \sum_{k \geq 1} \sum_{U \in \mathcal{W}_k} C\mu(U).$$

Thanks to i), we obtain ii). □

**Lemma 5.5.** *Under Condition (37), for all positive  $\lambda > 0$  and all non negative integrable functions  $f$ ,*

$$(39) \quad \mu\{M(f) > \lambda\} \leq C \frac{\|f\|_1}{\lambda}.$$

*Proof.* For all  $x \in A := \{M(f) > \lambda\}$ , there is an integer  $r(x) \geq 1$  such that

$$(40) \quad \int_{U_{r(x)}(x)} f d\mu > \lambda \mu(U_{r(x)}(x)).$$

Let  $\mathcal{V} = \{U_{r(x)}(x) : x \in A\}$ . By the previous lemma, there exists  $\mathcal{W} \subset \mathcal{V}$  such that i) and ii) hold. Therefore,

$$\|f\|_1 \geq \int_{\bigcup_{U \in \mathcal{W}} U} f d\mu \stackrel{\text{by i}}{=} \sum_{U \in \mathcal{W}} \int_U f d\mu > \lambda \sum_{U \in \mathcal{W}} \mu(U) \stackrel{\text{by ii}}{\geq} \frac{\lambda}{C} \mu(\bigcup_{V \in \mathcal{V}} V) \geq \frac{\lambda}{C} \mu(A). \quad \square$$

**Theorem 5.6.** *Under Conditions (37) and (38), for all  $f$  in  $L^1(\mu)$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{\mu(U_n(x))} \int_{U_n(x)} f d\mu = f(x), \text{ for } \mu\text{-a.e. } x \in X.$$

*In particular if  $B$  is a measurable set of positive measure, for every  $\varepsilon$ , there are  $n(\varepsilon)$  and  $B_\varepsilon \subset B$  of measure  $\geq \frac{1}{2}\mu(B)$  such that:*

$$\mu(B \cap U_n(x)) \geq (1 - \varepsilon)\mu(U_n(x)), \forall n \geq n(\varepsilon), \forall x \in B_\varepsilon.$$

*Proof.* It is enough to prove that, for all  $f \in L^1(\mu)$ , for almost all  $x$ ,

$$f^*(x) := \limsup_n \frac{1}{\mu(U_n(x))} \int_{U_n(x)} |f(y) - f(x)| d\mu(y) = 0.$$

For all  $\varepsilon > 0$ , there exists a continuous function  $g_\varepsilon \in L^1(\mu)$  such that  $\|f - g_\varepsilon\|_1 \leq \varepsilon$ .

By Condition (38), for all  $x$  there is an integer  $N(x)$  such that, for each  $n \geq N(x)$ , the variation of  $g_\varepsilon$  on  $U_n(x)$  is less than  $\varepsilon$ . Therefore, with  $h_\varepsilon = f - g_\varepsilon$ , we have for  $n \geq N(x)$ :

$$\begin{aligned} & \frac{1}{\mu(U_n(x))} \int_{U_n(x)} |f(y) - f(x)| d\mu(y) \\ & \leq \frac{1}{\mu(U_n(x))} \int_{U_n(x)} |h_\varepsilon(y) - h_\varepsilon(x)| d\mu(y) + \frac{1}{\mu(U_n(x))} \int_{U_n(x)} |g_\varepsilon(y) - g_\varepsilon(x)| d\mu(y) \\ & \leq M(|h_\varepsilon|)(x) + |h_\varepsilon(x)| + \varepsilon. \end{aligned}$$

Hence, for all  $x$  and  $\varepsilon > 0$ ,  $f^*(x) \leq M(|h_\varepsilon|)(x) + |h_\varepsilon(x)| + \varepsilon$ .

For  $\lambda > 0$ , taking  $\varepsilon$  such that  $0 < \varepsilon < \lambda$ , it follows by Lemma 5.5 that

$$\mu(f^* > \lambda) \leq \mu(M(|h_\varepsilon|) > \frac{\lambda - \varepsilon}{2}) + \mu(|h_\varepsilon| > \frac{\lambda - \varepsilon}{2}) \leq 2(1 + C) \frac{\|h_\varepsilon\|_1}{\lambda - \varepsilon} \leq \varepsilon \frac{2(1 + C)}{\lambda - \varepsilon} \rightarrow 0,$$

when  $\varepsilon$  goes to zero. As  $\lambda$  is arbitrary  $> 0$ , this implies  $f^* = 0$  a.e.

The last assertion follows by taking  $f = 1_B$ . □

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