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Solving numerically the two-dimensional time harmonic Maxwell problem with sign-changing coefficients

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Abstract

We are investigating the numerical solution to the 2D time-harmonic Maxwell equations in the presence of a classical medium and a metamaterial, that is with sign-changing coefficients. As soon as the problem has a unique solution, we are able to build a converging numerical approximation based on the finite element method, for which there is no constraint on the meshes related to the sign-changing behavior. Numerical examples illustrate the theory.

Keywords: Maxwell's equations, metamaterial, sign-changing coefficients, optimal control

1 Introduction

In electromagnetism, negative materials have been a major research focus over the past two decades, due to their potential applications. These materials can either be metals at optical frequencies or negative metamaterials, which are engineered materials with unique optical properties, see for instance [33, 34]. Unlike traditional materials, the real part of the permittivity and/or permeability of these materials is negative within certain frequency range. The combination of classical dielectrics with such negative materials offers exciting potential applications, such as plasmonic waveguides, perfect lenses [28, 31], invisibility cloaks [32]. However, due to the sign-changing coefficients, this leads to new challenges regarding the well-posedness and approximability of the resulting models. In this manuscript, we consider the numerical modeling of electromagnetic problems within a 2D geometry that involves structures combining both positive and negative materials. Such studies have already been carried out, both theoretically and numerically, see for instance [8, 24]. Compared to these references, the model that we study is slightly more general, in the sense that the geometrical setting and the equations governing the fields are identical, but we allow for non-zero charge densities. In those references, the mathematical tool for the analysis is T -coercivity in [8], and weak T -coercivity in [24]: we propose instead to use a control-based method in the spirit of [17]. Regarding the numerical methods, none is proposed in [8], while the one in [24] relies on T -conform meshes. In contrast, with our method, no constraint related to the sign-changing behavior of the coefficients is imposed on the meshes for the approximation to converge [1, 17].

The control-based method for solving scalar problems with one or two sign-changing coefficients was introduced in [17]. We recall that for solving those problems, there exist other methods, either relying

on the T -coercivity theory [9, 30, 11, 5, 23, 25] and T -conform meshes [15, 11, 5, 23], or on a posteriori error estimators [18].

The outline is as follows. First, in Section 2, we enumerate the main mathematical ingredients. Then, in Section 3, we introduce the 2D time-harmonic electromagnetic model, with the electric field as the main unknown. In Section 4, we propose a Helmholtz decomposition of the electric field, which allows to recast equivalently the model in a variational form, now with three scalar fields. One of the three variational formulations is completely standard, while the other two involve one or two sign-changing coefficients. To solve the latter variational formulations (one with Dirichlet boundary condition, the other one with Neumann boundary condition), we choose the control-based method. One of the two (Dirichlet bc) has already been dealt with in [1, 17]: in [1], a surface-based control was proposed, while a volume-based control is chosen in [17]. We develop the theory with a volume-based control for the second variational formulation (Neumann bc) in Sections 5 and 6 (see also the companion manuscript [13]): this is the main part of this manuscript, which covers both the theoretical solution, and a numerical approximation, of the control-based method. Importantly, convergence of discrete solutions to the exact electric field is proven. Then, in Section 7 we propose some numerical experiments, which we compare with the T -coercivity and plain methods. Since error estimates are not available for the control-based method, we provide some results regarding the approximation with T -conform meshes, and use them as a benchmark. Some concluding remarks are finally given.

2 Mathematical setting

Vector-valued function spaces are written in boldface characters. Unless otherwise specified, we consider spaces of real-valued functions. The index zmv indicates zero-mean-value fields. Duality brackets between a Banach space X and its topological dual X' are denoted by $\langle \cdot, \cdot \rangle_X$.

The symbol C is used to denote a generic positive constant which may depend only on the geometry, or on the coefficients defining the model. We use the notation $A \lesssim B$ for the inequality $A \leq CB$, where A and B are two scalar fields, and C is a generic constant independent of A and B .

In \mathbb{R}^d , a domain is a non-empty, open, connected, and bounded subset with a Lipschitz-continuous boundary. We recall that, in \mathbb{R}^2 , a domain is simply connected if, and only if, its boundary is connected. Given a non-empty open subset O of \mathbb{R}^2 with a Lipschitz boundary ∂O , we denote by $\boldsymbol{\nu} = (\nu_x, \nu_y)^t$ the unit outward normal vector field to ∂O , and by $\boldsymbol{\tau} = (\tau_x, \tau_y)^t = (\nu_y, -\nu_x)^t$ the vector such that $(\boldsymbol{\tau}, \boldsymbol{\nu})$ is an orthonormal basis (see Figure 1 for an example).

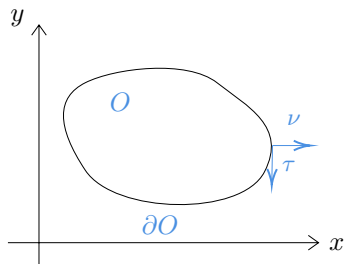


Figure 1 Example of an open subset O in 2D with notation.

Let $(\cdot, \cdot)_{0,O}$ and $\|\cdot\|_{0,O}$ denote the inner product and norm, respectively, for both Hilbert spaces $L^2(O)$ and $\mathbf{L}^2(O) := L^2(O)^2$. More generally, $(\cdot, \cdot)_{s,O}$ and $\|\cdot\|_{s,O}$ (resp. $|\cdot|_{s,O}$) refer to the inner product and norm (resp. semi-norm) of the Sobolev spaces $H^s(O)$ and $\mathbf{H}^s(O) := H^s(O)^2$ for $s \in \mathbb{R}$ (resp. for $s > 0$). Given Γ a Lipschitz submanifold of ∂O , we use the function space $\mathbf{H}_{0,\Gamma}^1(O) := \{u \in H^1(O) \text{ such that } u|_\Gamma = 0\}$.

In a domain O , we say that $\xi \in L^2(O)$ is piecewise smooth if there exists a partition¹ $(O_p)_{p=1,P}$ of O so that $\xi|_{O_p} \in W^{1,\infty}(O_p)$ for $p = 1, P$. In this case, the partition is called compatible with respect to ξ . Finally, for $s > 0$, we introduce

$$\mathbf{PH}^s(O) := \{u \in L^2(O) \text{ such that } u|_{O_p} \in H^s(O_p), \text{ for all } p = 1, P\}.$$

¹A partition of O is $(O_p)_{p=1,P}$ such that O_p is a domain, for $p = 1, P$; $O_p \cap O_q = \emptyset$ for $p \neq q$; $\overline{O} = \cup_{p=1,P} \overline{O_p}$.

We recall next that there are two 2D curl operators:

$$u \mapsto \mathbf{curl} u = \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x} \right)^t \text{ and } \mathbf{u} = (u_x, u_y)^t \mapsto \mathbf{curl} \mathbf{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y},$$

and that, given u, v and η sufficiently smooth in \bar{O} :

$$\begin{aligned} \mathbf{curl} u \cdot \mathbf{curl} v &= \nabla u \cdot \nabla v \text{ pointwise in } O; \\ \mathbf{curl}(\eta \mathbf{curl} u) &= -\operatorname{div}(\eta \nabla u) \text{ pointwise in } O; \\ \mathbf{curl} u \cdot \boldsymbol{\tau} &= \nabla u \cdot \boldsymbol{\nu} \text{ pointwise on } \partial O. \end{aligned}$$

We use classical functional spaces related to the 2D Maxwell's equations whose definitions are recalled below:

$$\begin{aligned} \mathbf{H}(\mathbf{curl}; O) &:= \{ \mathbf{u} \in \mathbf{L}^2(O) \mid \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(O) \}, \\ \mathbf{H}(\operatorname{div}; O) &:= \{ \mathbf{u} \in \mathbf{L}^2(O) \mid \operatorname{div} \mathbf{u} \in \mathbf{L}^2(O) \}, \\ \mathbf{H}_0(\mathbf{curl}; O) &:= \{ \mathbf{u} \in \mathbf{H}(\mathbf{curl}; O) \mid \mathbf{u} \cdot \boldsymbol{\tau} = 0 \text{ on } \partial O \}, \\ \mathbf{H}_0(\operatorname{div}; O) &:= \{ \mathbf{u} \in \mathbf{H}(\operatorname{div}; O) \mid \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial O \}, \\ \mathbf{H}(\operatorname{div} 0; O) &:= \{ \mathbf{u} \in \mathbf{H}(\operatorname{div}; O) \mid \operatorname{div} \mathbf{u} = 0 \text{ in } O \}. \end{aligned}$$

A priori, $\mathbf{H}(\mathbf{curl}; O)$ is endowed with the ‘‘natural’’ norm $\mathbf{u} \mapsto (\|\mathbf{u}\|_{0,O}^2 + \|\mathbf{curl} \mathbf{u}\|_{0,O}^2)^{1/2}$, etc. We refer to the monographs [21, 29, 27, 4] for details. Next, we recall a few useful results in 2D, namely integration by parts formulas and existence of scalar potentials (see eg. [21, 4]).

Theorem 2.1. *Let O be a domain of \mathbb{R}^2 . The following Green formulas hold:*

$$\langle \mathbf{v}, \nabla \varphi \rangle_{0,O} + \langle \operatorname{div} \mathbf{v}, \varphi \rangle_{0,O} = \langle \mathbf{v} \cdot \boldsymbol{\nu}, \varphi \rangle_{H^{1/2}(\partial O)}, \quad \forall (\mathbf{v}, \varphi) \in \mathbf{H}(\operatorname{div}; O) \times H^1(O). \quad (1)$$

$$\langle \mathbf{v}, \mathbf{curl} \varphi \rangle_{0,O} - \langle \mathbf{curl} \mathbf{v}, \varphi \rangle_{0,O} = \langle \mathbf{v} \cdot \boldsymbol{\tau}, \varphi \rangle_{H^{1/2}(\partial O)}, \quad \forall (\mathbf{v}, \varphi) \in \mathbf{H}(\mathbf{curl}; O) \times H^1(O). \quad (2)$$

Theorem 2.2. *Let O be a domain of \mathbb{R}^2 with a connected boundary. Let $\mathbf{v} \in \mathbf{L}^2(O)$, then*

$$\operatorname{div} \mathbf{v} = 0 \text{ in } O \iff \exists \psi \in H^1(O) \text{ such that } \mathbf{v} = \mathbf{curl} \psi \text{ in } O. \quad (3)$$

$$\mathbf{curl} \mathbf{v} = 0 \text{ in } O, \mathbf{v} \cdot \boldsymbol{\tau} = 0 \text{ on } \partial O \iff \exists \varphi \in H_0^1(O) \text{ such that } \mathbf{v} = \nabla \varphi \text{ in } O. \quad (4)$$

Moreover, the scalar potential ψ is unique up to a constant, and the scalar potential ϕ is unique.

3 Model and assumptions

We are interested in solving the 2D Maxwell's equations, assuming that the magnetic permeability and the electric permittivity have a sign-change. Broadly speaking, we are considering the case where there is a limiting amplitude principle, namely that if the data is harmonic in time, then so is the solution for large times. In particular, the electromagnetic energy remains bounded (in bounded regions) of \mathbb{R}^2 . For an extended discussion on the physical validity or relevance of this assumption, we refer to [12]. Note that the numerical solution of Maxwell's equations with sign changing coefficients using a classical discretization with Nédélec finite elements have difficulties to converge (see [16] for some numerical illustrations).

3.1 Equations

In this subsection, fields and function spaces are complex-valued, unless otherwise specified.

We study the time-harmonic Maxwell equations for a given pulsation $\omega > 0$, that is with known time

dependence $\exp(-i\omega t)$. The model is set in a domain Ω of \mathbb{R}^2 with a connected boundary,² where the medium is isotropic and surrounded by a perfect conductor. We assume that the electromagnetic fields and the current density are measurable and square integrable: $\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}, \mathbf{J} \in \mathbf{L}^2(\Omega)$, and that the magnetic permeability μ and the electric permittivity ε are real-valued, measurable and bounded as well as their inverse: $\mu, \mu^{-1}, \varepsilon, \varepsilon^{-1} \in L^\infty(\Omega)$. Consequently, one can use $\varepsilon \mathbf{E}$ for \mathbf{D} , or vice versa $\varepsilon^{-1} \mathbf{D}$ for \mathbf{E} , respectively $\mu \mathbf{H}$ for \mathbf{B} , or vice versa $\mu^{-1} \mathbf{B}$ for \mathbf{H} . Using Ampère's and Faraday's laws³ and the perfect conductor boundary conditions in terms of (\mathbf{E}, \mathbf{B}) :

$$\begin{cases} i\omega \varepsilon \mathbf{E} + \mathbf{curl}(\mu^{-1} \mathbf{B}) = \mathbf{J} \text{ in } \Omega, \\ -i\omega \mathbf{B} + \mathbf{curl} \mathbf{E} = 0 \text{ in } \Omega, \\ \mathbf{E} \cdot \boldsymbol{\tau} = 0 \text{ and } \mathbf{B} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega, \end{cases}$$

we find that $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$, and $\mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{E}) \in \mathbf{L}^2(\Omega)$. Furthermore, \mathbf{E} is governed by the vector second order PDE

$$\begin{cases} \text{Find } \mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \text{ such that:} \\ \mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{E}) - \omega^2 \varepsilon \mathbf{E} = i\omega \mathbf{J} \text{ in } \Omega. \end{cases} \quad (5)$$

This can be recast equivalently as

$$\begin{cases} \text{Find } \mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \text{ such that:} \\ (\mu^{-1} \mathbf{curl} \mathbf{E}, \mathbf{curl} \mathbf{v})_{0,\Omega} - \omega^2 (\varepsilon \mathbf{E}, \mathbf{v})_{0,\Omega} = i\omega (\mathbf{J}, \mathbf{v})_{0,\Omega}, \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega). \end{cases} \quad (6)$$

Then, we observe that for any $\varphi' \in \mathbf{H}_0^1(\Omega)$, one can choose the test-field $\mathbf{v} = \nabla \varphi'$ in (6). So, it holds that

$$-\omega^2 (\varepsilon \mathbf{E}, \nabla \varphi')_{0,\Omega} = i\omega (\mathbf{J}, \nabla \varphi')_{0,\Omega}, \quad \forall \varphi' \in \mathbf{H}_0^1(\Omega), \quad (7)$$

i.e. $\mathbf{div} \mathbf{J} \in \mathbf{H}^{-1}(\Omega)$, with the relation $\mathbf{div}(\varepsilon \mathbf{E}) = -i\omega^{-1} \mathbf{div} \mathbf{J}$. If $\mathbf{J} \in \mathbf{H}(\mathbf{div}; \Omega)$, we find that $\mathbf{div}(\varepsilon \mathbf{E}) = -i\omega^{-1} \mathbf{div} \mathbf{J}$ a.e. in Ω . In the manuscript, we do not assume that $\mathbf{div} \mathbf{J} = 0$: physically, this means that the charge density may not vanish.

3.2 Real-valued fields and function spaces

Note that one can split the problem into two parts, where $\Re e(\mathbf{E})$ is related to $-\Im m(\mathbf{J})$, respectively $-\Im m(\mathbf{E})$ is related to $-\Re e(\mathbf{J})$. So, we carry on with \mathbf{E} standing either for $\Re e(\mathbf{E})$ or $\Im m(\mathbf{E})$, respectively \mathbf{f} standing for $-\omega^{-1} \Im m(\mathbf{J})$ or $\omega^{-1} \Re e(\mathbf{J})$, that is with real-valued fields. For this reason, we use real-valued function spaces. With these notation, one can check that problem (5) can be recast as:

$$\begin{cases} \text{Find } \mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \text{ such that:} \\ \mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{E}) - \omega^2 \varepsilon \mathbf{E} = \omega^2 \mathbf{f} \text{ in } \Omega. \end{cases} \quad (8)$$

One has $\mathbf{f} \in \mathbf{L}^2(\Omega)$ in (8) with the relation $(\varepsilon \mathbf{E}, \nabla \varphi')_{0,\Omega} = -(\mathbf{f}, \nabla \varphi')_{0,\Omega}$ for all $\varphi' \in \mathbf{H}_0^1(\Omega)$. It is convenient to define $g := \mathbf{div} \mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ for later use.

3.3 Assumptions

Since we are dealing with 2D electromagnetic fields, we need assumptions on the companion scalar problems (see Section 4 for more details). Regarding the properties of the permeability μ and the permittivity ε , we assume in addition that they exhibit a sign-change within the domain. Precisely, we suppose that Ω can be divided into two disjoint subdomains Ω_+ and Ω_- ($\overline{\Omega} = \overline{\Omega_+} \cup \overline{\Omega_-}$, $\Omega_+ \cap \Omega_- = \emptyset$), which are such that μ and ε are positive in Ω_+ and negative in Ω_- . In other words, the subscripts '+' and '-' respectively refer to the positive and negative subdomains. Given a field z in Ω , we let $z_+ = z|_{\Omega_+}$, respectively $z_- = z|_{\Omega_-}$. We introduce the interface Σ , defined as $\Sigma := \partial\Omega_+ \cap \partial\Omega_-$ and subsequently, $\Gamma_\pm = \partial\Omega_\pm \setminus \Sigma$ and we assume that $\text{mes}_{\partial\Omega}(\Gamma_-) > 0$, and we denote by $\boldsymbol{\nu}$ the unit normal vector field to Σ , pointing to Ω_+ (see Figure 2 for an example that summarizes these notation).

²For the case of a boundary that is not connected, we refer to Section A.

³In the time-harmonic regime, Gauss' and magnetic Gauss' laws are consequences of Ampère's and Faraday's laws and the charge conservation equation.

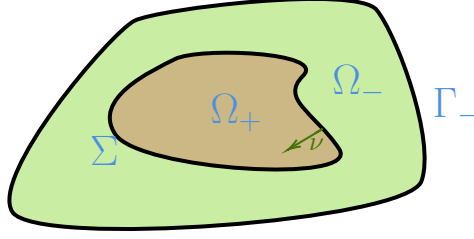


Figure 2 Example of a 2D geometry

Throughout the manuscript, we make some assumptions regarding the permittivity ε and the permeability μ . We make a first assumption on the companion scalar problem with homogeneous Dirichlet boundary condition:

$$\begin{cases} \text{Find } \varphi \in \mathbf{H}_0^1(\Omega) \text{ such that:} \\ (\varepsilon \nabla \varphi, \nabla \varphi')_{0,\Omega} = \langle g, \varphi' \rangle_{\mathbf{H}_0^1(\Omega)}, \quad \forall \varphi' \in \mathbf{H}_0^1(\Omega). \end{cases} \quad (9)$$

Let us call $A_D \in \mathcal{L}(\mathbf{H}_0^1(\Omega), (\mathbf{H}_0^1(\Omega))^*)$ the operator defined by

$$\langle A_D u, v \rangle_{(\mathbf{H}_0^1(\Omega))^*, \mathbf{H}_0^1(\Omega)} = (\varepsilon \nabla u, \nabla v)_{0,\Omega}, \quad \forall u, v \in \mathbf{H}_0^1(\Omega).$$

Assumption 1. *The coefficient ε is such that the operator A_D is an isomorphism.*

Assumption 1 amounts to saying that the companion scalar problem (9) is well-posed. For a characterization of classes of permittivities that lead to a well-posed companion scalar problem (9), we refer to [9, 30, 6].

We make a second assumption on the companion scalar problem with homogeneous Neumann boundary condition:

$$\begin{cases} \text{Find } \psi \in \mathbf{H}^1(\Omega) \text{ such that:} \\ (\varepsilon^{-1} \nabla \psi, \nabla \psi')_{0,\Omega} - \omega^2 (\mu \psi, \psi')_{0,\Omega} = \langle g, \psi' \rangle_{\mathbf{H}^1(\Omega)}, \quad \forall \psi' \in \mathbf{H}^1(\Omega). \end{cases} \quad (10)$$

Let us call $A_N(\omega^2) \in \mathcal{L}(\mathbf{H}^1(\Omega), (\mathbf{H}^1(\Omega))^*)$ the operator defined by

$$\langle A_N(\omega^2) u, v \rangle_{(\mathbf{H}^1(\Omega))^*, \mathbf{H}^1(\Omega)} = (\varepsilon^{-1} \nabla u, \nabla v)_{0,\Omega} - \omega^2 (\mu u, v)_{0,\Omega}, \quad \forall u, v \in \mathbf{H}^1(\Omega).$$

Assumption 2. *The coefficients ε , μ and the pulsation ω are such that the operator $A_N(\omega^2)$ is an isomorphism.*

Assumption 2 amounts to saying that the companion scalar problem (10) is well-posed. We recall that, according to Theorem 5.1 in [8], Assumption 1 implies that $A_N(0)$ is a Fredholm operator, so that $A_N(\omega^2)$ is also a Fredholm operator. In other words, Assumption 2 means that the kernel of $A_N(\omega^2)$ is reduced to $\{0\}$. Note that, Assumptions 1 and 2 can be relaxed (see Section 8 for more details).

Remark 3.1. *In [8], one proceeds by building an equivalent variational formulation to problem (8), and then by proving that the bilinear form is T -coercive. In [24], a similar approach is taken, except that it is proven that the form is weakly T -coercive. We proceed differently hereafter, using a decomposition of the electric field \mathbf{E} governed by (8).*

At some point, it is worthwhile to have some a priori knowledge of the smoothness of the quantities of interest (data, solution, auxiliary unknowns...). Also, when one is using finite element discretizations, error estimates, if any, are expected to be driven by the a priori piecewise regularity of the solution. So, we introduce the limit regularity exponents for the problems (9) and (10). We refer to Section 2 in [16] and References therein for more details.

We assume that a first shift theorem holds for problem (9). Namely, there exists $\tau_D(\varepsilon) \in]0, 1]$ depending only on the geometry and on ε such that

$$\begin{aligned} & \forall s \in [0, \tau_D(\varepsilon) \setminus \{1/2\}], \quad \forall g \in \mathbf{H}^{-1+s}(\Omega), \quad \varphi \in \mathbf{PH}^{1+s}(\Omega), \text{ and } \|\varphi\|_{\mathbf{PH}^{1+s}(\Omega)} \lesssim \|g\|_{-1+s,\Omega}; \\ & \text{if } \tau_D(\varepsilon) = 1, \quad \varphi \in \mathbf{PH}^2(\Omega), \text{ and } \|\varphi\|_{\mathbf{PH}^2(\Omega)} \lesssim \|g\|_{0,\Omega}. \end{aligned}$$

We also assume that a second shift theorem holds for problem (10) with $\omega^2 = 0$, regarding the a priori regularity of the solutions with $L^2(\Omega)$ data g . Namely, there exists $\tau_N(\varepsilon^{-1}) \in]0, 1]$ depending only on the geometry and on ε^{-1} such that

$$\begin{aligned} \forall \mathbf{s} \in [0, \tau_N(\varepsilon^{-1})[\setminus \{1/2\}, \forall g \in L^2(\Omega), \psi \in \text{PH}^{1+\mathbf{s}}(\Omega), \text{ and } \|\psi\|_{\text{PH}^{1+\mathbf{s}}(\Omega)} \lesssim \|g\|_{0,\Omega}; \\ \text{if } \tau_N(\varepsilon^{-1}) = 1, \psi \in \text{PH}^2(\Omega), \text{ and } \|\psi\|_{\text{PH}^2(\Omega)} \lesssim \|g\|_{0,\Omega}. \end{aligned}$$

Above, the constant hidden in \lesssim may depend on \mathbf{s} or \mathbf{s} , but not on g . In the analysis, we keep the largest value of $\tau_D(\varepsilon)$ and of $\tau_N(\varepsilon^{-1})$, which is called the limit regularity exponent.

One can define similarly limit regularity exponents for problems set in Ω or in subdomains of Ω , with either Dirichlet, Neumann or mixed boundary conditions, and possibly with other coefficients.

4 Helmholtz decompositions and characterization of the solution

As before, Ω is a domain of \mathbb{R}^2 with a connected boundary. One has a first Helmholtz decomposition, of the space $\mathbf{L}^2(\Omega)$.

Proposition 4.1. *It holds that*

$$\mathbf{L}^2(\Omega) = \nabla[\mathbf{H}_0^1(\Omega)] \oplus \mathbf{A}_0(\Omega, \varepsilon), \quad (11)$$

with $\mathbf{A}_0(\Omega, \varepsilon) := \{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \forall \varphi \in \mathbf{H}_0^1(\Omega), (\varepsilon \mathbf{w}, \nabla \varphi)_{0,\Omega} = 0\} = \{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \text{div}(\varepsilon \mathbf{w}) = 0 \text{ in } \Omega\}$. In addition, the linear map $\mathbf{v} \mapsto (\varphi, \mathbf{w})$ with $\mathbf{v} = \nabla \varphi + \mathbf{w}$ is continuous from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}_0^1(\Omega) \times \mathbf{A}_0(\Omega, \varepsilon)$.

Proof. It is clear that $\nabla[\mathbf{H}_0^1(\Omega)] + \mathbf{A}_0(\Omega, \varepsilon) \subseteq \mathbf{L}^2(\Omega)$. Let us now show the converse imbedding. Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$. Since problem (9) is well-posed (cf. Assumption 1), there exists $\varphi \in \mathbf{H}_0^1(\Omega)$ such that:

$$(\varepsilon \nabla \varphi, \nabla \varphi')_{0,\Omega} = (\varepsilon \mathbf{v}, \nabla \varphi')_{0,\Omega}, \quad \forall \varphi' \in \mathbf{H}_0^1(\Omega).$$

Then, we pose $\mathbf{w} = \mathbf{v} - \nabla \varphi$. By construction, we have $\mathbf{w} \in \mathbf{L}^2(\Omega)$ and for all $\varphi' \in \mathcal{D}(\Omega)$

$$\langle \text{div}(\varepsilon \mathbf{w}), \varphi' \rangle = -\langle \varepsilon \mathbf{w}, \nabla \varphi' \rangle = -(\varepsilon \mathbf{v} - \varepsilon \nabla \varphi, \nabla \varphi')_{0,\Omega} = 0,$$

i.e. $\text{div}(\varepsilon \mathbf{w}) = 0$ so that $\mathbf{w} \in \mathbf{A}_0(\Omega, \varepsilon)$. It follows that $\mathbf{L}^2(\Omega) = \nabla[\mathbf{H}_0^1(\Omega)] + \mathbf{A}_0(\Omega, \varepsilon)$.

Next, let $z \in \nabla[\mathbf{H}_0^1(\Omega)] \cap \mathbf{A}_0(\Omega, \varepsilon)$ be given. There exists $\varphi \in \mathbf{H}_0^1(\Omega)$ such that $z = \nabla \varphi$ and, by definition of $\mathbf{A}_0(\Omega, \varepsilon)$, φ is governed by (9) with zero right-hand side. By uniqueness of the solution, one has $\varphi = 0$ and so $z = 0$: the sum is direct.

Continuity of the linear map is a straightforward consequence of the well-posedness of the companion scalar problem (9). \square

Remark 4.2. *Because ε has a sign-change, the decomposition is not orthogonal, since $(\mathbf{u}, \mathbf{v}) \mapsto (\varepsilon \mathbf{u}, \mathbf{v})_{0,\Omega}$ does not define an inner product in $\mathbf{L}^2(\Omega)$.*

One can then write a second Helmholtz decomposition of the space $\mathbf{L}^2(\Omega)$.

Proposition 4.3. *It holds that*

$$\mathbf{L}^2(\Omega) = \nabla[\mathbf{H}_0^1(\Omega)] \oplus \varepsilon^{-1} \mathbf{curl}[\mathbf{H}_{zmv}^1(\Omega)]. \quad (12)$$

In addition, the linear map $\mathbf{v} \mapsto (\varphi, \psi)$ with $\mathbf{v} = \nabla \varphi + \varepsilon^{-1} \mathbf{curl} \psi$ is continuous from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_{zmv}^1(\Omega)$.

Proof. According to Theorem 2.2, one has $\mathbf{A}_0(\Omega, \varepsilon) = \varepsilon^{-1} \mathbf{curl}[\mathbf{H}_{zmv}^1(\Omega)]$, because $\mathbf{v} \in \mathbf{A}_0(\Omega, \varepsilon)$ is equivalent to $\varepsilon \mathbf{v} \in \mathbf{H}(\text{div } 0; \Omega)$. The decomposition (12) then follows from the first Helmholtz decomposition (11).

Also, (φ, ψ) is unique because φ is characterized as the solution to a well-posed variational formulation, and ψ is the unique scalar potential of $\varepsilon \mathbf{v}$ with zero mean-value. Continuity of the map $\mathbf{v} \mapsto (\varphi, \psi)$ stems from the fact that $\psi \mapsto \|\mathbf{curl} \psi\|_{0,\Omega}$ defines a norm on $\mathbf{H}_{zmv}^1(\Omega)$, which is equivalent to the $\mathbf{H}^1(\Omega)$ -norm. \square

Note that, for all $\varphi \in H_0^1(\Omega)$ and $\psi \in H_{zmv}^1(\Omega)$, using Green formula (2) we find that since $\nabla\varphi \cdot \boldsymbol{\tau} = 0$ on $\partial\Omega$:

$$(\varepsilon\nabla\varphi, \varepsilon^{-1}\mathbf{curl}\psi)_{0,\Omega} = (\nabla\varphi, \mathbf{curl}\psi)_{0,\Omega} = 0. \quad (13)$$

As a by-product, one can decompose the electric field \mathbf{E} that is governed by (8):

$$\exists!(\varphi_{\mathbf{E}}, \psi_{0,\mathbf{E}}) \in H_0^1(\Omega) \times H_{zmv}^1(\Omega) \text{ such that } \mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\mathbf{curl}\psi_{0,\mathbf{E}} \text{ in } \Omega. \quad (14)$$

Our next step is the characterization of the two scalar potentials $(\varphi_{\mathbf{E}}, \psi_{0,\mathbf{E}})$.

Proposition 4.4. *Let \mathbf{E} , a solution of (8), be decomposed as in (14). Then its scalar potential $\varphi_{\mathbf{E}} \in H_0^1(\Omega)$ is governed by the Dirichlet problem:*

$$\operatorname{div}(\varepsilon\nabla\varphi_{\mathbf{E}}) = -g \text{ in } \Omega. \quad (15)$$

The equivalent variational formulation writes:

$$\begin{cases} \text{Find } \varphi_{\mathbf{E}} \in H_0^1(\Omega) \text{ such that:} \\ (\varepsilon\nabla\varphi_{\mathbf{E}}, \nabla\varphi')_{0,\Omega} = (g, \varphi')_{H_0^1(\Omega)}, \quad \forall \varphi' \in H_0^1(\Omega). \end{cases} \quad (16)$$

Proof. Let $\varphi_{\mathbf{E}} \in H_0^1(\Omega)$ in the decomposition (14) of \mathbf{E} . According to the problem (8) and the definition of g , we find that $\varphi_{\mathbf{E}}$ satisfies (15).

The fact that (15) is equivalent to (16) is classical. \square

Proposition 4.5. *Let \mathbf{E} , a solution of (8), be decomposed as in (14). Then there exists a constant $c \in \mathbb{R}$ such that the shifted scalar potential $\psi_{c,\mathbf{E}} := (\psi_{0,\mathbf{E}} + c) \in H^1(\Omega)$ is governed by the Neumann problem*

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}}) - \omega^2\mu\psi_{c,\mathbf{E}} = \omega^2\mu s_0 \text{ in } \Omega, \\ \varepsilon^{-1}\nabla\psi_{c,\mathbf{E}} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega, \end{cases} \quad (17)$$

where the right-hand side is characterized by

$$s_0 \in H_{zmv}^1(\Omega) \text{ such that } \mathbf{curl}s_0 = \mathbf{f} + \varepsilon\nabla\varphi_{\mathbf{E}} \text{ in } \Omega. \quad (18)$$

The equivalent variational formulation writes:

$$\begin{cases} \text{Find } \psi_{c,\mathbf{E}} \in H^1(\Omega) \text{ such that:} \\ (\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}}, \nabla\psi')_{0,\Omega} - \omega^2(\mu\psi_{c,\mathbf{E}}, \psi')_{0,\Omega} = \omega^2(\mu s_0, \psi')_{0,\Omega}, \quad \forall \psi' \in H^1(\Omega). \end{cases} \quad (19)$$

Proof. Let $\varphi_{\mathbf{E}} \in H_0^1(\Omega)$ and $\psi_{0,\mathbf{E}} \in H_{zmv}^1(\Omega)$ be the two scalar potentials of a solution \mathbf{E} to (8), see (14). Since $\operatorname{div}(\mathbf{f} + \varepsilon\nabla\varphi_{\mathbf{E}}) = 0$ in Ω , we observe that the existence and uniqueness of $s_0 \in H_{zmv}^1(\Omega)$ fulfilling (18) is guaranteed by Theorem 2.2.

On the other hand, using problem (8), we find:

$$\begin{aligned} \mathbf{curl}(\mu^{-1}\mathbf{curl}(\varepsilon^{-1}\mathbf{curl}\psi_{0,\mathbf{E}})) - \omega^2\mathbf{curl}\psi_{0,\mathbf{E}} &= \mathbf{curl}(\mu^{-1}\mathbf{curl}\mathbf{E}) - \omega^2\varepsilon\mathbf{E} + \omega^2\varepsilon\nabla\varphi_{\mathbf{E}} \\ &= \omega^2(\mathbf{f} + \varepsilon\nabla\varphi_{\mathbf{E}}) \text{ in } \Omega. \end{aligned}$$

Hence, we obtain:

$$\mathbf{curl}(\mu^{-1}\mathbf{curl}(\varepsilon^{-1}\mathbf{curl}\psi_{0,\mathbf{E}})) - \omega^2\mathbf{curl}\psi_{0,\mathbf{E}} = \omega^2\mathbf{curl}s_0 \text{ in } \Omega.$$

Consequently, we derive:

$$\mathbf{curl}(\mu^{-1}\mathbf{curl}(\varepsilon^{-1}\mathbf{curl}\psi_{0,\mathbf{E}})) - \omega^2\psi_{0,\mathbf{E}} - \omega^2s_0 = 0 \text{ in } \Omega.$$

Because Ω is connected, the scalar field $\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \mathbf{curl} \psi_{0,\mathbf{E}}) - \omega^2 \psi_{0,\mathbf{E}} - \omega^2 s_0$ is equal to some constant, i.e. there exists a (unique) $c \in \mathbb{R}$ such that :

$$\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \mathbf{curl} \psi_{0,\mathbf{E}}) - \omega^2 \psi_{0,\mathbf{E}} - \omega^2 s_0 = \omega^2 c \text{ in } \Omega.$$

We deduce that:

$$\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \mathbf{curl}(\psi_{0,\mathbf{E}} + c)) - \omega^2 (\psi_{0,\mathbf{E}} + c) = \omega^2 s_0 \text{ in } \Omega.$$

Recall that $\mu \in L^\infty(\Omega)$, so we have:

$$\operatorname{curl}(\varepsilon^{-1} \mathbf{curl}(\psi_{0,\mathbf{E}} + c)) - \omega^2 \mu (\psi_{0,\mathbf{E}} + c) = \omega^2 \mu s_0 \text{ in } \Omega.$$

Let $\psi_{c,\mathbf{E}} := (\psi_{0,\mathbf{E}} + c) \in H^1(\Omega)$. We note that $\operatorname{curl}(\varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}}) = -\operatorname{div}(\varepsilon^{-1} \nabla \psi_{c,\mathbf{E}})$ in Ω , hence

$$-\operatorname{div}(\varepsilon^{-1} \nabla \psi_{c,\mathbf{E}}) - \omega^2 \mu \psi_{c,\mathbf{E}} = \omega^2 \mu s_0 \text{ in } \Omega.$$

According to (14), one has

$$\mathbf{E} = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl} \psi_{0,\mathbf{E}} = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl}(\psi_{0,\mathbf{E}} + c) = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}} \text{ in } \Omega,$$

and because \mathbf{E} is a solution to the problem (8), we infer the boundary condition:

$$\varepsilon^{-1} \nabla \psi_{c,\mathbf{E}} \cdot \boldsymbol{\nu} = \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}} \cdot \boldsymbol{\tau} = \mathbf{E} \cdot \boldsymbol{\tau} = 0 \text{ on } \partial\Omega.$$

We conclude that the equations governing $\psi_{c,\mathbf{E}} \in H^1(\Omega)$ are indeed (17). In particular $\varepsilon^{-1} \nabla \psi_{c,\mathbf{E}}$ belongs to $\mathbf{H}(\operatorname{div}; \Omega)$ with vanishing normal trace. Then, for $\psi' \in H^1(\Omega)$, we find by integration by parts (1) that

$$(\varepsilon^{-1} \nabla \psi_{c,\mathbf{E}}, \nabla \psi')_{0,\Omega} - \omega^2 (\mu \psi_{c,\mathbf{E}}, \psi')_{0,\Omega} = \omega^2 (\mu s_0, \psi')_{0,\Omega}, \quad \forall \psi' \in H^1(\Omega).$$

In other words, $\psi_{c,\mathbf{E}}$ is governed by the variational formulation (19).

The fact that (19) implies (17) is classical: in particular, one uses the fact that the trace mapping is surjective from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ to derive the boundary condition. \square

How to characterize the auxiliary potential s_0 variationally? We note that if s_0 is given by (18), then obviously it solves

$$\begin{cases} \text{Find } s_0 \in H_{zmv}^1(\Omega) \text{ such that:} \\ (\mathbf{curl} s_0, \mathbf{curl} \psi')_{0,\Omega} = (\mathbf{f} + \varepsilon \nabla \varphi_{\mathbf{E}}, \mathbf{curl} \psi')_{0,\Omega}, \quad \forall \psi' \in H_{zmv}^1(\Omega), \end{cases} \quad (20)$$

which is a well-posed variational formulation.

Lemma 4.6. *The auxiliary potential s_0 solves (18) if, and only if, it solves (20).*

Proof. We just proved that if s_0 solves (18), then it solves (20).

Reciprocally, if s_0 solves (20) which holds for all $\psi' \in H_{zmv}^1(\Omega)$, then choosing $\psi' \in \mathcal{D}(\Omega)$, we find first that $\operatorname{curl}(\mathbf{curl} s_0 - \mathbf{f} - \varepsilon \nabla \varphi_{\mathbf{E}}) = 0$ in $\mathcal{D}'(\Omega)$. Next, using $\psi' \in H^1(\Omega)$, we find after integration by parts (2) that:

$$\langle (\mathbf{curl} s_0 - \mathbf{f} - \varepsilon \nabla \varphi_{\mathbf{E}}) \cdot \boldsymbol{\tau}, \psi'_{|\partial\Omega} \rangle_{H^{1/2}(\partial\Omega)} = 0.$$

Since the trace mapping is surjective from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$, we conclude that $(\mathbf{curl} s_0 - \mathbf{f} - \varepsilon \nabla \varphi_{\mathbf{E}}) \cdot \boldsymbol{\tau} = 0$ on $\partial\Omega$. According to (4), there exists $z \in H_0^1(\Omega)$ such that $\mathbf{curl} s_0 - \mathbf{f} - \varepsilon \nabla \varphi_{\mathbf{E}} = \nabla z$ in Ω . Also,

$$\Delta z = \operatorname{div}(\mathbf{f} - \varepsilon \nabla \varphi_{\mathbf{E}}) = 0 \text{ in } \Omega,$$

and we conclude that $z = 0$ so that $\mathbf{curl} s_0 = \mathbf{f} + \varepsilon \nabla \varphi_{\mathbf{E}}$, i.e. s_0 is characterized by (18). \square

As we observed in the proof of Proposition 4.5, the presence of the constant c does not affect the resolution of the (8) problem. In fact, one has

$$\mathbf{E} = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}} \text{ in } \Omega.$$

We conclude that to solve the problem (8), one should be looking for the two scalar potentials $\varphi_{\mathbf{E}} \in H_0^1(\Omega)$ and $\psi_{c,\mathbf{E}} \in H^1(\Omega)$ respectively governed by (15) and (17), or their equivalent variational formulations (16) and (19). We summarize the results in the next theorem.

Theorem 4.7. *Let $\mathbf{E} \in \mathbf{L}^2(\Omega)$ be governed by (8). Then there exist two scalar potentials $\varphi_{\mathbf{E}} \in H_0^1(\Omega)$ and $\psi_{c,\mathbf{E}} \in H^1(\Omega)$ such that $\mathbf{E} = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}}$ in Ω , where $\varphi_{\mathbf{E}}$ is a solution to (15), resp. $\psi_{c,\mathbf{E}} \in H^1(\Omega)$ is a solution to (17).*

Conversely, Let $\varphi_{\mathbf{E}} \in H_0^1(\Omega)$ be governed by (15), the auxiliary field s_0 be characterized by (18), and $\psi_{c,\mathbf{E}} \in H^1(\Omega)$ be governed by (17). Then $\mathbf{E} = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}} \in \mathbf{L}^2(\Omega)$ is a solution to (8). Finally, the problem (8) is well-posed.

Proof. The first part of the theorem has already been established, see Propositions 4.4 and 4.5. Conversely, let $\mathbf{E} = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}}$, where $\varphi_{\mathbf{E}}$, s_0 , $\psi_{c,\mathbf{E}}$ are defined as in the statement. One finds that

$$\mathbf{curl} \mathbf{E} = \mathbf{curl}(\varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}}) = -\operatorname{div}(\varepsilon^{-1} \nabla \psi_{c,\mathbf{E}}) \stackrel{(17)}{=} \omega^2 \mu \psi_{c,\mathbf{E}} + \omega^2 \mu s_0 \text{ in } \Omega.$$

In particular, $\mathbf{E} \in \mathbf{H}(\mathbf{curl}; \Omega)$. Furthermore,

$$\mathbf{E} \cdot \boldsymbol{\tau} = \nabla \varphi_{\mathbf{E}} \cdot \boldsymbol{\tau} + \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}} \cdot \boldsymbol{\tau} = \nabla \varphi_{\mathbf{E}} \cdot \boldsymbol{\tau} + \varepsilon^{-1} \nabla \psi_{c,\mathbf{E}} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega,$$

that is $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \Omega)$. Finally,

$$\begin{aligned} \mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{E}) &= \omega^2 \mathbf{curl} \psi_{c,\mathbf{E}} + \omega^2 \mathbf{curl} s_0 \\ &\stackrel{(18)}{=} \omega^2 \mathbf{curl} \psi_{c,\mathbf{E}} + \omega^2 \mathbf{f} + \omega^2 \varepsilon \nabla \varphi_{\mathbf{E}} \\ &= \omega^2 \varepsilon \mathbf{E} + \omega^2 \mathbf{f} \text{ in } \Omega. \end{aligned}$$

So we conclude that \mathbf{E} is a solution to problem (8). Finally, we note that well-posedness of problem (8) is a direct consequence of Assumptions 1 and 2. \square

5 A volume optimal control method for solving scalar problems

In what follows, we will adopt *an optimal control method* to solve numerically the variational formulations (16) and (19), resp. corresponding to the Dirichlet problem, and the Neumann problem. As demonstrated in Section 4, these two scalar problems are coupled. As a matter of fact, to solve the variational formulation (19), we need to solve first the variational formulation (16), then to calculate the source term s_0 , as characterized by (18). For simplicity, we will use the notations φ and ψ in place of $\varphi_{\mathbf{E}}$ and $\psi_{c,\mathbf{E}}$, respectively. The method for solving the Dirichlet problem has been thoroughly investigated in [17]. In the subsequent sections, we will focus on the method for solving the Neumann problem.

To be able to apply the volume optimal control method, we will work under the following assumption regarding the Fredholm operator $A_- \in \mathcal{L}(H^1(\Omega_-), (H^1(\Omega_-))^*)$ defined by:

$$\langle A_- u, v \rangle_{(H^1(\Omega_-))^*, H^1(\Omega_-)} := (\varepsilon_-^{-1} \nabla u, \nabla v)_{0, \Omega_-} - \omega_-^2 (\mu_- u, v)_{0, \Omega_-} \quad \forall u, v \in H^1(\Omega_-).$$

Assumption 3. *The coefficients ε_- and μ_- are such that the operator A_- is an isomorphism.*

For further use, we introduce the related sesquilinear form $a_- : H^1(\Omega_-) \times H^1(\Omega_-) \rightarrow \mathbb{C}$

$$a_-(u, v) = \langle A_- u, v \rangle_{(H^1(\Omega_-))^*, H^1(\Omega_-)}, \quad \forall u, v \in H^1(\Omega_-).$$

5.1 A first reformulation of the Neumann problem

We propose below a reformulation of problem (17) with a right-hand side $f^N = \omega^2 \mu s_0$, in which the unknown $\psi \in \mathbf{H}^1(\Omega)$ is split into the two unknowns ψ_+ and ψ_- . Given that $f^N \in L^2(\Omega)$, the solution ψ to (17) is such that $\varepsilon^{-1} \nabla \psi$ belongs to $\mathbf{H}(\text{div}; \Omega)$. Consequently, finding ψ as a solution to problem (17) is equivalent to finding $(\psi_+, \psi_-) \in \mathbf{H}^1(\Omega_+) \times \mathbf{H}^1(\Omega_-)$, which is governed by

$$\begin{cases} -\text{div}(\varepsilon_+^{-1} \nabla \psi_+) - \omega^2 \mu_+ \psi_+ = f_+^N & \text{in } \Omega_+, \\ -\text{div}(\varepsilon_-^{-1} \nabla \psi_-) - \omega^2 \mu_- \psi_- = f_-^N & \text{in } \Omega_-, \\ \psi_+ = \psi_- \text{ and } \varepsilon_+^{-1} \partial_{\boldsymbol{\nu}} \psi_+ = \varepsilon_-^{-1} \partial_{\boldsymbol{\nu}} \psi_- & \text{on } \Sigma, \\ \varepsilon_+^{-1} \nabla \psi_+ \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega_+ \setminus \Sigma, \quad \varepsilon_-^{-1} \nabla \psi_- \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega_- \setminus \Sigma. \end{cases} \quad (21)$$

The equations governing by ψ_+ and ψ_- are elliptic. However, we cannot solve them independently, because they are coupled through the transmission conditions on Σ . To overcome this difficulty, we will apply the same strategy as in [17], by introducing an optimization-based formulation with volume control, which will be explained in the following section.

5.2 An optimal control reformulation of the Neumann problem

Instead of looking for $(\psi_+, \psi_-) \in \mathbf{H}^1(\Omega_+) \times \mathbf{H}^1(\Omega_-)$ as a solution to (21), the approach below focuses on finding a pair of functions $(\tilde{\psi}, \psi_-) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega_-)$, where $\tilde{\psi}$ is a continuous extension of the unknown ψ_+ to the entire domain Ω . This extension will be characterized by solving an optimal control problem, detailed below.

Remark 5.1. *It is worth noting that one could choose an extension $\tilde{\psi}$ from Ω_- to Ω so that $(\psi_+, \tilde{\psi}|_{\Omega_-})$ is a solution to (21). Or, one could follow the approach of [13] and extend the function ψ_+ to a strict sub-domain of Ω_- .*

To proceed, we denote by $\tilde{\varepsilon}$ and $\tilde{\mu}$ extensions of the functions ε_+ and μ_+ to Ω . Concerning their values in Ω_- , we choose them as follows:

$$\tilde{\varepsilon}, \tilde{\mu} \in L^\infty(\Omega), \text{ and } \exists c_1 > 0, \exists c_2 < 0 \text{ such that } \tilde{\varepsilon} > c_1 > 0 \text{ and } \tilde{\mu} < c_2 < 0 \text{ a.e. in } \Omega_-. \quad (22)$$

With this choice (22), we can define the following inner product on the space $\mathbf{H}^1(\Omega_-)$:

$$\begin{aligned} (\cdot, \cdot)_{\tilde{\varepsilon}, \tilde{\mu}} : \mathbf{H}^1(\Omega_-) \times \mathbf{H}^1(\Omega_-) &\rightarrow \mathbb{R} \\ (u, v) &\mapsto (u, v)_{\tilde{\varepsilon}, \tilde{\mu}} := (\tilde{\varepsilon}^{-1} \nabla u, \nabla v)_{0, \Omega_-} - \omega^2 (\tilde{\mu} u, v)_{0, \Omega_-}. \end{aligned}$$

The corresponding norm is denoted by $\|\cdot\|_{\tilde{\varepsilon}, \tilde{\mu}}$.

We then define the sesquilinear form $\tilde{a} : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \rightarrow \mathbb{C}$:

$$\tilde{a}(u, v) = (\tilde{\varepsilon}^{-1} \nabla u, \nabla v)_{0, \Omega} - \omega^2 (\tilde{\mu} u, v)_{0, \Omega}, \quad \forall u, v \in \mathbf{H}^1(\Omega),$$

and the operator $\tilde{\mathbf{A}} \in \mathcal{L}(\mathbf{H}^1(\Omega), (\mathbf{H}^1(\Omega))^*)$

$$\langle \tilde{\mathbf{A}} u, v \rangle_{(\mathbf{H}^1(\Omega))^*, \mathbf{H}^1(\Omega)} = \tilde{a}(u, v) \quad \forall u, v \in \mathbf{H}^1(\Omega). \quad (23)$$

Then, one has the

Lemma 5.2. *Except for a discrete set of $\omega \in \mathbb{R}_{>0}$, the operator $\tilde{\mathbf{A}}$ is an isomorphism.*

Proof. Let us consider the operator $\mathbf{A}_0 \in \mathcal{L}(\mathbf{H}^1(\Omega))$, and the related sesquilinear form, defined by

$$(\mathbf{A}_0 u, v)_{1, \Omega} = (\tilde{\varepsilon}^{-1} \nabla u, \nabla v)_{0, \Omega}, \quad \forall u, v \in \mathbf{H}^1(\Omega).$$

The sesquilinear form is coercive and thus \mathbf{A}_0 an isomorphism. On the other hand, introducing the compact operator $\mathbf{C} \in \mathcal{L}(\mathbf{H}^1(\Omega))$ defined by

$$(\mathbf{C}u, v)_{1, \Omega} = (\tilde{\mu}u, v)_{0, \Omega}, \quad \forall u, v \in \mathbf{H}^1(\Omega),$$

we note that $(\omega^2 \mathbf{C})_{\omega \in \mathbb{R}_{>0}}$ is a family of compact operators. By applying the analytic Fredholm theorem, we conclude that, except for a discrete set of $\omega \in \mathbb{R}_{>0}$, the operator $\tilde{\mathbf{A}} = \mathbf{A}_0 - \omega^2 \mathbf{C}$ is an isomorphism. \square

In other words, for any set of coefficients $\tilde{\varepsilon}$ and $\tilde{\mu}$ satisfying the condition (22), $\tilde{\mathbf{A}}$ is almost surely an isomorphism. We assume this is true from now on.

Now, let $E \in \mathcal{L}(\mathbf{H}^1(\Omega_+), \mathbf{H}^1(\Omega))$ be an arbitrary continuous extension operator. We have the

Lemma 5.3. *It holds that*

$$\begin{cases} \tilde{a}(E(\psi_+), v) = (f_+^N, v)_{0, \Omega_+} + (E(\psi_+), v)_{\tilde{\varepsilon}, \tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v \rangle_\Sigma, & \forall v \in \mathbf{H}^1(\Omega), \\ a_-(\psi_-, v_-) = (f_-^N, v_-)_{0, \Omega_-} + \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v_- \rangle_\Sigma, & \forall v_- \in \mathbf{H}^1(\Omega_-). \end{cases} \quad (24)$$

Proof. $\forall v \in \mathbf{H}^1(\Omega)$, we have

$$\begin{aligned} \tilde{a}(E(\psi_+), v) &= (\tilde{\varepsilon}^{-1} \nabla E(\psi_+), \nabla v)_{0, \Omega} - \omega^2 (\tilde{\mu} E(\psi_+), v)_{0, \Omega} \\ &= (\varepsilon_+^{-1} \nabla \psi_+, \nabla v)_{0, \Omega_+} - \omega^2 (\mu_+ \psi_+, v)_{0, \Omega_+} + (\tilde{\varepsilon}^{-1} \nabla E(\psi_+), \nabla v)_{0, \Omega_-} - \omega^2 (\tilde{\mu} E(\psi_+), v)_{0, \Omega_-} \\ &= -(\operatorname{div}(\varepsilon_+^{-1} \nabla \psi_+), v)_{0, \Omega_+} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v \rangle_\Sigma - \omega^2 (\mu_+ \psi_+, v)_{0, \Omega_+} \\ &\quad + (\tilde{\varepsilon}^{-1} \nabla E(\psi_+), \nabla v)_{0, \Omega_-} - \omega^2 (\tilde{\mu} E(\psi_+), v)_{0, \Omega_-} \\ &= (f_+^N, v)_{0, \Omega_+} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v \rangle_\Sigma + (\tilde{\varepsilon}^{-1} \nabla E(\psi_+), \nabla v)_{0, \Omega_-} - \omega^2 (\tilde{\mu} E(\psi_+), v)_{0, \Omega_-}. \end{aligned}$$

While, $\forall v_- \in \mathbf{H}^1(\Omega_-)$, we have

$$\begin{aligned} a_-(\psi_-, v_-) &= (\varepsilon_-^{-1} \nabla \psi_-, \nabla v_-)_{0, \Omega_-} - \omega^2 (\mu_- \psi_-, v_-)_{0, \Omega_-} \\ &= -(\operatorname{div}(\varepsilon_-^{-1} \nabla \psi_-), v_-)_{0, \Omega_-} + \langle \varepsilon_-^{-1} \partial_\nu \psi_-, v_- \rangle_\Sigma - \omega^2 (\mu_- \psi_-, v_-)_{0, \Omega_-} \\ &= (f_-^N, v_-)_{0, \Omega_-} + \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v_- \rangle_\Sigma \end{aligned}$$

\square

By design, $(\cdot, \cdot)_{\tilde{\varepsilon}, \tilde{\mu}}$ is an inner product in $\mathbf{H}^1(\Omega_-)$. Hence, for each $E(\psi_+) \in \mathbf{H}^1(\Omega)$, the Riesz representation theorem ensures the existence of a unique $z_{E(\psi_+)} \in \mathbf{H}^1(\Omega_-)$ such that

$$(z_{E(\psi_+)}, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (E(\psi_+), v_-)_{\tilde{\varepsilon}, \tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v_- \rangle_\Sigma, \quad \forall v_- \in \mathbf{H}^1(\Omega_-). \quad (25)$$

Consequently, we find that

$$\begin{cases} \tilde{a}(E(\psi_+), v) = (f_+^N, v)_{0, \Omega_+} + (z_{E(\psi_+)}, v)_{\tilde{\varepsilon}, \tilde{\mu}}, & \forall v \in \mathbf{H}^1(\Omega), \\ a_-(\psi_-, v_-) = (f_-^N, v_-)_{0, \Omega_-} + (E(\psi_+) - z_{E(\psi_+)}, v_-)_{\tilde{\varepsilon}, \tilde{\mu}}, & \forall v_- \in \mathbf{H}^1(\Omega_-). \end{cases} \quad (26)$$

Let us now proceed backwards. Given an auxiliary function $z \in \mathbf{H}^1(\Omega_-)$, we introduce the couple of functions $(\psi^z, \psi_-^z) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega_-)$ governed by

$$\begin{cases} \tilde{a}(\psi^z, v) = (f_+^N, v)_{0, \Omega_+} + (z, v)_{\tilde{\varepsilon}, \tilde{\mu}}, & \forall v \in \mathbf{H}^1(\Omega), \\ a_-(\psi_-^z, v_-) = (f_-^N, v_-)_{0, \Omega_-} + (\psi^z - z, v_-)_{\tilde{\varepsilon}, \tilde{\mu}}, & \forall v_- \in \mathbf{H}^1(\Omega_-). \end{cases} \quad (27)$$

Proposition 5.4. *The system of equations (27) is well-posed, and $\exists C' > 0$ (independent of z and f^N) such that:*

$$\|\psi^z\|_{\mathbf{H}^1(\Omega)} + \|\psi_-^z\|_{\mathbf{H}^1(\Omega_-)} \leq C'(\|f^N\|_{\mathbf{L}^2(\Omega)} + \|z\|_{\mathbf{H}^1(\Omega_-)}).$$

Proof. The operator $\tilde{\mathbf{A}}$ is an isomorphism. This implies that the first equation of system (27) is well-posed, i.e., $\exists C_1 > 0$ (independent of z and f^N) such that:

$$\|\psi^z\|_{\mathbf{H}^1(\Omega)} \leq C_1(\|f_+^N\|_{\mathbf{L}^2(\Omega_+)} + \|z\|_{\mathbf{H}^1(\Omega_-)}).$$

Regarding the second equation of this system, according to Assumption 3, this equation is well-posed, i.e. $\exists C_2 > 0$ (independent of z, ψ^z , and f_-^N) such that:

$$\begin{aligned} \|\psi_-^z\|_{\mathbf{H}^1(\Omega_-)} &\leq C_2(\|f_-^N\|_{\mathbf{L}^2(\Omega_-)} + \|\psi^z\|_{\mathbf{H}^1(\Omega)} + \|z\|_{\mathbf{H}^1(\Omega_-)}) \\ &\leq C_2(\|f_-^N\|_{\mathbf{L}^2(\Omega_-)} + C_1(\|f_+^N\|_{\mathbf{L}^2(\Omega_+)} + \|z\|_{\mathbf{H}^1(\Omega_-)}) + \|z\|_{\mathbf{H}^1(\Omega_-)}) \\ &\leq C(\|f_-^N\|_{\mathbf{L}^2(\Omega_-)} + \|f_+^N\|_{\mathbf{L}^2(\Omega_+)} + \|z\|_{\mathbf{H}^1(\Omega_-)}) \end{aligned}$$

Finally, we conclude that there exist $C' > 0$ such that:

$$\|\psi^z\|_{\mathbf{H}^1(\Omega)} + \|\psi_-^z\|_{\mathbf{H}^1(\Omega_-)} \leq C'(\|f^N\|_{\mathbf{L}^2(\Omega)} + \|z\|_{\mathbf{H}^1(\Omega_-)}).$$

□

As we proceed, we show that introducing a control $z \in \mathbf{H}^1(\Omega_-)$ allows us to construct a pair of functions that satisfy the same equation as ψ in both Ω_+ and Ω_- , with their respective boundary conditions and the vanishing jump of the normal derivative on the interface, although in general without vanishing jump of the trace on the interface.

Proposition 5.5. *$\forall z \in \mathbf{H}^1(\Omega_-)$, the functions $\psi_+^z := \psi^z|_{\Omega_+}$ and ψ_-^z are such that*

$$\begin{cases} -\operatorname{div}(\varepsilon_+^{-1} \nabla \psi_+^z) - \omega^2 \mu_+ \psi_+^z = f_+^N \text{ in } \Omega_+, \\ -\operatorname{div}(\varepsilon_-^{-1} \nabla \psi_-^z) - \omega^2 \mu_- \psi_-^z = f_-^N \text{ in } \Omega_-, \\ \varepsilon_+^{-1} \partial_\nu \psi_+^z = \varepsilon_-^{-1} \partial_\nu \psi_-^z \text{ on } \Sigma, \\ \varepsilon_+^{-1} \nabla \psi_+ \cdot \nu = 0 \text{ on } \partial\Omega_+ \setminus \Sigma, \quad \varepsilon_-^{-1} \nabla \psi_- \cdot \nu = 0 \text{ on } \partial\Omega_- \setminus \Sigma. \end{cases} \quad (28)$$

Proof. To carry out the proof, we will divide it into four steps.

First step: Show that: $-\operatorname{div}(\varepsilon_+^{-1} \nabla \psi_+^z) - \omega^2 \mu_+ \psi_+^z = f_+^N$ in Ω_+ .

Let's take $v_+ \in \mathcal{D}(\Omega_+)$ and extend it by 0 to the set Ω to obtain the function $v \in \mathcal{D}(\Omega)$. Using v in the first equation of the problem (27), we obtain

$$-\operatorname{div}(\varepsilon_+^{-1} \nabla \psi_+^z) - \omega^2 \mu_+ \psi_+^z = f_+^N \text{ in } \Omega_+.$$

Second step: Show that $-\operatorname{div}(\varepsilon_-^{-1} \nabla \psi_-^z) - \omega^2 \mu_- \psi_-^z = f_-^N$ in Ω_- .

Let's take $v_- \in \mathcal{D}(\Omega_-)$ extend it by 0 to the set Ω to obtain the function $v \in \mathcal{D}(\Omega)$. Using v and v_- in the problem (27), we find by replacing f_N^+ that

$$\begin{cases} (\tilde{\varepsilon}^{-1} \nabla \psi^z, \nabla v_-)_{0, \Omega_-} - \omega^2 (\tilde{\mu} \psi^z, v_-)_{0, \Omega_-} = (\tilde{\varepsilon}^{-1} \nabla z, \nabla v_-)_{0, \Omega_-} - \omega^2 (\tilde{\mu} z, v_-)_{0, \Omega_-}, \\ (\varepsilon_-^{-1} \nabla \psi_-^z, \nabla v_-)_{0, \Omega_-} - \omega^2 (\mu_- \psi_-^z, v_-)_{0, \Omega_-} = (f_-^N, v_-)_{0, \Omega_-} + (\varepsilon_-^{-1} \nabla (\psi^z - z), \nabla v_-)_{0, \Omega_-} \\ \quad - \omega^2 (\tilde{\mu} (\psi^z - z), v_-)_{0, \Omega_-}. \end{cases}$$

Summing the two formulations, we obtain that

$$(\varepsilon_-^{-1} \nabla \psi_-^z, \nabla v_-)_{0, \Omega_-} - \omega^2 (\mu_- \psi_-^z, v_-)_{0, \Omega_-} = (f_-^N, v_-)_{0, \Omega_-}.$$

Then we find that: $-\operatorname{div}(\varepsilon_-^{-1} \nabla \psi_-^z) - \omega^2 \mu_- \psi_-^z = f_-^N$ in Ω_- .

Third step: Show that $\varepsilon_+^{-1} \partial_\nu \psi_+^z = 0$ on $\partial\Omega_+ \setminus \Sigma$.

Let's take $v_+ \in H_{0,\Sigma}^1(\Omega_+)$ and extend it by 0 to the set Ω to obtain the function $v \in H^1(\Omega)$. Using now v in the first equation of the problem (27) and integrating by parts, we obtain that $\varepsilon_+^{-1} \partial_\nu \psi_+^z = 0$ on $\partial\Omega_+ \setminus \Sigma$ thanks to the surjectivity of the trace mapping on $\Omega_+ \setminus \Sigma$ (starting from $H_{0,\Sigma}^1(\Omega_+)$).

We obtain similarly that $\varepsilon_-^{-1} \partial_\nu \psi_-^z = 0$ on $\partial\Omega_- \setminus \Sigma$. In particular, both $\varepsilon_+^{-1} \partial_\nu \psi_+^z|_\Sigma$ and $\varepsilon_-^{-1} \partial_\nu \psi_-^z|_\Sigma$ belong to $(H^{1/2}(\Sigma))'$ according to Lemma 3.10 of [3].

Fourth step:

Let's show that $\varepsilon_+^{-1} \partial_\nu \psi_+^z = \varepsilon_-^{-1} \partial_\nu \psi_-^z$ on Σ . Let's take $v \in H^1(\Omega)$ and use $v_- = v|_{\Omega_-} \in H^1(\Omega_-)$ as test functions in (27), so

$$\begin{cases} \tilde{a}(\psi^z, v) = (f_+^N, v)_{0,\Omega_+} + (\tilde{\varepsilon}^{-1} \nabla z, \nabla v_-)_{0,\Omega_-} - \omega^2 (\tilde{\mu} z, v_-)_{0,\Omega_-}, \\ (\psi_-^z, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (f_-^N, v_-)_{0,\Omega_-} + (\tilde{\varepsilon}^{-1} \nabla (\psi^z - z), \nabla v_-)_{0,\Omega_-} - \omega^2 (\tilde{\mu} (\psi^z - z), v_-)_{0,\Omega_-}. \end{cases}$$

By summing the two formulations, integrating by parts in both formulations (see again Lemma 3.10 of [3]) and using the equations satisfied by ψ_+^z and ψ_-^z , we deduce that

$$-\langle \varepsilon_+ \partial_\nu \psi_+^z, v \rangle_\Sigma = -\langle \varepsilon_- \partial_\nu \psi_-^z, v \rangle_\Sigma, \quad \forall v \in H^1(\Omega).$$

Thanks to the surjectivity of the trace mapping on Σ we finally obtain that $\varepsilon_+^{-1} \partial_\nu \psi_+^z = \varepsilon_-^{-1} \partial_\nu \psi_-^z$ on Σ . \square

Consequently, it remains to prove that there exists $z^* \in H^1(\Omega_-)$ such that the solution of problem (27) satisfies $\psi^{z^*} = \psi_-^{z^*}$ on Σ , so that $(\psi_{|\Omega_+}^{z^*}, \psi_-^{z^*})$ solves (21). Next, we show the existence of such a control in $H^1(\Omega_-)$.

Lemma 5.6. *There exists $z^* \in H^1(\Omega_-)$ such that the solution of problem (27) satisfies $\psi^{z^*} = \psi_-^{z^*}$ on Σ .*

Proof. According to Assumption 2, the problem (21) is well posed. Therefore, the solutions ψ_+ and ψ_- are well defined. We choose the extension $\psi = E(\psi_+)$, in particular, its restriction to the domain Ω_- is ψ_- . With this choice, the system (24) can be written as

$$\begin{cases} \tilde{a}(\psi, v) = (f_+^N, v)_{0,\Omega_+} + (\psi_-, v)_{\tilde{\varepsilon}, \tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v \rangle_\Sigma, & \forall v \in H^1(\Omega), \\ a_-(\psi_-, v_-) = (f_-^N, v_-)_{0,\Omega_-} + \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v_- \rangle_\Sigma, & \forall v_- \in H^1(\Omega_-). \end{cases}$$

Since $\psi_- \in H^1(\Omega_-)$, we know that according to the Riesz representation theorem, there exists a unique $z^* = z_{\psi_-} \in H^1(\Omega_-)$ such that $\forall v_- \in H^1(\Omega_-)$

$$(z_{\psi_-}, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (\psi_-, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v_- \rangle_\Sigma, \quad \forall v_- \in H^1(\Omega_-).$$

By inspection, we notice that the solution to (27) is $(\psi^{z^*}, \psi_-^{z^*}) = (E(\psi_+), \psi_-) = (\psi, \psi_-)$. Using this definition, and since the traces on Σ of $\psi_{|\Omega_+}$ and ψ_- match, the claim is proven. \square

This leads us to the next result.

Lemma 5.7. *Let (ψ_+, ψ_-) be the solution of (21). Then, the set of $z^* \in H^1(\Omega_-)$ such that the solution of (27) satisfying $\psi^{z^*} = \psi_-^{z^*}$ on Σ in addition to (28) is isomorphic to the set of all possible continuous extensions of ψ_+ over Ω . Moreover, z^* and ψ^{z^*} are linked by the relation*

$$(z^*, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (\psi^{z^*}, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v_- \rangle_\Sigma, \quad \forall v_- \in H^1(\Omega_-). \quad (29)$$

Proof. Let $E : H^1(\Omega_+) \rightarrow H^1(\Omega)$ be an arbitrary continuous extension operator. According to Lemma 5.3, $(E(\psi_+), \psi_-)$ satisfies the following problem

$$\begin{cases} \tilde{a}(E(\psi_+), v) = (f_+^N, v)_{0, \Omega_+} + (E(\psi_+), v)_{\tilde{\varepsilon}, \tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v \rangle_\Sigma, & \forall v \in H^1(\Omega), \\ a_-(\psi_-, v_-) = (f_-^N, v_-)_{0, \Omega_-} + \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v_- \rangle_\Sigma, & \forall v_- \in H^1(\Omega_-). \end{cases}$$

We proceed as in the proof of Lemma 5.6. Since $E(\psi_+) \in H^1(\Omega)$, the Riesz representation theorem guarantees the existence of a unique $z^* = z_{E(\psi_+)} \in H^1(\Omega_-)$ such that

$$(z_{E(\psi_+)}, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (E(\psi_+), v_-)_{\tilde{\varepsilon}, \tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_\nu \psi_+, v_- \rangle_\Sigma, \quad \forall v_- \in H^1(\Omega_-).$$

Then, we conclude as before that $(\psi^{z^*}, \psi_-^{z^*}) = (E(\psi_+), \psi_-)$ (and that the traces on Σ match). Replacing $E(\psi_+)$ by z^* , we find (29). Finally, if one chooses two extensions such that $E_1(\psi_+) \neq E_2(\psi_+)$, looking once more at (29) we note that the corresponding controls z_1^* and z_2^* are different. Conversely, assume that one is given a z^* such that $\psi_+^{z^*}|_\Sigma = \psi_-^{z^*}|_\Sigma$. It follows that $\psi_+^{z^*}$ and $\psi_-^{z^*}$, which are already governed by (28), fulfill (21): one concludes by uniqueness that $\psi_+^{z^*} = \psi_+$, and $\psi_-^{z^*} = \psi_-$, so ψ^{z^*} is an extension of ψ_+ . \square

We will explain in the section below how to choose a control function $z^* \in H^1(\Omega_-)$.

5.3 Reformulation of the problem as a minimization problem

To find a function $z^* \in H^1(\Omega_-)$, such that $\psi^{z^*} = \psi_-^{z^*}$ on Σ , one can solve the following optimal control problem

$$\text{Find } z^* = \underset{z \in H^1(\Omega_-)}{\operatorname{argmin}} J(z) \quad \text{with} \quad J(z) = \frac{1}{2} \int_\Sigma |\psi^z - \psi_-^z|^2 \, d\sigma, \quad (30)$$

where (ψ^z, ψ_-^z) is the solution of (27).

Thanks to Lemma 5.7, one can show without any difficulty, that the optimal control problem (30) has an infinite number of solutions. Then, the set of solution of (30), denoted M_J , is infinite.

Below, we show some properties of the functional J and its set of minimizers M_J .

5.3.1 Properties of the functional J and its set of minimizers M_J

The proofs of the following propositions are similar to those presented in [17].

Proposition 5.8. *The functional J satisfies the following properties:*

1. Let $(z_n)_n$ be a sequence of elements in $H^1(\Omega_-)$ that weakly converges to $z_0 \in H^1(\Omega_-)$. Then, $(J(z_n))_n$ converges to $J(z_0)$.
2. The functional J is continuous and convex on $H^1(\Omega_-)$.

Proposition 5.9. *One has:*

1. $M_J = \{z \in H^1(\Omega_-) \text{ such that } J(z) = 0\}$.
2. The set M_J is a convex and closed subset of $H^1(\Omega_-)$.

Consequently, as a result of Proposition 5.9, the minimization problem

$$\text{Find } z_J^* = \underset{z \in M_J}{\operatorname{argmin}} \|z\|_{\tilde{\varepsilon}, \tilde{\mu}}^2, \quad (31)$$

has a unique solution.

The goal of the following is to find a characterization of $E_{z_J^*}(\psi_+)$, the continuous extension of ψ_+ associated with z_J^* , which is essential for the discretization process. To proceed, we define $E_{z_H}(\psi_+) \in$

$H^1(\Omega)$, the continuous extension of ψ_+ , that satisfies

$$\begin{cases} -\operatorname{div}(\tilde{\varepsilon}^{-1} \nabla E_{z_H}(\psi_+)) - \omega^2 \tilde{\mu} E_{z_H}(\psi_+) = 0, & \text{in } \Omega_-, \\ \tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_H}(\psi_+) = 0, & \text{on } \partial\Omega_- \setminus \Sigma, \end{cases} \quad (32)$$

and we denote by $z_H \in M_J$ the minimizer associated with $E_{z_H}(\psi_+)$ through (29).

Proposition 5.10. *The minimizers z_H and z_J^* coincide.*

Proof. Starting from (32), we have by integration by parts

$$(E_{z_H}(\psi_+), v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (\tilde{\varepsilon}^{-1} \nabla E_{z_H}(\psi_+), \nabla v_-)_{0, \Omega_-} - \omega^2 (\tilde{\mu} E_{z_H}(\psi_+), v_-)_{0, \Omega_-} = 0, \quad \forall v_- \in H_{0, \Sigma}^1(\Omega_-).$$

On the other hand, we know that

$$(z_H, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (\tilde{\varepsilon}^{-1} \nabla E_{z_H}(\psi_+), \nabla v_-)_{0, \Omega_-} - \omega^2 (\tilde{\mu} E_{z_H}(\psi_+), v_-)_{0, \Omega_-} - \langle \varepsilon_+^{-1} \partial_{\nu} \psi_+, v_- \rangle_{\Sigma}, \quad \forall v_- \in H^1(\Omega_-).$$

We infer that

$$(z_H, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = 0, \quad \forall v_- \in H_{0, \Sigma}^1(\Omega_-).$$

In particular, for $v_- = E_{z_H}(\psi_+) - E_{z_J^*}(\psi_+) \in H_{0, \Sigma}^1(\Omega_-)$, we have

$$(E_{z_H}(\psi_+) - E_{z_J^*}(\psi_+), z_H)_{\tilde{\varepsilon}, \tilde{\mu}} = 0.$$

However, according to (25), we also have

$$(z_J^* - z_H, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (E_{z_H}(\psi_+) - E_{z_J^*}(\psi_+), v_-)_{\tilde{\varepsilon}, \tilde{\mu}}.$$

Hence, taking $v_- = z_H$, we obtain $(z_J^* - z_H, z_H)_{\tilde{\varepsilon}, \tilde{\mu}} = 0$. It follows by orthogonality that

$$\|z_J^*\|_{\tilde{\varepsilon}, \tilde{\mu}}^2 = \|z_J^* - z_H + z_H\|_{\tilde{\varepsilon}, \tilde{\mu}}^2 = \|z_J^* - z_H\|_{\tilde{\varepsilon}, \tilde{\mu}}^2 + \|z_H\|_{\tilde{\varepsilon}, \tilde{\mu}}^2 \geq \|z_H\|_{\tilde{\varepsilon}, \tilde{\mu}}^2,$$

so that $z_H = z_J^*$, due to the characterization (31) of z_J^* . \square

The main goal of the rest of Section 5.3 is to apply a regularization technique to ensure the uniqueness of the solution to problem (30).

Before that, we observe that one can characterize z_J^* .

Corollary 5.11. *The minimizer $z_J^* \in H^1(\Omega_-)$ is governed by*

$$\begin{cases} -\operatorname{div}(\tilde{\varepsilon}^{-1} \nabla z_J^*) - \omega^2 \tilde{\mu} z_J^* = 0, & \text{in } \Omega_-, \\ \tilde{\varepsilon}^{-1} \partial_{\nu} z_J^* = 0, & \text{on } \partial\Omega_- \setminus \Sigma, \\ \tilde{\varepsilon}^{-1} \partial_{\nu} z_J^* = \tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_J^*}(\psi_+) - \varepsilon_+^{-1} \partial_{\nu} \psi_+, & \text{on } \Sigma. \end{cases} \quad (33)$$

Proof. We know that $z_J^* = z_H$, where $z_H \in H^1(\Omega_-)$ is associated with $E_{z_H}(\psi_+)$ through (29), and $E_{z_H}(\psi_+)$ is itself characterized by (32). So, $z_J^* \in H^1(\Omega_-)$ and, according to (29),

$$(z_J^*, v_-)_{\tilde{\varepsilon}, \tilde{\mu}} = (E_{z_H}(\psi_+), v_-)_{\tilde{\varepsilon}, \tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_{\nu} \psi_+, v_- \rangle_{\Sigma}, \quad \forall v_- \in H^1(\Omega_-). \quad (34)$$

First, choosing $v_- \in \mathcal{D}(\Omega_-)$ and differentiating in the sense of distributions, we find thanks to (32) that

$$-\operatorname{div}(\tilde{\varepsilon}^{-1} \nabla z_J^*) - \omega^2 \tilde{\mu} z_J^* = 0 \text{ in } \Omega_-.$$

Second, given $v_- \in H^1(\Omega_-)$, we now find by a double integration by parts that

$$\langle \tilde{\varepsilon}^{-1} \partial_{\nu} z_J^*, v_- \rangle_{\partial\Omega_-} = (\tilde{\varepsilon}^{-1} \nabla z_J^*, \nabla v_-)_{0, \Omega_-} + (\operatorname{div}(\tilde{\varepsilon}^{-1} \nabla z_J^*), v_-)_{0, \Omega_-}$$

$$\begin{aligned}
&= (\tilde{\varepsilon}^{-1} \nabla z_J^*, \nabla v_-)_{0, \Omega_-} - \omega^2 (\tilde{\mu} z_J^*, v_-)_{0, \Omega_-} \\
&= (z_J^*, v_-)_{\tilde{\varepsilon}, \tilde{\mu}}; \\
\langle \tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_J^*}(\psi_+), v_- \rangle_{\Sigma} &\stackrel{(32\text{-}bottom)}{=} \langle \tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_J^*}(\psi_+), v_- \rangle_{\partial \Omega_-} \\
&= (\tilde{\varepsilon}^{-1} \nabla E_{z_J^*}(\psi_+), \nabla v_-)_{0, \Omega_-} + (\operatorname{div}(\tilde{\varepsilon}^{-1} \nabla E_{z_J^*}(\psi_+)), v_-)_{0, \Omega_-} \\
&\stackrel{(32\text{-}top)}{=} (\tilde{\varepsilon}^{-1} \nabla E_{z_J^*}(\psi_+), \nabla v_-)_{0, \Omega_-} - \omega^2 (\tilde{\mu} E_{z_H}(\psi_+), v_-)_{0, \Omega_-} \\
&= (E_{z_H}(\psi_+), v_-)_{\tilde{\varepsilon}, \tilde{\mu}}.
\end{aligned}$$

So, using (34) we find that

$$\langle \tilde{\varepsilon}^{-1} \partial_{\nu} z_J^*, v_- \rangle_{\partial \Omega_-} = \langle \tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_J^*}(\psi_+) - \varepsilon_+^{-1} \partial_{\nu} \psi_+, v_- \rangle_{\Sigma}, \quad \forall v_- \in \mathbf{H}^1(\Omega_-).$$

Then, as in the proof of Proposition 5.5, we find first that $\tilde{\varepsilon}^{-1} \partial_{\nu} z_J^* = 0$ on $\partial \Omega_- \setminus \Sigma$, and then that $\tilde{\varepsilon}^{-1} \partial_{\nu} z_J^* = \tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_J^*}(\psi_+) - \varepsilon_+^{-1} \partial_{\nu} \psi_+$ on Σ . \square

An interesting consequence of Corollary 5.11 is that one can gather further knowledge on the minimizer $z_J^* \in \mathbf{H}^1(\Omega_-)$, namely determine its a priori regularity. Let us give an illustration (with simplifying assumptions). Specifically, let us assume that: ε_+ is constant in Ω_+ ; $\tilde{\varepsilon}$ is constant in Ω_- ; $\partial \Omega \cap \Sigma = \emptyset$; and $\tau_N(\varepsilon^{-1}) \in (1/2, 1)$. Starting from (33), and since $\tilde{\varepsilon}$ is constant, z_J^* solves a Laplace problem with a right-hand side $-\omega^2 \tilde{\varepsilon} \tilde{\mu} z_J^*$ that belongs to $L^2(\Omega_-)$, and Neumann boundary condition (equal to $(\tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_J^*}(\psi_+) - \varepsilon_+^{-1} \partial_{\nu} \psi_+)_{|\Sigma}$ on Σ , vanishing on $\partial \Omega_- \setminus \Sigma$). What is the a priori regularity of this boundary data ?

On the one hand, the regularity of $\partial_{\nu} \psi_+|_{\Sigma}$ is driven by the a priori regularity of ψ : going back to Section 3.3 and thanks to the above assumption on $\tau_N(\varepsilon^{-1})$, we find in particular that $\partial_{\nu} \psi_+|_{\Sigma} \in L^2(\Sigma)$, so that $\varepsilon_+^{-1} \partial_{\nu} \psi_+|_{\Sigma} \in L^2(\Sigma)$.

On the other hand, the regularity of $\tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_J^*}(\psi_+)$ is driven by the a priori regularity of $E_{z_J^*}(\psi_+)$: according to (32), it is governed by a Laplace problem with a right-hand side that belongs to $L^2(\Omega_-)$, and mixed boundary conditions, with Dirichlet data on Σ , where the trace of $E_{z_J^*}(\psi_+)$ is by definition equal to $\psi_+|_{\Sigma}$, and vanishing Neumann data on $\partial \Omega_- \setminus \Sigma$. According again to the above assumption on $\tau_N(\varepsilon^{-1})$, we find that $\psi_+|_{\Sigma}$ belongs to $\mathbf{H}^1(\Sigma)$. Adapting the result of [26] (specifically, using that $\partial \Omega \cap \Sigma = \emptyset$ to handle separately the two boundary conditions), we now find that $E_{z_J^*}(\psi_+) \in \mathbf{H}^{3/2}(\Omega_-)$, and $\tilde{\varepsilon}^{-1} \partial_{\nu} E_{z_J^*}(\psi_+)|_{\Sigma} \in L^2(\Sigma)$.

Hence, the Neumann data for the Laplace problem with unknown z_J^* belongs to $L^2(\Sigma)$: using again [26], we conclude that the minimizer z_J^* belongs to $\mathbf{H}^{3/2}(\Omega_-)$.

In the general case, that is without the above simplifying assumptions, we note that one can nevertheless derive improved regularity results with the help of other limit regularity exponents.

5.3.2 Tikhonov regularization of the minimization problem

To ensure a unique and well-posed solution, we will apply the classical Tikhonov regularization to the problem (30). Our objective is to analyze the convergence of this regularization. To achieve this, we consider for any $\lambda > 0$ the minimization problem

$$\min_{z \in \mathbf{H}^1(\Omega_-)} J^\lambda(z) := J(z) + \lambda \|z\|_{\tilde{\varepsilon}, \tilde{\mu}}^2. \quad (35)$$

Given that the function J is strictly convex and coercive, it follows that for all $\lambda > 0$, the problem (35) has a unique solution, which we denote z_λ .

Proposition 5.12. *As $\lambda \rightarrow 0^+$, the sequence $(z_\lambda)_\lambda$ converges to z_J^* in $\mathbf{H}^1(\Omega_-)$.*

Proof. By the definitions of z_λ and of z_J^* , we have

$$\forall \lambda > 0, \quad \lambda \|z_\lambda\|_{\tilde{\varepsilon}, \tilde{\mu}}^2 \leq J^\lambda(z_\lambda) \leq J^\lambda(z_J^*) = J(z_J^*) + \lambda \|z_J^*\|_{\tilde{\varepsilon}, \tilde{\mu}}^2 = \lambda \|z_J^*\|_{\tilde{\varepsilon}, \tilde{\mu}}^2. \quad (36)$$

So, for all $\lambda > 0$, we have $\|z_\lambda\|_{\varepsilon, \bar{\mu}}^2 \leq \|z_J^*\|_{\varepsilon, \bar{\mu}}^2$, and as a consequence $(z_\lambda)_\lambda$ is bounded in $H^1(\Omega_-)$. Using Rellich's theorem, there exists a subsequence that we also denote $(z_\lambda)_\lambda$ which converges weakly to some z_0 in $H^1(\Omega_-)$, when λ tends to 0. Let's now show that z_0 is a minimizer of J . According to (36), we have

$$\forall \lambda > 0, \quad 0 \leq J(z_\lambda) \leq \lambda \|z_J^*\|_{\varepsilon, \bar{\mu}}^2.$$

This shows that $(J(z_\lambda))_\lambda$ converges to zero when λ tends to zero.

On the other hand, using the result from Proposition 5.8, we know that $(J(z_\lambda))_\lambda$ converges to $J(z_0)$. Consequently, $J(z_0) = 0$ and then z_0 is a minimizer of J .

Let's now show that $(z_\lambda)_\lambda$ converges strongly to z_0 and that $z_0 = z_J^*$. First, we have

$$\begin{cases} \|z_\lambda\|_{\varepsilon, \bar{\mu}} \leq \|z_J^*\|_{\varepsilon, \bar{\mu}} \quad \forall \lambda > 0, & \implies \quad \limsup_{\lambda \rightarrow 0} \|z_\lambda\|_{\varepsilon, \bar{\mu}} \leq \|z_J^*\|_{\varepsilon, \bar{\mu}}, \\ z_\lambda \rightharpoonup z_0 \text{ in } H^1(\Omega_-), & \implies \quad \|z_0\|_{\varepsilon, \bar{\mu}} \leq \liminf_{\lambda \rightarrow 0} \|z_\lambda\|_{\varepsilon, \bar{\mu}}, \end{cases}$$

where the second assertion is proven e.g. in Proposition 3.13 of [10]. This implies that $\|z_0\|_{\varepsilon, \bar{\mu}} \leq \|z_J^*\|_{\varepsilon, \bar{\mu}}$. From the definition of z_J^* , we have that $z_0 = z_J^*$: in other words, $(z_\lambda)_\lambda$ converges weakly to z_J^* . Also,

$$\limsup_{\lambda \rightarrow 0} \|z_\lambda\|_{\varepsilon, \bar{\mu}} = \liminf_{\lambda \rightarrow 0} \|z_\lambda\|_{\varepsilon, \bar{\mu}} = \|z_J^*\|_{\varepsilon, \bar{\mu}}.$$

Now, as $H^1(\Omega_-)$ is a Hilbert space, we get that $z_\lambda \rightarrow z_J^*$ in $H^1(\Omega_-)$ (see e.g. Proposition 3.32 of [10]). Noting that z_J^* is independent of the subsequence considered, the result is then shown. \square

The purpose of the next section is to calculate the gradient of the functional J so that we can use it in the numerical part.

5.3.3 Gradient of the functional J

To do this, let's start by recalling that the functional J is differentiable, because it can be written as a composition of the two differentiable maps j_1 and j_2 defined by: $j_1 : H^1(\Omega_-) \rightarrow L^2(\Sigma)$ and $j_2 : L^2(\Sigma) \rightarrow \mathbb{R}$ such that for all $z \in H^1(\Omega_-)$, $g \in L^2(\Sigma)$, we have

$$\begin{cases} j_1(z) = (\psi^z - \psi_-^z)|_\Sigma, \text{ where } (\psi^z, \psi_-^z) \in H^1(\Omega) \times H^1(\Omega_-) \text{ is the solution to (27),} \\ j_2(g) = \frac{1}{2} \int_\Sigma |g|^2 \, d\sigma. \end{cases}$$

Now, since the functional J is scalar valued, its differential in $z \in H^1(\Omega_-)$ can be represented by its gradient $J'(z) \in H^1(\Omega_-)$

$$\forall h \in H^1(\Omega_-), \quad (J'(z), h)_{\varepsilon, \bar{\mu}} = \lim_{t \rightarrow 0} \frac{J(z + th) - J(z)}{t}. \quad (37)$$

Several methods allow us to compute the gradient of J , such as gradient methods or quasi-Newton methods (see [19]). But, the most appropriate method is the adjoint approach which was introduced in [14]. In particular, this will allow us to compute the gradient of the functional J which depends in a non-explicit way on the main variable of the problem.

Let us introduce the Lagrangian function associated with the constrained optimization problem from the variational formulation of (27). Let $\mathbf{v} := H^1(\Omega) \times H^1(\Omega_-) \times H^1(\Omega_-) \times H^1(\Omega) \times H^1(\Omega_-)$ and $\mathcal{L} : \mathbf{v} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \mathcal{L}(\psi, \psi_-, z, d, d_-) &= \frac{1}{2} \int_\Sigma |\psi - \psi_-|^2 \, d\sigma + \tilde{a}(\psi, d) - (f_+^N, d)_{0, \Omega_+} - (z, d)_{\varepsilon, \bar{\mu}} \\ &\quad + a_-(\psi_-, d_-) - (f_-^N, d_-)_{0, \Omega_-} - (\psi - z, d_-)_{\varepsilon, \bar{\mu}}. \end{aligned}$$

The functions $d \in \mathbf{H}^1(\Omega)$, $d_- \in \mathbf{H}^1(\Omega_-)$ are the adjoint variables associated to ψ, ψ_- respectively. Given $z \in \mathbf{H}^1(\Omega_-)$, we recall that (ψ^z, ψ_-^z) is the solution of the problem (27). By construction,

$$\begin{cases} \tilde{a}(\psi^z, d) - (f_+^N, d)_{0, \Omega_+} - (z, d)_{\varepsilon, \bar{\mu}} = 0, & \forall d \in \mathbf{H}^1(\Omega), \\ a_-(\psi_-^z, d_-) - (f_-^N, d_-)_{0, \Omega_-} - (\psi^z - z, d_-)_{\varepsilon, \bar{\mu}} = 0, & \forall d_- \in \mathbf{H}^1(\Omega_-). \end{cases}$$

So, we obtain:

$$\mathcal{L}(\psi^z, \psi_-^z, z, d, d_-) = J(z), \quad \forall d \in \mathbf{H}^1(\Omega), \quad \forall d_- \in \mathbf{H}^1(\Omega_-). \quad (38)$$

On the other hand, it is clear that the functional \mathcal{L} is differentiable with respect to all its variables. For all $\mathbf{v} = (\psi, \psi_-, z, d, d_-) \in \mathbf{V}$, the partial derivatives of \mathcal{L} in \mathbf{v} belong respectively to $\partial_\psi \mathcal{L}(\mathbf{v}) \in (\mathbf{H}^1(\Omega))^*$, $\partial_{\psi_-} \mathcal{L}(\mathbf{v}) \in (\mathbf{H}^1(\Omega_-))^*$, $\partial_z \mathcal{L}(\mathbf{v}) \in (\mathbf{H}^1(\Omega_-))^*$, $\partial_d \mathcal{L}(\mathbf{v}) \in (\mathbf{H}^1(\Omega))^*$ and $\partial_{d_-} \mathcal{L}(\mathbf{v}) \in (\mathbf{H}^1(\Omega_-))^*$.

Let $d \in \mathbf{H}^1(\Omega)$ and $d_- \in \mathbf{H}^1(\Omega_-)$ be given and $\mathbf{v}^z = (\psi^z, \psi_-^z, z, d, d_-) \in \mathbf{V}$. Taking the derivative of the relation (38) with respect to z and applying the formula of the chain rule, we obtain

$$(J'(z), h)_{\varepsilon, \bar{\mu}} = \langle \partial_\psi \mathcal{L}(\mathbf{v}^z), \frac{d\psi^z}{dz}(h) \rangle + \langle \partial_{\psi_-} \mathcal{L}(\mathbf{v}^z), \frac{d\psi_-^z}{dz}(h) \rangle + \langle \partial_z \mathcal{L}(\mathbf{v}^z), h \rangle, \quad \forall h \in \mathbf{H}^1(\Omega_-). \quad (39)$$

Now, if there exists $(d^z, d_-^z) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega_-)$ for which the equations

$$\partial_\psi \mathcal{L}(\psi^z, \psi_-^z, z, d^z, d_-^z) = 0 \quad \text{and} \quad \partial_{\psi_-} \mathcal{L}(\psi^z, \psi_-^z, z, d^z, d_-^z) = 0,$$

are satisfied for all $z \in \mathbf{H}^1(\Omega_-)$, then

$$(J'(z), h)_{\varepsilon, \bar{\mu}} = \langle \partial_z \mathcal{L}(\psi^z, \psi_-^z, z, d^z, d_-^z), h \rangle.$$

Our goal now is to study the existence of (d^z, d_-^z) . For that, we calculate $\partial_\psi \mathcal{L}(\mathbf{v}^z)$ and $\partial_{\psi_-} \mathcal{L}(\mathbf{v}^z)$:

$$\begin{aligned} \langle \partial_\psi \mathcal{L}(\mathbf{v}^z), v \rangle &= \int_{\Sigma} (\psi^z - \psi_-^z) v \, d\sigma + \tilde{a}(d, v) - (d_-, v)_{\varepsilon, \bar{\mu}}, \quad \forall v \in \mathbf{H}^1(\Omega), \\ \langle \partial_{\psi_-} \mathcal{L}(\mathbf{v}^z), v_- \rangle &= - \int_{\Sigma} (\psi^z - \psi_-^z) v_- \, d\sigma + a_-(d_-, v_-), \quad \forall v_- \in \mathbf{H}^1(\Omega_-). \end{aligned}$$

Consequently, the functions $(d^z, d_-^z) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega_-)$ should verify the system of equations

$$\begin{cases} \tilde{a}(d^z, v) = (d_-^z, v)_{\varepsilon, \bar{\mu}} - \int_{\Sigma} (\psi^z - \psi_-^z) v \, d\sigma, & \forall v \in \mathbf{H}^1(\Omega), \\ a_-(d_-^z, v_-) = \int_{\Sigma} (\psi^z - \psi_-^z) v_- \, d\sigma, & \forall v_- \in \mathbf{H}^1(\Omega_-). \end{cases} \quad (40)$$

Proposition 5.13. *The system of equations (40) is well posed, i.e. $\exists C > 0$ (independent of ψ^z and ψ_-^z) such that*

$$\|d^z\|_{\mathbf{H}^1(\Omega)} + \|d_-^z\|_{\mathbf{H}^1(\Omega_-)} \leq C(\|\psi^z\|_{\mathbf{H}^1(\Omega)} + \|\psi_-^z\|_{\mathbf{H}^1(\Omega_-)}).$$

Proof. First, we look at the second equation in the system of equations (40).

Thanks to (22), this equation is well-posed so, using the continuity of the trace mapping to $L^2(\Sigma)$, $\exists C_1 > 0$ such that

$$\|d_-^z\|_{\mathbf{H}^1(\Omega_-)} \leq C_1(\|\psi^z\|_{\mathbf{H}^1(\Omega)} + \|\psi_-^z\|_{\mathbf{H}^1(\Omega_-)}).$$

On the other hand, the operator $\tilde{\mathbf{A}}$ is an isomorphism. This implies that the first equation of system (40) is well-posed, i.e. $\exists C_2 > 0$ such that

$$\begin{aligned} \|d^z\|_{\mathbf{H}^1(\Omega)} &\leq C_2(\|\psi_-^z\|_{\mathbf{H}^1(\Omega_-)} + \|\psi^z\|_{\mathbf{H}^1(\Omega)} + \|d_-^z\|_{\mathbf{H}^1(\Omega_-)}) \\ &\leq C_2(1 + C_1)(\|\psi_-^z\|_{\mathbf{H}^1(\Omega_-)} + \|\psi^z\|_{\mathbf{H}^1(\Omega)}). \end{aligned}$$

The conclusion follows. \square

As a straightforward consequence of (38), (39) and (40), one has the

Lemma 5.14. $\forall z \in \mathbf{H}^1(\Omega_-)$, there holds $J'(z) = d_-^z - d_{|\Omega_-}^z$ where (d^z, d_-^z) solves (40).

5.4 A volume optimal control reformulation of the problem satisfied by φ

This problem has already been addressed in [17]. The steps are similar to those in the previous section, but differ in two ways: the use of Dirichlet boundary conditions and the absence of zero-order term. The main distinction is that for the scalar problem associated with φ , we simply extend ε_+ like in (22) and easily recover the results of Lemma 5.2.

6 Discretization of the volume optimal control method

We present a numerical approximation of our problems, based on Lagrange Finite Elements. To simplify the presentation, we consider that Ω , Ω_+ and Ω_- are polygons. The notation h denotes the discretisation parameter (classically, for the finite element method, h is equal to the meshsize). Next, we first propose an approximation of the minimisation problem (35), and prove that the sequence of discrete controls $(z_h^*)_h$ converges to z_J^* in $\mathbf{H}^1(\Omega_-)$. In addition, by approximating (27), we prove that one can build a sequence $(\psi_h^{z_h^*})_h$ that converges to $E_{z_J^*}(\psi_+)$ in $\mathbf{H}^1(\Omega)$, respectively a sequence $(\psi_{-,h}^{z_h^*})_h$ that converges to ψ_- in $\mathbf{H}^1(\Omega_-)$. We also refer to Section 7 for a comparison of convergence rates between the method and a method using T -conform meshes (for which convergence rates are known, cf. [15, 5]). *Further estimates on the errors between the exact and discrete solutions can be found in the paper [13].*

To discretize our problems, we use a shape regular family of meshes of $\bar{\Omega}$, made of triangles $(\mathcal{T}_h)_h$, which are conforming with the interface Σ : for all h , for all $K \in \mathcal{T}_h$, there exists $i \in \{+, -\}$ such that $K \subset \bar{\Omega}_i$. Additionally, we let S_h be the family of edges of \mathcal{T}_h such that $\bar{\Sigma} := \cup_{S \in S_h} \bar{S}$.

6.1 Discretization of the problem satisfied by ψ

We outline how to discretize the problem associated with ψ , using again an arbitrary source term $f^N \in L^2(\Omega)$, instead of the source term $\omega^2 \mu s_0$.

For any $k \in \mathbb{N}^*$, we define

$$\mathbf{V}_h^k(\Omega) := \{v_h \in \mathbf{H}^1(\Omega) \mid v_h|_T \in \mathbf{P}^k(T), \forall T \in \mathcal{T}_h\},$$

where $\mathbf{P}^k(T)$ represents the space of polynomials defined on T of degree at most k .

As a preamble, we recall that, even under Assumption 2, there is no guarantee that the problem

$$\text{Find } \psi_h \in \mathbf{V}_h^k(\Omega) \text{ such that } (\varepsilon^{-1} \nabla \psi_h, \nabla v_h)_{0,\Omega} - \omega^2 (\mu \psi_h, v_h)_{0,\Omega} = (f^N, v_h)_{0,\Omega}, \quad \forall v_h \in \mathbf{V}_h^k(\Omega),$$

is well-posed, even for h small enough. This is the reason why we proceed with the discretization of the volume optimal control approach. We refer to Section 7.1 for the T -coercivity method, that is a discretisation with specifically designed meshes. To that aim, we define

$$\begin{aligned} \mathbf{V}_h^k(\Omega_i) &:= \{v_{i,h} \in \mathbf{H}^1(\Omega_i) \mid v_{i,h}|_T \in \mathbf{P}^k(T), \forall T \in \mathcal{T}_h^i\}, i = \pm, \\ \mathbf{V}_h^k(\Sigma) &:= \{g \in L^2(\Sigma) \mid \exists v_h \in \mathbf{V}_h^k(\Omega), g = v_h|_\Sigma\}. \end{aligned}$$

Moreover, we recall the basic approximation property

$$\begin{cases} \forall v \in \mathbf{H}^1(\Omega), & \lim_{h \rightarrow 0} \left(\inf_{v_h \in \mathbf{V}_h^k(\Omega)} \|v - v_h\|_{\mathbf{H}^1(\Omega)} \right) = 0, \\ \forall v_- \in \mathbf{H}^1(\Omega_-), & \lim_{h \rightarrow 0} \left(\inf_{v_{-,h} \in \mathbf{V}_h^k(\Omega_-)} \|v_- - v_{-,h}\|_{\varepsilon, \bar{\mu}} \right) = 0. \end{cases} \quad (41)$$

Mimicking (27), for given $h > 0$ and $z \in \mathbf{H}^1(\Omega_-)$, we define the functions $\psi_h^z \in \mathbf{V}_h^k(\Omega)$ and $\psi_{-,h}^z \in \mathbf{V}_h^k(\Omega_-)$ as solutions to the following well-posed discrete problems: Find $(\psi_h^z, \psi_{-,h}^z) \in \mathbf{V}_h^k(\Omega) \times \mathbf{V}_h^k(\Omega_-)$ such that

$$\begin{cases} \tilde{a}(\psi_h^z, v_h) = (f_+^N, v_h)_{0, \Omega_+} + (z, v_h)_{\varepsilon, \bar{\mu}}, & \forall v_h \in \mathbf{V}_h^k(\Omega), \\ a_-(\psi_{-,h}^z, v_{-,h}) = (f_-^N, v_{-,h})_{0, \Omega_-} + (\psi_h^z - z, v_{-,h})_{\varepsilon, \bar{\mu}}, & \forall v_{-,h} \in \mathbf{V}_h^k(\Omega_-). \end{cases} \quad (42)$$

First, we introduce the projection operator $\pi_h^k : \mathbf{H}^1(\Omega_-) \rightarrow \mathbf{V}_h^k(\Omega_-)$ such that: $\forall z \in \mathbf{H}^1(\Omega_-)$, $\pi_h^k z$ is defined as the unique element of $\mathbf{V}_h^k(\Omega_-)$ that satisfies the problem

$$(\pi_h^k z, v_{-,h})_{\varepsilon, \bar{\mu}} = (z, v_{-,h})_{\varepsilon, \bar{\mu}}, \quad \forall v_{-,h} \in \mathbf{V}_h^k(\Omega_-). \quad (43)$$

In other words, π_h^k is the orthogonal projection operator from $\mathbf{H}^1(\Omega_-)$ to $\mathbf{V}_h^k(\Omega_-)$, with respect to the inner product $(\cdot, \cdot)_{\varepsilon, \bar{\mu}}$.

Lemma 6.1. $\forall z \in \mathbf{H}^1(\Omega_-) \setminus \{0\}$, there exists $(c_h)_h \geq 0$ such that

$$\|\pi_h^k z\|_{\varepsilon, \bar{\mu}} \leq c_h \|z\|_{\varepsilon, \bar{\mu}}, \quad (44)$$

with $\lim_{h \rightarrow 0} c_h = 1$.

Proof. By the definition of the orthogonal projection operator (43), it holds that

$$\|z - \pi_h^k z\|_{\varepsilon, \bar{\mu}} = \min_{v_{-,h} \in \mathbf{V}_h^k(\Omega_-)} \|z - v_{-,h}\|_{\varepsilon, \bar{\mu}}.$$

Thanks to the triangle inequality, we have

$$\|\pi_h^k z\|_{\varepsilon, \bar{\mu}} \leq \|\pi_h^k z - z\|_{\varepsilon, \bar{\mu}} + \|z\|_{\varepsilon, \bar{\mu}} \leq \|z\|_{\varepsilon, \bar{\mu}} \left(1 + \frac{1}{\|z\|_{\varepsilon, \bar{\mu}}} \min_{v_{-,h} \in \mathbf{V}_h^k(\Omega_-)} \|z - v_{-,h}\|_{\varepsilon, \bar{\mu}} \right).$$

Using the approximability property (41) leads to the conclusion. \square

From the definitions (43) and (42), we notice that $\forall z \in \mathbf{H}^1(\Omega_-)$, we have the identities

$$\psi_h^{\pi_h^k z} = \psi_h^z \quad \text{and} \quad \psi_{-,h}^{\pi_h^k z} = \psi_{-,h}^z. \quad (45)$$

Now, we are interested in the numerical approximation of the optimization problem (30) by the finite element method. We consider the discrete problem

$$\min_{z_h \in \mathbf{V}_h^k(\Omega_-)} J_h(z_h) \quad \text{with} \quad J_h(z_h) = \frac{1}{2} \int_{\Sigma} |\psi_h^{z_h} - \psi_{-,h}^{z_h}|^2 \, d\sigma. \quad (46)$$

We proceed as before to show that the functional $J_h : \mathbf{V}_h^k(\Omega_-) \rightarrow \mathbb{R}_+$ is convex and continuous. Next, we introduce the functional

$$J_h^{\lambda_h}(z_h) := \frac{1}{2} \int_{\Sigma} |\psi_h^{z_h} - \psi_{-,h}^{z_h}|^2 \, d\sigma + \lambda_h \|z_h\|_{\varepsilon, \bar{\mu}}^2, \quad \forall z_h \in \mathbf{V}_h^k(\Omega_-),$$

with λ_h a strictly positive function that depends on h such that $\lim_{h \rightarrow 0} \lambda_h = 0$.

As $\lambda_h > 0$ for all $h > 0$, the functional $J_h^{\lambda_h}$ is continuous, strictly convex and coercive. Therefore, the finite dimensional minimization problem

$$\text{Find } z_h^* = \operatorname{argmin}_{z_h \in \mathbf{V}_h^k(\Omega_-)} J_h^{\lambda_h}(z_h) \quad (47)$$

admits a unique solution.

6.2 Convergence of the method

In this section, our goal is to choose the parameter λ_h such that the sequence of solutions of (42) with $z = z_h^*$, more precisely $(\psi_h^{z_h^*}|_{\Omega_+}, \psi_{-,h}^{z_h^*})_h$, converges to the solution of (21), when $h \rightarrow 0$.

Lemma 6.2. *We have the following estimate*

$$J_h^{\lambda_h}(z_h^*) \leq \frac{1}{2} \int_{\Sigma} |\psi_h^{z_h^*} - \psi_{-,h}^{z_h^*}|^2 d\sigma + c_h \lambda_h \|z_J^*\|_{\bar{\varepsilon}, \bar{\mu}}^2, \quad (48)$$

where z_J^* is defined in (31).

Proof. On the one hand, since $\pi_h^k z_J^* \in V_h^k(\Omega_-)$ and using the fact that $z_{k,h}^*$ is the unique solution to the optimization problem (47), we deduce that $J_h^{\lambda_h}(z_h^*) \leq J_h^{\lambda_h}(\pi_h^k z_J^*)$. On the other hand, the identity (45) gives us that

$$\frac{1}{2} \int_{\Sigma} |\psi_h^{\pi_h^k z_J^*} - \psi_{-,h}^{\pi_h^k z_J^*}|^2 = \frac{1}{2} \int_{\Sigma} |\psi_h^{z_J^*} - \psi_{-,h}^{z_J^*}|^2.$$

Therefore,

$$J_h^{\lambda_h}(\pi_h^k z_J^*) = \frac{1}{2} \int_{\Sigma} |\psi_h^{z_J^*} - \psi_{-,h}^{z_J^*}|^2 d\sigma + \lambda_h \|\pi_h^k z_J^*\|_{\bar{\varepsilon}, \bar{\mu}}^2 \leq \frac{1}{2} \int_{\Sigma} |\psi_h^{z_J^*} - \psi_{-,h}^{z_J^*}|^2 d\sigma + c_h \lambda_h \|z_J^*\|_{\bar{\varepsilon}, \bar{\mu}}^2,$$

thanks to the estimate (44). The two bounds put together lead to (48). \square

To simplify the notations, for $h > 0$ and $z \in H^1(\Omega_-)$, we denote by $A_h(z)$ the real number

$$A_h(z) = \frac{1}{2} \int_{\Sigma} |\psi_h^z - \psi_{-,h}^z|^2 d\sigma.$$

Using (45)-(46), we know that for any $z \in H^1(\Omega_-)$, we have

$$A_h(z) = J_h(\pi_h^k z).$$

Theorem 6.3. *Suppose we can choose λ_h such that the sequences $(\lambda_h)_h$ and $(A_h(z_J^*)/\lambda_h)_h$ converge to 0 when $h \rightarrow 0$. Then, when h tends to 0, we have that*

- the sequence $(z_h^*)_h$ converges to z_J^* in $H^1(\Omega_-)$;
- the sequence $(\psi_h^{z_h^*})_h$ converges to $E_{z_J^*}(\psi_+)$ in $H^1(\Omega)$, respectively the sequence $(\psi_{-,h}^{z_h^*})_h$ converges to ψ_- in $H^1(\Omega_-)$, where (ψ_+, ψ_-) is the solution of (21).

Proof. To simplify notation, we refer to $\psi_h^* \in V_h^k(\Omega)$ and $\psi_{-,h}^* \in V_h^k(\Omega_-)$ as the functions

$$\psi_h^* = \psi_h^{z_h^*} \quad \text{and} \quad \psi_{-,h}^* = \psi_{-,h}^{z_h^*}.$$

First step: Weak convergence of $(z_h^*)_h$, $(\psi_h^*)_h$ and $(\psi_{-,h}^*)_h$

According to (48),

$$\|z_h^*\|_{\bar{\varepsilon}, \bar{\mu}}^2 \leq J_h^{\lambda_h}(z_h^*)/\lambda_h \leq A_h(z_J^*)/\lambda_h + c_h \|z_J^*\|_{\bar{\varepsilon}, \bar{\mu}}^2. \quad (49)$$

Furthermore, using the fact that $(A_h(z_J^*)/\lambda_h)_h$ goes to 0 and c_h goes to 1 when $h \rightarrow 0$, we deduce that $(z_h^*)_h$ is bounded in $H^1(\Omega_-)$.

Then, using Rellich's theorem, there exists a subsequence, also called $(z_h^*)_h$ that converges weakly to a $z_0 \in H^1(\Omega_-)$. As the system of equations (42) is uniformly elliptic with respect to h , we know that the sequence $(\psi_h^*)_h$ converges weakly in $H^1(\Omega)$ to a $\psi \in H^1(\Omega)$ and the sequence $(\psi_{-,h}^*)_h$ converges weakly in $H^1(\Omega_-)$ to a $\psi_- \in H^1(\Omega_-)$. Moreover, using the basic approximation property (41), we deduce that $\psi = \psi^{z_0}$ and $\psi_- = \psi_-^{z_0}$.

Second step: z_0 is a minimizer of J

The continuity of the trace operator and the compact inclusion $H^{1/2}(\Sigma) \subset L^2(\Sigma)$ gives us

$$\psi_h^*|_{\Sigma} - \psi_{-,h}^*|_{\Sigma} \rightarrow \psi^{z_0}|_{\Sigma} - \psi_{-}^{z_0}|_{\Sigma} \text{ in } L^2(\Sigma),$$

when $h \rightarrow 0$. Noting that

$$\frac{1}{2} \int_{\Sigma} |\psi_h^* - \psi_{-,h}^*|^2 d\sigma = J_h(z_h^*) \leq J_h^{\lambda_h}(z_h^*) \leq \lambda_h (A_h(z_J^*)/\lambda_h + c_h \|z_J^*\|_{\tilde{\varepsilon}, \tilde{\mu}}^2),$$

and using the fact that $\lambda_h, A_h(z_J^*)/\lambda_h \rightarrow 0$, and $c_h \rightarrow 1$ when $h \rightarrow 0$, we deduce that $\psi^{z_0}|_{\Sigma} - \psi_{-}^{z_0}|_{\Sigma} = 0$. Therefore, z_0 is a minimizer of J .

Third step: Strong convergence of $(z_h^*)_h$ to z_J^*

Since $A_h(z_J^*)/\lambda_h \rightarrow 0$ and $c_h \rightarrow 1$ when $h \rightarrow 0$ and using the inequality (49), we obtain that

$$\limsup_{h \rightarrow 0} \|z_h^*\|_{\tilde{\varepsilon}, \tilde{\mu}} \leq \|z_J^*\|_{\tilde{\varepsilon}, \tilde{\mu}}.$$

On the other hand, $z_h^* \rightharpoonup z_0$ in $H^1(\Omega_-)$ when $h \rightarrow 0$, so (cf. again Proposition 3.13 of [10])

$$\|z_0\|_{\tilde{\varepsilon}, \tilde{\mu}} \leq \liminf_{h \rightarrow 0} \|z_h^*\|_{\tilde{\varepsilon}, \tilde{\mu}},$$

This implies that $\|z_0\|_{\tilde{\varepsilon}, \tilde{\mu}} \leq \|z_J^*\|_{\tilde{\varepsilon}, \tilde{\mu}}$ and, since z_0 is a minimizer of J , we conclude thanks to (31) that $z_0 = z_J^*$. Furthermore, we deduce that

$$\lim_{h \rightarrow 0} \|z_h^*\|_{\tilde{\varepsilon}, \tilde{\mu}} = \|z_0\|_{\tilde{\varepsilon}, \tilde{\mu}}.$$

Thanks once more to Proposition 3.32 of [10], we conclude that $(z_h^*)_h$ converges strongly to $z_0 = z_J^*$ in $H^1(\Omega_-)$.

Fourth step: Strong convergence of $(\psi_h^*)_h$ and $(\psi_{-,h}^*)_h$

Using the basic approximation property (41), the fact that the system of equations (42) is elliptic and that $(z_h^*)_h$ converges strongly to z_J^* , we obtain the convergence of $(\psi_h^*)_h$ in $H^1(\Omega)$ to $\psi^{z_J^*}$ and of $(\psi_{-,h}^*)_h$ in $H^1(\Omega_-)$ to $\psi_{-}^{z_J^*}$. Observing that $\psi^{z_J^*} = E_{z_J^*}(\psi_+)$ ends the proof. \square

Now we have to show that we can find a family of parameters $(\lambda_h)_h$ such that $(\lambda_h)_h$ and $(A_h(z_J^*)/\lambda_h)_h$ converge to 0 when $h \rightarrow 0$.

Let us introduce limit regularity exponents (with values in $(0, 1]$) for some scalar problems with coefficient $\tilde{\varepsilon}^{-1}$: with homogeneous Neumann boundary condition, set either in Ω ($\tau_N(\tilde{\varepsilon}^{-1})$), or in Ω_- ($\tau_{-,N}(\tilde{\varepsilon}^{-1})$); in Ω_- , with mixed homogeneous Dirichlet (on Σ) and Neumann (on $\partial\Omega_- \setminus \Sigma$) boundary conditions ($\tau_{-,M}(\tilde{\varepsilon}^{-1})$). We introduce

$$\tau = \min(\tau_N(\tilde{\varepsilon}^{-1}), \tau_{-,M}(\tilde{\varepsilon}^{-1})) \text{ and } \tilde{\tau} := \min(\tau_N(\tilde{\varepsilon}^{-1}), \tau_{-,N}(\tilde{\varepsilon}^{-1})).$$

Following [22], if $\tilde{\varepsilon}$ is smooth, or if it is piecewise smooth without triple point in the domain and without double point at the boundary, it holds that $\tilde{\tau} > 1/2$. In this situation, $\tau_{-,M}(\tilde{\varepsilon}^{-1}) > 1/4$ and, if $\partial\Omega \cap \Sigma = \emptyset$, $\tau_{-,M}(\tilde{\varepsilon}^{-1}) > 1/2$.

Proposition 6.4. *For all $\mathbf{s}' \in [0, \tau)$, it holds that*

$$\|\psi^{z_J^*} - \psi_h^{z_J^*}\|_{H^1(\Omega)} \lesssim h^{\mathbf{s}'} \|f_N\|_{L^2(\Omega)} \text{ and } \|\psi_{-}^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{\tilde{\varepsilon}, \tilde{\mu}} \lesssim h^{\mathbf{s}'} \|f_N\|_{L^2(\Omega)}.$$

Moreover, for all $\mathbf{s}'' \in [0, \tau + \tilde{\tau})$, it holds that

$$\|\psi^{z_J^*} - \psi_h^{z_J^*}\|_{L^2(\Omega)} \lesssim h^{\mathbf{s}''} \|f_N\|_{L^2(\Omega)} \text{ and } \|\psi_{-}^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{L^2(\Omega_-)} \lesssim h^{\mathbf{s}''} \|f_N\|_{L^2(\Omega)}.$$

Proof. Let $\mathbf{s}' \in [0, \tau)$. Recall that $\psi^{z_j^*} = E_{z_H}(\psi_+)$: one has $\psi_+^{z_j^*} = \psi_+ \in \text{PH}^{1+\mathbf{s}'}(\Omega_+)$, because $\mathbf{s}' < \tau_N(\varepsilon^{-1})$. Also, $\psi_-^{z_j^*} \in \text{H}^1(\Omega_-)$ solves equation (32): it is governed by

$$\begin{cases} -\operatorname{div}(\tilde{\varepsilon}^{-1} \nabla \psi_-^{z_j^*}) = \omega^2 \tilde{\mu} \psi_-^{z_j^*} \text{ in } \Omega_-, \\ \tilde{\varepsilon}^{-1} \partial_\nu \psi_-^{z_j^*} = 0 \text{ on } \partial\Omega_- \setminus \Sigma, \\ \psi_-^{z_j^*} = \psi_+ \text{ on } \Sigma. \end{cases}$$

This is a scalar problem with mixed boundary conditions. Since $\omega^2 \tilde{\mu} \psi_-^{z_j^*} \in \text{L}^2(\Omega_-)$ and $(\psi_+)|_\Sigma \in \text{PH}^{1/2+\mathbf{s}'}(\Sigma)$, and because $\mathbf{s}' < \tau$, we find that $\psi_-^{z_j^*} \in \text{PH}^{1+\mathbf{s}'}(\Omega_-)$. Thanks to the regularity of $\psi_+^{z_j^*} = \psi_+$, one has $\psi^{z_j^*} \in \text{PH}^{1+\mathbf{s}'}(\Omega)$.

The system of equations in (27) being elliptic with smooth coefficients, we can compare the exact and discrete solutions with the estimates

$$\begin{aligned} \|\psi^{z_j^*} - \psi_h^{z_j^*}\|_{\text{H}^1(\Omega)} &\lesssim h^{\mathbf{s}'} \|\psi^{z_j^*}\|_{\text{PH}^{1+\mathbf{s}'}(\Omega)} \lesssim h^{\mathbf{s}'} \|f_N\|_{\text{L}^2(\Omega)} \text{ and} \\ \|\psi_-^{z_j^*} - \psi_{-,h}^{z_j^*}\|_{\tilde{\varepsilon}, \tilde{\mu}} &\lesssim h^{\mathbf{s}'} \|\psi_-^{z_j^*}\|_{\text{PH}^{1+\mathbf{s}'}(\Omega_-)} \lesssim h^{\mathbf{s}'} \|f_N\|_{\text{L}^2(\Omega)}. \end{aligned}$$

This proves the first part of the claim.

Let $\mathbf{s}'' \in [0, \tau + \tilde{\tau})$. Applying the classic Aubin-Nitsche lemma to (27) (scalar problems with homogeneous Neumann boundary condition), we find that

$$\|\psi^{z_j^*} - \psi_h^{z_j^*}\|_{\text{L}^2(\Omega)} \lesssim h^{\mathbf{s}''} \|f_N\|_{\text{L}^2(\Omega)} \text{ and } \|\psi_-^{z_j^*} - \psi_{-,h}^{z_j^*}\|_{\text{L}^2(\Omega_-)} \lesssim h^{\mathbf{s}''} \|f_N\|_{\text{L}^2(\Omega)},$$

which is the second part of the claim. \square

Corollary 6.5. *For all $\mathbf{s} \in [0, 2\tau + \tilde{\tau})$, it holds that*

$$A_h(z_j^*) \lesssim h^{\mathbf{s}} \|f_N\|_{\text{L}^2(\Omega)}^2.$$

Proof. Let $\mathbf{s} \in [0, 2\tau + \tilde{\tau})$: there exists $\mathbf{s}' \in [0, \tau)$ and $\mathbf{s}'' \in [0, \tau + \tilde{\tau})$ such that $\mathbf{s} = \mathbf{s}' + \mathbf{s}''$. By applying the multiplicative trace inequality and using the estimates of Proposition 6.4, we obtain

$$\|\psi^{z_j^*} - \psi_h^{z_j^*}\|_{\text{L}^2(\Sigma)}^2 \lesssim h^{\mathbf{s}} \|f_N\|_{\text{L}^2(\Omega)}^2 \text{ and } \|\psi_-^{z_j^*} - \psi_{-,h}^{z_j^*}\|_{\text{L}^2(\Sigma)}^2 \lesssim h^{\mathbf{s}} \|f_N\|_{\text{L}^2(\Omega)}^2.$$

Now, we have $\psi^{z_j^*}|_\Sigma = \psi_-^{z_j^*}|_\Sigma$. Using the triangle inequality

$$\|\psi_h^{z_j^*} - \psi_{-,h}^{z_j^*}\|_{\text{L}^2(\Sigma)}^2 \leq 2(\|\psi^{z_j^*} - \psi_h^{z_j^*}\|_{\text{L}^2(\Sigma)}^2 + \|\psi_-^{z_j^*} - \psi_{-,h}^{z_j^*}\|_{\text{L}^2(\Sigma)}^2)$$

proves the claim. \square

This enables us to choose $(\lambda_h)_h$ such that both $(\lambda_h)_h$ and $(A_h(z_j^*)/\lambda_h)_h$ converge to 0 as $h \rightarrow 0$.

Corollary 6.6. *Any sequence $(\lambda_h)_h$ such that $\lambda_h \lesssim h^q$ for some $q \in (0, 2\tau + \tilde{\tau})$ fulfills the conditions of Theorem 6.3: the sequences $(\lambda_h)_h$ and $(A_h(z_j^*)/\lambda_h)_h$ converge to 0 when $h \rightarrow 0$.*

We observe that the convergence to 0 of the traces $((\psi_h^{z_j^*} - \psi_{-,h}^{z_j^*})|_\Sigma)_h$ can be proven in a stronger norm than the $\text{L}^2(\Sigma)$ -norm. Let us give an illustration: if one can choose $q > 1$ in Corollary 6.6, then both sequences $(\lambda_h/h)_h$ and $(A_h(z_j^*)/h)_h$ converge to 0 when $h \rightarrow 0$. On the other hand, we recall that if the sequence of meshes of the interface Σ is quasi-uniform in the sense of Definition 22.20 in [20], one can use global inverse inequalities. In particular (Eq. (22.39) loc. cit.), it holds that

$$\forall h, \forall g_h \in \mathbf{V}_h^k(\Sigma), \quad \|g_h\|_{\text{H}^{1/2}(\Sigma)}^2 \lesssim \frac{1}{h} \|g_h\|_{\text{L}^2(\Sigma)}^2.$$

Since using the bound (48), we already know that

$$\|\psi_h^{z_h^*} - \psi_{-,h}^{z_h^*}\|_{L^2(\Sigma)}^2 \leq 2J_h^{\lambda_h}(z_h^*) \leq 2A_h(z_h^*) + 2c_h\lambda_h\|z_h^*\|_{\varepsilon,\bar{\mu}}^2,$$

it follows that $\|\psi_h^{z_h^*} - \psi_{-,h}^{z_h^*}\|_{H^{1/2}(\Sigma)}$ converges to 0 when $h \rightarrow 0$. More generally (using still Eq. (22.39) loc. cit.), depending on the admissible range of values for q given by Corollary 6.6, convergence in $H^\eta(\Sigma)$ -norm can always be recovered for some $\eta \in (0, 1]$.

6.3 Numerical approximation of the electric field \mathbf{E}

For the numerical approximation of the problem satisfied by φ , the steps are similar to the ones that we proposed to approximate the problem satisfied by ψ . The main difference lies in the choice of the inner product, which plays a crucial role in choosing the coefficient λ_h . Further details can be found in [17]. With these numerical tools, one can build an approximation of \mathbf{E} , called \mathbf{E}_h , in the form $\mathbf{E}_h = \nabla\varphi_h + \varepsilon^{-1} \mathbf{curl} \psi_h$, where $\varphi_h \in V_h^k(\Omega) \cap H_0^1(\Omega)$ is an approximation of $\varphi_{\mathbf{E}}$, and $\psi_h \in V_h^k(\Omega)$ is an approximation of $\psi_{c,\mathbf{E}}$. For that, we approximate (16) first. Then we compute an approximation $s_0^h \in V_h^k(\Omega)$ of s_0 by discretizing the variational formulation (20). To that aim we use the discrete spaces $(V_{h,zmv}^k(\Omega))_h$, where $V_{h,zmv}^k(\Omega) = V_h^k(\Omega) \cap H_{zmv}^1(\Omega)$:

$$\begin{cases} \text{Find } s_0^h \in V_{h,zmv}^k(\Omega) \text{ such that:} \\ (\mathbf{curl} s_0^h, \mathbf{curl} \psi_h')_{0,\Omega} = (\mathbf{f} + \varepsilon \nabla\varphi_h, \mathbf{curl} \psi_h')_{0,\Omega}, \quad \forall \psi_h' \in V_{h,zmv}^k(\Omega). \end{cases} \quad (50)$$

Finally we approximate (19) with s_0 replaced by s_0^h in the right-hand side.

7 Numerical illustrations

In this section, we present some numerical illustrations of the proposed method, generated using the Freefem++ library. From an implementation perspective, we can solve the discrete minimization problems associated to ψ (and φ) in two ways: either by applying iterative algorithms like BFGS (Broyden-Fletcher-Goldfarb-Shanno) and NLCG (Nonlinear Conjugate Gradient), as proposed by [17], or by using an exact solver [2]. We will employ the BFGS function to solve the minimization problems associated to $\varphi_{\mathbf{E}}$ and $\psi_{\mathbf{E}}$. In the chosen numerical experiments, we emphasize that both contributions $\nabla\varphi_{\mathbf{E}}$ and $\mathbf{curl} \psi_{\mathbf{E}}$ are non-vanishing.

A relevant question is why the direct approach, that is the discretization of problems (16) and (19), is not used to solve the problem. To address this issue, we will compare the results with the plain method (use of standard meshes) and with the T -coercivity method (use of specially designed meshes). In particular, the numerical analysis is available for the latter method [15, 5].

7.1 T -coercivity and plain methods

As mentioned above, both methods rely on the discretization of problems (16), (20) and (19).

For the the so-called T -coercivity method, that is with the use of *locally T -conform meshes*, one can derive estimates on the error between the exact and discrete electric fields. Locally T -conform meshes $(\mathcal{T}_h)_h$ fulfill the same assumptions on the meshes as in Section 6, with *additional geometrical constraints*: detailed meshing rules can be found in [15, 5], with illustrations; an example is presented in Figure 3. A limitation of the T -conform meshes is that they can be difficult to build in the presence of corners.

Let us assume that ε is piecewise smooth: let $(\Omega_p^\varepsilon)_{p=1,P}$ be a partition that is compatible with respect to ε . We observe that the partition is also compatible with ε^{-1} .

We note that if the data in (8) is such that $\mathbf{f} \in \mathbf{H}^s(\Omega)$, then $g := \text{div} \mathbf{f} \in H^{-1+s}(\Omega)$ with $\|g\|_{-1+s,\Omega} \leq \|\mathbf{f}\|_{s,\Omega}$. So one can replace the assumption $g \in H^{-1+s}(\Omega)$ by $\mathbf{f} \in \mathbf{H}^s(\Omega)$, and use $\|\mathbf{f}\|_{s,\Omega}$ in the bounds. To simplify further the presentation, we assume that $\mathbf{f} \in \mathbf{H}^{TD(\varepsilon)}(\Omega)$, with the notation $\|\mathbf{f}\| =$

$\|\mathbf{f}\|_{\tau_D(\varepsilon), \Omega}$.⁴

We use the discrete spaces $(\mathbf{V}_{h,0}^k(\Omega))_h$, where $\mathbf{V}_{h,0}^k(\Omega) = \mathbf{V}_h^k(\Omega) \cap \mathbf{H}_0^1(\Omega)$, to discretize (16):

$$\begin{cases} \text{Find } \varphi_h \in \mathbf{V}_{h,0}^k(\Omega) \text{ such that:} \\ (\varepsilon \nabla \varphi_h \nabla \varphi'_h)_{0,\Omega} = \langle g, \varphi'_h \rangle_{\mathbf{H}_0^1(\Omega)}, \quad \forall \varphi'_h \in \mathbf{V}_{h,0}^k(\Omega). \end{cases} \quad (51)$$

One can perform the numerical analysis [5, 16] to find that, for all $\mathfrak{t} \in [0, \tau_D(\varepsilon))$,

$$\exists h_0, \forall h < h_0, \quad \|\varphi_{\mathbf{E}} - \varphi_h\|_{1,\Omega} \lesssim h^{\mathfrak{t}} \|\mathbf{f}\|. \quad (52)$$

We turn next to problem (20). Regarding the data, we observe that because ε has a jump at the interface, one gets only that $\varepsilon \nabla \varphi_{\mathbf{E}}$ belongs to $\mathbf{H}^{\mathfrak{t}'}(\Omega)$ for $\mathfrak{t}' \in [0, \tau_D^-(\varepsilon))$, where $\tau_D^-(\varepsilon) = \min(\tau_D(\varepsilon), 1/2)$. In this setting, since $\mathbf{f} \in \mathbf{H}^{\mathfrak{t}'}(\Omega)$, one has $\mathbf{f} + \varepsilon \nabla \varphi_{\mathbf{E}} \in \mathbf{H}^{\mathfrak{t}'}(\Omega)$. Because the auxiliary potential s_0 governed by (20) is also characterized by (18), we infer that $s_0 \in \mathbf{H}^{1+\mathfrak{t}'}(\Omega)$. One is solving a classical Laplace problem with homogeneous Dirichlet boundary conditions, so the numerical analysis is completely standard. Comparing (50) to (20), for all $\mathfrak{t} \in [0, \tau_D(\varepsilon))$, the consistency error is bounded by

$$\|\varepsilon(\nabla \varphi_{\mathbf{E}} - \nabla \varphi_h)\|_{0,\Omega} \leq \|\varepsilon\|_{L^\infty(\Omega)} \|\nabla \varphi_{\mathbf{E}} - \nabla \varphi_h\|_{0,\Omega} \lesssim h^{\mathfrak{t}} \|\mathbf{f}\|.$$

Recall that $\tau_D^-(\varepsilon) \leq \tau_D(\varepsilon)$, so the consistency error in the right-hand side (order $\tau_D(\varepsilon)$) is dominated by the interpolation error on the solution (order $\tau_D^-(\varepsilon)$), and we find that, for all $\mathfrak{t}' \in [0, \tau_D^-(\varepsilon))$,

$$\forall h < h_0, \quad \|s_0 - s_0^h\|_{1,\Omega} \lesssim h^{\mathfrak{t}'}. \|\mathbf{f}\|.$$

Let $\tau_N \in]1/2, 1]$ be the limit regularity exponent for this classical problem with $L^2(\Omega)$ right-hand sides. Using the Aubin-Nitsche lemma, we conclude that, for all $\mathfrak{t}'' \in [0, \tau_N + \tau_D^-(\varepsilon))$,

$$\forall h < h_0, \quad \|s_0 - s_0^h\|_{0,\Omega} \lesssim h^{\mathfrak{t}''}. \|\mathbf{f}\|. \quad (53)$$

Finally, to discretize problem (19), we use the discrete spaces $(\mathbf{V}_h^k(\Omega))_h$:

$$\begin{cases} \text{Find } \psi_h \in \mathbf{V}_h^k(\Omega) \text{ such that:} \\ (\varepsilon^{-1} \nabla \psi_h, \nabla \psi'_h)_{0,\Omega} - \omega^2 (\mu \psi_h, \psi'_h)_{0,\Omega} = \omega^2 (\mu s_0^h, \psi'_h)_{0,\Omega}, \quad \forall \psi'_h \in \mathbf{V}_h^k(\Omega). \end{cases} \quad (54)$$

In the right-hand side s_0 is replaced by s_0^h , and the consistency error is bounded by, for all $\mathfrak{t}'' \in [0, \tau_N + \tau_D^-(\varepsilon))$,

$$\|\mu(s_0 - s_0^h)\|_{0,\Omega} \lesssim h^{\mathfrak{t}''}. \|\mathbf{f}\|.$$

One can perform the numerical analysis in the spirit of [5, 16] (the Neumann boundary condition replacing the Dirichlet one). We find that, for all $\mathfrak{t}''' \in [0, \min(\tau_N(\varepsilon^{-1}), \tau_N + \tau_D^-(\varepsilon))$,

$$\exists h'_0, \forall h < \min(h_0, h'_0), \quad \|\psi_{c,\mathbf{E}} - \psi_h\|_{1,\Omega} \lesssim h^{\mathfrak{t}''}. \|\mathbf{f}\|. \quad (55)$$

Finally, with the T -conform meshes, the approximation of \mathbf{E} is equal to $\mathbf{E}_h^T = \nabla \varphi_h + \varepsilon^{-1} \mathbf{curl} \psi_h \in \mathbf{L}^2(\Omega)$. Then it holds the

Theorem 7.1. *For all $\mathfrak{t} \in [0, \min(\tau_D(\varepsilon), \tau_N(\varepsilon^{-1}))$,*

$$\exists h''_0, \forall h < h''_0, \quad \|\mathbf{E} - \mathbf{E}_h^T\|_{0,\Omega} \lesssim h^{\mathfrak{t}} \|\mathbf{f}\|.$$

Proof. Observe that for all $\tau_D(\varepsilon) \in (0, 1]$, one has $\tau_N + \tau_D^-(\varepsilon) > \tau_D(\varepsilon)$. So one just gathers the previous error estimates (52) and (55) and the claim follows. \square

⁴Obviously, the numerical analysis is simpler if $g = 0$. In this case, $\varphi_{\mathbf{E}} = 0$, one chooses $\varphi_{\mathbf{E}}^h = 0$ as its (best) approximation, and one can consider that $\tau_D(\varepsilon) = 1$ in the results below.

To conclude, the expected order of convergence is $\min(\tau_D(\varepsilon), \tau_N(\varepsilon^{-1})) \in (0, 1]$.

Briefly, we call the plain method the same sequence of numerical algorithms with discrete spaces supported by meshes that are not necessarily T-conform (see Figure 3 below for an illustration in a practical case). Up to our knowledge, when the coefficients are sign-changing, the numerical analysis is not available "in all generality". In particular, it has not been proven that convergence is always achieved. This can be verified in some configurations that are proposed below.

7.2 Flat interface

We test the performance of our method in the case where the interface Σ is flat, setting the coefficients $\mu_- = -2$ and $\varepsilon_+ = \mu_+ = 1$, with their extensions defined as $\tilde{\varepsilon}|_{\Omega_-} = 10, \tilde{\mu}|_{\Omega_-} = -10$. We denote the contrast by $\kappa_\varepsilon = \varepsilon_-/\varepsilon_+$, and adjust ε_- to modify the value of κ_ε . We consider below the configuration where the domain Ω is a symmetric cavity, formed by the union of:

$$\Omega_+ = \{(x, y) \in (0; 1/2) \times (0; 1)\} \quad \text{and} \quad \Omega_- = \{(x, y) \in (1/2; 1) \times (0; 1)\}.$$

For such a configuration, problem (15) is well posed as soon as $\kappa_\varepsilon \neq -1$. We assume that is it also the case for problem (17) for the set of values $\kappa_\varepsilon \in \{-2, -1.01, -1.001\}$ and $\omega \in \{2, 9\}$ that we consider below for the experiments.

We define $\omega^2 \mathbf{f}$ as the source term associated to the solution:

$$\mathbf{E}(x, y)|_{\Omega_+} = \begin{bmatrix} (2x + b) \sin(\pi y) + y^3(1 - y)^3 \\ \pi(x^2 + bx) \cos(\pi y) + \sin(2\pi x)^3 \end{bmatrix}, \quad \mathbf{E}(x, y)|_{\Omega_-} = \begin{bmatrix} a \sin(\pi y) + y^3(1 - y)^3 \\ \pi a(x - 1) \cos(\pi y) + \sin(2\pi x)^3 \end{bmatrix},$$

where $a = \frac{1}{2(\kappa_\varepsilon + 1)}$ and $b = -\frac{\kappa_\varepsilon + 2}{2(\kappa_\varepsilon + 1)}$.

How to choose λ_h ? Taking into account the geometry of the domain and the piecewise constant value of $\tilde{\varepsilon}$, we note that the limit regularity exponents τ and $\tilde{\tau}$ defined in Section 6.2 are both equal to -1 . Hence, we observe that our discretized optimal control method should converge if one chooses $\lambda_h = c h^q$ with $q \in (0; 3)$ (cf. Corollary 6.6). Specifically, for $\omega = 2$, we set $\lambda_h = 0.05h^2, 0.05h^{2.3}, 0.05h^{2.5}$ respectively for $\kappa_\varepsilon = -2, -1.01, -1.001$; and for $\omega = 9$, we set $\lambda_h = 0.05h^{2.5}, 0.05h^2, 0.05h^2$ respectively for $\kappa_\varepsilon = -2, -1.01, -1.001$. The relative error between the exact and numerical solutions, equal to

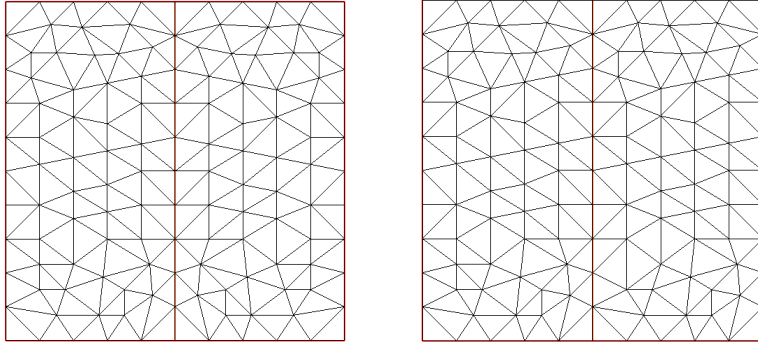


Figure 3 [Left] T -conform mesh (symmetric with respect to Σ); [Right] plain mesh.

$$e_h = \frac{\|\mathbf{E} - \mathbf{E}_h\|_{L^2(\Omega)}}{\|\mathbf{E}\|_{L^2(\Omega)}} \quad (56)$$

is plotted in log-log scale using the L^2 -norm in Figure 4.

For our method, we use the same meshes as in the plain method. On the other hand, T -conform meshes are meshes that are (at least locally) symmetric with respect to the interface Σ (see Figure 3 for an example). For this method, the expected order of convergence of Theorem 7.1 is equal to 1,

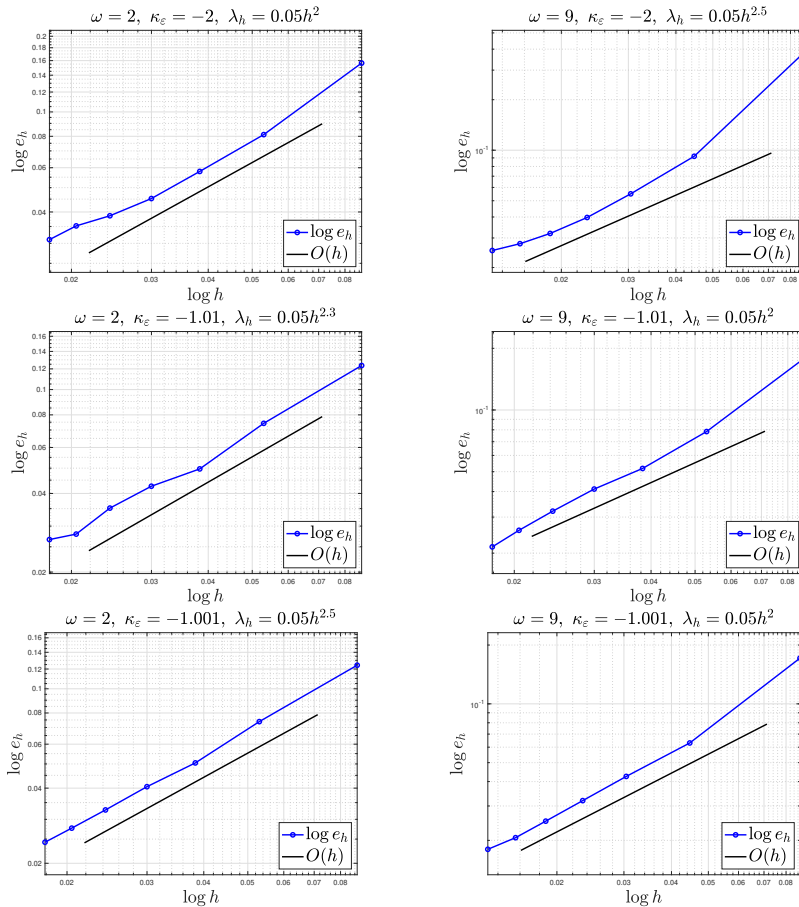


Figure 4 Behavior of the relative error with respect to the mesh size h .

since $\tau_D(\varepsilon) = \tau_N(\varepsilon^{-1}) = 1$, cf. [15].

For $\omega = 2$, we remark that our method converges at roughly the same rate as the T -conforming meshes method for $\kappa_\varepsilon = -2$, although with a slightly higher error level for as the contrast is closer to -1 (see Figure 5). In these cases (and although the meshes presented in Figure 3 look quite similar) we note that the plain method fails to converge for $\kappa_\varepsilon \in \{-1.01, -1.001\}$.

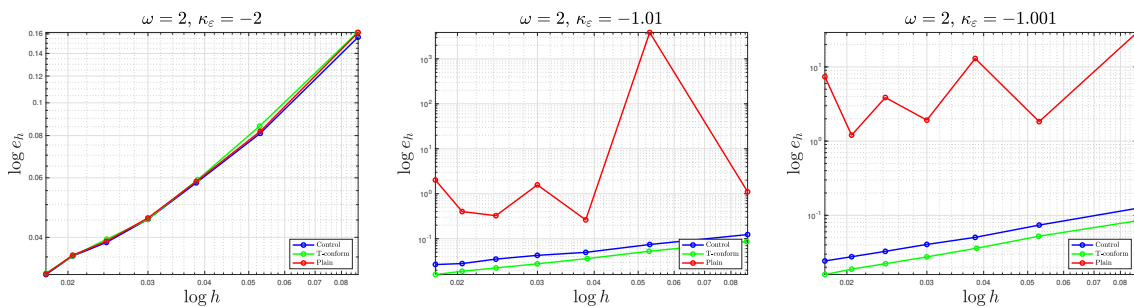


Figure 5 Comparison of the three relative errors (control, T -conform, plain) with respect to the mesh size h .

7.3 Circular interface

We now test the performance of our method in the case where the interface Σ is circular. We consider the case where the domains Ω_1, Ω_2 are equal to

$$\Omega_+ = \{(x, y) \in \mathbb{R}^2 \mid 1 < |(x, y)|_2 < 2\} \text{ and } \Omega_- = \{(x, y) \in \mathbb{R}^2 \mid 2 < |(x, y)|_2 < 3\}.$$

Specifically, Ω_1 refers to the smaller ring with radii ranging from 1 to 2, while Ω_2 represents the larger ring with radii between 2 and 3. We set again the coefficients $\mu_- = -2$ and $\varepsilon_+ = \mu_+ = 1$, with extensions defined as $\tilde{\varepsilon}|_{\Omega_-} = 10, \tilde{\mu}|_{\Omega_-} = -10$. In this case, the boundary $\partial\Omega$ is not connected. This situation is handled in Section A. We assume below that problem (15) and problem (17) (with $\omega = 2$) are well-posed for the set of values $\kappa_\varepsilon \in \{-2, -1.001\}$.

We define $\omega^2 \mathbf{f}$ as the source term associated to the solution:

$$\mathbf{E}(x, y)|_{\Omega_+} = \begin{bmatrix} (2a_1 r + a_2) \cos(\theta) - (r-1)(r-3)(r-2)^3 \sin(\theta) \\ (2a_1 r + a_2) \sin(\theta) + (r-1)(r-3)(r-2)^3 \cos(\theta) \end{bmatrix},$$

$$\mathbf{E}(x, y)|_{\Omega_-} = \begin{bmatrix} 2a_3(r-3) \cos(\theta) - (r-1)(r-3)(r-2)^3 \sin(\theta) \\ 2a_3(r-3) \sin(\theta) + (r-1)(r-3)(r-2)^3 \cos(\theta) \end{bmatrix},$$

where (r, θ) are the polar coordinates, and $a_1 = \frac{-1 - 2\kappa_\varepsilon}{3 + 4\kappa_\varepsilon}$, $a_2 = 1 - a_1$, and $a_3 = 4a_1 + 2a_2 - 1$. An

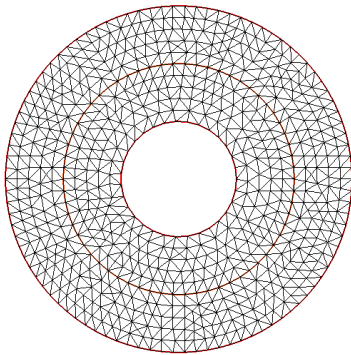


Figure 6 An example of (plain) mesh.

example of mesh is given in Figure 6. We note that, due to the geometry, triangular meshes are non-conforming with the two components of the boundary, nor with the interface.

How to choose λ_h ? Because of the smoothness of the boundary and interface, and because the coefficients are piecewise smooth, we find again that $\tau = \tilde{\tau} = 1$. Hence, $\lambda_h = ch^q$ with $q \in (0; 3)$ (cf. Corollary 6.6) is admissible. We choose $\lambda_h = 0.002h^2$ for $\kappa_\varepsilon = -2$, and $\lambda_h = 0.05h^{2.5}$ for $\kappa_\varepsilon = -1.001$. We report in Figure 7 the behavior of the relative error (56), with convergence rates exceeding expectations.

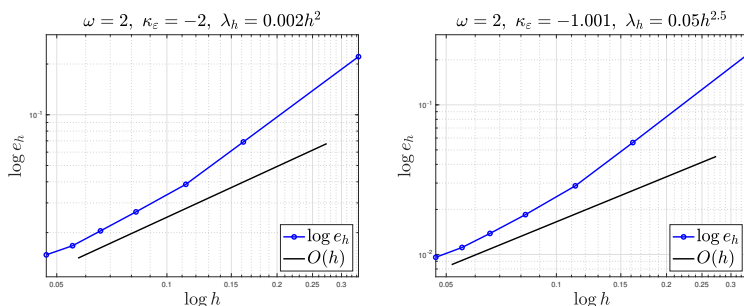


Figure 7 Behavior of the relative error with respect to the mesh size h .

8 Relaxing the assumptions

In this section, we show that the assumptions on the well-posedness of the scalar problems can be weakened. Specifically, the theory remains valid as long as we only require:

- [injectivity] uniqueness of solutions for problems (8) and (15), which means that, for a zero r.h.s., the only solution is the trivial one;
- existence of the solutions for problems (8) and (15), with the given data \mathbf{f} .

Let us work under these two assumptions.

Proposition 8.1. *For the given data \mathbf{f} , if problem (8) with r.h.s. $\omega^2 \mathbf{f}$ admits a solution \mathbf{E} , respectively problem (15) with r.h.s. $-\operatorname{div} \mathbf{f}$ admits a solution φ , then problem (17) with r.h.s. $\omega^2 \mu s_0$, where s_0 is characterized by (18), has a solution ψ .*

Proof. By construction: $\operatorname{div}(\varepsilon(\mathbf{E} - \nabla\varphi)) = 0$. According to Theorem 2.2, there exists $\psi_0 \in H_{zmv}^1(\Omega)$ such that $\varepsilon(\mathbf{E} - \nabla\varphi) = \operatorname{curl} \psi_0$. This implies $\mathbf{E} = \nabla\varphi + \varepsilon^{-1} \operatorname{curl} \psi_0$. Following the proof of Proposition 4.5 and defining the constant c as before, one then checks that $\psi_0 + c$ solves problem (17), with the data s_0 is given by (18). \square

Once the existence of ψ has been established, the result of Lemma 5.6 holds, and thus the theory of the volume-based control method is valid under those weakened assumptions. The numerical method of Section 6 can therefore still be applied.

On the other hand, we can demonstrate the equivalence between the injectivity of problem (8) and the injectivity of problem (17).

Proposition 8.2. *Problem (8) is injective if and only if problem (17) is injective.*

Proof. Suppose that problem (17) is injective. We aim to show that problem (8) is injective: Let $\mathbf{E} \in \mathbf{H}_0(\operatorname{curl}; \Omega)$ be such that:

$$\operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{E}) - \omega^2 \varepsilon \mathbf{E} = 0 \text{ in } \Omega.$$

Taking the divergence of both sides, we obtain: $\operatorname{div}(\varepsilon \mathbf{E}) = 0$ in Ω . By Theorem 2.2, $\exists \psi \in H^1(\Omega)$ such that $\mathbf{E} = \varepsilon^{-1} \operatorname{curl} \psi$. First, we have $\varepsilon^{-1} \operatorname{curl} \psi \cdot \boldsymbol{\tau} = 0$ on $\partial\Omega$. Then, it holds that $\operatorname{curl}(\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \operatorname{curl} \psi)) - \omega^2 \operatorname{curl} \psi = 0$ in Ω , i.e. $\operatorname{curl}(\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \operatorname{curl} \psi) - \omega^2 \psi) = 0$ in Ω . Then, there exists a unique constant $c \in \mathbb{R}$ such that: $\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \operatorname{curl} \psi) - \omega^2 \psi = \omega^2 c$ in Ω . Hence, $\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \operatorname{curl}(\psi + c)) - \omega^2(\psi + c) = 0$ in Ω . Noting that $\operatorname{curl}(\varepsilon^{-1} \operatorname{curl}(\psi + c)) = -\operatorname{div}(\varepsilon^{-1} \nabla(\psi + c))$ in Ω , then $\psi + c \in H^1(\Omega)$ is governed by:

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1} \nabla(\psi + c)) - \omega^2 \mu(\psi + c) = 0 \text{ in } \Omega, \\ \varepsilon^{-1} \nabla(\psi + c) \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

Since problem (17) is injective, it follows that $(\psi + c) = 0$ in Ω . Thus, $\mathbf{E} = \varepsilon^{-1} \operatorname{curl} \psi = 0$.

Conversely, assume that problem (8) is injective. Our goal is to prove the injectivity of problem (17). Let $\psi \in H^1(\Omega)$ satisfy the following problem:

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1} \nabla \psi) - \omega^2 \mu \psi = 0 \text{ in } \Omega, \\ \varepsilon^{-1} \nabla \psi \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

The equation satisfied by ψ in Ω can be written as: $\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \operatorname{curl} \psi) = \omega^2 \psi$. By applying the vector rotational to both sides, we obtain: $\operatorname{curl}(\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \operatorname{curl} \psi)) = \omega^2 \operatorname{curl} \psi$ in Ω . Then, $\varepsilon^{-1} \operatorname{curl} \psi \in \mathbf{H}(\operatorname{curl}; \Omega)$ is governed by:

$$\begin{cases} \operatorname{curl}(\mu^{-1} \operatorname{curl}(\varepsilon^{-1} \operatorname{curl} \psi)) - \omega^2 \varepsilon(\varepsilon^{-1} \operatorname{curl} \psi) = 0 \text{ in } \Omega, \\ \varepsilon^{-1} \operatorname{curl} \psi \cdot \boldsymbol{\tau} = 0 \text{ on } \partial\Omega. \end{cases}$$

Since problem (8) is injective, it follows that $\varepsilon^{-1} \operatorname{curl} \psi = 0$ in Ω . Then, there exists a constant $c' \in \mathbb{R}$ such that: $\psi = c'$ in Ω . Furthermore, $-\omega^2 \mu c' = 0$ in Ω , it follows that $\psi = 0$ in Ω . \square

The proof of Proposition 8.2 is independent of the assumptions on problem (15) with solution φ . Hence, one may replace the weakened assumptions on problem (8) by similar ones on problem (17). Namely, in addition to the injectivity of problem (15) and the existence of a solution with r.h.s. $-\operatorname{div} \mathbf{f}$, that:

- [injectivity] uniqueness of solution for problem (17);
- existence of a solution ψ with r.h.s. $\omega^2 \mu s_0$, where s_0 is characterized by (18).

In this case too, the theory of the volume-based control method remains valid, and the numerical method of Section 6 still applies.

9 Concluding remarks

In this paper, we generalized the volume based control method of [17] to 2D time harmonic Maxwell problem with sign-changing coefficients. Classical Lagrange finite elements are used instead of Nédélec finite elements, as the method involves solving three scalar problems.

In addition, one of the key advantages of the volume-based control method is that there is no a priori regularity assumption on the solutions, contrary to the surface-based control method of [1, 2].

For the numerical approximation, the claim that one can use any mesh is supported by the numerical experiments. In addition, the convergence rates match those obtained with a direct solver using T -conform meshes (for which the convergence rates are known).

Regarding the numerical analysis, it is still incomplete, although some partial results have been obtained (global convergence, convergence of the jump on the interface in $H^\eta(\Sigma)$ -norm for some $\eta > 0$, ...).

Finally, the natural continuation of this work is to apply this approach to the 3D time-harmonic Maxwell equations with sign changing coefficients. As a matter of fact, the numerical analysis has been carried out for one sign changing coefficient only [16]. However, unlike the 2D case, the decomposition of the electric field in 3D leads to a rotational problem with a vectorial unknown rather than a scalar one.

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A Case of a boundary that is not connected

Let us consider the case of domain Ω of \mathbb{R}^2 with a boundary $\partial\Omega$ that is not connected: in other words, there exist at least two distinct maximal connected components $(\Gamma_k)_{k=0,K}$ of $\partial\Omega$. Define

$$\mathbf{H}_{\partial\Omega}^1(\Omega) := \{\varphi \in \mathbf{H}^1(\Omega) \mid \varphi|_{\Gamma_0} = 0 \text{ and } \varphi|_{\Gamma_k} \in \mathbb{R} \text{ for } k = 1, K\},$$

and, for $k = 1, K$, we let $q_k \in \mathbf{H}_{\partial\Omega}^1(\Omega)$ such that $\Delta q_k = 0$ in Ω and $q_k|_{\Gamma_m} = \delta_{km}$ for $m = 0, K$. Classically,

$$\mathbf{H}_{\partial\Omega}^1(\Omega) = \mathbf{H}_0^1(\Omega) \oplus \text{span}_{k=1,K}(q_k).$$

For a boundary that is not connected, the statement of Theorem 2.2 is replaced by

Theorem A.1. *Let Ω be a domain of \mathbb{R}^2 .*

Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$, then

$$\text{div } \mathbf{v} = 0 \text{ in } \Omega, \langle \mathbf{v} \cdot \boldsymbol{\nu}, 1 \rangle_{\mathbf{H}^{1/2}(\Gamma_k)} = 0 \quad \forall k \iff \exists \psi \in \mathbf{H}^1(\Omega) \text{ s.t. } \mathbf{v} = \mathbf{curl} \psi \text{ in } \Omega. \quad (57)$$

$$\mathbf{curl} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \boldsymbol{\tau} = 0 \text{ on } \partial\Omega \iff \exists \varphi \in \mathbf{H}_{\partial\Omega}^1(\Omega) \text{ s.t. } \mathbf{v} = \nabla \varphi \text{ in } \Omega. \quad (58)$$

Moreover, the scalar potential ψ is unique up to a constant, and the scalar potential ϕ is unique.

In subsection 3.1, going back to the model, one can now use the equivalent variational formulation (6) with the test-field $\mathbf{v} = \nabla \varphi'$ for any $\varphi' \in \mathbf{H}_{\partial\Omega}^1(\Omega)$:

$$-\omega^2(\varepsilon \mathbf{E}, \nabla \varphi')_{0,\Omega} = \iota \omega (\mathbf{J}, \nabla \varphi')_{0,\Omega}, \quad \forall \varphi' \in \mathbf{H}_{\partial\Omega}^1(\Omega). \quad (59)$$

In other words, $\varphi' \mapsto (\mathbf{J}, \nabla \varphi')_{0,\Omega}$ is an element of $(\mathbf{H}_{\partial\Omega}^1(\Omega))'$.

If $\text{div } \mathbf{J} \in \mathbf{L}^2(\Omega)$, then taking $\varphi' \in \mathbf{H}_0^1(\Omega)$, we find $\text{div}(\varepsilon \mathbf{E}) = -\iota \omega^{-1} \text{div } \mathbf{J}$ a.e. in Ω . Next, for $k = 1, K$, taking $\varphi' = q_k$ and integrating by parts yields

$$\iota \omega^{-1} (\mathbf{J}, \nabla q_k)_{0,\Omega} = (\text{div}(\varepsilon \mathbf{E}), q_k)_{0,\Omega} - \langle (\varepsilon \mathbf{E}) \cdot \boldsymbol{\nu}, 1 \rangle_{\mathbf{H}^{1/2}(\Gamma_k)} = -\iota \omega^{-1} (\text{div } \mathbf{J}, q_k)_{0,\Omega} - \langle (\varepsilon \mathbf{E}) \cdot \boldsymbol{\nu}, 1 \rangle_{\mathbf{H}^{1/2}(\Gamma_k)},$$

that is $\langle (\varepsilon \mathbf{E}) \cdot \boldsymbol{\nu}, 1 \rangle_{\mathbf{H}^{1/2}(\Gamma_k)} = -\iota \omega^{-1} ((\text{div } \mathbf{J}, q_k)_{0,\Omega} + (\mathbf{J}, \nabla q_k)_{0,\Omega})$. In other words, there are K additional relations. From the physics point of view, for $k = 1, K$, this corresponds to the flux of $\mathbf{D} = \varepsilon \mathbf{E}$ through the connected component Γ_k of the boundary, equal to the total surface charges on Γ_k .

To proceed, the assumption on the companion scalar problem (9) remains the same, namely that it is well-posed (cf. Assumption 1). Under this assumption, one has the (see e.g. Proposition 8.9 in [7])

Lemma A.2. *Let $k = 1, K$, then there exists one, and only one, solution to the problem:*

$$\begin{cases} \text{Find } q_k^\varepsilon \in \mathbf{H}_{\partial\Omega}^1(\Omega) \text{ such that:} \\ \text{div}(\varepsilon \nabla q_k^\varepsilon) = 0 \text{ in } \Omega, \quad q_k^\varepsilon|_{\Gamma_m} = \delta_{km} \text{ for } m = 0, K. \end{cases} \quad (60)$$

In addition, $(q_k^\varepsilon)_{k=1,K}$ is a free family, and

$$\mathbf{H}_{\partial\Omega}^1(\Omega) = \mathbf{H}_0^1(\Omega) \oplus \text{span}_{k=1,K}(q_k^\varepsilon). \quad (61)$$

Note that by reformulating (60) variationally, we find that $(\varepsilon \nabla q_k^\varepsilon, \nabla q_0)_{0,\Omega} = 0$, for all $q_0 \in \mathbf{H}_0^1(\Omega)$ and all $k = 1, K$. To complement the Assumption 1 on problem (9), we assume that the so-called capacitance matrix of $\mathbb{R}^{K \times K}$ with entries $(\varepsilon \nabla q_m^\varepsilon, \nabla q_k^\varepsilon)_{0,\Omega}$ for $k, m = 1, K$, is invertible.

Under those two assumptions, one can still achieve Helmholtz decompositions in a domain with a boundary that is not connected.

Proposition A.3. *It holds that*

$$\mathbf{L}^2(\Omega) = \nabla[\mathbf{H}_{\partial\Omega}^1(\Omega)] \oplus \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon), \quad (62)$$

with $\mathbf{A}_{\partial\Omega}(\Omega, \varepsilon) := \{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \forall \varphi \in \mathbf{H}_{\partial\Omega}^1(\Omega), (\varepsilon\mathbf{w}, \nabla\varphi)_{0,\Omega} = 0\}$. In addition, the linear map $\mathbf{v} \mapsto (\varphi, \mathbf{w})$ with $\mathbf{v} = \nabla\varphi + \mathbf{w}$ is continuous from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}_0^1(\Omega) \times \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon)$.

Proof. Obviously, $\nabla[\mathbf{H}_{\partial\Omega}^1(\Omega)] + \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon) \subseteq \mathbf{L}^2(\Omega)$. Let us now show the converse imbedding. Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$. Since problem (9) is well-posed, there exists $\varphi_0 \in \mathbf{H}_0^1(\Omega)$ such that:

$$(\varepsilon\nabla\varphi_0, \nabla\varphi')_{0,\Omega} = (\varepsilon\mathbf{v}, \nabla\varphi')_{0,\Omega}, \quad \forall \varphi' \in \mathbf{H}_0^1(\Omega).$$

And, since the capacitance matrix is invertible, there exists $\varphi_{\partial\Omega} \in \text{span}_{k=1,K}(q_k^\varepsilon)$ such that:

$$(\varepsilon\nabla\varphi_{\partial\Omega}, \nabla q_m^\varepsilon)_{0,\Omega} = (\varepsilon\mathbf{v}, \nabla q_m^\varepsilon)_{0,\Omega}, \quad \forall m = 1, K.$$

Let $\varphi = \varphi_0 + \varphi_{\partial\Omega} \in \mathbf{H}_{\partial\Omega}^1(\Omega)$, we pose $\mathbf{w} = \mathbf{v} - \nabla\varphi$. Using the decomposition (61), we find that $\mathbf{w} \in \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon)$, and the end of the proof is similar to that of the proof of Proposition 4.1. \square

Using Theorem A.1, one obtains another Helmholtz decomposition.

Proposition A.4. *It holds that*

$$\mathbf{L}^2(\Omega) = \nabla[\mathbf{H}_{\partial\Omega}^1(\Omega)] \oplus \varepsilon^{-1} \mathbf{curl}[\mathbf{H}_{zmv}^1(\Omega)]. \quad (63)$$

In addition, the linear map $\mathbf{v} \mapsto (\varphi, \psi)$ with $\mathbf{v} = \nabla\varphi + \varepsilon^{-1} \mathbf{curl}\psi$ is continuous from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}_{\partial\Omega}^1(\Omega) \times \mathbf{H}_{zmv}^1(\Omega)$.

Proof. According to Theorem A.1, one has $\mathbf{A}_{\partial\Omega}(\Omega, \varepsilon) = \varepsilon^{-1} \mathbf{curl}[\mathbf{H}_{zmv}^1(\Omega)]$, because $\mathbf{w} \in \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon)$ is equivalent to $\text{div } \varepsilon\mathbf{w} = 0$ in Ω , and $\langle \varepsilon\mathbf{w} \cdot \boldsymbol{\nu}, 1 \rangle_{\mathbf{H}^{1/2}(\Gamma_k)} = 0$ for $k = 0, K$. The remainder of the proof is completely similar to that of the proof of Proposition 4.3. \square

We note that the property (13) still holds for all $\varphi \in \mathbf{H}_{\partial\Omega}^1(\Omega)$ and $\psi \in \mathbf{H}_{zmv}^1(\Omega)$.

Moreover, one can again decompose the electric field \mathbf{E} that is governed by (8):

$$\exists! (\varphi_{\mathbf{E}}, \psi_{0,\mathbf{E}}) \in \mathbf{H}_{\partial\Omega}^1(\Omega) \times \mathbf{H}_{zmv}^1(\Omega) \text{ such that } \mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl}\psi_{0,\mathbf{E}} \text{ in } \Omega. \quad (64)$$

To characterize the two scalar potentials $(\varphi_{\mathbf{E}}, \psi_{0,\mathbf{E}})$, we proceed as before. First using the property (13), we find the

Proposition A.5. *Let \mathbf{E} , a solution of (8), be decomposed as in (64). Then its scalar potential $\varphi_{\mathbf{E}} \in \mathbf{H}_{\partial\Omega}^1(\Omega)$ is governed by:*

$$\begin{cases} \text{Find } \varphi_{\mathbf{E}} \in \mathbf{H}_{\partial\Omega}^1(\Omega) \text{ such that:} \\ (\varepsilon\nabla\varphi_{\mathbf{E}}, \nabla\varphi')_{0,\Omega} = -(\mathbf{f}, \nabla\varphi')_{0,\Omega} \quad \forall \varphi' \in \mathbf{H}_{\partial\Omega}^1(\Omega). \end{cases} \quad (65)$$

By construction, $\mathbf{w} = \varepsilon\nabla\varphi_{\mathbf{E}} + \mathbf{f}$ is such that

$$\text{div } \mathbf{w} = 0 \text{ in } \Omega, \text{ and } \langle \mathbf{w} \cdot \boldsymbol{\nu}, 1 \rangle_{\mathbf{H}^{1/2}(\Gamma_k)} = 0 \text{ for } k = 0, K. \quad (66)$$

Then, to characterize the scalar potential $\psi_{0,\mathbf{E}} \in \mathbf{H}_{zmv}^1(\Omega)$, the result is identical to that of Proposition 4.5, with the auxiliary potential s_0 given by (18). Finally, one has to check that the scalar potential s_0 can still be characterized variationally when the boundary is not connected.

Lemma A.6. *The auxiliary potential s_0 solves (18) if, and only if, it solves (20).*

Proof. If s_0 solves (18), then it obviously solves (20).

Reciprocally, if s_0 solves (20) then proceeding as in the proof of Lemma 4.6, we find first that $\mathbf{curl}(\mathbf{curl} s_0 - \mathbf{f} - \varepsilon\nabla\varphi_{\mathbf{E}}) = 0$, and second $(\mathbf{curl} s_0 - \mathbf{f} - \varepsilon\nabla\varphi_{\mathbf{E}}) \cdot \boldsymbol{\tau} = 0$ on $\partial\Omega$. According to (58), there exists $z \in \mathbf{H}_{\partial\Omega}^1(\Omega)$ such that $\mathbf{curl} s_0 - \mathbf{f} - \varepsilon\nabla\varphi_{\mathbf{E}} = \nabla z$ in Ω . By construction,

$$\Delta z = \text{div}(\mathbf{f} - \varepsilon\nabla\varphi_{\mathbf{E}}) = 0 \text{ in } \Omega,$$

so that $z \in \text{span}_{k=1,K}(q_k)$. Integrating by parts (2) and (1), we find that for $k = 1, K$

$$\begin{aligned} (\nabla z, \nabla q_k)_{0,\Omega} &= (\mathbf{curl} s_0, \nabla q_k)_{0,\Omega} - (\mathbf{f} + \varepsilon \nabla \varphi_{\mathbf{E}}, \nabla q_k)_{0,\Omega} \\ &= 0 - \langle \mathbf{f} + \varepsilon \nabla \varphi_{\mathbf{E}} \cdot \boldsymbol{\nu}, 1 \rangle_{\text{H}^{1/2}(\Gamma_k)} = 0, \end{aligned}$$

where we used (66) to evaluate the last term. Hence, we conclude that $z = 0$ so that $\mathbf{curl} s_0 = \mathbf{f} + \varepsilon \nabla \varphi_{\mathbf{E}}$, i.e. s_0 remains characterized by (18). \square

We can conclude as before.

Theorem A.7. *Let $\mathbf{E} \in \mathbf{L}^2(\Omega)$ be governed by (8). Then there exist two scalar potentials $\varphi_{\mathbf{E}} \in \text{H}_{\partial\Omega}^1(\Omega)$ and $\psi_{c,\mathbf{E}} \in \text{H}^1(\Omega)$ such that $\mathbf{E} = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}}$ in Ω , where $\varphi_{\mathbf{E}}$ is a solution to (65), resp. $\psi_{c,\mathbf{E}} \in \text{H}^1(\Omega)$ is a solution to (17).*

Conversely, Let $\varphi_{\mathbf{E}} \in \text{H}_{\partial\Omega}^1(\Omega)$ be governed by (65), the auxiliary field s_0 be characterized by (18), and $\psi_{c,\mathbf{E}} \in \text{H}^1(\Omega)$ be governed by (17). Then $\mathbf{E} = \nabla \varphi_{\mathbf{E}} + \varepsilon^{-1} \mathbf{curl} \psi_{c,\mathbf{E}} \in \mathbf{L}^2(\Omega)$ is a solution to (8). Finally, the problem (8) is well-posed.

In a domain Ω with a boundary that is not connected, we note that we have been able to characterize the two scalar potentials of the electric field, and the auxiliary potential s_0 , variationally. It follows that one can discretize them as before, using either the volume optimal control method presented in Section 5, or the T -conform meshes presented in Section 7.1. The only difference lies in the definition of the discrete spaces for approximating $\varphi_{\mathbf{E}}$, now equal to $\text{V}_h^k(\Omega) \cap \text{H}_{\partial\Omega}^1(\Omega)$. The rest of the numerical analysis and implementation is as before.