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Mathai and Quillen's Thom Form of a Vector Bundle

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Abstract

This article gives a detailed mathematical construction of Mathai and Quillen's representative of a vector bundle's Thom class. The derivation is not original but completes the constructions described in existing references, and is intended as a complete mathematical reference. As an example we construct Mathai and Quillen's Thom form of the normal bundle of a smoothly embedded knot in a closed **3**-manifold.

Keywords: Thom Form,

1 Introduction

If $\pi : E \rightarrow B$ is a real oriented vector bundle of rank n over a d -dimensional manifold B and $j : B \hookrightarrow E$ is the natural inclusion of B as the zero section in E then for every closed differential d -form α on E a Thom form Φ is a differential n form on E satisfying

$$(2\pi)^{-\frac{n}{2}} \int_E \alpha \wedge \Phi = \int_B j^* \alpha.$$

The Euler number $\chi(M) \equiv \chi(TM)$ of (the tangent bundle) of a manifold M can be calculated, classically, in one of two ways: (1) choose a vector field v on M with isolated zeros and count these zeros with sign – this is the Hopf Theorem; (2) choose a connection ∇ on TM and calculate the integral of a certain normalised density determined by ∇ 's curvature over M – this is the Gauss-Bonnet Theorem.

Mathai and Quillen (1986) developed a general approach which generalizes the two classical views of the Euler number. They constructed a particular Thom form Φ_∇ of a vector bundle $E \rightarrow B$ based on a connection ∇ . This form satisfies

$$\chi(E) = \int_B s^* \Phi_\nabla(E)$$

and reduces to the Hopf Theorem, or Gauss-Bonnet-Chern Theorem for appropriate choices of s .

While this construction takes place in finite dimensions, Atiyah and Jeffrey (1990) noticed that Mathai and Quillen's form could be used to formally define Euler numbers for infinite dimensional vector bundles $H \rightarrow X$ parameterised by sections s :

$$\chi_s(H) = \int_X s^* \Phi_\nabla(H)$$

For natural choices of s these numbers should, intuitively, carry topological information about H .

This short review article gives a complete mathematical construction of Mathai and Quillen's Thom form based on the Berezin integral – a type of "fermionic" integration developed by Berezin, and which has now become a standard tool in particle physics. The derivation is not original but does complete the constructions described in several existing references (Berline et al., 2003; Zhang, 2001)). We also give an example of how to construct Mathai and Quillen's Thom form for smoothly embedded knots in a 3-manifold.

2 Fibre Bundles, Differential Forms, Connections, and Curvature

Let

$$\pi : \mathcal{E} \rightarrow M$$

be a smooth map from a manifold \mathcal{E} to a manifold M . We say (\mathcal{E}, π, M, F) is a *fibre bundle* with typical fibre F over M if there is a covering of M by open sets U_i and diffeomorphisms

$$\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times F$$

such that $\pi|_{\pi^{-1}(U_i)}$ can be written as

$$\pi|_{\pi^{-1}(U_i)} = pr_1 \circ \phi_i$$

where pr_1 is projection onto the first factor in $U_i \times F$. The space \mathcal{E} is called the *total space*, M the *base space*, and F the *typical fibre*. If the typical fibre F is a vector space then we call (\mathcal{E}, π, M, F) a *vector bundle*.

A section s of a fibre bundle over M is a map

$$s : M \rightarrow \mathcal{E}$$

such that $\pi \circ s(x) = x$ for all $x \in M$. The word smooth means infinitely differentiable and the space of smooth sections is denoted $\Gamma(M, \mathcal{E})$, while the space of smooth, compactly supported sections is denoted $\Gamma_c(M, \mathcal{E})$.

The tangent bundle TM of a smooth manifold M , and its dual T^*M are typical examples of vector bundles over M . Smooth vector fields on M are smooth sections $\Gamma(M, TM)$ of the tangent bundle. A 1-form on M is a smooth section $\alpha \in \Gamma(M, T^*M)$ of T^*M

$$M \ni x \rightarrow \alpha(x) \in T_x^*M.$$

The p -th exterior power of T^*M is a vector bundle $\Lambda^p T^*M$ together with a p -linear skew-symmetric map

$$\wedge^p : (T^*M, \dots, T^*M) \rightarrow \Lambda^p T^*M.$$

The bundle $\Lambda^p T^*M$ has fibre $\Lambda^p T_x^*M$ at $x \in M$ and the map \wedge^p is given by

$$\wedge^p(x, \alpha_1, \dots, \alpha_p) = \alpha_1 \wedge \dots \wedge \alpha_p, \quad \alpha_i \in T_x^*M,$$

which satisfies

$$\alpha_1 \wedge \dots \wedge \alpha_i \wedge \alpha_{i+1} \wedge \dots \wedge \alpha_p = -\alpha_1 \wedge \dots \wedge \alpha_{i+1} \wedge \alpha_i \wedge \dots \wedge \alpha_p$$

If $p = 0$ we set $\Lambda^0 T^*M = M \times \mathbb{R}$ and define the Whitney sum of bundles

$$\Lambda T^*M = \bigoplus_{p=0}^{\dim M} \Lambda^p T^*M.$$

The fibre of ΛT^*M is ΛT_x^*M and has rank $2^{\dim M}$. A *differential form* is a smooth section $\alpha \in \Gamma(M, \Lambda T^*M)$ of the vector bundle ΛT^*M , which, at each point $x \in M$, is a skew-symmetric p -linear function in $T_x M$. Denote the set of differential forms by $\mathcal{A}(M)$ and the set of differential p -forms by $\mathcal{A}^p = \Gamma(M, \Lambda^p T^*M)$. If $\alpha \in \mathcal{A}^p(M)$ and $\beta \in \mathcal{A}^q(M)$ then their exterior product is the $p+q$ form

$$\alpha \wedge \beta \in \mathcal{A}^{p+q}(M),$$

satisfying

$$\alpha \wedge \beta = (-1)^{p \cdot q} \beta \wedge \alpha.$$

The exterior derivative d is the unique operator on $\mathcal{A}(M)$ such that

1. $d : \mathcal{A}^*(M) \rightarrow \mathcal{A}^{*+1}(M)$ satisfies $d^2 = 0$;
2. if $f \in C^\infty(M)$ then $df \in \mathcal{A}^1(M)$ is the 1-form such that $df(X) = X(f)$ for a smooth vector field X ; and
3. d is a derivation; that is, if α and β are two p -forms then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

If V is a vector space then the contraction operator $\iota_v : \Lambda V^* \rightarrow \Lambda V^*$ satisfies

1. $\iota_v \alpha = \alpha(v)$ if $\alpha \in V^*$
2. $\iota_v (\alpha \wedge \beta) = \iota_v \alpha \wedge \beta + (-1)^p \alpha \wedge \iota_v \beta$ if α and β are degree p .

The operation of *contraction* with a vector field X is

$$\iota_X : \mathcal{A}^*(M) \rightarrow \mathcal{A}^{*-1}(M).$$

For a vector bundle \mathcal{E} over M , let $\mathcal{A}(M, \mathcal{E})$ denote the space of differential forms on M with values in \mathcal{E} ; specifically,

$$\mathcal{A}^p(M, \mathcal{E}) = \Gamma(M, \Lambda^p T^*M \otimes \mathcal{E}).$$

Any $\alpha \in \mathcal{A}^p(M, \mathcal{E})$ can be expressed in local coordinates $\mathcal{U} \subset M$ as

$$\alpha = \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

where the coefficients $\alpha_{i_1 \dots i_p} \in \Gamma(M, \mathcal{E})|_{\mathcal{U}}$ are sections of \mathcal{E} . We can naturally define wedge products $\alpha \wedge \beta \in \mathcal{A}^{p+q}(M, \mathcal{E})$ for $\alpha \in \mathcal{A}^p(M, \mathcal{E})$ and $\beta \in \mathcal{A}^q(M)$. But it makes no sense if both forms are bundle valued.

A connection on \mathcal{E} is essentially an extension of the exterior derivative d to include the coefficient from \mathcal{E} : A *connection* $\nabla^{\mathcal{E}}$ on \mathcal{E} is a linear operator

$$\nabla^{\mathcal{E}} : \Gamma(M, \mathcal{E}) \rightarrow \mathcal{A}^1(M, \mathcal{E})$$

such that for any $f \in C^\infty(M)$, $s \in \Gamma(M, \mathcal{E})$

$$\nabla^{\mathcal{E}} f \cdot s = (df) \otimes s + f \cdot \nabla^{\mathcal{E}} s.$$

So in words, $\nabla^{\mathcal{E}}$ takes as argument a section of \mathcal{E} and produces a 1-form on M which returns a section of \mathcal{E} when evaluated on a vector field. If $X \in \Gamma(M, TM)$ then a connection induces a map

$$\nabla_X^{\mathcal{E}} : \Gamma(M, \mathcal{E}) \rightarrow \Gamma(M, \mathcal{E}).$$

This is the covariant derivative along X .

In a local trivialisation of \mathcal{E} we can find a local orthonormal frame $\{e_1, \dots, e_n\}$ of sections. Any section s can be locally expressed as

$$s = \sum_i s^i e_i.$$

Each quantity $\nabla^{\mathcal{E}} e_i \in \mathcal{A}^1(M, \mathcal{E})$ can be written as

$$\nabla^{\mathcal{E}} e_i = \sum_j A_i^j \otimes e_j$$

for 1-forms A_i^j . The Leibniz rule now gives

$$\nabla^{\mathcal{E}} s = \sum_i \left(ds^i + A_i^j s^j \right) \otimes e_j \quad (1)$$

so that the connection can be expressed locally as $d + A$ for some matrix A of 1-forms on U .

The connection $\nabla^{\mathcal{E}}$ can be extended to a map

$$d_{\nabla^{\mathcal{E}}} : \mathcal{A}^*(M, \mathcal{E}) \rightarrow \mathcal{A}^{*+1}(M, \mathcal{E}) \quad (2)$$

defined in local coordinates by

$$\begin{aligned} d_{\nabla^{\mathcal{E}}} \alpha &= \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n} \nabla^{\mathcal{E}} [\alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}] \\ &= \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n} \nabla_j^{\mathcal{E}} \alpha_{i_1 \dots i_p} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \end{aligned}$$

where $\nabla_j^{\mathcal{E}}$ is the covariant derivative along the j^{th} coordinate vector field, which agrees with $\nabla^{\mathcal{E}}$ on $\Gamma(M, \mathcal{E})$, and which satisfies the Leibniz rule

$$d_{\nabla^{\mathcal{E}}} \alpha \cdot \theta = (d\alpha) \wedge \theta + (-1)^p \alpha \wedge d_{\nabla^{\mathcal{E}}} \theta$$

for any $\alpha \in \mathcal{A}^p(M)$ and any $\theta \in \mathcal{A}(M, \mathcal{E})$.

The curvature $R^{\mathcal{E}}$ of a connection $\nabla^{\mathcal{E}}$ is defined by

$$R^{\mathcal{E}} := d_{\nabla^{\mathcal{E}}} \circ d_{\nabla^{\mathcal{E}}} : \Gamma(M, \mathcal{E}) \rightarrow \mathcal{A}^2(M, \mathcal{E}).$$

This map is $C^\infty(M)$ linear; that is,

$$R^{\mathcal{E}}(f \cdot s) = f \cdot R^{\mathcal{E}} s$$

for any $s \in \Gamma(M, \mathcal{E})$. For any two vector fields X and Y in TM we have

$$R^{\mathcal{E}}(X, Y) = \nabla_X^{\mathcal{E}} \nabla_Y^{\mathcal{E}} - \nabla_Y^{\mathcal{E}} \nabla_X^{\mathcal{E}} - \nabla_{[X, Y]}^{\mathcal{E}}$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. In this way the curvature can be seen as a section of the vector bundle formed by the collection of fibre-wise endomorphisms of $\mathcal{E} - \Gamma(M, \text{End}(\mathcal{E}))$. Indeed, using the coordinate expression for $\omega \in \mathcal{A}^1(M, \mathcal{E})$ we find

$$d_{\nabla^{\mathcal{E}}} \omega = \nabla_j^{\mathcal{E}} \omega_i dx^j \wedge dx^i,$$

and the k - l component of $d_{\nabla^{\mathcal{E}}} \omega$ is then

$$(d_{\nabla^{\mathcal{E}}} \omega)_{kl} dx^k \wedge dx^l = (\nabla_k^{\mathcal{E}} \omega_l - \nabla_l^{\mathcal{E}} \omega_k) dx^k \wedge dx^l.$$

The expression

$$(d_{\nabla^{\mathcal{E}}} \omega)(X, Y) = \nabla_X^{\mathcal{E}}(\omega(Y)) - \nabla_Y^{\mathcal{E}}(\omega(X)) - \nabla_{[X, Y]}^{\mathcal{E}} \omega;$$

gives a well-defined bundle valued map and follows from the former in conjunction with the Leibniz rule. Now replacing ω by $\nabla^{\mathcal{E}} s$ for some section $s \in \Gamma(M, \mathcal{E})$ we obtain the formula

$$(d_{\nabla^{\mathcal{E}}} \circ d_{\nabla^{\mathcal{E}}} s)(X, Y) = R^{\mathcal{E}}(X, Y) s$$

as written above.

A metric $\langle \cdot, \cdot \rangle$ on a vector bundle is a smooth family of non-degenerate inner products on the fibres of \mathcal{E} . A connection $\nabla^{\mathcal{E}}$ preserves a metric if

$$d\langle s_1, s_2 \rangle = \langle \nabla^{\mathcal{E}} s_1, s_2 \rangle + \langle s_1, \nabla^{\mathcal{E}} s_2 \rangle.$$

If $\{e_1, \dots, e_n\}$ is a local frame for \mathcal{E} on a trivialisation then metric preservation implies that the matrix of 1-forms A in the local expression for $\nabla^{\mathcal{A}}$ is skew-symmetric. The connection $\nabla^{\mathcal{E}}$ induces a connection $\nabla^{\wedge \mathcal{E}}$ on the exterior algebra of \mathcal{E} via

$$\nabla_X^{\wedge \mathcal{E}}(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_k e_{i_1} \wedge \dots \wedge \nabla_X^{\mathcal{E}} e_k \wedge \dots \wedge e_{i_p}.$$

If all members of the local frame appear in the product and the connection preserves the metric then $\nabla_X^{\wedge \mathcal{E}}(e_1 \wedge \dots \wedge e_n) = 0$; indeed, for any vector field X

$$\begin{aligned} \nabla_X^{\mathcal{E}}(e_1 \wedge \dots \wedge e_n) &= \sum_k e_1 \wedge \dots \wedge \nabla_X^{\mathcal{E}} e_k \wedge \dots \wedge e_n \\ &= \sum_k e_1 \wedge \dots \wedge A_k^j(X) e_j \wedge \dots \wedge e_n \\ &= \sum_k e_1 \wedge \dots \wedge A_k^k(X) e_k \wedge \dots \wedge e_n \quad \text{since all indices appear in the product} \\ &= 0 \quad \text{since } A_k^k(X) \equiv 0 \text{ because of skew-symmetry.} \end{aligned}$$

Since this holds for any X we have $\nabla^{\wedge \mathcal{E}}(e_1 \wedge \dots \wedge e_n) = 0$.

3 The Berezin Integral and Mathai-Quillen's Thom Form

We'll present the construction of the Berezin Integral in stages:

1. First, for a vector space V thought of as a vector bundle over a point;
2. Then for a rank N vector bundle $\pi : \mathcal{E} \rightarrow M$;
3. Finally, for the pullback of a rank N vector bundle over itself: $p : \pi^* \mathcal{E} \rightarrow \mathcal{E}$.

3.1 Stage 1

Let V be an oriented N -dimensional Euclidean vector space with an inner product whose norm is denoted $\|\cdot\|$; we interpret this spaces as a vector bundle over a point. Let ΛV be the exterior algebra of V - monomials of the form $e_I = e_{i_1} \wedge \cdots \wedge e_{i_q}$, $1 \leq q \leq N$, $I\{i_1, \dots, i_q\}$. The *Berezin Integral* is a linear map

$$T^B : \Lambda V \rightarrow \mathbb{R}$$

defined by projecting $u \in \Lambda V$ onto the coefficient of the monomial $e_1 \wedge \cdots \wedge e_N$ via linearity:

$$T^B(a_I \cdot e_I) = \begin{cases} a_I & |I| = N \\ 0 & \text{otherwise} \end{cases}$$

If e_i is a basis for V with dual basis e^i for V^* , the elements $e^i \in \Lambda V^*$ play the role of coordinate functions on V . A coordinate system x^i on V is one in which $dx^i = e^i$ are precisely the coordinate functions. View the identity map

$$\begin{aligned} id : V &\rightarrow V \\ \mathbf{x} &\mapsto \mathbf{x} = x^1 e_1 + \cdots + x^p e_p \end{aligned}$$

as an element of $\mathcal{A}^0(V, \Lambda V) = \Gamma(V, \Lambda^0 V^* \otimes \Lambda V)$; indeed, the coordinate expression of id consists of a product of functions on V with elements of the exterior algebra of V . The exterior differential $d\mathbf{x}$ is then an element of $\mathcal{A}^1(V, \Lambda V) = \Gamma(V, \Lambda^1 V^* \otimes \Lambda V)$ and given by

$$d\mathbf{x} = dx^1 \otimes e_1 + \cdots + dx^N \otimes e_p.$$

Notice that the summands commute with respect to multiplication

$$\begin{aligned} (dx^i \otimes e_i) \cdot (dx^j \otimes e_j) &= (dx^i \wedge dx^j) \otimes (e_i \wedge e_j) = (dx^j \wedge dx^i) \otimes (e_j \wedge e_i) \\ &= (dx^j \otimes e_j) \cdot (dx^i \otimes e_i) \end{aligned}$$

The exponential $e^{-d\mathbf{x}}$, interpreted as a power series, is then an element of $\mathcal{A}(V, \Lambda V) = \Gamma(V, \Lambda V^* \otimes \Lambda V)$ and can be written using the commutativity of the exponents and skew-symmetry of the wedge product as

$$e^{-d\mathbf{x}} = e^{-dx^1 \otimes e_1} \cdots e^{-dx^N \otimes e_N} = \prod_{k=1}^p (1 - dx^k \otimes e_k)$$

Since $\frac{\|\mathbf{x}\|^2}{2}$ is a function and commutes with $d\mathbf{x}$ we can write

$$\mathcal{A}(V, \Lambda V) \ni e^{-\frac{\|\mathbf{x}\|^2}{2} - d\mathbf{x}} = e^{-\frac{\|\mathbf{x}\|^2}{2}} \prod_{k=1}^N (1 - dx^k \otimes e_k).$$

Extend the Berezin integral to a map

$$T^B : \mathcal{A}(V, \Lambda V) \rightarrow \mathcal{A}(V) := \Gamma(V, \Lambda V^*)$$

by

$$T^B(\alpha \otimes \xi) = \alpha \cdot T^B(\xi)$$

for $\alpha \in \Gamma(V, \Lambda V^*)$ and $\xi \in \Lambda V$. Now

$$\Phi(\mathbf{x}) = T^B \left(e^{-\frac{\|\mathbf{x}\|^2}{2} - d\mathbf{x}} \right)$$

is a differential N -form on V whose total integral over V is 1.

$$\int_V \Phi(\mathbf{x}) = (2\pi)^{\frac{N}{2}}.$$

To see this

$$\begin{aligned} T^B \left(e^{-\frac{\|\mathbf{x}\|^2}{2} - d\mathbf{x}} \right) &= T^B \left(e^{-\frac{\|\mathbf{x}\|^2}{2}} \prod_{k=1}^p (1 - dx^k \otimes e_k) \right) \\ &= e^{-\frac{\|\mathbf{x}\|^2}{2}} \cdot T^B \left(\prod_{k=1}^p (1 - dx^k \otimes e_k) \right) \quad \text{linearity of } T^B \\ &= e^{-\frac{\|\mathbf{x}\|^2}{2}} \cdot T^B \left((dx^1 \wedge \cdots \wedge dx^p) \otimes (e_1 \wedge \cdots \wedge e_p) \right) \\ &\quad \text{only this term has non-zero contribution} \\ &= e^{-\frac{\|\mathbf{x}\|^2}{2}} dx^1 \wedge \cdots \wedge dx^p \cdot T^B(e_1 \wedge \cdots \wedge e_p) \\ &= e^{-\frac{\|\mathbf{x}\|^2}{2}} dx^1 \wedge \cdots \wedge dx^p. \end{aligned}$$

The integral of Φ over V is now the integral of a Gaussian function which is $(2\pi)^{\frac{N}{2}}$.

3.2 Stage 2

Now let (M, \mathcal{E}, π, F) be a rank N vector bundle with a metric $\langle \cdot, \cdot \rangle$. The Berezin integral T^B defines a map

$$T^B : \mathcal{A}(M, \Lambda \mathcal{E}) \rightarrow \mathcal{A}(M)$$

applied fibre-wise. Choose a connection $\nabla^\mathcal{E}$ on \mathcal{E} which preserves the metric. This induces a connection $\nabla^{\Lambda \mathcal{E}}$ on the exterior $\Lambda \mathcal{E}$ as mentioned above, which, in turn, extends to a map $\nabla := d_{\nabla^{\Lambda \mathcal{E}}}$ on $\mathcal{A}(M, \Lambda \mathcal{E})$ analogously to (2).

Lemma 1. *For any $\omega \in \mathcal{A}(M, \Lambda \mathcal{E})$ we have*

$$dT^B(\omega) = T^B(\nabla \omega)$$

Proof. In local trivialisation for the vector bundle \mathcal{E} , let $\{e_1, \dots, e_N\}$ be a local orthonormal frame for \mathcal{E} . We can assume that $\omega = \alpha \otimes f_N \cdot e_1 \wedge \dots \wedge e_N = f_N \cdot \alpha \otimes e_1 \wedge \dots \wedge e_N$, for $\alpha \in \mathcal{A}(M)$ and $f_N \in C^\infty(M)$ since this gives the only non-zero contribution to T^B . Then

$$\begin{aligned} d_{\nabla^{\Lambda \mathcal{E}}} \omega &= d(f_N \cdot \alpha) \otimes e_1 \wedge \dots \wedge e_N + (-1)^{\deg(\alpha)} \alpha \otimes \nabla^{\Lambda \mathcal{E}} e_1 \wedge \dots \wedge e_N \\ &= d(f_N \cdot \alpha) \otimes e_1 \wedge \dots \wedge e_N. \end{aligned}$$

Applying T^B to both sides gives the result. \square

3.3 Stage 3

We're going to apply Lemma 1 to the bundle with $M = \mathcal{E}$ and $\mathcal{E} = \pi^* \mathcal{E}$ with connection $\nabla^{\pi^* \mathcal{E}} = \pi^* \nabla^{\mathcal{E}}$. The *tautological section* $\mathbf{x} \in \Gamma(\mathcal{E}, \pi^* \mathcal{E})$ assigns to each point $v \in \mathcal{E}$ the point $(v, v) \in \pi^* \mathcal{E}$. In a local trivialisation, we can write

$$\nabla^{\pi^* \mathcal{E}} \mathbf{x} = dv + A \cdot v$$

The space $\mathcal{A} = \mathcal{A}(\mathcal{E}, \Lambda \pi^* \mathcal{E})$ is now bi-graded since it is a direct sum of subspaces

$$\mathcal{A}^{i,j} = \mathcal{A}^i(\mathcal{E}, \Lambda^j \pi^* \mathcal{E}).$$

The chain of reasoning in stage 2 produces, from $\nabla^{\pi^* \mathcal{E}}$, a map $\nabla : \mathcal{A}^{i,j} \rightarrow \mathcal{A}^{i+1,j}$

If $s \in \Gamma(\mathcal{E}, \pi^* \mathcal{E})$, the contraction i_s is defined by the formula

$$i_s(\alpha \otimes (s_{i_1} \wedge \dots \wedge s_{i_k})) = \sum_{j=1}^k (-1)^{\deg \alpha + j - 1} (s, s_{i_j}) \cdot \alpha \otimes (s_{i_1} \wedge \dots \wedge \hat{s}_{i_j} \wedge \dots \wedge s_{i_k}),$$

where (\cdot, \cdot) is the metric on $\pi^* \mathcal{E}$, and gives a map $i_s : \mathcal{A}^{i,j} \rightarrow \mathcal{A}^{i,j-1}$. Now for any $\omega \in \mathcal{A}$, $i_s \omega$ has no components in $\mathcal{A}^{i,N}$ and therefore $T^B(i_s \omega) \equiv 0$. From Lemma 1

$$dT^B(\omega) = T^B((\nabla + i_s)\omega).$$

What's left is the curvature; we want to view this as an element of $\mathcal{A}^{2,2}$. To do this we view the curvature of \mathcal{E} as an $\mathfrak{so}(\mathcal{E})$ -valued 2-form which can be identified with $\Lambda^2 \mathcal{E}$ by the map

$$R \mapsto \sum_{i < j} (Re_i, e_j) \cdot e_i \wedge e_j.$$

Pulling this back by π^* , the curvature $R^{\pi^* \mathcal{E}}$ can be viewed as an element of $\mathcal{A}^{2,2} = \mathcal{A}^2(\mathcal{E}, \Lambda^2 \mathcal{E})$.

We work with the following data:

- The tautological section $\mathbf{x} \in \mathcal{A}^{0,1}$;
- The function $\|\mathbf{x}\|^2 \in \mathcal{A}^{0,0}$ and the section-valued 1-form $\nabla \mathbf{x} \in \mathcal{A}^{1,1}$;
- The pullback of the curvature $R^{\pi^* \mathcal{E}} \in \mathcal{A}^{2,2}$.

Lemma 2. If $\Omega = \frac{\|\mathbf{x}\|^2}{2} + \nabla \mathbf{x} - R^{\pi^* \mathcal{E}}$ then

$$(\nabla + i_x) \Omega = 0$$

Proof. We'll do the calculations and put them together. Since $\|\mathbf{x}\|^2 \in \mathcal{A}^{0,0}$ the contraction with \mathbf{x} is identically zero: $i_{\mathbf{x}} \|\mathbf{x}\|^2 \equiv 0$. Next, $\nabla \frac{\|\mathbf{x}\|^2}{2}$: since $\|\mathbf{x}\|^2 \in \mathcal{A}^{0,0}$ ∇ reduces to applying $\nabla^{\pi^* \mathcal{E}}$ which in turn reduces to applying d to $\|\mathbf{x}\|^2$; using metric compatibility

$$d \frac{\|\mathbf{x}\|^2}{2} = \left(\nabla^{\pi^* \mathcal{E}} \mathbf{x}, \mathbf{x} \right).$$

In a trivialisation with orthonormal frame of sections $\{e_1, \dots, e_n\}$ we have

$$\begin{aligned} \mathbf{x}(x, v) &= v^i e_i \\ \nabla^{\pi^* \mathcal{E}} \mathbf{x} &= (dv^j + v^i A_i^j) \otimes e_j \end{aligned}$$

so that

$$\left(\nabla^{\pi^* \mathcal{E}} \mathbf{x}, \mathbf{x} \right) = \sum_j (e_j, \mathbf{x}) \cdot (dv^j + v^i A_i^j)$$

Since the degree of the form is 1, this can be re-written in terms of the contraction operator $i_{\mathbf{x}}$ as

$$\nabla \frac{\|\mathbf{x}\|^2}{2} = -i_{\mathbf{x}} \nabla \mathbf{x}.$$

Now $\nabla \nabla \mathbf{x} = i_{\mathbf{x}} R^{\pi^* \mathcal{E}}$ by definition and $\nabla R^{\pi^* \mathcal{E}} \equiv 0$ by the Bianchi identity.

Putting these all together

$$(\nabla + i_x) \Omega = -i_{\mathbf{x}} \nabla \mathbf{x} + i_{\mathbf{x}} R^{\pi^* \mathcal{E}} + i_{\mathbf{x}} \nabla \mathbf{x} - 0 - i_{\mathbf{x}} R^{\pi^* \mathcal{E}} = 0$$

□

3.4 Mathai and Quillen's Thom Form

Theorem 1. Let $\pi : \mathcal{E} \rightarrow M$ be a rank N vector bundle and Ω the term given in Lemma 2. The form

$$\Phi = (2\pi)^{\frac{N}{2}} T^B(e^{-\Omega})$$

is a closed differential N -form on \mathcal{E} whose integral over each fibre is identically 1

$$\int_{\mathcal{E}_x} \Phi = 1$$

Proof. Closedness follows from Lemma 2 after performing a calculation identical to the one given in stage 1 and observing that $\nabla + i_{\mathbf{x}}$ is a derivation; the fibre integral property follows from what was seen in stage 1 as-well. □

Mathai and Quillen's Thom form is a Gaussian shaped differential form, peaked over the zero section. We can even modify this form to have compact support. Let $B_r = \{(x, v) \in E : \|v\| < r\}$, where $\|\cdot\|$ is the norm induced by the metric $\langle \cdot, \cdot \rangle$ on E , and define the diffeomorphism

$$\begin{aligned} \rho_r : E &\rightarrow E \\ (x, v) &\mapsto \left(x, \frac{v}{\sqrt{r^2 - \|v\|_g^2}}\right). \end{aligned}$$

Then

$$\rho_r^* \left(e^{-\frac{1}{2} \|v\|_g^2} \right) = e^{-\frac{\|v\|_g^2}{2(r^2 - \|v\|_g^2)}}$$

vanishes to infinite order as $\|v\|_g^2 \rightarrow r^2$ which is enough to kill all growth induced by denominators in $\rho_r^* dx^i$. Extending $\rho_r^* \Phi$ smoothly across the boundary of B_r by 0 we have smooth, compactly supported differential N form whose fibre integral is still 1 by the change of variables Theorem.

Theorem 2 (A localisation principle). *The support of Φ can be shrunk into any neighbourhood B_r of the zero section.*

4 The Mathai-Quillen Thom Form of a Smoothly Embedded Knot in a 3-Manifold

As an example we will consider an embedded loop in 3-manifold; for example, $M = \mathbb{S}^3$ with the standard round metric, or $M = \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ with the flat metric, or $M = \mathbb{R}^3$ with the flat metric. The Frenet frame along γ is described by the equations

$$\nabla_T T = \kappa \cdot N, \quad \nabla_T N = -\kappa \cdot T + \tau \cdot B, \quad \nabla_T B = -\tau \cdot N \quad (3)$$

where T is the tangent vector to γ , N the unit normal, B the unit binormal, and κ and τ are the curvature and torsion of γ . We'll compute the Thom form of γ 's normal bundle – a rank 2 vector bundle over γ .

In a trivialisation of ν around a point x_o on σ we can use the Frenet frame to define coordinates (x, t_1, t_2) . The induced connection on σ , and the induced normal connection on σ 's normal bundle are related by

$$\nabla = \nabla^T + \nabla^\nu = P^T(\nabla) + P^\nu(\nabla)$$

where P^* are the orthogonal projections onto the tangent and normal bundles of σ , respectively. The normal connection ∇^ν preserves the metric and since σ is one dimensional and the normal space two dimensional the matrix of connection 1-forms is $A_1^1 = A_2^2 = 0$ and $A_2^1 = -f dx = -A_1^2$ for some smooth function f on σ . Comparing with the Frenet formulas this function must be exactly the torsion τ of σ .

The pullback bundle of ν over itself is

$$\pi^* \nu = \{(v, w) \in \nu \times \nu : w \in \nu_{\pi(v)}\}$$

and a tangent vector on ν has the form $X = (p, v, p, \nabla_p^\nu w|_{t=0})$, for a curve $(p(t), w(t))$ with $p(0) = p, \dot{p}(0) = v \in T_p\sigma$. The differential of the projection map is then $d\pi \cdot X = v \in T_p\sigma$. To calculate the Thom form we need to calculate $\nabla^{\pi^*\nu} \mathbf{x}$ and the curvature $R^{\pi^*\nu}$, with \mathbf{x} the tautological section. Fortunately, since $T\sigma$ is one-dimensional the curvature of ∇^ν , and therefore of $\nabla^{\pi^*\nu}$ is identically zero. So it remains to calculate $\nabla^{\pi^*\nu} \mathbf{x}$ which can be done using the standard pullback formula

$$\nabla_X^{\pi^*\nu} \mathbf{x} = \pi^* (\nabla_{d\pi \cdot X}^\nu \mathbf{x}) = \pi^* (\nabla_v^\nu \mathbf{x})$$

for a vector field X on ν with projection $d\pi \cdot X = v \in T_p\sigma$. In coordinates, the tautological section and vector field take the form

$$\mathbf{x} = t_1 \partial_{t_1} + t_2 \partial_{t_2}, \quad X = g_1 \partial_x + g_2 \partial_{t_1} + g_3 \partial_{t_2}$$

Then using the coordinate expression (1) for the connection ∇^ν and the connection 1-forms we have

$$\left(\nabla^{\pi^*\nu} \pi^* \mathbf{x} \right) = (dt_1 - t_2 \tau dx) \otimes \partial_{t_1} + (dt_2 + t_1 \tau dx) \otimes \partial_{t_2}.$$

Plugging these quantities into the Thom form and collecting only those terms which are non-zero under the Berezin integral we get

$$\Phi_\gamma = \frac{1}{2\pi} e^{-\frac{t_1^2 + t_2^2}{2}} [dt_1 \wedge dt_2 + \tau \cdot (t_1 dt_1 + t_2 dt_2) \wedge dx].$$

This form clearly has fibre integral 1 and since the torsion depends only on $x \in \sigma$ it is also closed.

Let \mathcal{T}_r be an open neighbourhood of the zero section γ in N_γ , with uniform radius r , and consider the diffeomorphism $\rho_r : \mathcal{T}_r \rightarrow N_\gamma$ for $r \in \mathbb{R}$, given locally in Fermi coordinates (x, t_1, t_2) by:

$$(x, v) \mapsto \left(x, \frac{r}{(r^2 - \|v\|_g^2)^{\frac{1}{2}}} v \right)$$

with Jacobian matrix:

$$D\rho_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{r(r^2 - t_2^2)}{(r^2 - \|v\|_g^2)^{\frac{3}{2}}} & \frac{rt_1 t_2}{(r^2 - \|v\|_g^2)^{\frac{3}{2}}} \\ 0 & \frac{rt_1 t_2}{(r^2 - \|v\|_g^2)^{\frac{3}{2}}} & \frac{r(r^2 - t_1^2)}{(r^2 - \|v\|_g^2)^{\frac{3}{2}}} \end{bmatrix}.$$

In the same coordinates (x, t_1, t_2) the Thom form has the structure given in (??) and the pullback of the coefficient function is:

$$\rho_r^* e^{-\frac{\|v\|_g^2}{2}} = e^{-\frac{r^2 \cdot \|v\|_g^2}{2(r^2 - \|v\|^2)}},$$

which vanishes to infinite order as $\|v\| \rightarrow r$, and dominates any growth induced by the polynomial denominators of $\rho_r^*(dt_1 \wedge dt_2 + \tau \cdot (t_1 dt_1 + t_2 dt_2) \wedge dx)$ coming from determinant minors of the Jacobian. Using the above expressions for ρ_r and Φ_γ :

$$\rho_r^* \Phi_\gamma = \frac{r^4 \cdot e^{-\frac{r^2(t_1^2 + t_2^2)}{2(r^2 - t_1^2 - t_2^2)}}}{2\pi(r^2 - \|v\|_g^2)^2} [dt_1 \wedge dt_2 - \tau \cdot t_1 \cdot dx \wedge dt_1 - t_2 \cdot dx \wedge dt_2]. \quad (4)$$

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