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Small-time global controllability of the Burgers equation via bilinear controls

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Abstract

The aim of this work is to study the controllability of the viscous Burgers equation in the case of bilinear controls. We consider the problem on the one-dimensional flat torus and on bounded intervals equipped with Dirichlet or Neumann boundary conditions. The controls depend solely on time and act through a given family of spatial functions. We first prove the small-time global approximate controllability of the equation between states of the same sign. This result is ensured by a saturating geometric control approach with at least three controls that are localized in frequency. Afterward, we show the small-time global exact controllability to the non-zero constant states of the equation via at least four controls in the case of the flat torus and Neumann boundary conditions. For this second result, we proceed by studying the null-controllability of a suitable linearized system. Then, we infer the controllability for the initial bilinear Burgers equation via fixed-point arguments. Explicit examples of bilinear controls verifying our results are provided in the work.

Keywords: Bilinear control, viscous Burgers equation

2020 Mathematics Subject Classification: 35Q93, 93C20, 93C10, 35K91

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1 Introduction

Control theory is a branch of applied mathematics that investigates whether practical problems from fields such as engineering, physics, and other sciences can be controlled through specific external actions. Control problems involving partial differential equations are typically studied in the presence of internal or boundary controls due to the nature of the system evolution. A linear control term is added to the equation, and controllability is analyzed based on the reachable states of the modified system. Since the associated control problem is often linear, this facilitates its mathematical analysis. From a practical standpoint, internal or boundary controls

represent external influences on the system that interact with its evolution. However, in many applications across applied science, physics, and chemistry, the control mechanism often modifies the evolving variable itself rather than acting as an external influence. In such cases, it is relevant to consider control terms that depend on the evolving variable. A notable and significant example of this approach is the use of multiplicative controls.

The aim of this work is the study of the following viscous Burgers equation in the presence of a particular multiplicative control:

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi = (u \cdot Q) \psi & \text{in } (0, T), \\ \psi(0, \cdot) = \psi_0. \end{cases} \quad (1.1)$$

The equation is considered at first on the one-dimensional torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and later on the interval $(0, 2\pi)$ in the presence of Dirichlet and Neumann boundary conditions. In (1.1), the state ψ is controlled via the multiplicative control term $(u \cdot Q) \psi$. Here, the function

$$Q = (Q_0, \dots, Q_{q-1})$$

with $q \in \mathbb{N}^*$ is fixed and $u \in L^2(0, T; \mathbb{R}^q)$ is the control. This type of multiplicative control is also called bilinear, since only the time-dependent intensity is the actual control of the evolution and the term is linear with respect to it. Our aim is to study different controllability results for (1.1). Our theory is developed in the framework of the flat torus \mathbb{T} in order to simplify the presentation. We explain later how to deal with the Dirichlet and Neumann cases.

Small-time global approximate controllability

We recall that, with the above assumptions, the equation (1.1) admits a unique strong solution with values in $H^1(\mathbb{T})$ as soon as Q is smooth enough (see Proposition 2.1, Proposition 6.1 and Proposition 7.1 below). We consider first the framework of the one-dimensional torus and our first objective is to show that when

$$\{1, \cos(x), \sin(x)\} \subset \text{span}\{Q_0, \dots, Q_{q-1}\}, \quad (1.2)$$

we can approximately reach any element of

$$\mathcal{S}(\psi_0) := \{e^\varphi \psi_0 : \varphi \in H^1(\mathbb{T})\}.$$

The corresponding result states as follows:

Theorem 1.1. *Assume the condition (1.2). Then, for any $\psi_0 \in H^1(\mathbb{T})$, for any $\psi_1 \in \mathcal{S}(\psi_0)$, for any $\varepsilon > 0$ and $T > 0$ there exist $\tau \in (0, T]$ and $u \in L^2(0, \tau; \mathbb{R}^q)$ such that the solution ψ of (1.1) satisfies*

$$\|\psi(\tau, \cdot) - \psi_1\|_{H^1(\mathbb{T})} < \varepsilon.$$

As a corollary, if ψ_0 is positive (respectively negative), then one can reach any $\psi_1 \in H^1(\mathbb{T})$ with the same sign:

Corollary 1.2. *Assume the condition (1.2) and assume $\psi_0, \psi_1 \in H^1(\mathbb{T})$ with $\psi_0 \psi_1 > 0$ in \mathbb{T} . For any $\varepsilon > 0$ and $T > 0$, there exist $\tau \in (0, T]$ and $u \in L^2(0, \tau; \mathbb{R}^q)$ such that the solution ψ of (1.1) satisfies*

$$\|\psi(\tau, \cdot) - \psi_1\|_{H^1(\mathbb{T})} < \varepsilon.$$

More generally, the same result holds provided that $\ln(\psi_1/\psi_0) \in H^1(\mathbb{T})$. This allows us to consider the case where ψ_0 and ψ_1 can cancel, but in that case, they need to have the same behaviour around any (common) zero to avoid a singularity in $\ln(\psi_1/\psi_0)$. We can extend this condition, but we obtain an approximation in L^2 instead of H^1 :

Corollary 1.3. *Assume the condition (1.2) and assume $\psi_0, \psi_1 \in H^1(\mathbb{T})$ with $\text{sign}(\psi_0) = \text{sign}(\psi_1)$. For any $\varepsilon > 0$ and $T > 0$, there exist $\tau \in (0, T]$ and $u \in L^2(0, \tau; \mathbb{R}^q)$ such that the solution ψ of (1.1) satisfies*

$$\|\psi(\tau, \cdot) - \psi_1\|_{L^2(\mathbb{T})} < \varepsilon.$$

Corollary 1.2 and Corollary 1.3 state the global approximate controllability of the equation (1.1) in any positive time. The control can be performed by considering the three potentials

$$Q_0(x) = 1, \quad Q_1(x) = \cos(x), \quad Q_2(x) = \sin(x). \quad (1.3)$$

The method used to prove Theorem 1.1, the saturation limit developed in Section 3, is well-adapted to manifolds without boundaries, such as \mathbb{T} . Nevertheless, one can extend this method to the case of manifolds with boundaries. Here, we consider the Burgers equation in the interval $(0, 2\pi)$ with the “standard” Dirichlet or Neumann boundary conditions. The Dirichlet case is quite close to the torus, while we slightly need to adapt our techniques for the Neumann boundary conditions (see Section 6 and Section 7). In detail, we prove the following results.

- When Dirichlet boundary conditions are verified, an equivalent result to Theorem 1.1 can be ensured and we stated it in Theorem 6.4. In this case, we replace the hypothesis (1.2) with

$$\{1, \cos(x/2), \sin(x/2)\} \subset \text{span}\{Q_0, \dots, Q_{q-1}\}.$$

The result infers the small-time global approximate controllability with respect to the L^2 -topology which is presented in Corollary 6.5 (as Corollary 1.3 is implied by Theorem 1.1).

- The same property is true for the Neumann framework considered in Theorem 7.5, which adapts Theorem 1.1 under the hypothesis

$$\{1, \cos(x/2), \cos(x)\} \subset \text{span}\{Q_0, \dots, Q_{q-1}\}.$$

Corollary 7.7 and Corollary 7.8 present two subsequent approximate controllability results, respectively, with respect to the H^1 -topology and to the L^2 -topology (as Corollary 1.2 and Corollary 1.3).

To the best of our knowledge, this type of results was previously unknown in the literature. This may have been due to the uncertainty surrounding the study of bilinear global approximate controllability for parabolic equations before the recent work [17]. In that work, the authors introduced a successful technique for achieving this controllability in the heat equation with bilinear controls. The proof of Theorem 1.1 revisits the methods from [17] and demonstrates that this approach can be extended to the Burgers equation, despite its specific nonlinear dynamics. From this perspective, it is natural to ask whether the Navier-Stokes system can also be controlled via bilinear controls. This will be the subject of a future work.

Small-time exact controllability to the constant states

Our second objective is to show the small-time global exact controllability of (1.1) to the non-zero constant states associated with $u \equiv 0$ in the framework of the flat torus or in the presence of Neumann boundary conditions. As above, we focus at first on the torus \mathbb{T} . We denote by

$$\lambda_k := k^2, \quad \forall k \in \mathbb{Z} \tag{1.4}$$

the eigenvalues of the Laplace operator $A_0 : H^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$, $\xi \mapsto -\partial_{xx}\xi$. Up to $\lambda_0 = 0$, the eigenvalues are double ($\lambda_k = \lambda_{-k}$) and a corresponding family of orthonormalized eigenfunctions for A_0 is given by

$$w_k(x) := \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad \forall k \in \mathbb{Z}, x \in \mathbb{T}. \tag{1.5}$$

Theorem 1.4. *Assume the condition (1.2) and assume that $Q_0 \in H^2(\mathbb{T}; \mathbb{R}^q)$ verifies*

$$\exists b, d > 0, \quad \forall k \in \mathbb{Z}, \quad (\lambda_k^d + 1) \left| (Q_0, w_k)_{L^2(\mathbb{T})} \right| \geq b. \tag{1.6}$$

For any $T > 0$, $\psi_0 \in H^1(\mathbb{T})$ with $\psi_0 > 0$ and $\Psi \in \mathbb{R}_+^$, there exists $u \in L^2(0, T; \mathbb{R}^q)$ such that the solution ψ of (1.1) satisfies $\psi(T, \cdot) = \Psi$.*

Theorem 1.4 ensures the global exact controllability of the equation (1.1) to the non-zero stationary states associated with $u \equiv 0$ in any positive time. The result is obtained by coupling the small-time global approximate controllability ensured in Corollary 1.2 and a local exact controllability to the stationary states in any positive time. For this reason, we can control the equations by considering 4 control potentials : we need the 3 directions to ensure Corollary 1.2 (as in (1.3)), together with an addition control. An example of Q ensuring Theorem 1.4 is the following:

$$Q_0(x) = x^2(x - 2\pi)^2, \quad Q_1(x) = 1, \quad Q_2(x) = \cos(x), \quad Q_3(x) = \sin(x).$$

The result of Theorem 1.4 is also valid when considering the Burgers equation with Neumann boundary conditions, as presented in Theorem 7.9. Also in this case, we can give explicit potentials Q verifying the controllability. Two examples that can be deduced from Example 7.10 are the following

$$\begin{aligned} Q_0(x) &= x^2, \quad Q_1(x) = 1, \quad Q_2(x) = \cos(x/2), \quad Q_3(x) = \cos(x), \\ Q_0(x) &= x^3(x - 2\pi)^2, \quad Q_1(x) = 1, \quad Q_2(x) = \cos(x/2), \quad Q_3(x) = \cos(x). \end{aligned}$$

The question of whether exact controllability can be achieved within the Dirichlet framework remains open, as constant functions are not stationary solutions in this setting. Alternative non-constant states must therefore be considered, but our current techniques do not readily extend to this scenario. Further research will be essential to explore exact controllability under Dirichlet boundary conditions.

Some bibliography

The controllability of the Burgers equation has been widely studied via additive controls. Let us mention a few key results. One of the first results concerns the case of the non-viscous Burgers equation [23]. Subsequently, the local exact controllability of the Burgers equation with additive distributed controls was obtained using Carleman estimates in [20]. Several articles then addressed the global exact controllability, such as [22], [14], [25], [26], and others.

The study of bilinear controllability for parabolic-type equations has not been extensively explored until recently. The primary reason is the nature of the bilinear control, which is generally weaker compared to additive control. For instance, when we consider the simple linear heat equation

$$\partial_t \psi - \partial_{xx} \psi = (u \cdot Q) \psi, \quad (1.7)$$

with bilinear controls, it is well-known that exact controllability cannot be achieved in L^2 when $Q \in L^\infty$. This obstruction was first proved by Ball, Marsden, and Slemrod in the seminal work [6]. The authors demonstrated that the reachable set of the system (1.7) is contained in a countable union of compact subsets of L^2 , and its complement is dense. This property represents a natural obstruction to exact controllability in L^2 , and alternative types of controllability, such as approximate controllability or exact controllability to trajectories, must be considered. Of course, the results in [6] do not directly apply to (1.1), but they highlight the difficulty of achieving exact controllability via bilinear controls.

To the best of our knowledge, the approximate controllability of the Burgers equation via bilinear control has not been proven in the literature, and the result for the heat equation (1.7) (even in the presence of polynomial nonlinearities) was only established recently in [17]. In that work, the authors adapted a saturating geometric control approach, introduced by Agrachev and Sarychev in [1, 2] for internal controllability. This method was first applied to the bilinear controllability of the Schrödinger equation in [15], and further improvements were made in subsequent works such as [7, 8, 16, 27]. Our result on approximate controllability, presented in Theorem 1.1, shows that the saturating method can be extended to study the Burgers equation (1.1), despite its specific nonlinear behavior.

A second type of controllability introduced in the literature to overcome the obstructions proved in [6] is exact controllability to trajectories. This concept was first introduced by Alabau-Boussouira, Cannarsa, and Urbani in [3] to study heat-type equations like (1.7) on intervals. This approach was later applied to study the same equation in the presence of polynomial nonlinearities in [17], and on compact graphs in [11]. The first result of this kind in higher-dimensional settings is found in [10], where the controllability of (1.7) is proven for two-dimensional domains. A key aspect of the techniques developed in [3] to prove exact controllability to eigensolutions is ensuring the solvability of a suitable moment problem. This kind of result has been widely studied in the past few decades, starting with the classical works [18, 19], and extending to more recent ones such as [4, 5, 9, 11, 12, 13, 21], and others.

Outline of the work

The paper is organized as follows. In Section 2, we present some preliminary results on the problem as the well-posedness of the Burgers equation (1.1). In Section 3, we ensure an important saturation limit, which is used in the proof of Theorem 1.1. Section 4 is dedicated to the proof of Theorem 1.1 and Corollary 1.3. In Section 5, we finally develop the proof of Theorem 1.4. In Section 6 and Section 7, we present the controllability of the Burgers equation for Dirichlet and Neumann boundary conditions instead of the periodic boundary case.

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2 Preliminary results

The well-posedness of the Burgers equation is classical and can be done by adapting the proofs for the Navier-Stokes system (see, for instance, [28, Theorem 3.1, p. 282]): we use the Galerkin method where we show some a priori estimates and a compactness argument to pass to the limit in the nonlinear term. We have in particular the following result:

Proposition 2.1. Assume $\psi_0 \in H^1(\mathbb{T})$, $Q \in H^2(\mathbb{T}; \mathbb{R}^q)$, $T > 0$ and $u \in L^2(0, T; \mathbb{R}^q)$. Then, there exists a unique strong solution

$$\psi \in L^2(0, T; H^2(\mathbb{T})) \cap C^0([0, T]; H^1(\mathbb{T})) \cap H^1(0, T; L^2(\mathbb{T}))$$

of (1.1).

Assume $t_1 < t_2$. For $u \in L^2(t_1, t_2; \mathbb{R}^q)$, we denote by $\mathcal{R}_{t, t_1}(\psi_0; u)$ the solution $\psi(t, \cdot)$ of

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi = (u \cdot Q) \psi & \text{in } (t_1, t_2) \times \mathbb{T}, \\ \psi(t_1, \cdot) = \psi_0 & \text{in } \mathbb{T}. \end{cases} \quad (2.1)$$

In the particular case where $t_1 = 0$, we simply write $\mathcal{R}_t(\psi_0; u)$ instead of $\mathcal{R}_{t, 0}(\psi_0; u)$. Let us now show a continuity result with respect to the initial condition:

Proposition 2.2. Assume $\psi_0 \in H^1(\mathbb{T})$, $Q \in H^2(\mathbb{T}; \mathbb{R}^q)$, $T > 0$ and $u \in L^2(0, T; \mathbb{R}^q)$. Then, there exists a constant $C > 0$ such that for any $\psi_1 \in H^1(\mathbb{T})$,

$$\|\mathcal{R}(\psi_0; u) - \mathcal{R}(\psi_1; u)\|_{C^0([0, T]; H^1(\mathbb{T}))} \leq C \|\psi_0 - \psi_1\|_{H^1(\mathbb{T})} e^{C \|\psi_0 - \psi_1\|_{L^2(\mathbb{T})}^2}. \quad (2.2)$$

Proof. Assume $\psi_0, \psi_1 \in H^1(\mathbb{T})$, $Q \in H^2(\mathbb{T}; \mathbb{R}^q)$, $T > 0$ and $u \in L^2(0, T; \mathbb{R}^q)$. Let us set

$$\psi := \mathcal{R}(\psi_1; u) - \mathcal{R}(\psi_0; u), \quad \bar{\psi} := \mathcal{R}(\psi_0; u).$$

Then

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi + \bar{\psi} \partial_x \psi + \psi \partial_x \bar{\psi} = (u \cdot Q) \psi & \text{in } (0, T) \times \mathbb{T}, \\ \psi(0, \cdot) = \psi_1 - \psi_0 & \text{in } \mathbb{T}. \end{cases} \quad (2.3)$$

Let us first multiply the first equation of the above system by ψ and integrate by parts:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} |\psi|^2 dx + \int_{\mathbb{T}} |\partial_x \psi|^2 dx + \frac{1}{2} \int_{\mathbb{T}} (\partial_x \bar{\psi}) \psi^2 dx = \int_{\mathbb{T}} (u \cdot Q) \psi^2 dx.$$

Integrating in time and using the Grönwall lemma, we deduce the existence of a positive constant $C > 0$ such that

$$\|\psi\|_{L^\infty(0, T; L^2(\mathbb{T}))} + \|\psi\|_{L^2(0, T; H^1(\mathbb{T}))} \leq C \|\psi_0 - \psi_1\|_{L^2(\mathbb{T})} \exp\left(C \left(\|\bar{\psi}\|_{L^2(0, T; H^2(\mathbb{T}))} + \|Q\|_{L^\infty(\mathbb{T})^q} \|u\|_{L^2(0, T)^q}\right)\right).$$

Second, we multiply the first equation of (2.3) by $-\partial_{xx} \psi$ and integrate by parts:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} |\partial_x \psi|^2 dx + \int_{\mathbb{T}} |\partial_{xx} \psi|^2 dx &= \int_{\mathbb{T}} \psi \partial_x \psi \partial_{xx} \psi dx + \int_{\mathbb{T}} \bar{\psi} \partial_x \psi \partial_{xx} \psi dx \\ &\quad + \int_{\mathbb{T}} \psi \partial_x \bar{\psi} \partial_{xx} \psi dx - \int_{\mathbb{T}} (u \cdot Q) \psi \partial_{xx} \psi dx. \end{aligned} \quad (2.4)$$

By using the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}} |\partial_x \psi|^2 dx + \int_{\mathbb{T}} |\partial_{xx} \psi|^2 dx &\leq C \left(\|\psi\|_{L^\infty(\mathbb{T})}^2 + \|\bar{\psi}\|_{L^\infty(\mathbb{T})}^2 \right) \|\partial_x \psi\|_{L^2(\mathbb{T})}^2 \\ &\quad + C \left(\|\partial_x \bar{\psi}\|_{L^\infty(\mathbb{T})}^2 + \|Q\|_{L^\infty(\mathbb{T})^q}^2 |u|^2 \right) \|\psi\|_{L^2(\mathbb{T})}^2. \end{aligned} \quad (2.5)$$

Applying the Grönwall lemma, the above estimate yields

$$\|\partial_x \psi\|_{L^\infty(0, T; L^2(\mathbb{T}))}^2 + \|\partial_{xx} \psi\|_{L^2(0, T; H^1(\mathbb{T}))}^2 \leq C_1 \|\psi_0 - \psi_1\|_{H^1(\mathbb{T})}^2 \exp\left(C_1 \|\psi_0 - \psi_1\|_{L^2(\mathbb{T})}^2\right), \quad (2.6)$$

where $C_1 := C \exp\left(C \left(\|\bar{\psi}\|_{L^2(0, T; H^2(\mathbb{T}))}^2 + \|Q\|_{L^\infty(\mathbb{T})^q}^2 \|u\|_{L^2(0, T)^q}^2\right)\right)$ for some positive constant C . \square

3 The saturation limit

The aim of this section is to prove the following result

Proposition 3.1. Assume $\psi_0 \in H^1(\mathbb{T})$, $Q \in H^2(\mathbb{T}; \mathbb{R}^q)$, $u \in \mathbb{R}^q$ and $\varphi \in H^3(\mathbb{T})$, $\varphi > 0$. Then, the following limit holds

$$e^{\delta^{-1/2} \varphi} \mathcal{R}_\delta(e^{-\delta^{-1/2} \varphi} \psi_0, \delta^{-1} u) \rightarrow e^{(\varphi')^2 + u \cdot Q} \psi_0 \quad \text{in } H^1(\mathbb{T}) \text{ as } \delta \rightarrow 0^+.$$

Proof. To simplify, we assume in all the proof that $\delta \in (0, 1)$. By definition,

$$\psi := \mathcal{R} \left(e^{-\frac{\varphi}{\sqrt{\delta}}} \psi_0, \frac{u}{\delta} \right)$$

is the solution of

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi = \frac{1}{\delta} (u \cdot Q) \psi & \text{in } (0, T) \times \mathbb{T}, \\ \psi(0, \cdot) = e^{-\frac{\varphi}{\sqrt{\delta}}} \psi_0 & \text{in } \mathbb{T}. \end{cases} \quad (3.1)$$

We then denote

$$\psi_\delta(t, \cdot) := e^{\frac{\varphi}{\sqrt{\delta}}} \mathcal{R}_{\delta t} \left(e^{-\frac{\varphi}{\sqrt{\delta}}} \psi_0, \frac{u}{\delta} \right) \quad (t \geq 0), \quad (3.2)$$

so that for any $t \geq 0$,

$$\psi(t, \cdot) := e^{\frac{\varphi}{\sqrt{\delta}}} \psi_\delta \left(\frac{t}{\delta}, \cdot \right).$$

We deduce the following formulas:

$$\partial_t \psi = \frac{1}{\delta} e^{\frac{\varphi}{\sqrt{\delta}}} \partial_t \psi_\delta, \quad \partial_x \psi = e^{\frac{\varphi}{\sqrt{\delta}}} \left(\partial_x \psi_\delta - \frac{\varphi'}{\sqrt{\delta}} \psi_\delta \right), \quad (3.3)$$

$$\partial_{xx} \psi = e^{\frac{\varphi}{\sqrt{\delta}}} \left(\partial_{xx} \psi_\delta - 2 \frac{\varphi'}{\sqrt{\delta}} \partial_x \psi_\delta - \frac{\varphi''}{\sqrt{\delta}} \psi_\delta + \frac{(\varphi')^2}{\delta} \psi_\delta \right). \quad (3.4)$$

As a consequence, ψ_δ is solution of the system

$$\begin{aligned} \partial_t \psi_\delta - \delta \partial_{xx} \psi_\delta + 2\sqrt{\delta} \varphi' \partial_x \psi_\delta + \sqrt{\delta} \varphi'' \psi_\delta + \psi_\delta e^{\frac{\varphi}{\sqrt{\delta}}} \left(\delta \partial_x \psi_\delta - \sqrt{\delta} \varphi' \psi_\delta \right) \\ = \left((\varphi')^2 + u \cdot Q \right) \psi_\delta \quad \text{in } (0, T) \times \mathbb{T}, \end{aligned} \quad (3.5)$$

$$\psi_\delta(0, \cdot) = \psi_0 \quad \text{in } \mathbb{T}. \quad (3.6)$$

We consider the Laplace operator

$$A_0 : H^3(\mathbb{T}) \rightarrow H^1(\mathbb{T}), \quad \psi \mapsto -\partial_{xx} \psi. \quad (3.7)$$

This is a diagonalizable non-negative operator and thus (see, for instance, [29, Proposition 2.6.5, p.41]) the infinitesimal generator of semigroup an analytic semigroup $(e^{tA})_{t \geq 0}$. We set

$$\psi_0^\delta := e^{\delta^{1/4} A_0} \psi_0. \quad (3.8)$$

Then using a property of analytic semigroups, there exists a constant $C > 0$ such that

$$\|\psi_0^\delta\|_{H^1(\mathbb{T})} \leq C \|\psi_0\|_{H^1(\mathbb{T})}, \quad \|\psi_0^\delta\|_{H^3(\mathbb{T})} \leq \frac{C}{\delta^{1/4}} \|\psi_0\|_{H^1(\mathbb{T})}. \quad (3.9)$$

Moreover,

$$\psi_0^\delta \rightarrow \psi_0 \quad \text{in } H^1(\mathbb{T}), \quad \text{as } \delta \rightarrow 0^+. \quad (3.10)$$

Now let us set

$$g(t, \cdot) := e^{[(\varphi')^2 + u \cdot Q]t} \psi_0^\delta. \quad (3.11)$$

By using that $H^1(\mathbb{T})$ and $H^2(\mathbb{T})$ are algebras, we deduce that $g \in L^\infty(0, T; H^2(\mathbb{T}))$ and there exists a constant $C > 0$ (depending on u, Q, φ, ψ^0) such that

$$\|g\|_{L^\infty(0, T; H^1(\mathbb{T}))} \leq C, \quad \|g\|_{L^\infty(0, T; H^2(\mathbb{T}))} \leq \frac{C}{\delta^{1/4}} \quad (3.12)$$

Let us define

$$v := \psi_\delta - g. \quad (3.13)$$

We deduce from (3.5) that

$$\begin{aligned} \partial_t v - \delta \partial_{xx} v + 2\sqrt{\delta} \varphi' \partial_x v + \sqrt{\delta} \varphi'' v + \delta e^{\frac{\varphi}{\sqrt{\delta}}} (v \partial_x v + g \partial_x v + v \partial_x g) - \sqrt{\delta} e^{\frac{\varphi}{\sqrt{\delta}}} \varphi' (v^2 + 2vg) \\ = F + \left((\varphi')^2 + u \cdot Q \right) v \quad \text{in } (0, T) \times \mathbb{T}, \end{aligned} \quad (3.14)$$

with

$$F := \delta \partial_{xx}g - 2\sqrt{\delta}\varphi' \partial_xg - \sqrt{\delta}\varphi''g - \delta e^{\frac{-\varphi}{\sqrt{\delta}}}g \partial_xg + \sqrt{\delta}e^{\frac{-\varphi}{\sqrt{\delta}}}\varphi'g^2. \quad (3.15)$$

The estimates (3.12) and the above definition yield that

$$\|F\|_{L^\infty(0,T;L^2(\mathbb{T}))} \leq C\delta^{1/2}. \quad (3.16)$$

First, let us multiply (3.14) by v and integrate by parts:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{L^2(\mathbb{T})}^2 + \delta \|\partial_x v\|_{L^2(\mathbb{T})}^2 &= \frac{2}{3} \sqrt{\delta} \int_{\mathbb{T}} e^{\frac{-\varphi}{\sqrt{\delta}}} \varphi' v^3 dx - \frac{\delta}{2} \int_{\mathbb{T}} e^{\frac{-\varphi}{\sqrt{\delta}}} v^2 \partial_x g dx \\ &\quad + \frac{3}{4} \sqrt{\delta} \int_{\mathbb{T}} e^{\frac{-\varphi}{\sqrt{\delta}}} \varphi' v^2 g dx + \int_{\mathbb{T}} Fv dx + \int_{\mathbb{T}} \left((\varphi')^2 + u \cdot Q \right) v^2 dx \quad \text{in } (0, T). \end{aligned} \quad (3.17)$$

We deduce from (3.12) and (3.16) that

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(\mathbb{T})}^2 + \delta \|\partial_x v\|_{L^2(\mathbb{T})}^2 \leq C \|F\|_{L^2(\mathbb{T})}^2 + C\sqrt{\delta} \|v\|_{L^3(\mathbb{T})}^3 + C \|v\|_{L^2(\mathbb{T})}^2 \quad \text{in } (0, T). \quad (3.18)$$

Second, we multiply (3.14) by $-\partial_{xx}v$ and integrate by parts:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_{xx}v\|_{L^2(\mathbb{T})}^2 + \delta \|\partial_{xxx}v\|_{L^2(\mathbb{T})}^2 &= 2\sqrt{\delta} \int_{\mathbb{T}} \varphi' \partial_x v \partial_{xx}v dx + \sqrt{\delta} \int_{\mathbb{T}} \varphi'' v \partial_{xx}v dx \\ &\quad + \delta \int_{\mathbb{T}} e^{\frac{-\varphi}{\sqrt{\delta}}} (v \partial_x v + g \partial_x v + v \partial_x g) \partial_{xx}v dx - \sqrt{\delta} \int_{\mathbb{T}} e^{\frac{-\varphi}{\sqrt{\delta}}} \varphi' (v^2 + 2vg) \partial_{xx}v dx \\ &\quad - \int_{\mathbb{T}} F \partial_{xx}v dx + \int_{\mathbb{T}} (2\varphi' \varphi'' + u \cdot Q') v \partial_{xx}v dx + \int_{\mathbb{T}} \left((\varphi')^2 + u \cdot Q \right) (\partial_{xx}v)^2 dx \quad \text{in } (0, T). \end{aligned} \quad (3.19)$$

Then, using that $\varphi \in H^3(\mathbb{T})$ and (3.12), we have the following estimates

$$\left| 2\sqrt{\delta} \int_{\mathbb{T}} \varphi' \partial_x v \partial_{xx}v dx + \sqrt{\delta} \int_{\mathbb{T}} \varphi'' v \partial_{xx}v dx \right| \leq \frac{\delta}{5} \|\partial_{xx}v\|_{L^2(\mathbb{T})}^2 + C \|v\|_{H^1(\mathbb{T})}^2, \quad (3.20)$$

$$\begin{aligned} \delta \left| \int_{\mathbb{T}} e^{\frac{-\varphi}{\sqrt{\delta}}} (v \partial_x v + g \partial_x v + v \partial_x g) \partial_{xx}v dx \right| &\leq \frac{\delta}{5} \|\partial_{xx}v\|_{L^2(\mathbb{T})}^2 \\ &\quad + C\delta \left(\|v\|_{L^\infty(\mathbb{T})}^2 \|\partial_x v\|_{L^2(\mathbb{T})}^2 + \|g\|_{L^\infty(\mathbb{T})}^2 \|\partial_x v\|_{L^2(\mathbb{T})}^2 + \|v\|_{L^\infty(\mathbb{T})}^2 \|\partial_x g\|_{L^2(\mathbb{T})}^2 \right) \\ &\leq \frac{\delta}{5} \|\partial_{xx}v\|_{L^2(\mathbb{T})}^2 + C \|v\|_{H^1(\mathbb{T})}^2 + C\delta \|v\|_{H^1(\mathbb{T})}^4 \end{aligned} \quad (3.21)$$

and using that $e^{\frac{-\varphi}{\sqrt{\delta}}} \leq C\delta$,

$$\left| \sqrt{\delta} \int_{\mathbb{T}} e^{\frac{-\varphi}{\sqrt{\delta}}} \varphi' (v^2 + 2vg) \partial_{xx}v dx \right| \leq \frac{\delta}{5} \|\partial_{xx}v\|_{L^2(\mathbb{T})}^2 + C \|v\|_{H^1(\mathbb{T})}^2 + C\delta \|v\|_{H^1(\mathbb{T})}^4. \quad (3.22)$$

Finally, from (3.15),

$$\begin{aligned} \left| \int_{\mathbb{T}} F \partial_{xx}v dx \right| &\leq \int_{\mathbb{T}} \left| \delta \partial_{xx}g - \delta e^{\frac{-\varphi}{\sqrt{\delta}}} g \partial_xg + \sqrt{\delta} e^{\frac{-\varphi}{\sqrt{\delta}}} \varphi' g^2 \right| |\partial_{xx}v| dx \\ &\quad + \int_{\mathbb{T}} \left(3\sqrt{\delta} \varphi'' \partial_xg + 2\sqrt{\delta} \varphi' \partial_{xx}g + \sqrt{\delta} \varphi^{(3)} g \right) \partial_{xx}v dx. \end{aligned} \quad (3.23)$$

Using (3.12), we deduce that

$$\left| \int_{\mathbb{T}} F \partial_{xx}v dx \right| \leq \frac{\delta}{5} \|\partial_{xx}v\|_{L^2(\mathbb{T})}^2 + C\sqrt{\delta} + C \|v\|_{H^1(\mathbb{T})}^2. \quad (3.24)$$

Gathering (3.19), (3.20), (3.21), (3.22), and (3.24), we find

$$\frac{1}{2} \frac{d}{dt} \|\partial_{xx}v\|_{L^2(\mathbb{T})}^2 + \frac{\delta}{5} \|\partial_{xxx}v\|_{L^2(\mathbb{T})}^2 \leq C\sqrt{\delta} + C \|v\|_{H^1(\mathbb{T})}^2 + C\delta \|v\|_{H^1(\mathbb{T})}^4 \quad \text{in } (0, T).$$

The above estimate and (3.18) yield

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^1(\mathbb{T})}^2 + \frac{\delta}{5} \|v\|_{H^2(\mathbb{T})}^2 \leq C\sqrt{\delta} + C \|v\|_{H^1(\mathbb{T})}^2 + C\delta \|v\|_{H^1(\mathbb{T})}^4 \quad \text{in } (0, T). \quad (3.25)$$

Integrating in time the above relation and using (3.16), there exists a constant $C_0 > 0$ such that for $t \in [0, T]$,

$$\|v(t, \cdot)\|_{H^1(\mathbb{T})}^2 \leq \|\psi_0 - \psi_0^\delta\|_{H^1(\mathbb{T})}^2 + C_0\sqrt{\delta} + C_0 \int_0^t \left(\|v(s, \cdot)\|_{H^1(\mathbb{T})}^2 + \delta \|v(s, \cdot)\|_{H^1(\mathbb{T})}^4 \right) ds. \quad (3.26)$$

From (3.10), there exists $\delta_0 \in (0, 1)$ such that for $\delta \in (0, \delta_0)$,

$$\left(\|\psi_0 - \psi_0^\delta\|_{H^1(\mathbb{T})}^2 + C_0\sqrt{\delta} \right) e^{2C_0T} < 1. \quad (3.27)$$

By using the continuity of v and (3.27), we deduce the existence of a maximal interval $[0, T_\delta) \subset [0, T]$ such that

$$\|v(t, \cdot)\|_{H^1(\mathbb{T})}^2 < 1 \quad t \in [0, T_\delta).$$

In particular, using (3.26), the Grönwall lemma and the above relation, we obtain

$$\|v(t, \cdot)\|_{H^1(\mathbb{T})}^2 \leq \left(\|\psi_0 - \psi_0^\delta\|_{H^1(\mathbb{T})}^2 + C_0\sqrt{\delta} \right) e^{2C_0T} \quad (t \in [0, T_\delta),$$

and this shows that $T_\delta = T$. In particular, we can take $t = 1$ in the above relation and this implies that if $\delta \rightarrow 0$, then

$$\|v(1, \cdot)\|_{H^1(\mathbb{T})} = \left\| e^{\frac{\varphi}{\sqrt{\delta}}} \mathcal{R}_\delta \left(e^{-\frac{\varphi}{\sqrt{\delta}}} \psi_0, \frac{u}{\delta} \right) - e^{[(\varphi')^2 + u \cdot Q]} \psi_0^\delta \right\|_{H^1(\mathbb{T})} \rightarrow 0$$

Using again (3.10), we deduce the result. \square

4 Proof of the approximate controllability results

In this section, we prove Theorem 1.1 and Corollary 1.3 by using Proposition 3.1 proved in the previous section. Let us start with the following definition:

Definition 4.1. We denote by \mathcal{A} the set of $\varphi \in H^1(\mathbb{T})$ such that for any $\psi_0 \in H^1(\mathbb{T})$, for any $T > 0$ and $\varepsilon > 0$, there exist $\tau \in (0, T)$ and $u \in \mathcal{P}(0, \tau)$ such that

$$\|\mathcal{R}_\tau(\psi_0, u) - e^\varphi \psi_0\|_{H^1(\mathbb{T})} \leq \varepsilon. \quad (4.1)$$

Remark 4.2. From Proposition 3.1 (with $\varphi = 0$), we already have that

$$\text{span}\{Q_0, \dots, Q_{q-1}\} \subset \mathcal{A}.$$

Moreover, we remark that $\varphi_1, \varphi_2 \in \mathcal{A} \implies \varphi_1 + \varphi_2 \in \mathcal{A}$.

Proposition 4.3. Assume $\varphi \in H^3(\mathbb{T})$, $\varphi > 0$ such that $\text{span } \varphi \subset \mathcal{A}$. Then $(\varphi')^2 \in \mathcal{A}$.

Proof. Let us consider $\psi_0 \in H^1(\mathbb{T})$, $T > 0$ and $\varepsilon \in (0, 1)$. By Proposition 3.1, there exists $\tau_1 \in (0, T/3)$ such that

$$\left\| e^{\frac{\varphi}{\sqrt{\tau_1}}} \mathcal{R}_{\tau_1} \left(e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0, 0 \right) - e^{(\varphi')^2} \psi_0 \right\|_{H^1(\mathbb{T})} \leq \frac{\varepsilon}{3}.$$

We set

$$\psi_1 := \mathcal{R}_{\tau_1} \left(e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0, 0 \right) \in H^1(\mathbb{T})$$

so that the above estimate writes

$$\left\| e^{\frac{\varphi}{\sqrt{\tau_1}}} \psi_1 - e^{(\varphi')^2} \psi_0 \right\|_{H^1(\mathbb{T})} \leq \frac{\varepsilon}{3}. \quad (4.2)$$

Since $\varphi/\sqrt{\tau_1} \in \mathcal{A}$, there exists $\tau_2 \in (0, T/3)$, $\tilde{u}_2 \in \mathcal{P}(0, \tau_2)$ such that

$$\left\| \mathcal{R}_{\tau_2}(\psi_1, \tilde{u}_2) - e^{\frac{\varphi}{\sqrt{\tau_1}}} \psi_1 \right\|_{H^1(\mathbb{T})} \leq \frac{\varepsilon}{3}. \quad (4.3)$$

From Proposition 2.2, there exists $C_1 > 0$ such that for any $\psi_2 \in H^1(\mathbb{T})$ with

$$\left\| \psi_2 - e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0 \right\|_{H^1(\mathbb{T})} \leq 1, \quad (4.4)$$

then

$$\left\| \mathcal{R}(\psi_2, 0) - \mathcal{R}(e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0, 0) \right\|_{C^0([0, T]; H^1(\mathbb{T}))} \leq C_1 \left\| \psi_2 - e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0 \right\|_{H^1(\mathbb{T})}. \quad (4.5)$$

Similarly, there exists $C_2 > 0$ such that for any $\psi_2 \in H^1(\mathbb{T})$ with

$$\|\psi_2 - \psi_1\|_{H^1(\mathbb{T})} \leq 1, \quad (4.6)$$

then

$$\|\mathcal{R}(\psi_2, \tilde{u}_2) - \mathcal{R}(\psi_1, \tilde{u}_2)\|_{C^0([0, T]; H^1(\mathbb{T}))} \leq C_2 \|\psi_2 - \psi_1\|_{H^1(\mathbb{T})}. \quad (4.7)$$

Since $-\varphi/\sqrt{\tau_1} \in \mathcal{A}$, there exist $\tau_0 \in (0, T/3)$, $u_0 \in \mathcal{P}(0, \tau_0)$ such that

$$\left\| \mathcal{R}_{\tau_0}(\psi_0, u_0) - e^{-\frac{\varphi}{\sqrt{\tau_1}} \psi_0} \right\|_{H^1(\mathbb{T})} \leq \min\left(\frac{1}{3(1+C_1)(1+C_2)}\varepsilon, \frac{1}{1+C_1}\right). \quad (4.8)$$

Combining (4.5) and (4.8), we deduce that

$$\begin{aligned} & \left\| \mathcal{R}_{\tau_1+\tau_0, \tau_0}(\mathcal{R}_{\tau_0}(\psi_0, u_0), 0) - \mathcal{R}_{\tau_1+\tau_0, \tau_0}(e^{-\frac{\varphi}{\sqrt{\tau_1}} \psi_0}, 0) \right\|_{H^1(\mathbb{T})} \\ &= \left\| \mathcal{R}_{\tau_0+\tau_1}(\psi_0, u_0 \mathbf{1}_{(0, \tau_0)}) - \psi_1 \right\|_{H^1(\mathbb{T})} \leq \min\left(\frac{1}{3(1+C_2)}\varepsilon, 1\right). \end{aligned} \quad (4.9)$$

By translating \tilde{u}_2 , we deduce from (4.3) the existence of $u_2 \in \mathcal{P}(\tau_0 + \tau_1, \tau_0 + \tau_1 + \tau_2)$ such that

$$\left\| \mathcal{R}_{\tau_0+\tau_1+\tau_2, \tau_0+\tau_1}(\psi_1, u_2) - e^{\frac{\varphi}{\sqrt{\tau_1}} \psi_1} \right\|_{H^1(\mathbb{T})} \leq \frac{\varepsilon}{3}. \quad (4.10)$$

Combining (4.7) and (4.9), we deduce that

$$\left\| \mathcal{R}_{\tau_0+\tau_1+\tau_2}(\psi_0, u_0 \mathbf{1}_{(0, \tau_0)} + u_2 \mathbf{1}_{(\tau_0+\tau_1, \tau_0+\tau_1+\tau_2)}) - \mathcal{R}_{\tau_0+\tau_1+\tau_2, \tau_0+\tau_1}(\psi_1, u_2) \right\|_{H^1(\mathbb{T})} \leq \frac{\varepsilon}{3}. \quad (4.11)$$

Combining the above estimate with (4.10) and (4.2), we deduce that

$$\left\| \mathcal{R}_{\tau_0+\tau_1+\tau_2}(\psi_0, u_0 \mathbf{1}_{(0, \tau_0)} + u_2 \mathbf{1}_{(\tau_0+\tau_1, \tau_0+\tau_1+\tau_2)}) - e^{(\varphi')^2 \psi_0} \right\|_{H^1(\mathbb{T})} \leq \varepsilon. \quad (4.12)$$

□

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Using Remark 4.2 and (1.2), we already have that

$$\text{span}\{1, \sin, \cos\} \subset \mathcal{A}.$$

Then using a result proved in [15] and Proposition 4.3, we can show that

$$\text{span}\{1, x \mapsto \sin(kx), x \mapsto \cos(kx) : k \in \mathbb{N}^*\} \subset \mathcal{A}.$$

This implies that \mathcal{A} is dense in $H^1(\mathbb{T})$. Using this fact and that the mapping

$$H^1(\mathbb{T}) \rightarrow H^1(\mathbb{T}), \quad \varphi \mapsto e^\varphi$$

is continuous, there exists $\varphi_\varepsilon \in \mathcal{A}$ such that

$$\|e^\varphi \psi_0 - e^{\varphi_\varepsilon} \psi_0\|_{H^1(\mathbb{T})} \leq \frac{\varepsilon}{2}.$$

Since $\varphi_\varepsilon \in \mathcal{A}$, for any $T > 0$, there exist $\tau \in (0, T)$ and $u \in \mathcal{P}(0, \tau)$ such that

$$\|\mathcal{R}_\tau(\psi_0, u) - e^{\varphi_\varepsilon} \psi_0\|_{H^1(\mathbb{T})} \leq \frac{\varepsilon}{2}.$$

Combining the last two relations, we deduce the result. □

We also prove here Corollary 1.3:

Proof of Corollary 1.3. Assume $\psi_0, \psi_1 \in H^1(\mathbb{T})$ with $\text{sign}(\psi_0) = \text{sign}(\psi_1)$ and let us consider $\varepsilon > 0$. We denote by Z the closed set of zeros of ψ_0 and ψ_1 :

$$Z := \psi_0^{-1}(\{0\}) = \psi_1^{-1}(\{0\}).$$

For $\eta > 0$, we set

$$Z_\eta := \{x \in \mathbb{T} : \text{dist}(x, Z) > \eta\} \quad \text{and} \quad \varphi_\eta := 1_{Z_\eta} \ln \left(\frac{\psi_1}{\psi_0} \right).$$

Since $\psi_0, \psi_1 \in H^1(\mathbb{T}) \hookrightarrow C^0(\mathbb{T})$, we deduce that $\varphi_\eta \in L^\infty(\mathbb{T})$. In Z_η , $e^{\varphi_\eta} \psi_0 - \psi_1 = 0$, whereas for $x \in \mathbb{T} \setminus Z_\eta$, we have

$$\left| e^{\varphi_\eta(x)} \psi_0(x) - \psi_1(x) \right| = |\psi_0(x) - \psi_1(x)| \leq \sqrt{\eta} \|\psi_0 - \psi_1\|_{H^1(\mathbb{T})}.$$

We thus deduce that for $\eta > 0$ small enough,

$$\|e^{\varphi_\eta} \psi_0 - \psi_1\|_{L^2(\mathbb{T})} \leq \frac{\varepsilon}{3}.$$

On the other hand, since \mathcal{A} is dense in $H^1(\mathbb{T})$, it is dense in $L^2(\mathbb{T})$. Thus, there exists $\varphi \in \mathcal{A}$ such that

$$\|e^\varphi \psi_0 - e^{\varphi_\eta} \psi_0\|_{L^2(\mathbb{T})} \leq \frac{\varepsilon}{3}.$$

Finally, using the definition of \mathcal{A} , for any $T > 0$, there exist $\tau \in (0, T)$ and $u \in \mathcal{P}(0, \tau)$ such that

$$\|\mathcal{R}_\tau(\psi_0, u) - e^\varphi \psi_0\|_{H^1(\mathbb{T})} \leq \frac{\varepsilon}{3}.$$

Combining the last three relations yields the conclusion of the corollary. \square

5 Exact controllability to the stationary states

The proof of Theorem 1.4 is a direct consequence of the combination of Corollary 1.2 and of the following local exact controllability result:

Theorem 5.1. *Assume that $q = 1$ and that Q_0 verifies (1.6). For any $T > 0$ and for any $\Psi > 0$, there exists $\varepsilon > 0$ such that for any*

$$\psi_0 \in H^1(\mathbb{T}), \quad \|\psi_0 - \Psi\|_{H^1(\mathbb{T})} < \varepsilon,$$

there exists $u_0 \in L^2(0, T; \mathbb{R})$ so that the solution ψ of (1.1) satisfies

$$\psi(T, \cdot) = \Psi.$$

Remark 5.2. Note that we only consider here the case of one control ($q = 1$) but if we have more controls, it is sufficient to take them equal to 0.

Let us give the short proof of Theorem 1.4:

Proof of Theorem 1.4. In order to prove Theorem 1.4, we divide the time interval in $[0, T/2]$ and $[T/2, T]$. Then, we consider the size $\varepsilon > 0$ of Theorem 5.1 associated with Ψ and $T/2$ and apply Corollary 1.2 to obtain a control $u \in L^2(0, \tau; \mathbb{R}^q)$ and a time $\tau \in (0, T/2)$ such that

$$\|\psi(\tau, \cdot) - \Psi\|_{H^1(\mathbb{T})} < \varepsilon.$$

Then applying Theorem 5.1, we obtain a control $u \in L^2(0, \tau + T/2; \mathbb{R}^q)$ such that

$$\psi(\tau + T/2, \cdot) = \Psi.$$

Then, in the time $(\tau + T/2, T)$, we take the control $u \equiv 0$ and use that $\psi = \Psi$ is the corresponding solution of (1.1) so that $\psi(T, \cdot) = \Psi$. \square

Before giving the proof of Theorem 5.1, let us first show the existence of Q_0 satisfying (1.6):

Example 5.3. Let us defined

$$Q_0(x) = x^2(x - 2\pi)^2 \quad x \in [0, 2\pi],$$

extended by 2π -periodicity so that $Q_0 \in H^2(\mathbb{T})$. Some standard computation yields

$$(Q_0, w_k)_{L^2(\mathbb{T})} = -\frac{24\sqrt{2\pi}}{k^4}, \quad \forall k \in \mathbb{Z}^*,$$

and $(Q_0, w_0)_{L^2(\mathbb{T})} > 0$. In particular, using (1.4), we deduce the existence of $b > 0$ such that

$$\forall k \in \mathbb{Z}, \quad (\lambda_k^2 + 1) \left| (Q_0, w_k)_{L^2(\mathbb{T})} \right| \geq b,$$

that is (1.6) for $d = 2$.

We now focus on the proof Theorem 5.1. If ψ is the solution of (1.1) associated with the control u_0 , then $\xi := \psi - \Psi$ satisfies the system

$$\begin{cases} \partial_t \xi - \partial_{xx} \xi + \Psi \partial_x \xi = (u_0 Q_0) \Psi - \xi \partial_x \xi + (u_0 Q_0) \xi & \text{in } (0, T) \times \mathbb{T}, \\ \xi(0, \cdot) = \xi_0 & \text{in } \mathbb{T}, \end{cases} \quad (5.1)$$

with $\xi_0 := \psi_0 - \Psi$ and our aim is to show the local null controllability of the above system.

5.1 Control of the linearized system

In order to prove Theorem 5.1, we first linearize (5.1) and consider the null-controllability of the linear system

$$\begin{cases} \partial_t \xi - \partial_{xx} \xi + \Psi \partial_x \xi = (u_0 Q_0) \Psi & \text{in } (0, T) \times \mathbb{T}, \\ \xi(0, \cdot) = \xi_0 & \text{in } \mathbb{T}, \end{cases} \quad (5.2)$$

with $Q_0 \in L^2(\mathbb{T})$ satisfying (1.6). Let us define

$$\mathcal{D}(A) := H^2(\mathbb{T}), \quad A : \mathcal{D}(A) \rightarrow L^2(\mathbb{T}), \quad \xi \mapsto \partial_{xx} \xi - \Psi \partial_x \xi. \quad (5.3)$$

It is well-known that A is the infinitesimal generator of an analytic semigroup. We also define the control operator

$$B \in \mathcal{L}(\mathbb{R}, L^2(\mathbb{T})), \quad Bu_0 := (u_0 Q_0) \Psi.$$

With the above notation, we can write (5.2) in the following abstract form

$$\partial_t \xi = A\xi + Bu_0 \quad \text{in } (0, T), \quad \xi(0) = \xi_0 \in L^2(\mathbb{T}). \quad (5.4)$$

Our result is the following one:

Proposition 5.4. *Assume that Q_0 verifies (1.6). Then, there exist $M, \nu \in \mathbb{R}_+^*$ such that the problem (5.2) is null controllable in any time $T > 0$ with a cost of the control less than $Me^{\nu/T}$. More precisely, for any $\xi_0 \in L^2(\mathbb{T})$, there exists a control $u_0 \in L^2(0, T)$ such that*

$$\|u_0\|_{L^2(0, T)} \leq Me^{\nu/T} \|\xi_0\|_{L^2(\mathbb{T})}$$

and such that the solution of (5.2) satisfies $\xi(T, \cdot) = 0$.

Proof. The adjoint A^* of the operator A defined by (5.3) is given by

$$\mathcal{D}(A^*) := H^2(\mathbb{T}), \quad A^* : \mathcal{D}(A^*) \rightarrow L^2(\mathbb{T}), \quad \xi \mapsto \partial_{xx} \xi + \Psi \partial_x \xi. \quad (5.5)$$

Note that the family $\{w_k\}_{k \in \mathbb{Z}}$ defined by (1.5) is a family of eigenvectors of $-A^*$ and the corresponding eigenvalues are

$$\tilde{\lambda}_k := k^2 - ik\Psi \quad (k \in \mathbb{Z}).$$

We then use the standard moment method, that consists in formally multiplying (5.2) by $W_k(t) := e^{-\tilde{\lambda}_k(T-t)} w_k$, that is the solution of the adjoint system

$$-\partial_t W_k = A^* W_k \quad \text{in } (0, T), \quad W_k(T) = w_k.$$

After some integrations by parts, we deduce that the solution ξ of (5.2) satisfies $\xi(T, \cdot) = 0$ if and only if

$$-\frac{(\xi_0, w_{-k})_{L^2(\mathbb{T})}}{\Psi(Q_0, w_{-k})_{L^2(\mathbb{T})}} e^{-\tilde{\lambda}_k T} = \int_0^T u_0(t) e^{-\tilde{\lambda}_k(T-t)} dt \quad (k \in \mathbb{Z}). \quad (5.6)$$

In order to solve the above moment system, we apply Theorem 1.5 in [9]. More precisely, let us consider the bijection

$$\Xi : \mathbb{Z} \rightarrow \mathbb{N}^*, \quad \begin{cases} k \mapsto 2k & \text{if } k \geq 1, \\ k \mapsto -2k + 1 & \text{if } k \leq 0. \end{cases}$$

Then we set

$$\Lambda_k := \tilde{\lambda}_{\Xi^{-1}(k)} + 1.$$

We can check that the sequence $\{\Lambda_k\}_{k \geq 1}$ satisfies the following relations

1. For all $k \geq 1$, $\text{Re } \Lambda_k > 0$.
2. For all $k \geq 1$, $|\text{Im } \Lambda_k| \leq |\Psi| \sqrt{\text{Re } \Lambda_k}$.

3. For all $k \geq 1$, $|\Lambda_k| \leq |\Lambda_{k+1}|$.
4. For all $k, n \geq 1$, $|\Lambda_k - \Lambda_n| \geq \min\left(\frac{2|\Psi|}{5}, \frac{1}{9}\right) |k^2 - n^2|$.
5. By setting for $r > 0$,

$$\mathcal{N}(r) := \text{card} \{k \geq 1 : |\Lambda_k| \leq r\},$$

there exists a constant $C > 0$ such that for any $r > 0$, $|\mathcal{N}(r) - 2\sqrt{r}| \leq C$.

Theorem 1.5 in [9] states that if the sequence $\{\Lambda_k\}_{k \geq 1}$ satisfies the above conditions, then there exists a sequence

$$\{E_k\}_{k \geq 1} \subset L^2(-T/2, T/2),$$

such that

$$\forall k, j \geq 1, \quad \int_{-T/2}^{T/2} E_k(t) e^{-\Lambda_j t} dt = \delta_{k,j}$$

and there exists a constant $C > 0$ such that

$$\forall k \geq 1, \quad \|E_k\|_{L^2(-T/2, T/2)} \leq C e^{C(\sqrt{\text{Re} \Lambda_k} + 1/T)}. \quad (5.7)$$

We set for $k \in \mathbb{Z}$ and $t \in [0, T]$,

$$e_k(t) := E_{\Xi(k)} \left(\frac{T}{2} - t \right) e^{-(T-t) + \Lambda_{\Xi(k)} T/2}.$$

For any $k, j \in \mathbb{Z}$,

$$\int_0^T e_k(t) e^{-\tilde{\lambda}_j(T-t)} dt = e^{(\Lambda_{\Xi(k)} - \Lambda_{\Xi(j)})T/2} \int_{-T/2}^{T/2} E_{\Xi(k)}(t) e^{-\Lambda_{\Xi(j)} t} dt = \delta_{k,j}.$$

Moreover, we deduce from (5.7) that

$$\forall k \in \mathbb{Z}, \quad \|e_k\|_{L^2(0, T)} \leq C e^{C(\sqrt{k^2+1} + 1/T) + (k^2+1)T/2}. \quad (5.8)$$

Our aim is to use this family $\{e_k\}_{k \in \mathbb{Z}}$ to construct a solution for (5.6): if the series

$$u_0 := \sum_{k \in \mathbb{Z}} -\frac{(\xi_0, w_{-k})_{L^2(\mathbb{T})}}{\Psi(Q_0, w_{-k})_{L^2(\mathbb{T})}} e^{-\tilde{\lambda}_k T} e_k. \quad (5.9)$$

is convergent in $L^2(0, T)$, then it is a solution of (5.6). Combining (5.8) and (1.6), we can see that

$$\left\| -\frac{(\xi_0, w_{-k})_{L^2(\mathbb{T})}}{\Psi(Q_0, w_{-k})_{L^2(\mathbb{T})}} e^{-\tilde{\lambda}_k T} e_k \right\|_{L^2(0, T)} \leq C k^{2d} \left| (\xi_0, w_{-k})_{L^2(\mathbb{T})} \right| e^{C(k+1/T) - k^2 T/2 + T/2} \quad (5.10)$$

Assume that $T \in (0, 1)$. Then

$$\sum_{k \in \mathbb{Z}} \left(k^{2d} e^{C(k+1/T) - k^2 T/2 + T/2} \right)^2 \leq C e^{C/T} \sum_{k \geq 1} k^{4d} e^{-k^2 T/2}. \quad (5.11)$$

In order to estimate this series, we introduce the function $f(x) := x^{4d} e^{-x^2 T/2}$ for $x \geq 1$. We can check that f is increasing for $x \in [1, 2\sqrt{d/T}]$ and decreasing for $x \geq 2\sqrt{d/T}$. In particular, there exists a constant $C = C(d) > 0$ such that

$$\forall x \geq 1, \quad f(x) \leq \frac{C}{T^{2d}}$$

Let us set

$$k_1 := \left\lfloor 2\sqrt{\frac{d}{T}} \right\rfloor.$$

Then

$$\sum_{k \geq 1} k^{4d} e^{-k^2 T/2} = \sum_{1 \leq k \leq k_1-1} k^{4d} e^{-k^2 T/2} + \sum_{k=k_1, k_1+1} k^{4d} e^{-k^2 T/2} + \sum_{k \geq k_1+2} k^{4d} e^{-k^2 T/2}$$

so that

$$\sum_{k \geq 1} k^{4d} e^{-k^2 T/2} \leq \int_0^\infty x^{4d} e^{-x^2 T/2} dx + \frac{C}{T^{2d}} \leq \frac{C}{T^{2d+1/2}} + \frac{C}{T^{2d}}.$$

Combining the above estimate with (5.11), we deduce that

$$\sum_{k \in \mathbb{Z}} \left(k^{2d} e^{C(k+1/T) - k^2 T/2 + T/2} \right)^2 \leq C e^{C/T}$$

and using the Cauchy-Schwarz inequality on (5.10), we obtain that u_0 is well-defined by (5.9), is in $L^2(0, T)$ with

$$\|u_0\|_{L^2(0, T)} \leq C e^{C/T} \|\xi_0\|_{L^2(\mathbb{T})}.$$

This concludes the proof. \square

5.2 Controllability of the nonlinear system

From the previous section, we have the obtain the null-controllability of (5.4) with an estimate of the cost of the control of the form $\gamma(T) := M e^{\nu/T}$. We now use a method developed in [24] to show the null-controllability of

$$\partial_t \xi = A\xi + B u_0 + f \quad \text{in } (0, T), \quad \xi(0) = \xi_0 \in L^2(\mathbb{T}). \quad (5.12)$$

More precisely, let us set

$$\rho_{\mathcal{F}}(t) := e^{-72\nu/(T-t)}, \quad \rho_0(t) := M e^{-45\nu/(T-t)}, \quad \rho(t) := e^{-44\nu/(T-t)} \quad (5.13)$$

and

$$\mathcal{F} := \left\{ f \in L^2(0, T; L^2(\mathbb{T})) : \frac{f}{\rho_{\mathcal{F}}} \in L^2(0, T; L^2(\mathbb{T})) \right\}, \quad (5.14)$$

$$\mathcal{U} := \left\{ u_0 \in L^2(0, T) : \frac{u}{\rho_0} \in L^2(0, T) \right\}. \quad (5.15)$$

Then combining Proposition 2.3 and Proposition 2.8 in [24], we obtain the following result.

Proposition 5.5. *There exists an operator $\mathcal{E} \in \mathcal{L}(H^1(\mathbb{T}) \times \mathcal{F}, \mathcal{U})$ such that for any $(\xi_0, f) \in H^1(\mathbb{T}) \times \mathcal{F}$, the solution ξ of (5.12) associated with $u_0 = \mathcal{E}(\xi_0, f)$ satisfies*

$$\frac{\xi}{\rho} \in L^2(0, T; H^2(\mathbb{T})) \cap C^0([0, T]; H^1(\mathbb{T})) \cap H^1(0, T; L^2(\mathbb{T})). \quad (5.16)$$

Moreover, there exists a constant $C > 0$ such that

$$\left\| \frac{\xi}{\rho} \right\|_{L^2(0, T; H^2(\mathbb{T})) \cap C^0([0, T]; H^1(\mathbb{T})) \cap H^1(0, T; L^2(\mathbb{T}))} + \|u_0\|_{\mathcal{U}} \leq C \left(\|\xi_0\|_{H^1(\mathbb{T})} + \|f\|_{\mathcal{F}} \right). \quad (5.17)$$

Remark 5.6. The exponents in the definitions of the weight functions in (5.13) are neither unique nor optimal. They correspond to the construction used in [24]. Once γ and $\rho_{\mathcal{F}}$ are chosen, the weight function ρ_0 is given by

$$\rho_0(t) := \rho_{\mathcal{F}}(T + r^2(t - T)) \gamma((r - 1)(T - t)) \quad \left(t \in \left[T \left(1 - \frac{1}{r^2} \right), T \right] \right)$$

for some $r > 1$. Here we take $r = 6/5$. Then ρ is chosen so that there exists a positive constant such that

$$\rho_0 \leq C\rho, \quad \rho_{\mathcal{F}} \leq C\rho, \quad |\rho'| \rho_0 \leq C\rho^2 \quad \text{in } (0, T).$$

We can now prove Theorem 5.1 in a standard way by using a fixed-point argument:

Proof of Theorem 5.1. Assume $\xi_0 \in H^1(\mathbb{T})$ and let us consider the mapping

$$\mathcal{Z} : \mathcal{F} \rightarrow \mathcal{F}, \quad f \mapsto -\xi \partial_x \xi + (u_0 Q_0) \xi,$$

where $u_0 = \mathcal{E}(\xi_0, f)$ and where ξ is the solution of (5.12), that satisfies in particular (5.16) and (5.17). If we show that \mathcal{Z} admits a fixed point f , then the corresponding solution ξ of (5.12) associated with $u_0 = \mathcal{E}(\xi_0, f)$ is a solution of (5.1) such that $\xi(T, \cdot) = 0$.

In order to show that \mathcal{Z} admits a fixed-point, we use the Banach fixed-point theorem. First,

$$\|\mathcal{Z}(f)\|_{\mathcal{F}} \leq \left\| \frac{\xi \partial_x \xi}{\rho_{\mathcal{F}}} \right\|_{L^2(0,T;L^2(\mathbb{T}))} + \left\| \frac{(u_0 Q_0) \xi}{\rho_{\mathcal{F}}} \right\|_{L^2(0,T;L^2(\mathbb{T}))}.$$

With our choice in (5.13), we have $\rho^2 \leq C \rho_{\mathcal{F}}$ and $\rho \rho_0 \leq C \rho_{\mathcal{F}}$. Consequently

$$\begin{aligned} \left\| \frac{\xi \partial_x \xi}{\rho_{\mathcal{F}}} \right\|_{L^2(0,T;L^2(\mathbb{T}))} + \left\| \frac{(u_0 Q_0) \xi}{\rho_{\mathcal{F}}} \right\|_{L^2(0,T;L^2(\mathbb{T}))} &\leq C \left(\left\| \frac{\xi \partial_x \xi}{\rho} \right\|_{L^2(0,T;L^2(\mathbb{T}))} + \left\| Q_0 \frac{u_0 \xi}{\rho_0 \rho} \right\|_{L^2(0,T;L^2(\mathbb{T}))} \right) \\ &\leq C_0 \left(\|\xi_0\|_{H^1(\mathbb{T})} + \|f\|_{\mathcal{F}} \right)^2 \end{aligned}$$

for some positive constant C_0 . Let us consider $R \in \left(0, \frac{1}{4C_0}\right)$. Then, we deduce from the above estimate that for any $\xi_0 \in H^1(\mathbb{T})$, $\|\xi_0\|_{H^1(\mathbb{T})} \leq R$, the closed ball

$$B_R := \{f \in \mathcal{F} : \|f\|_{\mathcal{F}} \leq R\}$$

is invariant by \mathcal{Z} . With a similar argument, one can show that the restriction of \mathcal{Z} on B_R is a strict contraction. This concludes the proof of Theorem 5.1. \square

6 Controllability of the Burger equation with Dirichlet boundary conditions

In the previous sections, we have obtained different small-time controllability results for the Burgers equation in the torus \mathbb{T} , which corresponds to periodic boundary conditions. Here, we show how to adapt some of these results in the case of another classical boundary conditions: the Dirichlet boundary conditions

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi = (u \cdot Q) \psi & \text{in } (0, T) \times (0, 2\pi), \\ \psi(\cdot, 0) = \psi(\cdot, 2\pi) = 0 & \text{in } (0, T), \\ \psi(0, \cdot) = \psi_0 & \text{in } (0, 2\pi). \end{cases} \quad (6.1)$$

More precisely, we show that the small-time global approximate controllability of Theorem 1.1 is also valid for the problem (6.1). The Neumann case is studied in the next section. In these two new frameworks, several proofs are similar to the ones of the torus case, and we only underline the main differences.

We start by noticing that, for (6.1), we have similar properties to Proposition 2.1 and Proposition 2.2, ensuring the well-posedness of the system and the continuity of the solutions with respect to the initial data.

Proposition 6.1. *Assume $\psi_0 \in H_0^1(0, 2\pi)$, $Q \in H^2(0, 2\pi; \mathbb{R}^q)$, $T > 0$ and $u \in L^2(0, T; \mathbb{R}^q)$. Then, there exists a unique strong solution*

$$\psi \in L^2(0, T; H^2(0, 2\pi)) \cap C^0([0, T]; H_0^1(0, 2\pi)) \cap H^1(0, T; L^2(0, 2\pi))$$

of (6.1).

We keep the same notation as in the case of the torus: for $t_1 < t_2$ and for $u \in L^2(t_1, t_2; \mathbb{R}^q)$, we denote by $\mathcal{R}_{t, t_1}(\psi_0; u)$ the solution $\psi(t, \cdot)$ of

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi = (u \cdot Q) \psi & \text{in } (t_1, t_2) \times (0, 2\pi), \\ \psi(\cdot, 0) = \psi(\cdot, 2\pi) = 0 & \text{in } (t_1, t_2), \\ \psi(t_1, \cdot) = \psi_0 & \text{in } (0, 2\pi). \end{cases} \quad (6.2)$$

In the particular case where $t_1 = 0$, we simply write $\mathcal{R}_t(\psi_0; u)$ instead of $\mathcal{R}_{t, 0}(\psi_0; u)$.

Proposition 6.2. *Assume $\psi_0 \in H_0^1(0, 2\pi)$, $Q \in H^2(0, 2\pi; \mathbb{R}^q)$, $T > 0$ and $u \in L^2(0, T; \mathbb{R}^q)$. Then, there exists a constant $C > 0$ such that for any $\psi_1 \in H_0^1(0, 2\pi)$,*

$$\|\mathcal{R}(\psi_0; u) - \mathcal{R}(\psi_1; u)\|_{C^0([0, T]; H^1(0, 2\pi))} \leq C \|\psi_0 - \psi_1\|_{H^1(0, 2\pi)} e^{C \|\psi_0 - \psi_1\|_{L^2(0, 2\pi)}^2}. \quad (6.3)$$

The proof of Proposition 6.1 can be obtained by adapting the proofs for the Navier-Stokes system (see, for instance, [28, Theorem 3.1, p. 282]) to the Burgers equation and the proof of Proposition 6.2 is almost identical to the proof of Proposition 2.2 since the boundary terms in the integration by parts disappear with the Dirichlet boundary conditions.

Now, we start the study of the controllability of (6.1) and the result associated with the saturation limit can be stated similarly as Proposition 3.1:

Proposition 6.3. *Assume $\psi_0 \in H_0^1(0, 2\pi)$, $Q \in H^2(0, 2\pi; \mathbb{R}^q)$, $u \in \mathbb{R}^q$ and $\varphi \in H^3(0, 2\pi)$, $\varphi > 0$. Then, the following limit holds*

$$e^{\delta^{-1/2}\varphi}\mathcal{R}_\delta(e^{-\delta^{-1/2}\varphi}\psi_0, \delta^{-1}u) \rightarrow e^{(\varphi')^2+u\cdot Q}\psi_0 \quad \text{in } H^1(0, 2\pi) \text{ as } \delta \rightarrow 0^+.$$

Proof. The proof of Proposition 6.3 is completely similar to the proof of Proposition 3.1. We note in particular that ψ_δ defined by (3.2) satisfies the Dirichlet boundary conditions at the boundaries $\{0, 2\pi\}$. For the definition of g , we still use the formula (3.11) with ψ_0^δ given by (3.8), but we modify the definition of A_0 . Instead of (3.7), we set

$$\begin{aligned} \mathcal{D}(A_0) &:= \{\psi \in H^4(0, 2\pi) : \psi(0) = \psi(2\pi) = \psi'(0) = \psi'(2\pi) = 0\}, \\ A_0 : \mathcal{D}(A_0) &\rightarrow L^2(0, 2\pi), \quad \psi \mapsto \psi^{(4)}. \end{aligned}$$

One can check that A_0 is a positive self-adjoint operator so that $(e^{tA_0})_{t \geq 0}$ is an analytic semigroup on $L^2(0, 2\pi)$. We can also prove that

$$\mathcal{D}\left(A_0^{\frac{1}{2}}\right) = H_0^2(0, 2\pi) := \{\psi \in H^2(0, 2\pi) : \psi(0) = \psi(2\pi) = \psi'(0) = \psi'(2\pi) = 0\}$$

and

$$\mathcal{D}\left(A_0^{\frac{1}{4}}\right) = H_0^1(0, 2\pi).$$

This shows that $(e^{tA_0})_{t \geq 0}$ is an analytic semigroup on $H_0^1(0, 2\pi)$ and thus that

$$\psi_0^\delta := e^{\delta^{1/4}A_0}\psi_0 \tag{6.4}$$

satisfies, for some constant $C > 0$,

$$\|\psi_0^\delta\|_{H^1(0, 2\pi)} \leq C \|\psi_0\|_{H^1(0, 2\pi)}, \quad \|\psi_0^\delta\|_{H^5(0, 2\pi)} \leq \frac{C}{\delta^{1/4}} \|\psi_0\|_{H^1(0, 2\pi)}. \tag{6.5}$$

Moreover,

$$\psi_0^\delta \rightarrow \psi_0 \quad \text{in } H^1(0, 2\pi), \quad \text{as } \delta \rightarrow 0^+. \tag{6.6}$$

Finally, with the definition of $\mathcal{D}(A_0)$, we see that g defined by (3.11) satisfies

$$g(t, 0) = g(t, 2\pi) = \partial_x g(t, 0) = \partial_x g(t, 2\pi) = 0 \quad t \geq 0. \tag{6.7}$$

In particular, we have

$$v(t, 0) = v(t, 2\pi) = 0 \quad t \geq 0. \tag{6.8}$$

Then, with the boundary conditions (6.7) and (6.8), we can see that all the computations in the proof of Proposition 3.1 are the same here, all the boundary terms in the integration by parts cancel. This yields the result. \square

We are finally ready to state our approximate controllability results for (6.1). As for Theorem 1.1, we set

$$\mathcal{S}(\psi_0) := \{e^\varphi \psi_0 : \varphi \in H^1(0, 2\pi)\}$$

and we replace (1.2) by the following condition

$$\left\{1, \cos(x/2), \sin(x/2)\right\} \subset \text{span}\{Q_0, \dots, Q_{q-1}\}. \tag{6.9}$$

Then, we obtain the following result (see Theorem 1.1):

Theorem 6.4. *Assume the condition (6.9). Then, for any $\psi_0 \in H_0^1(0, 2\pi)$, for any $\psi_1 \in \mathcal{S}(\psi_0)$, for any $\varepsilon > 0$ and $T > 0$ there exist $\tau \in (0, T]$ and $u \in L^2(0, \tau; \mathbb{R}^q)$ such that the solution ψ of (6.1) satisfies*

$$\|\psi(\tau, \cdot) - \psi_1\|_{H^1(0, 2\pi)} < \varepsilon.$$

Proof. The proof of Theorem 6.4 is similar to the proof of Theorem 1.1, see Section 4. One can show the validity of an equivalent result to Proposition 4.3 and combining it with [15, Proposition 2.5], we obtain that if we have (6.9), then

$$\text{span}\{1, x \mapsto \sin(kx/2), x \mapsto \cos(kx/2) : k \in \mathbb{N}^*\} \subset \mathcal{A}.$$

Then, we note that $\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{x \mapsto \frac{1}{\sqrt{\pi}} \cos(kx/2)\right\}_{k \in \mathbb{N}^*}$ is a Hilbert basis made by the eigenfunctions of the Laplace operator A_N with Neumann boundary conditions. Since $\mathcal{D}\left(A_N^{\frac{1}{2}}\right) = H^1(0, 2\pi)$, we deduce that \mathcal{A} is dense in $H^1(0, 2\pi)$ and we conclude as in the proof of Theorem 1.1. \square

Notice now that Corollary 1.2 can not be adapted here because the boundary conditions of ψ_0 yield that ψ_0 vanishes at the boundaries. We can however obtain from Theorem 6.4 a similar result to Corollary 1.3:

Corollary 6.5. *Assume the condition (6.9) and assume $\psi_0, \psi_1 \in H_0^1(0, 2\pi)$ with $\text{sign}(\psi_0) = \text{sign}(\psi_1)$. For any $\varepsilon > 0$ and $T > 0$, there exist $\tau \in (0, T]$ and $u \in L^2(0, \tau; \mathbb{R}^q)$ such that the solution ψ of (6.1) satisfies*

$$\|\psi(\tau, \cdot) - \psi_1\|_{L^2(0, 2\pi)} < \varepsilon.$$

Proof. The proof Corollary 6.5 follows the same steps of the one of Corollary 1.3 by using Theorem 6.4. \square

7 Controllability of the Burger equation with Neumann boundary conditions

In this section, we study the controllability of the Burgers equation with the Neumann boundary conditions, that is the system:

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi = (u \cdot Q) \psi & \text{in } (0, T) \times (0, 2\pi), \\ \partial_x \psi(\cdot, 0) = \partial_x \psi(\cdot, 2\pi) = 0 & \text{in } (0, T), \\ \psi(0, \cdot) = \psi_0 & \text{in } (0, 2\pi). \end{cases} \quad (7.1)$$

As in the previous framework, we ensure the small-time global approximate controllability of Theorem 1.1. Then, we also prove that the exact controllability to the constant stationary states of Theorem 1.4 holds for this framework.

7.1 Well-posedness result

Let us start by studying the well-posedness and the continuity of the solutions with respect to the initial data. There is an additional difficulty with respect to the torus case and the Dirichlet case. Here, when multiplying formally the first equation of (7.1) with ψ and integrate by parts, the term corresponding to $\psi \partial_x \psi$ does not disappear as in the previous cases. In particular, the energy estimate is more complicated to obtain. As a consequence, we only obtain a local well-posedness:

Proposition 7.1. *Assume $\psi_0 \in H^1(0, 2\pi)$, $Q \in H^2(0, 2\pi; \mathbb{R}^q)$, and $u \in L_{\text{loc}}^2([0, \infty); \mathbb{R}^q)$. Then, there exists a maximal $T_* \in (0, \infty]$ and a unique strong solution ψ of (7.1) on $(0, T_*)$ such that for any $T \in (0, T_*)$*

$$\psi \in L^2(0, T; H^2(0, 2\pi)) \cap C^0([0, T]; H^1(0, 2\pi)) \cap H^1(0, T; L^2(0, 2\pi)).$$

Proof. The proof is quite standard and we only give the idea: it can be done by a fixed point argument on the mapping

$$f \in L^2(0, T; L^2(0, 2\pi)) \mapsto -\psi \partial_x \psi,$$

where ψ is the solution of

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi = (u \cdot Q) \psi + f & \text{in } (0, T) \times (0, 2\pi), \\ \partial_x \psi(\cdot, 0) = \partial_x \psi(\cdot, 2\pi) = 0 & \text{in } (0, T), \\ \psi(0, \cdot) = \psi_0 & \text{in } (0, 2\pi). \end{cases} \quad (7.2)$$

One can show that for $T > 0$ small enough the above mapping is a strict contraction and this shows the existence and the uniqueness of a solution for small times. \square

We keep the same notation as in the case of the torus and of the Dirichlet boundary conditions: for $t_1 < t_2$ and for $u \in L^2(t_1, t_2; \mathbb{R}^q)$, we denote by $\mathcal{R}_{t_1, t_2}(\psi_0; u)$ the solution $\psi(t, \cdot)$ of

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi = (u \cdot Q) \psi & \text{in } (t_1, t_2) \times (0, 2\pi), \\ \partial_x \psi(\cdot, 0) = \partial_x \psi(\cdot, 2\pi) = 0 & \text{in } (t_1, t_2), \\ \psi(t_1, \cdot) = \psi_0 & \text{in } (0, 2\pi). \end{cases} \quad (7.3)$$

This implies in particular that the time of existence of the solution is larger than $t_2 - t_1$. As usual, if $t_1 = 0$, we simply write $\mathcal{R}_t(\psi_0; u)$ instead of $\mathcal{R}_{t, 0}(\psi_0; u)$. Similarly as Proposition 2.2 and Proposition 6.2, we have here the following result:

Proposition 7.2. *Assume $\psi_0 \in H^1(0, 2\pi)$, $Q \in H^2(0, 2\pi; \mathbb{R}^q)$ and $u \in L_{\text{loc}}^2([0, \infty); \mathbb{R}^q)$. Assume that the time of existence of $\mathcal{R}(\psi_0; u)$ is larger than $T > 0$. Then there exist $\alpha > 0$ and $C > 0$ such that for any $\psi_1 \in H^1(0, 2\pi)$*

$$\|\psi_0 - \psi_1\|_{H^1(0, 2\pi)} \leq \alpha, \quad (7.4)$$

the time of existence of $\mathcal{R}(\psi_1; u)$ is larger than T and

$$\|\mathcal{R}(\psi_0; u) - \mathcal{R}(\psi_1; u)\|_{C^0([0, T]; H^1(0, 2\pi))} \leq C \|\psi_0 - \psi_1\|_{H^1(0, 2\pi)}. \quad (7.5)$$

Proof. Assume $\psi_0, \psi_1 \in H^1(0, 2\pi)$, $Q \in H^2(0, 2\pi; \mathbb{R}^q)$, $T > 0$ and $u \in L^2(0, T; \mathbb{R}^q)$. Let us set

$$\psi := \mathcal{R}(\psi_1; u) - \mathcal{R}(\psi_0; u), \quad \bar{\psi} := \mathcal{R}(\psi_0; u).$$

Then

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi + \psi \partial_x \psi + \bar{\psi} \partial_x \psi + \psi \partial_x \bar{\psi} = (u \cdot Q) \psi & \text{in } (0, T) \times (0, 2\pi), \\ \partial_x \psi(\cdot, 0) = \partial_x \psi(\cdot, 2\pi) = 0 & \text{in } (0, T), \\ \psi(0, \cdot) = \psi_1 - \psi_0 & \text{in } (0, 2\pi). \end{cases} \quad (7.6)$$

Let us first multiply the first equation of the above system by ψ and integrate by parts:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{(0, 2\pi)} |\psi|^2 dx + \int_{(0, 2\pi)} |\partial_x \psi|^2 dx &\leq \int_{(0, 2\pi)} |\partial_x \psi| \psi^2 dx + \int_{(0, 2\pi)} |\partial_x \psi| |\psi| |\bar{\psi}| dx \\ &+ C \left(\|\bar{\psi}\|_{H^2(0, 2\pi)} + \|u\| \|Q\|_{L^\infty(0, 2\pi)^q} \right) \int_{(0, 2\pi)} \psi^2 dx. \end{aligned} \quad (7.7)$$

We have

$$\int_{(0, 2\pi)} |\partial_x \psi| \psi^2 dx \leq \|\psi\|_{L^\infty(0, 2\pi)} \|\psi\|_{L^2(0, 2\pi)} \|\partial_x \psi\|_{L^2(0, 2\pi)}$$

and using the Sobolev embedding $H^1(0, 2\pi) \hookrightarrow L^\infty(0, 2\pi)$, we deduce

$$\int_{(0, 2\pi)} |\partial_x \psi| \psi^2 dx \leq C \|\psi\|_{H^1(0, 2\pi)}^3.$$

Thus we deduce from (7.7) that

$$\begin{aligned} \frac{d}{dt} \int_{(0, 2\pi)} |\psi|^2 dx + \int_{(0, 2\pi)} |\partial_x \psi|^2 dx &\leq C \|\psi\|_{H^1(0, 2\pi)}^3 \\ &+ C \left(1 + \|\bar{\psi}\|_{H^2(0, 2\pi)}^2 + \|u\|^2 \|Q\|_{L^\infty(0, 2\pi)^q}^2 \right) \|\psi\|_{L^2(0, 2\pi)}^2. \end{aligned} \quad (7.8)$$

Second, we multiply the first equation of (7.6) by $-\partial_{xx} \psi$ and integrate by parts:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{(0, 2\pi)} |\partial_x \psi|^2 dx + \int_{(0, 2\pi)} |\partial_{xx} \psi|^2 dx &= \int_{(0, 2\pi)} \psi \partial_x \psi \partial_{xx} \psi dx + \int_{(0, 2\pi)} \bar{\psi} \partial_x \psi \partial_{xx} \psi dx \\ &+ \int_{(0, 2\pi)} \psi \partial_x \bar{\psi} \partial_{xx} \psi dx - \int_{(0, 2\pi)} (u \cdot Q) \psi \partial_{xx} \psi dx. \end{aligned} \quad (7.9)$$

By using the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} \frac{d}{dt} \int_{(0, 2\pi)} |\partial_x \psi|^2 dx + \int_{(0, 2\pi)} |\partial_{xx} \psi|^2 dx &\leq C \left(\|\psi\|_{L^\infty(0, 2\pi)}^2 + \|\bar{\psi}\|_{L^\infty(0, 2\pi)}^2 \right) \|\partial_x \psi\|_{L^2(0, 2\pi)}^2 \\ &+ C \left(\|\partial_x \bar{\psi}\|_{L^\infty(0, 2\pi)}^2 + \|Q\|_{L^\infty(0, 2\pi)^q}^2 \|u\|^2 \right) \|\psi\|_{L^2(0, 2\pi)}^2. \end{aligned} \quad (7.10)$$

Combining the above relation with (7.8), we deduce the existence of $C_1, C_2 > 0$ such that

$$\frac{d}{dt} \|\psi\|_{H^1(0, 2\pi)}^2 \leq C_1 \|\psi\|_{H^1(0, 2\pi)}^4 + C_2 \left(1 + \|\bar{\psi}\|_{H^2(0, 2\pi)}^2 + \|u\|^2 \|Q\|_{L^\infty(0, 2\pi)^q}^2 \right) \|\psi\|_{H^1(0, 2\pi)}^2. \quad (7.11)$$

Let us now consider $\alpha \in (0, 1)$ such that

$$\alpha^2 \exp \left((C_1 + C_2)T + \|\bar{\psi}\|_{L^2(0, T; H^2(0, 2\pi))}^2 + \|u\|_{L^2(0, T)^q}^2 \|Q\|_{L^\infty(0, 2\pi)^q}^2 \right) < 1. \quad (7.12)$$

Assuming (7.4) with this α , we have that for a maximal time $T_* \in (0, T]$,

$$\|\psi(t, \cdot)\|_{H^1(0, 2\pi)} < 1 \quad (t \in [0, T_*]).$$

Using this estimate and applying the Grönwall lemma in (7.11), we find that for $t \in [0, T_*]$,

$$\|\psi(t, \cdot)\|_{H^1(0, 2\pi)}^2 \leq \|\psi(0, \cdot)\|_{H^1(0, 2\pi)}^2 \exp \left((C_1 + C_2)T + \|\bar{\psi}\|_{L^2(0, T; H^2(0, 2\pi))}^2 + \|u\|_{L^2(0, T)^q}^2 \|Q\|_{L^\infty(0, 2\pi)^q}^2 \right). \quad (7.13)$$

In particular, we deduce with (7.12) that $T_* = T$ and we obtain (7.5). \square

7.2 Small-time controllability results

We now present the controllability results for (7.1) and we start by studying the approximate controllability. As for the case of the torus and of the Dirichlet boundary conditions, it is associated with a saturation limit stated as follows:

Proposition 7.3. *Assume $\psi_0 \in H_0^1(0, 2\pi)$, $Q \in H^2(0, 2\pi; \mathbb{R}^q)$, $u \in \mathbb{R}^q$ and $\varphi \in H^3(0, 2\pi)$, $\varphi > 0$, with*

$$Q'(0) = Q'(2\pi) = 0, \quad \varphi'(0) = \varphi'(2\pi) = 0. \quad (7.14)$$

Then, the following limit holds

$$e^{\delta^{-1/2}\varphi} \mathcal{R}_\delta(e^{-\delta^{-1/2}\varphi} \psi_0, \delta^{-1}u) \rightarrow e^{(\varphi')^2 + u \cdot Q} \psi_0 \quad \text{in } H^1(0, 2\pi) \text{ as } \delta \rightarrow 0^+.$$

We note in particular that, with respect to Proposition 3.1 and to Proposition 6.3, we need here the condition (7.14), used in the proof to cancel the boundary terms appearing in the integration by parts. Let us sketch the proof of Proposition 7.3.

Proof of Proposition 7.3. The proof is similar to the proof of Proposition 3.1 with some small differences that we point out. We define ψ and ψ_δ as in the proof of Proposition 3.1. We have in particular (3.5) and (3.6) with \mathbb{T} replaced by $(0, 2\pi)$. Moreover, with (7.14), we can check that ψ_δ satisfies the Neumann boundary conditions:

$$\partial_x \psi_\delta(\cdot, 0) = \partial_x \psi_\delta(\cdot, 2\pi) = 0 \quad \text{in } (0, T).$$

Then we define g by (3.11), where ψ_0^δ is given by

$$\psi_0^\delta := e^{\delta^{1/4} A_N} \psi_0, \quad (7.15)$$

where

$$\mathcal{D}(A_N) := \{\psi \in H^2(0, 2\pi) : \psi'(0) = \psi'(2\pi) = 0\}, \quad A_N : \mathcal{D}(A_N) \rightarrow L^2(0, 2\pi), \quad \psi \mapsto -\psi''. \quad (7.16)$$

The operator A_N is the infinitesimal generator of an analytic semigroup $(e^{tA_N})_{t \geq 0}$ in $L^2(0, 2\pi)$ and $\mathcal{D}(A_N^{1/2}) = H^1(0, 2\pi)$. Thus, there exists a constant $C > 0$ such that

$$\|\psi_0^\delta\|_{H^1(0, 2\pi)} \leq C \|\psi_0\|_{H^1(0, 2\pi)}, \quad \|\psi_0^\delta\|_{H^3(0, 2\pi)} \leq \frac{C}{\delta^{1/4}} \|\psi_0\|_{H^1(0, 2\pi)}. \quad (7.17)$$

Moreover,

$$\psi_0^\delta \rightarrow \psi_0 \quad \text{in } H^1(0, 2\pi), \quad \text{as } \delta \rightarrow 0^+. \quad (7.18)$$

With this choice and (7.14), we have that

$$\partial_x g(t, 0) = \partial_x g(t, 2\pi) = 0 \quad (t \geq 0).$$

In particular, v defined by (3.13) satisfies (3.14) with \mathbb{T} replaced by $(0, 2\pi)$ and with the Neumann boundary conditions:

$$\partial_x v(\cdot, 0) = \partial_x v(\cdot, 2\pi) = 0 \quad \text{in } (0, T).$$

The first estimates used when multiplying (3.14) by v are the same as in the proof of Proposition 3.1, except for the term

$$\int_{(0, 2\pi)} \delta e^{\frac{-\varphi}{\sqrt{\delta}}} v^2 \partial_x v \, dx$$

that we need to estimate directly since integration by parts:

$$\left| \int_{(0, 2\pi)} \delta e^{\frac{-\varphi}{\sqrt{\delta}}} v^2 \partial_x v \, dx \right| \leq \frac{\delta}{4} \|\partial_x v\|_{L^2(0, 2\pi)}^2 + C\delta \|v\|_{L^4(0, 2\pi)}^4.$$

Consequently, instead of (3.18), we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(0, 2\pi)}^2 + \delta \|\partial_x v\|_{L^2(0, 2\pi)}^2 \leq C \|F\|_{L^2(0, 2\pi)}^2 + C\delta \|v\|_{L^4(0, 2\pi)}^4 + C \|v\|_{L^2(0, 2\pi)}^2 \quad \text{in } (0, T). \quad (7.19)$$

The remaining part of the proof is the same and in particular all the integration by parts yields the same results since the boundary terms disappear. We obtain in particular the same differential inequality (3.25) (with \mathbb{T} replaced by $(0, 2\pi)$). \square

In order to ensure our approximate controllability result, we define \mathcal{A} similarly as in Definition 4.1 and we show a result similar to Proposition 4.3:

Proposition 7.4. *Assume $\varphi \in H^3(0, 2\pi)$, $\varphi > 0$ and $\varphi'(0) = \varphi'(2\pi) = 0$ such that $\text{span } \varphi \subset \mathcal{A}$. Then $(\varphi')^2 \in \mathcal{A}$.*

Proof. Let us consider $\psi_0 \in H^1(0, 2\pi)$, $T > 0$ and $\varepsilon \in (0, 1)$. By Proposition 7.3, there exists $\tau_1 \in (0, T/3)$ such that

$$\left\| e^{\frac{\varphi}{\sqrt{\tau_1}}} \mathcal{R}_{\tau_1} \left(e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0, 0 \right) - e^{(\varphi')^2} \psi_0 \right\|_{H^1(0, 2\pi)} \leq \frac{\varepsilon}{3}.$$

We set

$$\psi_1 := \mathcal{R}_{\tau_1} \left(e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0, 0 \right) \in H^1(0, 2\pi)$$

so that the above estimate writes

$$\left\| e^{\frac{\varphi}{\sqrt{\tau_1}}} \psi_1 - e^{(\varphi')^2} \psi_0 \right\|_{H^1(0, 2\pi)} \leq \frac{\varepsilon}{3}. \quad (7.20)$$

Since $\varphi/\sqrt{\tau_1} \in \mathcal{A}$, there exists $\tau_2 \in (0, T/3)$, $\tilde{u}_2 \in \mathcal{P}(0, \tau_2)$ such that

$$\left\| \mathcal{R}_{\tau_2}(\psi_1, \tilde{u}_2) - e^{\frac{\varphi}{\sqrt{\tau_1}}} \psi_1 \right\|_{H^1(0, 2\pi)} \leq \frac{\varepsilon}{3}. \quad (7.21)$$

From Proposition 7.2, there exist α_1 and $C_1 > 0$ such that for any $\psi_2 \in H^1(0, 2\pi)$ with

$$\left\| \psi_2 - e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0 \right\|_{H^1(0, 2\pi)} \leq \alpha_1, \quad (7.22)$$

then

$$\left\| \mathcal{R}(\psi_2, 0) - \mathcal{R}(e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0, 0) \right\|_{C^0([0, \tau_1]; H^1(0, 2\pi))} \leq C_1 \left\| \psi_2 - e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0 \right\|_{H^1(0, 2\pi)}. \quad (7.23)$$

Similarly, there exist $\alpha_2 > 0$ and $C_2 > 0$ such that for any $\psi_2 \in H^1(0, 2\pi)$ with

$$\|\psi_2 - \psi_1\|_{H^1(0, 2\pi)} \leq \alpha_2, \quad (7.24)$$

then

$$\|\mathcal{R}(\psi_2, \tilde{u}_2) - \mathcal{R}(\psi_1, \tilde{u}_2)\|_{C^0([0, \tau_2]; H^1(0, 2\pi))} \leq C_2 \|\psi_2 - \psi_1\|_{H^1(0, 2\pi)}. \quad (7.25)$$

Since $-\varphi/\sqrt{\tau_1} \in \mathcal{A}$, there exist $\tau_0 \in (0, T/3)$, $u_0 \in \mathcal{P}(0, \tau_0)$ such that

$$\left\| \mathcal{R}_{\tau_0}(\psi_0, u_0) - e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0 \right\|_{H^1(0, 2\pi)} \leq \min \left(\alpha_1, \frac{\alpha_2}{C_1}, \frac{\varepsilon}{3C_1C_2} \right). \quad (7.26)$$

Combining (7.23) and (7.26), we deduce that

$$\begin{aligned} & \left\| \mathcal{R}_{\tau_1+\tau_0, \tau_0}(\mathcal{R}_{\tau_0}(\psi_0, u_0), 0) - \mathcal{R}_{\tau_1+\tau_0, \tau_0}(e^{-\frac{\varphi}{\sqrt{\tau_1}}} \psi_0, 0) \right\|_{H^1(0, 2\pi)} \\ &= \left\| \mathcal{R}_{\tau_0+\tau_1}(\psi_0, u_0 1_{(0, \tau_0)}) - \psi_1 \right\|_{H^1(0, 2\pi)} \leq \min \left(\alpha_2, \frac{\varepsilon}{3C_2} \right). \end{aligned} \quad (7.27)$$

By translating \tilde{u}_2 , we deduce from (7.21) the existence of $u_2 \in \mathcal{P}(\tau_0 + \tau_1, \tau_0 + \tau_1 + \tau_2)$ such that

$$\left\| \mathcal{R}_{\tau_0+\tau_1+\tau_2, \tau_0+\tau_1}(\psi_1, u_2) - e^{\frac{\varphi}{\sqrt{\tau_1}}} \psi_1 \right\|_{H^1(0, 2\pi)} \leq \frac{\varepsilon}{3}. \quad (7.28)$$

Combining (7.25) and (7.27), we deduce that

$$\left\| \mathcal{R}_{\tau_0+\tau_1+\tau_2}(\psi_0, u_0 1_{(0, \tau_0)} + u_2 1_{(\tau_0+\tau_1, \tau_0+\tau_1+\tau_2)}) - \mathcal{R}_{\tau_0+\tau_1+\tau_2, \tau_0+\tau_1}(\psi_1, u_2) \right\|_{H^1(0, 2\pi)} \leq \frac{\varepsilon}{3}. \quad (7.29)$$

Combining the above estimate with (7.28) and (7.20), we deduce that

$$\left\| \mathcal{R}_{\tau_0+\tau_1+\tau_2}(\psi_0, u_0 1_{(0, \tau_0)} + u_2 1_{(\tau_0+\tau_1, \tau_0+\tau_1+\tau_2)}) - e^{(\varphi')^2} \psi_0 \right\|_{H^1(0, 2\pi)} \leq \varepsilon. \quad (7.30)$$

□

We are finally ready to state the small-time global approximate controllability for the system (7.1). First we replace (1.2) by the following condition

$$\left\{1, x \mapsto \cos\left(\frac{x}{2}\right), x \mapsto \cos(x)\right\} \subset \text{span}\{Q_0, \dots, Q_{q-1}\}. \quad (7.31)$$

We also set

$$\mathcal{S}(\psi_0) := \{e^\varphi \psi_0 : \varphi \in H^1(0, 2\pi)\}.$$

Theorem 7.5. *Assume the condition (7.31). Then, for any $\psi_0 \in H^1(0, 2\pi)$, for any $\psi_1 \in \mathcal{S}(\psi_0)$, for any $\varepsilon > 0$ and $T > 0$ there exist $\tau \in (0, T]$ and $u \in L^2(0, \tau; \mathbb{R}^q)$ such that the solution ψ of (6.1) exists in $[0, \tau]$ and satisfies*

$$\|\psi(\tau, \cdot) - \psi_1\|_{H^1(0, 2\pi)} < \varepsilon.$$

The proof of Theorem 7.5 is similar to the proof of Theorem 1.1 and is based on Proposition 7.4, but with a difference that we discuss now. The idea is to show that

$$\text{span} \left\{1, x \mapsto \cos\left(\frac{kx}{2}\right) : k \in \mathbb{N}^*\right\} \subset \mathcal{A} \quad (7.32)$$

(with \mathcal{A} defined as in Definition 4.1) and since $\text{span}\{1, x \mapsto \cos(kx/2) : k \in \mathbb{N}^*\}$ is dense in $H^1(0, 2\pi)$, we can conclude as in the proof of Theorem 1.1. Nevertheless, from (7.31) and Proposition 7.4, it seems difficult to obtain $x \mapsto \cos(3x/2)$ but only an approximation of it. We thus need to extend Proposition 7.4 in order to show that an approximation of $(\varphi')^2$ can be obtained. We show this result only in our case here to simplify:

Proposition 7.6. *Assume*

$$\text{span}\{1, x \mapsto \cos(kx/2) : k \in \{1, \dots, K\}\} \subset \mathcal{A}. \quad (7.33)$$

Then

$$\text{span}\{x \mapsto \cos((K+1)x/2)\} \subset \mathcal{A}.$$

Proof. To simplify, we denote by c_k and s_k the functions $x \mapsto \cos(kx/2)$ and $x \mapsto \sin(kx/2)$. Assume $\beta \in \mathbb{R}^*$, $\psi_0 \in H^1(0, 2\pi)$, $T > 0$ and $\varepsilon > 0$. Assume also $\gamma \in [0, 1]$. Then, some standard computations yield the existence of $C > 0$ such that

$$\left\|e^{\gamma^2 s_K^2} - 1\right\|_{H^1(0, 2\pi)} \leq C\gamma^2.$$

Using the Sobolev embeddings, there exists a constant $C > 0$ such that

$$\left\|e^{\gamma^2 s_K^2 + \beta c_{K+1}} \psi_0 - e^{\beta c_{K+1}} \psi_0\right\|_{H^1(0, 2\pi)} \leq C \left\|e^{\gamma^2 s_K^2} - 1\right\|_{H^1(0, 2\pi)} \left\|e^{\beta c_{K+1}} \psi_0\right\|_{H^1(0, 2\pi)}$$

In particular, there exists $\gamma \in (0, 1]$ such that

$$\left\|e^{\gamma^2 s_K^2 + \beta c_{K+1}} \psi_0 - e^{\beta c_{K+1}} \psi_0\right\|_{H^1(0, 2\pi)} \leq \frac{\varepsilon}{2}. \quad (7.34)$$

Now, let us set

$$\varphi := \left(1 + \frac{2|\beta|}{\gamma} + \frac{2\gamma}{K}\right) - \frac{2\beta}{\gamma} c_1 + \frac{2\gamma}{K} c_K.$$

We have $\varphi \in H^3(0, 2\pi)$, $\varphi > 0$, $\varphi'(0) = \varphi'(2\pi) = 0$ and $\text{span } \varphi \subset \mathcal{A}$. Applying (7.4), we deduce that $(\varphi')^2 \in \mathcal{A}$. However,

$$(\varphi')^2 = \frac{\beta^2}{2\gamma^2} (1 - c_2) + \gamma^2 s_K^2 + \beta (c_{K+1} - c_{K-1}).$$

Using (7.33), this implies that $\gamma^2 s_K^2 + \beta c_{K+1} \in \mathcal{A}$: thus, there exists $\tau \in (0, T)$ and $u \in \mathcal{P}(0, \tau)$ such that

$$\left\|\mathcal{R}_\tau(\psi_0, u) - e^{\gamma^2 s_K^2 + \beta c_{K+1}} \psi_0\right\|_{H^1(0, 2\pi)} \leq \frac{\varepsilon}{2}.$$

Combining this with (7.34), we deduce the result. \square

Proof of Theorem 7.5. As discussed before, Theorem 7.5 is ensured via the same techniques leading to Theorem 1.1 and it is based on Proposition 7.4 and Proposition 7.6. \square

As a consequence of Theorem 7.5, we have the following corollaries.

Corollary 7.7. *Assume the condition (7.31) and assume $\psi_0, \psi_1 \in H^1(0, 2\pi)$ with $\psi_0\psi_1 > 0$ in $(0, 2\pi)$. For any $\varepsilon > 0$ and $T > 0$, there exist $\tau \in (0, T]$ and $u \in L^2(0, \tau; \mathbb{R}^q)$ such that the solution ψ of (7.1) exists in $[0, \tau]$ and satisfies*

$$\|\psi(\tau, \cdot) - \psi_1\|_{H^1(0, 2\pi)} < \varepsilon.$$

Proof. The proof of Corollary 7.7 is the same as the one of Corollary 1.2 and exploits Theorem 7.5. \square

Corollary 7.8. *Assume the condition (7.31) and assume $\psi_0, \psi_1 \in H^1(0, 2\pi)$ with $\text{sign}(\psi_0) = \text{sign}(\psi_1)$. For any $\varepsilon > 0$ and $T > 0$, there exist $\tau \in (0, T]$ and $u \in L^2(0, \tau; \mathbb{R}^q)$ such that the solution ψ of (6.1) exists in $[0, \tau]$ and it satisfies*

$$\|\psi(\tau, \cdot) - \psi_1\|_{L^2(0, 2\pi)} < \varepsilon.$$

Proof. The proof of Corollary 7.8 is the same as the one of Corollary 1.3 and exploits Theorem 7.5. \square

We ensure now the small-time global exact controllability to the non-zero constant states by combining Theorem 7.5 with a local exact controllability. First, to write the hypotheses, we recall the Laplace operator A_N with Neumann boundary conditions is defined by (7.16) and its eigenvalues are

$$\lambda_k := \frac{k^2}{4}, \quad \forall k \in \mathbb{N} \quad (7.35)$$

and a corresponding family of eigenfunctions is

$$w_0(x) := \frac{1}{\sqrt{2\pi}}, \quad w_k(x) := \frac{1}{\sqrt{\pi}} \cos\left(\frac{kx}{2}\right), \quad \forall k \in \mathbb{N}^*, \quad x \in (0, 2\pi). \quad (7.36)$$

Theorem 7.9. *Assume the condition (7.31) and assume that Q_0 verifies*

$$\exists b, d > 0, \quad \forall k \in \mathbb{N}, \quad (\lambda_k^d + 1) \left| (Q_0, w_k)_{L^2(0, 2\pi)} \right| \geq b. \quad (7.37)$$

For any $T > 0$, $\psi_0 \in H^1(0, 2\pi)$ with $\psi_0 > 0$ and $\Psi \in \mathbb{R}_+^$, there exists $u \in L^2(0, T; \mathbb{R}^q)$ such that the solution ψ of (7.1) satisfies $\psi(T, \cdot) = \Psi$.*

Proof. The proof of Theorem 7.9 follows the proof of Theorem 1.4: it is direct consequence of Theorem 7.5 and a local exact controllability to the non-zero constant states. \square

Example 7.10. Let us give some examples of functions Q_0 satisfying (7.37). First, if $Q_0(x) = x^2$, then one can check that

$$(Q_0, w_k)_{L^2(0, 2\pi)} = \frac{16\sqrt{\pi}(-1)^k}{k^2} \quad (k \neq 0)$$

and that $(Q_0, w_0)_{L^2(0, 2\pi)} > 0$ so that (7.37) holds for $d = 1$. Second, we can take $Q_0(x) = x^3(x - 2\pi)^2$. We have

$$(Q_0, w_k)_{L^2(0, 2\pi)} = \frac{-128}{\sqrt{\pi}k^6} (60(1 + (-1)^{k+1}) + k^2\pi^2(-3 + 9(-1)^k)) \quad (k \in \mathbb{N}^*).$$

We also have $(Q_0, w_0)_{L^2(0, 2\pi)} > 0$. Thus, (7.37) holds for $d = 2$.

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