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Evelyne Hubert, Martin Jalard. Orbit separation and stratification by isotropy classes of piezoelectricity tensors. *Journal of Pure and Applied Algebra*, 2025, pp.108034. <10.1016/j.jpaa.2025.108034>. <hal-04905290>

HAL Id: hal-04905290

<https://hal.science/hal-04905290v1>

Submitted on 22 Jan 2025

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Orbit separation and stratification by isotropy classes of piezoelectricity tensors

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January 22, 2025

Abstract

We explore an innovative method to compute separating invariants in a real G -variety \mathcal{V} . A refinement of Seshadri slice Lemma enables us to decompose \mathcal{V} into a union of stable subsets $G\mathcal{Z}_1 \sqcup \dots \sqcup G\mathcal{Z}_r$. It then reduces the problem to separating the orbits in the slices \mathcal{Z}_i for their normalizers $N_i < G$. This sequencing allows also to identify efficiently the isotropy class of any point. After the presentation of three types of Seshadri slices for representations of $SO_3(\mathbb{R})$, we apply the method to the space of piezoelectricity tensors $\mathbb{P}iez$. It provides a separating set of low cardinality and a complete stratification of $\mathbb{P}iez$ by isotropy classes.

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1 Introduction

Mathematical objects often need to be understood modulo a group action. Such is the case of the space $\mathbb{P}\text{iez}$ of piezoelectric tensors, that characterizes to some extent the properties of a material in mechanics. The natural action of rotations of $\text{SO}_3(\mathbb{R})$ on the material induces an action on its piezoelectric response. But the study of some properties of this material should not depend on its orientation. This necessitates separating invariants, *i.e.*, invariant functions determining a piezoelectric tensor up to rotation. Their evaluation determines in particular the isotropy class, that is the self symmetry of the object.

[GHP19] and [HJ24] identify Sashadri slices (\mathcal{Z}, N) for suitable representations \mathcal{V} of the real group $G = \text{SO}_3(\mathbb{R})$. \mathcal{Z} is an affine set intersecting generic orbits normalized by $N < G$. These slices help to compute a generic separating set, that is set of invariant separating orbits outside a residual G -stable subvariety \mathcal{Q} . Inspired by [Pop92], we refine the notion of Seshadri slice such that a set of N -invariants separating N -orbits in \mathcal{Z} can be unequivocally lifted to a set of $\text{SO}_3(\mathbb{R})$ -invariants separating $\text{SO}_3(\mathbb{R})$ -orbits in $G\mathcal{Z}$. The residual variety is thus proved to be $\mathcal{Q} = \mathcal{V} \setminus G\mathcal{Z}$. Furthermore, the isotropy class of orbits in $G\mathcal{Z}$ is the isotropy class in the normalizer N of their representants on \mathcal{Z} . By repeating the process on the successive residual varieties \mathcal{Q} , we shall produce a *separating system* for (\mathcal{V}, G) , that is a sequence of pairs $(Q_1, \mathcal{F}_1), \dots, (Q_r, \mathcal{F}_r)$ such that each Q_i is G -invariant, $\mathcal{V} = Q_1 \cup \dots \cup Q_r$ a decomposition of \mathcal{V} , $\mathcal{F}_i \subset \mathbb{R}(Q_i)^G$ and any two distinct orbits on \mathcal{V} are separated by some $f \in \bigcup_{i=1}^r \mathcal{F}_i$. We furthermore exhibit the relationships on the elements of \mathcal{F}_i that cut the isotropy strata in Q_i . We apply the method to $\mathbb{P}\text{iez}$ and achieve convincing results compared to what has been done before.

A traditional approach in invariant theory is to compute generators of the invariant algebra. The textbooks [Stu08; DK15] are entry points for this purpose. A set of invariant is said *separating* if it separates orbits that can be separated by the invariant algebra [DK15, Section 2.4]. For compact groups, it is equivalent to separate all the orbits. Thus generating implies separating. Yet a separating set can have considerably lower cardinality than a generating set. For the representation of $\text{SO}_3(\mathbb{R})$ on $\mathbb{P}\text{iez}$, [Oli14] provides a generating set of the invariant algebra of cardinality 495, from which [Che+19] selects a separating set of cardinality 260. Our separating system for the space of piezoelectric tensors contains a total of 102 invariants, and at most 40 of them need to be evaluated to decide whether two tensors are in the same orbit.

Determining the isotropy of a piezo-electric tensor has been achieved by Maxwell multipole in [ZTE13]. The advantage of obtaining polynomial equations for the isotropy strata is nonetheless emphasized in [Azz+22]. Thanks to our separating system we can further stratify $\mathbb{P}\text{iez}$ by isotropy classes, giving equalities, of low degree and with small number of terms, defining each isotropy stratum. If separating sets of invariants had already been computed, it is the first time that simple enough invariants allow to construct a complete stratification of the space of piezoelectric tensors according to their isotropy.

The $\text{SO}_3(\mathbb{R})$ -representation space $\mathbb{P}\text{iez}$ decomposes into as $\mathbb{P}\text{iez} \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where \mathcal{H}_d is the space of totally symmetric trace free tensors of order d or, equivalently, the space of harmonic homogeneous polynomials of degree d . We identify three types of $\text{SO}_3(\mathbb{R})$ -slices in $\mathbb{P}\text{iez}$. We first refer to the slices in \mathcal{H}_1 [HJ24] and \mathcal{H}_2 [GHP19]. It provides stratifications failing respectively where the component in \mathcal{H}_1 vanishes, and on the variety $\mathcal{S}^\circ \subset \mathcal{H}_2$ of matrices with a repeated eigenvalue. We thus introduce a new slice in the $\text{SO}_3(\mathbb{R})$ -variety \mathcal{S}° . When finally all the components of a tensor pertaining to \mathcal{H}_1 and \mathcal{H}_2 are zero we deal with \mathcal{H}_3 in a similar way to [AKP14].

The method is developed over \mathbb{R} , addressing the larger part of applications (as on $\mathbb{P}\text{iez}$). There is no theoretical obstacle to do the same over \mathbb{C} where Seshadri Lemma still holds. Nevertheless, the decomposition $\hat{\mathcal{V}} = \hat{\mathcal{Q}}_0 \cup \dots \cup \hat{\mathcal{Q}}_n$ is less efficient among complex numbers, as discussed in Remark 3.5. As in other circumstances [Duf09; Jal24], the separation for a real G -variety requires an invariant set of lower cardinality than for its complexification.

Our approach could also be applied to the space of elasticity tensors $\mathbb{E}la \cong \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2 \oplus \mathcal{H}_4$. Yet a different angle to isotropy stratification and separation of orbits is offered in [Oli14; Des+21]. Polynomial equations for the isotropy strata are given in terms of *covariants* and then separating sets of invariants are provided on each of these isotropy stratum.

The paper is organized as follows: in Section 2, we define closed Seshadri slices and shows how they help separating orbits and stratyfing by isotropy type. In Section 3, we present three slices appearing in suitable representations of $SO_3(\mathbb{R})$. In Section 4, we decompose the representation of $SO_3(\mathbb{R})$ on $\mathbb{P}ie_z$ thanks to these three types of slices. We next use this decomposition to compute a separating system of invariants of low cardinality and a complete stratification by isotropy type. Appendices A and B provide complementary computational clues. We note \hat{A} the complexification of a real algebraic group or variety A .

2 Orbit separation and stratification with Seshadri slices

Seshadri slice lemma has been mostly used to prove the rationality of the invariant field of representations of algebraic groups [CS05]. In this Section we introduce *closed* slices inspired by [Pop92], and get a refinement of this theorem. It provides a constructive method to compute invariants separating the orbits (Proposition 2.4). We show that the obtained invariants are relevant to determine the orbit type of a point (Theorem 2.9).

2.1 Seshadri slices

Theorem 2.1. *Seshadri slice Lemma [CS05]*

Let G be a real algebraic group acting on a geometrically irreducible real variety \mathcal{V} . Consider \mathcal{Z} an open subset of a geometrically irreducible subvariety $\overline{\mathcal{Z}} \subset \mathcal{V}$, and N its normalizer. Assume that:

- (1) The closure $\overline{G\mathcal{Z}}$ is \mathcal{V} .
 - (2) For all $g \in \hat{G}$ and $z \in \hat{\mathcal{Z}}$ satisfying $g(z) \in \hat{\mathcal{Z}}$, then there exists $h \in \hat{N}$ such that $h(z) = g(z)$.
- Then, the restriction to \mathcal{Z} induces a natural isomorphism $\mathbb{R}(\mathcal{V})^G \cong \mathbb{R}(\overline{\mathcal{Z}})^N$.

We say then that (\mathcal{Z}, N) is a Seshadri slice. The point (2) indicates the existence of a quotient map

$$\phi : G\mathcal{Z} \rightarrow \frac{\mathcal{Z}}{N} \cong \frac{G\mathcal{Z}}{G}.$$

We think of the restriction isomorphism in Theorem 2.1 as the pullback $\phi^* : \mathbb{R}(\mathcal{Z})^N \rightarrow \mathbb{R}(\mathcal{V})^G$. This restriction can fail at some points of $\overline{\mathcal{Z}}$. Even for a polynomial $P \in \mathbb{R}[\overline{\mathcal{Z}}]^N$, $\phi^*P \in \mathbb{R}(\mathcal{V})^G$ can have singularities in \mathcal{Z} . [Pop92, Corollary 2] describes the regularity of the pulled back function under stronger hypotheses:

Definition 2.2. (\mathcal{Z}, N) is called a closed Seshadri slice if the points $z \in \hat{\mathcal{Z}}$ are normal in \mathcal{V} and satisfy the following stronger statement instead of (2):

- (2°) For all $g \in \hat{G}$ satisfying $g(z) \in \hat{\mathcal{Z}}$, then $g \in \hat{N}$.

Theorem 2.3. Let (\mathcal{Z}, N) be a closed Seshadri slice. For any $f \in \mathbb{R}[\mathcal{Z}]^N$, $\phi^*f \in \mathbb{R}(\mathcal{V})^G$ is regular on $G\mathcal{Z}$.

2.2 Orbit separation

We first notice that a closed Seshadri slice helps computing functions taking different values on two distinct orbits in $G\mathcal{Z}$:

Proposition 2.4. Let \mathcal{V} be a G -variety admitting a closed Seshadri slice (\mathcal{Z}, N) . Assume that a set of polynomials $\mathcal{F} \subset \mathbb{R}[\mathcal{Z}]^N$ separates the orbits in \mathcal{Z} . Then, $\phi^*\mathcal{F} \subset \mathbb{R}(\mathcal{V})^G$ separates the orbits in $G\mathcal{Z}$.

Proof. Let v and v' be two points in $G\mathcal{Z}$ such that $v' \notin Gv$. Take z and z' representants of their respective orbits in \mathcal{Z} . Then $z' \notin Nz$. Since $\phi^*\mathcal{F}$ separate N -orbits on \mathcal{Z} , there exists $f \in \phi^*\mathcal{F}$ with $f(z) \neq f(z')$. Since we have the restriction $\mathcal{F} = (\phi^*\mathcal{F})|_{\mathcal{Z}}$, $\phi^*f(v) = \phi^*f(z) \neq \phi^*f(z') = \phi^*f'v'$. We proved that $\phi^*\mathcal{F}$ separates orbits in $G\mathcal{Z}$. \square

The separating set $\phi^*\mathcal{F}$ consists of rational functions admitting possibly singularities in $\mathcal{V} \setminus G\mathcal{Z}$. For all the slices studied in this paper, $\hat{\mathcal{V}} \setminus \hat{G}\hat{\mathcal{Z}}$ is an irreducible hypersurface defined by an irreducible polynomial $Q \in \mathbb{R}[\mathcal{V}]^G$. The denominators in $\phi^*\mathcal{F}$ are thus a power of Q . Selecting only the numerators, we deduce a set of polynomials with the same separation property:

Proposition 2.5. *Suppose that $\hat{\mathcal{V}} \setminus \hat{G}\hat{\mathcal{Z}}$ is an irreducible hypersurface $\mathcal{V}(Q)$, $Q \in \mathbb{R}[\mathcal{V}]^G$. Then the pulled back invariant algebra $\phi^*(\mathbb{R}[\mathcal{Z}]^N)$ is included in the localisation $\mathbb{R}[\mathcal{V}]_Q^G$. We can note $\phi^*\mathcal{F} = \left\{ \frac{P_i}{Q^{k_i}}, 1 \leq i \leq \#\mathcal{F} \right\}$ for some positive integer exponents k_i . Then, the set $\mathcal{P} = \{Q, P_1, \dots, P_{\#\mathcal{F}}\} \subset \mathbb{R}[\mathcal{V}]^G$ is a set of invariant polynomials separating orbits in $G\mathcal{Z}$.*

Proof. Take two point z_1, z_2 in $G\mathcal{Z}$ which are not in the same orbit. Then, there is some $1 \leq i \leq \#\mathcal{E}$ such that $\frac{P_i}{Q^{k_i}}(z_1) \neq \frac{P_i}{Q^{k_i}}(z_2)$. Note that $Q \in \mathcal{P}$ does not vanish at z_1 and z_2 . Suppose that $Q(z_1) = Q(z_2) \neq 0$. By multiplication by $Q(z_1)^{k_i} = Q(z_2)^{k_i} \neq 0$ one has $P_i(z_1) \neq P_i(z_2)$. \square

The problem is that orbits of G in $\mathcal{Q} = \mathcal{V} \setminus G\mathcal{Z}$ are not separated by $\phi^*\mathcal{F}$. We need a complementary set of invariants separating orbits in \mathcal{Q} . The method we shall apply to piezoelectricity tensor is to find recursively, as long as possible, another slices in the residual non separated variety. It cuts the representations in $\mathcal{V} = \mathcal{Q}_0 \supset \dots \supset \mathcal{Q}_n \supset \mathcal{Q}_{n+1}$ where for all $0 \leq i \leq n-1$, $\mathcal{Q}_{i+1} = \mathcal{Q}_i \setminus G\mathcal{Z}_i$:

$$\begin{array}{ccccccc} \mathcal{V} = \mathcal{Q}_0 & \supset & \mathcal{Q}_1 & \supset & \dots & \supset & \mathcal{Q}_n & \supset & \mathcal{Q}_{n+1} \\ | & & | & & & & | & & \\ (\mathcal{Z}_0, N_0) & & (\mathcal{Z}_1, N_1) & & \dots & & (\mathcal{Z}_n, N_n) & & \\ \downarrow & & \downarrow & & & & \downarrow & & \\ (G\mathcal{Z}_0 & \sqcup & G\mathcal{Z}_1 & \sqcup & \dots & \sqcup & G\mathcal{Z}_n & \sqcup & \mathcal{Q}_{n+1}) = \mathcal{V} \end{array}$$

For each $0 \leq i \leq n$, we compute \mathcal{F}_i a set of polynomials in $\mathbb{R}[\mathcal{Z}_i]^{N_i}$ separating orbits in \mathcal{Z}_i . If the last subvariety \mathcal{Q}_{n+1} is simple enough (as for Piez where we manage to restrict until $\mathcal{Q}_{n+1} = \mathcal{H}_3$), traditional methods provide a final set $\mathcal{F}_{n+1} \subset \mathbb{R}[\mathcal{Q}_{n+1}]^G$ separating orbits of G in \mathcal{Q}_{n+1} . We obtain finally a separating system for $G \curvearrowright \mathcal{V}$

$$\mathcal{F} = \{(\mathcal{Q}_0, \phi_0^*\mathcal{F}_0), \dots, (\mathcal{Q}_n, \phi_n^*\mathcal{F}_n), (\mathcal{Q}_{n+1}, \mathcal{F}_{n+1})\}$$

in the sense of the following definition:

Definition 2.6. *Let \mathcal{V} be a G -variety. A separating system is a set $\{(\mathcal{Q}_i, \mathcal{F}_i), 0 \leq i \leq n\}$ such that*

- $\forall 0 \leq i \leq n$, \mathcal{Q}_i is a G -stable subvariety in \mathcal{V} and $\mathcal{F}_i \subset \mathbb{R}(\mathcal{Q}_i)^G$.
- $\mathcal{V} = \bigcup_{i=0}^n \mathcal{Q}_i$.
- For any two points $x, y \in \mathcal{V}$ which are not in the same orbit, there is some $P \in \bigcup_{i=0}^n \mathcal{F}_i$ defined at x and y , with $P(x) \neq P(y)$.

Remark 2.7. *For any $i \geq 1$, \mathcal{Q}_i is a subvariety of codimension greater than 1. In general, functions in $\phi_i^*\mathcal{F}_i \subset \mathbb{R}(\mathcal{Q}_i)^G$ cannot be extended in invariant functions in $\mathbb{R}(\mathcal{V})^G$. The notion of separating system is thus different than the separating sets computed in [Che+19; Oli14], where all invariants are defined on the whole representation.*

2.3 Stratification by isotropy classes

In this section we show how a closed slice helps stratifying a representation by orbit type. Let \mathcal{V} be a G -variety with (\mathcal{Z}, N) a closed Seshadri slice. We prove that the stratification of $G\mathcal{Z}$ by G -orbit type is given by the stratification of \mathcal{Z} by N -orbit type (Corollary 2.9). This stratification with respect to the normalizer is far easier to compute, as it appears in Section 4.

Definition 2.8. Let G be a compact group and ρ a continuous representation on a vector space \mathcal{V} . Take v a point in \mathcal{V} . The isotropy group of v is defined as $G_v = \{g \in G \mid \rho(g)(v) = v\}$. For $J < G$ an isotropy group, the isotropy stratum Σ_J^G is the subset of points of isotropy group conjugated to J :

$$\Sigma_J^G = \{v \in \mathcal{V} \text{ such that } G_v \in [J]\}.$$

When there is no ambiguity about G , we note simply Σ_J . Two points in the same stratum, that is with conjugated isotropy groups, are said of same orbit type. There is an inclusion partial order among isotropy classes. Namely, for two isotropy groups J_1, J_2 ,

$$[J_1] < [J_2] \Leftrightarrow \exists K \in [J_1] \text{ such that } K < J_2.$$

The isotropy stratum closure is the set of points stabilized by at least a member of $[J]$:

$$\overline{\Sigma_J} = \bigcup_{[J] < [K]} \Sigma_K.$$

The closed strata are semialgebraic sets in \mathcal{V} . It has been recently proven that for compact groups, closed strata are in fact algebraic varieties [Azz+23]. The second aim of our article is to compute polynomials defining the stratum of each isotropy class.

Theorem 2.9. Let \mathcal{V} be a G -variety admitting a closed Seshadri slice (\mathcal{Z}, N) , $\mathcal{F}_1 = \{f_1, \dots, f_k\} \in \mathbb{R}[\overline{\mathcal{Z}}]^N$ separating orbits in \mathcal{Z} and J an isotropy group in N . Suppose that the polynomials $P_1, \dots, P_r \in \mathbb{R}[f_1, \dots, f_k]$ define the closed stratum of $J < N$ in \mathcal{Z} :

$$\forall z \in \mathcal{Z}, z \in \overline{\Sigma_J^N} \Leftrightarrow \forall 1 \leq i \leq r, P_i(f_1, \dots, f_k)(z) = 0.$$

Then, they define the closed stratum of $J < G$ in $G\mathcal{Z}$:

$$\forall v \in G\mathcal{Z}, v \in \overline{\Sigma_J^G} \Leftrightarrow \forall 1 \leq i \leq r, P_i(\phi^* f_1, \dots, \phi^* f_k)(v) = 0.$$

Proof. Let v be a point in $G\mathcal{Z}$. Then, there is some $g \in G$ such that $g(v) = z \in \mathcal{Z}$. Since $N < G$ we know that the isotropy groups verify $N_z < G_z$. On the opposite, for any $g' \in G_z$, we have $g'(z) = z \in \mathcal{Z}$ and by the assumption (2^o), $g' \in N_z$. Then $G_z = N_z$. Since $G_v = g^{-1}G_zg$, the isotropy class of v in G is $[N_z]$. \square

It appears in particular that for an isotropy class $[J]$ not $\leq [N]$, $\overline{\Sigma_J} \cap G\mathcal{Z} = \emptyset$. In Section 2.2 we announced successive sets of invariants $\phi_0^* \mathcal{F}_0, \dots, \phi_n^* \mathcal{F}_n, \mathcal{F}_{n+1}$ separating the orbits of G in $\mathcal{V} = G\mathcal{Z}_0 \sqcup \dots \sqcup G\mathcal{Z}_n \sqcup \mathcal{Q}_{n+1}$. It allows to sequence efficiently the work. Let v be a point in \mathcal{V} . For $0 \leq i \leq n$, the non separated varieties \mathcal{Q}_{i+1} are defined by the irreducible polynomials $Q_{i+1} \in \phi_i^* \mathcal{F}_i$. Thus the first index $0 \leq i \leq n$ such that $\phi_{i+1}^* Q_{i+1}(v) \neq 0$ indicates that $v \in G\mathcal{Z}_i = \mathcal{Q}_i \setminus \mathcal{Q}_{i+1}$. Then the stratification of $G\mathcal{Z}_i$ by N_i isotropy class, provided by polynomials of \mathcal{F}_i , gives the orbit type of v .

3 Closed Seshadri slices for some actions of $SO_3(\mathbb{R})$

For the rest of the article, we consider $G = SO_3(\mathbb{R})$. In this section we present three different closed Seshadri slices appearing in representations of $SO_3(\mathbb{R})$, in order to apply the method developed in the previous

sections. For d an integer, let \mathcal{H}_d be the vector space of degree d real harmonic homogeneous polynomials on \mathbb{R}^3 :

$$\mathcal{H}_d := \left\{ P \in \mathbb{R}[U, V, W]_d \mid \nabla^2(P) = \frac{\partial^2 P}{\partial U^2} + \frac{\partial^2 P}{\partial V^2} + \frac{\partial^2 P}{\partial W^2} = 0 \right\}$$

The standard $\mathrm{SO}_3(\mathbb{R})$ -action on \mathbb{R}^3 induces the linear action on polynomials $\rho_d(g) : P \mapsto P \circ g^{-1}$, which stabilises the vector spaces \mathcal{H}_d . The obtained representations (ρ_d, \mathcal{H}_d) are the only irreducible representations of $\mathrm{SO}_3(\mathbb{R})$ [GSS88]. Note that $\mathcal{H}_1 = \mathbb{R}[U, V, W]_1 \cong \mathbb{R}^3$ is the standard representation, and \mathcal{H}_2 is isomorphic to the space of traceless symmetric 3×3 matrices. [HJ24] and [GHP19] provide closed Seshadri slices for representations of $\mathrm{SO}_3(\mathbb{R})$ containing \mathcal{H}_1 (Theorem 3.1) or \mathcal{H}_2 (Theorem 3.3). We also identify a slice in the variety $\mathcal{S}^\circ \subset \mathcal{H}_2$ (Theorem 3.6), that corresponds to trace free symmetric matrices with a repeated eigenvalue.

3.1 The slice with \mathcal{H}_1

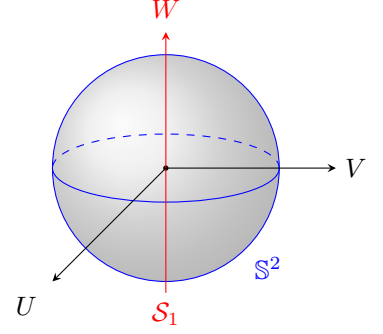
In this section, we consider a representation $\mathcal{V} := \mathcal{H}_1 \oplus \mathcal{H}$ of $\mathrm{SO}_3(\mathbb{R})$. [HJ24] exhibits a closed slice (\mathcal{Z}_1, N_1) in \mathcal{V} , using the first irreducible representation \mathcal{H}_1 :

Theorem 3.1. *Consider the subset*

$$\mathcal{Z}_1 = \mathbb{R}^*W \oplus \mathcal{H}.$$

Then, (\mathcal{Z}_1, N_1) is a closed Seshadri slice with normalizer $N_1 \cong \mathrm{O}_2(\mathbb{R})$ composed by rotations fixing W or mapping W to $-W$:

$$N_1 = \left\{ \left(\begin{array}{c|c} g & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & \det(g) \end{array} \right), g \in \mathrm{O}_2(\mathbb{R}) \right\}$$



Proof. Since \mathcal{V} is a vector space, all its points are normal. Let us next prove the two points to be a closed Seshadri slice.

(1) Let $v := h_1 + h$ be a generic vector in $\mathcal{V} = \mathcal{H}_1 \oplus \mathcal{H}$. The action of $\mathrm{SO}_3(\mathbb{R})$ on \mathcal{H}_1 is isomorphic to the standard action which is transitive on each sphere. Hence there exists a rotation $g \in \mathrm{SO}_3(\mathbb{R})$ such that $\rho_1(g)(h_1) = \|h_1\|W \in \mathbb{R}^*W$. That is, $\rho(g)(v) \in \mathbb{R}^*W \oplus \mathcal{H} = \mathcal{Z}_1$.

(2°) \mathcal{Z}_1 is N_1 -stable. Furthermore, if one takes some $z \in \hat{\mathcal{Z}}_1$ and $g \in \mathrm{SO}_3(\mathbb{C})$ such that $\rho(g)(z) \in \hat{\mathcal{Z}}_1$, then $\rho_1(g) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ a & b & c \end{pmatrix}$ with $a, b, c \in \mathbb{C}$. However we have $\rho_1(g)^t = g^t \in \mathrm{SO}_3(\mathbb{C})$ and $\rho_1(g)^t \rho_1(g) = I_3$.

$$\Rightarrow \rho_1(g)^t \rho_1(g) = \begin{pmatrix} * & * & a \\ * & * & b \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ a & b & c \end{pmatrix} \begin{pmatrix} * & * & ac \\ * & * & bc \\ ac & bc & c^2 \end{pmatrix} = I_3$$

This is $a = b = 0$ and $c = \pm 1$. Then, $g \in \hat{N}_1$. □

Since $N_1 \cong \mathrm{O}_2(\mathbb{R})$, a minimal separating set \mathcal{F}_1 for \mathcal{Z}_1 is provided by [Jal24]. Then the extension $\phi_1^* \mathcal{F}_1$ is a set of $\mathrm{SO}_3(\mathbb{R})$ -invariant rational functions separating orbits in $\mathrm{G}\mathcal{Z}_1$, this is everywhere but in $\mathcal{Q}_2 = \{0\} \oplus \mathcal{H}$.

3.2 The slice with \mathcal{H}_2

In this section, we consider a representation of $\mathrm{SO}_3(\mathbb{R})$ containing at least once the irreducible representation \mathcal{H}_2 :

$$\mathcal{V} = \mathcal{H}_2 \oplus \mathcal{H}$$

where \mathcal{H} is an arbitrary representation of $\mathrm{SO}_3(\mathbb{R})$. [GHP19] exhibits a closed Seshadri slice $(\mathcal{Z}_2, \mathcal{N}_2)$ in \mathcal{V} . It uses the isomorphism between the quadratic harmonics \mathcal{H}_2 and the space \mathcal{S} of trace free symmetric matrices.

Proposition 3.2. *Let $\mathrm{SO}_3(\mathbb{R})$ act on \mathcal{S} by conjugation and on the space of quadratic harmonics \mathcal{H}_2 by composition. The following morphism is an isomorphism of $\mathrm{SO}_3(\mathbb{R})$ -varieties:*

$$\begin{cases} \mathcal{S} & \longrightarrow \mathcal{H}_2 \\ M & \longmapsto \{x \mapsto x^t M x\} \end{cases}$$

The representation space $\mathcal{V} \cong \mathcal{S} \oplus \mathcal{H}$ admits a Seshadri slice, corresponding to diagonal matrices:

Theorem 3.3. *Note $\mathcal{D} \subset \mathcal{S}$ the space of trace free diagonal matrices, and $\mathcal{D}^\circ \subset \mathcal{D}$ the subset of diagonal matrices with two identical eigenvalues. Consider $\mathcal{Z}_2 = (\mathcal{D} \setminus \mathcal{D}^\circ) \oplus \mathcal{H}$ and $\mathcal{N}_2 = \mathcal{O}$ the group of rotations preserving the cube. The pair $(\mathcal{Z}_2, \mathcal{N}_2)$ is a closed Seshadri slice for $\mathrm{SO}_3(\mathbb{R}) \curvearrowright \mathcal{V}$.*

Proof. Since \mathcal{V} is a vector space, all its points are normal. We next prove the two conditions to be a closed Seshadri slice:

(1) Take M a trace free symmetric matrix. By the spectral theorem, there exists some $g \in \mathrm{SO}_3(\mathbb{R})$ such that $g^t M g$ is diagonal.

(2^o) $\mathcal{D} \setminus \mathcal{D}^\circ$ is a Zariski open set stable by \mathcal{O} . Take some $D + H \in \hat{\mathcal{Z}}_2$ with $D = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$, λ_1, λ_2 and λ_3 being distinct numbers in \mathbb{C} . Choose $g \in \mathrm{SO}_3(\mathbb{C})$ such that $\rho(g)(D + H) \in \hat{\mathcal{Z}}_2$, ie $g^t D g$ is diagonal. Since conjugation by orthogonal matrices preserves the eigenvalues, $g^t D g = \mathrm{Diag}(\sigma(\lambda_1), \sigma(\lambda_2), \sigma(\lambda_3))$ for a permutation σ . If we note P_σ the permutation matrix associated to σ and $\varepsilon(\sigma)$ its determinant, the element $o = \varepsilon(\sigma) P_\sigma \in \mathcal{O}$ satisfies $o^t D o = g^t D g$. We have $g \in o \times G_D$, where G_D is the isotropy group of D . Since λ_1, λ_2 and λ_3 are distinct, G_D is the subgroup of diagonal matrices of determinant 1, with only ± 1 on the diagonal. This forces $g \in \mathcal{O}$. \square

We note $\mathcal{S}^\circ = \mathrm{SO}_3(\mathbb{R}) \times \mathcal{D}^\circ$ the variety of symmetric trace free matrices with two identical eigenvalues. Among \mathcal{S} , it is defined by $Q_3 = \prod_{1 \leq i, j \leq 3} (\lambda_i - \lambda_j)$ the discriminant of the characteristic polynomial of the matrix. If one computes a set \mathcal{F}_2 of \mathcal{N}_2 -invariant polynomials separating orbits in \mathcal{Z}_2 , then the extension $\phi_2^* \mathcal{F}_2$ is a set of $\mathrm{SO}_3(\mathbb{R})$ -invariant rational functions separating orbits in $\mathcal{G}\mathcal{Z}_2$, this is everywhere but in $\mathcal{Q}_3 \cong \mathcal{S}^\circ \oplus \mathcal{H}$.

3.3 The slice with \mathcal{S}°

In Section 3.2 arises the variety $\mathcal{Q}_3 \cong \mathcal{S}^\circ \oplus \mathcal{H}$, where \mathcal{S}° is the set of trace free symmetric matrices with a repeated eigenvalue. In this section, we identify a new Seshadri slice $(\mathcal{Z}_3, \mathcal{N}_3)$ in \mathcal{Q}_3 (Theorem 3.6). We first check that \mathcal{Q}_3 is geometrically irreducible, that is $\hat{\mathcal{Q}}_3$ is irreducible:

Proposition 3.4. *The variety $\hat{\mathcal{Q}}_3 = \hat{\mathcal{S}}^\circ \oplus \hat{\mathcal{H}}$ is irreducible.*

Proof. Over real numbers, the spectral theorem claims that $\mathcal{S}^\circ = \mathrm{SO}_3(\mathbb{R}) \times \mathbb{R}\mathcal{S}$ where $\mathbb{R}\mathcal{S}$ is the vector line

spanned by the matrix $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. The complexification of a real variety is its Zariski closure over

\mathbb{C} . Then $\hat{\mathcal{S}}^\circ = \overline{\mathcal{S}^\circ} = \overline{\mathrm{SO}_3(\mathbb{C}) \times \mathbb{C}\mathcal{S}}$. Since $\mathrm{SO}_3(\mathbb{C})$ and $\mathbb{C}\mathcal{S}$ are irreducible, $\mathrm{SO}_3(\mathbb{C}) \times \mathbb{C}\mathcal{S}$ is an irreducible affine set. Thus, $\overline{\mathrm{SO}_3(\mathbb{C}) \times \mathbb{C}\mathcal{S}}$ is an irreducible variety, and so is $\hat{\mathcal{Q}}_3 = \hat{\mathcal{S}}^\circ \oplus \hat{\mathcal{H}}$. \square

Remark 3.5. The real variety \mathcal{S}° can be defined by an ideal \mathcal{J}_5 generated by only five polynomials: the trace, the three differences between opposite coefficients of the matrix and Q_3 the discriminant of the characteristic polynomial. But \mathcal{J}_5 is not sufficient to define the complexification $\hat{\mathcal{S}}^\circ$. For instance, $\hat{\mathcal{V}}(\mathcal{J}_5)$ contains the affine set of greater dimension $\mathrm{SO}_3(\mathbb{C}) \times (\mathbb{C}S + \mathbb{C}N)$, where $N = \begin{pmatrix} 1 & \mathbf{i} & 0 \\ \mathbf{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a nilpotent symmetric

matrix of Jordan form $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We need additional polynomials to define $\hat{\mathcal{S}}^\circ$. Matrices $M \in \hat{\mathcal{S}}^\circ$ have a minimal polynomial of degree two:

$$M \in \hat{\mathcal{S}}^\circ \Leftrightarrow \Pi_{\lambda S}(M) = (M - \lambda I_3)(M + 2\lambda I_3) = 0.$$

It is equivalent to say that I_3, M and M^2 are linearly dependent. That can be pointed out by the 3×3 minors in the entries of I_3, M and M^2 .

The affine set of dimension two $\mathcal{N} = \mathrm{SO}_3(\mathbb{C}) \times \mu N$ of symmetric nilpotent matrices of order two is also annihilated by these minors. It is in the adherence of $\mathrm{SO}_3(\mathbb{C}) \times \lambda S$:

$$\mathcal{N} = \overline{(\mathrm{SO}_3(\mathbb{C}) \times \lambda S)} \setminus (\mathrm{SO}_3(\mathbb{C}) \times \lambda S).$$

$\hat{\mathcal{V}}(\mathcal{J}_5)$ corresponds to the complex set in $\hat{\mathcal{S}}$ which cannot be separated through the slice $(\mathcal{Z}_2, \mathcal{N}_2)$. It requires more invariants to separate orbits in $\hat{\mathcal{V}}(\mathcal{J}_5)$ than in $\mathcal{Q}_3 \ll \hat{\mathcal{V}}(\mathcal{J}_5)$. As in other circumstances [Jal24; Duf09], separating sets over \mathbb{R} have lower cardinalities than over \mathbb{C} .

Theorem 3.6. Consider $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, $\mathcal{Z}_3 = \mathbb{R}^*S \oplus \mathcal{H}$ and

$$\mathcal{N}_3 = \left\{ \left(\begin{array}{c|c} g & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & \det(g) \end{array} \right), g \in \mathrm{O}_2(\mathbb{R}) \right\}.$$

Then, $(\mathcal{Z}_3, \mathcal{N}_3)$ is a closed Seshadri slice for the action of $\mathrm{SO}_3(\mathbb{R})$ on $\mathcal{S}^\circ \oplus \mathcal{H}$.

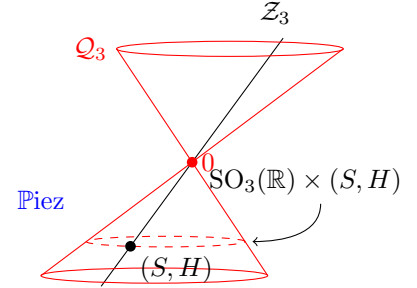


Figure 1: The Seshadri slice in \mathcal{Q}_3 .

Proof. We first check the normality of points of $\hat{\mathcal{Z}}_3$ in $\hat{\mathcal{Q}}_3 = \mathcal{S}^\circ \oplus \hat{\mathcal{H}}$. Suppose that S is a singular point in \mathcal{S}° . Since \mathcal{S}° is homogeneous, λS is singular for any $\lambda \in \mathbb{C}^*$. Thus $\hat{\mathcal{Z}}_3 = \mathbb{C}^*S \oplus \hat{\mathcal{H}}$ is included in singular points of $\hat{\mathcal{Q}}_3$. Since connected algebraic groups have smooth orbits, the whole $\mathrm{SO}_3(\mathbb{C})\hat{\mathcal{Z}}_3$ are singular points of $\hat{\mathcal{Q}}_3$. This is not possible since $\mathrm{SO}_3(\mathbb{C})\hat{\mathcal{Z}}_3$ is dense in $\hat{\mathcal{Q}}_3$. Hence points of $\hat{\mathcal{Z}}_3$ are regular, and thus normal, in $\hat{\mathcal{Q}}_3$. We next prove the two conditions to be a closed Seshadri slice:

(1) Take some $Q = (M, H)$ in $\mathcal{Q}_3 = \mathcal{S}^\circ \oplus \mathcal{H}$. There exists $O \in \mathrm{SO}_3(\mathbb{R})$ diagonalizing M . Since M has two identical eigenvalues, a permutation matrix $P \in \mathrm{O}_3(\mathbb{R})$ gives $POMOP^t = \lambda M$ for some $\lambda \in \mathbb{R}$.

(2^o) Choose $(\lambda S, H)$ a point in the complexification $\hat{\mathcal{Z}}_3$ (this is $\lambda \in \mathbb{C}^*$). Take some $O \in \mathrm{SO}_3(\mathbb{C})$ such that still $\rho(O)(S, H) \in \hat{\mathcal{Z}}_3$. That is, O stabilises S by conjugation. It then necessarily stabilises the eigenspaces of S associated to 1 and -2 . One has thus $O \in \hat{\mathcal{N}}_3$. \square

If \mathcal{F}_3 is a set of \mathcal{N}_3 -invariant polynomials separating orbits in \mathcal{Z}_3 , then the extension $\phi_3^*\mathcal{F}_3$ is a set of $\mathrm{SO}_3(\mathbb{R})$ -invariant rational functions separating orbits in $\mathrm{G}\mathcal{Z}_3$, this is everywhere but in $\mathcal{Q}_4 = \{0\} \oplus \mathcal{H}$. Such a set is provided by [Jal24].

4 Application to the space of piezoelectricity tensors

Piezoelectricity tensors characterize the piezoelectricity response of materials to some constraints and are studied in mechanics [Azz+22; Che+19; Oli14]. $\mathbb{P}iez$ is defined as the set of order three tensors verifying the symmetry $\mathbf{P}_{ijk} = \mathbf{P}_{ikj}$, endowed with the natural action of $SO_3(\mathbb{R})$. We rather use its decomposition into irreducible representations, given by [Oli14]:

$$\mathbb{P}iez = \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3.$$

In the previous sections we presented slices for representations of $SO_3(\mathbb{R})$ containing $\mathcal{H}_1, \mathcal{H}_2$ of \mathcal{S}° . It decomposes $\mathbb{P}iez$ into the following sequence:

$$\begin{array}{ccccccccc} \mathbb{P}iez = \mathcal{Q}_0 & \supset & \mathcal{Q}_1 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 & \supset & \mathcal{Q}_2 = \mathcal{H}_2 \oplus \mathcal{H}_3 & \supset & \mathcal{Q}_3 = \mathcal{S}^\circ \oplus \mathcal{H}_3 & \supset & \mathcal{Q}_4 = \mathcal{H}_3 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\mathcal{Z}_0, \mathcal{N}_0) & & (\mathcal{Z}_1, \mathcal{N}_1) & & (\mathcal{Z}_2, \mathcal{N}_2) & & (\mathcal{Z}_3, \mathcal{N}_3) & & \\ \downarrow & \sqcup & \downarrow & \sqcup & \downarrow & \sqcup & \downarrow & \sqcup & \\ (\mathcal{G}\mathcal{Z}_0) & & \mathcal{G}\mathcal{Z}_1 & & \mathcal{G}\mathcal{Z}_2 & & \mathcal{G}\mathcal{Z}_3 & & \mathcal{H}_3 = \mathbb{P}iez \end{array}$$

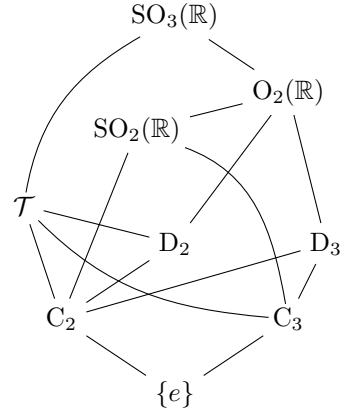
In this section, we use this decomposition to get a separating system of invariants for $\mathbb{P}iez$ (Theorem 4.16). We also stratify each set $\mathcal{G}\mathcal{Z}_i$ by isotropy classes thanks to Theorem 2.9.

The possible isotropy groups of irreducible representations $SO_3(\mathbb{R}) \curvearrowright \mathcal{H}_d$ for $d \in \mathbb{N}$ are given in [IG84, Theorem 3.2]. The isotropy group of a point in a representation is the intersection of the isotropy group of its projections on each irreducible representation. It provides the poset of all the isotropy classes present in $\mathbb{P}iez$ (Figure 2). We note C_k the cyclic group of cardinal k , D_k the dihedral group of cardinal $2k$, \mathcal{T} the group of rotations preserving the tetrahedron, \mathcal{O} the group of rotations preserving the cube and \mathcal{I} the group of rotations preserving the isocahedron.

Proposition 4.1. [IG84] *The isotropy classes for the irreducible representation of $SO_3(\mathbb{R})$ on \mathcal{H}_d are*

- $O_2(\mathbb{R})$ for d even.
- $SO_2(\mathbb{R})$ for d odd.
- \mathcal{I} for $d = 6, 10, 12, 15, 18$ or $d \geq 20$ such that $d \neq 23, 29$.
- \mathcal{O} for $d = 4, 6, 8, 9, 10$ or $d \geq 12$.
- \mathcal{T} for $d = 3, 6, 7$ or $d \geq 9$.
- D_k whenever $k \leq d$.
- C_k whenever $\begin{cases} k \leq d, d \text{ odd.} \\ k \leq \frac{d}{2}, d \text{ even.} \end{cases}$
- $\{e\}$ whenever $d \geq 3$.

Figure 2: Poset of isotropy classes for $\mathbb{P}iez$



4.1 Orbit separation and stratification on $\mathcal{G}\mathcal{Z}_0$ and $\mathcal{G}\mathcal{Z}_1$

Firstly, Theorem 3.1 provides in $\mathbb{P}iez$ the slice $(\mathcal{Z}_0, \mathcal{N}_0)$ where $\mathcal{Z}_0 = \mathbb{R}^*W \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ and

$$\mathcal{N}_0 = \left\{ \left(\left(\begin{array}{c|c} & 0 \\ g & 0 \\ \hline 0 & 0 & \det(g) \end{array} \right), g \in O_2(\mathbb{R}) \right) \right\} \cong O_2(\mathbb{R}).$$

In this section, we first decompose $\overline{\mathcal{Z}}_0$ into irreducible representations of $N_0 \cong O_2(\mathbb{R})$. Then [Jal24] provides a set of invariant polynomials $\mathcal{F}_0 \in \mathbb{R}[\mathcal{Z}_0]^{N_0}$ separating the orbits of N_0 in \mathcal{Z}_0 (Proposition 4.3). We next deduce the inequalities stratifying $G\mathcal{Z}_0$ by orbit type (Proposition 4.5). We also give these results for the second slice (\mathcal{Z}_1, N_1) , which is of same type.

We consider the spherical harmonic bases $\mathcal{B}_{sp}^{(d)} = \{Y_{m,d}, -d \leq m \leq d\}$ (Definition B.1) of the irreducible representations \mathcal{H}_d . They endow $\overline{\mathcal{Z}}_0$ with the coordinates

$$(w, \underbrace{y_{-1,1}, y_{0,1}, y_{1,1}}_{\mathcal{H}_1}, \underbrace{y_{-2,2}, y_{-1,2}, y_{0,2}, y_{1,2}, y_{2,2}}_{\mathcal{H}_2}, \underbrace{y_{-3,3}, y_{-2,3}, y_{-1,3}, y_{0,3}, y_{1,3}, y_{2,3}, y_{3,3}}_{\mathcal{H}_3})$$

These fit very well with the action of $N_0 \cong O_2(\mathbb{R})$. For $\alpha \in \mathbb{R}$, we note

$$g_\alpha^+ = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad g_\alpha^- = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$$

the rotation of angle α and the symmetry of axis of angle α . Then for each $1 \leq m \leq d$, the representation of N_0 on $\text{Vect}(Y_d^{-m}, Y_d^m)$ is isomorphic to the representation (ϕ_m, \mathcal{V}_m) of $O_2(\mathbb{R})$ defined by

$$\phi_m(g_\alpha^+) = \begin{pmatrix} e^{im\alpha} & 0 \\ 0 & e^{-im\alpha} \end{pmatrix} \quad \text{and} \quad \phi_m(g_\alpha^-) = \begin{pmatrix} 0 & e^{-im\alpha} \\ e^{im\alpha} & 0 \end{pmatrix}$$

The action of N_0 on $\text{Vect}(Y_d^0)$ is isomorphic to the trivial representation $(\text{id}, \mathcal{V}_0)$ when d is even or the determinant (\det, \mathcal{V}_{-1}) when d is odd. It provides the following decomposition into irreducible representations:

Lemma 4.2. *The representation of $N_0 \cong O_2(\mathbb{R})$ on $\overline{\mathcal{Z}}_0 = \mathbb{R}W \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ decomposes into*

	$3\mathcal{V}_{-1} \oplus$	$\mathcal{V}_0 \oplus$	$3\mathcal{V}_1 \oplus$	$2\mathcal{V}_2 \oplus$	\mathcal{V}_3	$= \overline{\mathcal{Z}}_0$
<i>wrt. the coordinates</i>	(w)					$\rightarrow \mathbb{R}W$
	$(y_{0,1})$		$(y_{-1,1}, y_{1,1})$			$\rightarrow \mathcal{H}_1$
		$(y_{0,2})$	$(y_{-1,2}, y_{1,2})$	$(y_{-2,2}, y_{2,2})$		$\rightarrow \mathcal{H}_2$
	$(y_{0,3})$		$(y_{-1,3}, y_{1,3})$	$(y_{-2,3}, y_{2,3})$	$(y_{-3,3}, y_{3,3})$	$\rightarrow \mathcal{H}_3$

Corollary 5.11 in [Jal24] provides then a separating set of polynomials. We put an order on the six possible pairs (m, d) , $d = 1, 2, 3$ and $1 \leq m \leq d$:

$$(m, d) < (m', d') \Leftrightarrow d < d' \text{ or } \begin{cases} d = d' \\ m < m' \end{cases}$$

Proposition 4.3. *The following set in $\mathbb{R}[\overline{\mathcal{Z}}_0]^{N_0}$ of cardinality 40 separates the orbits in \mathcal{Z}_0 :*

$$\mathcal{F}_0 = \left\{ \begin{array}{l} T_2 = y_{0,2}, \\ D_{00} = w^2, \\ D_{0d} = w y_{0,d}, \\ P_{m',d}^{m,d} = y_{-m,d}^{\frac{m \wedge m'}{m}} y_{m',d'}^{\frac{m \wedge m'}{m'}} + y_{m,d}^{\frac{m \wedge m'}{m}} y_{-m',d'}^{\frac{m \wedge m'}{m'}}, \\ S_{0,m',d}^{m,d} = i w \left(y_{-m,d}^{\frac{m \wedge m'}{m}} y_{m',d'}^{\frac{m \wedge m'}{m'}} - y_{m,d}^{\frac{m \wedge m'}{m}} y_{-m',d'}^{\frac{m \wedge m'}{m'}} \right), \end{array} \quad \begin{array}{l} d \in \{1, 3\} \\ \left\{ \begin{array}{l} d \in \{1, 2, 3\}, \\ m \geq 1, \end{array} \right. (m, d) \leq (m', d') \\ \left\{ \begin{array}{l} d \in \{1, 2, 3\}, \\ m \geq 1, \end{array} \right. (m, d) < (m', d') \end{array} \right\}$$

Remark 4.4. *Having $w \neq 0$ in \mathcal{Z}_0 reduces considerably the cardinality of the separating set. To separate orbits in $\overline{\mathcal{Z}}_0$ (where we have possibly $w = 0$), [Jal24] proposes a separating set of cardinality 88.*

Although we give the expressions of polynomials of \mathcal{F}_0 only on \mathcal{Z}_0 , Seshadri slice lemma ensures that they are the restrictions of invariant rational functions $\phi_0^* \mathcal{F}_0$ defined on $G\mathcal{Z}_0 = \text{SO}_3(\mathbb{R}) \times \mathcal{Z}$. A technique involving Wigner matrices is provided in [HJ24, Section 3.4] to evaluate them at any point in $G\mathcal{Z}_0$. We use these rational functions to stratify $G\mathcal{Z}_0$ by isotropy classes:

Proposition 4.5. *For any tensor $\mathbf{P} \in \text{Piez}$, we have $\mathbf{P} \in \text{GZ}_0 \Leftrightarrow \phi_0^* D_0(\mathbf{P}) \neq 0$. In that case we have the following stratification:*

- $\mathbf{P} \in \overline{\Sigma_{\text{SO}_2(\mathbb{R})}} \Leftrightarrow \forall(m, d), \phi_0^* P_{m,d}^{m,d}(\mathbf{P}) = 0$.
- $\mathbf{P} \in \overline{\Sigma_{C_3}} \Leftrightarrow \phi_0^* P_{1,1}^{1,1}(\mathbf{P}) = \phi_0^* P_{1,2}^{1,2}(\mathbf{P}) = \phi_0^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_0^* P_{2,2}^{2,2}(\mathbf{P}) = \phi_0^* P_{2,3}^{2,3}(\mathbf{P}) = 0$.
- $\mathbf{P} \in \overline{\Sigma_{C_2}} \Leftrightarrow \phi_0^* P_{1,1}^{1,1}(\mathbf{P}) = \phi_0^* P_{1,2}^{1,2}(\mathbf{P}) = \phi_0^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_0^* P_{3,3}^{3,3}(\mathbf{P}) = 0$.
- *Otherwise, $\mathbf{P} \in \Sigma_{\{e\}}$.*

Proof. First, the polynomial $\phi_0^* D_0$ is the squared norm of the projection on the first subspace $\mathcal{H}_1 \subset \text{Piez}$. Then, $\mathbf{P} \in \text{GZ}_0$ iff $\phi_0^* D_0(\mathbf{P}) \neq 0$.

In that case, since the polynomials are invariants on the orbits, we shall assume that $\mathbf{P} \in \mathcal{Z}_0$. By Theorem 2.9, it is sufficient to determine its isotropy class in the normalizer N_0 . However, the stratification of the orbit space of actions of $N_0 \cong \text{O}_2(\mathbb{R})$ is given in [Jal24, Theorem 8.3]:

- (1) For any $k \in \mathbb{N}^*$, $\mathbf{P} \in \overline{\Sigma_{C_k}}$ iff for all (m, d) such that k does not divide m , $P_{m,d}^{m,d}(\mathbf{P}) = 0$.
- (2) $\mathbf{P} \in \overline{\Sigma_{D_k}}$ iff furthermore, $\forall d, D_{0d}(\mathbf{P}) = 0$.
- (3) $\mathbf{P} \in \overline{\Sigma_{\text{SO}_2(\mathbb{R})}}$ iff for all (m, d) , $P_{m,d}^{m,d}(\mathbf{P}) = 0$.
- (4) $\mathbf{P} \in \overline{\Sigma_{\text{O}_2(\mathbb{R})}}$ iff furthermore, $\forall d, D_{0d}(\mathbf{P}) = 0$.

Note that the cases (2) and (4) are deleted since $D_{00}(\mathbf{P}) \neq 0$. □

Remark 4.6. *The case where $\phi_0^* D_{00} \neq 0 \Leftrightarrow \mathbf{P} \in \text{GZ}_0$ is the "worst". Indeed it requires to evaluate 40 rational functions to determine the orbit of \mathbf{P} , and 7 to determine its isotropy class. Otherwise, if $\phi_0^* D_{00} = 0$, lower number of functions provide these informations in the following GZ_i .*

Next, the non separated variety $\mathcal{Q}_1 = \text{Piez} \setminus \text{GZ}_0$ is the vector space $\{0\} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. We thus identify a second Sechadri slice (\mathcal{Z}_1, N_1) of the same type, where $\overline{\mathcal{Z}}_1 = \mathbb{R}W \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \cong \mathcal{V}_{-1}^2 \oplus \mathcal{V}_0 \oplus \mathcal{V}_1^2 \oplus \mathcal{V}_2^2 \oplus \mathcal{V}_3$. [Jal24, Corollary 5.11] provides a second set of invariants separating the orbits of N_1 in \mathcal{Z}_1 :

$$\mathcal{F}_1 = \left\{ \begin{array}{l} T_2 = y_{0,2}, \\ D_{1d} = y_{0,1} y_{0,d}, \\ P_{m',d'}^{m,d} = y_{-m,d}^{\frac{m \wedge m'}{m}} y_{m',d'}^{\frac{m \wedge m'}{m'}} + y_{m,d}^{\frac{m \wedge m'}{m}} y_{-m',d'}^{\frac{m \wedge m'}{m'}}, \\ S_{1,m',d'}^{m,d} = i y_{0,1} \left(y_{-m,d}^{\frac{m \wedge m'}{m}} y_{m',d'}^{\frac{m \wedge m'}{m'}} - y_{m,d}^{\frac{m \wedge m'}{m}} y_{-m',d'}^{\frac{m \wedge m'}{m'}} \right), \end{array} \quad \begin{array}{l} d \in \{1, 3\} \\ \left\{ \begin{array}{l} d \in \{2, 3\}, \\ m \geq 1, \end{array} \right. (m, d) \leq (m', d') \\ \left\{ \begin{array}{l} d \in \{2, 3\}, \\ m \geq 1, \end{array} \right. (m, d) \leq (m', d') \end{array} \right\}$$

Theorem 4.7. *For any tensor $\mathbf{P} \in \text{Piez}$, we have $\mathbf{P} \in \text{GZ}_1 \Leftrightarrow \left\{ \begin{array}{l} \phi_0^* D_{00}(\mathbf{P}) = 0. \\ \phi_1^* D_{11}(\mathbf{P}) \neq 0. \end{array} \right.$ In that case we have the following stratification:*

- $\mathbf{P} \in \overline{\Sigma_{\text{SO}_2(\mathbb{R})}} \Leftrightarrow \forall(m, d), \phi_1^* P_{m,d}^{m,d}(\mathbf{P}) = 0$.
- $\mathbf{P} \in \overline{\Sigma_{C_3}} \Leftrightarrow \phi_1^* P_{1,2}^{1,2}(\mathbf{P}) = \phi_1^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_1^* P_{2,3}^{2,3}(\mathbf{P}) = 0$.
- $\mathbf{P} \in \overline{\Sigma_{C_2}} \Leftrightarrow \phi_1^* P_{1,2}^{1,2}(\mathbf{P}) = \phi_1^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_1^* P_{3,3}^{3,3}(\mathbf{P}) = 0$.
- *Otherwise, $\mathbf{P} \in \Sigma_{\{e\}}$.*

Remark 4.8. *Some polynomials belong to both \mathcal{F}_0 and \mathcal{F}_1 , as for instance T_2 . However, their pullbacks $\phi_0^* T_2$ and $\phi_1^* T_2$ are different functions. Indeed $\phi_0^* T_2$ has a pole along \mathcal{Q}_1 , and not $\phi_1^* T_2$.*

4.2 Orbit separation and stratification on GZ_2

Here it remains to separate orbits in $\mathcal{Q}_2 = \mathcal{H}_2 \oplus \mathcal{H}_3$. Theorem 3.3 provides the slice (\mathcal{Z}_2, N_2) , where $\mathcal{Z}_2 \cong (\mathcal{D} \setminus \mathcal{D}^o) \oplus \mathcal{H}_3$ and $N_2 = \mathcal{O}$. $\mathcal{D} \setminus \mathcal{D}^o$ is the set of traceless diagonal matrices with *distinct* eigenvalues. In this section we compute a set of polynomials $\mathcal{F}_2 \in \mathbb{R}[\overline{\mathcal{Z}}_2]^{N_2}$ separating orbits in \mathcal{Z}_2 (Theorem 4.10). We next give the polynomial equalities stratifying GZ_2 by orbit type (Proposition 4.11).

For this purpose, we need the cubic harmonics, an alternative basis on \mathcal{H}_3 which fits to its decomposition in irreducible representations of \mathcal{O} .

Proposition 4.9. [FK77] *The following set is an orthonormal basis for \mathcal{H}_3*

$$\mathcal{B}_{cu}^{(3)} = \left\{ \begin{array}{l} \alpha_1 = \frac{\sqrt{7}}{4\pi}U(2U^2 - 3V^2 - 3W^2) \quad \frac{\sqrt{105}}{4\pi}\beta_1 = U(V^2 - W^2) \\ \alpha_2 = \frac{\sqrt{7}}{4\pi}V(2V^2 - 3U^2 - 3W^2) \quad \frac{\sqrt{105}}{4\pi}\beta_2 = V(W^2 - U^2) \quad \frac{\sqrt{105}}{2\pi}\gamma = UVW \\ \alpha_3 = \frac{\sqrt{7}}{4\pi}W(2W^2 - 3V^2 - 3U^2) \quad \frac{\sqrt{105}}{4\pi}\beta_3 = W(U^2 - V^2) \end{array} \right\}$$

The base change matrix from the spherical harmonic basis $\mathcal{B}_{sp}^{(3)}$ is provided in Appendix B. For $Z \in \mathcal{Z}_2$, we note λ, a, b vectors in \mathbb{R}^3 and $c \in \mathbb{R}$ such that

$$Z = \lambda_1 U^2 + \lambda_2 V^2 + \lambda_3 W^2 + a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 + b_1 \beta_1 + b_2 \beta_2 + b_3 \beta_3 + c \gamma$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Let σ be a permutation in \mathfrak{S}_3 and $\delta = (\delta_1, \delta_2, \delta_3) \in \{-1, +1\}^3$. We note P_σ the matrix associated to σ and $\varepsilon(\sigma) = \det(P_\sigma)$ the signature of σ , $\text{Diag}(\delta)$ is the matrix $\begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix}$. The element $(\sigma, \delta) = P_\sigma \text{Diag}(\delta)$ in the semidirect product $\mathcal{O}^- \cong \mathfrak{S}_3 \rtimes \mathbb{C}_2^3$ acts as

$$\begin{aligned} \rho(\sigma, \delta)(\lambda) &= P_\sigma \lambda \\ \rho(\sigma, \delta)(a) &= P_\sigma \text{Diag}(\delta) a \\ \rho(\sigma, \delta)(b) &= \varepsilon(\sigma) P_\sigma \text{Diag}(\delta) b \\ \rho(\sigma, \delta)(c) &= \delta_1 \delta_2 \delta_3 c \end{aligned}$$

Theorem 4.10. *Consider the set $\mathcal{F}_2 \subset \mathbb{R}[\mathcal{S}_2]$ composed with the five polynomials*

$$L_0 = b_1 b_2 b_3, \quad L_1 = [\lambda] (a_1 b_2 b_3 + a_2 b_1 b_3 + a_3 b_1 b_2), \quad L_2 = (b_1 a_2 a_3 + b_2 a_1 a_3 + b_3 a_1 a_2), \quad L_3 = [\lambda] a_1 a_2 a_3$$

$$L_c = [\lambda] c$$

and the twelve entries of the following matrix:

$$R = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}}_{W(\lambda)} \begin{pmatrix} \lambda_1 & a_1^2 & b_1^2 & [\lambda] a_1 b_1 \\ \lambda_2 & a_2^2 & b_2^2 & [\lambda] a_2 b_2 \\ \lambda_3 & a_3^2 & b_3^2 & [\lambda] a_3 b_3 \end{pmatrix}$$

where $[\lambda] = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. Then, \mathcal{F}_2 is invariant for N_2 and separates the orbits of N_2 in \mathcal{Z}_2 .

Proof. We choose $Z = (\lambda, a, b, c)$ and $\tilde{Z} = (\tilde{\lambda}, \tilde{a}, \tilde{b}, \tilde{c})$ two points in \mathcal{Z}_2 where polynomials of \mathcal{F}_2 agree. We then prove that they lie in the same orbit.

1. Separation for \mathcal{O}^- .

- The polynomials $R_{1,1}$, $R_{2,1}$ and $R_{3,1}$ are the Newton sums of λ_1, λ_2 and λ_3 . These generate the symmetric polynomials in λ , that is the invariant ring for the action of \mathfrak{S}_3 on \mathcal{D} . Thus they separate orbits in \mathcal{D} , and there exists a permutation σ such that for any δ , $\rho(\sigma, \delta)(\lambda) = \tilde{\lambda}$.
- We notice furthermore that

$$W(\lambda) \begin{pmatrix} a_1^2 \\ a_2^2 \\ a_3^2 \end{pmatrix} = W(\tilde{\lambda}) \begin{pmatrix} \tilde{a}_1^2 \\ \tilde{a}_2^2 \\ \tilde{a}_3^2 \end{pmatrix} = W(P_\sigma \lambda) \begin{pmatrix} \tilde{a}_1^2 \\ \tilde{a}_2^2 \\ \tilde{a}_3^2 \end{pmatrix} = W(\lambda) P_\sigma^t \begin{pmatrix} \tilde{a}_1^2 \\ \tilde{a}_2^2 \\ \tilde{a}_3^2 \end{pmatrix}$$

Since $Z \in \mathcal{Z}_2$, λ has distinct coefficients and $W(\lambda)$ is invertible. It follows that $\begin{pmatrix} \tilde{a}_1^2 \\ \tilde{a}_2^2 \\ \tilde{a}_3^2 \end{pmatrix} = P_\sigma \begin{pmatrix} a_1^2 \\ a_2^2 \\ a_3^2 \end{pmatrix}$, and there exists some $\delta \in \{-1, 1\}^3$ such that $\rho(\sigma, \delta)(a) = P_\sigma \text{Diag}(\delta) a = \tilde{a}$.

- Similarly, we compute

$$W(\lambda) \begin{pmatrix} b_1^2 \\ b_2^2 \\ b_3^2 \end{pmatrix} = W(\tilde{\lambda}) \begin{pmatrix} \tilde{b}_1^2 \\ \tilde{b}_2^2 \\ \tilde{b}_3^2 \end{pmatrix} = W(P_\sigma \lambda) \begin{pmatrix} \tilde{b}_1^2 \\ \tilde{b}_2^2 \\ \tilde{b}_3^2 \end{pmatrix} = W(\lambda) P_\sigma^t \begin{pmatrix} \tilde{b}_1^2 \\ \tilde{b}_2^2 \\ \tilde{b}_3^2 \end{pmatrix}$$

It follows that $\begin{pmatrix} \tilde{b}_1^2 \\ \tilde{b}_2^2 \\ \tilde{b}_3^2 \end{pmatrix} = P_\sigma \begin{pmatrix} b_1^2 \\ b_2^2 \\ b_3^2 \end{pmatrix}$, and there exists some δ' such that $\rho(\sigma, \delta')(b) = \tilde{b}$.

- We need to check that $\delta = \delta'$. Let us compute

$$[\lambda]W(\lambda) \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{pmatrix} = [\tilde{\lambda}]W(\tilde{\lambda}) \begin{pmatrix} \tilde{a}_1 \tilde{b}_1 \\ \tilde{a}_2 \tilde{b}_2 \\ \tilde{a}_3 \tilde{b}_3 \end{pmatrix} = \varepsilon(\sigma)[\lambda]W(\lambda) P_\sigma^t P_\sigma \begin{pmatrix} \delta_1 a_1 & 0 & 0 \\ 0 & \delta_2 a_2 & 0 \\ 0 & 0 & \delta_3 a_3 \end{pmatrix} P_\sigma^t \varepsilon(\sigma) P_\sigma \begin{pmatrix} \delta'_1 b_1 \\ \delta'_2 b_2 \\ \delta'_3 b_3 \end{pmatrix}$$

It follows the identity $\begin{pmatrix} \delta_1 a_1 & 0 & 0 \\ 0 & \delta_2 a_2 & 0 \\ 0 & 0 & \delta_3 a_3 \end{pmatrix} \begin{pmatrix} \delta'_1 b_1 \\ \delta'_2 b_2 \\ \delta'_3 b_3 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, that is

$$\forall 1 \leq i \leq 3, \delta_i \delta'_i a_i b_i = a_i b_i.$$

For each i , $a_i b_i \neq 0 \Rightarrow \delta_i = \delta'_i$. But if $a_i = 0$, then $\delta_i = \pm 1$ both satisfy $\rho(\sigma, \delta)(a) = \tilde{a}$. Similarly $b_i = 0$ implies that $\delta'_i = \pm 1$ both satisfy $\rho(\sigma, \delta')(b) = \tilde{b}$. Anyway we can obtain $\delta = \delta'$.

2. Separation for $\mathcal{O} = \mathcal{O}^- \cap \text{SO}_3(\mathbb{R})$

Now we want $(\sigma, \delta) \in \mathcal{O} = \text{N}_2$, this is $\det(\sigma, \delta) = \varepsilon(\sigma) \delta_1 \delta_2 \delta_3 = 1$.

- If $a_i = b_i = 0$ for some $i = 1, 2, 3$, then $\delta_i = \pm 1$ satisfy both $\rho(\sigma, \delta)(a, b) = (\tilde{a}, \tilde{b})$. Then we choose δ_i such that $\varepsilon(\sigma) \delta_1 \delta_2 \delta_3 = 1$.
- Otherwise, there is some $j \leq 4$ such that the polynomial L_j does not vanish. We write $L_j(Z) = [\lambda]^{j \bmod 2} \mathbf{m}(a, b)$ where the degree of \mathbf{m} in b is $3 - j$. Then,

$$[\lambda]^{j \bmod 2} \mathbf{m}(a, b) = [\tilde{\lambda}]^{j \bmod 2} \mathbf{m}(\tilde{a}, \tilde{b}) = \varepsilon(\sigma)^j [\lambda]^{j \bmod 2} \delta_1 \delta_2 \delta_3 \varepsilon(\sigma)^{3-j} \mathbf{m}(a, b)$$

We obtain also $\delta_1 \delta_2 \delta_3 = \varepsilon(\sigma)$.

The last polynomial L_c provides the identities $\begin{cases} [\lambda]c = [\tilde{\lambda}]\tilde{c} = \varepsilon(\sigma)[\lambda]\tilde{c} \\ [\lambda] \neq 0 \end{cases} \Rightarrow \tilde{c} = \varepsilon(\sigma)c$. We thus verify that $\rho(\sigma, \delta)c = \delta_1 \delta_2 \delta_3 c = \varepsilon(\sigma)c = \tilde{c}$. Its finally gives $\rho(\sigma, \delta)(Z) = \tilde{Z}$. \square

The polynomial $[\lambda] = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$ is equivariant and $[\lambda]^2$ is invariant. $Q_3 = \phi_2^*[\lambda]^2$ is the discriminant of the characteristic polynomial of matrices of \mathcal{S} . It vanishes iff two eigenvalues are identical and defines the variety $\mathcal{Q}_3 = \mathcal{S}^o \oplus \mathcal{H}_3$.

Although we give the expressions of polynomials of \mathcal{F}_2 only on \mathcal{Z}_2 , Seshadri slice lemma ensures that they are the restrictions of invariant rational functions $\phi_2^* \mathcal{F}_2$ defined on $\text{G}\mathcal{Z}_2$. [GHP19, Section 6.1] describes a procedure based on linear algebra to evaluate them at a point outside \mathcal{Z}_2 . We use these rational functions to stratify $\text{G}\mathcal{Z}_2$ by isotropy classes:

Theorem 4.11. *For any tensor $\mathbf{P} \in \mathbb{P}\text{iez}$, we have $\mathbf{P} \in \text{G}\mathcal{Z}_2 \Leftrightarrow \begin{cases} \phi_0^* D_{00}(\mathbf{P}) = \phi_1^* D_{11}(\mathbf{P}) = 0. \\ Q_3(\mathbf{P}) \neq 0. \end{cases}$ In that case we have the following stratification:*

$$\bullet \mathbf{P} \in \overline{\Sigma_{\text{D}_2}} \Leftrightarrow \begin{cases} \phi_2^* R_{1,2}(\mathbf{P}) = 0 \\ \phi_2^* R_{1,3}(\mathbf{P}) = 0 \end{cases}$$

- $\mathbf{P} \in \overline{\Sigma_{C_2}} \Leftrightarrow \begin{cases} \phi_2^* L_0(\mathbf{P}) = \phi_2^* L_1(\mathbf{P}) = \phi_2^* L_2(\mathbf{P}) = \phi_2^* L_3(\mathbf{P}) = 0 \\ \phi_2^* (R_{1,2}R_{3,2} - R_{2,2}^2)(\mathbf{P}) = 0 \\ \phi_2^* (R_{1,3}R_{3,3} - R_{2,3}^2)(\mathbf{P}) = 0 \\ \phi_2^* ([\lambda]^2 R_{1,2}R_{1,3} - R_{1,4}^2)(\mathbf{P}) = 0 \end{cases}$
- Otherwise, $\mathbf{P} \in \Sigma_{\{e\}}$.

Proof. According to Proposition 4.5 and 4.7, $\mathbf{P} \in \mathcal{Q}_2$ if and only if $\phi_0^* D_{00}(\mathbf{P}) = \phi_1^* D_{11}(\mathbf{P}) = 0$. Then, recall that $\mathcal{Q}_3(\mathbf{P}) \neq 0$ iff λ_1, λ_2 and λ_3 are distinct. That is $\mathbf{P} \in \mathbf{G}\mathcal{Z}_2 = \mathcal{Q}_2 \setminus \mathcal{Q}_3$.

In that case, since functions are invariants on the orbits, we shall assume that $\mathbf{P} \in \mathcal{Z}_2$. We then write $\mathbf{P} = (\lambda, a, b, c)$ with respect to the coordinates introduced in Section 3.2. By Theorem 2.9, it is sufficient to determine its isotropy in the normalizer N_2 . Let $g \in N_2$ be an element stabilizing \mathbf{P} . We note $g = (\sigma, \delta)$ in the semi direct product $\mathfrak{S}_3 \rtimes \{\pm 1\}^3$. Since $g \text{Diag}(\lambda_1, \lambda_2, \lambda_3)g^{-1} = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$, we have $g = \text{Diag}(\delta_1, \delta_2, \delta_3)$ where $\begin{cases} |\delta_1| = |\delta_2| = |\delta_3| = 1 \\ \delta_1 \delta_2 \delta_3 = 1 \end{cases}$. Yet we see that $\mathbf{G}_{\mathbf{P}}$ is isomorphic to C_2 or D_2 .

Suppose that for any $i = 1, 2, 3$, $a_i = b_i = 0$ (equivalently, $R_{1,1}(\mathbf{P}) = R_{1,2}(\mathbf{P}) = 0$). Then, for any $\delta \in \{\pm 1\}^3$ with $\delta_1 \delta_2 \delta_3 = 1$, $\rho(g)(\mathbf{P}) = \mathbf{P}$. This gives $\mathbf{G}_{\mathbf{P}} \cong D_2$.

Suppose that $L_0(\mathbf{P}) = L_1(\mathbf{P}) = L_2(\mathbf{P}) = L_3(\mathbf{P}) = 0$. As noticed in the proof of Theorem 4.10, there is some $i \in \{1, 2, 3\}$ such that $a_i = b_i = 0$. Note j, k integers with $\{i, j, k\} = \{1, 2, 3\}$. Then,

$$\begin{aligned} (R_{1,2}R_{3,2} - R_{2,2}^2)(\mathbf{P}) = 0 &\Leftrightarrow (a_j^2 + a_k^2)(\lambda_j^2 a_j^2 + \lambda_k^2 a_k^2) - (\lambda_j a_j^2 + \lambda_k a_k^2)^2 = 0 \\ &\Leftrightarrow (\lambda_k^2 + \lambda_j^2) a_j^2 a_k^2 - 2\lambda_j \lambda_k a_j^2 a_k^2 = 0 \\ &\Leftrightarrow (\lambda_k - \lambda_j)^2 a_j^2 a_k^2 = 0. \end{aligned}$$

Since λ_j and λ_k are distinct, $a_j a_k = 0$. Similarly, $(R_{1,3}R_{3,3} - R_{2,3}^2)(\mathbf{P}) = 0 \Leftrightarrow b_j b_k = 0$. However, the last equation gives

$$\begin{aligned} ([\lambda]^2 R_{1,2}R_{1,3} - R_{1,4}^2)(\mathbf{P}) = 0 &\Leftrightarrow [\lambda]^2 (a_j^2 + a_k^2)(b_j^2 + b_k^2) - [\lambda]^2 (a_j b_j + a_k b_k)^2 = 0 \\ &\Leftrightarrow a_j^2 b_k^2 + a_k^2 b_j^2 - 2a_j b_j a_k b_k = 0 \\ &\Leftrightarrow (a_j b_k - a_k b_j)^2 = 0. \end{aligned}$$

It ensures that $a_j = b_j = 0$ or $a_k = b_k = 0$. In the first case, put $\delta_k = 1$ and $\delta_i = \delta_j = -1$. We can check that $g = \text{Diag}(\delta_1, \delta_2, \delta_3)$ stabilizes \mathbf{P} . In the second case, it is the same with $\delta_j = 1$ and $\delta_i = \delta_k = -1$.

Otherwise, if $L_i(\mathbf{P}) \neq 0$ for some $0 \leq j \leq 3$, then for any $1 \leq i \leq 3$ one has $a_i \neq 0$ or $b_i \neq 0$. This forces $\delta_i = 1$. The only element g stabilizing \mathbf{P} is I_3 . \square

4.3 Orbit separation and stratification on $\mathbf{G}\mathcal{Z}_3$

Here it remains to separate orbits in $\mathcal{Q}_3 = \mathcal{S}^o \oplus \mathcal{H}_3$ where \mathcal{S}^o is the variety of trace free symmetric matrices with two identical eigenvalues. Theorem 3.6 provides the slice (\mathcal{Z}_3, N_3) , where

$$\mathcal{Z}_3 \cong \underbrace{\mathbb{R}^* \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_S \oplus \mathcal{H}_3, \quad N_3 = \left\{ \left(\begin{array}{c|c} & 0 \\ g & 0 \\ \hline 0 & 0 & \det(g) \end{array} \right), g \in O_2(\mathbb{R}) \right\}.$$

In this section we provide a set of polynomials $\mathcal{F}_3 \in \mathbb{R}[\overline{\mathcal{Z}_3}]^{N_3}$ separating orbits in $\mathcal{Z}_3 = \mathbb{R}^* S \oplus \mathcal{H}_3$ (Proposition 4.12). We next use it to stratify $\mathbf{G}\mathcal{Z}_3$ by isotropy classes (Proposition 4.13).

The action of N_3 on the vector line $\mathbb{R}S$ is trivial. Hence by Proposition 4.2, the action of N_3 on $\overline{\mathcal{Z}_3} = \mathbb{R}S \oplus \mathcal{H}_3$ is isomorphic to the linear representation $\mathcal{V}_0 \oplus \mathcal{V}_{-1} \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$. We still consider the coordinates $(y_{0,2}, y_{-3,3}, y_{-2,3}, y_{-1,3}, y_{0,3}, y_{1,3}, y_{2,3}, y_{3,3})$ introduced in Section 3.1. A separating set is directly provided in [Jal24, Corollary 5.11]:

Proposition 4.12. *The following set $\mathcal{F}_3 \in \mathbb{R}[\overline{\mathcal{Z}}_3]^{\mathbb{N}_3}$ of cardinal 12 separates orbits in \mathcal{Z}_3 :*

$$\mathcal{F}_3 = \left\{ \begin{array}{l} T_2 = y_{0,2}, \\ D_{33} = y_{0,3} y_{0,3}, \\ P_{m',3}^{m,3} = y_{-m,3}^{\frac{m\Delta m'}{m}} y_{m',3}^{\frac{m\Delta m'}{m'}} + y_{m,3}^{\frac{m\Delta m'}{m}} y_{-m',3}^{\frac{m\Delta m'}{m'}}, \\ S_{3,m',3}^{m,3} = i y_{0,3} \left(y_{-m,3}^{\frac{m\Delta m'}{m}} y_{m',3}^{\frac{m\Delta m'}{m'}} - y_{m,3}^{\frac{m\Delta m'}{m}} y_{-m',3}^{\frac{m\Delta m'}{m'}} \right), \\ T_{231} = (y_{2,3} y_{-1,3}^2 - y_{-2,3} y_{1,3}^2)(y_{3,3} y_{-1,3}^3 - y_{-3,3} y_{1,3}^3) \end{array} \right. \quad \begin{array}{l} m \leq m' \\ \\ \\ m < m' \end{array}$$

Although we give only their expression on \mathcal{Z}_3 , Wigner matrices enable us to evaluate them as rational functions on the whole \mathcal{Q}_3 [HJ24, Section 3.4]. We use these rational functions to stratify $\text{G}\mathcal{Z}_3$ by isotropy classes:

Theorem 4.13. *For any tensor $\mathbf{P} \in \text{Piez}$, we have $\mathbf{P} \in \text{G}\mathcal{Z}_3 \Leftrightarrow \begin{cases} \phi_0^* D_{00}(\mathbf{P}) = \phi_1^* D_{11}(\mathbf{P}) = Q_3(\mathbf{P}) = 0. \\ \phi_3^* T_2(\mathbf{P}) \neq 0. \end{cases}$*

In that case we have the following stratification:

- $\mathbf{P} \in \overline{\Sigma_{\text{O}_2(\mathbb{R})}} \Leftrightarrow \begin{cases} \phi_3^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_3^* P_{2,3}^{2,3}(\mathbf{P}) = \phi_3^* P_{3,3}^{3,3}(\mathbf{P}) = 0 \\ \phi_3^* D_{33}(\mathbf{P}) = 0 \end{cases}$
- $\mathbf{P} \in \overline{\Sigma_{\text{SO}_2(\mathbb{R})}} \Leftrightarrow \phi_3^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_3^* P_{2,3}^{2,3}(\mathbf{P}) = \phi_3^* P_{3,3}^{3,3}(\mathbf{P}) = 0.$
- $\mathbf{P} \in \overline{\Sigma_{\text{D}_3}} \Leftrightarrow \begin{cases} \phi_3^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_3^* P_{2,3}^{2,3}(\mathbf{P}) = 0 \\ \phi_3^* D_{33}(\mathbf{P}) = 0 \end{cases}$
- $\mathbf{P} \in \overline{\Sigma_{\text{D}_2}} \Leftrightarrow \begin{cases} \phi_3^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_3^* P_{3,3}^{3,3}(\mathbf{P}) = 0 \\ \phi_3^* D_{33}(\mathbf{P}) = 0 \end{cases}$
- $\mathbf{P} \in \overline{\Sigma_{\text{C}_3}} \Leftrightarrow \phi_3^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_3^* P_{2,3}^{2,3}(\mathbf{P}) = 0.$
- $\mathbf{P} \in \overline{\Sigma_{\text{C}_2}} \Leftrightarrow \phi_3^* P_{1,3}^{1,3}(\mathbf{P}) = \phi_3^* P_{3,3}^{3,3}(\mathbf{P}) = 0.$
- *Otherwise, $\mathbf{P} \in \Sigma_{\{e\}}$.*

To prove it, we refer to the proof of Proposition 4.5, based on the stratification of representations of $\text{O}_2(\mathbb{R}) \cong \mathbb{N}_3$. Let us focus on the stratum $\overline{\Sigma_{\text{C}_2}}$, which is quite particular. In $\mathbb{N}_3 \cong \text{O}_2(\mathbb{R})$, two distinct isotropy classes whose representants are isomorphic to C_2 are identified in [Jal24]. Two generators are for

instance $g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Yet g_1 and g_2 are not conjugated in \mathbb{N}_3 , they are

conjugated in $\text{SO}_3(\mathbb{R})$. The two distinct branches described in [Jal24] are thus included in a single stratum.

Proposition 2.5 deduces from the rational functions in $\phi_3^* \mathcal{F}_3$ a set of invariant polynomials \mathcal{P}_3 , which still separates orbits of $\text{SO}_3(\mathbb{R})$ in \mathcal{Q}_3 . Here a problem arises. Invariant polynomials in \mathcal{P}_3 cannot be algebraically extended on $\mathcal{H}_2 \oplus \mathcal{H}_3 \supset \mathcal{Q}_3$. If one only wants to separate orbits, then the result is sufficient. Indeed, the orbits outside \mathcal{Q}_3 have already been separated in the previous sections. Nevertheless, it might be necessary to use invariants functions which are continuous on the whole representation Piez . It is needed, for instance, to construct a surjection $\mathcal{H} \rightarrow \mathcal{H}/\text{SO}_3(\mathbb{R})$ and design an invariant neural network [DG24; BV23]. For this purpose, a technique to extend polynomials in \mathcal{P}_3 to continuous maps on Piez is provided in Appendix A.

4.4 Orbit separation and stratification on \mathcal{H}_3

Then, it remains only to separate orbits in $\mathcal{Q}_4 = \mathcal{H}_3$. This problem is already answered in the literature. Indeed, a set generating the invariant algebra $\mathbb{R}[\mathcal{H}_3]^{\text{SO}_3(\mathbb{R})}$ (and thus separating) is provided in [SB97]. It uses the isomorphism between homogeneous polynomials of degree three and the symmetric tensors of order three. Among them, the harmonic polynomials corresponds to the trace free tensors.

Proposition 4.14. [SB97] *For $(A_{ijk})_{i,j,k}$ a tensor and $i, j \in \{1, 2, 3\}$, we note*

$$B_{ij} = \sum_{p,q} A_{ipq} A_{jpq}, \quad C_i = \sum_{j,k} A_{ijk} B_{jk} \quad \text{and} \quad \varepsilon_{ijk} := \begin{cases} 1 & \text{if } ijk = 123, 231, 312. \\ -1 & \text{if } ijk = 132, 213, 321. \\ 0 & \text{otherwise.} \end{cases}$$

The following set of polynomials separates the orbits of $\text{SO}_3(\mathbb{R})$ in the space of symmetric trace free tensors:

$$\mathcal{F}_4 = \left\{ \begin{array}{l} K_2 = \sum_{i,j,k} A_{ijk}^2 \quad K_4 = \sum_{i,j} B_{ij}^2 \quad K_6 = \sum_i C_i^2 \\ K_{10} = \sum_{i,j,k} A_{ijk} C_i C_j C_k \quad K_{15} = \sum_{i,j,k,p,q} \varepsilon_{ijk} C_i B_{jp} C_p A_{kqr} C_q C_r \end{array} \right\}$$

The representation of $\text{SO}_3(\mathbb{R})$ on \mathcal{H}_3 is also simple enough to stratify it by classical techniques:

Theorem 4.15. *For any tensor $\mathbf{P} \in \mathbb{P}\text{iez}$, we have $\mathbf{P} \in \mathcal{H}_3 \Leftrightarrow \phi_0^* D_{00}(\mathbf{P}) = \phi_1^* D_{11}(\mathbf{P}) = Q_3(\mathbf{P}) = \phi_3^* T_2 = 0$. In that case we have the following stratification:*

- The strata $\overline{\Sigma_{\text{SO}_2(\mathbb{R})}}$ is defined by the system
$$\begin{cases} -11K_2^2 + 25K_4 = 0 \\ -8K_2^3 + 125K_6 = 0 \\ -32K_2^5 + 3125K_{10} = 0 \end{cases}$$
- The strata $\overline{\Sigma_{\mathcal{T}}}$ is defined by the system
$$\begin{cases} -K_2^2 + 3K_4 = 0 \\ K_6 = 0 \\ K_{10} = 0 \end{cases}$$
- The strata $\overline{\Sigma_{\text{D}_3}}$ is defined by the system
$$\begin{cases} -K_2^2 + 2K_4 = 0 \\ K_6 = 0 \\ K_{10} = 0 \end{cases}$$
- The strata $\overline{\Sigma_{\text{C}_3}}$ is defined by the system

$$\begin{cases} K_2^6 - 8K_2^4 K_4 + 6K_2^3 K_6 + 21K_2^2 K_4^2 - 18K_2 K_4 K_6 - 18K_4^3 + 27K_6^2 = 0 \\ -K_2^5 + 5K_2^3 K_4 + 3K_2^2 K_6 - 6K_2 K_4^2 - 18K_4 K_6 + 27K_{10} = 0 \end{cases}$$

- The strata $\overline{\Sigma_{\text{C}_2}}$ is defined by the system

$$\begin{cases} -2K_2^6 + 14K_2^4 K_4 - 6K_2^3 K_6 - 32K_2^2 K_4^2 + 12K_2 K_4 K_6 + 24K_4^3 + 9K_6^2 = 0 \\ -K_2^5 + 5K_2^3 K_4 - 6K_2^2 K_6 - 6K_2 K_4^2 + 9K_4 K_6 + 9K_{10} = 0 \end{cases}$$

- The rest is in $\Sigma_{\{e\}}$.

Proof. Propositions 4.5, 4.7, 4.11 and 4.13 give $\mathbf{P} \in \mathcal{H}_3 \Leftrightarrow \phi_0^* D_{00}(\mathbf{P}) = \phi_1^* D_{11}(\mathbf{P}) = Q_3(\mathbf{P}) = \phi_3^* T_2 = 0$.

First, we notice the polynomial K_{15} (whose degree and size were the worst) can be ignored. K_2, K_4, K_6 and K_{10} are in fact $\text{O}_3(\mathbb{R})$ -invariants and separate the orbits of $\text{O}_3(\mathbb{R})$ in \mathcal{H}_3 [OA14, Theorem II.I]. Thus the evaluation of K_2, K_4, K_6 and K_{10} characterizes the isotropy class in $\text{O}_3(\mathbb{R})$. However, take $\text{J} < \text{O}_3(\mathbb{R})$ an isotropy group. It determines a unique isotropy group in $\text{SO}_3(\mathbb{R})$ by $\text{J}^+ = \text{J} \cap \text{SO}_3(\mathbb{R}) < \text{SO}_3(\mathbb{R})$. Take $\text{J}_1, \text{J}_2 < \text{O}_3(\mathbb{R})$ two isotropy groups with $g\text{J}_1 g^{-1} = \text{J}_2$ for some $g \in \text{O}_3(\mathbb{R})$. Suppose that $\det(g) = -1$. Then $\det(-g) = 1$ and $-g \in \text{SO}_3(\mathbb{R})$ satisfies still $-g\text{J}_1^+ (-g)^{-1} = \text{J}_2^+$. Hence, the isotropy class in $\text{O}_3(\mathbb{R})$ characterizes the isotropy class in $\text{SO}_3(\mathbb{R})$. We deduce that the polynomials K_2, K_4, K_6 and K_{10} suffice to stratify the orbit space.

Next, we apply for each isotropy group J the same reasoning that can be found in [AKP14]. We find a subvariety \mathcal{X} representing all the orbits of isotropy group J , with a nice expression in the spherical coordinates $(y_{-3,3}, y_{-2,3}, y_{-1,3}, y_{0,3}, y_{1,3}, y_{2,3}, y_{3,3})$ on \mathcal{H}_3 . Then the ideal of relation of the restrictions of K_2, K_4, K_6 and K_{10} on \mathcal{X} defines Σ_{J} . Generators of this ideal are computed by an algorithm involving Groebner bases provided in [CLO15, Section 2.8].

- Assume that $\text{G}_{\mathbf{P}} \cong \text{C}_2$. Up to $\text{SO}_3(\mathbb{R})$, $\text{G}_{\mathbf{P}}$ is generated by $\left(\begin{array}{c|c} g & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & \begin{matrix} 0 \\ 1 \end{matrix} \end{array} \right)$ where $g \in \text{SO}_2(\mathbb{R})$ is

the rotation of angle $\pi/2$. Then $\text{G}_{\mathbf{P}}$ stabilizes \mathbf{P} iff $\mathbf{P} \in \mathcal{X} = \{y_{-1,3} = y_{1,3} = y_{-3,3} = y_{3,3} = 0\}$. The explicit expressions of the restrictions of K_2, K_4, K_6 and K_{10} on \mathcal{X} are

$$\begin{cases} K_2|_{\mathcal{X}} = 10y_{0,3}^2 + 6y_{-2,3}y_{2,3} & K_4|_{\mathcal{X}} = 44y_{0,3}^4 + 72y_{0,3}^2 y_{2,3} y_{-2,3} + 12y_{2,3}^2 y_{-2,3}^2 \\ K_6|_{\mathcal{X}} = 64y_{0,3}^2 (y_{0,3}^2 - y_{-2,3}y_{2,3})^2 & K_{10}|_{\mathcal{X}} = 1024y_{0,3}^4 (y_{0,3}^2 - y_{-2,3}y_{2,3})^3 \end{cases}$$

The ideal of relations of $\{K_2|_{\mathcal{X}}, K_4|_{\mathcal{X}}, K_6|_{\mathcal{X}}, K_{10}|_{\mathcal{X}}\}$ is generated by

$$\left\{ \begin{array}{l} -2K_2^6 + 14K_2^4K_4 - 6K_2^3K_6 - 32K_2^2K_4^2 + 12K_2K_4K_6 + 24K_4^3 + 9K_6^2 \\ -K_2^5 + 5K_2^3K_4 - 6K_2^2K_6 - 6K_2K_4^2 + 9K_4K_6 + 9K_{10} \end{array} \right\}.$$

• Assume that $G_{\mathbf{P}} \cong C_3$. Up to $SO_3(\mathbb{R})$, $\mathbf{P} \in \mathcal{X} = \{y_{0,3} = y_{1,3} = y_{-1,3} = y_{2,3} = y_{-2,3} = 0\}$. We obtain the restrictions

$$\begin{array}{ll} K_2|_{\mathcal{X}} = 10y_{0,3}^2 + 4y_{-3,3}y_{3,3} & K_4|_{\mathcal{X}} = 44y_{0,3}^4 + 16y_{0,3}^2y_{3,3}y_{-3,3} + 8y_{3,3}^2y_{-3,3}^2 \\ K_6|_{\mathcal{X}} = 16y_{0,3}^2(2y_{0,3}^2 - y_{-3,3}y_{3,3})^2 & K_{10}|_{\mathcal{X}} = 128y_{0,3}^4(2y_{0,3}^2 - y_{-3,3}y_{3,3})^3 \end{array}$$

• The dimensions of the spaces of points stabilized by D_2 and \mathcal{T} in \mathcal{H}_3 are both equal to one [IG84, Theorem 3.2]. Whereas, $D_2 < \mathcal{T}$. We deduce that $G_{\mathbf{P}} < \mathcal{T} \Leftrightarrow D_2 < G_{\mathbf{P}}$. Up to conjugation, we then have $\mathbf{P} \in \mathcal{X} = \{y_{0,3} = y_{1,3} = y_{-1,3} = y_{3,3} = y_{-3,3} = 0\}$. We obtain the restrictions

$$\begin{array}{ll} K_2|_{\mathcal{X}} = 6y_{-2,3}y_{2,3} & K_4|_{\mathcal{X}} = 12y_{2,3}^2y_{-2,3}^2 \\ K_6|_{\mathcal{X}} = 0 & K_{10}|_{\mathcal{X}} = 0 \end{array}$$

• Assume that $G_{\mathbf{P}} \cong D_3$. Up to $SO_3(\mathbb{R})$, $\mathbf{P} \in \mathcal{X} = \{y_{0,3} = y_{1,3} = y_{-1,3} = y_{2,3} = y_{-2,3} = 0\}$. We obtain the restrictions

$$\begin{array}{ll} K_2|_{\mathcal{X}} = 4y_{-3,3}y_{3,3} & K_4|_{\mathcal{X}} = 8y_{3,3}^2y_{-3,3}^2 \\ K_6|_{\mathcal{X}} = 0 & K_{10}|_{\mathcal{X}} = 0 \end{array}$$

• Assume that $G_{\mathbf{P}} \cong SO_2(\mathbb{R})$. Up to $SO_3(\mathbb{R})$, $\mathbf{P} \in \mathcal{X} = \{y_{1,3} = y_{-1,3} = y_{2,3} = y_{-2,3} = y_{3,3} = y_{-3,3} = 0\}$. We obtain the restrictions

$$\begin{array}{ll} K_2|_{\mathcal{X}} = 10y_{0,3}^2 & K_4|_{\mathcal{X}} = 44y_{0,3}^4 \\ K_6|_{\mathcal{X}} = 64y_{0,3}^6 & K_{10}|_{\mathcal{X}} = 1024y_{0,3}^{10} \end{array}$$

All the computations have been made with Maple. □

4.5 Conclusion

Finally, the union of separating sets provides a system of invariants separating orbits in the whole of $\mathbb{P}ie_z$:

Theorem 4.16. *The system of invariants $\mathcal{F} = \{(\mathcal{Q}_0, \phi_0^*\mathcal{F}_0), (\mathcal{Q}_1, \phi_1^*\mathcal{F}_1), (\mathcal{Q}_2, \phi_2^*\mathcal{F}_2), (\mathcal{Q}_3, \phi_3^*\mathcal{F}_3), (\mathcal{Q}_4, \mathcal{F}_4)\}$ separates the orbits of the representation of $SO_3(\mathbb{R})$ on $\mathbb{P}ie_z$. Its final cardinality is $40 + 28 + 17 + 12 + 5 = 102$.*

By Proposition 2.5, one can deduce a separating system of polynomials $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{F}_4)$ of cardinality 102. This is a significant progress compared to the results already published in the literature. For the representation on $\mathbb{P}ie_z$, [Oli14] provides a generating set (a set of invariant generating the algebra $\mathbb{R}[\mathbb{P}ie_z]^{SO_3(\mathbb{R})}$) of cardinal 495. By elimination techniques, [Che+19] selects a subset of 260 invariants separating the orbits.

The low cardinality obtained in this paper allows new computations. For instance, it is the first time that separating polynomials are simple enough to deduce an entire stratification of $\mathbb{P}ie_z$.

A Expand polynomials on \mathcal{Q}_3 to continuous maps on \mathcal{V}

In Section 3.3, a Seshadri slice is provided for the action of $SO_3(\mathbb{R})$ on $\mathcal{Q}_3 = \mathcal{S}^o \oplus \mathcal{H} \subset \mathcal{H}_2 \oplus \mathcal{H}$. \mathcal{S}^o is the variety of trace free symmetric matrices with two identic eigenvalues and \mathcal{H} is an arbitrary representation of $SO_3(\mathbb{R})$. It allows to construct a set $\phi_3^*\mathcal{F}_3$ of rational functions defined on \mathcal{Q}_3 and separating orbits in $G\mathcal{Z}_2$, where

$$\mathcal{Z}_2 = \mathbb{R}^* \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_S \oplus \mathcal{H}.$$

Selecting only the numerators, Proposition 2.5 deduces a set of separating polynomials $\mathcal{P}_3 \subset \mathbb{R}[\mathcal{Q}_3]^{\text{SO}_3(\mathbb{R})}$. In this section, we expand them to continuous invariant functions defined on $\mathcal{V} = \mathcal{H}_2 \oplus \mathcal{H}$ which still separate orbits in GZ_2 .

This construction can be quite important. It is needed to obtain, by concatenation, a continuous separating map $\mathcal{V} \rightarrow \mathcal{V}/\text{SO}_3(\mathbb{R})$. Such a map allows in particular to design invariant neural networks.

Proposition A.1. *For $M \in \mathcal{S}$ a traceless symmetric matrix, take $O \in \text{SO}_3(\mathbb{R})$ such that*

- *If $\det(M) = 0$, then $O = I_3$.*
- *Else, OMO^t is diagonal and its two last coefficients are of the same sign.*

Then the map $\Phi : \begin{cases} \mathcal{S} \oplus \mathcal{H} & \rightarrow \overline{\mathcal{Z}}_3/\text{SO}_3(\mathbb{R}) \\ (M, H) & \mapsto [\det(M)(M, OH)] \end{cases}$ is well defined, continuous, invariant for $\text{SO}_3(\mathbb{R})$ and its restriction to \mathcal{Q}_3 separates orbits in \mathcal{Q}_3 .

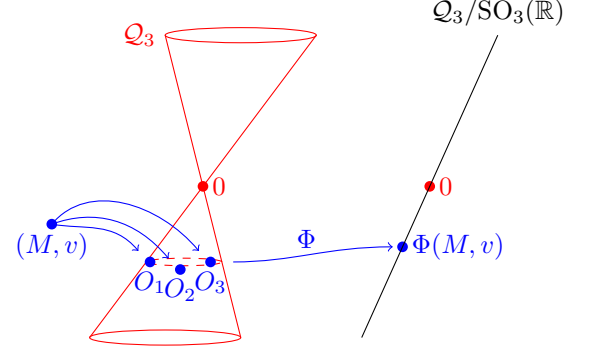


Figure 3: Application Φ

Proof. Let (M, H) be a point in $\mathcal{S} \oplus \mathcal{H}$.

• Assume that O_1, O_2 are two matrices in $\text{SO}_3(\mathbb{R})$ diagonalizing M such that the two last eigenvalues have the same sign. Note $O_1MO_1^t = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$. Then, $O_2MO_2^t = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$ or $\text{Diag}(\lambda_1, \lambda_3, \lambda_2)$. However, $O_2MO_2^t = O_2O_1^tO_1MO_1^tO_1O_2^t = O_2O_1^t\text{Diag}(\lambda_1, \lambda_2, \lambda_3)O_1O_2^t$. Recall that λ_1 cannot be equal to λ_2 or λ_3 , having the opposite sign. We thus have necessarily $O_2O_1^t \in \text{N}$. This is, $O_2O_1^t$ stabilises S by conjugation, and one has

$$\begin{aligned} \rho(O_2O_1^t)(\det(M)(S, O_1H)) &= \det(M)(O_2O_1^tSO_1O_2^t, O_2O_1^tO_1H) \\ &= \det(M)(M, O_2H) \end{aligned}$$

Then $\det(M)(S, O_1H) \in [\det(M)(S, O_2H)]$ and Φ is well defined.

• Let O be an element of $\text{SO}_3(\mathbb{R})$ diagonalizing M such that the two last eigenvalues have the same sign, and $g \in \text{SO}_3(\mathbb{R})$. Then, Og^t diagonalizes gMg^t such that the two last eigenvalues have the same sign. We thus have $\Phi \circ \rho(g)(M, H) = [\Phi(gMg^t, gH)] = [\det(M)(S, Og^t gH)] = [\det(M)(S, OH)] = \Phi(M, H)$. This is, Φ is invariant for $\text{SO}_3(\mathbb{R})$.

• Take Q_1, Q_2 two point in \mathcal{Q}_3 . They have $\lambda_1(S, H_1), \lambda_2(S, H_2)$ representants of their orbits in $\overline{\mathcal{Z}}_3$. Since Φ is invariant, $\Phi(Q_1) = \Phi(\lambda_1(S, H_1)) = [\lambda_1(S, H_1)]$ and $\Phi(Q_2) = [\lambda_2(S, H_2)]$. Thus, Q_1 and Q_2 are in the same orbit iff $\Phi(Q_1) = \Phi(Q_2)$. \square

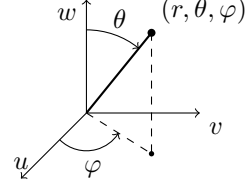
Corollary A.2. *Suppose that $\mathcal{P} = \{P_1, \dots, P_n\} \in \mathbb{R}[\mathcal{Q}_3]$ are invariant polynomials separating orbits of G in \mathcal{Q}_3 . Then $\{P_1 \circ \Phi, \dots, P_n \circ \Phi\}$ are invariant continuous functions on $\mathcal{H}_2 \oplus \mathcal{H}$, which separates orbits in \mathcal{Q}_3 .*

An explicit formula of these functions is difficult to find. However, there is no difficulty to evaluate them at any point in \mathcal{V} , computing a matrix O diagonalizing M .

B Explicit expression of cubic and spherical harmonic bases

For the needs of the paper, we introduced two different bases of the harmonic spaces \mathcal{H}_d , namely the spherical and cubic harmonics. Spherical harmonics are defined through spherical coordinates:

$$\begin{aligned}
u &= r \sin(\theta) \cos(\varphi) \\
v &= r \sin(\theta) \sin(\varphi) \\
w &= r \cos(\theta)
\end{aligned}$$



Definition B.1. [Wig59; HJ24] For $0 \leq m \leq d$, note $c_{m,d} = \sqrt{\frac{(2d+1)(d-m)!}{4\pi(d+m)!}}$ a normalisation constant and $P_{m,d} = (1-t^2)^{\frac{m}{2}} \frac{1}{2^d d!} \frac{\partial^{d+m}}{\partial t^{d+m}} (t^2-1)^d$ is the $(d,m)^{th}$ associated Legendre polynomial. A basis for $\hat{\mathcal{H}}_d$ is

$$\mathcal{B}_{sp}^{(d)} = \left\{ \begin{array}{l} Y_{-m,d}(r, \theta, \varphi) = i^{-d} c_{m,d} r^d e^{-im\varphi} P_{m,d}(\cos(\theta)), \quad 1 \leq m \leq d, \\ Y_{0,d}(r, \theta, \varphi) = c_d^0 r^d P_d^0(\cos(\theta)), \\ Y_{m,d}(r, \theta, \varphi) = i^d c_{m,d} r^d e^{im\varphi} P_{m,d}(\cos(\theta)), \quad 1 \leq m \leq d. \end{array} \right\}$$

Example B.2. The vector spaces $\hat{\mathcal{H}}_1$, $\hat{\mathcal{H}}_2$ and $\hat{\mathcal{H}}_3$ are endowed with the bases

$$\begin{aligned}
\mathcal{B}_{sp}^{(1)} &= \left\{ Y_{-1,1} = -\sqrt{\frac{3}{2\pi}} (iU + V), Y_{0,1} = \sqrt{\frac{3}{2\pi}} W, Y_{1,1} = \sqrt{\frac{3}{2\pi}} (iU - V) \right\}, \\
\mathcal{B}_{sp}^{(2)} &= \left\{ \begin{array}{l} Y_{-2,2} = \frac{\sqrt{15}}{4\sqrt{\pi}} (i2UV - V^2 + U^2), \quad Y_{2,2} = -\frac{\sqrt{15}}{4\sqrt{\pi}} (i2UV + V^2 - U^2), \\ Y_{-1,2} = \frac{\sqrt{15}}{2\sqrt{\pi}} (iVW - UW), \quad Y_{1,2} = -\frac{\sqrt{15}}{2\sqrt{\pi}} (iVW + UW), \\ Y_{0,2} = \frac{\sqrt{5}}{4\sqrt{\pi}} (2W^2 - U^2 - V^2) \end{array} \right\}, \\
\mathcal{B}_{sp}^{(3)} &= \left\{ \begin{array}{l} Y_{-3,3} = \frac{\sqrt{35}}{4\sqrt{\pi}} (3U^2V - V^3 + i(U^3 - 3UV^2)), \\ Y_{3,3} = \frac{\sqrt{35}}{4\sqrt{\pi}} (3U^2V - V^3 - i(U^3 - 3UV^2)), \\ Y_{-2,3} = \sqrt{\frac{105}{4\pi}} (2UVW + iW(U^2 - V^2)), \\ Y_{2,3} = \sqrt{\frac{105}{4\pi}} (2UVW - iW(U^2 - V^2)), \\ Y_{-1,3} = \frac{\sqrt{21}}{8\sqrt{\pi}} (4W^2 - U^2 - V^2) (V + iU), \\ Y_{1,3} = \frac{\sqrt{21}}{8\sqrt{\pi}} (4W^2 - U^2 - V^2) (V - iU), \\ Y_{0,3} = \frac{\sqrt{7}}{4\sqrt{\pi}} (2W^2 - 3U^2 - 3V^2) W \end{array} \right\}.
\end{aligned}$$

These bases fit particularly well with the action of $N_0 \cong O_2(\mathbb{R})$ in Section 4.1. However, in Section 4.2, we need another basis to fit with the action of $N_2 \cong \mathcal{O}$: the cubic harmonic basis. Only the cubic basis of \mathcal{H}_3 is required, but one finds a general way to construct them in [FK77].

Proposition B.3. The following set is an orthonormal basis for \mathcal{H}_3

$$\mathcal{B}_{cu}^{(3)} = \left\{ \begin{array}{l} \alpha_1 = \frac{\sqrt{7}}{4\pi} U(2U^2 - 3V^2 - 3W^2) \quad \frac{\sqrt{105}}{4\pi} \beta_1 = U(V^2 - W^2) \\ \alpha_2 = \frac{\sqrt{7}}{4\pi} V(2V^2 - 3U^2 - 3W^2) \quad \frac{\sqrt{105}}{4\pi} \beta_2 = V(W^2 - U^2) \quad \frac{\sqrt{105}}{2\pi} \gamma = UVW \\ \alpha_3 = \frac{\sqrt{7}}{4\pi} W(2W^2 - 3V^2 - 3U^2) \quad \frac{\sqrt{105}}{4\pi} \beta_3 = W(U^2 - V^2) \end{array} \right\}$$

The polynomials of Theorem 4.10 are given through cubic coordinates. If one wants to get them in spherical coordinates to be coherent through the whole paper, computations with Maple provides the base

change from $\mathcal{B}_{sp}^{(3)}$:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma \end{pmatrix} = \begin{pmatrix} -\frac{i}{4}\sqrt{5} & 0 & \frac{i}{4}\sqrt{3} & 0 & -\frac{i}{4}\sqrt{3} & 0 & \frac{i}{4}\sqrt{5} \\ -\frac{\sqrt{5}}{4} & 0 & -\frac{\sqrt{3}}{4} & 0 & -\frac{\sqrt{3}}{4} & 0 & -\frac{\sqrt{5}}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{i}{4}\sqrt{3} & 0 & \frac{i}{4}\sqrt{5} & 0 & -\frac{i}{4}\sqrt{5} & 0 & -\frac{i}{4}\sqrt{3} \\ -\frac{\sqrt{3}}{4} & 0 & \frac{\sqrt{5}}{4} & 0 & \frac{\sqrt{5}}{4} & 0 & -\frac{\sqrt{3}}{4} \\ 0 & -\frac{i}{2}\sqrt{2} & 0 & 0 & 0 & \frac{i}{2}\sqrt{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{pmatrix} Y_{-3,3} \\ Y_{-2,3} \\ Y_{-1,3} \\ Y_{0,3} \\ Y_{1,3} \\ Y_{2,3} \\ Y_{3,3} \end{pmatrix}$$

In Section 4.4, \mathcal{H}_3 is also described as the space of trace free symmetric tensors of order three A_{ijk} . The expressions of spherical harmonics with respect to these tensors is given by

$$\begin{aligned} Y_{-3,3} &= \frac{\sqrt{35}}{4\sqrt{\pi}} (3A_{112} - A_{222} + i(A_{111} - 3A_{122})), \\ Y_{3,3} &= \frac{\sqrt{35}}{4\sqrt{\pi}} (3A_{122} - A_{222} - i(A_{111} - 3A_{122})), \\ Y_{-2,3} &= \sqrt{\frac{105}{4\pi}} (2A_{123} + i(A_{113} - A_{223})), \\ Y_{2,3} &= \sqrt{\frac{105}{4\pi}} (2A_{123} - i(A_{113} - A_{223})), \\ Y_{-1,3} &= \frac{\sqrt{21}}{8\sqrt{\pi}} (4A_{233} - A_{112} - A_{222} + i(A_{111} + A_{122} - 4A_{133})), \\ Y_{1,3} &= \frac{\sqrt{21}}{8\sqrt{\pi}} (4A_{233} - A_{112} - A_{222} - i(A_{111} + A_{122} - 4A_{133})), \\ Y_{0,3} &= \frac{\sqrt{7}}{4\sqrt{\pi}} (2A_{333} - 3A_{113} - 3A_{223}) \end{aligned}$$

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