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# Continuous-time modeling and bootstrap for chain-ladder reserving

Nicolas Baradel\*

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## Abstract

We revisit the famous Mack's model which gives an estimate for the conditional mean squared error of prediction of the chain-ladder claims reserves. We introduce a stochastic differential equation driven by a Brownian motion to model the accumulated total claims amount for the chain-ladder method. Within this continuous-time framework, we propose a bootstrap technique for estimating the distribution of claims reserves. It turns out that our approach leads to inherently capturing asymmetry and non-negativity, eliminating the necessity for additional assumptions. We conclude with a case study and comparative analysis against alternative methodologies based on Mack's model.

## 1 Introduction

The chain-ladder technique is a cornerstone of reserving in non-life insurance. Several stochastic frameworks have been developed to derive the chain-ladder reserve estimator. One notable approach employs maximum likelihood estimation within an over-dispersed Poisson generalized linear model, yielding equivalent reserve estimates [7], [10], and [8].

Mack's model [11] provides a distribution-free approach to obtain estimators and claims reserves similar to the chain-ladder method, relying on minimal assumptions about conditional moments. This framework introduces a stochastic model that also enables estimation of the conditional Mean Squared Error of Prediction (MSEP), a measure of prediction uncertainty.

Several studies have explored Mack's model, some adopting stronger assumptions aligned with Mack's framework. For example, [2] introduced a time series for claims development with independent and identically distributed noise satisfying Mack's assumptions. However, this approach may produce negative values for incurred claims, an issue that cannot be resolved by conditioning the noise without compromising increment independence, as noted in [12].

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In this paper, we introduce a continuous-time model for claims development based on a well chosen stochastic differential equation driven by Brownian motion. We demonstrate that our model adheres to Mack's assumptions, and in a specific scenario, we can leverage all of Mack's estimators. The primary advantage lies in our ability to simulate total claims reserves using a parametric bootstrap method, which inherently incorporates asymmetry and non-negativity without the need for residual computation or additional assumptions.

Several studies have investigated continuous-time frameworks for the chain-ladder method. For instance, [14] proposes a continuous framework for loss reserving in non-life insurance, utilizing a bivariate density approach with a kernel-based method to model individual claims within the development triangle, aiming to estimate the distribution in the lower triangle. [1] uses a continuous-time approach with marked point processes to capture the timing and magnitude of individual claim payments. In contrast, our work focuses on aggregated data, introducing a continuous-time stochastic process with continuous paths to model the evolution of aggregated incurred claims, satisfying Mack's assumptions.

The paper is organized as follows. Section 2 presents Mack's general model, with a review of key estimators. Section 3 introduces our continuous-time model for the accumulated total claims amount, for which we derive several properties and establishing its connection to Mack's model. Section 4 describes a bootstrap procedure tailored to the continuous-time model, addressing uncertainty in parameter estimation. Finally, Section 5 provides a case study that assesses the impact of the continuous-time framework and compares it with alternative approaches based on Mack's model.

## 2 Mack's model

Mack's model provides a probabilistic framework that aligns with the chain-ladder method. It calculates the conditional MSEP for reserves without requiring a specific distribution: it imposes constraints on the conditional moments of the underlying process.

The model introduces the process  $(C_{i,j})_{1 \leq i,j \leq n}$  which represents the accumulated total claims amount for both occurrence year  $i$  and development year  $j$  across  $n$  years of observations. For each  $1 \leq k \leq n$ , we define:

$$\mathcal{F}_k^i := \sigma(C_{i,j}, j \leq k), \quad 1 \leq i \leq n.$$

We make the following assumption:

### Assumption 2.1.

*H1 The random variables  $(C_{i_1,j})_{1 \leq j \leq n}$  and  $(C_{i_2,j})_{1 \leq j \leq n}$  are independent for  $i_1 \neq i_2$ .*

*H2 For  $1 \leq j \leq n - 1$ , there exists  $F_j > 0$  (deterministic constant) such that*

$$\mathbb{E}(C_{i,j+1} \mid \mathcal{F}_j^i) = F_j C_{i,j}, \quad 1 \leq i \leq n.$$

H3 For  $1 \leq j \leq n-1$ , there exists  $\Sigma_j \geq 0$  (deterministic constant) such that

$$\text{Var}(C_{i,j+1} \mid \mathcal{F}_j^i) = \Sigma_j^2 C_{i,j}, \quad 1 \leq i \leq n.$$

From the above assumption, we can derive the general expressions for the first two moments across all dates:

**Lemma 2.2.** For all  $1 \leq i \leq n$  and  $s \leq j \leq n$ ,

$$\begin{aligned} \mathbb{E}(C_{i,j} \mid \mathcal{F}_s^i) &= \left( \prod_{k=s}^{j-1} F_k \right) C_{i,s}, \\ \text{Var}(C_{i,j} \mid \mathcal{F}_s^i) &= \left( \sum_{k=s}^{j-1} \left[ \left( \prod_{\ell=k+1}^{j-1} F_\ell^2 \right) \Sigma_k^2 \left( \prod_{\ell=s}^{k-1} F_\ell \right) \right] \right) C_{i,s}. \end{aligned}$$

Mack provides accurate estimators for both the  $F$ 's and the  $\Sigma^2$ 's:

$$\begin{aligned} \widehat{F}_j &:= \frac{\sum_{i=1}^{n-j} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{i,j}}, & 1 \leq j \leq n-1, \\ \widehat{\Sigma}_j^2 &:= \frac{1}{n-j-1} \sum_{i=1}^{n-j} C_{i,j} \left( \frac{C_{i,j+1}}{C_{i,j}} - \widehat{F}_j \right)^2, & 1 \leq j \leq n-2. \end{aligned} \tag{2.1}$$

Several methods exist for estimating  $\Sigma_{n-1}^2$ , as discussed in [11]. Mack also provides an unbiased estimator for the ultimate value:

$$\widehat{C}_{i,n} := C_{i,n-i+1} \left( \prod_{k=n-i+1}^{n-1} \widehat{F}_k \right), \quad 2 \leq i \leq n,$$

which consequently leads to the reserve estimator:

$$\widehat{R} := \sum_{i=2}^n \widehat{C}_{i,n} - C_{i,n-i+1}.$$

Moreover, he presents an estimator for the conditional MSEP of the reserves, accounting for uncertainty arising from parameter estimation. An alternative method to assess the conditional MSEP involves employing a bootstrap approach. For a comprehensive introduction to this technique in the realm of insurance reserving, refer to [4]. Unlike solely estimating the conditional MSEP of the reserve, bootstrap analysis offers insight into the entire distribution.

The aim of this paper is to establish a continuous-time framework using stochastic differential equations that fulfill Assumption 2.1.

In [2], a time series methodology was employed, yielding the following model:

$$C_{i,j+1} = F_j C_{i,j} + \Sigma_j \sqrt{C_{i,j}} \varepsilon_{i,j}, \tag{2.2}$$

where the  $\varepsilon$ 's represent independent variables with a mean of zero and a variance of one. A key limitation of the formulation above is that sampling from (2.2) may yield negative values for the  $C$ 's terms, which is inconsistent. While the authors propose

using a conditional distribution for  $\varepsilon_{i,j}$  based on  $C_{i,j-1}$ , this approach, as noted in [12], compromises the independence assumption for the  $\varepsilon$ 's terms, rendering it unsuitable.

Our framework, detailed in the following section, extends the yearly-based model in (2.2) to a continuous-time setting, effectively addressing the issue of negative values.

### 3 A continuous-time model

Let  $\Omega_W := C([1, 2n], \mathbb{R}^n)$  denote the space of continuous functions mapping  $[1, 2n]$  to  $\mathbb{R}^n$ , where functions start with value 0 at 1. We denote by  $W(\omega) = \omega$  the canonical process and let  $\mathbb{P}_W$  be the Wiener measure defined on the Borel sets of  $\Omega_W$ . Consequently,  $W = (W^i)_{1 \leq i \leq n}$  comprises  $n$  independent Brownian motions. Let  $\Omega_1$  be a Polish space and  $\mathbb{P}_1$  a Borel measure on  $\Omega_1$ . Finally, we define  $\Omega := \Omega_1 \times \Omega_W$  and the product measure  $\mathbb{P} := \mathbb{P}_1 \otimes \mathbb{P}_W$  on the Borel sets of  $\Omega$ .

We introduce the following filtrations, which represent the knowledge at development time  $t$  for an occurrence year  $i$ :

$$\mathcal{F}_t^i := \sigma(C_1^i; W_s^i, s \leq t), \quad 1 \leq i \leq n, 1 \leq t \leq n,$$

in which the  $(C_1^i)_{1 \leq i \leq n}$  are random variables defined on  $\Omega_1$  and valued in  $\mathbb{R}_+$ . We define the filtration of the entire knowledge at time  $t \in [1, n]$ .

$$\mathcal{F}_t := \sigma(C_1^i, i \leq t; W_s^i, i + s \leq t + 1, s \leq n), \quad 1 \leq t \leq 2n.$$

Hereafter, all random variables are considered within the probability space  $(\Omega, \mathcal{F}_{2n})$ . Let  $(C_t^i)_{t \in [1, n]}^{i \in \{1, \dots, n\}}$  represent the processes of accumulated total claims amount for occurrence year  $i$  at development date  $t$ .

#### Assumption 3.1.

*H1'* The random variables  $(C_1^i)_{1 \leq i \leq n}$  are square integrable and independent.

*H2'* There exist two measurable and bounded functions  $f : [1, n] \rightarrow \mathbb{R}$  and  $\sigma : [1, n] \rightarrow \mathbb{R}_+$  such that, for all  $1 \leq i \leq n$ ,  $(C_t^i)_{1 \leq t \leq n}$  is the unique strong solution of the stochastic differential equation:

$$C_t^i = C_s^i + \int_s^t f_u C_u^i du + \int_s^t \sigma_u \sqrt{C_u^i} dW_u^i, \quad 1 \leq s \leq t \leq n. \quad (3.1)$$

The processes  $C^i$  are well-defined by (3.1) since these stochastic differential equations possess a unique (non-negative) strong solution, as established in, for instance, [17] or [13, Theorem 4.6.11]. Furthermore, they satisfy:

$$\mathbb{E} \left[ \sup_{1 \leq t \leq n} (C_t^i)^2 \right] < +\infty, \quad 1 \leq i \leq n. \quad (3.2)$$

**Remark 3.2.** In the specific case where the coefficients  $f$  and  $\sigma$  are constant, this process is referred to as the Feller diffusion, originally introduced in [5], see for example: [15], [6], and [13].

The above process bears resemblance to the *Cox-Ingersoll-Ross* process commonly employed in finance, yet it distinguishes itself by lacking mean reversion. Notably, it is well-known in population dynamics studies, as it can be interpreted as the limit of the *Galton-Watson branching process*. Its primary characteristic is the *branching property*. This property is also satisfied in the classical Mack chain-ladder model and we find it again in the continuous-time model in a general form.

**Lemma 3.3.** *The processes  $(C_t^i)_{1 \leq t \leq n}$  satisfy the branching property: if for  $1 \leq i \leq n$ ,  $(C_t^{i'})_{1 \leq t \leq n}$  is another independent process satisfying (3.1) with a different Brownian motion, then  $(C_t^i + C_t^{i'})_{1 \leq t \leq n}$  also satisfies (3.1) with yet another Brownian motion.*

*Proof.* This property is standard when considering constant coefficients. For instance, refer to [13, Proposition 4.7.1] or, for a more general approach, refer to [9]. With bounded time-dependent coefficients, the proof remains straightforward, without any significant differences.  $\square$

**Remark 3.4.** *The branching property of Lemma 3.3 above implies the following consequence: if we consider a portfolio consisting of  $d$  independent components, each governed by the dynamics defined in (3.1) with identical parameters  $f$  and  $\sigma$ , then the aggregation of these  $d$  components will also exhibit the dynamics described by (3.1). Consequently, it will yield the same aggregated reserve distribution. Similarly, dividing a portfolio into two homogeneous independent sub-portfolios maintains the same dynamics and, consequently, the same aggregated reserve distribution. Implicit in this assertion is the assumption that all constituents of a portfolio are independent.*

**Remark 3.5.** *We began defining the process at  $t = 1$ , with the initial condition  $(C_1^i)_{1 \leq i \leq n}$  as random variable. This approach aligns with the Mack's general framework, as we make no assumptions about  $(C_1^i)$  other than its implicit squared integrability. Additionally, extending the process  $(C_t^i)_{1 \leq t \leq n}$  defined in (3.1) back to  $t = 0$  would require  $C_0^i > 0$ , which is not relevant. Implicitly, the randomness of  $(C_1^i)_{1 \leq i \leq n}$ , corresponding to the year of occurrence, follows a different process. This process does not need to be defined for the chain-ladder technique to derive the reserves and their conditional MSEP or distribution, conditional on the current information. However, it should be defined in order to simulate  $C_n^{n+1}$ .*

We now derive the first two conditional moments of the  $C$ 's to verify Assumption 2.1.

**Proposition 3.6.** *The first two conditional moments of the processes  $(C_t^i)_{1 \leq t \leq n}$  are, for all  $1 \leq s \leq t \leq n$  and  $1 \leq i \leq n$ ,*

$$\begin{aligned}\mathbb{E}(C_t^i | \mathcal{F}_s^i) &= C_s^i e^{\int_s^t f_u du}, \\ \text{Var}(C_t^i | \mathcal{F}_s^i) &= C_s^i \int_s^t \sigma_u^2 e^{\int_s^u f_z dz + \int_u^t 2f_z dz} du.\end{aligned}$$

*Proof.* Fix  $i \in \{1, \dots, n\}$ . For convenience, denote  $C^i$  as  $C$ ,  $\mathcal{F}^i$  for  $\mathcal{F}$ , and  $W^i$  as  $W$  throughout this proof.

1. Applying the expected value operator  $\mathbb{E}$  to (3.1) and utilizing (3.2) for the local martingale yields:

$$\mathbb{E}(C_t | \mathcal{F}_s) = C_s + \int_s^t f_u \mathbb{E}(C_u | \mathcal{F}_s) du.$$

This forms a simple linear homogeneous ordinary differential equation with the unique solution:

$$\mathbb{E}(C_t | \mathcal{F}_s) = C_s e^{\int_s^t f_u du}. \quad (3.3)$$

2. Itô's formula gives:

$$C_t^2 = C_s^2 + 2 \int_s^t f_u C_u^2 du + 2 \int_s^t \sigma_u C_u \sqrt{C_u} dW_u + \int_s^t \sigma_u^2 C_u du.$$

Introducing the stopping times  $T_m := \inf\{t \geq s : C_t = m\}$ , which tends to infinity a.s. as  $m \rightarrow +\infty$ , we apply the expected value operator:

$$\mathbb{E}(C_{t \wedge T_m}^2 | \mathcal{F}_s) = C_s^2 + 2\mathbb{E}\left(\int_s^{t \wedge T_m} f_u C_u^2 du | \mathcal{F}_s\right) + \mathbb{E}\left(\int_s^{t \wedge T_m} \sigma_u^2 C_u du | \mathcal{F}_s\right). \quad (3.4)$$

Taking the limit as  $m \rightarrow +\infty$ , and using (3.2) along with the dominated convergence theorem, we obtain:

$$\mathbb{E}(C_t^2 | \mathcal{F}_s) = C_s^2 + 2 \int_s^t f_u \mathbb{E}(C_u^2 | \mathcal{F}_s) du + \int_s^t \sigma_u^2 \mathbb{E}(C_u | \mathcal{F}_s) du. \quad (3.5)$$

From (3.3), we have

$$\mathbb{E}(C_t | \mathcal{F}_s)^2 = C_s^2 e^{2 \int_s^t f_u du}$$

thus,  $t \mapsto \mathbb{E}(C_t | \mathcal{F}_s)^2$  satisfies the following ordinary differential equation:

$$\mathbb{E}(C_t | \mathcal{F}_s)^2 = C_s^2 + 2 \int_s^t f_u \mathbb{E}(C_u | \mathcal{F}_s)^2 du.$$

Combining it with (3.5) and (3.3) gives:

$$\text{Var}(C_t | \mathcal{F}_s) = 2 \int_s^t f_u \text{Var}(C_u | \mathcal{F}_s) du + C_s \int_s^t \sigma_u^2 e^{\int_s^u f_z dz} du.$$

It is a linear non-homogeneous ordinary differential equation whose unique solution is:

$$\text{Var}(C_t | \mathcal{F}_s) = C_s \int_s^t \sigma_u^2 e^{\int_s^u f_z dz + \int_u^t 2f_z dz} du.$$

□

**Corollary 3.7.** *The processes  $(C_t^i)_{1 \leq t \leq n}$  for  $1 \leq i \leq n$  satisfy assumptions H1, H2 and H3 of Assumption 2.1 by setting, for  $1 \leq j \leq n$ :*

$$F_j := e^{\int_j^{j+1} f_u du},$$

$$\Sigma_j^2 := \int_j^{j+1} \sigma_u^2 e^{\int_j^u f_z dz + \int_u^{j+1} 2f_z dz} du.$$

**Remark 3.8.** *We directly obtain the expected values of the continuous process  $C_t^i$  conditional on  $\mathcal{F}_s^i$ , which correspond to the discrete ones stated in Lemma 2.2.*

There is no need to precisely track the  $(f, \sigma) : t \rightarrow (f_t, \sigma_t)$  function continuously. To simplify matters, we introduce an additional assumption: that the function remains constant over each one-year interval. Consequently, we establish a connection between the estimators derived from the classic framework and our continuous-time framework.

**Assumption 3.9.** *The functions  $f$  and  $\sigma$  are constant on each  $[t, t + 1)$ , i.e., for  $1 \leq t < n$ :*

$$f_t := \sum_{j=1}^{n-1} f_j \mathbf{1}_{[j, j+1)}(t),$$

$$\sigma_t := \sum_{j=1}^{n-1} \sigma_j \mathbf{1}_{[j, j+1)}(t).$$

**Lemma 3.10.** *Under the additional Assumption 3.9, the relation in Corollary 3.7 simplifies to*

$$F_j = e^{f_j} \qquad f_j = \log(F_j)$$

$$\Sigma_j^2 = \frac{\sigma_j^2}{f_j} (e^{2f_j} - e^{f_j}) \iff \sigma_j^2 = \frac{\Sigma_j^2 \log(F_j)}{F_j(F_j - 1)}$$

*Proof.* The proof follows straightforwardly from computing the simple integrals.  $\square$

Note that in (3.1), as  $C_t^i$  approaches zero, both the term preceding  $dt$  and the one preceding  $dW_t^i$  vanish. We will now discuss the conditional distribution of  $C_t^i$ , particularly emphasizing that while it is possible for  $C_t^i$  to reach zero, this occurrence is practically negligible.

**Lemma 3.11.** *Under Assumption 3.9, for  $j \leq t \leq j + 1$  and  $z > -\frac{2f_j}{\sigma_j^2(e^{f_j(t-j)} - 1)}$ , the Laplace transform of  $C_t^i$  conditional on  $\mathcal{F}_j^i$  is given by:*

$$g_{i,j,t}(z) = \mathbb{E} \left( e^{-zC_t^i} \middle| \mathcal{F}_j^i \right) = \exp \left( -\frac{2f_j e^{f_j(t-j)} C_j^i z}{2f_j + \sigma_j^2 (e^{f_j(t-j)} - 1) z} \right).$$

*Proof.* This result is standard; see, for example, [13, Proposition 4.7.1] or [3, Proposition 4.4]. Note that the latter contains a typographical error, omitting factors of 2 in the expression, though its proof remains correct.  $\square$

**Remark 3.12.** *For  $j \leq t \leq j + 1$ , under Assumption 3.9, we have*

$$\mathbb{P}(C_t^i = 0 \mid \mathcal{F}_j^i) = \lim_{z \rightarrow +\infty} g_{i,j,t}(z) = \exp \left( -\frac{2f_j e^{f_j(t-j)} C_j^i}{\sigma_j^2 (e^{f_j(t-j)} - 1)} \right).$$

*This implies that the processes  $(C_t^i)_{t \geq 1}$  can reach 0 (and remain there). However, in practice, as we will observe, this probability is often numerically close to 0, signifying the scenario where all claims ultimately cost 0. Moreover, the distribution of  $C_t^i$ , conditioned to be positive, is continuous.*

In a Cox-Ingersoll-Ross framework, the conditional marginal distributions of the process follow a continuous distribution, specifically a non-central chi-squared distribution. Within our framework, these distributions assign positive probability to zero, yet they are fully characterized as described below.

**Lemma 3.13.** Let  $N \sim \mathcal{P}(\lambda)$  with  $\lambda > 0$  and  $(X_k)_{k \geq 1} \stackrel{i.i.d.}{\sim} \mathcal{E}(\beta)$  with  $\beta > 0$ . Define

$$S := \sum_{k=1}^N X_k.$$

For  $j \leq t \leq j+1$ , if

$$\lambda = \frac{2f_j e^{f_j(t-j)} C_j^i}{\sigma_j^2 (e^{f_j(t-j)} - 1)}, \quad \beta = \frac{2f_j}{\sigma_j^2 (e^{f_j(t-j)} - 1)},$$

then the conditional distribution of  $C_t^i$  given  $\mathcal{F}_j^i$  is identical to that of  $S$  given  $\mathcal{F}_j^i$ .

*Proof.* For  $z > -\beta$ , the Laplace transform of  $S$  is

$$g_S(z) := \mathbb{E}(e^{-zS}) = g_N(-\log(g_X(z))),$$

where  $g_N(z) = e^{\lambda(e^{-z}-1)}$  is the Laplace transform of  $N$  and  $g_X(z) = \frac{\beta}{\beta+z}$  is the Laplace transform of  $X_1$ . Thus,

$$g_S(z) = e^{\frac{-\lambda z}{\beta+z}}.$$

Equating  $g_S$  to  $g_{i,j,t}$  and solving for  $\lambda$  and  $\beta$  yields the result.  $\square$

**Remark 3.14.** The preceding lemma provides an exact method for simulations of  $S$ , eliminating the need for an Euler scheme with fine discretization. Since

$$S | N \sim \mathcal{G}(N, \beta),$$

where  $\mathcal{G}$  denotes the Gamma distribution, one can simulate  $S$  by first generating  $N$  and then sampling from the conditional distribution  $S | N$ . This approach ensures both accuracy and computational efficiency in the simulation process.

This property is highly valuable for bootstrap simulations. Since Mack's assumptions are satisfied, we obtain the same estimators for the reserves and can compute the same conditional MSEF. Our goal is to propose a bootstrap methodology, tailored for our continuous-time framework, which will enable the estimation of the distribution of the reserves.

## 4 The bootstrap methodology

There are the two classical steps:

1. Bootstrapping the parameters: the  $F$ 's and the  $\Sigma$ 's, to account for the *estimation error*;
2. Simulating the lower part of the triangle using the bootstrapped coefficients to incorporate the *process error*.

We adapt the bootstrap approach described in [4] to our framework.

1. For bootstrapping the coefficients,

$$C_{j+1}^{i,m} = C_j^i + \int_j^{j+1} \widehat{f}_j C_u^{i,m} du + \int_j^{j+1} \widehat{\sigma}_j \sqrt{C_u^{i,m}} dW_u^{i,m}, \quad i+j \leq n, \quad 1 \leq m \leq M.$$

Note that the above stochastic differential equation uses  $C_j^i$  as its initial condition, not  $C_j^{i,m}$ . By Lemma 3.13 and Remark 3.14, the conditional simulation of  $C_{j+1}^{i,m}$  is done with:

$$C_{j+1}^{i,m} \sim \mathcal{G}(N_{i,j}^m, \widehat{\beta}_j) \text{ with } N_{i,j}^m \sim \mathcal{P}(\widehat{\lambda}_{i,j}), \quad i+j \leq n, \quad 1 \leq m \leq M,$$

where  $\widehat{\beta}_j := \frac{2\widehat{f}_j}{\widehat{\sigma}_j^2(e^{\widehat{f}_j} - 1)}$  and  $\widehat{\lambda}_{i,j} := \widehat{\beta}_j e^{\widehat{f}_j} C_j^i$ .

We obtain the new estimators  $(\widehat{F}_j^m)$  and  $(\widehat{\Sigma}_j^m)$  defined as, for all  $1 \leq m \leq M$ :

$$\begin{aligned} \widehat{F}_j^m &:= \frac{\sum_{i=1}^{n-j} C_{j+1}^{i,m}}{\sum_{i=1}^{n-j} C_j^i}, & 1 \leq j \leq n-1, \\ (\widehat{\Sigma}_j^m)^2 &:= \frac{1}{n-j-1} \sum_{i=1}^{n-j} C_j^i \left( \frac{C_{j+1}^{i,m}}{C_j^i} - \widehat{F}_j^m \right)^2, & 1 \leq j \leq n-2, \end{aligned} \quad (4.1)$$

and  $\widehat{\Sigma}_{n-1}^m := \min\left(\frac{(\widehat{\Sigma}_{n-2}^m)^2}{\widehat{\Sigma}_{n-3}^m}, \widehat{\Sigma}_{n-3}^m, \widehat{\Sigma}_{n-2}^m\right)$  as proposed in [11]. We then derive  $(\widehat{f}_j^m)$  and  $(\widehat{\sigma}_j^m)$  using Lemma 3.10.

2. For bootstrapping the process error:

$$C_n^{i,m} = C_{n-i+1}^i + \int_{n-i+1}^n \widehat{f}_n^m C_u^{i,m} du + \int_{n-i+1}^n \widehat{\sigma}_n^m \sqrt{C_u^{i,m}} dW_u^{i,m}, \quad 2 \leq i \leq n, \quad 1 \leq m \leq M.$$

Again, by Lemma 3.13 and Remark 3.14, the conditional simulation of  $C_{j+1}^{i,m}$  is done with:

$$C_{j+1}^{i,m} \sim \mathcal{G}(N_{i,j}^m, \widehat{\beta}_j^m) \text{ with } N_{i,j}^m \sim \mathcal{P}(\widehat{\lambda}_{i,j}^m), \quad i+j > n, \quad 1 \leq m \leq M,$$

where  $\widehat{\beta}_j^m := \frac{2\widehat{f}_j^m}{(\widehat{\sigma}_j^m)^2(e^{\widehat{f}_j^m} - 1)}$  and  $\widehat{\lambda}_{i,j}^m := \widehat{\beta}_j^m e^{\widehat{f}_j^m} C_j^{i,m}$ , and where  $C_{n-i+1}^{i,m} := C_{n-i+1}^i$ .

3. It yields to the simulation of the reserves:

$$R^m := \sum_{i=2}^n C_n^{i,m} - C_{n-i+1}^i, \quad 1 \leq m \leq M. \quad (4.2)$$

The vector  $(R^m)_{1 \leq m \leq M}$  approximates the distribution of the reserves, conditional on our observations.

**Remark 4.1.** We described a bootstrap procedure to simulate the reserves. This method can be adapted to simulate  $C_n^{n+1}$ . Given that the only assumption on  $C_1^{n+1}$  is its square integrability, an additional assumption is needed to simulate it. One approach is to use the exposure and a corresponding parametric distribution, such as  $\mathcal{G}(\alpha E_{n+1}, \beta)$ , where  $E_{n+1} > 0$  represents the exposure at year  $n+1$ , and  $\alpha > 0$  and  $\beta > 0$  are parameters to be fitted using the observations  $(C_1^i)_{1 \leq i \leq n}$ , assuming the exposure information is available. Once this is done, we can combine the simulations of  $C_1^{n+1}$  with the  $(\widehat{f}_j^m)$  and  $(\widehat{\sigma}_j^m)$ , and then apply the bootstrap to obtain the simulations of  $C_n^{n+1}$ .

## 5 Examples

We implement our bootstrap method within a continuous-time framework, applying it to two data examples provided by [11]. The first dataset, initially introduced by [16], is presented in Table 1. The second dataset, pertaining to the mortgage guarantee business, is shown in Table 2.

$i$	$C_{i,1}$	$C_{i,2}$	$C_{i,3}$	$C_{i,4}$	$C_{i,5}$	$C_{i,6}$	$C_{i,7}$	$C_{i,8}$	$C_{i,9}$	$C_{i,10}$
1	357848	1124788	1735330	2218270	2745596	3319994	3466336	3606286	3833515	3901463
2	352118	1236139	2170033	3353322	3799067	4120063	4647867	4914039	5339085	
3	290507	1292306	2218525	3235179	3985995	4132918	4628910	4909315		
4	310608	1418858	2195047	3757447	4029929	4381982	4588268			
5	443160	1136350	2128333	2897821	3402672	3873311				
6	396132	1333217	2180715	2985752	3691712					
7	440832	1288463	2419861	3483130						
8	359480	1421128	2864494							
9	376686	1363294								
10	344014									

Table 1: The first dataset used in [11] and originally presented by [16].

$i$	$C_{i,1}$	$C_{i,2}$	$C_{i,3}$	$C_{i,4}$	$C_{i,5}$	$C_{i,6}$	$C_{i,7}$	$C_{i,8}$	$C_{i,9}$
1	58046	127970	476599	1027692	1360489	1647310	1819179	1906852	1950105
2	24492	141767	984288	2142656	2961978	3683940	4048898	4115760	
3	32848	274682	1522637	3203427	4445927	5158781	5342585		
4	21439	529828	2900301	4999019	6460112	6853904			
5	40397	763394	2920745	4989572	5648563				
6	90748	951994	4210640	5866482					
7	62096	868480	1954797						
8	24983	284441							
9	13121								

Table 2: The second dataset used in [11].

We then compare our results with those obtained by Mack and the distribution derived from traditional bootstrap procedures.

We evaluate the conditional MSEF and the bootstrap distribution across the following models:

- Mack’s model [11], using the classic Mack’s conditional MSEF formula and assuming a Log-normal parameterized distribution for the reserves’ distribution;
- Mack’s model with the bootstrap method;
- The time series model [2] with the bootstrap technique;

- Our continuous-time model incorporating the bootstrap approach.

Let us briefly review the first three models. Our continuous-time model with bootstrap was described in the previous section.

## 5.1 Mack's model with a parameterized Log-normal distribution

We employ the classic estimator  $\widehat{C}_{i,n} := C_{i,n-i+1} \prod_{k=n-i+1}^{n-1} \widehat{F}_k$ , which leads to the estimation of the expected value for the total reserve:

$$\widehat{\mu}_R := \widehat{R}.$$

We denote by  $\widehat{\sigma}_R^2$  the conditional MSEP of [11]. Finally, we approximate the distribution of the reserve with a Log-normal distribution by matching the moments:

$$\mathcal{LN} \left( \log(\widehat{\mu}_R) - \frac{1}{2} \log \left( 1 + \frac{\widehat{\sigma}_R^2}{\widehat{\mu}_R} \right), \log \left( 1 + \frac{\widehat{\sigma}_R^2}{\widehat{\mu}_R} \right) \right).$$

## 5.2 Mack's model with Bootstrap

For comparison purposes, we calculate both the conditional MSEP and its distribution using a bootstrap method, where the conditional MSEP is the variance of the bootstrap distribution.

We employ the procedure outlined in [4], which we briefly summarize here. First, we compute the Pearson residuals  $(r_{i,j})$ .

1. We simulate:

$$C_{i,j+1}^m := \widehat{F}_j C_{i,j} + \widehat{\Sigma}_j \sqrt{C_{i,j}} r_{i,j}^m, \quad 1 \leq m \leq M,$$

where each  $r_{i,j}^m$  is chosen from the  $(r_{i,j})$  uniformly and independently. Using (2.1), we compute  $(\widehat{F}_j^m, \widehat{\Sigma}_j^m)_{1 \leq j \leq n-1}$  for  $1 \leq m \leq M$ .

2. We initiate the simulation with  $C_{i,n-i+1}^m := C_{i,n-i+1}$ , and then iteratively simulate the lower triangle for  $2 \leq i \leq n$  as follows:

$$C_{i,j+1}^m \sim \mathcal{N} \left( \widehat{F}_j^m C_{i,j}^m, (\widehat{\Sigma}_j^m)^2 C_{i,j}^m \right),$$

and we deduce the bootstrap distribution of the total reserve with the formula (4.2).

## 5.3 Time series with Bootstrap

The model developed in [2] is based on the relation defined in (2.2), which is:

$$C_{i,j+1} = F_j C_{i,j} + \Sigma_j \sqrt{C_{i,j}} \varepsilon_{i,j},$$

where the  $\varepsilon$ 's are independent and centered with unit variance. We introduce the following hypothesis:

$$(\varepsilon_{i,j})_{1 \leq i,j \leq n} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1).$$

Now, we describe the bootstrap method for this model.

1. We simulate:

$$C_{i,j+1}^m \sim \mathcal{N}\left(\widehat{F}_j C_{i,j}, \widehat{\Sigma}_j^2 C_{i,j}\right), \quad 1 \leq m \leq M.$$

Using (2.1), we derive  $(\widehat{F}_j^m, \widehat{\Sigma}_j^m)_{1 \leq j \leq n-1}$  for  $1 \leq m \leq M$ . Additionally, it is noteworthy that when  $C_{i,j}$  is fixed,

$$\begin{aligned} \widehat{F}_j^m &\sim \mathcal{N}\left(\widehat{F}_j, \frac{\widehat{\Sigma}_j^2}{\sum_{i=1}^{n-j} C_{i,j}}\right), \\ (\widehat{\Sigma}_j^m)^2 &\sim \widehat{\Sigma}_j^2 \frac{\chi_{n-j-1}^2}{n-j-1}, \end{aligned} \tag{5.1}$$

and both are independent. We can simulate the  $(\widehat{F}_j^m, \widehat{\Sigma}_j^m)_{1 \leq j \leq n-1}$  directly.

2. As in Section 5.2, we begin with  $C_{i,n-i+1}^m := C_{i,n-i+1}$ , we simulate iteratively, for  $2 \leq i \leq n$  the lower triangle:

$$C_{i,j+1}^m \sim \mathcal{N}\left(\widehat{F}_j^m C_{i,j}^m, (\widehat{\Sigma}_j^m)^2 C_{i,j}^m\right),$$

and we deduce the bootstrap distribution of the total reserve with the formula (4.2).

## 5.4 Comparison and conclusion

All bootstraps hereafter use  $M = 10^7$  simulations. We begin by computing the conditional MSEP of the different methods using the first dataset from Table 1, along with the 99.5% quantile, all expressed relative to the common reserve estimator  $\widehat{R}$ . The conditional MSEP for Mack's Log-normal model is computed using the classical estimator from [11], while those for other methods are derived from the empirical bootstrap distributions of the reserves.

Method	$\sqrt{\widehat{MSEP}}$ (in % of $\widehat{R}$ )	$Q(R; 99.5\%) - \widehat{R}$ (in % of $\widehat{R}$ )
Mack Log-normal	13.0995	38.7466
Mack Bootstrap	11.7585	33.0675
Time series Bootstrap	13.1030	36.2963
Continuous-time Bootstrap	13.1039	37.0219

Table 3: Conditional MSEP for the three other methods introduced and our *continuous-time Bootstrap* from Section 4 with the dataset from Table 1.

Our results closely align with Mack's original formula for the conditional MSEP, as seen in the Time Series Bootstrap approach. The Mack Bootstrap yields a lower conditional MSEP, differing from the Time Series Bootstrap only in the simulations of  $(\widehat{F}_j^m, \widehat{\Sigma}_j^m)_{1 \leq j \leq n-1}$ . This discrepancy primarily arises from the Pearson residuals being more *regular*, indicating smaller values. Regarding quantiles, our analysis reveals a modest decrease relative to Mack Log-normal approach, a slight increase compared to the Time Series Bootstrap, and a substantial increase relative to Mack Bootstrap. For Mack's classic approach, which assumes a distribution-free framework, we adopted a log-normal distribution for the quantiles. This choice has notable implications: when a Gamma distribution is used instead, the quantile excess decreases to 36.95%.

In our simulations, both the Mack Bootstrap and Time Series Bootstrap methods occasionally yield  $C_j^i < 0$ . Although rare in this example due to the data's regularity, occurring roughly once every  $10^5$  simulations, such occurrences have been removed, with the introduced bias being negligible. The processes  $(C_t^i)_{1 \leq t \leq n}$  remain non-negative in the continuous-time model, by construction. Nevertheless, in Remark 3.12, we noted that  $\mathbb{P}(C_j^i = 0) > 0$  and asserted it to be numerically negligible. The highest probabilities arise for  $j = 2$ , and we have:

$$\mathbb{P}(C_2^n = 0) = \exp(-52.3031) \approx 1.9277 \times 10^{-23}.$$

With less regularly structured data, the Mack or Times series Bootstrap methods might more frequently yield  $C_j^i < 0$ , potentially introducing bias if the corresponding simulations are removed or set to 0. However, such occurrences never arise within our continuous-time framework.

In Figure 1, we display the complete distributions associated with the various models and our framework.

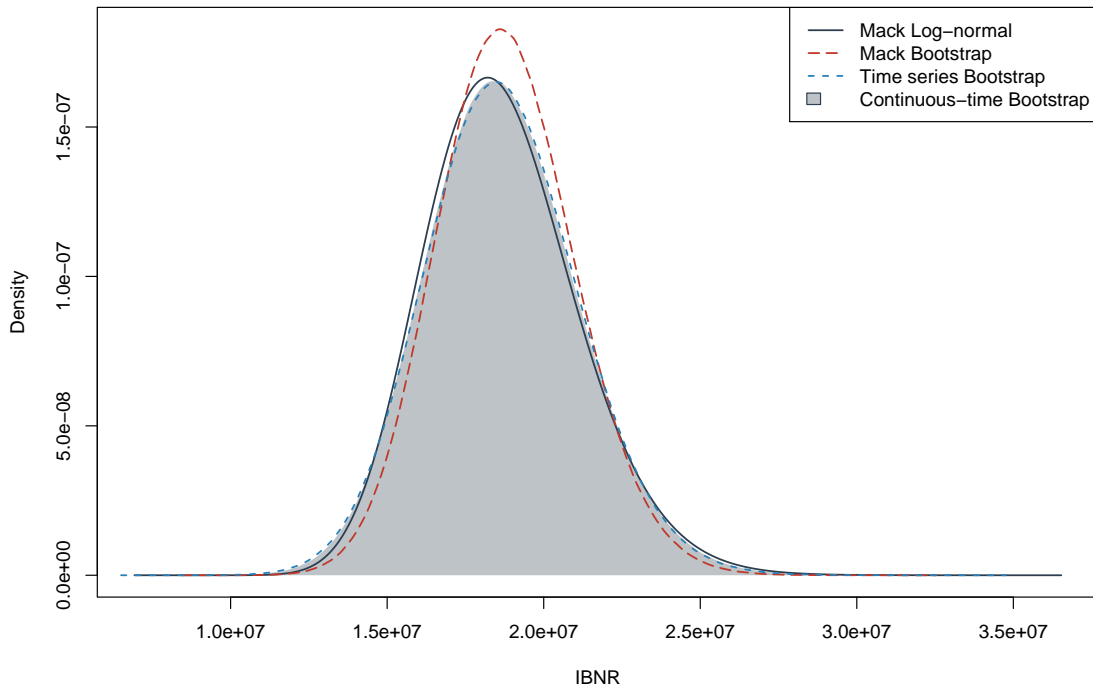


Figure 1: Estimated conditional distributions of the total reserve with the dataset from Table 1.

As observed in Table 3, the distribution within our framework closely resembles that of Mack with the Log-normal parameterized distribution and the Time Series Bootstrap. Our approach offers a significant advantage: it employs continuous simulation to eliminate negative values without introducing bias while maintaining the integrity of moment assumptions.

Finally, in Figure 2, we illustrate the bootstrap distribution of  $C_2^n$  derived from the estimated  $\widehat{F}_1$  and  $\widehat{\Sigma}_1^2$ , alongside comparisons with a Gaussian distribution and a Gamma distribution possessing equivalent moments.

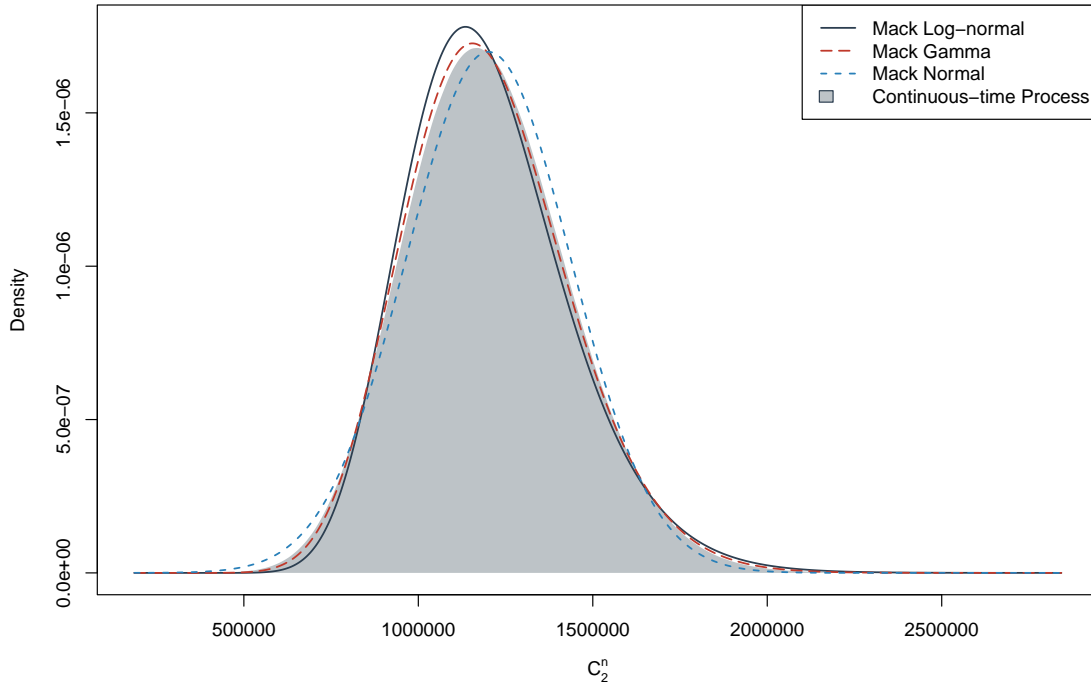


Figure 2: Estimated distributions of  $C_2^n$  in our continuous-time model,  $\mathbb{P}(C_2^n = 0)$  is neglected.

We observe that our continuous-time model provides a slightly asymmetric distribution, closely resembling that of a Gamma distribution.

We now examine the second dataset introduced in Table 2. This dataset exhibits less regularity, featuring a development factor of approximately 11 in the first year. As with the previous analysis, we compute the conditional MSEF for the various methods, this time using the new dataset, alongside the 99.5% quantile, all expressed relative to the common reserve estimator  $\widehat{R}$ . Notably, the bootstrap simulations yield non-negligible negative values, with 18.9% of Mack's bootstrap simulations and 26.2% of Time Series bootstrap simulations encountering such outcomes. Whenever  $C_j^i < 0$ , we replace these with 0, a choice we will discuss further at the end.

Method	$\sqrt{\widehat{MSEP}}$ (in % of $\widehat{R}$ )	$Q(R; 99.5\%) - \widehat{R}$ (in % of $\widehat{R}$ )
Mack Log-normal	25.6337	85.5185
Mack Bootstrap	22.9662	77.2303
Time series Bootstrap	24.6414	76.9349
Continuous-time Bootstrap	25.7493	88.3811

Table 4: The conditional MSEF of the three other methods introduced and our continuous-time bootstrap from Section 4 with the dataset from Table 2.

With this new dataset, the results diverge from previous findings. The Mack Bootstrap and Time Series Bootstrap yield lower conditional MSEF values, whereas the continuous-time bootstrap produces a conditional MSEF comparable to that of the classic Mack method. Similarly, quantile analysis reveals consistent patterns: the continuous-time bootstrap exhibits the highest quantile, surpassing the Mack Log-normal by 3%. However, adjusting the latter to employ a Gamma distribution reduces the quantile to 78.2503%, highlighting the critical influence of the chosen distribution.

In Figure 3, we display the complete distributions associated with the various models and our framework.

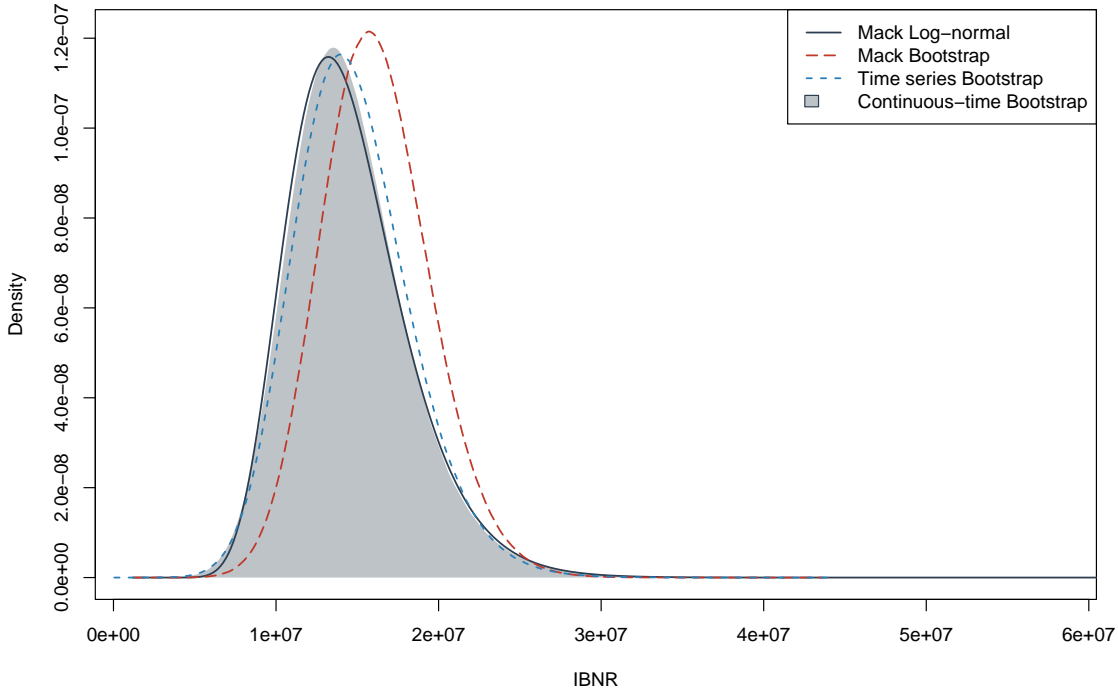


Figure 3: Estimated conditional distributions of the total reserve with the dataset from Table 2.

This time, the empirical distributions exhibit clear differences.

For this dataset, across all bootstrap methods, 20% to 40% of the simulations produced at least one  $C_j^i < 0$ , which we replaced with 0. In the continuous-time bootstrap, such values were inherently zero, requiring no adjustment. This correction may introduce bias into the Mack Bootstrap and Time Series Bootstrap procedures. For the continuous-time bootstrap, we calculate:

$$\mathbb{P}(C_2^n = 0) = \exp(-1.8102) \approx 0.1636.$$

This probability is no longer negligible. This is perfectly taken into account within the simulations and does not bias the methodology; it preserves the moments and remains tailored to the case. However, a probability of  $\mathbb{P}(C_2^n = 0) \approx 0.1636$  seems unrealistic in practice. With a highly irregular triangle, we approach the limits of the method. The small value of  $C_1^n$  plays a significant role here, being notably low relative to its column, especially in combination with the irregular triangle. If we replace with  $C_1^{n-1}$  (24983 instead of 13121), we obtain  $\mathbb{P}(C_2^{n-1} = 0) \approx 0.03184$ , which is more reasonable but still not negligible.

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