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L^p ASYMPTOTIC STABILITY OF 1D DAMPED WAVE EQUATION WITH NONLINEAR DAMPING

Y. CHITOUR, M. KAFNEMER, P. MARTINEZ, AND B. MEBKHOUT

ABSTRACT. In this paper, we study the one-dimensional wave equation with localized nonlinear damping and Dirichlet boundary conditions, in the L^p framework, with $p \in [1, \infty)$.

We begin by addressing the well-posedness problem, establishing the existence and uniqueness of weak and strong solutions for $p \in [1, \infty)$, under suitable assumptions on the damping function.

Next, we study the asymptotic behaviour of the associated energy when $p \in (1, \infty)$, and we provide decay estimates that appear to be almost optimal compared to similar problems with boundary damping.

Our work is motivated by earlier studies, particularly, those by Chitour, Marx and Prieur [Journal of Differential Equations, 269 (2020)], and Haraux [C.R.A.S Paris, 287 (1978)]. The proofs combine arguments from Kafnemer, Mebkhout and Chitour [ESAIM: COCV, 28 (2022)] for wave equation in the L^p framework with a linear damping, techniques of weighted energy estimates introduced in Martinez [ESAIM: COCV, 4 (1999)], new integral inequalities for $p > 2$, and convex analysis tools when $p \in (1, 2)$.

1. INTRODUCTION

In this paper, we consider the one-dimensional wave equation with nonlinear and localized damping, in a functional framework relying on L^p spaces, with $p \in [1, +\infty)$. The problem is the following:

$$(1.1) \quad \begin{cases} z_{tt} - z_{xx} + a(x)g(z_t) = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ z(t, 0) = 0 = z(t, 1) & \text{for } t \in \mathbb{R}_+, \\ z(0, x) = z_0(x), z_t(0, x) = z_1(x) & \text{for } x \in (0, 1), \end{cases}$$

where the initial data (z_0, z_1) belong to an L^p functional space (defined later), with $p \in [1, \infty)$. The function a is a continuous nonnegative function on $[0, 1]$, bounded from below by a positive constant on some non-empty open interval ω of $(0, 1)$, which represents the region of the domain where the damping term is active. The nonlinearity $g : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 nondecreasing function such that $g(0) = 0$ and

$$(1.2) \quad \forall x \in \mathbb{R}, \quad x g(x) \geq 0,$$

the classical condition ensuring that the usual energy is nonincreasing in the L^2 framework.

For many years, the asymptotic behaviour of the solutions of (1.1) has been investigated in the classical functional space $H_0^1(0, 1) \times L^2(0, 1)$. Our paper is

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devoted to the case of the following functional spaces

$$(1.3) \quad X_p := W_0^{1,p}(0,1) \times L^p(0,1),$$

$$(1.4) \quad Y_p := \left(W^{2,p}(0,1) \cap W_0^{1,p}(0,1) \right) \times W_0^{1,p}(0,1),$$

both equipped with their natural norm, and with $p \in [1, +\infty)$.

To the best of our knowledge, very few is known about the global asymptotic stability of (1.1) in those functional spaces. This is partly due to the fact that for linear wave equations on \mathbb{R}^n , $n \geq 2$, the “semigroup” associated with the d’Alembertian operator $\square z$ (equal to $z_{tt} - \Delta z$) is not, in general, a well-defined bounded operator for an L^p space framework when $p \neq 2$, see Peral [20]. However Haraux [8] succeeded to study the problem in the one-dimensional case, and this is one of the motivations of our work.

Let us give our main results, and then we will compare them to the already existing results on this subject.

2. MAIN RESULTS

2.1. Assumptions.

We will work under the following assumptions:

(H₁) $a : [0, 1] \rightarrow \mathbb{R}_+$ is continuous, and satisfies

$$(2.1) \quad \exists a_0 > 0, \quad a \geq a_0 \quad \text{on} \quad \omega = (b, c) \subset [0, 1].$$

(H₂) $g : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 nondecreasing function such that $g(0) = 0$, and

$$(2.2) \quad \forall x \neq 0, \quad xg(x) > 0.$$

(Note that hypothesis **(H₂)** implies that $g'(0) \geq 0$.)

(H₃) $g : \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz function, (2.2) holds true and there exists an increasing and odd function $g_0 : [-1, 1] \rightarrow \mathbb{R}$ and $0 < \alpha \leq \beta$ such that

$$(2.3) \quad \begin{cases} \forall s \in [-1, 1], & |g_0(s)| \leq |g(s)| \leq \beta|s|, \\ \forall |s| \geq 1, & \alpha|s| \leq |g(s)| \leq \beta|s|. \end{cases}$$

2.2. Well posedness: Weak/strong solutions of Problem (1.1).

2.2.a. Functional framework.

Given $p \in [1, +\infty)$, the space X_p (defined in (1.3)) is equipped with the norm

$$(2.4) \quad \|(u, v)\|_{X_p} := \left(\frac{1}{p} \int_0^1 |u'|^p + |v|^p dx \right)^{\frac{1}{p}}.$$

which is equivalent to the natural norm $\|u'\|_{L^p} + \|v\|_{L^p}$.

Given $p \in [1, +\infty)$, the space Y_p (defined in (1.4)) is equipped with the norm

$$(2.5) \quad \|(u, v)\|_{Y_p} := \left(\frac{1}{p} \int_0^1 |u''|^p + |v'|^p dx \right)^{\frac{1}{p}}.$$

which is equivalent to the natural norm $\|u''\|_{L^p} + \|v'\|_{L^p}$.

2.2.b. Weak/Strong solutions of (1.1).

We consider the following definitions:

Definition 2.1. (*Weak solutions of Problem (1.1)*)

Given $p \in [1, +\infty)$, and $(z_0, z_1) \in X_p$, the function

$$z \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0,1) \cap W^{1,\infty}(\mathbb{R}_+, L^p(0,1)))$$

is said to be a weak solution of Problem (1.1) if it satisfies the problem in the dual sense (meaning that the equalities are taken in the weak topology of X_p).

Definition 2.2. (Strong solutions of Problem (1.1))

Given $p \in [1, +\infty)$, and $(z_0, z_1) \in Y_p$, the function

$$z \in L^\infty(\mathbb{R}_+, W^{2,p}(0,1) \cap W_0^{1,p}(0,1)) \cap W^{1,\infty}(\mathbb{R}_+, W_0^{1,p}(0,1))$$

is said to be a strong solution of Problem (1.1) if it satisfies the problem in the classical sense.

Then we prove the following results, that complete the studies in [8, 6] in the L^∞ framework:

Theorem 2.1. Weak solutions for $p \in [2, +\infty)$.

Assume that Hypotheses (\mathbf{H}_1) and (\mathbf{H}_2) are satisfied, and take $2 \leq p < \infty$. Then for every initial conditions $(z_0, z_1) \in X_p$, there exists a unique weak solution z of Problem (1.1), satisfying additionally

$$(2.6) \quad z_t \in L^\infty(\mathbb{R}_+, L^p(0,1)).$$

Theorem 2.2. Strong solutions for $p \in [1, +\infty)$.

Assume that Hypotheses (\mathbf{H}_1) and (\mathbf{H}_2) are satisfied, and take $1 \leq p < \infty$. Then for every $(z_0, z_1) \in Y_p$, there exists a unique strong solution z of Problem (1.1), satisfying additionally

$$(2.7) \quad z_t \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0,1)).$$

Theorem 2.3. Weak solutions for $p \in [1, +\infty)$.

Assume that Hypotheses (\mathbf{H}_1) and (\mathbf{H}_3) are satisfied, and take $1 \leq p < \infty$. Then, for every initial condition $(z_0, z_1) \in X_p$, there exists a unique weak solution z verifying (2.6).

Remark 2.1. The argument based on D'Alembert formula and fixed point theory that is used in [6] cannot be used in the nonlinear case without imposing the extra assumption that g is linearly bounded.

2.3. Stability results.

In the classical L^2 framework, the natural energy is defined by

$$\mathcal{E}_2(t) := \frac{1}{2} \int_0^1 |z_x|^2 + |z_t|^2 dx,$$

and (1.2) implies that \mathcal{E}_2 is nonincreasing along solutions of (1.1). We will recall in section 3 the main stability results in this context.

In the L^p framework, the natural energy is defined by

$$(2.8) \quad \mathcal{E}_p(t) := \|(z, z_t)(t)\|_{X_p}^p = \frac{1}{p} \int_0^1 |z_x|^p + |z_t|^p dx,$$

and is equivalent to the following one, which will reveal to be more useful:

$$(2.9) \quad E_p(t) := \frac{1}{p} \int_0^1 |z_x + z_t|^p + |z_x - z_t|^p dx.$$

It is well known ([8]) that E_p is nonincreasing along solutions of (1.1). The main stability result of this paper is to estimate the decay of that energy for a large class of damping functions g .

2.3.a. Stability results when $p \in (2, +\infty)$.

When $p \in (2, +\infty)$, we prove the following.

Theorem 2.4. *Fix $p \in (2, +\infty)$. Assume that (2.1) and (2.3) hold. Then we have the following:*

(i) *If $g'(0) > 0$, the energy E_p decays exponentially to 0, i.e., there exists $C > 0$, $\omega > 0$ such that, for every initial condition $(z_0, z_1) \in X_p$, we have*

$$(2.10) \quad \forall t \geq 0, \quad E_p(t) \leq C E_p(0) e^{-\omega t}.$$

(ii) *If $g'(0) = 0$, assume that there exists $\eta > 0$ such that the function*

$$H(s) := \frac{g_0(s)}{s}$$

is increasing on $[0, \eta]$. Then, for every initial condition $(z_0, z_1) \in X_p$ and $\delta > 0$, there exists $C_\delta(E_p(0))$ such that

$$(2.11) \quad \forall t \geq 1, \quad E_p(t) \leq C_\delta(E_p(0)) \left(g_0^{-1}\left(\frac{1}{t}\right)\right)^{p-\delta}.$$

Moreover, fix $m \geq 1$, and denote

$$t_m := \frac{\frac{2}{\eta^{1/m}}}{H\left(\frac{\eta}{2^m}\right)}.$$

Then there exists $C_{m,\delta}(E_p(0))$ such that

$$(2.12) \quad \forall t \geq t_m, \quad E_p(t) \leq \frac{C_{m,\delta}(E_p(0))}{s(t)^{m(p-\delta)}} \quad \text{where} \quad t = \frac{2s(t)}{H\left(\frac{1}{(2s(t))^m}\right)}.$$

Two typical examples of application:

- Assume that there exists $\alpha > 0$ and $q > 1$ such that

$$(2.13) \quad \forall s \in [0, 1], \quad g_0(s) = \alpha s^q.$$

The estimate (2.11) gives that

$$(2.14) \quad \forall t \geq 1, \quad E_p(t) \leq \frac{C_\delta(E_p(0))}{t^{\frac{p}{q}-\delta}},$$

and this can be improved using the estimate (2.12), which is not ideal, but gives almost optimal decay rates here: indeed, taking m large enough, (2.12) gives that, for every $\delta > 0$, there exists $C_\delta(E_p(0))$ such that

$$(2.15) \quad \forall t \geq 1, \quad E_p(t) \leq \frac{C_\delta(E_p(0))}{t^{\frac{p}{q-1}-\delta}}.$$

- Assume that there exists $\alpha > 0$ and $q > 0$ such that

$$(2.16) \quad \forall s \in [0, 1], \quad g_0(s) = \alpha e^{-\alpha/s^q}.$$

Then (2.11) (and (2.12)) gives that, for every $\delta > 0$, there exists $C_\delta(E_p(0))$ such that

$$(2.17) \quad \forall t \geq 2, \quad E_p(t) \leq \frac{C_\delta(E_p(0))}{(\ln t)^{\frac{p}{q}-\delta}}.$$

The estimates (2.15) and (2.17) are almost optimal in the following sense: consider the wave equation with boundary damping, acting at the boundary point:

$$(2.18) \quad \begin{cases} z_{tt} - z_{xx} = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ z(t, 0) = 0 & \text{for } t \in \mathbb{R}_+, \\ z_x(t, 1) + g(z_t(t, 1)) = 0 & \text{for } t \in \mathbb{R}_+, \\ z(0, x) = z_0(x), z_t(0, x) = z_1(x) & \text{for } x \in (0, 1). \end{cases}$$

Then, using [21] (see also [5]), there exist initial conditions (z_0, z_1) such that

- in the case (2.14), the solution z satisfies

$$E_p(t) \sim \frac{C_\delta(E_p(0))}{t^{\frac{p}{q-1}}} \quad \text{as } t \rightarrow +\infty,$$

- in the case (2.16), the solution z satisfies

$$E_p(t) \sim \frac{C_\delta(E_p(0))}{(\ln t)^{\frac{p}{q}}} \quad \text{as } t \rightarrow +\infty.$$

([21] gives lower bounds of energy for several classes of damping function g in the L^2 framework, but the proofs are immediately adaptable to the L^p framework.) This is why we say that (2.15) and (2.17) are almost optimal. The optimal ones would probably be the same ones with $\delta = 0$, but we were not able to obtain such estimates.

The proof of Theorem 2.4 is based on the multiplier method with suitably chosen multipliers from [12] (done in the linear case), on the method developed in [17] (to study the nonlinear stabilization in the L^2 framework), and on new integral inequalities. We refer the reader to Lemmas 5.8 and 5.9.

2.3.b. Stability results when $p \in (1, 2)$.

When $p \in (1, 2)$, we need to combine previous arguments with convex analysis tools, in particular, the convex conjugate of some suitably chosen convex function \tilde{F} (that will replace the function $F(y) = |y|^p$), and Fenchel's inequality, see section 6.4.

If g is linear, [12] proves that there exists $C, \omega > 0$ such that, for every initial condition $(z_0, z_1) \in X_p$, we have

$$(2.19) \quad \forall t \geq 0, \quad E_p(t) \leq CE_p(0) e^{-\omega t}.$$

Here we complete this result, in the following way:

Theorem 2.5. *Fix $p \in (1, 2)$. Assume that (2.1) and (2.3) hold. Then we have the following:*

(i) *If $g'(0) > 0$, the energy E_p decays exponentially to 0, i.e., for every initial condition $(z_0, z_1) \in X_p$, there exists $C^{(0)} > 0$, $\omega^{(0)} > 0$ depending on $E_p(0)$ such that,*

$$(2.20) \quad \forall t \geq 0, \quad E_p(t) \leq C^{(0)} E_p(0) e^{-\omega^{(0)} t}.$$

(ii) *If $g'(0) = 0$, assume that there exists $\eta > 0$ such that the function*

$$H(s) := \frac{g_0(s)}{s}$$

is increasing on $[0, \eta]$. Then, for every initial condition $(z_0, z_1) \in X_p$, there exists $C(E_p(0))$ such that

$$(2.21) \quad \forall t \geq 1, \quad E_p(t) \leq C(E_p(0)) \left(g_0^{-1}\left(\frac{1}{t}\right) \right)^p.$$

Moreover, we also have the following: then, given $m \geq 1$, we have

$$(2.22) \quad \forall t \geq t_m, \quad E_p(t) \leq \frac{C_\delta(E_p(0))}{s(t)^{pm}} \quad \text{where} \quad t = \frac{2s(t)}{H\left(\frac{1}{(2s(t))^m}\right)}$$

and t_m is given in Theorem 2.4.

Two typical examples of application:

- if g_0 satisfies (2.13), then the estimate (2.22) (with m large enough) gives that, for every $\delta > 0$, there exists $C_\delta(E_p(0))$ such that (2.15) holds;

- if (2.16) holds, then (2.21) gives that there exists $C(E_p(0))$ such that

$$(2.23) \quad \forall t \geq 2, \quad E_p(t) \leq \frac{C(E_p(0))}{(\ln t)^{\frac{p}{a}}}.$$

2.4. Organization of the paper.

The paper is organized as follows:

- In Section 3, we compare our results with the already existing ones,
- In Section 4, we prove Theorems 2.1-2.3, starting by rewriting (1.1) using the Riemann invariants.
- In Section 5, we prove Theorem 2.4.
- In Section 6, we prove Theorem 2.5.

3. COMPARISON WITH EARLIER RESULTS

3.1. The L^2 framework.

The nonlinear problem (1.1) has already been hugely studied in the Hilbertian framework, i.e., with $p = 2$. The well-posedness is a classical result and is a consequence of the theory of maximal monotone operators. Decay estimates have been known for a long time, and we refer in particular to Nakao [19], Haraux [7], Lagnese [15], Zuazua [22], Komornik [13] for estimates when the damping function has a polynomial behaviour near 0, to Lasiecka and Tataru [16], Martinez [17], Alabau-Boussouira [1, 2] under more general conditions on the damping function, and to Vancostenoble and Martinez [21] for lower bound estimates, essentially proving the optimality of the previous upper estimates. These upper estimates are mainly obtained thanks to the multiplier method that allows one to obtain integral estimates on the energy, and then to some integral inequalities of the Gronwall type, or comparison with ODE, for which the asymptotic behaviour of the solutions can be more easily estimated. In particular, Alabau-Boussouira [1, 2] succeeded into proving optimal estimates under very few assumptions, and for a large class of problems, combining clever convexity arguments with weighted integral inequalities.

3.2. The L^p framework.

The non-Hilbertian framework has only been considered recently, even if, from an applied point of view, the L^∞ setting is particularly interesting (and challenging). The usual well-posedness Hilbertian techniques (maximal monotone operators for instance) are either not easy to handle or no longer applicable in this framework.

The main results that exist and are relevant to our context come primarily from Haraux [8], and were extended in several directions in Chitour, Marx, and Prieur [6], and Kafnemer, Mebkhout, Jean, and Chitour [12].

Haraux [8] proves that the semigroup associated with the d'Alembertian operator in one space dimension and with Dirichlet boundary conditions is well-posed for every X_p , $p \in [2, +\infty]$. Haraux also provides some decay rate estimates of an energy equivalent to the natural one for solutions associated with smooth initial data, and when $a \equiv 1$ and g behaves polynomially near 0 and at infinity: if there exists $k_0, k_1 > 0$, and $R \geq r > 0$ such that

$$(3.1) \quad \forall s \in \mathbb{R}, \quad k_0 |s|^r \leq g'(s) \leq k_1 (|s|^r + |s|^R),$$

and $(z_0, z_1) \in Y_\infty$, then

$$(3.2) \quad \forall t \geq 0, \quad \|z\|_{W_0^{1,p}(0,1)}^p + \|z_t\|_{L^p(0,1)}^p \leq \frac{C(z_0, z_1)}{(1+t)^{\frac{2(p+1)}{3r}}}.$$

In [3], Amadori, Aqel, and Dal Santo provide an asymptotic analysis when a is positive and g has a linear behaviour:

$$\begin{cases} 0 < k_1 \leq a(x) \leq k_2, \\ \forall x \in (0, 1), \end{cases} \quad \text{and} \quad \begin{cases} 0 < g_1 \leq g'(s) \leq g_2, \\ \forall s \in \mathbb{R}. \end{cases}$$

The techniques used in this paper are based on the theory of scalar conservation laws, which makes the analysis in those L^p spaces really natural.

In [6], Chitour, Marx and Prieur study (1.1) when $p \in [2, +\infty]$, and provide well-posedness results in the associated L^p framework (relying on the d'Alembert formula); they also provide semi-global exponential stability results when the damping function satisfies mainly $g'(0) > 0$: given $R > 0$, there exists $K(R), \beta(R) > 0$ such that

$$\|(z_0, z_1)\|_{X_p} \leq R \quad \implies \quad \|(z(t), z_t(t))\|_{X_p} \leq K(R)e^{-\beta(R)t}, \quad \forall t \geq 0.$$

Finally, in [12], Kafnemer, Mebkhout, Jean and Chitour study (1.1) when $g(s) \equiv k_0 s$ with some $k_0 > 0$, extend the well-posedness results of [6] to the case $p > 1$, obtain exponential decay for all $1 < p < +\infty$ using a generalized multiplier method and exponential stability when $p = \infty$ in some cases of a constant in space global damping.

Here we use as a basis [10, 12] alongside with some techniques from [17] and [6] to prove our stability results. However, combining the existing arguments was not sufficient, and we needed new integral inequalities necessary to conclude when $p > 2$, see Lemmas 5.8 and 5.9, and to extend or precise some estimates when $p \in (1, 2)$.

3.3. Open questions.

There are some natural open questions associated with our work:

- Optimality of the decay estimates of Theorem 2.4: when $p > 2$, we believe that the optimal decay estimates would be the ones of Theorem 2.4, with $\delta = 0$. It would be interesting to know if such decay estimates hold true.
- Optimality of the decay estimates of Theorem 2.5: in case (ii), we do not know if the decay rate can be uniform with respect to the norm of the initial conditions. It would also be interesting to know if the estimate (2.15) holds with $\delta = 0$.
- Our proofs are not valid in the case $p \in \{1, \infty\}$. However, there are some results in some particular cases in [10], when the function $a(\cdot)$ is constant on $(0, 1)$, and with some constraints on the value of that constant. It would be interesting to go further in that direction and to obtain estimates in X_p , $p \in \{1, \infty\}$, with less restrictive assumptions on the feedback term.

4. WELL-POSEDNESS

In this section, we study the well-posedness of Problem (1.1). Before, we introduce a crucial tool, namely, the Riemann invariants.

4.1. Riemann invariants.

We define the Riemann invariants for all $(t, x) \in \mathbb{R}_+ \times (0, 1)$ by

$$(4.1) \quad \rho(t, x) = z_x(t, x) + z_t(t, x),$$

$$(4.2) \quad \xi(t, x) = z_x(t, x) - z_t(t, x).$$

Along strong solutions of (1.1), we deduce that

$$(4.3) \quad \begin{cases} \begin{cases} \rho_t - \rho_x = -a(x)g\left(\frac{\rho-\xi}{2}\right) & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \\ \xi_t + \xi_x = a(x)g\left(\frac{\rho-\xi}{2}\right) & \text{for } (t, x) \in \mathbb{R}_+ \times (0, 1), \end{cases} \\ \begin{cases} \rho(t, 0) - \xi(t, 0) = 0 & \text{for } t \in \mathbb{R}_+, \\ \rho(t, 1) - \xi(t, 1) = 0 & \text{for } t \in \mathbb{R}_+, \end{cases} \\ \begin{cases} \rho(0, x) = z'_0(x) + z_1(x) =: \rho_0(x) & \text{for } x \in (0, 1), \\ \xi(0, x) = z'_0(x) - z_1(x) =: \xi_0(x) & \text{for } x \in (0, 1). \end{cases} \end{cases}$$

Note that $(\rho_0, \xi_0) \in W^{1,p}(0, 1) \times W^{1,p}(0, 1)$.

Before proving Theorems 2.1-2.3, we need to study the monotonicity of the associated energy.

4.2. The energy E_p of strong solutions is nonincreasing.

For $r \geq 0$, we introduce the following notation

$$(4.4) \quad \lfloor x \rfloor^r := \text{sgn}(x)|x|^r, \quad \forall x \in \mathbb{R},$$

where $\text{sgn}(x) = \frac{x}{|x|}$ for nonzero $x \in \mathbb{R}$ and $\text{sgn}(0) = [-1, 1]$. We have the following obvious formulas, which will be repeatedly used later on:

$$(4.5) \quad \frac{d}{dx}(\lfloor x \rfloor^r) = r|x|^{r-1}, \quad \forall r \geq 1, x \in \mathbb{R},$$

$$(4.6) \quad \frac{d}{dx}(|x|^r) = r\lfloor x \rfloor^{r-1}, \quad \forall r > 1, x \in \mathbb{R}.$$

First, we recall that the p th-energy E_p of a solution z is defined in (2.9). The Riemann invariants allow us to have the following expression of E_p :

$$(4.7) \quad E_p(t) = \frac{1}{p} \int_0^1 (|\rho|^p + |\xi|^p) dx.$$

Here is the natural extension of Proposition 2.1 of [12].

Proposition 4.1. *Let $p \in [1, \infty)$ and suppose that a strong solution z of (1.1) exists and is defined on a non trivial interval $I \subset \mathbb{R}_+$ containing 0, for some initial conditions $(z_0, z_1) \in Y_p$. Let \mathcal{F} be a C^1 convex function. For $t \in I$, define*

$$(4.8) \quad \Phi(t) := \int_0^1 [\mathcal{F}(\rho) + \mathcal{F}(\xi)] dx,$$

where ρ, ξ are the Riemann invariants, defined in (4.1). Then Φ is well defined for $t \in I$ and satisfies

$$(4.9) \quad \Phi'(t) = - \int_0^1 a(x)g\left(\frac{\rho-\xi}{2}\right)(\mathcal{F}'(\rho) - \mathcal{F}'(\xi))dx.$$

As a consequence, the function Φ is nonincreasing on I .

As an immediate consequence of Proposition 4.1, we will have the following

Corollary 4.1. *Let $p \in (1, \infty)$. Assume that z is a strong solution of (1.1). Denote ρ and ξ the Riemann invariants, defined in (4.1). Then*

$$(4.10) \quad E'_p(t) = - \int_0^1 a(x)g\left(\frac{\rho-\xi}{2}\right) (\lfloor \rho \rfloor^{p-1} - \lfloor \xi \rfloor^{p-1}) dx.$$

As a consequence, the energy E_p is nonincreasing on its existence interval.

Proof of Corollary 4.1 assuming Proposition 4.1. If $p > 1$, it is sufficient to apply Proposition 4.1 with $\mathcal{F}(x) = |x|^p$, and it is clear that \mathcal{F} is C^1 on \mathbb{R} and convex. \square

Proof of Proposition 4.1. The proof of the formula of $\Phi'(t)$ is similar to the proof of [12, Proposition 2.1], but for the sake of completeness, we provide it. For $t \geq 0$, it holds

$$\Phi'(t) = \int_0^1 \mathcal{F}'(\rho)\rho_t + \mathcal{F}'(\xi)\xi_t dx.$$

Using (4.3), we have $\rho_t = \rho_x - a(x)g\left(\frac{\rho-\xi}{2}\right)$, and $\xi_t = -\xi_x + a(x)g\left(\frac{\rho-\xi}{2}\right)$. Therefore

$$\begin{aligned} \Phi'(t) &= \int_0^1 \mathcal{F}'(\rho)\rho_x - \mathcal{F}'(\xi)\xi_x dx - \int_0^1 a(x)g\left(\frac{\rho-\xi}{2}\right) (\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx \\ &= \left[\mathcal{F}(\rho) - \mathcal{F}(\xi) \right]_0^1 - \int_0^1 a(x)g\left(\frac{\rho-\xi}{2}\right) (\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx. \end{aligned}$$

Thanks to the Dirichlet boundary conditions, we have $z_t = 0$ at the boundary, and then

$\mathcal{F}(\rho)|_{x=1} - \mathcal{F}(\xi)|_{x=1} = \mathcal{F}(z_x + z_t)|_{x=1} - \mathcal{F}(z_x - z_t)|_{x=1} = \mathcal{F}(z_x)|_{x=1} - \mathcal{F}(z_x)|_{x=1} = 0$, and the same thing at $x = 0$. This implies that

$$\Phi'(t) = - \int_0^1 a(x)g\left(\frac{\rho-\xi}{2}\right) (\mathcal{F}'(\rho) - \mathcal{F}'(\xi)) dx.$$

To conclude, since \mathcal{F}' is nondecreasing, the sign of $\mathcal{F}'(\rho) - \mathcal{F}'(\xi)$ is the same than the sign of $\rho - \xi$; and the sign of $g\left(\frac{\rho-\xi}{2}\right)$ is also the same than the sign of $\rho - \xi$. This implies that $g\left(\frac{\rho-\xi}{2}\right) (\mathcal{F}'(\rho) - \mathcal{F}'(\xi))$ is nonnegative, and then that $\Phi'(t) \leq 0$. \square

Remark 4.1. The previous proposition was first introduced in [7] and reused in [6] to prove that the energy functional is nonincreasing. Then it was improved in [12] by omitting the hypothesis that function \mathcal{F} should be even on top of being convex.

4.3. Proof of Theorem 2.1.

Fix $2 \leq p < +\infty$ and let $(z_0, z_1) \in X_p$. Since Y_p is dense in X_p for all $2 \leq p < \infty$ and $Z_0 = (z_0, z_1) \in X_p$, there exists a sequence $(Z_0^{(n)} = (z_0^{(n)}, z_1^{(n)}))_n$ such that $Z_0^{(n)} \in Y_p$ and $Z_0^{(n)} \rightarrow Z_0$ in X_p .

Now, since $Y_p \subset X_\infty$, we have $Z_0^{(n)} \in X_\infty$, and [6, Theorem 1] gives us that there exists a unique solution

$$z^{(n)} \in L^\infty(\mathbb{R}_+; W_0^{1,\infty}(0,1)) \cap W^{1,\infty}(\mathbb{R}_+; L^\infty(0,1))$$

of (1.1) associated to the initial condition $(z_0^{(n)}, z_1^{(n)})$, and we have, for all $t \geq 0$,

$$(4.11) \quad \|(z^{(n)}, z_t^{(n)})\|_{X_\infty} \leq 2 \max \left(\|z_0^{(n)'}\|_{L^\infty(0,1)}, \|z_1^{(n)}\|_{L^\infty(0,1)} \right).$$

Denote $Z^{(n)} := (z^{(n)}, z_t^{(n)})$. We prove now that, for every $t_0 \geq 0$, the sequence $(Z^{(n)}(t_0, \cdot))$ is a Cauchy sequence in X_p . This will follow from the following observation:

$$\begin{aligned} \|Z^{(n)}(t_0) - Z^{(m)}(t_0)\|_{X_p}^p &= \left\| \left(z^{(n)}(t_0) - z^{(m)}(t_0), z_t^{(n)}(t_0) - z_t^{(m)}(t_0) \right) \right\|_{X_p}^p \\ &= \frac{1}{p} \int_0^1 \left| z_x^{(n)}(t_0) - z_x^{(m)}(t_0) \right|^p + \left| z_t^{(n)}(t_0) - z_t^{(m)}(t_0) \right|^p dx. \end{aligned}$$

Define for all $(t, x) \in \mathbb{R}_+ \times (0,1)$ the quantity $e^{(n,m)}$ as

$$(4.12) \quad e^{(n,m)} = z^{(n)} - z^{(m)}.$$

Then

$$\|Z^{(n)}(t_0) - Z^{(m)}(t_0)\|_{X_p}^p = \frac{1}{p} \int_0^1 \left| e_x^{(n,m)}(t_0) \right|^p + \left| e_t^{(n,m)}(t_0) \right|^p dx = \mathcal{E}_p(e^{(n,m)})(t_0),$$

where \mathcal{E}_p is defined in (2.8). But \mathcal{E}_p is equivalent to the energy defined in (2.9): there exists $C^* > C'_*$ such that

$$C_* E_p \leq \mathcal{E}_p \leq C^* E_p,$$

and therefore we obviously have

$$(4.13) \quad \|Z^{(n)}(t_0) - Z^{(m)}(t_0)\|_{X_p}^p \leq C^* E_p(e^{(n,m)})(t_0).$$

Then the property that the sequence $(Z^{(n)}(t_0, \cdot))_n$ is a Cauchy sequence in X_p will derive from a suitable estimate on $E_p(e^{(n,m)})(t_0)$, and this will be a consequence of the problem satisfied by the function $e^{(n,m)}$, which is the following one:

$$(4.14) \quad \begin{cases} e_{tt}^{(n,m)} - e_{xx}^{(n,m)} + a(x) \left(g(z_t^{(n)}) - g(z_t^{(m)}) \right) = 0, & (t, x) \in \mathbb{R}_+ \times (0, 1), \\ e^{(n,m)}(t, 0) = e^{(n,m)}(t, 1) = 0, & t \geq 0, \\ e^{(n,m)}(0, \cdot) = z_0^{(n)} - z_0^{(m)}, \quad e_t^{(n,m)}(0, \cdot) = z_1^{(n)} - z_1^{(m)}. \end{cases}$$

Since

$$g(y) - g(x) = \int_x^y g'(s) ds = \left(\int_0^1 g'(\tau y + (1-\tau)x) d\tau \right) (y-x),$$

we have

$$\begin{aligned} g(z_t^{(n)}) - g(z_t^{(m)}) &= \left(\int_0^1 g'(\tau z_t^{(n)} + (1-\tau)z_t^{(m)}) d\tau \right) (z_t^{(n)} - z_t^{(m)}) \\ &= \left(\int_0^1 g'(\tau z_t^{(n)} + (1-\tau)z_t^{(m)}) d\tau \right) e_t^{(n,m)}. \end{aligned}$$

Denoting

$$A^{(n,m)}(t, x) := a(x) \left(\int_0^1 g'(\tau z_t^{(n)} + (1-\tau)z_t^{(m)}) d\tau \right),$$

we see that $e^{(n,m)}$ satisfies

$$(4.15) \quad \begin{cases} e_{tt}^{(n,m)} - e_{xx}^{(n,m)} + A^{(n,m)}(t, x) e_t^{(n,m)} = 0, & (t, x) \in \mathbb{R}_+ \times (0, 1), \\ e^{(n,m)}(t, 0) = e^{(n,m)}(t, 1) = 0, & t \geq 0, \\ e^{(n,m)}(0, \cdot) = z_0^{(n)} - z_0^{(m)}, \quad e_t^{(n,m)}(0, \cdot) = z_1^{(n)} - z_1^{(m)}. \end{cases}$$

Proceeding as in Proposition 4.1, one has in the same way

$$\frac{d}{dt} E_p(e^{(n,m)})(t) = -\frac{1}{2} \int_0^1 A^{(n,m)}(t, x) \left(\rho^{(n,m)} - \xi^{(n,m)} \right) \left(\mathcal{F}'(\rho^{(n,m)}) - \mathcal{F}'(\xi^{(n,m)}) \right) dx,$$

where $\mathcal{F}(x) = |x|^p$, and $\rho^{(n,m)}$ and $\xi^{(n,m)}$ are the Riemann invariants associated to $e^{(n,m)}$: $\rho^{(n,m)} = e_x^{(n,m)} + e_t^{(n,m)}$, and $\xi^{(n,m)} = e_x^{(n,m)} - e_t^{(n,m)}$. Since g is non-increasing, g' is nonnegative, and thus $A^{(n,m)} \geq 0$. This implies that $E_p(e^{(n,m)})$ is nonincreasing on $[0, +\infty)$. It follows that

$$\forall t \geq 0, \quad E_p(e^{(n,m)})(t) \leq E_p(e^{(n,m)})(0),$$

and using (4.13), we have

$$\forall t_0 \geq 0, \quad \|Z^{(n)}(t_0) - Z^{(m)}(t_0)\|_{X_p}^p \leq C^* E_p(e^{(n,m)})(0) \leq \frac{C^*}{C_*} \|Z_0^{(n)} - Z_0^{(m)}\|_{X_p}^p.$$

Since $(Z_0^{(n)})_n$ is a Cauchy sequence in X_p , we obtain that for all $t_0 \geq 0$, the sequence $(Z^{(n)}(t_0, \cdot))_n$ is also a Cauchy sequence in X_p . It follows then that $(Z^n(t_0, \cdot))_n$ converges to a limit in X_p that we denote by $Z(t_0, \cdot) = (z(t_0, \cdot), z_t(t_0, \cdot))$. In particular,

for $t_0 = 0$, we have that $Z(0, \cdot) = Z_0(\cdot)$. Note also that the convergence is uniform with respect to $t_0 \geq 0$. We define the function $(t, x) \mapsto Z(t, x)$, where $Z(t_0, \cdot)$ is the limit of $Z^n(t_0, \cdot)$ in X_p for every $t_0 \geq 0$.

It remains to prove now that the limit Z is a weak solution of (1.1). Fix $T > 0$ and denote ψ a test function that belongs to $C^1([0, T] \times [0, 1])$ also verifying $\psi(T, \cdot) = \psi(0, \cdot) \equiv 0$ and $\psi(\cdot, 0) = \psi(\cdot, 1) \equiv 0$. Define

$$(4.16) \quad B_T^{(n)}(\psi) = \int_0^T \int_0^1 (z_{tt}^{(n)} - z_{xx}^{(n)}) \psi \, dx dt.$$

We have that

$$B_T^{(n)}(\psi) - B_T^{(m)}(\psi) = \int_0^T \int_0^1 (z_{tt}^{(n)} - z_{xx}^{(n)}) \psi \, dx dt - \int_0^T \int_0^1 (z_{tt}^{(m)} - z_{xx}^{(m)}) \psi \, dx dt.$$

By integrating by parts, it follows that

$$B_T^{(n)}(\psi) - B_T^{(m)}(\psi) = - \int_0^T \int_0^1 (z_t^{(n)} - z_t^{(m)}) \psi_t \, dx dt + \int_0^T \int_0^1 (z_x^{(m)} - z_x^{(n)}) \psi_x \, dx dt.$$

Using Hölder's inequality,

$$\begin{aligned} |B_T^{(n)}(\psi) - B_T^{(m)}(\psi)| &\leq \left(\int_0^T \int_0^1 |z_t^{(n)} - z_t^{(m)}|^p \, dx dt \right)^{\frac{1}{p}} \left(\int_0^T \int_0^1 |\psi_t|^q \, dx dt \right)^{\frac{1}{q}} \\ &\quad + \left(\int_0^T \int_0^1 |z_x^{(n)} - z_x^{(m)}|^p \, dx dt \right)^{\frac{1}{p}} \left(\int_0^T \int_0^1 |\psi_x|^q \, dx dt \right)^{\frac{1}{q}}, \end{aligned}$$

which means that

$$\left| B_T^{(n)}(\psi) - B_T^{(m)}(\psi) \right| \leq T^{\frac{1}{p}} E_p(e^{(n,m)})^{\frac{1}{p}}(0) (\|\psi_t\|_{L^q((0,T) \times (0,1))} + \|\psi_x\|_{L^q((0,T) \times (0,1))}).$$

By a density argument, we obtain that for all ψ in the space

$$\begin{aligned} \mathcal{X}_q^T &= \{ \psi : [0, T] \times [0, 1] \mapsto \mathbb{R}_+, (\psi, \psi_t) \in W^{1,q}((0, T) \times (0, 1)) \times L^q((0, T) \times (0, 1)), \\ &\quad \psi(T, \cdot) = \psi(0, \cdot) \equiv 0 \text{ and } \psi(\cdot, 0) = \psi(\cdot, 1) \equiv 0 \}, \end{aligned}$$

where q is the conjugate exponent of p and is equal to $\frac{p}{p-1}$, we have that

$$\left| B_T^{(n)}(\psi) - B_T^{(m)}(\psi) \right| \leq T^{\frac{1}{p}} E_p(e^{(n,m)})^{\frac{1}{p}}(0) (\|\psi_t\|_{L^q((0,T) \times (0,1))} + \|\psi_x\|_{L^q((0,T) \times (0,1))}).$$

Then

$$\left| B_T^{(n)}(\psi) - B_T^{(m)}(\psi) \right| \leq T^{\frac{1}{p}} E_p(e^{(n,m)})^{\frac{1}{p}}(0) \|\psi\|_{\mathcal{X}_q^T},$$

which gives that $(B_T^{(n)})_n$ is a Cauchy sequence in $(\mathcal{X}_q^T)'$, the dual of \mathcal{X}_q^T , since $(Z_0^{(n)})_n$ is in X_p . We conclude then that $(B_T^{(n)})_n$ converges in $(\mathcal{X}_q^T)'$, i.e., $(z_{tt}^{(n)} - z_{xx}^{(n)})_n$ converges to $z_{tt} - z_{xx}$ as a linear functional on \mathcal{X}_q^T . We also know that for all $n \in \mathbb{N}$,

$$z_{tt}^{(n)} - z_{xx}^{(n)} = -a(x)g(z_t^{(n)}), \text{ on } \mathbb{R}_+ \times (0, 1),$$

which means that, the sequence of linear functionals $(-a(x)g(z_t^{(n)}))_n$ defined on \mathcal{X}_q^T also converges $z_{tt}(t, \cdot) - z_{xx}(t, \cdot)$ in $(\mathcal{X}_q^T)'$. We now use the existence of weak solutions in L^2 framework (see [18], [11]). For $(z_0, z_1) \in X_p$ with $p \geq 2$, we have the existence of a unique weak solution $z \in W^{1,\infty}(\mathbb{R}_+, L^2(0, 1)) \cap L^\infty(\mathbb{R}_+, H_0^1(0, 1))$. The solution z satisfies that for almost every $t \in \mathbb{R}_+$

$$(4.17) \quad z_{tt}(t, \cdot) - z_{xx}(t, \cdot) = -a(x)g(z_t(t, \cdot)) \quad \text{in } H^{-1}(0, 1).$$

In particular, for every $T > 0$, $z_{tt}(t, \cdot) - z_{xx}(t, \cdot) = -a(x)g(z_t)$ belongs to $(\mathcal{X}_2^T)'$. Since $q \leq 2$, one has that $(\mathcal{X}_q^T)' \subset (\mathcal{X}_2^T)'$, yielding in particular that z is a weak solution of System (1.1) in X_p . \square

4.4. Proof of Theorem 2.2.

Take $Z_0 = (z_0, z_1) \in Y_p$ for all $1 \leq p < \infty$. Using Sobolev embeddings classical results, we have that $Z_0 \in X_\infty$. Then we can use once again [6, Theorem 1], and we have the existence of a unique solution z of (1.1) such that

$$(z, z_t) \in L^\infty(\mathbb{R}_+; W_0^{1,\infty}(0,1)) \times W^{1,\infty}(\mathbb{R}_+; L^\infty(0,1)),$$

and, for all $t \geq 0$,

$$(4.18) \quad \|(z, z_t)\|_{X_\infty} \leq 2 \max(\|z_0'\|_{L^\infty(0,1)}, \|z_1\|_{L^\infty(0,1)}).$$

To have more information on z_t , we note that $w := z_t$ satisfies

$$(4.19) \quad \begin{cases} w_{tt} - w_{xx} = -a(x)g'(w)w_t & \text{in } \mathbb{R}_+ \times (0,1), \\ w(t,0) = w(t,1) = 0 & \forall t \in \mathbb{R}_+, \\ w(0,\cdot) = z_1, \quad w_t(0,\cdot) = z_0'' - g(z_1). \end{cases}$$

We define for all $(t, x) \in \mathbb{R}_+ \times (0,1)$ the associated Riemann invariants for (4.19):

$$(4.20) \quad u = w_x + w_t,$$

$$(4.21) \quad v = w_x - w_t.$$

Along strong solutions of (4.19), we have

$$(4.22) \quad \begin{cases} u_t - u_x = -\frac{a(x)}{2}g'(w)(u-v) & \text{in } \mathbb{R}_+ \times (0,1), \\ v_t + v_x = \frac{a(x)}{2}g'(w)(u-v) & \text{in } \mathbb{R}_+ \times (0,1), \\ u(t,0) - v(t,0) = u(t,1) - v(t,1) = 0 & \forall t \in \mathbb{R}_+. \end{cases}$$

Introducing

$$\tilde{a}(t, x) := a(x)g'(w),$$

Problem (4.22) can be written

$$(4.23) \quad \begin{cases} u_t - u_x = -\tilde{a}(t, x)\frac{u-v}{2} & \text{in } \mathbb{R}_+ \times (0,1), \\ v_t + v_x = \tilde{a}(t, x)\frac{u-v}{2} & \text{in } \mathbb{R}_+ \times (0,1), \\ u(t,0) - v(t,0) = u(t,1) - v(t,1) = 0 & \forall t \in \mathbb{R}_+. \end{cases}$$

Then, we consider the p -th energy associated with w :

$$(4.24) \quad E_p(w)(t) = \frac{1}{p} \int_0^1 (|u|^p + |v|^p) dx,$$

and proceeding as in Corollary 4.1, we see that $E_p(w)$ is nonincreasing, the fundamental remark to obtain that property being that the function \tilde{a} is nonnegative, since $g' \geq 0$. Hence

$$E_p(w)(t) \leq E_p(w)(0), \quad \text{for a.e. } t \geq 0.$$

Then, using the fact that $E_p(w)$ is an equivalent energy to $\mathcal{E}_p(w)$, it follows that

$$\int_0^1 |w_x|^p dx \leq pC_p E_p(w)(0).$$

Therefore

$$\|z_t\|_{W^{1,p}(0,1)} \leq (pC^* E_p(w)(0))^{\frac{1}{p}},$$

and this implies that $(z, z_t) \in Y_p$ for all $t \geq 0$, and yields the required regularity for a strong solution. \square

Remark 4.2. For strong solutions in the case $p \geq 2$, we can easily use the results that have been proved in [8] for $p \geq 2$ to prove the well-posedness. Indeed, let $(z_0, z_1) \in Y_p$, with $p \geq 2$ this implies that $(z_0, z_1) \in (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$. We have then the existence of a unique strong solution

$$z \in C(\mathbb{R}_+, H_0^1(0,1)) \cap C^1(\mathbb{R}_+, L^2(0,1)),$$

such that

$$(z(t, \cdot), z_t(t, \cdot)) \in (H^2(0, 1) \cap H_0^1(0, 1)) \times H_0^1(0, 1), \quad \forall t \in \mathbb{R}_+,$$

which means that $z_t(t, \cdot) \in L^\infty(0, 1)$ we can then use [8, Corollary 2.3, item (ii)] that implies in our context that if $(z_0, z_1) \in Y_p$ then the solution $z \in L^\infty(\mathbb{R}_+, W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1))$ and $z_t \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0, 1))$ which guarantees the well-posedness in Y_p .

4.5. Proof of Theorem 2.3.

Now assume Hypotheses (\mathbf{H}_1) and (\mathbf{H}_3) . In that case, the argument follows closely [7, Corollary 2.3] or [12, Theorem 3.3] and we refer the reader to these references. \square

5. PROOF OF THEOREM 2.4

5.1. Asymptotic stability: the key integral inequality on the energy.

For the rest of the paper, we will manipulate (essentially) only strong solutions of (1.1) and a standard density will complete the arguments in order to obtain conclusions relative to weak solutions of (1.1).

In the following we assume that $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is increasing and concave. The goal of this section is to prove the next lemma, that will be useful to determine the asymptotic behaviour of the energy. The function ϕ will be chosen later.

Lemma 5.1. *There exists $C_u > 0$ (independent of the initial conditions) such that, for every pair of times $0 \leq S \leq T$ and, it holds*

$$(5.1) \quad \int_S^T E_p(t)^2 \phi'(t) dt \leq C_u E_p(S)^2 \phi'(S) \\ + C_u \int_S^T E_p(t) \phi'(t) \int_0^1 a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt.$$

The proof of Lemma 5.1 is based on the estimates (5.4), (5.6) and (5.9), that are consequences of the identities (5.3), (5.5) and (5.8). These identities will be obtained using the multiplier method, and we will use some cut-off functions in order to localize some computations in the damping region.

In order to make the paper easier to read, we first give the main identities (obtained by the multiplier method) and their consequences (see Lemmas 5.2-5.4). We next prove that Lemmas 5.2-5.4 imply Lemma 5.1 and then show how Lemma 5.1 implies Theorem 2.4. Finally, we prove Lemmas 5.2-5.4 with the multiplier method and suitable estimates.

5.2. Multiplier method: useful estimates.

5.2.a. Some notations.

Consider

$$(5.2) \quad F(s) := \frac{|s|^p}{p}, \quad f(s) := F'(s) = [s]^{p-1}.$$

Choose $a_3 < a_2 < a_1 < a_0 < b_0 < b_1 < b_2 < b_3$ such that $(a_3, b_3) \subset\subset \omega$, and $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3$ and introduce the following cut-off functions:

- function $\psi_1 : \mathbb{R} \rightarrow [0, 1]$, smooth, equal to 1 on $(-\infty, a_1] \cup [b_1, +\infty)$ and equal to 0 on $[a_0, b_0]$;
- function $\psi_2 : \mathbb{R} \rightarrow [0, 1]$, smooth, equal to 0 on $(-\infty, a_2] \cup [b_2, +\infty)$ and equal to 1 on $[a_1, b_1]$;
- function $\psi_3 : \mathbb{R} \rightarrow [0, 1]$, smooth, equal to 0 on $(-\infty, a_3] \cup [b_3, +\infty)$ and equal to 1 on $[a_2, b_2]$.

Of course, we have that there exists $C > 0$ such that $\psi_3 \leq Ca$.

5.2.b. First multiplier, first identity, first estimate.

Multiplying the first equation of (4.3) by $E_p \phi' x \psi_1(x) f(\rho)$ and the second equation of (4.3) by $E_p \phi' x \psi_1(x) f(\xi)$, we obtain the following first identity: along strong solutions of (1.1), it holds

$$\begin{aligned}
(5.3) \quad \int_S^T E_p(t)^2 \phi'(t) dt &= \left[E_p(t) \phi'(t) \int_0^1 x \psi_1(x) (F(\xi) - F(\rho)) dx \right]_S^T \\
&+ \int_S^T E_p(t) \phi'(t) \int_0^1 (1 - (x \psi_1(x))') (F(\rho) + F(\xi)) dx dt \\
&+ \int_S^T (E_p(t) \phi'(t))' \int_0^1 x \psi_1(x) (F(\rho) - F(\xi)) dx dt \\
&- \int_S^T E_p(t) \phi'(t) \int_0^1 x a(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) + f(\xi)) dx dt.
\end{aligned}$$

This first identity (5.3) allows us to obtain the following first estimate:

Lemma 5.2. *There exists $C_1 > 0$ independent of the initial conditions such that*

$$\begin{aligned}
(5.4) \quad \int_S^T E_p(t)^2 \phi'(t) dt &\leq \frac{7}{2} E_p(S)^2 \phi'(S) \\
&+ C_1 \int_S^T E_p(t) \phi'(t) \int_0^1 \psi_2(x) (\rho f(\rho) + \xi f(\xi)) dx dt \\
&- \int_S^T E_p(t) \phi'(t) \int_0^1 x a(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) + f(\xi)) dx dt.
\end{aligned}$$

5.2.c. Second multiplier, second identity, second estimate.

Multiplying the first equation of (4.3) by $E_p \phi' \psi_2(x) f'(\rho) z$ and the second equation of (4.3) by $E_p \phi' \psi_2(x) f'(\xi) z$, we obtain the following second identity:

$$\begin{aligned}
(5.5) \quad \int_S^T E_p(t) \phi'(t) \int_0^1 \psi_2(x) (\rho f(\rho) + \xi f(\xi)) dx dt \\
&= \left[E_p(t) \phi'(t) \int_0^1 \psi_2(x) z (f(\xi) - f(\rho)) dx \right]_S^T \\
&+ \int_S^T (E_p(t) \phi'(t))' \int_0^1 \psi_2(x) z (f(\rho) - f(\xi)) dx dt \\
&- \int_S^T E_p(t) \phi'(t) \int_0^1 \psi_2'(x) z (f(\rho) + f(\xi)) dx dt \\
&- \int_S^T E_p(t) \phi'(t) \int_0^1 \psi_2(x) a(x) z (f'(\rho) + f'(\xi)) dx dt \\
&\quad + 2 \int_S^T E_p(t) \phi'(t) \int_0^1 \psi_2(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt.
\end{aligned}$$

This second identity (5.5) allows us to obtain the following second estimate:

Lemma 5.3. *There exists $C'_2 > 0$ independent of the initial conditions such that*

$$(5.6) \quad \int_S^T E_p(t)^2 \phi'(t) dt \leq C'_2 E_p(S)^2 \phi'(S) + C'_2 \int_S^T E_p(t) \phi'(t) \left(\int_{a_2}^{b_2} |z|^p dx \right) dt \\ + 2C_1 \int_S^T E_p(t) \phi'(t) \int_0^1 \psi_2(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt \\ - 2 \int_S^T E_p(t) \phi'(t) \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) + f(\xi)) dx dt.$$

5.2.d. **Third multiplier, third identity, third estimate.**

Given $t > 0$, consider the solution $v^{(t)}$ of the elliptic problem

$$(5.7) \quad \begin{cases} v_{xx} = \psi_3(x) f(z(t, x)) & x \in (0, 1), \\ v(0) = 0 = v(1). \end{cases}$$

Multiplying the first equation and the second equation of (4.3) by $E_p \phi' v$, we obtain the following third identity:

$$(5.8) \quad 2 \int_S^T E_p \phi' \int_0^1 \psi_3(x) |z|^p = \left[E_p \phi' \int_0^1 v(\rho - \xi) \right]_S^T \\ - \int_S^T (E_p \phi')' \int_0^1 v(\rho - \xi) - \int_S^T E_p \phi' \int_0^1 v_t(\rho - \xi) \\ + 2 \int_S^T E_p \phi' \int_0^1 a(x) v g\left(\frac{\rho - \xi}{2}\right).$$

This third identity (5.8) allows us to obtain the following third estimate:

Lemma 5.4. *There exists $C_4 > 0$ independent of the initial conditions such that*

$$(5.9) \quad \int_S^T E_p^2 \phi' \leq C_4 E_p(S)^2 \phi'(S) + C_4 \int_S^T E_p \phi' \int_0^1 \psi_3(x) |\rho - \xi|^p dx \\ + 4C_3 \int_S^T E_p \phi' \int_0^1 a(x) v g\left(\frac{\rho - \xi}{2}\right) + C_4 \int_S^T E_p \phi' \int_0^1 \psi_2(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) \\ - 4 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) + f(\xi)).$$

We also recall the following

Lemma 5.5. ([12]) *Consider v the solution of (5.7). Then there exist positive constants C, C_p such that we have the following estimates:*

$$(5.10) \quad \forall t > 0, \quad \int_0^1 |v(t, x)|^q dx \leq CC_p E_p(t),$$

and, for all $\sigma > 0$,

$$(5.11) \quad \forall t > 0, \quad \int_0^1 |v_t(t, x)|^q dx \leq C_p(p-2)\sigma E_p(t) + \frac{C_p}{\sigma^{p-2}} \int_0^1 \psi_3(x) |z_t(t, x)|^p dx.$$

5.3. Proof of Lemma 5.1 assuming Lemmas 5.2-5.4.

The starting point is (5.9), and the goal is to estimate the second, third, and fifth terms.

The third term of the right hand side of (5.9): using (5.10) and the fact that g is globally Lipschitz, we have

$$\begin{aligned} \left| \int_0^1 a(x) v(t, x) g\left(\frac{\rho - \xi}{2}\right) dx \right| &\leq C \left(\int_0^1 |v(t, x)|^q dx \right)^{1/q} \left(\int_0^1 a(x)^p |g\left(\frac{\rho - \xi}{2}\right)|^p dx \right)^{1/p} \\ &\leq C' \left(CC_p E_p(t) \right)^{1/q} \left(\int_0^1 a(x) |\rho - \xi|^p dx \right)^{1/p}. \end{aligned}$$

Then, as we did before, choosing $\sigma > 0$, we have

$$\left| \int_0^1 a(x) v(t, x) g\left(\frac{\rho - \xi}{2}\right) dx \right| \leq C' \left[\frac{\sigma CC_p}{q} E_p(t) + \frac{1}{p\sigma^{p/q}} \int_0^1 a(x) |\rho - \xi|^p dx \right].$$

Therefore

$$\begin{aligned} C_4 \int_S^T E_p \phi' \int_0^1 a(x) v g\left(\frac{\rho - \xi}{2}\right) &\leq C_4 C' \frac{\sigma CC_p}{q} \int_S^T E_p^2 \phi' dt \\ &\quad + \frac{C_4 C'}{p\sigma^{p/q}} \int_S^T E_p \phi' \int_0^1 a(x) |\rho - \xi|^p dx dt, \end{aligned}$$

and choosing σ such that $C_4 C' \frac{\sigma CC_p}{q} = \frac{1}{2}$, and remembering that $\psi_3 \leq Ca$, we obtain that

$$\begin{aligned} (5.12) \quad \int_S^T E_p^2 \phi' &\leq 2C_4 E_p(S)^2 \phi'(S) + C_4' \int_S^T E_p \phi' \int_0^1 a(x) |\rho - \xi|^p dx \\ &\quad + C_4 \int_S^T E_p \phi' \int_0^1 \psi_2(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) \\ &\quad - 4 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) + f(\xi)). \end{aligned}$$

The fifth term of the right hand side of (5.9): We note that

$$\begin{aligned} -4 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) + f(\xi)) & \\ = 4 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi) - 2f(\rho)) & \\ = 4 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) & \\ + 4 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (-2f(\rho)) & \\ \leq 4 \int_S^T E_p \phi' \int_0^1 a(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) & \\ + 8 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) |g\left(\frac{\rho - \xi}{2}\right)| |\rho|^{p-1}. & \end{aligned}$$

On the other hand, one has

$$\begin{aligned} 4 \int_S^T E_p \phi' \int_0^1 a(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) &= 4 \int_S^T E_p \phi' (-E_p') dt \\ &\leq 4\phi'(S) \int_S^T -E_p' E_p dt = 4\phi'(S) \left[\frac{-1}{2} E_p(t)^2 \right]_S^T \leq 2E_p(S)^2 \phi'(S), \end{aligned}$$

and also

$$\begin{aligned}
8 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) |g\left(\frac{\rho - \xi}{2}\right)| |\rho|^{p-1} &\leq 8C_g \int_S^T E_p \phi' \int_0^1 a(x) \left|\frac{\rho - \xi}{2}\right| |\rho|^{p-1} \\
&\leq 4C_g \int_S^T E_p \phi' \left(\int_0^1 a(x)^p |\rho - \xi|^p\right)^{1/p} \left(\int_0^1 |\rho|^p\right)^{1/q} dt \\
&\leq 4C_g \int_S^T E_p \phi' \left(\int_0^1 a(x)^p |\rho - \xi|^p\right)^{1/p} \left(E_p(t)\right)^{1/q} dt \\
&\leq \frac{1}{4} \int_S^T E_p^2 \phi' dt + C \int_S^T E_p \phi' \int_0^1 a(x) |\rho - \xi|^p dx dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
-4 \int_S^T E_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) + f(\xi)) \\
\leq 2E_p(S)^2 \phi'(S) + \frac{1}{4} \int_S^T E_p^2 \phi' dt + C \int_S^T E_p \phi' \int_0^1 a(x) |\rho - \xi|^p dx dt,
\end{aligned}$$

and with (5.12), we obtain that

$$\begin{aligned}
(5.13) \quad \int_S^T E_p^2 \phi' &\leq 3C_4 E_p(S)^2 \phi'(S) + C_4'' \int_S^T E_p \phi' \int_0^1 a(x) |\rho - \xi|^p dx \\
&\quad + C_4 \int_S^T E_p \phi' \int_0^1 \psi_2(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)).
\end{aligned}$$

The second term of the right hand side of (5.9): we claim that there exists $C > 0$ such that

$$(5.14) \quad \forall a, b \in \mathbb{R}, \quad |a - b|^p \leq C(a - b)(f(a) - f(b)).$$

Indeed, (5.14) is obvious if $ab = 0$: assume that $b = 0$, then $|a - b|^p = |a|^p = af(a) = (a - b)(f(a) - f(b))$.

Next, (5.14) also holds if $ab < 0$: assume that $b < 0 < a$, and denote $d := -b$; then

$$|a - b|^p = (a + d)^p \leq C(a^p + d^p),$$

and

$$(a - b)(f(a) - f(b)) = (a + d)(f(a) + f(d)) = (a + d)(a^{p-1} + d^{p-1}) \geq a^p + d^p,$$

which implies that (5.14) holds if $ab < 0$.

Last, (5.14) also holds if $ab > 0$. One can assume that a and b are positive and that $b < a$. Then $b = \theta a$ with some $\theta \in [0, 1]$; and

$$|a - b|^p = a^p(1 - \theta)^p, \quad (a - b)(f(a) - f(b)) = a^p(1 - \theta)(1 - \theta^{p-1}).$$

Denote $s := 1 - \theta$. The function $s \in (0, 1] \mapsto \frac{s^{p-1}}{1 - (1-s)^{p-1}}$ is continuous on $(0, 1]$ and goes to 0 as $s \rightarrow 0^+$, therefore this function is bounded on $(0, 1]$. Hence there exists $C > 0$ such that

$$\forall s \in (0, 1], \quad \frac{s^{p-1}}{1 - (1-s)^{p-1}} \leq C,$$

hence

$$\forall s \in (0, 1], \quad s^{p-1} \leq C(1 - (1-s)^{p-1}).$$

Coming back to $\theta = 1 - s$:

$$\forall \theta \in [0, 1), \quad (1 - \theta)^{p-1} \leq C(1 - \theta^{p-1}).$$

Therefore

$$\forall \theta \in [0, 1), \quad (1 - \theta)^p \leq C(1 - \theta)(1 - \theta^{p-1}),$$

which gives that

$$|a - b|^p = a^p(1 - \theta)^p \leq Ca^p(1 - \theta)(1 - \theta^{p-1}) = (a - b)(f(a) - f(b)).$$

Therefore (5.14) holds.

And then, we deduce from (5.14) that

$$(5.15) \quad C_4'' \int_S^T E_p \phi' \int_0^1 a(x) |\rho - \xi|^p dx \\ \leq CC_4'' \int_S^T E_p \phi' \int_0^1 a(x) (\rho - \xi)(f(\rho) - f(\xi)) dx.$$

Since $\psi_3 \leq Ca$ on $(0, 1)$, (5.13) and (5.15) give (5.1). This concludes the proof of Lemma 5.1. \square

5.4. Asymptotic stability: Proof of Theorem 2.4, part (i).

We recall that we know from (4.10) that

$$E_p'(t) = - \int_0^1 a(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) dx.$$

Now assume that there exists $0 < \alpha < \beta$ such that

$$\forall s \in \mathbb{R}, \quad \alpha s \leq g(s) \leq \beta s.$$

Then choose $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\phi(t) = t$: Lemma 5.1 gives that

$$(5.16) \quad \int_S^T E_p(t)^2 dt \leq C_u E_p(S)^2 + C_u \int_S^T E_p(t) \int_0^1 a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt.$$

If $\rho > \xi$, we have

$$0 < \frac{\rho - \xi}{2} \leq \frac{1}{\alpha} g\left(\frac{\rho - \xi}{2}\right) \quad \text{and} \quad (f(\rho) - f(\xi)) > 0,$$

hence

$$\frac{\rho - \xi}{2} (f(\rho) - f(\xi)) \leq \frac{1}{\alpha} g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi));$$

if $\rho \leq \xi$, we have

$$\frac{\rho - \xi}{2} \geq \frac{1}{\beta} g\left(\frac{\rho - \xi}{2}\right) \quad \text{and} \quad (f(\rho) - f(\xi)) < 0,$$

hence

$$\frac{\rho - \xi}{2} (f(\rho) - f(\xi)) \leq \frac{1}{\beta} g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) \leq \frac{1}{\alpha} g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)).$$

Therefore (5.16) gives that

$$\int_S^T E_p^2 \leq C_u E_p(S)^2 + \frac{C_u}{\alpha} \int_S^T E_p \int_0^1 a(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) \\ = C_u E_p(S)^2 + \frac{C_u}{\alpha} \int_S^T E_p (-E_p') \\ = C_u E_p(S)^2 + \frac{C_u}{\alpha} \left[\frac{-1}{2} E_p(t)^2 \right]_S^T \leq \left(C_u + \frac{C_u}{2\alpha} \right) E_p(S)^2,$$

and then a classical Gronwall type inequality ([14]) gives that E_p^2 decays exponentially to 0, as proved in [12] in the linear case. \square

5.5. Asymptotic stability: Proof of Theorem 2.4, part (ii).

Consider the assumptions of Theorem 2.4, part (ii): $g'(0) = 0$, and there is an increasing and odd function $g_0 : [-1, 1] \rightarrow \mathbb{R}$ and $0 < \alpha \leq \beta$ such that (2.3) holds. Moreover, the function

$$(5.17) \quad H(s) := \frac{g_0(s)}{s},$$

is C^1 , satisfies $H(0) = 0$ and is increasing on $[0, \eta]$ for some $\eta > 0$.

Now, assume that $\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is C^1 , positive, $\varepsilon(0) \leq \eta$ and decreasing to 0 as $t \rightarrow +\infty$. The function ε will be chosen later. And we choose

$$(5.18) \quad \phi'(t) = H(\varepsilon(t)),$$

which is well defined and decreasing on \mathbb{R}_+ .

5.5.a. Decomposition in subdomains and application.

Given $t \geq 0$, consider the following partition of $(0, 1)$:

$$(5.19) \quad \begin{cases} \Omega_1^{(t)} := \{x \in (0, 1), |\rho(t, x) - \xi(t, x)| \leq 2\varepsilon(t)\}, \\ \Omega_2^{(t)} := \{x \in (0, 1), 2\varepsilon(t) < |\rho(t, x) - \xi(t, x)| \leq 2\eta\}, \\ \Omega_3^{(t)} := \{x \in (0, 1), |\rho(t, x) - \xi(t, x)| > 2\eta\}. \end{cases}$$

In the following, we decompose the integral appearing on the right-hand side of (5.1) according to the above domain decomposition,

$$\begin{aligned} & \int_S^T E_p \phi' \int_0^1 a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt \\ &= \int_S^T E_p \phi' \int_{\Omega_1^{(t)}} a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt \\ &+ \int_S^T E_p \phi' \int_{\Omega_2^{(t)}} a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt \\ &+ \int_S^T E_p \phi' \int_{\Omega_3^{(t)}} a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt. \end{aligned}$$

Estimate on $\Omega_3^{(t)}$. Here we have

$$\frac{\rho - \xi}{2} (f(\rho) - f(\xi)) = \left| \frac{\rho - \xi}{2} \right| |f(\rho) - f(\xi)|.$$

There exists $\alpha' \in (0, \alpha]$ such that

$$\forall |s| \geq \eta, \quad |g(s)| \geq \alpha' |s|.$$

Therefore

$$\left| \frac{\rho - \xi}{2} \right| |f(\rho) - f(\xi)| \leq \frac{1}{\alpha'} \left| g\left(\frac{\rho - \xi}{2}\right) \right| |f(\rho) - f(\xi)| = \frac{1}{\alpha'} g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)).$$

Therefore

$$\begin{aligned}
& \int_S^T E_p \phi' \int_{\Omega_3^{(t)}} a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt \\
& \leq \int_S^T E_p \phi' \int_{\Omega_3^{(t)}} a(x) \frac{1}{\alpha'} g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) dx dt \\
& \leq \frac{1}{\alpha'} \int_S^T E_p \phi' \int_0^1 a(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) dx dt \\
& = \frac{1}{\alpha'} \int_S^T E_p \phi' (-E_p') dt \leq \frac{1}{\alpha'} \phi'(S) \left[\frac{-1}{2} E_p(t)^2 \right]_S^T,
\end{aligned}$$

hence

$$(5.20) \quad \int_S^T E_p \phi' \int_{\Omega_3^{(t)}} a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt \leq \frac{1}{2\alpha'} E_p(S)^2 \phi'(S).$$

Estimate on $\Omega_2^{(t)}$. If $x \in \Omega_2^{(t)}$, then $\left| \frac{\rho - \xi}{2} \right| \in [\varepsilon(t), \eta]$, hence

$$H\left(\left| \frac{\rho - \xi}{2} \right|\right) \geq H(\varepsilon(t)) = \phi'(t),$$

which brings

$$g_0\left(\left| \frac{\rho - \xi}{2} \right|\right) \geq \phi'(t) \left| \frac{\rho - \xi}{2} \right|.$$

We deduce that

$$\forall x \in \Omega_2^{(t)}, \quad \phi'(t) \left| \frac{\rho - \xi}{2} \right| \leq g_0\left(\left| \frac{\rho - \xi}{2} \right|\right) = \left| g_0\left(\frac{\rho - \xi}{2}\right) \right| \leq \left| g\left(\frac{\rho - \xi}{2}\right) \right|.$$

Then

$$\begin{aligned}
& \int_S^T E_p \phi' \int_{\Omega_2^{(t)}} a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt \\
& = \int_S^T E_p \int_{\Omega_2^{(t)}} a(x) \phi'(t) \left| \frac{\rho - \xi}{2} \right| |f(\rho) - f(\xi)| dx dt \\
& \leq \int_S^T E_p \int_{\Omega_2^{(t)}} a(x) \left| g\left(\frac{\rho - \xi}{2}\right) \right| |f(\rho) - f(\xi)| dx dt \\
& = \int_S^T E_p \int_{\Omega_2^{(t)}} a(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) dx dt \\
& \leq \int_S^T E_p \int_0^1 a(x) g\left(\frac{\rho - \xi}{2}\right) (f(\rho) - f(\xi)) dx dt = \int_S^T E_p (-E_p') dt.
\end{aligned}$$

Therefore

$$(5.21) \quad \int_S^T E_p \phi' \int_{\Omega_1^{(t)}} a(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) dx dt \leq \frac{1}{2} E_p(S)^2.$$

Estimate on $\Omega_1^{(t)}$. Assume that $|\rho(t, x) - \xi(t, x)| \leq 2\varepsilon(t)$. If $\rho \geq \xi$, then the mean value theorem gives us that

$$\left| f(\rho) - f(\xi) \right| \leq \left(\sup_{(\xi, \xi + 2\varepsilon(t))} |f'| \right) |\rho - \xi| \leq (p-1) \left(|\xi| + 2\varepsilon(t) \right)^{p-2} 2\varepsilon(t).$$

And if $\rho \leq \xi$, then

$$\left| f(\rho) - f(\xi) \right| \leq (p-1) \left(|\rho| + 2\varepsilon(t) \right)^{p-2} 2\varepsilon(t).$$

Then, in any case, we have

$$\left| f(\rho) - f(\xi) \right| \leq (p-1) \left(|\rho| + |\xi| + 2\varepsilon(t) \right)^{p-2} 2\varepsilon(t).$$

Then we obtain that

$$\begin{aligned} & \int_S^T E_p \phi' \int_{\Omega_1^{(t)}} a(x) \frac{\rho - \xi}{2} \left(f(\rho) - f(\xi) \right) dx dt \\ & \leq \int_S^T E_p \phi' \int_{\Omega_1^{(t)}} 2(p-1) a(x) \varepsilon(t)^2 \left(|\rho| + |\xi| + 2\varepsilon(t) \right)^{p-2} dx dt \\ & \leq C \int_S^T E_p(t) \phi'(t) \varepsilon(t)^2 \left(\int_{\Omega_1^{(t)}} \left(|\rho| + |\xi| + 2\varepsilon(t) \right)^p dx \right)^{(p-2)/p} dt \\ & \leq C' \int_S^T E_p(t) \phi'(t) \varepsilon(t)^2 \left(\int_{\Omega_1^{(t)}} |\rho|^p + |\xi|^p + \varepsilon(t)^p dx \right)^{(p-2)/p} dt \\ & \leq C' \int_S^T E_p(t) \phi'(t) \varepsilon(t)^2 \left(p E_p(t) + \varepsilon(t)^p \right)^{(p-2)/p} dt \\ & \leq C'' \int_S^T E_p(t) \phi'(t) \varepsilon(t)^2 \left(E_p(t)^{(p-2)/p} + \varepsilon(t)^{p-2} \right) dt. \end{aligned}$$

Therefore

$$(5.22) \quad \int_S^T E_p \phi' \int_{\Omega_1^{(t)}} a(x) \frac{\rho - \xi}{2} \left(f(\rho) - f(\xi) \right) dx dt \leq C'' \int_S^T E_p(t)^{1+(p-2)/p} \phi'(t) \varepsilon(t)^2 dt + C'' \int_S^T E_p(t) \phi'(t) \varepsilon(t)^p dt.$$

Consequence on Lemma 5.1. Using (5.20)-(5.22), we deduce from (5.1) that

$$(5.23) \quad \int_S^T E_p^2 \phi' \leq C'_u E_p(S)^2 + C'_u \int_S^T E_p(t)^{1+(p-2)/p} \phi'(t) \varepsilon(t)^2 dt + C'_u \int_S^T E_p(t) \phi'(t) \varepsilon(t)^p dt.$$

In the following, we choose the function ε and we deduce some decay of the energy.

5.5.b. Some possible choices for the functions ϕ and ε and their properties.

A way to obtain an estimate on the decay of the energy is to note that since E_p is nonincreasing, we can derive from (5.23) that

$$(5.24) \quad \int_S^T E_p^2 \phi' \leq C'_u E_p(S)^2 + C'_u E_p(S)^{1+(p-2)/p} \int_S^T \phi'(t) \varepsilon(t)^2 dt + C'_u E_p(S) \int_S^T \phi'(t) \varepsilon(t)^p dt.$$

The trick is now to choose the function ε such that we can compute the integrals appearing on the right-hand side of (5.24): using (5.18), we have

$$\varepsilon(t) = H^{-1}(\phi'(t)),$$

which gives

$$\begin{aligned} \int_S^T \phi'(t) \varepsilon(t)^2 dt &= \int_S^T \left[H^{-1}(\phi'(t)) \right]^2 \phi'(t) dt \\ &= \int_{\phi(S)}^{\phi(T)} \left[H^{-1}(\phi'(\phi^{-1}(s))) \right]^2 ds = \int_{\phi(S)}^{\phi(T)} \left[H^{-1}\left(\frac{1}{(\phi^{-1})'(s)}\right) \right]^2 ds. \end{aligned}$$

Then, computations would be easy if, for some $m \geq 1$, we had

$$H^{-1}\left(\frac{1}{(\phi^{-1})'(s)}\right) = \frac{1}{(s+A)^m},$$

where A is a constant. With that idea,

- first, given $m \geq 1$, let us consider

$$(5.25) \quad \psi_m(s) := \int_0^s \frac{1}{H\left(\frac{1}{(\sigma+A)^m}\right)} d\sigma :$$

the function ψ_m is well defined, of class C^2 , and

$$\psi'_m(s) = \frac{1}{H\left(\frac{1}{(s+A)^m}\right)} > 0.$$

Hence ψ_m and ψ'_m are increasing, hence ψ_m is convex; and $\psi'_m(s) \rightarrow +\infty$ as $s \rightarrow +\infty$.

- Next, let us consider

$$(5.26) \quad \phi_m := \psi_m^{-1}.$$

Of course ϕ_m is C^2 , increasing and concave. Moreover, fix $A = \frac{1}{\eta^{1/m}}$, take $t \geq 0$ and denote $s := \phi_m(t)$; then $t = \phi_m^{-1}(s) = \psi_m(s)$, and

$$\phi'_m(t) = \phi'_m(\psi_m(s)) = \frac{1}{\psi'_m(s)} = H\left(\frac{1}{(s+A)^m}\right).$$

In particular, $\phi'_m(0) = H\left(\frac{1}{A^m}\right) = H(\eta)$, and $\phi'_m(t) \rightarrow 0$ as $t \rightarrow +\infty$, hence $\phi'_m([0, +\infty)) = (0, H(\eta)]$.

- Finally, let us choose

$$(5.27) \quad \varepsilon_m(t) := H^{-1}(\phi'_m(t)) :$$

from the previous remarks, ϕ'_m takes its values in the region where H^{-1} is well-defined, hence ε_m is well-defined on $[0, +\infty)$, and C^1 by composition, and decreasing, with $\varepsilon_m(0) = \eta$, and $\varepsilon_m(t) \rightarrow 0$ as $t \rightarrow +\infty$.

Therefore, these choices have all the requirements we assumed to derive (5.24), and are going to allow us to estimate the decay of the energy E_p .

To conclude, we will need to have an estimate of the asymptotic behaviour of ϕ_m as $t \rightarrow +\infty$: it is related to the behaviour of H (and thus of g_0) as $x \rightarrow 0^+$:

$$\forall s \geq A, \quad \psi_m(s) \leq \frac{s}{H\left(\frac{1}{(s+A)^m}\right)} \leq \frac{s}{H\left(\frac{1}{(2s)^m}\right)} \leq \frac{2s}{H\left(\frac{1}{(2s)^m}\right)},$$

hence

$$(5.28) \quad \forall s \geq A, \quad s \leq \phi_m(t), \quad \text{where} \quad t = \frac{2s}{H\left(\frac{1}{(2s)^m}\right)}.$$

Then,

- With $m = 1$, this brings

$$t = \frac{2s}{\frac{g_0(\frac{1}{2s})}{\frac{1}{2s}}} = \frac{1}{g_0(\frac{1}{2s})},$$

hence

$$(5.29) \quad \phi_1(t) \geq s = \frac{1}{2g_0^{-1}(\frac{1}{t})}.$$

- If there is some $\alpha > 0$ and some $q > 1$ such that

$$(5.30) \quad \forall x \in [0, \eta], \quad g_0(x) \geq \alpha x^q,$$

then

$$t = \frac{2s}{H(\frac{1}{(2s)^m})} \leq \frac{1}{\alpha} (2s)^{m(q-1)+1},$$

therefore

$$(5.31) \quad \phi_m(t) \geq s \geq \frac{1}{2} (\alpha t)^{1/[m(q-1)+1]}.$$

We end this section computing the two integrals that were the motivation for our choice of ϕ_m and ε_m : given $m \geq 1$, we have

$$(5.32) \quad \begin{aligned} \int_S^{+\infty} \phi'_m(t) \varepsilon_m(t)^2 dt &= \int_{\phi_m(S)}^{+\infty} \left[H^{-1} \left(\frac{1}{(\phi_m^{-1})'(s)} \right) \right]^2 ds \\ &= \int_{\phi_m(S)}^{+\infty} \left(\frac{1}{(s+A)^m} \right)^2 ds = \left[\frac{(s+A)^{-2m+1}}{-2m+1} \right]_{\phi_m(S)}^{+\infty} \\ &= \frac{1}{2m-1} \frac{1}{(\phi_m(S)+A)^{2m-1}}. \end{aligned}$$

In the same way,

$$(5.33) \quad \int_S^{+\infty} \phi'_m(t) \varepsilon_m(t)^p dt = \frac{1}{pm-1} \frac{1}{(\phi_m(S)+A)^{pm-1}}.$$

5.5.c. A first estimate on the decay of the energy, using ϕ_1 .

Let us choose $m = 1$, $\phi := \phi_1$ and $\varepsilon := \varepsilon_1$. Then we derive from (5.24), (5.32) (with $m=1$) and (5.33) (with $m=1$) that

$$(5.34) \quad \int_S^T E_p^2 \phi' \leq C'_u E_p(S)^2 + C'_u \frac{E_p(S)^{1+(p-2)/p}}{\phi(S)+A} + C'_u \frac{E_p(S)}{(\phi(S)+A)^{p-1}}.$$

This implies the following (weakened) estimate

$$(5.35) \quad \begin{aligned} \int_S^T E_p(t)^2 \phi'(t) dt &\leq C'_u E_p(S)^2 + C'_u E_p(0)^{(p-2)/p} \frac{E(S)}{\phi(S)+A} + C'_u \frac{E_p(S)}{\phi(S)+A} \\ &= C'_u E_p(S)^2 + \left(C'_u E_p(0)^{(p-2)/p} + C'_u \right) \frac{E(S)}{\phi(S)+A}, \end{aligned}$$

and we are in position to apply an integral estimate of the Gronwall type (see [17]):

Lemma 5.6. (Lemma 2 in [17]) *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nonincreasing continuous function. Assume that there exists $\sigma > 0$, $\sigma' > 0$ and $c > 0$ such that*

$$(5.36) \quad \forall t \geq 0, \quad \int_t^{+\infty} f(\tau)^{1+\sigma} d\tau \leq c f(t)^{1+\sigma} + \frac{c}{(1+t)^{\sigma'}} f(0)^\sigma f(t).$$

Then there exists $C > 0$ such that

$$(5.37) \quad \forall t \geq 0, \quad f(t) \leq f(0) \frac{C}{(1+t)^{(1+\sigma')/\sigma}}.$$

As an immediate corollary, we have the following.

Lemma 5.7. (Lemma 3 in [17]) *Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nonincreasing continuous function and $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty.$$

Assume that there exists $\sigma > 0$, $\sigma' > 0$ and $c > 0$ such that

$$(5.38) \quad \forall S \geq 0, \quad \int_S^{+\infty} E(t)^{1+\sigma} \phi'(t) dt \leq cE(S)^{1+\sigma} + \frac{c}{(1+\phi(S))^{\sigma'}} E(0)^\sigma E(S).$$

Then there exists $C > 0$ such that

$$(5.39) \quad \forall t \geq 0, \quad E(t) \leq E(0) \frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}}.$$

Then Lemma 5.7 gives that there exists $C(E_p(0))$ that depends on $E_p(0)$ such that

$$(5.40) \quad \forall t \geq 0, \quad E_p(t) \leq \frac{C(E_p(0))}{(\phi(S) + A)^2}.$$

Using (5.29), we obtain that

$$(5.41) \quad \forall t \geq 0, \quad E_p(t) \leq C(E_p(0)) \left(g_0^{-1} \left(\frac{1}{t} \right) \right)^2.$$

This is a first estimate of the decay of the energy.

5.5.d. A new integral inequality and its application.

We claim that we have the following more adapted result:

Lemma 5.8. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nonincreasing continuous function. Assume that there exists $p > 2$, and $c > 0$ such that*

$$(5.42) \quad \int_t^{+\infty} f(\tau)^2 d\tau \leq cf(t)^2 + c \frac{f(t)^{1+(p-2)/p}}{1+t} + c \frac{f(t)}{(1+t)^{p-1}}.$$

Then, for all $\delta > 0$, there exists $C_\delta > 0$ such that

$$(5.43) \quad \forall t \geq 0, \quad f(t) \leq f(0) \frac{C_\delta}{(1+t)^{p-\delta}}.$$

Remark 5.1. We were not able to prove that (5.43) is true for $\delta = 0$. This would be interesting in itself, and would of course give a better estimate of the decay of the energy E_p .

Proof of Lemma 5.8. Denote $\theta := \frac{p-2}{p}$. Assumption (5.42) implies that

$$(5.44) \quad \int_t^{+\infty} f(\tau)^2 d\tau \leq cf(t)^2 + cf(0)^\theta \frac{f(t)}{1+t} + c \frac{f(t)}{1+t},$$

which allows one to apply Lemma 5.6 with $\sigma = 1 = \sigma'$, and we obtain that there exists C_1 such that

$$(5.45) \quad f(t) \leq \frac{C_1}{(1+t)^2}.$$

This is a first estimate, that we can plug into (5.42) to obtain that

$$(5.46) \quad \int_t^{+\infty} f(\tau)^2 d\tau \leq cf(t)^2 + cC_1^\theta \frac{f(t)}{(1+t)^{1+2\theta}} + c \frac{f(t)}{(1+t)^{p-1}}.$$

Before using Lemma 5.6, we need to compare $1 + 2\theta$ and $p - 1$; we see that

$$1 + 2\theta < p - 1 :$$

indeed,

$$(p - 1) - (1 + 2\theta) = p - 2 - 2\frac{p - 2}{p} = \frac{p^2 - 4p + 4}{p} = \frac{(p - 2)^2}{p} > 0.$$

Therefore we deduce from (5.46) that

$$\int_t^{+\infty} f(\tau)^2 d\tau \leq cf(t)^2 + (cC_1^\theta + c)\frac{f(t)}{(1+t)^{1+2\theta}},$$

allowing one to apply Lemma 5.6 with $\sigma = 1$, and $\sigma' = 1 + 2\theta$, and we obtain that there exists C_2 such that

$$(5.47) \quad f(t) \leq \frac{C_2}{(1+t)^{2+2\theta}},$$

which improves the decay estimate (5.45).

Let us prove by induction that, for all $n \in \mathbb{N}$, there exists C_n such that

$$(5.48) \quad (H_n) : \quad f(t) \leq \frac{C_n}{(1+t)^{2\sum_{k=0}^n \theta^k}}.$$

(H_0) is true, thanks to (5.46), and (H_1) is true, thanks to (5.47). If (H_n) is true, then we derive from (5.34) to obtain that

$$(5.49) \quad \int_t^{+\infty} f(\tau)^2 d\tau \leq cf(t)^2 + cC_n^\theta \frac{f(t)}{(1+t)(1+t)^{2\theta\sum_{k=0}^n \theta^k}} + c\frac{f(t)}{(1+t)^{p-1}} \\ = cf(t)^2 + cC_n^\theta \frac{f(t)}{(1+t)^{1+2\sum_{k=1}^{n+1} \theta^k}} + c\frac{f(t)}{(1+t)^{p-1}}.$$

In the same way, we need to compare the exponents, and we remark that

$$(p - 1) - \left(1 + 2\sum_{k=1}^{n+1} \theta^k\right) = p - 2\sum_{k=0}^{n+1} \theta^k = p - 2\frac{1 - \theta^{n+2}}{1 - \theta} \\ = p - 2\frac{1 - \theta^{n+2}}{1 - \frac{p-2}{p}} = p - 2\frac{1 - \theta^{n+2}}{\frac{2}{p}} = p - p(1 - \theta^{n+2}) = p\theta^{n+2} > 0.$$

Therefore we deduce from (5.46) that

$$\int_t^{+\infty} f(\tau)^2 d\tau \leq cf(t)^2 + (cC_n^\theta + c)\frac{f(t)}{(1+t)^{1+2\sum_{k=1}^{n+1} \theta^k}},$$

therefore we are in position to apply Lemma 5.6 with $\sigma = 1$, and $\sigma' = 1 + 2\sum_{k=1}^{n+1} \theta^k$, and we obtain that there exists C_{n+1} such that

$$(5.50) \quad f(t) \leq \frac{C_{n+1}}{(1+t)^{2+2\sum_{k=1}^{n+1} \theta^k}} = \frac{C_{n+1}}{(1+t)^{2\sum_{k=0}^{n+1} \theta^k}}$$

which is (5.48) at the rank $n + 1$. This concludes the end of the induction and proves the validity of (5.48) for all $n \in \mathbb{N}$. Since we already noted that

$$2\sum_{k=0}^n \theta^k = 2\frac{1 - \theta^{n+1}}{1 - \theta} = 2\frac{1 - \theta^{n+1}}{1 - \frac{p-2}{p}} = 2\frac{1 - \theta^{n+1}}{\frac{2}{p}} = p(1 - \theta^{n+1}) \rightarrow p \quad \text{as } n \rightarrow \infty,$$

we obtain (5.43). \square

Then Lemma 5.8 can be applied to (5.34), and we obtain that

$$\forall \delta > 0, \exists C_\delta, \forall t \geq 0, \quad E_p(t) \leq \frac{C_\delta(E_p(0))}{(\phi_1(S) + A)^{p-\delta}} \leq C_\delta(E_p(0)) \left(g_0^{-1}\left(\frac{1}{t}\right)\right)^{p-\delta},$$

hence (2.11). \square

5.5.e. An extension of the new integral inequality and its application.

Let us now choose $m \geq 1$, $\phi := \phi_m$ (defined in (5.26)), and $\varepsilon := \varepsilon_m$ (defined in (5.27)). We derive from (5.24), (5.32) and (5.33) that

$$(5.51) \quad \int_S^T E_p^2 \phi' \leq C'_u E_p(S)^2 + C'_u \frac{E_p(S)^{1+(p-2)/p}}{(\phi(S) + A)^{2m-1}} + C'_u \frac{E_p(S)}{(\phi(S) + A)^{mp-1}}.$$

We claim that we have the following extension of Lemma 5.8:

Lemma 5.9. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nonincreasing continuous function. Assume that there exist $m \geq 1$, $p > 2$, and $c > 0$ such that*

$$(5.52) \quad \int_t^{+\infty} f(\tau)^2 d\tau \leq c f(t)^2 + c \frac{f(t)^{1+(p-2)/p}}{(1+t)^{2m-1}} + c \frac{f(t)}{(1+t)^{mp-1}}.$$

Then, for all $\delta > 0$, there exists $C_{m,\delta} > 0$ such that

$$(5.53) \quad \forall t \geq 0, \quad f(t) \leq f(0) \frac{C_{m,\delta}}{(1+t)^{m(p-\delta)}}.$$

Proof of Lemma 5.9. The proof is similar to the proof of Lemma 5.8, the only difference being the induction formula: one has to replace (5.48) by

$$(5.54) \quad (H_n) : \quad f(t) \leq \frac{C_n}{(1+t)^{2m \sum_{k=0}^n \theta^k}},$$

and (5.53) follows in the same way. \square

Then Lemma 5.9 can be applied to (5.51), and we obtain that

$$(5.55) \quad \forall \delta > 0, \exists C_\delta, \forall t \geq 0, \quad E_p(t) \leq \frac{C_{m,\delta}(E_p(0))}{(\phi_m(t) + A)^{m(p-\delta)}}.$$

Then (2.12) follows from the estimate (5.28) on the growth of ϕ_m . \square

Finally, assume that (5.30) holds true. Then we deduce from the growth estimate of ϕ_m given in (5.31) that

$$(5.56) \quad E_p(t) \leq \frac{C_{m,\delta}(E_p(0))}{(1+t)^{\frac{m(p-\delta)}{m(q-1)+1}}}$$

Taking m large enough, we find (2.15). \square

5.6. Proof of Lemmas 5.2-5.4.

The proof of Lemmas 5.2-5.4 can be easily obtained following [12], see also [4] on pages 26-32 for complete computations. The argument relies on inequalities involving the functions f, F and g as well as their incremental ratios.

6. PROOF OF THEOREM 2.5

Here we prove stability results when $1 < p < 2$, adapting the proof done in the case $p \geq 2$. Due to the presence of f' (and therefore of the power $p - 2$) in the second set of multipliers in the case $p \geq 2$ (see Subsection 5.2.c), it is not clear if one has the right to use them directly in the case $1 < p < 2$. However, we are able to adapt our arguments, mainly following and completing [12, Section 4.2].

6.1. A modified energy.

The first thing is to modify the functions f and F just like it was done in [12, Section 4.2]. Therefore, we consider, for $p \in (1, 2)$, the functions \tilde{f} and \tilde{F} defined on \mathbb{R} , by

$$(6.1) \quad \tilde{f}(y) = (p-1) \int_0^y (|s|+1)^{p-2} ds = \operatorname{sgn}(y) [(|y|+1)^{p-1} - 1],$$

$$(6.2) \quad \tilde{F}(y) = \int_0^y \tilde{f}(s) ds = \frac{1}{p} [(|y|+1)^p - 1] - |y|.$$

This drives us to modify the energy E_p by considering, for every $t \in \mathbb{R}_+$ and every solution of (1.1), the function \tilde{E}_p defined by

$$(6.3) \quad \tilde{E}_p(t) = \int_0^1 \left(\tilde{F}(\rho) + \tilde{F}(\xi) \right) dx.$$

First, we notice that \tilde{E}_p is nonincreasing: indeed, the function \tilde{f} is continuous on \mathbb{R} , increasing and odd, therefore the function \tilde{F} is C^1 on \mathbb{R} , convex (and even), and then Proposition 4.1 ensures that \tilde{E}_p is nonincreasing.

6.2. The key integral inequality on the modified energy.

We are going to prove the following extension of Lemma 5.1:

Lemma 6.1. *Let $p \in (1, 2)$. Then there exists $C^{(0)} > 0$ (depending on the norm of the initial conditions) such that*

$$(6.4) \quad \int_S^T \tilde{E}_p(t)^2 \phi'(t) dt \leq C^{(0)} \tilde{E}_p(S)^2 \phi'(S) \\ + C^{(0)} \int_S^T \tilde{E}_p(t) \phi'(t) \int_0^1 a(x) \frac{\rho - \xi}{2} \left(\tilde{f}(\rho) - \tilde{f}(\xi) \right) dx dt.$$

Before proving Lemma 6.1, let us prove the decay rate estimates of Theorem 2.5.

6.3. Decay rate estimates of \tilde{E}_p and E_p assuming Lemma 6.1.

The key integral inequality (6.4) will allow us to prove decay rate estimates of \tilde{E}_p when $p \in (1, 2)$. But E_p is a more natural energy, and we can note that

$$\left(\int_{|\rho| \leq 1} |\rho|^p \right)^{2/p} \leq \int_{|\rho| \leq 1} |\rho|^2 \leq \tilde{E}_p(t),$$

and therefore

$$E_p(t) \leq c \tilde{E}_p(t)^{p/2} + c \tilde{E}_p(t) \\ \leq c'(1 + \tilde{E}_p(0)^{\frac{2-p}{2}}) \tilde{E}_p(t)^{p/2} \leq c''(1 + E_p(0)^{\frac{2-p}{2}}) \tilde{E}_p(t)^{p/2},$$

therefore a decay rate estimate on \tilde{E}_p gives immediately a decay rate estimate for E_p .

6.3.a. The case where g linear ([12]).

If $g(s) = \alpha s$ with some $\alpha > 0$, then we choose $\phi(t) = t$, and (6.4) gives us that

$$\int_S^T \tilde{E}_p(t)^2 dt \leq C^{(0)} \tilde{E}_p(S)^2 \\ + \frac{C^{(0)}}{2\alpha} \int_S^T \tilde{E}_p(t) \int_0^1 a(x) g(\rho - \xi) \left(\tilde{f}(\rho) - \tilde{f}(\xi) \right) dx dt \\ = \left(1 + \frac{1}{4\alpha} \right) C^{(0)} \tilde{E}_p(S)^2,$$

and we deduce that \tilde{E}_p decays exponentially to 0. Since the problem is linear, the decay is uniform. \square

6.3.b. Theorem 2.5, (i): $g'(0) > 0$.

The situation is almost the same, the main difference being that the decay rate depends on $C^{(0)}$, hence on $E_p(0)$. \square

6.3.c. Theorem 2.5, (ii): $g'(0) = 0$.

Here we proceed as in subsection 5.5. the only difference lies in $\Omega_1^{(t)}$, where now \tilde{f} is bounded. Therefore

$$|\tilde{f}(\rho) - \tilde{f}(\xi)| \leq 2(p-1)\varepsilon(t),$$

and

$$\int_S^T \tilde{E}_p \phi' \int_{\Omega_1^{(t)}} a(x) (\rho - \xi) (\tilde{f}(\rho) - \tilde{f}(\xi)) \leq C'' \int_S^T \tilde{E}_p(t) \phi'(t) \varepsilon(t)^2 dt.$$

Choosing $\phi := \phi_m$ defined in (5.26), and $\varepsilon := \varepsilon_m$ defined in (5.27), we have

$$\int_S^T \tilde{E}_p(t)^2 \phi'(t) dt \leq C^{(0)} \tilde{E}_p(S)^2 + C^{(0)} \frac{\tilde{E}_p(S)}{(\phi_m(S) + A)^{2m-1}}.$$

Then Lemma 5.7 gives that

$$\tilde{E}_p(t) \leq \frac{C_m(E_p(0))}{(\phi_m(t) + A)^{2m}}.$$

Then, as in Theorem 2.4, we have

$$\tilde{E}_p(t) \leq \frac{C_m(E_p(0))}{s(t)^{2m}} \quad \text{where } t = \frac{2s(t)}{H\left(\frac{1}{(2s(t))^m}\right)}.$$

Concerning the natural energy, we have

$$E_p(t) \leq \frac{C_m(E_p(0))}{s(t)^{pm}} \quad \text{where } t = \frac{2s(t)}{H\left(\frac{1}{(2s(t))^m}\right)}.$$

If g_0 satisfies (5.30), then we deduce from (5.31) that

$$E_p(t) \leq \frac{C_m(E_p(0))}{t^{(mp)/[m(q-1)+1]}.$$

Letting $m \rightarrow \infty$, we find a decay of the form

$$E_p(t) \leq \frac{C_\delta(E_p(0))}{t^{\frac{p}{q-1}-\delta}},$$

hence as in the case $p > 2$. \square

Therefore it only remains to prove Lemma 6.1. Its proof follows from convex analysis arguments, combined with identities given by the multiplier method. In the following, we start by explaining the tools that we will use later, in particular some extensions of Hölder's inequality that will be adapted to our setting.

6.4. Fenchel's inequality and applications.

It is easy to see that the asymptotic behavior of \tilde{F} as $y \rightarrow 0^+$ and $y \rightarrow +\infty$ is the following:

$$\tilde{F}(y) \sim_{y \rightarrow 0} \frac{p-1}{2} y^2, \quad \text{and} \quad \tilde{F}(y) \sim_{y \rightarrow +\infty} \frac{y^p}{p}.$$

Then one can consider the convex conjugate of \tilde{F} , denoted \tilde{F}^* and defined by

$$(6.5) \quad \forall b \in \mathbb{R}, \quad \tilde{F}^*(b) := \sup_{y \in \mathbb{R}} (by - \tilde{F}(y)).$$

This function has the following properties, that we will use later:

- \tilde{F}^* takes finite values, thanks to the asymptotic behavior of \tilde{F} ,
- \tilde{F}^* is positive on $(0, +\infty)$, equal to 0 in 0 (since \tilde{F} is nonnegative),
- \tilde{F}^* is even, since \tilde{F} is even,
- \tilde{F}^* is convex, since the derivative of \tilde{F}^* is \tilde{f}^{-1} , that is increasing.

We also recall the classical Fenchel's inequality ([9]):

$$(6.6) \quad \forall a, b \in \mathbb{R}, \quad |ab| \leq \tilde{F}(|a|) + \tilde{F}^*(|b|),$$

that is an immediate consequence of the definition of \tilde{F}^* (taking $y = a$) but will have nontrivial consequences. (When $\tilde{F}(y) = \frac{|y|^p}{p}$, then $\tilde{F}^*(b) = \frac{|b|^q}{q}$ where q is the conjugate exponent of p , and (6.6) is the classical Hölder inequality.)

In particular, modifying a little some arguments of [12, Section 4.2], we have the following

Lemma 6.2. *There exists $C_p > 0$ such that*

$$(6.7) \quad \forall a, y \in \mathbb{R}, \forall \eta \in (0, 1), \quad |a\tilde{f}(y)| \leq \frac{C_p}{\eta^2} \tilde{F}(a) + C_p \eta^2 \tilde{F}(y),$$

and

$$(6.8) \quad \forall a, y \in \mathbb{R}, \forall \eta \in (0, 1), \quad |a\tilde{f}(y)| \leq C_p \eta^p \tilde{F}(a) + \frac{C_p}{\eta^q} \tilde{F}(y).$$

Note that (6.7) and (6.8) are very close, and allow one to have a small coefficient in front of $\tilde{F}(y)$ (using (6.7)) or in front of $\tilde{F}(a)$ (using (6.8)).

Proof of Lemma 6.2. First, we prove (6.7). Using (6.6), we have

$$|a\tilde{f}(y)| = \left| \left(\frac{a}{\eta} \right) (\eta\tilde{f}(y)) \right| \leq \tilde{F}\left(\left| \frac{a}{\eta} \right|\right) + \tilde{F}^*(|\eta\tilde{f}(y)|).$$

Next, thanks to the asymptotic behavior of \tilde{F} as $y \rightarrow 0^+$ and $y \rightarrow +\infty$:

$$\tilde{F}(y) \sim_{y \rightarrow 0} \frac{p-1}{2} y^2, \quad \tilde{F}(y) \sim_{y \rightarrow +\infty} \frac{y^p}{p},$$

we see that there exists $C_p > C'_p > 0$ such that

$$(6.9) \quad \forall y \in \mathbb{R}, \quad C'_p \min(y^2, y^p) \leq \tilde{F}(y) \leq C_p \min(y^2, y^p).$$

This yields

$$(6.10) \quad \begin{aligned} \tilde{F}\left(\left| \frac{a}{\eta} \right|\right) &\leq C_p \min\left(\frac{a^2}{\eta^2}, \frac{a^p}{\eta^p}\right) \leq C_p \min\left(\frac{a^2}{\eta^2}, \frac{a^p}{\eta^2}\right) \\ &= \frac{C_p}{\eta^2} \min(a^2, a^p) \leq \frac{C_p}{\eta^2 C'_p} \tilde{F}(|a|). \end{aligned}$$

On the other hand, looking at the definition of \tilde{F}^* , it is clear that, given $b \geq 0$, the function

$$y \in \mathbb{R} \mapsto by - \tilde{F}(y)$$

attains its maximum at $y := \tilde{f}^{-1}(b)$ (remember that \tilde{f} is increasing), and therefore we always have

$$\forall b \geq 0, \quad \tilde{F}^*(b) = b \tilde{f}^{-1}(b) - \tilde{F}(\tilde{f}^{-1}(b)),$$

or, equivalently,

$$(6.11) \quad \forall y \in \mathbb{R}, \quad y \tilde{f}(y) = \tilde{F}(y) + \tilde{F}^*(\tilde{f}(y)).$$

This allows us to obtain the asymptotic behavior of $\tilde{F}^*(b)$ as $b \rightarrow 0$ and $b \rightarrow +\infty$: first, we have

$$\tilde{F}^*(\tilde{f}(y)) \sim_{y \rightarrow 0} (p-1)y^2 - \frac{p-1}{2}y^2 = \frac{p-1}{2}y^2,$$

therefore

$$\tilde{F}^*(b) \sim \frac{b^2}{2(p-1)} \quad \text{as } b \rightarrow 0;$$

next, we have

$$\tilde{F}^*(\tilde{f}(y)) \sim_{y \rightarrow +\infty} y^p - \frac{y^p}{p} = \frac{p-1}{p} y^p,$$

therefore

$$\tilde{F}^*(b) \sim \frac{p-1}{p} b^{\frac{p}{p-1}} \quad \text{as } b \rightarrow +\infty.$$

Since $\frac{p}{p-1} \geq 2$ and since \tilde{F}^* is increasing on \mathbb{R}_+ (and even on \mathbb{R}), we see that there exists $C_p^* > C_p^{*'} > 0$ such that

$$(6.12) \quad \forall b \in \mathbb{R}, \quad C_p^{*'} \max(b^2, b^{\frac{p}{p-1}}) \leq \tilde{F}^*(b) \leq C_p^* \max(b^2, b^{\frac{p}{p-1}}).$$

Now we can conclude:

$$\begin{aligned} \tilde{F}^*(|\eta \tilde{f}(y)|) &\leq C_p^* \max((|\eta \tilde{f}(y)|)^2, (|\eta \tilde{f}(y)|)^{\frac{p}{p-1}}) \\ &= C_p^* \max(\eta^2 \tilde{f}(|y|)^2, \eta^{\frac{p}{p-1}} \tilde{f}(|y|)^{\frac{p}{p-1}}) \leq C_p^* \max(\eta^2 \tilde{f}(|y|)^2, \eta^2 \tilde{f}(|y|)^{\frac{p}{p-1}}) \\ &= C_p^* \eta^2 \max(\tilde{f}(|y|)^2, \tilde{f}(|y|)^{\frac{p}{p-1}}). \end{aligned}$$

But

$$\max(\tilde{f}(|y|)^2, \tilde{f}(|y|)^{\frac{p}{p-1}}) = \tilde{f}(|y|)^2 \sim (p-1)^2 y^2 \quad \text{as } y \rightarrow 0,$$

and

$$\max(\tilde{f}(|y|)^2, \tilde{f}(|y|)^{\frac{p}{p-1}}) = \tilde{f}(|y|)^{\frac{p}{p-1}} \sim y^p \quad \text{as } y \rightarrow +\infty,$$

therefore there exists $\tilde{C}_p > \tilde{C}_p' > 0$ such that

$$(6.13) \quad \forall y \geq 0, \quad \tilde{C}_p' \min(y^2, y^p) \leq \max(\tilde{f}(|y|)^2, \tilde{f}(|y|)^{\frac{p}{p-1}}) \leq \tilde{C}_p \min(y^2, y^p).$$

Then, using (6.9), we obtain that

$$(6.14) \quad \tilde{F}^*(|\eta \tilde{f}(y)|) \leq C_p^* \eta^2 \tilde{C}_p \min(y^2, y^p) \leq \frac{C_p^* \tilde{C}_p}{C_p'} \eta^2 \tilde{F}(|y|).$$

The proof of (6.7) follows from (6.10) and (6.14). \square

In the same spirit, we prove (6.8): first we can assume that $a, y > 0$, and using (6.6), we can write

$$a \tilde{f}(y) = (\eta a) \left(\frac{\tilde{f}(y)}{\eta} \right) \leq \tilde{F}(\eta a) + \tilde{F}^* \left(\frac{\tilde{f}(y)}{\eta} \right).$$

Using (6.9), we have

$$\tilde{F}(\eta a) \leq C_p \min(\eta^2 a^2, \eta^p a^p) \leq C_p \eta^p \min(a^2, a^p) \leq \frac{C_p}{C_p'} \eta^p \tilde{F}(a).$$

Using successively (6.12), (6.13) and (6.9), we have

$$\begin{aligned} \tilde{F}^* \left(\frac{\tilde{f}(y)}{\eta} \right) &\leq C_p^* \max \left(\frac{\tilde{f}(y)^2}{\eta^2}, \frac{\tilde{f}(y)^{\frac{p}{p-1}}}{\eta^{\frac{p}{p-1}}} \right) \\ &\leq \frac{C_p^*}{\eta^{\frac{p}{p-1}}} \max(\tilde{f}(y)^2, \tilde{f}(y)^{\frac{p}{p-1}}) \\ &\leq \frac{C_p^* \tilde{C}_p}{\eta^{\frac{p}{p-1}}} \min(y^2, y^p) \leq \frac{C_p^* \tilde{C}_p}{\eta^{\frac{p}{p-1}} C_p'} \tilde{F}(y). \end{aligned}$$

This concludes the proof of (6.8) and of Lemma 6.2. \square

As an application, we have the following generalized Poincaré's inequality:

Lemma 6.3. [10, Section 4.2] *Let $p \in (1, 2)$. Then there exists a positive constant C_p such that, for every absolutely continuous function $z : [0, 1] \rightarrow \mathbb{R}$ satisfying $z(0) = 0$, one has*

$$(6.15) \quad \int_0^1 \tilde{F}(z(s)) ds \leq C_p \int_0^1 \tilde{F}(z'(s)) ds.$$

Proof of Lemma 6.3. We have

$$\tilde{F}(z(x)) = \tilde{F}(z(x)) - \tilde{F}(z(0)) = \int_0^x \tilde{f}(z(s)) z'(s) ds.$$

Then using (6.7), we have

$$\left| \tilde{f}(z(s)) z'(s) \right| \leq \frac{C_p}{\eta^2} \tilde{F}(z'(s)) + C_p \eta^2 \tilde{F}(z(s)).$$

Therefore

$$\tilde{F}(z(x)) \leq \int_0^1 \left| \tilde{f}(z(s)) z'(s) \right| ds \leq \frac{C_p}{\eta^2} \int_0^1 \tilde{F}(z'(s)) ds + C_p \eta^2 \int_0^1 \tilde{F}(z(s)) ds.$$

Integrating with respect to x on $(0, 1)$, we get

$$\int_0^1 \tilde{F}(z(x)) dx \leq \frac{C_p}{\eta^2} \int_0^1 \tilde{F}(z'(s)) ds + C_p \eta^2 \int_0^1 \tilde{F}(z(s)) ds,$$

and we obtain (6.15) choosing η small enough. \square

6.5. Proof of Lemma 6.1.

The proof of Lemma 6.1 follows adapting the arguments given by [12, Section 4.2.4], using the following modified multipliers:

- modified first multiplier: we replace f by \tilde{f} , and E_p by \tilde{E}_p ;
- modified second multiplier: we replace f' by \tilde{f}' , and E_p by \tilde{E}_p ;
- modified third multiplier: we replace f by \tilde{f} in the elliptic problem (5.7), and v and E_p by \tilde{v} (the solution of the modified elliptic problem, now associated to \tilde{f}) and \tilde{E}_p .

6.5.a. Estimate given by the modified first multiplier.

When $p \in (1, 2)$, and with the modified first multiplier, we have the following estimate, which is a modified version of (5.4):

Lemma 6.4. *There exists $C_1 > 0$ independent of the initial conditions such that*

$$(6.16) \quad \int_S^T \tilde{E}_p(t)^2 \phi'(t) dt \leq C_1 \tilde{E}_p(S)^2 \phi'(S) \\ + C_1 \int_S^T \tilde{E}_p(t) \phi'(t) \int_0^1 \psi_2(x) (\rho \tilde{f}(\rho) + \xi \tilde{f}(\xi)) dx dt \\ - \int_S^T \tilde{E}_p(t) \phi'(t) \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (\tilde{f}(\rho) + \tilde{f}(\xi)) dx dt.$$

6.5.b. Estimate given by the modified second multiplier.

When $p \in (1, 2)$, and with the modified second multiplier, we have the following estimate, which is a modified version of (5.6):

Lemma 6.5. Denote $k^{(0)} := \max(1, E_p(0)^{\frac{2-p}{p}})$. Then there exists $C'_2 > 0$ independent of the initial conditions such that

$$(6.17) \quad \int_S^T \tilde{E}_p(t)^2 \phi'(t) dt \leq \tilde{C}'_2 k^{(0)} \tilde{E}_p(S)^2 \phi'(S) \\ + \tilde{C}'_2 k^{(0)q} \int_S^T \tilde{E}_p(t) \phi'(t) \left(\int_{a_2}^{b_2} \tilde{F}(z) dx \right) dt \\ + 2C_1 \int_S^T \tilde{E}_p(t) \phi'(t) \int_0^1 \psi_2(x) \frac{\rho - \xi}{2} (\tilde{f}(\rho) - \tilde{f}(\xi)) dx dt \\ - 2 \int_S^T \tilde{E}_p(t) \phi'(t) \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (\tilde{f}(\rho) + \tilde{f}(\xi)) dx dt.$$

(The only difference with (5.6) is the fact that the coefficients in (6.17) depend on $E_p(0)$, hence on the norm of the initial condition.)

6.5.c. Estimate given by the modified third multiplier.

Consider the solution \tilde{v} of

$$(6.18) \quad \begin{cases} \tilde{v}_{xx} = \psi_3(x) \tilde{f}(z(t, x)) & x \in (0, 1), \\ \tilde{v}(0) = 0 = \tilde{v}(1). \end{cases}$$

When $p \in (1, 2)$, and with the modified third multiplier, we have the following estimate, which is a modified version of (5.9):

Lemma 6.6. There exist $C_1^{(0)}, C_2^{(0)}, C_3^{(0)}$ that depend on $E_p(0)$ such that

$$(6.19) \quad \int_S^T \tilde{E}_p^2 \phi' \leq C_1^{(0)} \tilde{E}_p(S)^2 \phi'(S) + C_2^{(0)} \int_S^T \tilde{E}_p \phi' \int_0^1 \psi_3(x) \tilde{F}(\rho - \xi) dx \\ + C_3^{(0)} \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) \tilde{v} g\left(\frac{\rho - \xi}{2}\right) + 4C_1 \int_S^T \tilde{E}_p \phi' \int_0^1 \psi_2(x) \frac{\rho - \xi}{2} (f(\rho) - f(\xi)) \\ - 4 \int_S^T \tilde{E}_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (\tilde{f}(\rho) + \tilde{f}(\xi)).$$

(Note that, if it has some utility, we have

$$C_1^{(0)} = \tilde{C}'_2 \max(1, E_p(0)^{\frac{2-p}{p}}) + \max(1, E_p(0)^{2\frac{2-p}{p-1}}), \\ C_2^{(0)} = \tilde{C}'_2 \max(1, E_p(0)^{(q^2+q)\frac{2-p}{p}}) \max(E_p(0)^{\frac{2-p}{p}}, E_p(0)^{\frac{2-p}{p-1}}), \\ C_3^{(0)} = \tilde{C}'_3 \max(1, E_p(0)^{\frac{2-p}{p-1}}).$$

We will also need the following extension of Lemma 5.5:

Lemma 6.7. There exists C independent of the initial conditions such that

$$(6.20) \quad \forall t \geq 0, \quad \int_0^1 \tilde{F}^*(\tilde{v}(t, x)) dx \leq C \max(1, E_p(0)^{\frac{2-p}{p-1}}) \tilde{E}_p(t),$$

and

$$(6.21) \quad \forall t \geq 0, \quad \int_0^1 \tilde{F}(\tilde{v}_t) dx \leq C \max(E_p(0)^{\frac{2-p}{p}}, E_p(0)^{\frac{2-p}{p-1}}) \int_0^1 \psi_3 \tilde{F}(z_t) dx.$$

6.5.d. **Proof of Lemma 6.1 assuming Lemmas 6.4-6.7.**

We proceed as for the proof of Lemma 5.1, the starting point being now (6.19). First, as we did in the previous section, using (6.6), (6.9), (6.12), and (6.20), there exists $C > 0$ such that, for all $\sigma \in (0, 1)$, we have

$$\begin{aligned} & \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) \tilde{v} g\left(\frac{\rho - \xi}{2}\right) \\ & \leq C\sigma^2 \int_S^T \tilde{E}_p \phi' \int_0^1 \tilde{F}^*(v) + \frac{C}{\sigma^2} \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) \tilde{F}(\rho - \xi) \\ & \leq C\sigma^2 \max(1, E_p(0)^{\frac{2-p}{p-1}}) \int_S^T \tilde{E}_p^2 \phi' + \frac{C}{\sigma^2} \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) \tilde{F}(\rho - \xi), \end{aligned}$$

and that allows us to get rid of the third term of the right-hand side of (6.19) (taking $\sigma > 0$ small enough).

Next, as we did in the proof of Lemma 5.1, we have

$$\begin{aligned} & -4 \int_S^T \tilde{E}_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (\tilde{f}(\rho) + \tilde{f}(\xi)) \\ & = 4 \int_S^T \tilde{E}_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (\tilde{f}(\rho) - \tilde{f}(\xi)) \\ & \quad + 4 \int_S^T \tilde{E}_p \phi' \int_0^1 xa(x) \psi_1(x) g\left(\frac{\rho - \xi}{2}\right) (-2\tilde{f}(\rho)) \\ & \leq 2\tilde{E}_p(S)^2 \phi'(S) + 8C_g \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) \frac{\rho - \xi}{2} \tilde{f}(\rho), \end{aligned}$$

and using (6.7), we have

$$\begin{aligned} \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) \frac{\rho - \xi}{2} \tilde{f}(\rho) & \leq \int_S^T \tilde{E}_p \phi' \int_0^1 \frac{C_p}{\eta^2} \tilde{F}(a(x)(\rho - \xi)) + C_p \eta^2 \tilde{F}(\rho) \\ & \leq \frac{C'_p}{\eta^2} \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) \tilde{F}(\rho - \xi) + C_p \eta^2 \int_S^T \tilde{E}_p^2 \phi'. \end{aligned}$$

Combining with the previous estimate and taking $\eta > 0$ small enough, we can get rid of the fifth term of the right hand side of (6.19), and we obtain

$$(6.22) \quad \begin{aligned} \int_S^T \tilde{E}_p^2 \phi' & \leq C_4^{(0)} \tilde{E}_p(S)^2 \phi'(S) + C_5^{(0)} \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) \tilde{F}(\rho - \xi) dx \\ & \quad + 2C_1 \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) (\rho - \xi) (\tilde{f}(\rho) - \tilde{f}(\xi)), \end{aligned}$$

with $C_4^{(0)}$ and $C_5^{(0)}$ for which we can have an explicit estimate in terms of $E_p(0)$. It remains to estimate the second term of the right-hand side of (6.22), and this follows from the following result, extracted from [10, Lemma 3.2.7] but for which we give an alternative proof.

Lemma 6.8. *There exists C_p uniform such that, for all $M \geq 1$, we have*

$$(6.23) \quad \forall \rho, \xi \in \mathbb{R}, \quad \begin{aligned} \tilde{F}(\rho - \xi) & \leq \frac{C_p}{M^p} (\tilde{F}(\rho) + \tilde{F}(\xi)) \\ & \quad + C_p M^{2-p} (\rho - \xi) (\tilde{f}(\rho) - \tilde{f}(\xi)). \end{aligned}$$

End of the proof of Lemma 6.1 assuming Lemma 6.8. It follows from (6.22) and (6.23) that

$$(6.24) \quad \int_S^T \tilde{E}_p^2 \phi' \leq C_4^{(0)} \tilde{E}_p(S)^2 \phi'(S) + C_5^{(0)} \frac{C_p}{M^p} \int_S^T \tilde{E}_p^2 \phi' \\ + (C_5^{(0)} C_p M^{2-p} + 2C_1) \int_S^T \tilde{E}_p \phi' \int_0^1 a(x) (\rho - \xi) (\tilde{f}(\rho) - \tilde{f}(\xi)).$$

Then it is sufficient to take M large enough, and we obtain (6.4). \square

Proof of Lemma 6.8. Take $M \geq 1$, and divide the first quadrant of the plane $\{(\rho, \xi), \rho, \xi \in \mathbb{R}\}$ into the following four regions:

$$\mathcal{R}_1 := \{(\rho, \xi), \rho \geq 0, \xi \geq \frac{M+1}{M} \rho\}, \quad \mathcal{R}_2 := \{(\rho, \xi), \rho \geq 0, \rho \leq \xi \leq \frac{M+1}{M} \rho\}, \\ \mathcal{R}_3 := \{(\rho, \xi), \xi \geq 0, \frac{M}{M+1} \rho \leq \xi \leq \rho\}, \quad \mathcal{R}_4 := \{(\rho, \xi), \xi \geq 0, \xi \leq \frac{M}{M+1} \rho\}.$$

Then, using (6.9),

$$(\rho, \xi) \in \mathcal{R}_2 \implies 0 \leq \xi - \rho \leq \frac{\rho}{M} \implies \tilde{F}(\xi - \rho) \leq \tilde{F}\left(\frac{\rho}{M}\right) \leq \frac{\tilde{F}(\rho)}{C'_p M^p}.$$

In a symmetric way,

$$(\rho, \xi) \in \mathcal{R}_3 \implies 0 \leq \rho - \xi \leq \frac{\xi}{M} \implies \tilde{F}(\rho - \xi) \leq \tilde{F}\left(\frac{\xi}{M}\right) \leq \frac{\tilde{F}(\xi)}{C'_p M^p}.$$

Now, we note that

$$\mathcal{R}_1 = \mathcal{R}_{1,-} \cup \mathcal{R}_{1,+},$$

where

$$\mathcal{R}_{1,-} = \{(\rho, \xi) \in \mathcal{R}_1, \xi - \rho \leq 1\}, \quad \mathcal{R}_{1,+} = \{(\rho, \xi) \in \mathcal{R}_1, \xi - \rho \geq 1\}.$$

It is clear that $\mathcal{R}_{1,-}$ is compact:

$$(\rho, \xi) \in \mathcal{R}_{1,-} \implies \frac{\rho}{M} \leq \xi - \rho \leq 1 \implies \begin{cases} 0 \leq \rho \leq M, \\ 0 \leq \xi \leq 1 + \rho \leq 1 + M. \end{cases}$$

Therefore, if $(\rho, \xi) \in \mathcal{R}_{1,-}$, we see from (6.9) that

$$\tilde{F}(\xi - \rho) \leq C_p (\xi - \rho)^2,$$

and, on the other hand, there exists $c_{\rho, \xi} \in (\rho, \xi) \subset [0, M+1]$ such that

$$(\xi - \rho)(\tilde{f}(\xi) - \tilde{f}(\rho)) = (\xi - \rho)^2 \frac{\tilde{f}(\xi) - \tilde{f}(\rho)}{\xi - \rho} = (\xi - \rho)^2 \tilde{f}'(c_{\rho, \xi}) \geq (\xi - \rho)^2 \tilde{f}'(M+1).$$

Therefore

$$(\rho, \xi) \in \mathcal{R}_{1,-} \implies \tilde{F}(\xi - \rho) \leq \frac{C_p}{\tilde{f}'(M+1)} (\xi - \rho)(\tilde{f}(\xi) - \tilde{f}(\rho)).$$

Since $\tilde{f}'(x) = (p-1)(1+x)^{p-2}$, there exists $c_p > 0$ independent of $M \geq 1$ such that

$$(\rho, \xi) \in \mathcal{R}_{1,-} \implies \tilde{F}(\xi - \rho) \leq c_p M^{2-p} (\xi - \rho)(\tilde{f}(\xi) - \tilde{f}(\rho)).$$

Now, if $(\rho, \xi) \in \mathcal{R}_{1,+}$, introducing $\tau := \xi - \rho - 1$, we see from (6.9) that

$$\tilde{F}(\xi - \rho) \leq C_p (\xi - \rho)^p, \quad \text{hence} \quad \frac{\tilde{F}(\xi - \rho)}{\xi - \rho} \leq C_p (\xi - \rho)^{p-1} = C_p (1 + \tau)^{p-1},$$

and, on the other hand, we have

$$\begin{aligned}\tilde{f}(\xi) - \tilde{f}(\rho) &= (1 + \xi)^{p-1} - (1 + \rho)^{p-1} = \left([1 + \tau] + [1 + \rho]\right)^{p-1} - (1 + \rho)^{p-1} \\ &= (1 + \tau)^{p-1} \left((1 + \sigma)^{p-1} - \sigma^{p-1}\right) \quad \text{where } \sigma = \frac{1 + \rho}{1 + \tau}.\end{aligned}$$

To conclude, it is sufficient to note that, first, the function

$$\sigma \in \mathbb{R}_+ \mapsto (1 + \sigma)^{p-1} - \sigma^{p-1}$$

is nonincreasing, because $p < 2$, and that $\sigma \leq 2$ if $\rho \leq 1$, and

$$\sigma = \frac{1 + \rho}{1 + \tau} = \frac{1 + \rho}{\xi - \rho} \leq \frac{1 + \rho}{\frac{\rho}{M}} = M \frac{1 + \rho}{\rho} \leq 2M \quad \text{if } \rho \geq 1,$$

hence, in any case, $\sigma \leq 2M$, since $M \geq 1$. Therefore, we obtain that, if $(\rho, \xi) \in \mathcal{R}_{1,+}$, we have

$$\tilde{f}(\xi) - \tilde{f}(\rho) \geq \left((1 + 2M)^{p-1} - (2M)^{p-1}\right)(1 + \tau)^{p-1},$$

which gives that

$$(\rho, \xi) \in \mathcal{R}_{1,+} \implies \tilde{F}(\xi - \rho) \leq \frac{C_p}{(1 + 2M)^{p-1} - (2M)^{p-1}} (\xi - \rho)(\tilde{f}(\xi) - \tilde{f}(\rho)).$$

Since we have

$$(1 + 2M)^{p-1} - (2M)^{p-1} \sim \frac{p-1}{(2M)^{2-p}} \quad \text{as } p \rightarrow \infty,$$

there exists some $c_p > 0$ independent of $M \geq 1$ such that

$$(\rho, \xi) \in \mathcal{R}_1 \implies \tilde{F}(\xi - \rho) \leq c_p M^{2-p} (\xi - \rho)(\tilde{f}(\xi) - \tilde{f}(\rho)).$$

The situation is similar in \mathcal{R}_4 .

If $\rho\xi \leq 0$, the proof is much easier: assume that $\rho \leq 0 \leq \xi$. Then, either $\xi - \rho \leq 1$, or $\xi - \rho \geq 1$. The sub-region where $\xi - \rho \leq 1$ is exactly the triangle bounded by the horizontal axis ($O\rho$), the vertical axis ($O\xi$), and the line $\xi = \rho + 1$. Then $|\rho|, \xi \leq 1$, and we have

$$\tilde{F}(\xi - \rho) \leq C_p (\xi - \rho)^2 = C_p (\xi + |\rho|)^2,$$

and on the other hand,

$$\begin{aligned}(\xi - \rho)(\tilde{f}(\xi) - \tilde{f}(\rho)) &= (\xi + |\rho|) \left(\tilde{f}(\xi) + \tilde{f}(|\rho|)\right) \\ &\geq (\xi + |\rho|) \left(\tilde{f}'(1)\xi + \tilde{f}'(1)|\rho|\right) = \tilde{f}'(1) (\xi + |\rho|)^2 \geq \frac{\tilde{f}'(1)}{C_p} \tilde{F}(\xi - \rho).\end{aligned}$$

If $\xi - \rho \geq 1$, then

$$\frac{\tilde{F}(\xi - \rho)}{\xi - \rho} \leq C_p (\xi - \rho)^p,$$

while

$$\tilde{f}(\xi) - \tilde{f}(\rho) = \tilde{f}(\xi) + \tilde{f}(|\rho|) \geq c_p \xi^{p-1} + c_p |\rho|^{p-1},$$

and once again there is a uniform C such that

$$\tilde{F}(\xi - \rho) \leq C (\xi - \rho)(\tilde{f}(\xi) - \tilde{f}(\rho)).$$

This concludes the proof of Lemma 6.8. \square

6.5.e. Proof of Lemmas 6.4-6.7.

The proofs of Lemmas 6.4-6.7 are similar to the ones of Lemmas 5.2-5.4. We refer to [12], see also [4] on pages 41-46 for complete computations. Similarly to the proof of Lemmas 5.2-5.4, the complete argument relies on inequalities involving the functions f , F and g as well as their incremental ratios.

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