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► **To cite this version:**

P Cannarsa, P Martinez, C Urbani. BILINEAR CONTROL OF A DEGENERATE HYPERBOLIC EQUATION. SIAM Journal on Mathematical Analysis, 2023, 55 (6), pp.6517-6553. hal-04903146

HAL Id: hal-04903146

<https://hal.science/hal-04903146v1>

Submitted on 21 Jan 2025

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BILINEAR CONTROL OF A DEGENERATE HYPERBOLIC EQUATION

P. CANNARSA, P. MARTINEZ, AND C. URBANI

ABSTRACT. We consider the linear degenerate wave equation, on the interval $(0, 1)$

$$w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w,$$

with bilinear control p and Neumann boundary conditions. We study the controllability of this nonlinear control system, locally around a constant reference trajectory, the “ground state”.

Under some classical and generic assumption on μ , we prove that there exists a threshold value for time, $T_\alpha = \frac{4}{2-\alpha}$, such that the reachable set is

- a neighborhood of the ground state if $T > T_\alpha$,
- contained in a C^1 -submanifold of infinite codimension if $T < T_\alpha$,
- a C^1 -submanifold of codimension 1 if $\alpha \in [0, 1)$, and a neighborhood of the ground state if $\alpha \in (1, 2)$ if $T = T_\alpha$, the case $\alpha = 1$ remaining open.

This extends to the degenerate case the work [K. Beauchard, *Local controllability and non-controllability for a 1D wave equation with bilinear control*, J. Differential Equations, 250 (2011), pp. 2064-2098], and adapts to bilinear controls the work [F. Alabau-Boussouira, P. Cannarsa and G. Leugering, *Control and stabilization of degenerate wave equations*, SIAM J. Control Optim., 55 (2017), pp. 2052-2087].

Our proofs are based on a careful analysis of the spectral problem, and on Ingham type results, which are extensions of Kadec’s $\frac{1}{4}$ theorem.

1. INTRODUCTION

1.1. The context and the problem we study.

Degenerate partial differential equations appear in many domains, in particular physics, climate dynamics, biology, economics (see, e.g., [11, 16, 21]). For instance, when studying propagation of waves on a spherical domain (e.g. the Earth), see [30, Sections 8.3 and 7.3], the Legendre’s equation [30, (7.3.1)] plays a central role. Such equation is degenerate at both boundary points of the space domain $(-1, 1)$, with the same kind degeneracy that we shall consider in this paper.

Control of degenerate parabolic equations is, by now, a fairly well-developed subject (see, for instance, [10, 11, 12, 13, 18]), but very few results are available in the case of degenerate hyperbolic equations. To our best knowledge, a class of degenerate wave equations has been studied from the point of view of control theory in [1], where boundary control is studied using HUM and multiplier methods, and in [35, 34], where locally distributed control are considered.

1991 *Mathematics Subject Classification.* 35L80, 93B03, 93B60, 33C10, 42C40.

Key words and phrases. degenerate hyperbolic equations, bilinear control, Bessel functions, ground state, Riesz basis.

This research was partly supported by the Institut Mathematique de Toulouse and Istituto Nazionale di Alta Matematica. Part of this work was done during the conference “VIII Partial differential equations, optimal design and numerics 2019”, held in Benasque and supported by LIA COPDESC. We thank the Centro de Ciencias Pedro Pascual, in Benasque, Spain, for this opportunity. Moreover, the first author acknowledges support by the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

On the other hand, in many applications, one is naturally led to use bilinear controls, as such controls are more realistic than additive ones to govern the evolution of certain systems, see in particular [27] for several models coming from physics, and [3, 2, 8, 14, 17, 19, 20] for parabolic equations. This is why, in this paper, we address a bilinear control problem for the degenerate hyperbolic equation

$$(1.1) \quad \begin{cases} w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w, & x \in (0, 1), t \in (0, T), \\ (x^\alpha w_x)(x=0) = 0, \quad w_x(x=1) = 0, & t \in (0, T), \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), & x \in (0, 1). \end{cases}$$

Here,

- $\alpha \in [0, 2)$ is the degeneracy parameter ($\alpha = 0$ for the classical wave equation and $\alpha \in (0, 2)$ in the degenerate case),
- $p \in L^2(0, T)$ is a multiplicative control,
- μ is an admissible potential (and a key feature will be to analyze to which class μ has to belong in order to prove a controllability result.)

System (1.1) describes the vibration of a linear elastic string, made of a “smart material” (see [27]), having free ends with a prescribed slope. The displacement is due to an external force (e.g. temperature, electric or magnetic field) which alters the axial load $\nu(t, x) = p(t)\mu(x)$. The coefficient $a(x) = x^\alpha$ can be interpreted as the spatially varying stiffness of the string which exhibits a defect at the point of degeneracy $x = 0$ (when $\alpha > 0$), see [28].

Let us recall that the action of bilinear controls is weaker than the one of additive controls, in the sense that, with bilinear controls, one cannot expect the same kind of controllability results that can be proved with additive controls. This fact is described by the negative result obtained by Ball-Marsden-Slemrod in [5], where it is shown that the attainable set of any abstract linear system, subject to a bilinear control, has a dense complement.

For the Schrödinger equation, and then for the classical wave equation, attainability results with bilinear controls were obtained by Beauchard and Laurent in [7] and Beauchard in [6]. In particular in [6]

- it is proved that, starting from the “ground state” (which is the constant state associated to the first eigenvalue, 0), the solution is more regular than expected:

$$(w(T), w_t(T)) \in H^3(0, 1) \times H^2(0, 1),$$

- controllability properties have been established, distinguishing the cases $T > 2$, $T = 2$ and $T < 2$.

The goal of our paper is to extend these results to the degenerate case $\alpha \in (0, 2)$.

1.2. Main results, novelties and open questions.

First, we show that the solution starting from the ground state has additional regularity properties, see Proposition 3.2.

Moreover, starting from the the ground state, we analyse the reachable targets close to the ground state in the $H_\alpha^3(0, 1) \times H_\alpha^2(0, 1)$ topology. If μ verifies some natural conditions generically satisfied in some suitable Banach space (see Proposition 3.3), we prove that the threshold value for the controllability time is

$$T_\alpha = \frac{4}{2 - \alpha} :$$

- if $T > T_\alpha$: any target which is close to the ground state is reachable in time T (see Theorem 3.1),

- if $T = T_\alpha$ and $\alpha \in (1, 2)$: any target state close to the ground state is reachable in time T_α , except for a countable set of values of α (see Theorem 3.2),
- if $T = T_\alpha$ and $\alpha \in [0, 1)$, the set of reachable targets close to the ground state is a C^1 -manifold of codimension 1 (see Theorem 3.2)
- if $T < T_\alpha$, the set of reachable targets close to the ground state is contained in a C^1 -manifold of infinite dimension and of infinite codimension (see Theorem 3.3).

Our approach follows the strategy proposed by Beauchard [6] for the (nondegenerate) wave equation. However, it is worth noting that several new difficulties appear in the degenerate case:

- the spectral analysis is entirely new, nontrivial and crucial, see Proposition 3.1; moreover, the loss of regularity of the eigenfunctions influences the choice of the potential μ and the analysis of the problem,
- we prove our controllability results by using the moment method, coupled to an inversion theorem. The moment problem is studied
 - in a classical way when $T > T_\alpha$, using Ingham’s arguments,
 - thanks to some extensions of the classical Kadec $\frac{1}{4}$ Theorem ([33]), that allow us to treat the case $T = T_\alpha$ (see Lemma 3.1),
 - combining general results on non-minimal families of exponentials (see [4]) with density properties of the eigenvalues of our problem when $T < T_\alpha$,
- when $T = T_\alpha$, there exists a countable set of values of α for which our methods do not apply in order to solve the controllability issue. Among them, the case $\alpha = 1$ is the most interesting open problem, see Section 8.3.

To conclude, let us observe that, as the reader may have noticed, we have assumed the degeneracy exponent α to be in the interval $[0, 2)$: this restriction is partly due to our method, but on the other hand, it is known that problem (1. 1), with (additive) boundary controls, fails to be controllable for $\alpha \geq 2$ (see [1]).

1.3. Outline of the paper.

- In section 2, we recall the functional setting and the well-posedness results for the degenerate wave equation.
- In section 3, we state our main results:
 - the analysis of the eigenvalue problem associated to (1. 1), see Proposition 3.1,
 - a hidden regularity result of the value map, which is a fundamental observation to study the bilinear control problem, see Proposition 3.2,
 - a positive bilinear control result, see Theorem 3.1, when $T > \frac{4}{2-\alpha}$,
 - a bilinear control result, see Theorem 3.2, when $T = \frac{4}{2-\alpha}$,
 - a “negative” bilinear control result, see Theorem 3.3, when $T < \frac{4}{2-\alpha}$.
- In section 4, we prove the well posedness results.
- In section 5, we prove Proposition 3.1.
- In section 6, we prove Proposition 3.2.
- In section 7, we prove Theorem 3.1.
- In section 8, we prove Theorem 3.2.
- In section 9, we prove Theorem 3.3.
- In section 10, we prove Proposition 3.3.

Due to editorial requirements, some parts of the proof have only been sketched. The reader will find all technical details in the Supplementary material or in extended version of this paper [9], available on arXiv.

2. FUNCTIONAL SETTING AND WELL-POSEDNESS

2.1. **Weighted Sobolev spaces.**

We treat separately the cases of weak degeneracy, $\alpha \in [0, 1)$, and strong degeneracy, $\alpha \in [1, 2)$:

- for $0 \leq \alpha < 1$, we define

$$(2.1) \quad H_\alpha^1(0, 1) := \left\{ u \in L^2(0, 1), u \text{ abs. cont. in } [0, 1], x^{\alpha/2} u_x \in L^2(0, 1) \right\},$$

- for $1 \leq \alpha < 2$, we define

$$(2.2) \quad H_\alpha^1(0, 1) := \left\{ u \in L^2(0, 1), u \text{ loc. abs. cont. in } (0, 1], x^{\alpha/2} u_x \in L^2(0, 1) \right\},$$

where by abs. cont. and we mean absolutely continuous and by loc. abs. cont. we mean locally absolutely continuous. For any $\alpha \in [0, 2)$ we define the spaces

$$(2.3) \quad H_{\alpha,0}^1(0, 1) := \{ u \in H_\alpha^1(0, 1) : (x^\alpha u_x)(0) = 0, u_x(1) = 0 \}$$

and

$$(2.4) \quad H_\alpha^2(0, 1) := \left\{ u \in H_\alpha^1(0, 1), x^\alpha u_x \in H^1(0, 1) \right\},$$

and we consider the operator $A : D(A) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ defined by

$$(2.5) \quad \begin{cases} \forall u \in D(A), & Au := (x^\alpha u_x)_x, \\ D(A) := \{ u \in H_\alpha^2(0, 1), (x^\alpha u_x)(0) = 0, u_x(1) = 0 \}. \end{cases}$$

Then, the following results hold:

Proposition 2.1. *Let $\alpha \in [0, 2)$. Then,*

- $H_\alpha^1(0, 1)$ is a Hilbert space;
- $A : D(A) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ is a self-adjoint negative operator with dense domain.

We deduce that, for any $\alpha \in [0, 2)$, A is the infinitesimal generator of an analytic semigroup of contractions e^{At} on $L^2(0, 1)$.

The proof of Proposition 2.1 is similar to those of [12, Proposition 1] and [13, Proposition 2.1] and can be found in the Supplementary material or in [9, Proposition 2.1 and 2.2].

2.2. **Well posedness of the problem.**

Consider the non-homogeneous problem

$$(2.6) \quad \begin{cases} w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w + f(x, t), & x \in (0, 1), t \in (0, T), \\ (x^\alpha w_x)(x=0) = 0, \quad w_x(x=1) = 0, & t \in (0, T), \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), & x \in (0, 1). \end{cases}$$

We introduce the spaces

$$(2.7) \quad V_\alpha^{(1,\infty)} := \{ \mu \in H_\alpha^1(0, 1), x^{\alpha/2} \mu_x \in L^\infty(0, 1) \},$$

and

$$(2.8) \quad V_\alpha^1 := \begin{cases} H_\alpha^1(0, 1) & \text{if } \alpha \in [0, 1), \\ V_\alpha^{(1,\infty)} & \text{if } \alpha \in [1, 2). \end{cases}$$

Then, we have the following well-posedness result:

Proposition 2.2. *Let $T > 0$, $p \in L^2(0, T)$, $f \in L^2((0, T), H_\alpha^1(0, 1))$, and $\mu \in V_\alpha^1$. Then, for all $w_0 \in D(A)$, $w_1 \in H_\alpha^1(0, 1)$, there exists a unique classical solution of (2.6), i.e. a function*

$$(w, w_t) \in C^0([0, T], D(A) \times H_\alpha^1(0, 1)).$$

The proof of Proposition 2.2 is contained in Section 4.

3. MAIN RESULTS

3.1. The associated spectral problem.

We investigate the eigenvalues and eigenfunctions of the operator $-A$, hence we look for solutions (λ, Φ) of the problem

$$(3.1) \quad \begin{cases} -(x^\alpha \Phi')' = \lambda \Phi, & x \in (0, 1), \\ x^\alpha \Phi'(x) = 0, & x = 0, \\ \Phi'(1) = 0. \end{cases}$$

The difference with the spectral analysis of [12, 13] is the boundary condition at the point $x = 1$ which leads to new difficulties.

Proposition 3.1. *Given $\alpha \in [0, 2)$, set*

$$\kappa_\alpha := \frac{2-\alpha}{2}, \quad \nu_\alpha := \frac{\alpha-1}{2-\alpha},$$

and consider the Bessel function J_{ν_α} of order ν_α and of first kind, and the positive zeros $(j_{\nu_\alpha+1, n})_{n \geq 1}$ of the Bessel function $J_{\nu_\alpha+1}$. Then, the set of solutions (λ, Φ) of problem (3.1) is

$$\mathcal{S} = \{(\lambda_{\alpha, n}, \rho \Phi_{\alpha, n}), n \in \mathbb{N}, \rho \in \mathbb{R}\},$$

where

- for $n = 0$,

$$(3.2) \quad \lambda_{\alpha, 0} = 0, \quad \Phi_{\alpha, 0}(x) = 1,$$

- and for $n \geq 1$,

$$(3.3) \quad \lambda_{\alpha, n} = \kappa_\alpha^2 j_{\nu_\alpha+1, n}^2, \quad \Phi_{\alpha, n}(x) = K_{\alpha, n} x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}(j_{\nu_\alpha+1, n} x^{\frac{2-\alpha}{2}}),$$

where the positive constant $K_{\alpha, n}$ is chosen such that $\|\Phi_{\alpha, n}\|_{L^2(0,1)} = 1$.

Moreover, the sequence $(\Phi_{\alpha, n})_{n \geq 0}$ forms an orthonormal basis of $L^2(0, 1)$. Additionally, the sequence $(\sqrt{\lambda_{\alpha, n+1}} - \sqrt{\lambda_{\alpha, n}})_{n \geq 1}$ is decreasing and

$$(3.4) \quad \sqrt{\lambda_{\alpha, n+1}} - \sqrt{\lambda_{\alpha, n}} \rightarrow \frac{2-\alpha}{2} \pi \quad \text{as } n \rightarrow \infty.$$

The proof of Proposition 3.1 is contained in Section 5.

3.2. Hidden regularity.

Let us start by introducing some notation which will be used in the proofs of our results. To avoid possible problems generated by the eigenvalue 0, we define

$$(3.5) \quad \lambda_{\alpha, n}^* := \begin{cases} 1 & \text{for } n = 0, \\ \lambda_{\alpha, n} & \text{for } n \geq 1. \end{cases}$$

It will be useful to introduce the following intermediate Sobolev spaces for any $s > 0$:

$$(3.6) \quad H_{(\alpha)}^s := D((-A)^{s/2}) = \left\{ \psi \in L^2(0, 1), \sum_{k=0}^{\infty} (\lambda_{\alpha, k}^*)^s \langle \psi, \Phi_{\alpha, k} \rangle_{L^2(0,1)}^2 < \infty \right\},$$

equipped with the norm

$$\|\psi\|_{H_{(\alpha)}^s} := \left(\sum_{k=0}^{\infty} (\lambda_{\alpha, k}^*)^s \langle \psi, \Phi_{\alpha, k} \rangle_{L^2(0,1)}^2 \right)^{1/2}.$$

It can be checked that the spaces $H_{\alpha, 0}^1(0, 1)$ and $H_{(\alpha)}^1(0, 1)$ coincides as sets and the associated norms are equivalent, as well as the spaces $D(A)$ and $H_{(\alpha)}^2(0, 1)$ (see the Supplementary material).

We also define the following spaces

$$(3.7) \quad V_\alpha^{(2,\infty)} := \{\mu \in H_\alpha^2(0,1), x^{\alpha/2}\mu_x \in L^\infty(0,1)\},$$

$$(3.8) \quad V_\alpha^{(2,\infty,\infty)} := \{\mu \in H_\alpha^2(0,1), x^{\alpha/2}\mu_x \in L^\infty(0,1), (x^\alpha\mu_x)_x \in L^\infty(0,1)\}.$$

Given $(w_0, w_1) \in H_\alpha^1(0,1) \times L^2(0,1)$ and $p \in L^2(0,T)$, we will denote by $w^{(w_0, w_1; p)}$ the solution of (1.1) associated to initial conditions w_0, w_1 and control p . In particular, when $(w_0, w_1) = (1, 0)$ and $p = 0$, we note that the constant function equal to 1 satisfies (1.1), hence $w^{(1,0;0)} = 1$, and $w_t^{(1,0;0)} = 0$.

In the following, we will be interested in the regularity of the solution of (1.1) starting from the ground state $(1, 0)$, that is, the solution $w^{(1,0;p)}$ (or simply $w^{(p)}$) of

$$(3.9) \quad \begin{cases} w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w, & x \in (0,1), t \in (0,T), \\ (x^\alpha w_x)(x=0) = 0, \quad w_x(x=1) = 0, & t \in (0,T), \\ w(x,0) = 1, \quad w_t(x,0) = 0, & x \in (0,1). \end{cases}$$

We will prove the following result, that extends the regularity result of [6, Theorem 3] to the degenerate case:

Proposition 3.2. *Let $T > 0$ and*

$$(3.10) \quad \mu \in V_\alpha^2 := \begin{cases} V_\alpha^{(2,\infty)} & \text{if } \alpha \in [0,1), \\ V_\alpha^{(2,\infty,\infty)} & \text{if } \alpha \in [1,2). \end{cases}$$

Then, for all $p \in L^2(0,T)$, the solution $w^{(p)}$ of (3.9) satisfies

$$(3.11) \quad (w^{(p)}(T), w_t^{(p)}(T)) \in H_{(\alpha)}^3 \times D(A).$$

The proof of Proposition 3.2 is contained in Section 6.

3.3. Main controllability results.

Because of the negative result contained in [5], one could not expect any controllability property to hold in the spaces $H_{(\alpha)}^2(0,1) \times H_\alpha^1(0,1)$. However, since the multiplication operator $Bu := \mu u$ does not preserve the space $H_{(\alpha)}^3(0,1)$, the chance to achieve controllability results in $H_{(\alpha)}^3 \times H_{(\alpha)}^2(0,1)$ is still open. For this purpose, we will need additional assumptions on the admissible potential μ . Furthermore, we observe that controllability properties will depend on a threshold value for the controllability time because of the finite speed of propagation, as it always happens for hyperbolic equations.

We will show that the value

$$(3.12) \quad T_\alpha := \frac{4}{2-\alpha},$$

is the threshold time for controllability.

Let us define the following subclass of admissible potentials μ

$$(3.13) \quad V^{(adm)} := \{\mu \in V_\alpha^2, \text{ such that } \exists c > 0 : \lambda_{\alpha,n}^* |\langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)}| \geq c, \forall n \geq 0\}.$$

We observe that the space $V^{(adm)}$ is not empty. Indeed, in the proposition that follows we exhibit an admissible potential $\mu \in V^{(adm)}$.

Proposition 3.3. *The function $\mu(x) = x^{2-\alpha}$ belongs to $V^{(adm)}$. Moreover, the space $V^{(adm)}$ is dense in V_α^2 .*

The proof of Proposition 3.3 is contained in Section 10. We refer to the recent work of Alabau-Boussouira, Urbani [31, Chapter 5] where sufficient conditions for building polynomial functions that fulfill the last condition in (3.13) are given.

Let us state our controllability results.

Theorem 3.1. Given $\alpha \in [0, 2)$, let $\mu \in V^{(adm)}$ (defined in (3. 13)) and

$$(3. 14) \quad T > T_\alpha.$$

Then, there exists a neighbourhood $\mathcal{V}(1, 0)$ of $(1, 0)$ in $H_{(\alpha)}^3(0, 1) \times D(A)$ such that for all $(w_0^f, w_1^f) \in \mathcal{V}(1, 0)$ there exists a unique $p^f \in L^2(0, T)$ close to 0 such that

$$(w^{(p^f)}(T), w_t^{(p^f)}(T)) = (w_0^f, w_1^f).$$

Moreover, the application

$$\Gamma_{\alpha, T} : \mathcal{V}(1, 0) \rightarrow L^2(0, T), \quad (w_0^f, w_1^f) \mapsto p^f$$

is of class C^1 .

The proof of Theorem 3.1 can be found in Section 7.

Theorem 3.2. Let $\mu \in V^{(adm)}$ (defined in (3. 13)) and

$$(3. 15) \quad T = T_\alpha.$$

Then,

- a) for $\alpha \in [0, 1)$, the reachable set is locally a C^1 -submanifold of $H_{(\alpha)}^3(0, 1) \times D(A)$ of codimension 1,
- b) for $\alpha \in (1, 2)$ and $\frac{1}{2-\alpha} \notin \mathbb{N}$, the reachable set is a whole neighborhood of $(1, 0)$ in $H_{(\alpha)}^3(0, 1) \times D(A)$.

The proof of Theorem 3.2 is contained in Section 8.

Theorem 3.3. Let $\mu \in V^{(adm)}$ (defined in (3. 13)) and

$$(3. 16) \quad T < T_\alpha.$$

Then, the reachable set is locally contained in a C^1 -submanifold of $H_{(\alpha)}^3(0, 1) \times D(A)$ of infinite dimension and of infinite codimension.

The proof of Theorem 3.3 is in Section 9.

The proof of Theorem 3.2 ($T = T_\alpha$) is based on the following extension of the Kadec's $\frac{1}{4}$ Theorem ([26], [33, Theorem 1.14 p. 42]), which is close to results of Joó [25, Theorem F p. 149].

Lemma 3.1. Consider an odd sequence of real numbers $\{x_n, n \in \mathbb{Z}\}$, that is $x_{-n} = -x_n$, such that there exist $k \in \mathbb{N}^*$, $\delta \in (0, \frac{1}{4})$ and $N_0 \geq 0$ for which

$$(3. 17) \quad n \geq N_0 \quad \implies \quad \left| x_n - n - \frac{k}{2} \right| \leq \frac{1}{4} - \delta.$$

Then, choosing distinct real numbers $x'_1, \dots, x'_k \notin \{x_n, n \in \mathbb{Z}\}$, the set

$$\{e^{ix'_1 t}, \dots, e^{ix'_k t}\} \cup \{e^{ix_n t}, n \in \mathbb{Z}\}$$

is a Riesz basis of $L^2(-\pi, \pi)$.

The proof of Lemma 3.1 is contained in Section 8.2.b.

3.3.a. Remarks and open problems.

When $T > T_\alpha$ and $T < T_\alpha$, Theorems 3.1 and 3.3 describe the sets of reachable states. The case $T = T_\alpha$ is the most delicate and we would like to draw the attention to the different nature of the reachable set for the weak ($\alpha \in [0, 1)$) and strong ($\alpha \in [1, 2)$) degeneracy: while in the first case it is a submanifold of codimension 1, in the latter case it is a complete neighborhood of $(1, 0)$, except for a countable number of values of α . These exceptional values $\alpha_k = 2 - \frac{1}{k}$, $k \in \mathbb{N}^*$, are the points where the nature of the set $\{e^{i\omega_{\alpha, n} t}, n \in \mathbb{Z}\}$ (where $\omega_{\alpha, n} = \sqrt{\lambda_{\alpha, n}}$ for $n \geq 0$) changes:

- for $\alpha \in [0, 1)$ it is a Riesz basis of $L^2(0, T_\alpha)$,
- for $\alpha \in (1, \frac{3}{2})$ it has a deficiency equal to 1 in $L^2(0, T_\alpha)$,
- for $\alpha \in (\frac{3}{2}, \frac{5}{3})$ it has a deficiency equal to 2 in $L^2(0, T_\alpha)$,

and so on (see Lemma 8.4 for a detailed study, that derives from the Kadec's $\frac{1}{4}$ Theorem). What happens for these exceptional values of α is still an open problem, and the case $\alpha = 1$ is particularly challenging (see Section 8.3).

4. FUNCTIONAL SETTING: PROOF OF PROPOSITION 2.2

In order to recast (2. 6) into the a first order problem, we introduce

$$\mathcal{W} := \begin{pmatrix} w \\ w_t \end{pmatrix}, \quad \mathcal{W}_0 := \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \quad \mathcal{F}(x, t) := \begin{pmatrix} 0 \\ f(x, t) \end{pmatrix},$$

the state space

$$\mathcal{X} := H_\alpha^1(0, 1) \times L^2(0, 1),$$

and the operators

$$(4. 1) \quad \mathcal{A} := \begin{pmatrix} 0 & \text{Id} \\ A & 0 \end{pmatrix}, \quad D(\mathcal{A}) := D(A) \times H_\alpha^1(0, 1),$$

and

$$(4. 2) \quad \mathcal{B} := \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \quad D(\mathcal{B}) := H_\alpha^1(0, 1) \times L^2(0, 1).$$

So, problem (2. 6) can be rewritten as

$$(4. 3) \quad \begin{cases} \mathcal{W}'(t) = \mathcal{A}\mathcal{W}(t) + p(t)\mathcal{B}\mathcal{W}(t) + \mathcal{F}(t), \\ \mathcal{W}(0) = \mathcal{W}_0. \end{cases}$$

Let us now prove the following regularity result:

Lemma 4.1. *Let $\mu \in V_\alpha^1$. Then, the operator \mathcal{B} defined in (4. 2) satisfies*

$$\mathcal{B} \in \mathcal{L}_c(D(\mathcal{A}), D(\mathcal{A})).$$

Proof of Lemma 4.1. We have to prove that $w_0 \in D(A) \implies \mu w_0 \in H_\alpha^1(0, 1)$, and that there exists $C > 0$ such that

$$(4. 4) \quad \|\mu w_0\|_{H_\alpha^1(0, 1)} \leq C \|w_0\|_{D(A)}.$$

We distinguish the cases $\alpha \in [0, 1)$ and $\alpha \in [1, 2)$.

$\alpha \in [0, 1)$: we can decompose w_0' as follows $w_0' = (x^{\alpha/2} w_0')(x^{-\alpha/2})$. Since $w_0 \in H_\alpha^1(0, 1)$, we deduce that $w_0' \in L^1(0, 1)$ and thus $w_0 \in L^\infty(0, 1)$. The same holds for μ because $V_\alpha^1 = H_\alpha^1(0, 1)$ for $\alpha \in [0, 1)$. Hence, $(\mu w_0)' = \mu' w_0 + \mu w_0' \in L^1(0, 1)$ and therefore μw_0 is absolutely continuous on $[0, 1]$. Furthermore, we have that $x^{\alpha/2}(\mu w_0)' = (x^{\alpha/2} \mu') w_0 + \mu (x^{\alpha/2} w_0') \in L^2(0, 1)$ and so we infer that $\mu w_0 \in H_\alpha^1(0, 1)$. Finally, there exists $C > 0$ such that

$$\forall w \in H_\alpha^1(0, 1), \quad \|w\|_{L^\infty(0, 1)} \leq C \|w\|_{H_\alpha^1(0, 1)},$$

and this implies that (4. 4) holds.

$\alpha \in [1, 2)$: first, we observe that $\mu \in V_\alpha^{1, \infty}(0, 1)$ implies that $|\mu_x| \leq \frac{C}{x^{\alpha/2}}$. Therefore, we get that $\mu_x \in L^1(0, 1)$, and so $\mu \in L^\infty(0, 1)$ and $\mu w_0 \in L^2(0, 1)$. Moreover, $x^{\alpha/2}(\mu w_0)' = (x^{\alpha/2} \mu_x) w_0 + (x^{\alpha/2} w_0') \mu$ and since $x^{\alpha/2} \mu_x \in L^\infty(0, 1)$ and $w_0 \in L^2(0, 1)$, we have that $(x^{\alpha/2} \mu_x) w_0 \in L^2(0, 1)$. Furthermore, since $x^{\alpha/2} w_0' \in L^2(0, 1)$ and $\mu \in L^\infty(0, 1)$, we deduce that $(x^{\alpha/2} w_0') \mu \in L^2(0, 1)$ and hence $x^{\alpha/2}(\mu w_0)' \in L^2(0, 1)$. By reasoning as in the case $\alpha \in [0, 1)$, we deduce that (4. 4) is verified. \square

The proof of Proposition 2.2 follows in a classical way from a fixed point argument, considering the map

$$\mathcal{K} : C^0([0, T], D(\mathcal{A})) \rightarrow C^0([0, T], D(\mathcal{A}))$$

defined by

$$\forall t \in [0, T], \quad \mathcal{K}(\mathcal{W})(t) := e^{At}\mathcal{W}_0 + \int_0^t e^{A(t-s)}(p(s)\mathcal{B}\mathcal{W}(s) + \mathcal{F}(s)) ds.$$

Moreover, using Gronwall's lemma, one can prove that there exists $C = C(\alpha, T, p) > 0$ such that the fixed point \mathcal{W} satisfies

$$(4.5) \quad \|\mathcal{W}\|_{C^0([0, T], D(\mathcal{A}))} \leq C \left(\|\mathcal{W}_0\|_{D(\mathcal{A})} + \|F\|_{L^2(0, T; D(\mathcal{A}))} \right).$$

(See the Supplementary material or [9, Proposition 2.3] for a complete proof.) \square

5. SPECTRAL PROBLEM: PROOF OF PROPOSITION 3.1

We split the proof of Proposition 3.1 into five steps, each of which is treated in one of the subsections below.

5.1. A classical change of variables.

First we note that if (λ, Φ) solves (3.1), then $\lambda \geq 0$: indeed, multiplying by Φ , we obtain

$$\lambda \int_0^1 \Phi^2 = \int_0^1 -(x^\alpha \Phi')' \Phi = [-x^\alpha \Phi']_0^1 + \int_0^1 x^\alpha (\Phi')^2 = \int_0^1 x^\alpha (\Phi')^2.$$

Moreover, if $\lambda = 0$, then $x \mapsto x^\alpha \Phi'$ is constant and by imposing the boundary conditions we find that it is actually equal to 0. Thus, the constant functions are the only ones associated to the eigenvalue $\lambda = 0$.

We now investigate the positive eigenvalues: if $\lambda > 0$, we introduce the function ψ and the associated new space variable defined by the relations

$$\Phi(x) = x^{\frac{1-\alpha}{2}} \psi\left(\frac{2}{2-\alpha} \sqrt{\lambda} x^{\frac{2-\alpha}{2}}\right), \quad y = \frac{2}{2-\alpha} \sqrt{\lambda} x^{\frac{2-\alpha}{2}},$$

(see, e.g., [13, 22]). After some classical computations, we obtain that ψ satisfies the following problem:

$$(5.1) \quad \begin{cases} y^2 \psi''(y) + y \psi'(y) + \left(y^2 - \left(\frac{1-\alpha}{2-\alpha}\right)^2\right) \psi(y) = 0, & y \in (0, \frac{2}{2-\alpha} \sqrt{\lambda}), \\ y^{\frac{1-\alpha}{2-\alpha}} \psi'(y) + \frac{1-\alpha}{2-\alpha} y^{\frac{\alpha-1}{2-\alpha}} \psi(y) \rightarrow 0 \text{ as } y \rightarrow 0, \\ \sqrt{\lambda} \psi'\left(\frac{2}{2-\alpha} \sqrt{\lambda}\right) + \frac{1-\alpha}{2} \psi\left(\frac{2}{2-\alpha} \sqrt{\lambda}\right) = 0. \end{cases}$$

The first equation in (5.1) is the Bessel equation of order $\nu = |\nu_\alpha|$ (see [32] or [30]). Our study will be based on well-known properties of Bessel functions, and it is similar to the one in [13], the only difference, which makes the analysis interesting, lying in the boundary condition at point $x = 1$. In the following, we solve (3.1) using (5.1) when $\alpha \in [0, 1)$. The case $\alpha \in [1, 2)$ is similar and has been analyzed in [9, Section 5.3].

5.2. Analysis of the ODE for $\alpha \in [0, 1)$.

In this section we solve

$$(5.2) \quad y^2 \psi''(y) + y \psi'(y) + (y^2 - \nu^2) \psi(y) = 0, \quad y \in I \subset (0, +\infty)$$

with $\nu = |\nu_\alpha| \in (0, \frac{1}{2})$ and $I = (0, \frac{2}{2-\alpha} \sqrt{\lambda})$. The above equation is called *Bessel's equation for functions of order ν* . The fundamental theory of ordinary differential

equations establishes that the solutions of (5. 2) generate a vector space S_ν of dimension 2. Consider the Bessel function of order ν and of the first kind J_ν

$$(5. 3) \quad J_\nu(y) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left(\frac{y}{2}\right)^{2m+\nu} = \sum_{m=0}^{\infty} c_{\nu,m}^+ y^{2m+\nu}, \quad y > 0.$$

and $J_{-\nu}$

$$(5. 4) \quad J_{-\nu}(y) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m - \nu + 1)} \left(\frac{y}{2}\right)^{2m-\nu} = \sum_{m=0}^{\infty} c_{\nu,m}^- y^{2m-\nu}, \quad y > 0.$$

Since $\nu \notin \mathbb{N}$, the functions J_ν and $J_{-\nu}$ are linearly independent and therefore the pair $(J_\nu, J_{-\nu})$ forms a fundamental system of solutions of (5. 2), (see [32, section 3.1, eq. (8), p. 40], [32, section 3.12, eq. (2), p. 43] or [30, eq. (5.3.2), p. 102]). Hence,

$$(5. 5) \quad \psi \text{ solution of (5. 2) on } I \implies \exists C_+, C_- \in \mathbb{R}, \psi = C_+ J_{|\nu_\alpha|} + C_- J_{-|\nu_\alpha|}.$$

Thus, going back to the original variables, we obtain that

$$(5. 6) \quad -(x^\alpha \Phi')' = \lambda \Phi \text{ on } (0, 1) \implies \exists C_+, C_- \in \mathbb{R}, \Phi = C_+ \Phi_+ + C_- \Phi_-,$$

with

$$(5. 7) \quad \Phi_+(x) = x^{\frac{1-\alpha}{2}} J_{|\nu_\alpha|} \left(\frac{2}{2-\alpha} \sqrt{\lambda} x^{\frac{2-\alpha}{2}} \right) = \sum_{m=0}^{\infty} \tilde{c}_{\alpha,\lambda,m}^+ x^{1-\alpha+(2-\alpha)m}$$

and

$$(5. 8) \quad \Phi_-(x) = x^{\frac{1-\alpha}{2}} J_{-|\nu_\alpha|} \left(\frac{2}{2-\alpha} \sqrt{\lambda} x^{\frac{2-\alpha}{2}} \right) = \sum_{m=0}^{\infty} \tilde{c}_{\alpha,\lambda,m}^- x^{(2-\alpha)m}.$$

Observe that thanks to the above series development, it is possible to check that $\Phi_+, \Phi_- \in H_\alpha^2(0, 1)$. Hence, any linear combination of Φ_+ and Φ_- belongs to $H_\alpha^2(0, 1)$.

5.3. Information given by the boundary condition at $x = 0$ for $\alpha \in [0, 1)$.

An eigenfunction must satisfy additionally the boundary conditions. In particular, we should have that $x^\alpha \Phi'(x) \rightarrow 0$ as $x \rightarrow 0$. We observe that

$$x^\alpha \Phi'_-(x) \rightarrow 0 \quad \text{as } x \rightarrow 0,$$

while

$$x^\alpha \Phi'_+(x) \rightarrow \tilde{c}_{\alpha,\lambda,0}^+ (1-\alpha) \neq 0 \quad \text{as } x \rightarrow 0.$$

Therefore, we conclude that

$$\begin{cases} -(x^\alpha \Phi')' = \lambda \Phi, & x \in (0, 1) \\ (x^\alpha \Phi')(0) = 0 \end{cases} \implies \exists C_- \in \mathbb{R}, \quad \begin{cases} \Phi(x) = C_- \Phi_-(x), \\ x \in (0, 1), \end{cases}$$

5.4. Information given by the boundary condition at $x = 1$ for $\alpha \in [0, 1)$.

As regards the boundary condition at $x = 1$, Φ has to solve $\Phi'(1) = 0$. We compute Φ'_- :

$$\Phi'_-(x) = \frac{1-\alpha}{2} x^{\frac{1-\alpha}{2}} J_{\nu_\alpha} \left(\frac{2}{2-\alpha} \sqrt{\lambda} x^{\frac{2-\alpha}{2}} \right) + x^{\frac{1-\alpha}{2}} \sqrt{\lambda} x^{-\alpha/2} J'_{\nu_\alpha} \left(\frac{2}{2-\alpha} \sqrt{\lambda} x^{\frac{2-\alpha}{2}} \right).$$

Hence, if Φ is an eigenfunction, $C_- \neq 0$ and $\Phi'_-(1) = 0$, therefore

$$(5. 9) \quad \frac{1-\alpha}{2} J_{\nu_\alpha} \left(\frac{2}{2-\alpha} \sqrt{\lambda} \right) + \sqrt{\lambda} J'_{\nu_\alpha} \left(\frac{2}{2-\alpha} \sqrt{\lambda} \right) = 0.$$

This is the equation that characterizes the eigenvalues λ . Multiplying by $\frac{2}{2-\alpha}$ and introducing the variable $X_\lambda = \frac{2}{2-\alpha}\sqrt{\lambda}$, (5. 9) becomes

$$(5. 10) \quad -\nu_\alpha J_{\nu_\alpha}(X_\lambda) + X_\lambda J'_{\nu_\alpha}(X_\lambda) = 0.$$

We now consider the following well-known relation (see [32, p. 45, formula (4)])

$$(5. 11) \quad z J'_\nu(z) - \nu J_\nu(z) = z J_{\nu+1}(z).$$

Combining (5. 10) and (5. 11), we obtain that

$$(5. 12) \quad 0 = -\nu_\alpha J_{\nu_\alpha}(X_\lambda) + X_\lambda J'_{\nu_\alpha}(X_\lambda) = X_\lambda J_{\nu_\alpha+1}(X_\lambda),$$

which implies that

$$(5. 13) \quad J_{\nu_\alpha+1}(X_\lambda) = 0.$$

Thus, the possible values for X_λ are the positive zeros of $J_{\nu_\alpha+1}$:

$$\exists n \geq 1 : \frac{2}{2-\alpha}\sqrt{\lambda} = X_\lambda = j_{\nu_\alpha+1,n}.$$

We obtain that the eigenvalues of (3. 1) have the following form:

$$\exists n \geq 1 : \lambda_n = \kappa_\alpha^2 j_{\nu_\alpha+1,n}^2.$$

Vice-versa, given $n \geq 1$, consider

$$\lambda_n := \kappa_\alpha^2 j_{\nu_\alpha+1,n}^2 \quad \text{and} \quad \Phi_n(x) = x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(j_{\nu_\alpha+1,n} x^{\frac{2-\alpha}{2}}\right).$$

From the previous analysis we deduce that $\Phi_n \in H_\alpha^2(0, 1)$ and that (λ_n, Φ_n) solves (3. 1).

Finally, the proof of (3. 4) follows directly from [29, p. 135]: since $\nu_\alpha + 1 \geq \frac{1}{2}$, the sequence $(j_{\nu_\alpha+1,n+1} - j_{\nu_\alpha+1,n})_{n \geq 1}$ is nonincreasing and

$$j_{\nu_\alpha+1,n+1} - j_{\nu_\alpha+1,n} \rightarrow \pi \quad \text{as } n \rightarrow \infty.$$

This concludes the proof of Proposition 3.1 for $\alpha \in [0, 1)$.

When $\alpha \in [1, 2)$, several cases have to be distinguished depending on whether ν_α is an integer or not. The study involves Bessel's functions of order ν and of second kind when $\nu_\alpha \in \mathbb{N}$, see [32, section 3.54, eq. (1)-(2), p. 64] or [30, eq. (5.4.5)-(5.4.6), p. 104]). We refer to [9, Propositions 3.1 and 3.2] for a complete discussion of the spectral problem.

5.5. Additional information on the eigenfunctions.

Lemma 5.1. *Let $\alpha \in [0, 2)$ and $n \geq 1$. Then, $\Phi_{\alpha,n}$ has finite limits as $x \rightarrow 0^+$ and $x \rightarrow 1^-$, and satisfies*

$$(5. 14) \quad |\Phi_{\alpha,n}(1)| = \sqrt{2-\alpha},$$

and

$$(5. 15) \quad \Phi_{\alpha,n}(0) \sim \frac{1}{\Gamma(\nu_\alpha + 1)2^{\nu_\alpha}} \sqrt{\frac{(2-\alpha)\pi}{2}} (j_{\nu_\alpha+1,n})^{\frac{1}{2}+\nu_\alpha} \quad \text{as } n \rightarrow +\infty.$$

In particular, the sequence $(\Phi_{\alpha,n}(0))_{n \geq 1}$ is unbounded for any $\alpha > 0$.

Proof of Lemma 5.1. First, we note that $j_{\nu_\alpha+1,n}$ is not a zero of J_{ν_α} . Indeed, if $J_{\nu_\alpha}(j_{\nu_\alpha+1,n}) = 0$, we derive from (5. 11) that $J'_{\nu_\alpha}(j_{\nu_\alpha+1,n}) = 0$, and then the Cauchy problem satisfied by J_{ν_α} would imply that J_{ν_α} is constantly equal to zero.

We also derive from (5. 11) that

$$(5. 16) \quad J'_{\nu_\alpha}(j_{\nu_\alpha+1,n}) = \frac{\nu_\alpha}{j_{\nu_\alpha+1,n}} J_{\nu_\alpha}(j_{\nu_\alpha+1,n}).$$

We compute the value of the constant $K_{\alpha,n}$ that appears in (3. 2)

$$1 = K_{\alpha,n}^2 \int_0^1 x^{1-\alpha} J_{\nu_\alpha} \left(j_{\nu_\alpha+1,n} x^{\frac{2-\alpha}{2}} \right)^2 dx = \frac{2 K_{\alpha,n}^2}{2-\alpha} \int_0^1 y J_{\nu_\alpha} \left(j_{\nu_\alpha+1,n} y \right)^2 dy,$$

and using [30, formula (5.14.5) p.129] and (5. 16), we obtain

$$(5. 17) \quad \forall \alpha \in [0, 1), \forall n \geq 1, \quad K_{\alpha,n} = \frac{\sqrt{2-\alpha}}{|J_{\nu_\alpha}(j_{\nu_\alpha+1,n})|}.$$

Hence, (5. 14) is satisfied and we have that

$$(5. 18) \quad \Phi_{\alpha,n}(0) = \frac{\sqrt{2-\alpha}}{|J_{\nu_\alpha}(j_{\nu_\alpha+1,n})|} \frac{1}{\Gamma(\nu_\alpha+1)2^{\nu_\alpha}} (j_{\nu_\alpha+1,n})^{\nu_\alpha}.$$

Moreover, from the following classical asymptotic development ([30, formula (5.11.6) p. 122])

$$(5. 19) \quad J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \left(1 + O\left(\frac{1}{z^2}\right)\right) + O\left(\frac{1}{z}\right) \right] \quad \text{as } z \rightarrow \infty$$

we deduce that

$$(5. 20) \quad zJ_\nu(z)^2 + zJ_{\nu+1}(z)^2 \rightarrow \frac{2}{\pi} \quad \text{as } z \rightarrow +\infty.$$

Applying (5. 20) to $z = j_{\nu_\alpha+1,n}$ and using (5. 18), we finally obtain (5. 15). \square

6. PROOF OF PROPOSITION 3.2

The proof of Proposition 3.2 is based on several steps, the first one consists in analyzing the eigenvalues and eigenfunctions of the operator \mathcal{A} .

6.1. Eigenvalues and eigenfunctions of \mathcal{A} .

Let us give the following preliminary result.

Lemma 6.1. *Consider, for all $n \in \mathbb{Z}$,*

$$(6. 1) \quad \omega_{\alpha,n} := \begin{cases} -\sqrt{\lambda_{\alpha,|n|}}, & n \leq -1, \\ 0, & n = 0, \\ \sqrt{\lambda_{\alpha,n}}, & n \geq 1, \end{cases} \quad \text{and} \quad \Psi_{\alpha,n} := \begin{pmatrix} 1 \\ i\omega_{\alpha,n} \end{pmatrix} \Phi_{\alpha,|n|}.$$

Then, $\{\omega_{\alpha,n}\}_{n \in \mathbb{Z}}$ and $\{\Psi_{\alpha,n}\}_{n \in \mathbb{Z}}$ fulfill

$$(6. 2) \quad \forall n \in \mathbb{Z}, \quad \mathcal{A}\Psi_{\alpha,n} = i\omega_{\alpha,n} \Psi_{\alpha,n}.$$

The proof of Lemma 6.1 is immediate, see the Supplementary material or [9, Lemma 6.1].

6.2. Two integral expressions for the solution of (3. 9).

Since the family $\{\Phi_{\alpha,n}\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(0, 1)$, we can decompose the solution $w^{(p)}$ of (3. 9) under the form

$$w^{(p)}(x, t) = \sum_{n=0}^{\infty} w_n(t) \Phi_{\alpha,n}(x).$$

Introducing

$$(6. 3) \quad r_n(t) = \langle p(t)\mu(\cdot)w^{(p)}(\cdot, t), \Phi_{\alpha,n} \rangle_{L^2(0,1)},$$

we deduce from (3. 9) that the sequence $(w_n(t))_{n \geq 0}$ satisfies

$$\begin{cases} w_0''(t) = r_0(t), \\ w_0(0) = 1, \quad w_0'(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} w_n''(t) + \lambda_{\alpha,n} w_n(t) = r_n(t), \\ w_n(0) = 0, \quad w_n'(0) = 0, \end{cases} \quad \forall n \geq 1.$$

Solving these Cauchy problems, we obtain that the solution of (3. 9) satisfies

$$w^{(p)}(x, t) = \left(1 + \int_0^t r_0(s)(t-s) ds\right) + \sum_{n=1}^{\infty} \left(\int_0^t r_n(s) \frac{\sin \sqrt{\lambda_{\alpha,n}}(t-s)}{\sqrt{\lambda_{\alpha,n}}} ds\right) \Phi_{\alpha,n}(x),$$

and

$$w_t^{(p)}(x, t) = \left(\int_0^t r_0(s) ds\right) + \sum_{n=1}^{\infty} \left(\sqrt{\lambda_{\alpha,n}} \int_0^t r_n(s) \frac{\cos \sqrt{\lambda_{\alpha,n}}(t-s)}{\sqrt{\lambda_{\alpha,n}}} ds\right) \Phi_{\alpha,n}(x).$$

Denoting

$$(6. 4) \quad \Gamma_0^{(p)}(T) = \begin{pmatrix} 1 + \int_0^T r_0(s)(T-s) ds \\ \int_0^T r_0(s) ds \end{pmatrix} = \begin{pmatrix} \gamma_{00}^{(p)}(T) \\ \gamma_{01}^{(p)}(T) \end{pmatrix},$$

and

$$(6. 5) \quad \forall n \in \mathbb{Z}^*, \quad \gamma_n^{(p)}(T) = \int_0^T r_{|n|}(s) e^{-i\omega_{\alpha,n}s} ds,$$

one can check (see [9, Section 6.2]) that

$$(6. 6) \quad \begin{pmatrix} w^{(p)}(x, T) \\ w_t^{(p)}(x, T) \end{pmatrix} = \Gamma_0^{(p)}(T) + \sum_{n \in \mathbb{Z}^*} \frac{1}{2i\omega_{\alpha,n}} \gamma_n^{(p)}(T) \Psi_{\alpha,n}(x) e^{i\omega_{\alpha,n}T}.$$

Formula (6. 6) shows the role of the functions $(x, t) \mapsto \Psi_{\alpha,n}(x) e^{i\omega_{\alpha,n}t}$, which are solution of the homogeneous equation

$$(\Psi_{\alpha,n}(x) e^{i\omega_{\alpha,n}t})_t = \mathcal{A}(\Psi_{\alpha,n}(x) e^{i\omega_{\alpha,n}t}).$$

6.3. A sufficient condition to prove Proposition 3.2.

We derive from (6. 6) that $w^{(p)}(T) \in H_{(\alpha)}^3(0, 1)$ if and only if

$$\sum_{n=1}^{\infty} \lambda_{\alpha,n}^3 \left| \frac{1}{2i\omega_{\alpha,n}} \left(\gamma_n^{(p)}(T) e^{i\omega_{\alpha,n}T} - \gamma_{-n}^{(p)}(T) e^{-i\omega_{\alpha,n}T} \right) \right|^2 < \infty,$$

and that $w_t^{(p)}(T) \in H_{(\alpha)}^2(0, 1)$ if and only if

$$\sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 \left| \frac{1}{2} \left(\gamma_n^{(p)}(T) e^{i\omega_{\alpha,n}T} + \gamma_{-n}^{(p)}(T) e^{-i\omega_{\alpha,n}T} \right) \right|^2 < \infty.$$

Therefore,

$$(6. 7) \quad \sum_{n \in \mathbb{Z}} \lambda_{\alpha,|n|}^2 |\gamma_n^{(p)}(T)|^2 < \infty \implies (w^{(p)}(T), w_t^{(p)}(T)) \in H_{(\alpha)}^3(0, 1) \times D(A),$$

that is, the hidden regularity stated in Proposition 3.2. Hence, to prove Proposition 3.2, it is sufficient to prove that

$$(6. 8) \quad \sum_{n \in \mathbb{Z}} \lambda_{\alpha,|n|}^2 |\gamma_n^{(p)}(T)|^2 < \infty,$$

or, equivalently, using (6. 5) and (6. 3), that

$$(6. 9) \quad \sum_{n=1}^{\infty} \lambda_{\alpha,|n|}^2 \left| \int_0^T p(s) \langle \mu(\cdot) w^{(p)}(\cdot, s), \Phi_{\alpha,n} \rangle_{L^2(0,1)} e^{i\omega_{\alpha,n}s} ds \right|^2 < \infty.$$

This kind of result is related to the regularity in (x, t) of the function $(x, t) \mapsto p(t)\mu(x)w^{(p)}(x, t)$ (through its Fourier coefficients in x and its nonharmonic Fourier coefficients in t). The proof of (6. 9) will be a direct consequence of the following regularity results (see Lemmas 6.2 and 6.3).

6.4. Regularity results for proving Proposition 3.2.

We introduce the following closed subspace of $H_\alpha^2(0, 1)$

$$(6.10) \quad W_\alpha^{(2,0)} := \{w \in H_\alpha^2(0, 1), (x^\alpha w_x)(0) = 0\},$$

and we will prove the following regularity results:

Lemma 6.2. *Let $T > 0$, $p \in L^2(0, T)$, $g \in C^0([0, T], W_\alpha^{(2,0)}(0, 1))$. Consider the sequence $(S_n^{(p,g)})_{n \geq 1}$ defined by*

$$(6.11) \quad \forall n \geq 1, \quad S_n^{(p,g)} = \int_0^T p(s) \langle g(\cdot, s), \Phi_{\alpha,n} \rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}}s} ds.$$

Then, there exists a constant $C(\alpha, T) > 0$ independent of $p \in L^2(0, T)$ and $g \in C^0([0, T], W_\alpha^{(2,0)}(0, 1))$ such that

$$(6.12) \quad \left(\sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 |S_n^{(p,g)}|^2 \right)^{1/2} \leq C(\alpha, T) \|p\|_{L^2(0,T)} \|g\|_{C^0([0,T], W_\alpha^{(2,0)}(0,1))}.$$

Lemma 6.3. *If $\mu \in V_\alpha^2$ (defined in (3.10)) and $w \in C^0([0, T], D(A))$, then $\mu w \in C^0([0, T], W_\alpha^{(2,0)}(0, 1))$. Moreover, there exists $C(\alpha, T) > 0$, independent of $\mu \in V_\alpha^2$ and $w \in C^0([0, T], D(A))$, such that*

$$(6.13) \quad \|\mu w\|_{C^0([0,T], W_\alpha^{(2,0)}(0,1))} \leq C(\alpha, T) \|\mu\|_{V_\alpha^2} \|w\|_{C^0([0,T], D(A))}.$$

In the following section, we will prove Lemma 6.2. The proof of Lemma 6.3 is classical, see the Supplementary material or [9, Lemma 6.7].

Proof of Proposition 3.2 assuming Lemmas 6.2 and 6.3. By applying Lemma 6.3 with $w = w^{(p)}$, we obtain that $\mu w^{(p)} \in C^0([0, T], W_\alpha^{(2,0)}(0, 1))$. Then, from Lemma 6.2, with $g = \mu w^{(p)}$, we deduce that (6.9) holds true and then (6.7) shows that $(w^{(p)}(T), w_t^{(p)}(T)) \in H_{(\alpha)}^3(0, 1) \times D(A)$, which completes the proof of Proposition 3.2. \square

6.5. Proof of Lemma 6.2.

We proceed as in [6], and the properties of the space $W_\alpha^{(2,0)}(0, 1)$ will be crucial to overcome some new difficulties. (Note that $W_\alpha^{(2,0)}(0, 1) = H_\alpha^2(0, 1)$ for $\alpha \in [1, 2)$.)

First, using that $-\lambda_{\alpha,n} \Phi_{\alpha,n} = (x^\alpha \Phi'_{\alpha,n})'$, we derive that

$$(6.14) \quad -\lambda_{\alpha,n} S_n^{(p,g)} = \int_0^T p(s) \langle g(\cdot, s), (x^\alpha \Phi'_{\alpha,n})' \rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}}s} ds.$$

Then, integrating twice by parts, we have

$$\begin{aligned} \langle g(\cdot, s), (x^\alpha \Phi'_{\alpha,n})' \rangle_{L^2(0,1)} &= [g(x, s) x^\alpha \Phi'_{\alpha,n}(x)]_0^1 \\ &\quad - [x^\alpha g'(x, s) \Phi_{\alpha,n}(x)]_0^1 + \int_0^1 (x^\alpha g'(x, s))' \Phi_{\alpha,n}(x) dx. \end{aligned}$$

Using the above expression of the scalar product in (6.14), we get

$$(6.15) \quad -\lambda_{\alpha,n} S_n^{(p,g)} = S_n^{(1)} - S_n^{(2)} + S_n^{(3)},$$

with

$$(6.16) \quad \forall i \in \{1, 2, 3\}, \quad S_n^{(i)} = \int_0^T h_n^{(i)}(s) e^{i\sqrt{\lambda_{\alpha,n}}s} ds,$$

and associated functions

$$\begin{aligned}
(6.17) \quad h_n^{(1)}(s) &= p(s) [g(x, s) x^\alpha \Phi'_{\alpha, n}(x)]_{x=0}^{x=1}, \\
h_n^{(2)}(s) &= p(s) [x^\alpha g_x(x, s) \Phi_{\alpha, n}(x)]_{x=0}^{x=1}, \\
h_n^{(3)}(s) &= p(s) \langle (x^\alpha g_x)_x, \Phi_{\alpha, n} \rangle_{L^2(0,1)}.
\end{aligned}$$

6.5.a. *Study of $h_n^{(1)}$ and $(S_n^{(1)})_n$.*

We claim that

$$(6.18) \quad \forall n \geq 1, \quad h_n^{(1)} = 0 \text{ in } L^2(0, T),$$

which will imply that

$$(6.19) \quad \forall n \geq 1, \quad S_n^{(1)} = 0.$$

Indeed, since $g(\cdot, s) \in H_\alpha^2(0, 1)$, then $g(\cdot, s) \in H^1(\frac{1}{2}, 1)$ and has a finite limit as $x \rightarrow 1$. Therefore, thanks to the Neumann boundary condition $\Phi'_{\alpha, n}(1) = 0$, we deduce that $g(x, s) x^\alpha \Phi'_{\alpha, n}(x) \rightarrow 0$ as $x \rightarrow 1$. For $x \rightarrow 0$, we have to distinguish the cases of weak and strong degeneracy:

- $\alpha \in [0, 1)$: we observe that $g_x(x, s) = (x^{\alpha/2} g(x, s)) x^{-\alpha/2}$, which implies that $g_x(\cdot, s) \in L^1(0, 1)$. Therefore, $g(\cdot, s)$ has a finite limit as $x \rightarrow 0$. Furthermore, since $x^\alpha \Phi'_{\alpha, n}(x) \rightarrow 0$ as $x \rightarrow 0$, (6.18) holds true when $\alpha \in [0, 1)$,
- $\alpha \in [1, 2)$: from (3.3) we deduce that there exists $C_{\alpha, n} > 0$ such that $|x^\alpha \Phi'_{\alpha, n}(x)| \leq C_{\alpha, n} x$. Moreover, it is possible to check that $(xg(x, s))_x \in L^1(0, 1)$ and thus $x \mapsto xg(x, s)$ has a finite limit as $x \rightarrow 0$. Such a limit has to be equal to 0 since $g(\cdot, s) \in L^2(0, 1)$. Hence, (6.18) holds true.

We refer to [9, Lemma 6.4] for a more detailed proof.

6.5.b. *Study of $h_n^{(3)}$ and $(S_n^{(3)})_n$.*

Notice that

$$|S_n^{(3)}|^2 \leq \left(\int_0^T |h_n^{(3)}(s)| ds \right)^2 \leq \|p\|_{L^2(0, T)}^2 \left(\int_0^T |\langle (x^\alpha g_x)_x, \Phi_{\alpha, n} \rangle_{L^2(0,1)}|^2 ds \right),$$

hence, applying Parseval's equality, we obtain that

$$\begin{aligned}
\sum_{n=1}^{\infty} |S_n^{(3)}|^2 &\leq \|p\|_{L^2(0, T)}^2 \left(\int_0^T \sum_{n=1}^{\infty} |\langle (x^\alpha g_x)_x, \Phi_{\alpha, n} \rangle_{L^2(0,1)}|^2 ds \right) \\
&\leq \|p\|_{L^2(0, T)}^2 \left(\int_0^T \|(x^\alpha g_x)_x\|_{L^2(0,1)}^2 ds \right) \leq \|p\|_{L^2(0, T)}^2 T \|g\|_{C^0([0, T], W_\alpha^{(2,0)}(0,1))}^2,
\end{aligned}$$

which will be useful to prove (6.12).

6.5.c. *Study of $h_n^{(2)}$ and $(S_n^{(2)})_n$.*

From the definition (6.17) of $h_n^{(2)}$, using the fact that $x^\alpha g_x(x, s) \rightarrow 0$ as $x \rightarrow 0$ ($g(\cdot, s) \in W_\alpha^{(2,0)}(0, 1)$) and that $\Phi_{\alpha, n}$ has finite limits in 0 and 1 (Lemma 5.1), we obtain that

$$h_n^{(2)}(s) = \pm \sqrt{2 - \alpha} g_x(1, s) p(s),$$

where the sign \pm is determined by the sign of $\Phi'_{\alpha, n}(1)$. Therefore,

$$\sum_{n=1}^{\infty} |S_n^{(2)}|^2 = \sum_{n=1}^{\infty} \left| \int_0^T h_n^{(2)}(s) e^{i\sqrt{\lambda_{\alpha, n} s}} ds \right|^2 = (2 - \alpha) \sum_{n=1}^{\infty} \left| \int_0^T g_x(1, s) p(s) e^{i\sqrt{\lambda_{\alpha, n} s}} ds \right|^2,$$

Observe that Proposition 3.1 implies that

$$\forall \alpha \in [0, 2), \forall n \geq 1, \quad \sqrt{\lambda_{\alpha, n+1}} - \sqrt{\lambda_{\alpha, n}} \geq \frac{2-\alpha}{2}\pi,$$

and so we can apply a general result of Ingham (see, e.g., [29, Theorem 4.3], generalized by Haraux [23], see also [7, Theorem 6]), and we deduce that given

$$T_1 > \frac{2\pi}{\frac{2-\alpha}{2}\pi} = \frac{4}{2-\alpha} = T_\alpha,$$

there exist $C_1(\alpha, T_1), C_2(\alpha, T_1) > 0$ such that, for every sequence $(c_n)_{n \geq 1}$ with finite support and complex values, it holds that

$$(6.20) \quad C_1 \sum_{n=1}^{\infty} |c_n|^2 \leq \int_0^{T_1} \left| \sum_{n=1}^{\infty} c_n e^{i\sqrt{\lambda_{\alpha, n}t}} \right|^2 dt \leq C_2 \sum_{n=1}^{\infty} |c_n|^2.$$

If $T_1 > T_\alpha$, (6.20) implies that the sequence $(e^{i\sqrt{\lambda_{\alpha, n}t}})_{n \geq 1}$ is a Riesz basis of $\text{Vect} \{e^{i\sqrt{\lambda_{\alpha, n}t}}, n \geq 1\} \subset L^2(0, T_1)$ (see [7, Proposition 19, point (2)]). Thus, we infer that for all $T > 0$, there exists a positive constant $C_I(\alpha, T)$ such that

$$(6.21) \quad \forall f \in L^2(0, T), \quad \sum_{n=1}^{\infty} \left| \int_0^T f(t) e^{i\sqrt{\lambda_{\alpha, n}t}} dt \right|^2 \leq C_I(\alpha, T) \|f\|_{L^2(0, T)}^2.$$

Indeed, if $T > T_\alpha$ then (6.21) comes directly by applying [7, Proposition 19, point (3)]. Otherwise, if $T \leq T_\alpha$, it is sufficient to extend f by 0 on (T, T_α) , see [7, Corollary 4].

Finally, from (6.21) with $f(t) = g_x(1, t)p(t)$, we obtain that

$$\sum_{n=1}^{\infty} |S_n^{(2)}|^2 \leq (2-\alpha)C_I(\alpha, T) \|g_x(1, \cdot)p\|_{L^2(0, T)}^2.$$

By the continuous injection of $H^1(0, 1)$ into $C^0([0, 1])$, there exists a positive constant C_∞ such that

$$|g_x(1, s)| \leq C_\infty \|g(\cdot, s)\|_{H_\alpha^2(0, 1)} \leq C_\infty \|g\|_{C^0([0, T], W_\alpha^{(2, 0)}(0, 1))},$$

which gives

$$\sum_{n=1}^{\infty} |S_n^{(2)}|^2 \leq C' \|g_x(1, \cdot)p\|_{L^2(0, T)}^2 \|g\|_{C^0([0, T], W_\alpha^{(2, 0)}(0, 1))}^2.$$

Therefore, the above results on $S_n^{(1)}, S_n^{(2)}$ and $S_n^{(3)}$ yield (6.12) and conclude the proof of Lemma 6.2. \square

6.6. Complement to Proposition 3.2.

Thanks to Proposition 3.2, if $\mu \in V_\alpha^2$, one can consider the application

$$(6.22) \quad \Theta_T : L^2(0, T) \rightarrow H_{(\alpha)}^3 \times D(A), \quad \Theta_T(p) := (w^{(p)}(T), w_t^{(p)}(T)).$$

We prove the following

Lemma 6.4. *The map Θ_T (defined in (6.22)) is of class C^1 , and, fixed $p \in L^2(0, T)$, the map $D\Theta_T(p) : L^2(0, T) \rightarrow H_{(\alpha)}^3(0, 1) \times D(A)$ is given by*

$$D\Theta_T(p) \cdot q = (W^{(p, q)}(T), W_t^{(p, q)}(T)),$$

where $W^{(p, q)}$ is the solution of

$$(6.23) \quad \begin{cases} W_{tt}^{(p, q)} - (x^\alpha W_x^{(p, q)})_x = p(t)\mu(x)W^{(p, q)} + q(t)\mu(x)w^{(p)}, & x \in (0, 1), t \in (0, T), \\ (x^\alpha W_x^{(p, q)})(x=0, t) = 0, \quad W_x^{(p, q)}(x=1, t) = 0, & t \in (0, T), \\ W^{(p, q)}(x, 0) = 0, \quad W_t^{(p, q)}(x, 0) = 0, & x \in (0, 1). \end{cases}$$

Proof of Lemma 6.4. We start by proving that Θ_T is differentiable at every $p \in L^2(0, T)$. Let $p_0, q \in L^2(0, T)$. Consider $w^{(p_0)}$ solution of (3. 9) with $p = p_0$, and $w^{(p_0+q)}$ solution of (3. 9) with $p = p_0 + q$. Formally, let us write a limited development of $w^{(p_0+q)}$ with respect to q :

$$w^{(p_0+q)} = w^{(p_0)} + W_1(q) + \dots .$$

We use this development in (3. 9) to find the equation satisfied by the supposed first order term $W_1(q)$: denoting

$$Pw := w_{tt} - (x^\alpha w_x)_x,$$

we have

$$P(w^{(p_0)} + W_1(q) + \dots) = (p_0(t) + q(t)) \mu(x) (w^{(p_0)} + W_1(q) + \dots),$$

hence we deduce that $W_1(q)$ is solution of

$$PW_1(q) = p_0(t)\mu(x)W_1(q) + q(t)\mu(x)w^{(p_0)},$$

which is the motivation to consider $W_1(q)$ as the solution of (6. 23) with $p = p_0$, that is, $W_1(q) = W^{(p_0, q)}$. So, we introduce

$$(6. 24) \quad v^{(p_0, q)} := w^{(p_0+q)} - w^{(p_0)} - W^{(p_0, q)},$$

and we have that

$$\Theta_T(p_0 + q) = \Theta_T(p_0) + (W^{(p_0, q)}(T), W_t^{(p_0, q)}(T)) + (v^{(p_0, q)}(T), v_t^{(p_0, q)}(T)).$$

By using Lemma 6.2, one can prove the following lemmas (see the Supplementary material or [9, Lemmas 6.8 and 6.9] for the proofs):

Lemma 6.5. *The application*

$$\begin{aligned} L^2(0, T) &\rightarrow H_{(\alpha)}^3(0, 1) \times D(A) \\ q &\mapsto (W^{(p_0, q)}(T), W_t^{(p_0, q)}(T)) \end{aligned}$$

is well-defined, linear and continuous.

Lemma 6.6. *The application*

$$\begin{aligned} L^2(0, T) &\rightarrow H_{(\alpha)}^3(0, 1) \times D(A) \\ q &\mapsto (v^{(p_0, q)}(T), v_t^{(p_0, q)}(T)) \end{aligned}$$

is well-defined, and satisfies

$$(6. 25) \quad \frac{\|(v^{(p_0, q)}(T), v_t^{(p_0, q)}(T))\|_{H_{(\alpha)}^3(0, 1) \times D(A)}}{\|q\|_{L^2(0, T)}} \rightarrow 0, \quad \text{as } \|q\|_{L^2(0, T)} \rightarrow 0.$$

Then, we conclude that Θ_T is differentiable at p_0 and that

$$D\Theta_T(p_0) \cdot q = (W^{(p_0, q)}(T), W_t^{(p_0, q)}(T)).$$

With the same argument, one can prove that Θ_T is of class C^1 , and this concludes the proof of Lemma 6.4. \square

7. REACHABILITY WHEN $T > T_\alpha$: PROOF OF THEOREM 3.1

The proof follows from the classical inverse mapping theorem applied to the function $\Theta_T : L^2(0, T) \rightarrow H_{(\alpha)}^3 \times D(A)$ at the point $p_0 = 0$. We recall that $\Theta_T(p_0 = 0) = (1, 0)$. In what follows we study $D\Theta_T(0)$.

7.1. Surjectivity of $D\Theta_T(0)$: study of the associated moment problem.

The key point of the proof of Theorem 3.1 is the following result:

Lemma 7.1. *Let $\mu \in V^{(adm)}$ (defined in (3. 13)) and $T > T_\alpha$. Then, the linear application*

$$\begin{aligned} D\Theta_T(0) : L^2(0, T) &\rightarrow H_{(\alpha)}^3 \times D(A) \\ q &\mapsto (W^{(0,q)}(T), W_t^{(0,q)}(T)) \end{aligned}$$

is surjective, and

$$D\Theta_T(0) : \overline{\text{Vect} \{1, t, \cos \sqrt{\lambda_{\alpha,n}}t, \sin \sqrt{\lambda_{\alpha,n}}t, n \geq 1\}} \rightarrow H_{(\alpha)}^3 \times D(A)$$

is invertible.

Proof of Lemma 7.1. Since $w^{(0)} = 1$, (6. 23) implies that $W^{(0,q)}$ is solution of the following linear problem

$$(7. 1) \quad \begin{cases} W_{tt}^{(0,q)} - (x^\alpha W_x^{(0,q)})_x = q(t)\mu(x), & x \in (0, 1), t \in (0, T), \\ (x^\alpha W_x^{(0,q)})(x=0, t) = 0, \quad W_x^{(0,q)}(x=1, t) = 0, & t \in (0, T), \\ W^{(0,q)}(x, 0) = 0, \quad W_t^{(0,q)}(x, 0) = 0, & x \in (0, 1). \end{cases}$$

Following the procedure of section 6.2, we introduce

$$(7. 2) \quad r_n(s) = \langle q(s)\mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)} = \mu_{\alpha,n}q(s) \quad \text{with} \quad \mu_{\alpha,n} = \langle \mu, \Phi_{\alpha,n} \rangle_{L^2(0,1)},$$

and we can express the solution of (7. 1) at time T , $(W^{(0,q)}(T), W_t^{(0,q)}(T))$ as follows

$$W^{(0,q)}(x, T) = \int_0^T r_0(s)(T-s) ds + \sum_{n=1}^{\infty} \left(\int_0^T r_n(s) \frac{\sin \sqrt{\lambda_{\alpha,n}}(T-s)}{\sqrt{\lambda_{\alpha,n}}} ds \right) \Phi_{\alpha,n}(x),$$

and

$$W_t^{(0,q)}(x, T) = \int_0^T r_0(s) ds + \sum_{n=1}^{\infty} \left(\sqrt{\lambda_{\alpha,n}} \int_0^T r_n(s) \frac{\cos \sqrt{\lambda_{\alpha,n}}(T-s)}{\sqrt{\lambda_{\alpha,n}}} ds \right) \Phi_{\alpha,n}(x).$$

To prove the surjectivity of $D\Theta_T(0)$, we choose any pair $(Y^f, Z^f) \in H_{(\alpha)}^3 \times D(A)$, and we have to show that there exists $q \in L^2(0, T)$ such that

$$(7. 3) \quad (W^{(0,q)}(T), W_t^{(0,q)}(T)) = (Y^f, Z^f).$$

Introducing the Fourier coefficients of the target state

$$Y_{\alpha,n}^f = \langle Y^f, \Phi_{\alpha,n} \rangle_{L^2(0,1)} \quad \text{and} \quad Z_{\alpha,n}^f = \langle Z^f, \Phi_{\alpha,n} \rangle_{L^2(0,1)},$$

we have that (7. 3) is satisfied if and only if

$$(7. 4) \quad \begin{cases} \int_0^T r_0(s) ds = Z_{\alpha,0}^f, \\ \sqrt{\lambda_{\alpha,n}} \int_0^T r_n(s) \frac{\cos \sqrt{\lambda_{\alpha,n}}(T-s)}{\sqrt{\lambda_{\alpha,n}}} ds = Z_{\alpha,n}^f, & \text{for all } n \geq 1, \\ \int_0^T r_n(s) \frac{\sin \sqrt{\lambda_{\alpha,n}}(T-s)}{\sqrt{\lambda_{\alpha,n}}} ds = Y_{\alpha,n}^f, & \text{for all } n \geq 1, \\ \int_0^T r_0(s)(T-s) ds = Y_{\alpha,0}^f. \end{cases}$$

We define the function

$$Q(s) := q(T-s),$$

and, using (7. 2), we rewrite problem (7. 4) as

$$(7. 5) \quad \begin{cases} \mu_{\alpha,0} \int_0^T Q(t) dt = Z_{\alpha,0}^f, \\ \mu_{\alpha,n} \int_0^T Q(t) \cos \sqrt{\lambda_{\alpha,n}} t dt = Z_{\alpha,n}^f, & \text{for all } n \geq 1, \\ \mu_{\alpha,n} \int_0^T Q(t) \sin \sqrt{\lambda_{\alpha,n}} t dt = \sqrt{\lambda_{\alpha,n}} Y_{\alpha,n}^f, & \text{for all } n \geq 1, \\ \mu_{\alpha,0} \int_0^T Q(t) t dt = Y_{\alpha,0}^f. \end{cases}$$

System (7. 5) is usually called *moment problem*. Observe that (7. 5) can only be solved in full generality if $\mu_{\alpha,n} \neq 0$ for all $n \geq 0$, which is contained in assumption (3. 13).

Let us introduce the coefficients

$$(7. 6) \quad \begin{cases} A_{\alpha,0}^f := \frac{Z_{\alpha,0}^f}{\mu_{\alpha,0}}, \\ B_{\alpha,0}^f := \frac{Y_{\alpha,0}^f}{\mu_{\alpha,0}}, \end{cases} \quad \text{and} \quad \begin{cases} A_{\alpha,n}^f := \frac{Z_{\alpha,n}^f}{\mu_{\alpha,n}}, \\ B_{\alpha,n}^f := \frac{\sqrt{\lambda_{\alpha,n}} Y_{\alpha,n}^f}{\mu_{\alpha,n}}, \end{cases} \quad \forall n \geq 1,$$

and the functions defined on $(0, T)$:

$$(7. 7) \quad \begin{cases} c_{\alpha,0} : t \mapsto 1, \\ \tilde{s}_{\alpha,0} : t \mapsto t, \end{cases} \quad \text{and} \quad \begin{cases} c_{\alpha,n} : t \mapsto \cos \sqrt{\lambda_{\alpha,n}} t \\ s_{\alpha,n} : t \mapsto \sin \sqrt{\lambda_{\alpha,n}} t \end{cases} \quad \forall n \geq 1.$$

So, (7. 5) becomes

$$(7. 8) \quad \begin{cases} \langle Q, c_{\alpha,0} \rangle_{L^2(0,T)} = A_{\alpha,0}^f, \\ \langle Q, c_{\alpha,n} \rangle_{L^2(0,T)} = A_{\alpha,n}^f & \text{for all } n \geq 1, \\ \langle Q, s_{\alpha,n} \rangle_{L^2(0,T)} = B_{\alpha,n}^f & \text{for all } n \geq 1, \\ \langle Q, \tilde{s}_{\alpha,0} \rangle_{L^2(0,T)} = B_{\alpha,0}^f. \end{cases}$$

We are going to prove that system (7. 8) has (at least) a solution Q in two steps:

- we prove that the reduced system

$$(7. 9) \quad \begin{cases} \langle Q, c_{\alpha,0} \rangle_{L^2(0,T)} = A_{\alpha,0}^f, \\ \langle Q, c_{\alpha,n} \rangle_{L^2(0,T)} = A_{\alpha,n}^f & \text{for all } n \geq 1, \\ \langle Q, s_{\alpha,n} \rangle_{L^2(0,T)} = B_{\alpha,n}^f & \text{for all } n \geq 1 \end{cases}$$

has at least a solution Q_α ,

- using Q_α , we construct a solution Q of the full system (7. 8) (by using a suitable orthogonal projection).

We use results stated in [7, Appendix] (see also [15]).

Step 1: Existence of a solution of the reduced system (7. 9). We consider the space

$$E_\alpha := \overline{\text{Vect} \{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}},$$

which is a closed subspace of $L^2(0, T)$. To solve the reduced system, we use the following characterization of Riesz Basis (see ([7, Prop. 19]): the family $\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}$ is a Riesz basis of E_α if and only if there exist $C_1(\alpha, T), C_2(\alpha, T) > 0$ such that, for all $N \geq 1$ and for any $(a_n)_{0 \leq n \leq N}, (b_n)_{1 \leq n \leq N}$ it holds that

$$(7. 10) \quad C_1 \left(a_0^2 + \sum_{n=1}^N a_n^2 + b_n^2 \right) \leq \int_0^T \left| S_N^{(a,b)}(t) \right|^2 dt \leq C_2 \left(a_0^2 + \sum_{n=1}^N a_n^2 + b_n^2 \right),$$

where

$$(7. 11) \quad S_N^{(a,b)}(t) = a_0 c_{\alpha,0}(t) + \sum_{n=1}^N a_n c_{\alpha,n}(t) + b_n s_{\alpha,n}(t).$$

Now, recalling the definition (6. 1) of $(\omega_{\alpha,n})_{n \in \mathbb{Z}}$, since $\omega_{\alpha,n+1} - \omega_{\alpha,n} > 0$ for all $n \in \mathbb{Z}$ and $\omega_{\alpha,n+1} - \omega_{\alpha,n} \geq \frac{2-\alpha}{2} \pi$ for all $|n| \geq 2$, we can apply a general result of Haraux

[23] (see also [7, Theorem 6]) that ensures that if $T > \frac{2\pi}{2-\alpha} = \frac{4}{2-\alpha} = T_\alpha$, then there exist $C_1^{(T)}, C_2^{(T)} > 0$ independent of N and of the coefficients $(d_n)_{-N \leq n \leq N}$ such that

$$(7.12) \quad C_1^{(T)} \sum_{n=-N}^N |d_n|^2 \leq \int_0^T \left| \sum_{n=-N}^N d_n e^{i\omega_{\alpha,n}t} \right|^2 dt \leq C_2^{(T)} \sum_{n=-N}^N |d_n|^2.$$

For any real valued sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 1}$, we can always define the complex valued sequence $(d_n)_{n \in \mathbb{Z}}$ defined by

$$d_n = \frac{a_{-n}}{2} - \frac{b_{-n}}{2i}, \quad \text{for } n \leq -1, \quad d_0 = a_0, \quad d_n = \frac{a_n}{2} + \frac{b_n}{2i} \quad \text{for } n \geq 1.$$

Then we have that

$$S_N^{(a,b)}(t) := a_0 c_{\alpha,0}(t) + \sum_{n=1}^N a_n \cos \omega_{\alpha,n}(t) + b_n \sin \omega_{\alpha,n}(t) = \sum_{n=-N}^N d_n e^{i\omega_{\alpha,n}t},$$

and

$$\sum_{n=-N}^N |d_n|^2 = a_0^2 + 2 \sum_{n=1}^N \frac{a_n^2 + b_n^2}{4}.$$

Therefore we get that (7.12) implies that (7.10) holds true, and so the family $\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}$ is a Riesz basis of E_α . Thus, the application $\mathcal{F} : E_\alpha \rightarrow \ell^2(\mathbb{N})$:

$$\mathcal{F}(f) = (\langle f, c_{\alpha,0} \rangle_{L^2(0,T)}, \langle f, c_{\alpha,1} \rangle_{L^2(0,T)}, \langle f, s_{\alpha,1} \rangle_{L^2(0,T)}, \langle f, c_{\alpha,2} \rangle_{L^2(0,T)}, \dots)$$

is an isomorphism (see, e.g., [7, Proposition 20]). Finally, we note that (3.13) ensures us that if $Y^f \in H_{(\alpha)}^3(0,1)$ and $Z^f \in D(A)$, then the sequences $(A_{\alpha,n}^f)_n$ and $(B_{\alpha,n}^f)_n$ belong to $\ell^2(\mathbb{N})$. Therefore, there exists a unique $Q_\alpha \in E_\alpha$ such that

$$\mathcal{F}(Q_\alpha) = (A_{\alpha,0}^f, A_{\alpha,1}^f, B_{\alpha,1}^f, A_{\alpha,2}^f, \dots),$$

and the application

$$\ell^2(\mathbb{N}) \rightarrow E_\alpha, \quad (A_{\alpha,0}^f, A_{\alpha,1}^f, B_{\alpha,1}^f, A_{\alpha,2}^f, \dots) \mapsto Q_\alpha$$

is continuous. This proves the continuous solvability of the reduced system (7.9).

Step 2: Existence of a solution of the full system (7.8). We claim that $\tilde{s}_{\alpha,0} \notin E_\alpha$: indeed, if $t \mapsto t$ was the limit of a sequence of linear combinations of $c_{\alpha,0}, c_{\alpha,n}$ and $s_{\alpha,n}$, the same would be true for the function $t \mapsto t^2$, by integration. Then, by iterating this procedure, we deduce that all the polynomials could be written in this form. Therefore, $L^2(0,T)$ would be equal to E_α and (7.9) would have a unique solution. However, this is not the case. Indeed, suppose that $E_\alpha = L^2(0,T)$ and consider the sequences

$$\begin{aligned} \forall n \geq 0, \quad \tilde{A}_{\alpha,n}^f &= \langle 1, c_{\alpha,n} \rangle_{L^2(0,T)} = \int_0^T c_{\alpha,n}(s) ds, \\ \forall n \geq 1, \quad \tilde{B}_{\alpha,n}^f &= \langle 1, s_{\alpha,n} \rangle_{L^2(0,T)} = \int_0^T s_{\alpha,n}(s) ds. \end{aligned}$$

By integration, we obtain that $\sum_{n \geq 1} |\tilde{A}_{\alpha,n}^f|^2 + |\tilde{B}_{\alpha,n}^f|^2 < \infty$ (thanks to (3.3) and (8.4)). Let $T' \in (T_\alpha, T)$. We can choose, for example, $T' = \frac{T+T_\alpha}{2}$. From *Step 1*, the following problem

$$(7.13) \quad \begin{cases} \langle \tilde{Q}_\alpha, c_{\alpha,0} \rangle_{L^2(0,T')} = \tilde{A}_{\alpha,0}^f, \\ \langle \tilde{Q}_\alpha, c_{\alpha,n} \rangle_{L^2(0,T')} = \tilde{A}_{\alpha,n}^f & \text{for all } n \geq 1, \\ \langle \tilde{Q}_\alpha, s_{\alpha,n} \rangle_{L^2(0,T')} = \tilde{B}_{\alpha,n}^f & \text{for all } n \geq 1, \end{cases}$$

has at least a solution $\tilde{Q}_\alpha \in L^2(0, T')$. We extend function \tilde{Q}_α to 0 on (T', T) . Then, such extended function satisfies

$$(7.14) \quad \begin{cases} \langle \tilde{Q}_\alpha, c_{\alpha,0} \rangle_{L^2(0,T)} = \tilde{A}_{\alpha,0}^f, \\ \langle \tilde{Q}_\alpha, c_{\alpha,n} \rangle_{L^2(0,T)} = \tilde{A}_{\alpha,n}^f & \text{for all } n \geq 1, \\ \langle \tilde{Q}_\alpha, s_{\alpha,n} \rangle_{L^2(0,T)} = \tilde{B}_{\alpha,n}^f & \text{for all } n \geq 1. \end{cases}$$

Hence, $\tilde{Q}_\alpha - 1$ is orthogonal in $L^2(0, T)$ to the family $\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}$, that generates E_α , and therefore $L^2(0, T)$. This would imply $\tilde{Q}_\alpha - 1$ equal to 0, which is not true on (T', T) . Thus, $\tilde{s}_{\alpha,0} \notin E_\alpha$.

If we denote $p_{\alpha,0}^\perp$ the orthogonal projection of $\tilde{s}_{\alpha,0}$ on E_α , then $\tilde{s}_{\alpha,0} - p_{\alpha,0}^\perp \neq 0$, and

$$Q_\alpha^\perp := \frac{\tilde{s}_{\alpha,0} - p_{\alpha,0}^\perp}{\|\tilde{s}_{\alpha,0} - p_{\alpha,0}^\perp\|_{L^2(0,T)}}^2$$

is orthogonal to E_α , and furthermore

$$\langle Q_\alpha^\perp, \tilde{s}_{\alpha,0} \rangle_{L^2(0,T)} = 1.$$

Thus,

$$Q := Q_\alpha + (B_{\alpha,0}^f - \langle Q_\alpha, \tilde{s}_{\alpha,0} \rangle) Q_\alpha^\perp$$

solves (7.8). Moreover,

$$\begin{aligned} \|Q\|_{L^2(0,T)}^2 &= \|Q_\alpha\|_{L^2(0,T)}^2 + \|(B_{\alpha,0} - \langle Q_\alpha, \tilde{s}_{\alpha,0} \rangle) Q_\alpha^\perp\|_{L^2(0,T)}^2 \\ &\leq C \sum_{n=0}^{\infty} |A_{\alpha,n}^f|^2 + |B_{\alpha,n}^f|^2 = C \sum_{n=0}^{\infty} \frac{1}{\mu_{\alpha,n}^2} |Z_{\alpha,n}^f|^2 + \frac{\lambda_{\alpha,n}}{\mu_{\alpha,n}^2} |Y_{\alpha,n}^f|^2 \\ &\leq C' \sum_{n=0}^{\infty} |\lambda_{\alpha,n}^*|^2 |Z_{\alpha,n}^f|^2 + |\lambda_{\alpha,n}^*|^3 |Y_{\alpha,n}^f|^2 = C' \|(Y^f, Z^f)\|_{H_{(\alpha)}^3 \times D(A)}^2, \end{aligned}$$

where we used the assumption on μ . This completes the proof of Lemma 7.1. \square

7.2. Proof of Theorem 3.1 (inverse mapping argument).

We define the space

$$F_\alpha := \overline{\text{Vect} \{1, t, \cos \sqrt{\lambda_{\alpha,n}} t, \sin \sqrt{\lambda_{\alpha,n}} t, n \geq 1\}}.$$

Then, the restriction of Θ_T to F_α

$$\begin{aligned} \Theta_{\alpha,T} : F_\alpha &\rightarrow H_{(\alpha)}^3(0,1) \times D(A), \\ p &\mapsto \Theta_{\alpha,T}(p) := \Theta_T(p) \end{aligned}$$

is C^1 (Lemma 6.3) and $D\Theta_{\alpha,T}(0)$ is invertible (Lemma 7.1). Thus, the inverse mapping theorem ensures the existence of a neighborhood $\mathcal{V}(0) \subset F_\alpha$ and a neighborhood $\mathcal{V}(1,0) \subset H_{(\alpha)}^3(0,1) \times D(A)$ such that

$$\Theta_{\alpha,T} : \mathcal{V}(0) \rightarrow \mathcal{V}(1,0)$$

is a C^1 -diffeomorphism. Hence, given $(w_0^f, w_1^f) \in \mathcal{V}(1,0)$, we set $p^f := \Theta_{\alpha,T}^{-1}(w_0^f, w_1^f)$, and so the solution of (3.9) with $p = p^f$ satisfies

$$(w(T), w_t(T)) = \Theta_T(p^f) = \Theta_T(\Theta_{\alpha,T}^{-1}(w_0^f, w_1^f)) = (w_0^f, w_1^f).$$

This concludes the proof of Theorem 3.1. \square

8. REACHABILITY FOR $T = T_\alpha$: PROOF OF THEOREM 3.28.1. **Proof of Theorem 3.2 first part: Reachability when $\alpha \in [0, 1)$.**

The proof combines classical arguments concerning families of exponentials ([4]) in the space $L^2(0, T)$ and the strategy of Beauchard [6]:

- we study the solvability of the moment problem (7. 4) (or equivalently (7. 5)),
- we conclude using the inverse mapping theorem.

8.1.a. *Main tools to study the solvability of the moment problem* (7. 5).

We introduce the notation

$$(8. 1) \quad C_{\alpha,0}^f := \frac{Z_{\alpha,0}^f}{\mu_{\alpha,0}}, \quad \text{and} \quad \begin{cases} C_{\alpha,n}^f := \frac{Z_{\alpha,n}^f - i\sqrt{\lambda_{\alpha,n}} Y_{\alpha,n}^f}{\mu_{\alpha,n}}, \\ C_{\alpha,-n}^f := \frac{Z_{\alpha,n}^f + i\sqrt{\lambda_{\alpha,n}} Y_{\alpha,n}^f}{\mu_{\alpha,n}}, \end{cases} \quad \forall n \geq 1.$$

Then, from the definition of the natural scalar product in $L^2(0, T; \mathbb{C})$, (7. 5) becomes

$$(8. 2) \quad \begin{cases} \langle Q, e^{i\omega_{\alpha,0}t} \rangle_{L^2(0,T;\mathbb{C})} = C_{\alpha,0}^f, \\ \langle Q, e^{i\omega_{\alpha,n}t} \rangle_{L^2(0,T;\mathbb{C})} = C_{\alpha,n}^f \quad \text{for all } n \geq 1, \\ \langle Q, e^{i\omega_{\alpha,n}t} \rangle_{L^2(0,T;\mathbb{C})} = C_{\alpha,n}^f \quad \text{for all } n \leq -1, \\ \langle Q, \tilde{s}_{\alpha,0} \rangle_{L^2(0,T;\mathbb{C})} = B_{\alpha,0}^f, \end{cases}$$

(where $\tilde{s}_{\alpha,0}$ and $B_{\alpha,0}^f$ have been defined in (7. 7) and (7. 6), respectively). We are going to study first the solvability of the subsystem composed by the first three equations, that is, the following moment problem

$$(8. 3) \quad \forall n \in \mathbb{Z}, \quad \langle Q, e^{i\omega_{\alpha,n}t} \rangle_{L^2(0,T;\mathbb{C})} = C_{\alpha,n}^f.$$

8.1.b. *Main solvability results.*

Lemma 8.1. *Let $\alpha \in [0, 1)$. Then, the sequence $(e^{i\omega_{\alpha,n}t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T_\alpha)$.*

Lemma 8.2. *Let $\alpha \in [0, 1)$ and $T = T_\alpha$. Then, the moment problem (8. 3) has a unique solution $Q \in L^2(0, T_\alpha; \mathbb{R})$.*

Lemma 8.3. *Let $\alpha \in [0, 1)$ and $T = T_\alpha$. Then, there exists a closed hyperplane of $H_{(\alpha)}^3 \times D(A)$, denoted by P_α^f and defined in (8. 8), such that the moment problem (8. 2) has a solution if and only if $(Y^f, Z^f) \in P_\alpha^f$.*

8.1.c. *Proofs of Lemmas 8.1, 8.2, 8.3.*

Proof of Lemma 8.1. The proof follows from the Kadec's $\frac{1}{4}$ Theorem ([26], [33, Theorem 1.14 p. 42]). First, we note that the sequence $(\omega_{\alpha,n})_{n \in \mathbb{Z}}$ is odd, that is $\omega_{\alpha,-n} = -\omega_{\alpha,n}$, and

$$\forall n \geq 1, \quad \omega_{\alpha,n} = \kappa_\alpha j_{\nu_\alpha+1,n} = \frac{2-\alpha}{2} j_{\frac{1}{2-\alpha},n}.$$

Mac Mahon's formula (Watson [32, p. 506]) provides the following asymptotic development of $j_{\nu,n}$ as $n \rightarrow \infty$:

$$(8. 4) \quad j_{\nu,n} = \pi\left(n + \frac{\nu}{2} - \frac{1}{4}\right) + O\left(\frac{1}{n}\right),$$

from which we deduce that

$$(8. 5) \quad \frac{2}{\pi(2-\alpha)} \omega_{\alpha,n} = n + \frac{\alpha}{4(2-\alpha)} + O\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow +\infty,$$

and, in particular,

$$\frac{2}{\pi(2-\alpha)} \omega_{\alpha,n} - n \rightarrow \frac{\alpha}{4(2-\alpha)} \quad \text{as } n \rightarrow +\infty.$$

Since $\alpha \in [0, 1)$, it holds that

$$\frac{\alpha}{4(2-\alpha)} = \frac{1}{4} \frac{\alpha}{2-\alpha} < \frac{1}{4},$$

thus, we deduce that for any $L \in (\frac{1}{4} \frac{\alpha}{2-\alpha}, \frac{1}{4})$ there exists N_0 such that

$$\forall n \geq N_0, \quad \left| \frac{2}{\pi(2-\alpha)} \omega_{\alpha,n} - n \right| \leq L < \frac{1}{4}.$$

Since the sequence $(\frac{2}{\pi(2-\alpha)} \omega_{\alpha,n})_{n \in \mathbb{Z}}$ is odd, the above bound holds also for negative indices, hence

$$\forall |n| \geq N_0, \quad \left| \frac{2}{\pi(2-\alpha)} \omega_{\alpha,n} - n \right| \leq L < \frac{1}{4}.$$

We now consider the sequence $(\tilde{\omega}_{\alpha,n})_{n \in \mathbb{Z}}$ defined by

$$\tilde{\omega}_{\alpha,n} := \begin{cases} n, & \forall |n| < N_0, \\ \frac{2}{\pi(2-\alpha)} \omega_{\alpha,n}, & \forall |n| \geq N_0, \end{cases}$$

and we observe that

$$\forall n \in \mathbb{Z}, \quad \left| \tilde{\omega}_{\alpha,n} - n \right| \leq L < \frac{1}{4}.$$

Hence, we can apply Kadec's $\frac{1}{4}$ Theorem which implies that $(e^{i\tilde{\omega}_{\alpha,n}\tau})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(-\pi, \pi; \mathbb{C})$ (where τ is the variable in $(-\pi, \pi)$), see [33, Theorem 1.14 p. 42]. Thanks to the change of variables

$$[-\pi, \pi] \rightarrow [0, T_\alpha], \quad \tau \mapsto t = \frac{\tau}{\pi} \frac{T_\alpha}{2} + \frac{T_\alpha}{2},$$

we get that $(e^{i\tilde{\omega}_{\alpha,n} \frac{2\pi}{T_\alpha} t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T_\alpha; \mathbb{C})$. To conclude, we note that

$$\forall |n| \geq N_0, \quad \tilde{\omega}_{\alpha,n} \frac{2\pi}{T_\alpha} = \tilde{\omega}_{\alpha,n} \frac{2\pi}{\frac{4}{2-\alpha}} = \frac{2}{\pi(2-\alpha)} \omega_{\alpha,n} \frac{\pi(2-\alpha)}{2} = \omega_{\alpha,n},$$

and since modifying a finite number of terms does not affect the fact of being a Riesz basis ([4, Lemma II.4.11 p. 105]), we deduce that $(e^{i\omega_{\alpha,n}t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T_\alpha; \mathbb{C})$. \square

Proof of Lemma 8.2. Since $(e^{i\omega_{\alpha,n}t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T_\alpha; \mathbb{C})$, there exists a unique biorthogonal sequence $(\sigma_m(t))_{m \in \mathbb{Z}}$ satisfying

$$\langle \sigma_m, e^{i\omega_{\alpha,n}t} \rangle_{L^2(0, T_\alpha; \mathbb{C})} = \int_0^{T_\alpha} \sigma_m(t) e^{-i\omega_{\alpha,n}t} dt = \delta_{mn}.$$

Taking the conjugate, we obtain that

$$\int_0^{T_\alpha} \overline{\sigma_m(t)} e^{i\omega_{\alpha,n}t} dt = \delta_{mn};$$

Recalling that $\omega_{\alpha,n} = -\omega_{\alpha,-n}$, we have

$$\langle \overline{\sigma_m}, e^{i\omega_{\alpha,-n}t} \rangle_{L^2(0, T_\alpha; \mathbb{C})} = \delta_{mn} = \langle \sigma_{-m}, e^{i\omega_{\alpha,-n}t} \rangle_{L^2(0, T_\alpha; \mathbb{C})},$$

which implies that for every $m \in \mathbb{Z}$ it holds that $\overline{\sigma_m} = \sigma_{-m}$. Now, using that $(e^{i\omega_{\alpha,n}t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T_\alpha; \mathbb{C})$, the moment problem (8. 3) has a unique solution, given by

$$(8. 6) \quad Q(t) = \sum_{m \in \mathbb{Z}} C_{\alpha,m}^f \sigma_m(t).$$

It remains to verify that Q is real valued: taking the conjugate, we have

$$\overline{Q(t)} = \sum_{m \in \mathbb{Z}} \overline{C_{\alpha, m}^f \sigma_m(t)} = \sum_{m \in \mathbb{Z}} C_{\alpha, -m}^f \sigma_{-m}(t) = Q(t).$$

Hence $Q \in L^2(0, T_\alpha; \mathbb{R})$. \square

Proof of Lemma 8.3. We have proved in Lemma 8.2 that the subsystem (8. 3) admits a unique solution $Q \in L^2(0, T_\alpha; \mathbb{R})$, given by (8. 6). Therefore, the moment problem (8. 2) is satisfied if and only if the solution Q satisfies also the last equation in (8. 2). Since $(e^{i\omega_{\alpha, n} t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T_\alpha; \mathbb{C})$, there exists a unique sequence $(\beta_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\tilde{s}_{\alpha, 0}(t) = \sum_{m \in \mathbb{Z}} \beta_m e^{i\omega_{\alpha, m} t} \quad \text{in } L^2(0, T_\alpha; \mathbb{C}).$$

Since $\tilde{s}_{\alpha, 0}$ is real-valued, we have that

$$\sum_{m \in \mathbb{Z}} \beta_m e^{i\omega_{\alpha, m} t} = \tilde{s}_{\alpha, 0}(t) = \overline{\tilde{s}_{\alpha, 0}(t)} = \sum_{m \in \mathbb{Z}} \overline{\beta_m} e^{-i\omega_{\alpha, m} t} = \sum_{m \in \mathbb{Z}} \overline{\beta_{-m}} e^{i\omega_{\alpha, m} t},$$

from which we deduce that for every $m \in \mathbb{Z}$ it holds that $\overline{\beta_m} = \beta_{-m}$. Thus,

$$\begin{aligned} \langle Q, \tilde{s}_{\alpha, 0} \rangle_{L^2(0, T_\alpha; \mathbb{C})} &= \langle Q, \sum_{m \in \mathbb{Z}} \beta_m e^{i\omega_{\alpha, m} t} \rangle_{L^2(0, T_\alpha; \mathbb{C})} \\ &= \sum_{m \in \mathbb{Z}} \overline{\beta_m} \langle Q, e^{i\omega_{\alpha, m} t} \rangle_{L^2(0, T_\alpha; \mathbb{C})} = \sum_{m \in \mathbb{Z}} \overline{\beta_m} C_{\alpha, m}^f. \end{aligned}$$

Hence, the solution Q given by (8. 6) solves (8. 2) if and only if

$$B_{\alpha, 0}^f = \sum_{m \in \mathbb{Z}} \overline{\beta_m} C_{\alpha, m}^f,$$

or, equivalently, if and only if

$$(8. 7) \quad \frac{Y_{\alpha, 0}^f}{\mu_{\alpha, 0}} = \beta_0 \frac{Z_{\alpha, 0}^f}{\mu_{\alpha, 0}} + \sum_{m \geq 1} \left(2(\operatorname{Re} \beta_m) \frac{Z_{\alpha, m}^f}{\mu_{\alpha, m}} - 2(\operatorname{Im} \beta_m) \omega_{\alpha, m} \frac{Y_{\alpha, m}^f}{\mu_{\alpha, m}} \right).$$

where the relation of $B_{\alpha, 0}^f$ and $C_{\alpha, m}^f$ with (Y^f, Z^f) are given in (7. 6) and (8. 1). We now introduce the closed hyperplane P_α^f of $H_{(\alpha)}^3 \times D(A)$ defined by

$$(8. 8) \quad P_\alpha^f := \{(Y^f, Z^f) \in H_{(\alpha)}^3 \times D(A) \text{ such that (8. 7) is satisfied}\}.$$

Therefore, (8. 2) has a solution if and only if $(Y^f, Z^f) \in P_\alpha^f$. \square

8.1.d. *Proof of Theorem 3.2 first part (inverse mapping argument).*

As in section 7.2, we consider the application

$$(8. 9) \quad \Theta_{T_\alpha} : L^2(0, T_\alpha) \rightarrow H_{(\alpha)}^3 \times D(A), \quad \Theta_{T_\alpha}(p) := (w^{(p)}(T_\alpha), w_t^{(p)}(T_\alpha)).$$

We recall that P_α^f is the closed hyperplane of $H_{(\alpha)}^3 \times D(A)$ defined by (8. 8).

From the previous section it follows that $D\Theta_{T_\alpha}(0)(L^2(0, T_\alpha)) \subset P_\alpha^f$, and moreover $D\Theta_{T_\alpha}(0) : L^2(0, T_\alpha) \rightarrow P_\alpha^f$ is invertible (see Lemma 8.2 and formula (8. 6)). Now, consider $(Y^\perp, Z^\perp) \neq 0$ and orthogonal to P_α^f : this allows us to decompose the space $H_{(\alpha)}^3 \times D(A)$ into

$$H_{(\alpha)}^3 \times D(A) = P_\alpha^f \oplus (P_\alpha^f)^\perp$$

where $(P_\alpha^f)^\perp = \mathbb{R}(Y^\perp, Z^\perp)$ is one dimensional. Consider the associated orthogonal projections $\text{proj}_{P_\alpha^f}$ and $\text{proj}_{(P_\alpha^f)^\perp}$. Any $(Y, Z) \in H_{(\alpha)}^3(0, 1) \times D(A)$ can be decomposed as

$$(Y, Z) = \text{proj}_{P_\alpha^f}(Y, Z) + \text{proj}_{(P_\alpha^f)^\perp}(Y, Z) \text{ with } \begin{cases} \text{proj}_{P_\alpha^f}(Y, Z) \in P_\alpha^f, \\ \text{proj}_{(P_\alpha^f)^\perp}(Y, Z) \in (P_\alpha^f)^\perp. \end{cases}$$

The application

$$\tilde{\Theta}_{\alpha, T_\alpha} : L^2(0, T_\alpha) \rightarrow P_\alpha^f, \quad \tilde{\Theta}_{\alpha, T_\alpha}(q) = \text{proj}_{P_\alpha^f}(\Theta_{T_\alpha}(q))$$

satisfies

$$D\tilde{\Theta}_{\alpha, T_\alpha}(0) : L^2(0, T_\alpha) \rightarrow P_\alpha^f, \quad D\tilde{\Theta}_{\alpha, T_\alpha}(0) = \text{proj}_{P_\alpha^f}(D\Theta_{T_\alpha}(0)).$$

Hence, $D\tilde{\Theta}_{\alpha, T_\alpha}(0) : L^2(0, T_\alpha) \rightarrow P_\alpha^f$ is invertible, and therefore the inverse mapping theorem implies that there exists a neighborhood $\mathcal{V}(0) \subset L^2(0, T_\alpha)$ and a neighborhood $\mathcal{V}(\text{proj}_{P_\alpha^f}((1, 0))) \subset P_\alpha^f$ such that

$$\tilde{\Theta}_{\alpha, T_\alpha} : \mathcal{V}(0) \subset L^2(0, T_\alpha) \rightarrow \mathcal{V}(\text{proj}_{P_\alpha^f}((1, 0))) \subset P_\alpha^f$$

is a C^1 -diffeomorphism. Therefore,

$$\Theta_{T_\alpha}(\mathcal{V}(0)) = \{(Y, Z) + \text{proj}_{(P_\alpha^f)^\perp}(\Theta_{T_\alpha}(\tilde{\Theta}_{\alpha, T_\alpha}^{-1}(Y, Z))), (Y, Z) \in \mathcal{V}(\text{proj}_{P_\alpha^f}((1, 0)))\},$$

which means that $\Theta_{T_\alpha}(\mathcal{V}(0))$ is the graph of the application

$$\mathcal{V}(\text{proj}_{P_\alpha^f}((1, 0))) \rightarrow (P_\alpha^f)^\perp, \quad (Y, Z) \mapsto \text{proj}_{(P_\alpha^f)^\perp}(\Theta_{T_\alpha}(\tilde{\Theta}_{\alpha, T_\alpha}^{-1}(Y, Z))),$$

hence $\Theta_{T_\alpha}(\mathcal{V}(0))$ is a submanifold of codimension 1. This concludes the proof of Theorem 3.3 in the case $T = T_\alpha$ and $\alpha \in [0, 1)$. \square

8.2. Proof of Theorem 3.2 second part: Reachability when $\alpha \in [1, 2)$.

When $\alpha \in [1, 2)$, we derive from (8. 5) that

$$\frac{2}{\pi(2-\alpha)} \omega_{\alpha, n} - n \rightarrow \frac{\alpha}{4(2-\alpha)} \quad \text{as } n \rightarrow \infty.$$

However, we notice that

$$\alpha \in [1, 2) \implies \frac{\alpha}{4(2-\alpha)} \geq \frac{1}{4}.$$

This fact represents the main difference with respect to the analysis of the solvability of moment problem (8. 2) of section 8.1 (Lemma 8.1-8.3).

8.2.a. Main solvability results when $\alpha \in [1, 2)$.

In this section will prove the extension of the Kadec's $\frac{1}{4}$ Theorem ([26], [33, Theorem 1.14 p. 42]) stated in Lemma 3.1. Our results are similar to those of [25, Theorem F p. 149], however, thanks to our assumptions, we are able to give very simple statements and proofs.

Lemma 8.4. *Let $\alpha \in [1, 2)$. Assume that*

$$(8. 10) \quad \frac{1}{2-\alpha} \notin \mathbb{N}.$$

Denote by k_α the integer part of $\frac{1}{2-\alpha}$. Then, the set $\{e^{i\omega_{\alpha, n} t}, n \in \mathbb{Z}\}$ can be complemented by k_α exponentials to form a Riesz basis of $L^2(0, T_\alpha)$.

Lemma 8.5. *Let $\alpha \in [1, 2)$ satisfy (8. 10) and let $T = T_\alpha$. Then, moment problem (8. 2) has a unique solution*

$$Q \in L^2(0, T_\alpha; \mathbb{R}) \in F_\alpha := \overline{\text{Vect} \{ \tilde{s}_{\alpha, 0}, e^{i\omega_{\alpha, n} t}, n \in \mathbb{Z} \}}.$$

To prove the above lemmas, we will use Lemma 3.1. So, let us first give the proof of such result.

8.2.b. *Proof of Lemma 3.1.*

We are going to prove it for $k = 1$ and $k = 2$, and then the other cases are easily deduced.

Case k even. We consider $k = 2$, however the method applies similarly for all k even. For $k = 2$, the assumption reads as: there exist $\delta \in (0, \frac{1}{4})$ and $N_0 \geq 0$ such that

$$(8.11) \quad n \geq N_0 \implies |x_n - (n+1)| \leq \frac{1}{4} - \delta.$$

Then, let us consider the following sequence:

$$(8.12) \quad \forall n \in \mathbb{Z}, \quad x_n^{(mod)} = \begin{cases} x_{n-1} & \text{if } n \geq N_0 + 1, \\ n & \text{if } |n| \leq N_0, \\ x_{n+1} & \text{if } n \leq -N_0 - 1. \end{cases}$$

We claim that

$$(8.13) \quad \forall n \in \mathbb{Z}, \quad |x_n^{(mod)} - n| \leq \frac{1}{4} - \delta.$$

Indeed, (8.13) is straightforward for any $|n| \leq N_0$. Moreover, if $n \geq N_0 + 1$ we get that

$$|x_n^{(mod)} - n| = |x_{n-1} - n| \leq \frac{1}{4} - \delta$$

thanks to (8.11). Finally, if $n \leq -N_0 - 1$,

$$|x_n^{(mod)} - n| = |x_{n+1} - n| = |-x_{|n+1|} + |n| = |-x_{|n|-1} + |n| = |x_{|n|-1} - |n|| \leq \frac{1}{4} - \delta$$

once again using (8.11). Then (8.13) is satisfied. We deduce from the Kadec's $\frac{1}{4}$ Theorem ([26], [33, Theorem 1.14 p. 42]) that the set $\{e^{ix_n^{(mod)}t}, n \in \mathbb{Z}\}$ is a Riesz basis of $L^2(-\pi, \pi)$. However, we can reorder the family $\{e^{ix_n^{(mod)}t}, n \in \mathbb{Z}\}$ as follows

$$\begin{aligned} \{e^{ix_n^{(mod)}t}, n \in \mathbb{Z}\} &= \{e^{ix_{n+1}t}, n \leq -N_0 - 1\} \cup \{e^{ix_n^{(mod)}t}, |n| \leq N_0\} \cup \{e^{ix_{n-1}t}, n \geq N_0 + 1\} \\ &= \{e^{ix_m t}, m \leq -N_0\} \cup \{e^{ix_n^{(mod)}t}, |n| \leq N_0\} \cup \{e^{ix_m t}, m \geq N_0\} \\ &= \{e^{ix_m t}, |m| \geq N_0\} \cup \{e^{ix_n^{(mod)}t}, |n| \leq N_0 - 1\} \cup \{e^{ix_n^{(mod)}t}, |n| = N_0\}. \end{aligned}$$

In order to keep the property to be a Riesz basis, we are allowed to modify a finite number of the elements of the family $\{e^{ix_n^{(mod)}t}, n \in \mathbb{Z}\}$ (see [4, Lemma II.4.11 p. 105]), if we do not consider twice the same element. Therefore, we can transform the set of $2N_0 + 1$ elements

$$\{e^{ix_n^{(mod)}t}, |n| \leq N_0 - 1\} \cup \{e^{ix_n^{(mod)}t}, |n| = N_0\}$$

into

$$\{e^{ix_n t}, |n| \leq N_0 - 1\} \cup \{e^{ix'_0 t}, e^{ix''_0 t}\}$$

with $x'_0 \neq x''_0$ and $x'_0, x''_0 \notin \{x_n, n \in \mathbb{Z}\}$. Thus,

$$\{e^{ix_m t}, m \in \mathbb{Z}\} \cup \{e^{ix'_0 t}, e^{ix''_0 t}\}$$

is a Riesz basis of $L^2(-\pi, \pi)$. Therefore Lemma 3.1 is proved for $k = 2$ and for every even k greater than 2 it is easy to adapt the above method.

Case k odd. We now consider the case $k = 1$, which can be easily extended to any k odd. For $k = 1$, the assumption reads as: that there exist $\delta \in (0, \frac{1}{4})$ and $N_0 \geq 0$ such that

$$(8.14) \quad n \geq N_0 \implies \left| x_n - n - \frac{1}{2} \right| \leq \frac{1}{4} - \delta.$$

Consider the following sequence

$$(8.15) \quad \forall n \in \mathbb{Z}, \quad x_n^{(mod)} = \begin{cases} x_n - \frac{1}{2} & \text{if } n \geq N_0, \\ n & \text{if } -N_0 \leq n \leq N_0 - 1, \\ x_{n+1} - \frac{1}{2} & \text{if } n \leq -N_0 - 1. \end{cases}$$

We claim that

$$(8.16) \quad \forall n \in \mathbb{Z}, \quad \left| x_n^{(mod)} - n \right| \leq \frac{1}{4} - \delta.$$

Indeed, for $-N_0 \leq n \leq N_0 - 1$ (8.16) is trivially true. Moreover, for $n \geq N_0$ we have that

$$\left| x_n^{(mod)} - n \right| = \left| x_n - \frac{1}{2} - n \right| \leq \frac{1}{4} - \delta$$

thanks to (8.14). Finally, for $n \leq -N_0 - 1$,

$$\begin{aligned} \left| x_n^{(mod)} - n \right| &= \left| x_{n+1} - \frac{1}{2} - n \right| = \left| -x_{|n+1|} - \frac{1}{2} + |n| \right| \\ &= \left| -x_{|n|-1} - \frac{1}{2} + |n| \right| = \left| x_{|n|-1} - |n| + \frac{1}{2} \right| = \left| x_{|n|-1} - (|n| - 1) - \frac{1}{2} \right| \leq \frac{1}{4} - \delta \end{aligned}$$

using that $\{x_n\}_{n \in \mathbb{Z}}$ is odd and, once again, thanks to (8.14). Then, (8.16) is satisfied. We deduce from Kadec's $\frac{1}{4}$ Theorem ([26], [33, Theorem 1.14 p. 42]) that the set $\{e^{ix_n^{(mod)}t}, n \in \mathbb{Z}\}$ is a Riesz basis of $L^2(-\pi, \pi)$. Now we shift this basis. To this purpose, we observe that if $f \in L^2(-\pi, \pi)$, then $g : t \mapsto g(t) = f(t)e^{-it/2}$ is still a function of $L^2(-\pi, \pi)$. Hence, it can be decomposed as

$$f(t)e^{-it/2} = \sum_{n \in \mathbb{Z}} c_n e^{ix_n^{(mod)}t} \quad \text{with} \quad A \sum_{n \in \mathbb{Z}} |c_n|^2 \leq \|g\|_{L^2(-\pi, \pi)}^2 \leq B \sum_{n \in \mathbb{Z}} |c_n|^2,$$

since $\{e^{ix_n^{(mod)}t}, n \in \mathbb{Z}\}$ is a Riesz basis of $L^2(-\pi, \pi)$. Therefore, we have that

$$f(t) = \sum_{n \in \mathbb{Z}} c_n e^{i(x_n^{(mod)} + \frac{1}{2})t} \quad \text{with} \quad A \sum_{n \in \mathbb{Z}} |c_n|^2 \leq \|f\|_{L^2(-\pi, \pi)}^2 \leq B \sum_{n \in \mathbb{Z}} |c_n|^2.$$

Hence, the set $\{e^{i(x_n^{(mod)} + \frac{1}{2})t}, n \in \mathbb{Z}\}$ is another Riesz basis of $L^2(-\pi, \pi)$ and can be rewritten as

$$\begin{aligned} &\{e^{i(x_n^{(mod)} + \frac{1}{2})t}, n \in \mathbb{Z}\} \\ &= \{e^{ix_{n+1}t}, n \leq -N_0 - 1\} \cup \{e^{i(x_n^{(mod)} + \frac{1}{2})t}, -N_0 \leq n \leq N_0 - 1\} \cup \{e^{ix_n t}, n \geq N_0\} \\ &= \{e^{ix_m t}, |m| \geq N_0\} \cup \{e^{ix_n^{(mod)}t}, -N_0 \leq n \leq N_0 - 1\}. \end{aligned}$$

The last set on the right-hand side of the above formula contains $2N_0$ elements which can be modified without changing the Riesz basis property as follows

$$\{e^{ix_m t}, |m| \leq N_0 - 1\} \cup \{e^{ix'_0 t}\},$$

where $x'_0 \notin \{x_n, n \in \mathbb{Z}\}$. Therefore, Lemma 3.1 is proved for $k = 1$ and, similarly, for any k odd. \square

8.2.c. Proof of Lemma 8.4.

We know from (8.5) that

$$\frac{2}{\pi(2-\alpha)} \omega_{\alpha, n} - n \rightarrow \frac{\alpha}{4(2-\alpha)} =: \ell_\alpha \quad \text{as } n \rightarrow +\infty.$$

Hence, we introduce

$$\forall n \in \mathbb{Z}, \quad x_n := \frac{2}{\pi(2-\alpha)} \omega_{\alpha, n}.$$

Since $\frac{1}{2-\alpha} \notin \mathbb{N}$, we can decompose it into the sum of its integer part k_α and its fractional part θ_α :

$$\frac{1}{2-\alpha} = \left[\frac{1}{2-\alpha} \right] + \left\{ \frac{1}{2-\alpha} \right\} = k_\alpha + \theta_\alpha, \quad \text{where } \theta_\alpha \in (0, 1).$$

Then, we can rewrite ℓ_α as

$$\ell_\alpha = \frac{\alpha}{4(2-\alpha)} = \frac{1}{4}(k_\alpha + \theta_\alpha) \left(2 - \frac{1}{k_\alpha + \theta_\alpha} \right) = \frac{1}{4}(2k_\alpha + 2\theta_\alpha - 1) = \frac{k_\alpha}{2} + \frac{2\theta_\alpha - 1}{4}.$$

Hence,

$$x_n - n - \frac{k_\alpha}{2} \rightarrow \ell_\alpha - \frac{k_\alpha}{2} = \frac{2\theta_\alpha - 1}{4} \in \left(-\frac{1}{4}, \frac{1}{4} \right),$$

and (3. 17) is satisfied with $k = k_\alpha$. Therefore, the set $\{e^{ix_n t}, n \in \mathbb{Z}\}$ can be complemented by k_α exponentials to form a Riesz basis of $L^2(-\pi, \pi)$. Consequently, the set $\{e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}$ can be complemented by k_α exponentials to form a Riesz basis of $L^2(0, T_\alpha)$ (as we have seen in the proof of Lemma 8.1). This concludes the proof of Lemma 8.4. \square

8.2.d. Proof of Lemma 8.5.

As a consequence of Lemma 8.4, the set of solutions of the moment problem (8. 3) is an affine space generated by a vectorial space of dimension k_α . To solve the whole moment problem (8. 2), first we note that $\tilde{s}_{\alpha,0} \notin \overline{\text{Vect}} \{e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}$. Indeed, if this was not the case, then $\overline{\text{Vect}} \{e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}$ would contain all the polynomials (by integration, as shown in step 2 of section 7.1). Hence, it would contain $L^2(0, T_\alpha)$. However, this is contradiction with the fact that $\overline{\text{Vect}} \{e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}$ is of codimension k_α in $L^2(0, T_\alpha)$ (from Lemma 8.4).

Thus, proceeding as in *Step 2* of Section 7.1, the moment problem (8. 2) admits a unique solution Q in $\overline{\text{Vect}} \{\tilde{s}_{\alpha,0}, e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}$ which is real valued. Indeed, we derive from (8. 2) that

$$\int_0^T Q(t) \cos \omega_{\alpha,n} t dt \in \mathbb{R}, \quad \int_0^T Q(t) \sin \omega_{\alpha,n} t dt \in \mathbb{R}, \quad \text{and} \quad \int_0^T Q(t) t dt \in \mathbb{R}.$$

Hence, denoting by Q_2 the imaginary part of Q , we have

$$\int_0^T Q_2(t) \cos \omega_{\alpha,n} t dt = 0, \quad \int_0^T Q_2(t) \sin \omega_{\alpha,n} t dt = 0, \quad \text{and} \quad \int_0^T Q_2(t) t dt = 0.$$

This gives that

$$\begin{cases} \int_0^T Q_2(t) e^{-i\omega_{\alpha,n} t} dt = 0 & \text{for all } n \in \mathbb{Z}, \\ \int_0^T Q_2(t) t dt = 0, \end{cases}$$

which implies that Q_2 is orthogonal to $\overline{\text{Vect}} \{\tilde{s}_{\alpha,0}, e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}$. On the other hand $2iQ_2 = Q - \overline{Q} \in \overline{\text{Vect}} \{\tilde{s}_{\alpha,0}, e^{i\omega_{\alpha,n} t}, n \in \mathbb{Z}\}$, thus $Q_2 = 0$ and Q is real-valued. \square

8.2.e. Proof of Theorem 3.2 second part part (inverse mapping argument).

Thanks to the previous results, we can conclude the proof Theorem 3.2 with the same procedure explained in section 7.2.

8.3. The case of the exceptional values.

As already noted, the values of α such that $\frac{1}{2-\alpha} \in \mathbb{N}$ are particular, because the Kadec's Theorem can no more be applied.

Let us consider, for instance, the case $\alpha = 1$. We observe that, for $\alpha \in [0, 1)$, the family $\{e^{i\omega_{\alpha,n}t}, n \in \mathbb{Z}\}$ is a Riesz basis of $L^2(0, \frac{4}{2-\alpha})$. However, it shows a deficiency equal to 1 if α is greater and close to 1. The stability of Riesz bases ([33, Corollary, p. 191]) implies that the family $\{e^{i\omega_{1,n}t}, n \in \mathbb{Z}\}$ is *not* a Riesz basis of $L^2(0, 4)$. Indeed, if it was a Riesz basis, the family $\{e^{i\omega_{\alpha,n}t}, n \in \mathbb{Z}\}$ would also be a Riesz basis of $L^2(0, \frac{4}{2-\alpha})$ for all α sufficiently close to 1, which is not true. This can be proved by performing a change of variables to work in the fixed space $L^2(-\pi, \pi)$. Therefore, the solvability of the moment problem is not clear. The analysis of the control problem for these special values α such that $\frac{1}{2-\alpha} \in \mathbb{N}$ is still a work in progress.

9. REACHABILITY WHEN $T < T_\alpha$: PROOF OF THEOREM 3.3

9.1. Proof of Theorem 3.3 for $\alpha \in [0, 1)$.

We recall that in Lemma 8.1 we have proved that $(e^{i\omega_{\alpha,n}t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T_\alpha)$.

9.1.a. *The moment problem (8. 2) is overdetermined when $T < T_\alpha$.*

Let $T < T_\alpha$. Following [4, p. 100], we introduce

$$\forall r > 0, \quad n(r) := \text{card} \{n \in \mathbb{Z}, |\omega_{\alpha,n}| < r\}.$$

We recall from Propositions 3.1 that the sequence $(\omega_{\alpha,n+1} - \omega_{\alpha,n})_n$ is nonincreasing and goes to $\kappa_\alpha \pi$ as $n \rightarrow \infty$. Hence, we deduce that

$$\forall n \geq 0, \quad \omega_{\alpha,n} \geq \kappa_\alpha \pi (n - 1).$$

Therefore, if $n - 1 \geq \frac{r}{\kappa_\alpha \pi}$ we have $\omega_{\alpha,n} \geq r$. This gives that

$$(9. 1) \quad n(r) \leq 2 \frac{r}{\kappa_\alpha \pi} + 1,$$

where the factor 2 comes from the negatives indices.

On the other hand, given $\varepsilon > 0$, there exists $n_0 \geq 0$ such that

$$\forall n \geq n_0, \quad \kappa_\alpha \pi \leq \omega_{\alpha,n+1} - \omega_{\alpha,n} \leq \kappa_\alpha \pi + \varepsilon.$$

Hence,

$$\forall n \geq n_0, \quad \omega_{\alpha,n_0} + \kappa_\alpha \pi (n - n_0) \leq \omega_{\alpha,n} \leq \omega_{\alpha,n_0} + (\kappa_\alpha \pi + \varepsilon)(n - n_0).$$

Thus, given $r > 0$

$$0 \leq n < n_0 + \frac{r - \omega_{\alpha,n_0}}{\kappa_\alpha \pi + \varepsilon} \implies \omega_{\alpha,n} < r.$$

We derive that for all r large enough,

$$(9. 2) \quad n(r) \geq 2 \left(n_0 + \frac{r - \omega_{\alpha,n_0}}{\kappa_\alpha \pi + \varepsilon} \right) - 1,$$

Then, we obtain from (9. 1) and (9. 2) that

$$(9. 3) \quad \lim_{r \rightarrow +\infty} \frac{n(r)}{r} = \frac{2}{\kappa_\alpha \pi} = \frac{4}{(2-\alpha)\pi} = \frac{T_\alpha}{\pi}.$$

Since $T < T_\alpha$, we have

$$\limsup_{r \rightarrow +\infty} \frac{n(r)}{r} > \frac{T}{\pi},$$

and so we can apply [4, Corollary II.4.2 p. 100], and we deduce that the family $\{e^{i\omega_{\alpha,n}t}, n \in \mathbb{Z}\}$ is not minimal in $L^2((0, T), \mathbb{C})$.

9.1.b. *How much overdetermined the moment problem (8. 2) is when $T < T_\alpha$.*

As proved in Lemma 8.1, $\{e^{i\omega_{\alpha,n}t}, n \in \mathbb{Z}\}$ is a Riesz basis of $L^2((0, T_\alpha), \mathbb{C})$. From Horváth-Joó [24] (see also [4, Theorem II.4.16 p. 107]) we deduce that there exists a subfamily $\{e^{i\omega_{\alpha,\varphi(n)}t}, n \in \mathbb{Z}\}$ which is a Riesz basis of $L^2((0, T), \mathbb{C})$. Then, consider

$$\forall r > 0, \quad n_\varphi(r) := \text{card} \{n \in \mathbb{Z}, |\omega_{\alpha,\varphi(n)}| < r\}.$$

Since $\{e^{i\omega_{\alpha,\varphi(n)}t}, n \in \mathbb{Z}\}$ is minimal in $L^2((0, T), \mathbb{C})$, we derive from [4, Corollary II.4.2 p. 100], that

$$\limsup_{r \rightarrow +\infty} \frac{n_\varphi(r)}{r} \leq \frac{T}{\pi},$$

hence,

$$(9. 4) \quad \liminf_{r \rightarrow +\infty} \frac{n(r) - n_\varphi(r)}{r} \geq \frac{T_\alpha - T}{\pi}.$$

Since

$$n(r) - n_\varphi(r) = \text{card} \{n \in \mathbb{Z}, |\omega_{\alpha,n}| < r \text{ and } n \notin \text{Im } \varphi\},$$

the asymptotic behaviour (9. 4) gives an idea of how much overdetermined the moment problem (8. 2) is.

9.1.c. *Solvability of the moment problem.*

We consider (8. 2). First, assume that (8. 2) has a solution $Q \in L^2(0, T; \mathbb{R})$. This implies that

$$(9. 5) \quad \langle Q, e^{i\omega_{\alpha,\varphi(n)}t} \rangle_{L^2(0,T;\mathbb{C})} = C_{\alpha,\varphi(n)}^f \quad \text{for all } n \in \mathbb{Z}.$$

Now, consider $m \notin \text{Im } \varphi$. Then, $e^{i\omega_{\alpha,m}t}$ can be decomposed as follows

$$e^{i\omega_{\alpha,m}t} = \sum_{n \in \mathbb{Z}} \Omega_{\alpha,\varphi(n)}^{(m)} e^{i\omega_{\alpha,\varphi(n)}t} \quad \text{in } L^2(0, T; \mathbb{C}).$$

Therefore, we have that

$$C_{\alpha,m}^f = \langle Q, e^{i\omega_{\alpha,m}t} \rangle_{L^2(0,T;\mathbb{C})} = \sum_{n \in \mathbb{Z}} \overline{\Omega_{\alpha,\varphi(n)}^{(m)}} C_{\alpha,\varphi(n)}^f.$$

In the same way, $\tilde{s}_{\alpha,0}$ can be decomposed as follows

$$\tilde{s}_{\alpha,0} = \sum_{n \in \mathbb{Z}} \tilde{S}_{\alpha,\varphi(n)} e^{i\omega_{\alpha,\varphi(n)}t} \quad \text{in } L^2(0, T; \mathbb{C}),$$

which implies that

$$B_{\alpha,0}^f = \sum_{n \in \mathbb{Z}} \overline{\tilde{S}_{\alpha,\varphi(n)}} C_{\alpha,\varphi(n)}^f.$$

Hence,

$$Q \text{ solution of (8. 2)} \quad \implies \quad \begin{cases} C_{\alpha,m}^f = \sum_{n \in \mathbb{Z}} \overline{\Omega_{\alpha,\varphi(n)}^{(m)}} C_{\alpha,\varphi(n)}^f & \forall m \notin \text{Im } \varphi, \\ B_{\alpha,0}^f = \sum_{n \in \mathbb{Z}} \overline{\tilde{S}_{\alpha,\varphi(n)}} C_{\alpha,\varphi(n)}^f. \end{cases}$$

This leads to consider the space

$$(9. 6) \quad H_\alpha^f := \left\{ (Y^f, Z^f) \in H_{(\alpha)}^3 \times D(A), \begin{cases} C_{\alpha,m}^f = \sum_{n \in \mathbb{Z}} \overline{\Omega_{\alpha,\varphi(n)}^{(m)}} C_{\alpha,\varphi(n)}^f & \forall m \notin \text{Im } \varphi, \\ B_{\alpha,0}^f = \sum_{n \in \mathbb{Z}} \overline{\tilde{S}_{\alpha,\varphi(n)}} C_{\alpha,\varphi(n)}^f \end{cases} \right\},$$

where the relations between (Y^f, Z^f) and $B_{\alpha,0}^f, C_{\alpha,m}^f, m \geq 1$ are given in (7. 6) and (8. 1). Thus, we have proved that

$$(9. 7) \quad D\Theta_T(0)(L^2(0, T; \mathbb{R})) \subset H_\alpha^f.$$

Now, let us prove the following reverse inclusion.

Lemma 9.1. *Let $\alpha \in [0, 1)$, $T < T_\alpha$ and H_α^f be defined in (9. 6). Then, the following identity holds*

$$(9. 8) \quad D\Theta_T(0)(L^2(0, T; \mathbb{R})) = H_\alpha^f.$$

Proof of Lemma 9.1. Since we already proved (9. 7), it is sufficient to prove that $H_\alpha^f \subset D\Theta_T(0)(L^2(0, T; \mathbb{R}))$. Let $(Y^f, Z^f) \in H_\alpha^f$. Since $(e^{i\omega_{\alpha, \varphi(n)}t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T)$, the moment problem (9. 5) has one and only one solution Q (which can be expressed using the unique biorthogonal family to $(e^{i\omega_{\alpha, \varphi(n)}t})_{n \in \mathbb{Z}}$). Then, for all $m \notin \text{Im } \varphi$, we have

$$\begin{aligned} \langle Q, e^{i\omega_{\alpha, m}t} \rangle_{L^2(0, T; \mathbb{C})} &= \langle Q, \sum_{n \in \mathbb{Z}} \Omega_{\alpha, \varphi(n)}^{(m)} e^{i\omega_{\alpha, \varphi(n)}t} \rangle_{L^2(0, T; \mathbb{C})} \\ &= \sum_{n \in \mathbb{Z}} \overline{\Omega_{\alpha, \varphi(n)}^{(m)}} \langle Q, e^{i\omega_{\alpha, \varphi(n)}t} \rangle_{L^2(0, T; \mathbb{C})} = \sum_{n \in \mathbb{Z}} \overline{\Omega_{\alpha, \varphi(n)}^{(m)}} C_{\alpha, \varphi(n)}^f = C_{\alpha, m}^f, \end{aligned}$$

where the last equality derives from the fact that $(Y^f, Z^f) \in H_\alpha^f$. In the same way, we get

$$\langle Q, \tilde{s}_{\alpha, 0} \rangle_{L^2(0, T; \mathbb{C})} = \sum_{n \in \mathbb{Z}} \overline{\tilde{S}_{\alpha, \varphi(n)}} \langle Q, e^{i\omega_{\alpha, \varphi(n)}t} \rangle_{L^2(0, T; \mathbb{C})} = \sum_{n \in \mathbb{Z}} \overline{\tilde{S}_{\alpha, \varphi(n)}} C_{\alpha, \varphi(n)}^f = B_{\alpha, 0}^f.$$

Hence, Q solves the whole moment problem (8. 2). It remains to prove that Q is real-valued: this follows easily from the fact that

$$\forall n \geq 0, \quad \begin{cases} \int_0^T Q(t) e^{-i\omega_{\alpha, n}t} dt = C_{\alpha, n}^f = \frac{Z_{\alpha, n}^f - i\sqrt{\lambda_{\alpha, n}} Y_{\alpha, n}^f}{\mu_{\alpha, n}}, \\ \int_0^T Q(t) e^{i\omega_{\alpha, n}t} dt = \overline{C_{\alpha, n}^f} = \frac{Z_{\alpha, n}^f + i\sqrt{\lambda_{\alpha, n}} Y_{\alpha, n}^f}{\mu_{\alpha, n}}. \end{cases}$$

By adding (subtracting) one to each other the above equations, we obtain

$$\forall n \geq 0, \quad \begin{cases} \int_0^T Q(t) \cos \omega_{\alpha, n}t dt = 2 \frac{Z_{\alpha, n}^f}{\mu_{\alpha, n}}, \\ \int_0^T Q(t) 2i \sin \omega_{\alpha, n}t dt = \frac{2i\sqrt{\lambda_{\alpha, n}} Y_{\alpha, n}^f}{\mu_{\alpha, n}}. \end{cases}$$

Thus, the real part of Q solves (7. 5), and its imaginary part Q_2 satisfies

$$\forall n \geq 0, \quad \int_0^T Q_2(t) \cos \omega_{\alpha, n}t dt = 0 = \int_0^T Q_2(t) \sin \omega_{\alpha, n}t dt.$$

Therefore, for every $n \in \mathbb{Z}$ it holds that $\langle Q_2, e^{i\omega_{\alpha, \varphi(n)}t} \rangle_{L^2(0, T; \mathbb{C})} = 0$, which implies $Q_2 = 0$ since $(e^{i\omega_{\alpha, \varphi(n)}t})_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0, T)$. So, we have proved that Q is real-valued and this completes the proof of (9. 8). \square

We conclude by proving the following

Lemma 9.2. *H_α^f is a closed vectorial space of $H_{(\alpha)}^3 \times D(A)$ of infinite dimension and infinite codimension.*

Proof of Lemma 9.2. Let us consider

$$\forall m \notin \text{Im } \varphi, \quad L_T^m : H_{(\alpha)}^3 \times D(A) \rightarrow \mathbb{C}, \quad L_T^m(Y^f, Z^f) := C_{\alpha, m}^f - \sum_{n \in \mathbb{Z}} \overline{\Omega_{\alpha, \varphi(n)}^{(m)}} C_{\alpha, \varphi(n)}^f,$$

and

$$\ell_T^0 : H_{(\alpha)}^3 \times D(A) \rightarrow \mathbb{C}, \quad \ell_T^0(Y^f, Z^f) := B_{\alpha, 0}^f - \sum_{n \in \mathbb{Z}} \overline{\tilde{S}_{\alpha, \varphi(n)}} C_{\alpha, \varphi(n)}^f.$$

Observe that ℓ_T^0 and L_T^m are linear continuous forms and furthermore $H_\alpha^f = \text{Ker } \ell_T^0 \cap \left(\bigcap_{m \notin \text{Im } \varphi} \text{Ker } L_T^m \right)$. Hence, H_α^f is a closed vectorial space of $H_{(\alpha)}^3 \times D(A)$, and is of infinite dimension. To prove that H_α^f has infinite codimension, we use the fact

that $\mathbb{Z} \setminus (\text{Im } \varphi)$ is infinite (see section 9.1.b). Then, fix $N \geq 1$, and $n_1, \dots, n_N \notin (\text{Im } \varphi \cup \{0\})$, and consider

$$L_T^{n_1, \dots, n_N} : H_{(\alpha)}^3 \times D(A) \rightarrow \mathbb{C}^N, \quad L_T^{n_1, \dots, n_N}(Y^f, Z^f) := \begin{pmatrix} L_T^{n_1}(Y^f, Z^f) \\ \dots \\ L_T^{n_N}(Y^f, Z^f) \end{pmatrix}.$$

Observe that

$$L_T^{n_1, \dots, n_N}(\Phi_{\alpha, n_1}, 0) = -i \frac{\omega_{\alpha, n_1}}{\mu_{\alpha, n_1}} \varepsilon_1, \quad \dots, \quad L_T^{n_1, \dots, n_N}(\Phi_{\alpha, n_N}, 0) = -i \frac{\omega_{\alpha, n_N}}{\mu_{\alpha, n_N}} \varepsilon_N,$$

where $\varepsilon_1, \dots, \varepsilon_N$ is the canonical basis of \mathbb{C}^N . Hence, $L_T^{n_1, \dots, n_N}$ is surjective, and its kernel is of codimension N . Since this is true for all $N \geq 1$, this implies that H_{α}^f is a closed vectorial space of $H_{(\alpha)}^3 \times D(A)$ of infinite codimension. \square

9.1.d. *Proof of Theorem 3.3 for $\alpha \in [0, 1)$ (inverse mapping argument).*

The proof follows the scheme of section 8.1.d, replacing the hyperplane P_{α}^f by H_{α}^f . Therefore, $\Theta_T(\mathcal{V}(0))$ turns out to be a submanifold of infinite dimension and infinite codimension. \square

9.2. Proof of Theorem 3.3: modifications when $\alpha \in [1, 2)$.

Given $x \in \mathbb{R}$, $r > 0$, we define

$$N(x, r) := \text{card } \{\omega_{\alpha, n}, x \leq \omega_{\alpha, n} < x + r\}.$$

Thanks to (8. 5), one can prove that

$$\frac{N(x, r)}{r} \rightarrow \frac{T_{\alpha}}{2\pi}, \quad \text{as } r \rightarrow +\infty$$

uniformly with respect to $x \in \mathbb{R}$ (see [9, Section 9]).

Thus, from [4, Theorem II.4.18 p. 109] we deduce that given $T < T_{\alpha}$, the family $(e^{i\omega_{\alpha, n}t})_{n \in \mathbb{Z}}$ contains a subfamily that forms a Riesz basis of $L^2(0, T)$. And then, once again, we can proceed as in section 9.1 to prove Theorem 3.3.

10. PROOF OF PROPOSITION 3.3

First, it is easy to check that $\mu(x) = x^{2-\alpha}$ satisfies all the regularity assumptions: $\mu \in V_{\alpha}^{(2, \infty)}(0, 1)$ if $\alpha \in [0, 1)$, and $\mu \in V_{\alpha}^{(2, \infty, \infty)}(0, 1)$ if $\alpha \in [1, 2)$. We refer to [9, Section 10] for a detailed proof. It remains to prove the validity of (3. 13). Direct computations show that

$$\langle \mu, \Phi_{\alpha, 0} \rangle_{L^2(0, 1)} = \int_0^1 x^{2-\alpha} dx = \frac{1}{3-\alpha},$$

and, for all $n \geq 1$, we develop the scalar product as follows

$$\langle \mu, \Phi_{\alpha, n} \rangle_{L^2(0, 1)} = \int_0^1 \mu(x) \Phi_{\alpha, n} dx = \frac{1}{\lambda_{\alpha, n}} \int_0^1 \mu(x) (-x^{\alpha} \Phi'_{\alpha, n})' dx.$$

Integrating twice by parts and thanks to Lemma 5.1, we obtain that

$$\langle \mu, \Phi_{\alpha, n} \rangle_{L^2(0, 1)} = \frac{2-\alpha}{\lambda_{\alpha, n}} [x \Phi_{\alpha, n}(x)]_0^1 = \frac{(2-\alpha)^{3/2}}{\lambda_{\alpha, n}}.$$

Hence, (3. 13) is satisfied. Finally, one can prove that the set of functions μ satisfying (3. 13) is dense in V_{α}^2 , adapting suitably the method of [6], based on the Baire theorem (see [9, Section 10], and [31, Chapter 5] for general explicit constructions of such μ). \square

Acknowledgements. The authors would like to thank A. Duca and V. Komornik for interesting discussions.

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