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1 **LIPSCHITZ STABILITY FOR THE GROWTH RATE**
2 **COEFFICIENTS IN A NONLINEAR FISHER-KPP EQUATION**

3 P. MARTINEZ AND J. VANCOSTENOBLE

ABSTRACT. We consider a reaction-diffusion model of biological invasion in which the evolution of the population is governed by several parameters among them the intrinsic growth rate $\mu(x)$. The knowledge of this growth rate is essential to predict the evolution of the population, but it is a priori unknown for exotic invasive species. We prove uniqueness and unconditional Lipschitz stability for the corresponding inverse problem, taking advantage of the positivity of the solution inside the spatial domain and studying its behaviour near the boundary with maximum principles. Our results complement previous works by Cristofol and Roques [10, 12].

4 1. INTRODUCTION

5 1.1. **Biological invasion model.**

6 Throughout this work, we consider the following biological invasion model: let
7 Ω be a bounded domain of \mathbb{R}^d whose boundary $\partial\Omega$ is assumed to be smooth; the
8 population density u of the considered invasive specie u is governed by the following
9 reaction-diffusion equation of Fisher-Kolmogorov-Petrovsky-Piskunov type:

$$(1.1) \quad (P_{\mu,\gamma}) : \begin{cases} u_t - D\Delta u = u(\mu(x) - \gamma(x)u) & (t, x) \in (0, +\infty) \times \Omega, \\ u(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_i(x) & x \in \Omega. \end{cases}$$

10 In the above problem, $D > 0$ is the diffusion coefficient, $\mu : \Omega \rightarrow \mathbb{R}$ represents the
11 intrinsic growth rate (i.e. the birth rate minus the death rate in the absence of
12 competition) and $\gamma \geq 0$ measures the effects of competition between individuals.
13 (When $\gamma > 0$, the quantity μ/γ is called the environment's carrying capacity.)

14 1.2. **Inverse problem arising in the case of newly invasive species.**

15 In models like (1.1), the knowledge of the parameters μ and γ is essential
16 to predict the evolution of the population (see for example [28, 4, 9, 29, 31] for
17 qualitative results). For instance, the success of an invasion and its rate of spread
18 depend on the coefficient μ , see [29, 17, 18].

19 In the case of invasive species which have recently been introduced in a new
20 environment, these parameters are unknown (and for exotic invasive species, it may
21 strongly differ from that of the native species). So we are interested in determining
22 μ and γ using only partial measurements of the population density.

23 In the following, we investigate some inverse coefficient problem that consists in
24 recovering μ given γ (respectively recovering γ given μ) from partial measurements
25 of the solution u over $(t_0, t_1) \times \omega$ and over $\{T'\} \times \Omega$ where $0 < t_0 < t_1$, $\omega \subset\subset \Omega$
26 are given and where $T' \in (t_0, t_1)$. More precisely, we prove a Lipschitz stability
27 result that ensures that the coefficient μ (respectively γ) is identifiable from the
28 chosen set of observation and that close observations of the solutions lead to close

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1 estimates of the coefficient. This stability result follows from the combination of
2 several parabolic equations techniques, namely

- 3 • maximum principles, and here we will use the weak and strong maximum
- 4 principles, and the Hopf's lemma to obtain suitable estimates near the
- 5 boundary, where the Dirichlet condition holds,
- 6 • some consequences of the regularizing effect of the heat equation,
- 7 • and Carleman estimates to obtain the Lipschitz stability estimates.

8 This complements several earlier results of the literature, in particular of Cristofol
9 and Roques [10, 30, 12] (see section 3.2).

10 2. ASSUMPTIONS AND MAIN RESULTS

11
12 We fix several parameters

- 13 • $\alpha \in (0, 1)$,
- 14 • $M > 0$, $\gamma^- \geq 0$, $\gamma^+ > \gamma^-$,
- 15 • $\underline{u}_i > 0$, $\overline{u}_i > \underline{u}_i$,
- 16 • $x_i \in \Omega$ and $\varepsilon > 0$ such that $B_\varepsilon := B(x_i, \varepsilon) \subset \Omega$,

17 and then we consider the associated functions spaces:

- 18 • we will take the function μ in the set

$$(2. 1) \quad \mathcal{M}_{\alpha, M} := \{\rho \in C^{0, \alpha}(\overline{\Omega}), \|\rho\|_{C^{0, \alpha}(\overline{\Omega})} \leq M\}$$

- 19 • we will take the function γ in the set

$$(2. 2) \quad \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+} := \{\rho \in C^{0, \alpha}(\overline{\Omega}), \gamma^- \leq \rho(x) \leq \gamma^+ \text{ on } \overline{\Omega} \text{ and } \|\rho\|_{C^{0, \alpha}(\overline{\Omega})} \leq M\},$$

- 20 • we will take the initial condition u_i in the set

$$(2. 3) \quad \mathcal{D}_{\underline{u}_i, \overline{u}_i} := \{\phi \geq 0, \phi \in C^0(\overline{\Omega}), \phi = 0 \text{ on } \partial\Omega, \|\phi\|_{L^\infty(\overline{\Omega})} \leq \overline{u}_i, \phi \geq \underline{u}_i \text{ in } B_\varepsilon\}.$$

21 As usual we denote $d(x, \partial\Omega)$ the distance from $x \in \Omega$ to the boundary $\partial\Omega$.

22 We investigate here some inverse coefficient problem that consists in recovering μ
23 (respectively γ) in (1. 1). For any $0 < t_0 < t_1$ and any $\omega \subset\subset \Omega$, we wish to estimate
24 the function μ (respectively γ) in Ω from partial measurements of the solution u
25 over $(t_0, t_1) \times \omega$ and over $\{T'\} \times \Omega$ where $T' = (t_0 + t_1)/2$ (the case $T' \in (t_0, t_1)$ is
26 similar). We prove the following Lipschitz stability estimates:

27 **Theorem 2.1. [Stability of the intrinsic growth rate μ]**

28 *Let $t_1 > t_0 > 0$ and $\omega \subset\subset \Omega$ be given, and consider*

- 29 • $\mu, \tilde{\mu} \in \mathcal{M}_{\alpha, M}$,
- 30 • $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$,
- 31 • and $u_i \in \mathcal{D}_{\underline{u}_i, \overline{u}_i}$.

Let u, \tilde{u} be the solutions of

$$(P_{\mu, \gamma}) : \begin{cases} u_t - D\Delta u = u(\mu(x) - \gamma(x)u) & (t, x) \in (0, +\infty) \times \Omega, \\ u(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_i(x) & x \in \Omega, \end{cases}$$

$$(P_{\tilde{\mu}, \gamma}) : \begin{cases} \tilde{u}_t - D\Delta \tilde{u} = \tilde{u}(\tilde{\mu}(x) - \gamma(x)\tilde{u}) & (t, x) \in (0, +\infty) \times \Omega, \\ \tilde{u}(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ \tilde{u}(0, x) = u_i(x) & x \in \Omega. \end{cases}$$

*Then there exists $C = C(D, \alpha, M, \gamma^-, \gamma^+, \underline{u}_i, \overline{u}_i, x_i, \varepsilon, t_0, t_1, \omega)$, hence some (ex-
plicit) constant which is uniform with respect to $\mu, \tilde{\mu} \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and*

$u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$, such that the following inequality holds :

$$(2.4) \quad \|d(x, \partial\Omega)(\mu - \tilde{\mu})\|_{L^2(\Omega)} \leq C \left(\|u_t - \tilde{u}_t\|_{L^2((t_0, t_1) \times \omega)} + \|u(T') - \tilde{u}(T')\|_{H^2(\Omega) \cap H_0^1(\Omega)} \right).$$

1 **Theorem 2.2. [Stability of the susceptibility to crowding effect γ]**

2 Let $t_1 > t_0 > 0$ and $\omega \subset\subset \Omega$ be given, and consider

- 3
 - $\mu \in \mathcal{M}_{\alpha, M}$,
 - 4 • $\gamma, \gamma^* \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$,
 - 5 • and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$.

Let u, u^* be the solutions of

$$(P_{\mu, \gamma}) : \begin{cases} u_t - D\Delta u = u(\mu(x) - \gamma(x)u) & (t, x) \in (0, +\infty) \times \Omega, \\ u(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_i(x) & x \in \Omega, \end{cases}$$

$$(P_{\mu, \gamma^*}) : \begin{cases} u_t^* - D\Delta u^* = u^*(\mu(x) - \gamma^*(x)u^*) & (t, x) \in (0, +\infty) \times \Omega, \\ u^*(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u^*(0, x) = u_i(x) & x \in \Omega. \end{cases}$$

Then there exists $C = C(D, \alpha, M, \gamma^-, \gamma^+, \underline{u}_i, \bar{u}_i, x_i, \varepsilon, t_0, t_1, \omega)$, hence some (explicit) constant which is uniform with respect to $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma, \gamma^* \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$, such that the following inequality holds :

$$(2.5) \quad \|d(x, \partial\Omega)^2(\gamma - \gamma^*)\|_{L^2(\Omega)} \leq C \left(\|u_t - u_t^*\|_{L^2((t_0, t_1) \times \omega)} + \|u(T') - u^*(T')\|_{H^2(\Omega) \cap H_0^1(\Omega)} \right).$$

The existence, uniqueness and regularity of the solution u (and \tilde{u}, u^*) are classical, see, e.g. Pao [25] (Theorem 5.2 page 66). In particular

$$u \in C^0([0, +\infty) \times \bar{\Omega}) \cap C_1^2((0, +\infty) \times \Omega),$$

6 where $C_i^j((0, +\infty) \times \Omega)$ denotes the set of functions whose i -times derivatives in t
7 and j -times derivatives in x are continuous in $(0, +\infty) \times \Omega$.

8 The proof of Theorems 2.1 and 2.2 is based on Carleman estimates and maximum
9 principles for parabolic equations, to deal with the nonlinear terms. In the following
10 we recall some of the major results and techniques on these questions, in order to
11 precise what are the novelties of our results:

- 12
 - Lipschitz stability under weakened assumptions on the parameters μ and
13 γ , in particular avoiding any assumption on their values near the boundary,
 - 14 • Lipschitz estimates with weights, coming from the study of the solution
15 near the boundary $\partial\Omega$,

16 and we conclude by some open questions, mainly

- 17
 - the validity of such stability estimates under lower regularity on μ and γ
18 (only $C^0(\bar{\Omega})$ for example),
 - 19 • and the simultaneous reconstruction of μ and γ , with additionnal measures.

20 2.1. Plan of the paper.

21 The rest of the paper is organized as follows :

- 22
 - In section 3, we compare our results to the related literature;
 - 23 • in section 4, we give some preliminary estimates following from maximum
24 principles;
 - 25 • in section 5, we prove Theorem 2.1;
 - 26 • in section 6, we give the ideas to prove Theorem 2.2.

1

3. RELATION TO LITERATURE

2 Let us recall that many references deal with parabolic inverse problems, see for
 3 instance [6, 20, 21, 22] and the references therein. To our knowledge, few results
 4 of that kind (unconditional Lipschitz stability) are known in *nonlinear* situations.
 5 We follow here a methodology introduced in [33] (and then used also in [24]) to
 6 recover the insulation coefficient in the nonlinear Sellers climate model. But before
 7 proving Theorems 2.1 and 2.2 (in section 5), let us recall and compare our result
 8 with inspiring earlier results.

9 **3.1. The determination of a source term.**

10 Imanuvilov and Yamamoto [19] proposed and developed a useful technique to
 11 obtain unconditional Lipschitz stability for some standard inverse source problem,
 12 using global Carleman estimates. More precisely, they consider the problem of
 13 recovering the source term h in

$$(3. 1) \quad \begin{cases} y_t - \Delta y = h(t, x) & (t, x) \in (0, T) \times \Omega, \\ y(t, x) = 0 & (t, x) \in (0, T) \times \partial\Omega, \\ y(0, x) = y_0(x) & x \in \Omega, \end{cases}$$

14 where $T > 0$ and $y_0 \in L^2(\Omega)$. (Let us mention that the problem considered in [19]
 15 is more general : the case of a general uniformly parabolic operator with mixed
 16 Neumann-Dirichlet boundary conditions is treated). The problem concerns the
 17 determination of h using partial measurements of the solution y over $(t_0, t_1) \times \omega$
 18 and over $\{T'\} \times \Omega$ where $0 < t_0 < t_1$, $\omega \subset\subset \Omega$ and $T' = (t_0 + t_1)/2$. Under some
 19 assumption on h (see (5. 3) later), the authors prove Lipschitz stability for this
 20 inverse problem:

$$(3. 2) \quad \|h\|_{L^2((0,T)\times\Omega)}^2 \leq C\|y_t\|_{L^2((t_0,t_1)\times\omega)}^2 + C\|y(T')\|_{L^2(\Omega)}^2 + C\|\Delta y(T')\|_{L^2(\Omega)}^2.$$

21 Later on, this idea of using global Carleman estimates to solve inverse problem and
 22 obtain Lipschitz stability results has proved its efficiency in many situations where
 23 the considered systems is *linear*, (see for instance [1, 2, 3, 7, 8, 13, 27, 35]). However
 24 the tools (especially the Carleman estimates) are not well adapted or at least are
 25 not sufficient to study nonlinear models.

26 **3.2. The determination of some coefficients in nonlinear models.**

27 3.2.a. *In nonlinear population dynamics models.* The inverse problem that consists
 28 in recovering some coefficients in nonlinear population dynamics models as (1. 1)
 29 has been considered in a series of papers. Cristofol and Roques [10] compare the
 30 solution u of $(P_{\mu,\gamma})$ to the solution of the linearized problem

$$(3. 3) \quad (P_{\mu,0}) : \begin{cases} v_t - D\Delta v = \mu(x)v & (t, x) \in (0, +\infty) \times \Omega, \\ v(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ v(0, x) = u_i(x) & x \in \Omega, \end{cases}$$

for which they obtain Lipschitz stability in the reconstruction of the coefficient μ
 (see [10, Theorem 2.1]) (developping the Imanuvilov-Yamamoto approach); then
 they deduce that, if u is the solution of the nonlinear problem $(P_{\mu,\gamma})$ and \tilde{v} is the
 solution of the linear problem $(P_{\tilde{\mu},0})$, then

$$\begin{aligned} \|\mu - \tilde{\mu}\|_{L^2(\Omega)}^2 &\leq \frac{C}{\bar{u}_i^2} \left(\|u_t - \tilde{v}_t\|_{L^2((t_0,t_1)\times\omega)}^2 \right. \\ &\quad \left. + \|u(T') - \tilde{v}(T')\|_{L^2(\Omega)}^2 + \|\Delta u(T') - \Delta \tilde{v}(T')\|_{L^2(\Omega)}^2 \right) + \mathcal{O}(\bar{u}_i). \end{aligned}$$

1 They deduce that if the initial population density is far from the environment
 2 carrying capacity i.e. $\bar{u}_i \ll \mu/\gamma$ (see the comments after Theorem 2.4 in [10]), and
 3 if the solution \tilde{v} of the linear problem $(P_{\tilde{\mu},0})$ takes values on the observation regions
 4 that are close to the measurements of the solution u of the nonlinear problem $(P_{\mu,\gamma})$,
 5 then $\tilde{\mu}$ is an accurate estimate of μ . From this result, they propose some explicit
 6 algorithm to recover μ .

7 In Remark 2.5 in [10], the authors also mention the fact that $\mathcal{O}(\bar{u}_i)$ increases
 8 exponentially with time t_1 . So, obtaining accurate estimates of μ requires in practice
 9 to work with small times, hence at the beginning of the invasion.

In [12], Cristofol and Roques provide Lipschitz estimates for μ and γ , for a
 problem similar to $(P_{\mu,\gamma})$ but with Neumann boundary conditions and provided
 that the values of the solution at the boundary $u(t, x)$ for $(t, x) \in (0, +\infty) \times \partial\Omega$ are
 a priori known. More precisely, they consider

$$(P_{\mu,\gamma})(N) : \begin{cases} u_t - D\Delta u = u(\mu(x) - \gamma(x)u) & (t, x) \in (0, +\infty) \times \Omega, \\ \partial_\nu u(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_i(x) & x \in \Omega, \end{cases}$$

and

$$(P_{\tilde{\mu},\tilde{\gamma}})(N) : \begin{cases} \tilde{u}_t - D\Delta \tilde{u} = u(\tilde{\mu}(x) - \tilde{\gamma}(x)\tilde{u}) & (t, x) \in (0, +\infty) \times \Omega, \\ \tilde{u}(t, x) = u(t, x) & (t, x) \in (0, +\infty) \times \partial\Omega, \\ \tilde{u}(0, x) = u_i(x) & x \in \Omega, \end{cases}$$

and they prove the following Lipschitz estimate (see Theorem 2.2 in [12]):

$$\|(\mu, \gamma) - (\tilde{\mu}, \tilde{\gamma})\| \leq C(\|u - \tilde{u}\|_{H^2(t_0 - \delta, t_1 + \delta), L^2(\omega)} + \|(u - \tilde{u})(t_0)\|_{H^2(\Omega)} + \|(u - \tilde{u})(t_1)\|_{H^2(\Omega)}),$$

10 under some assumptions

- on the initial condition: u_i has to satisfy $u_i > 0$ in $\bar{\Omega}$, $\partial_\nu u_i = 0$ on $\partial\Omega$, and some smallness assumptions:

$$6 \max_{\bar{\Omega}} u_i < \frac{\min_{\bar{\Omega}} \mu^-}{\max_{\bar{\Omega}} \gamma^+}, \quad -D\Delta u_i - u_i(\mu^- - \gamma^+ u_i) < 0 \text{ in } \bar{\Omega};$$

- on the time interval (t_0, t_1) : it has to be sufficiently large;
- and on the parameters $(\mu, \tilde{\mu})$, $(\gamma, \tilde{\gamma})$: they are quite smooth ($C^5(\bar{\Omega})$), and moreover

$$\begin{cases} \mu, \tilde{\mu} \in \tilde{\mathcal{M}} & := \{\rho \in C^5(\bar{\Omega}), \mu^- \leq \rho \leq \mu^+ \text{ on } \bar{\Omega}\}, \\ \gamma, \tilde{\gamma} \in \tilde{\mathcal{C}} & := \{\rho \in C^5(\bar{\Omega}), \gamma^- \leq \rho \leq \gamma^+ \text{ on } \bar{\Omega}\}, \end{cases}$$

with the following important assumption:

$$\mu^- = \mu^+, \quad \text{and} \quad \gamma^- = \gamma^+ \quad \text{on a neighborhood of } \partial\Omega.$$

12 Let us comment these assumptions:

- The restrictions on the initial condition are quite natural: indeed, when
 13 $\mu(x) = \gamma(x)$, then the constant solution $u = 1$ satisfies always the reaction-
 14 diffusion equation and the Neumann boundary condition, hence if u_i is
 15 constant equal to 1, this is the solution of the problem, whatever the function
 16 μ is. This implies that if $u_i = 1$, and when $\mu = \gamma$, there is no hope to
 17 determine μ from some measurements of the solution. Hence, some restric-
 18 tions on the initial condition are natural when one wants to determine the
 19 couple (μ, γ) .
 20

- On the other hand, the definition of the set of the parameters implies that

$$\mu = \tilde{\mu}, \quad \text{and} \quad \gamma = \tilde{\gamma} \quad \text{on a neighborhood of } \partial\Omega,$$

1 and from a mathematical point of view, this is an important assumption:
 2 indeed, it allows one to get rid of what happens near the boundary.

3 These works complete [30] where uniqueness results for μ and γ are obtained
 4 in the one-dimensional setting, using analyticity techniques. More precisely, in
 5 [10], the authors prove the uniqueness of $\mu(x)$ (when $\gamma(x)$ is known) under the
 6 assumption that the initial density $u(0, x)$ is known over the spatial domain (a, b)
 7 and from measurements of the solution u and its spatial derivative u_x at some point
 8 $x_0 \in (a, b)$ and for all $t \in (0, \varepsilon)$.

The result in [10] was later extended to the case of several coefficients : in [11],
 the authors consider the following problem also in the one-dimensional setting

$$\begin{cases} u_t - Du_{xx} = \mu_1(x)u + \mu_2(x)u^2 + \dots + \mu_N(x)u^N & (t, x) \in (0, +\infty) \times (a, b), \\ u(t, a) = 0 = u(t, b) & t \in (0, +\infty), \\ u(0, x) = u^0(x) & x \in \Omega. \end{cases}$$

9 They establish a uniqueness result for the N -uple $(\mu_1(x), \dots, \mu_N(x))$ from mea-
 10 surements of the N solutions u_j and their spatial derivatives $u_{j,x}$ in $(0, \varepsilon) \times \{x_0\}$
 11 starting from N nonintersecting initial condition u_j^0 . These are measurements only
 12 at a point x_0 , during some small interval of time (before explosion occurs, if it has
 13 to occur).

3.2.b. *In the Sellers model (climate dynamics).* The same kind of inverse problem
 question for a nonlinear parabolic equation was in fact already studied in [33],
 more precisely for an energy balance model, that appears in climate dynamics
 when one wants to understand the past and future climate and its sensitivity to
 some relevant parameters on large time scales. There, the goal was to recover the
 so-called 'insolation function' $q(x)$ in the nonlinear parabolic equation

$$u_t - ((1 - x^2)u_x)_x = r(t)q(x)\beta(u) - |u|^3u, \quad x \in (-1, 1)$$

14 where u is the mean temperature, the function β is the co-albedo (the fraction of
 15 the energy received from the sun and absorbed by the Earth, according to the aver-
 16 age temperature), and with suitable boundary conditions (a generalized Neumann
 17 one, associated with the degenerate diffusion operator). The same kind of Lips-
 18 chitz stability estimate was proved, based on a suitable combination of Carleman
 19 estimates and maximum principles. In the present paper, we follow the strategy of
 20 [33], but new difficulties appear: in [33], the coefficient $r(t)\beta(u)$ remains positive
 21 and bounded from below by a positive constant, while this is no more the case
 22 here (because of the Dirichlet boundary condition); to overcome this difficulty and
 23 provide good stability estimates, we need to study the behaviour of the solution
 24 near the boundary.

25 3.3. Some comments related to our results: comparison, extensions and 26 open questions.

27 3.3.a. *Comparison.* Theorems 2.1 and 2.2 complement several results of Cristofol
 28 and Roques:

- 29 • The theoretical results of [10], by proving the Lipschitz stability of the
 30 inverse coefficient problem
 - 31 – directly for the nonlinear system $(P_{\mu, \gamma})$,
 - 32 – eliminating the assumptions of smallness of the initial condition \bar{u}_i and
 33 on the time interval of observation;

34 this is a natural mathematical improvement, but it is also an improvement
 35 with respect to the biological problem, it says that one can consider also

1 an invasion that occurred at some known period, not only at the beginning
2 of the invasion.

3 • Some of the results of [12], by eliminating the assumptions on the imposed
4 values of the parameters μ and γ near the boundary $\partial\Omega$. The cost is then
5 the apparition of some weight in the estimate:

- 6 – $d(x, \partial\Omega)$ in (2. 4),
7 – $d(x, \partial\Omega)^2$ in (2. 5);

8 the apparition of these weights derive from maximum principles that al-
9 low us to analyze the behaviour of u near the boundary (see in particular
10 Lemma 4.3), and is also natural, since the equation gives μu (respectively
11 γu^2). Once again this is a natural mathematical improvement, since it al-
12 lows to get rid of the knowledge of μ (respectively γ) near the boundary,
13 and also an improvement with respect to the biological problem since the
14 goal is the identification of these parameters.

15 3.3.b. *Extensions.* Let us mention that the result of this paper could also be written
16 with a boundary observation on a sub-part $\partial\Omega_0$ of $\partial\Omega$ instead of the observation on
17 ω . In this case, the term $\|u_t - \tilde{u}_t\|_{L^2((t_0, t_1) \times \omega)}^2$ in Theorem 2.1 has to be replaced
18 by $\|\partial_\nu u_t - \partial_\nu \tilde{u}_t\|_{L^2((t_0, t_1) \times \partial\Omega_0)}^2$.

Motivated by [11], one may also consider

$$\begin{cases} u_t - D\Delta u = u(\gamma_1(x) - \gamma_2(x)u - \dots - \gamma_N(x)u^{N-1}) & (t, x) \in (0, +\infty) \times \Omega, \\ u(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_i(x) & x \in \Omega. \end{cases}$$

19 In population dynamics model, it is natural to assume that the coefficients $\gamma_k(x)$
20 are positive and bounded from below. Then the method used to prove Theorems
21 2.1 and 2.2 easily extends to this case and give Lipschitz estimates of $\gamma_j - \tilde{\gamma}_j$
22 (of course with the weight $d(x, \partial\Omega)^j$) knowing the other coefficients, under similar
23 assumptions on the coefficients γ_k .

24 3.3.c. *Open questions.* Some questions are directly related to our results:

- 25 • The result [12] enables to simultaneously recover both coefficient μ and γ
26 (provided that the initial condition lies in the required set). Here we were
27 able to recover the coefficient μ without restriction on the initial condition.
28 It would be interesting to investigate the question of recovering simulta-
29 neously both coefficients, restricting the initial condition to some peculiar
30 subset if necessary.
31 • The regularity $C^{0, \alpha}(\bar{\Omega})$ allowed us to use classical results for parabolic equa-
32 tions, in order to get the regularity that we needed. Maybe this regularity
33 could be weakened.
34 • Besides some other kind of inverse problems may be of interest. For example
35 the inverse problem concerning the reconstruction of the initial condition
36 may be of interest. In the case of the heat equation, it has been studied in
37 [23, 32] and here again it would be interested to see if nonlinear problems
38 like (1. 1) could be handled.

39 4. PRELIMINARY ESTIMATES FOLLOWING FROM MAXIMUM PRINCIPLES

40 4.1. Lower and upper estimates for u .

41 4.1.a. *The classical upper estimate.*

42 **Lemma 4.1.** *Consider $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \overline{u}_i}$. Then*

- 1 • if $\gamma^- = 0$: the corresponding solution u of (1. 1) satisfies

$$(4. 1) \quad \forall t \geq 0, \forall x \in \Omega, \quad 0 \leq u(t, x) \leq \bar{u}_i e^{Mt}.$$

- 2 • if $\gamma^- > 0$: the corresponding solution u of (1. 1) satisfies

$$(4. 2) \quad \forall t \geq 0, \forall x \in \Omega, \quad 0 \leq u(t, x) \leq \max(\bar{u}_i, \frac{M}{\gamma^-}).$$

3 *Proof of Lemma 4.1.* The estimates (4. 1) and (4. 2) follow directly from the weak
4 maximum principle ([26], Theorem 12, p. 187). \square

5 In the following, we will denote

$$(4. 3) \quad M_{0,0}^\infty := \begin{cases} \bar{u}_i e^{Mt_1} & \text{if } \gamma^- = 0, \\ \max(\bar{u}_i, \frac{M}{\gamma^-}) & \text{if } \gamma^- > 0, \end{cases}$$

6 and then

$$(4. 4) \quad \forall t \in [t_0, t_1], \forall x \in \Omega, \quad 0 \leq u(t, x) \leq M_{0,0}^\infty.$$

7 4.1.b. *A local in time - local in space lower estimate.*

8 **Lemma 4.2.** *Take Ω_1 a compact subset of Ω . Then there is some positive uniform*
9 *constant $m_{0,0}^{loc-loc}$ depending only on $D, M, \gamma^+, \underline{u}_i, B_\varepsilon, t_0, t_1$ and Ω_1 such that, for*
10 *all $\mu \in \mathcal{M}_{\alpha, M}, \gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}, u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$, the corresponding solution u of (1. 1)*
11 *satisfies*

$$(4. 5) \quad \forall t \in [t_0, t_1], \forall x \in \Omega_1, \quad u(t, x) \geq m_{0,0}^{loc-loc}.$$

12 *Proof of Lemma 4.2.* Consider the solution \underline{u} of

$$(4. 6) \quad \begin{cases} \underline{u}_t - D\Delta \underline{u} = \underline{u}(-M - \gamma^+ \underline{u}) & (t, x) \in (0, +\infty) \times \Omega, \\ \underline{u}(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ \underline{u}(0, x) = \underline{u}_{i,\varepsilon}(x) & x \in \Omega, \end{cases}$$

where $\underline{u}_{i,\varepsilon} : \bar{\Omega} \rightarrow \mathbb{R}$ is nonnegative, continuous on $\bar{\Omega}$, compactly supported in B_ε ,
and

$$\underline{u}_{i,\varepsilon}(x) = \underline{u}_i \quad \text{on } B_{\varepsilon/2}.$$

13 First, classical parabolic regularity results and comparison principles (see Pao [25],
14 Theorem 5.2 page 66)) imply that

$$(4. 7) \quad \underline{u} \in C^0([0, +\infty) \times \bar{\Omega}),$$

and the weak maximum principle implies that \underline{u} is nonnegative. Moreover the
difference $u - \underline{u}$ satisfies

$$(u - \underline{u})_t - D\Delta(u - \underline{u}) = u(\mu + M) - M(u - \underline{u}) - \gamma(u^2 - \underline{u}^2) + (\gamma^+ - \gamma)\underline{u}^2,$$

hence

$$\begin{aligned} (u - \underline{u})_t - D\Delta(u - \underline{u}) + M(u - \underline{u}) + \gamma(u + \underline{u})(u - \underline{u}) \\ = u(\mu + M) + (\gamma^+ - \gamma)\underline{u}^2. \end{aligned}$$

Therefore

$$\begin{cases} (u - \underline{u})_t - D\Delta(u - \underline{u}) + M(u - \underline{u}) + \gamma(u + \underline{u})(u - \underline{u}) \geq 0, \\ (u - \underline{u}) \geq 0 \text{ on } \partial\Omega, \\ (u - \underline{u})(t = 0) \geq 0 \text{ on } \Omega. \end{cases}$$

Then the weak maximum principle implies that $u \geq \underline{u}$ for all $t > 0$. Now consider
the parabolic operator

$$Pv := v_t - D\Delta v + (M + \gamma^+ \underline{u})v.$$

\underline{u} is a nonnegative solution of the parabolic operator P (since $P\underline{u} = 0$). Then the strong maximum principle ([25] Lemma 2.1 p. 54) implies that

$$\forall t > 0, \forall x \in \Omega_1, \quad \underline{u}(t, x) > 0.$$

1 Since \underline{u} is continuous on $[0, +\infty) \times \overline{\Omega}$, \underline{u} is bounded from below on the compact set
 2 $[t_0, t_1] \times \overline{\Omega}_1$ by a positive constant m_0 . Finally, since $u \geq \underline{u}$ on that set, this implies
 3 that (4. 5) holds. \square

4 4.1.c. *A local in time - global in space lower estimate.*

5 We prove the following

6 **Lemma 4.3.** *There is some positive uniform constant $m_{0,0}^{loc-glob}$ depending only*
 7 *on $D, \alpha, M, \gamma^+, \underline{u}_i, B_\varepsilon, t_0, t_1$ such that, for all $\mu \in \mathcal{M}_{\alpha, M}, \gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and*
 8 *$u_i \in \mathcal{D}_{\underline{u}_i, \overline{u}_i}$, the corresponding solution u of (1. 1) satisfies*

$$(4. 8) \quad \forall t \in [t_0, t_1], \forall x \in \Omega, \quad u(t, x) \geq m_{0,0}^{loc-glob} d(x, \partial\Omega).$$

Proof of Lemma 4.3. Once again the starting point is that

$$\forall t > 0, \quad u \geq \underline{u},$$

9 where \underline{u} is defined in (4. 6). We also recall that $\underline{u} > 0$ in $(0, +\infty) \times \Omega$. Then the
 10 Hopf's lemma for parabolic equations ([26] Theorem 7 p. 174) implies that

$$(4. 9) \quad \forall t > 0, \forall x \in \partial\Omega, \quad \partial_\nu \underline{u}(t, x) < 0.$$

11 We are going to use the following consequence of the regularizing effect of the
 12 heat equation:

13 **Lemma 4.4.** *The function \underline{u} satisfies*

$$(4. 10) \quad \underline{u} \in C^0((0, +\infty); C^1(\overline{\Omega})).$$

Proof of Lemma 4.4. (4. 10) follows from classical regularity results. First we decompose \underline{u} in the following way:

$$\underline{u} = \underline{U} + \underline{V},$$

14 where \underline{U} is the solution of

$$(4. 11) \quad \begin{cases} \underline{U}_t - D\Delta \underline{U} = \underline{u}(-M - \gamma^+ \underline{u}) & (t, x) \in (0, +\infty) \times \Omega, \\ \underline{U}(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ \underline{u}(0, x) = 0 & x \in \Omega, \end{cases}$$

15 and \underline{V} is the solution of

$$(4. 12) \quad \begin{cases} \underline{V}_t - D\Delta \underline{V} = 0 & (t, x) \in (0, +\infty) \times \Omega, \\ \underline{V}(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ \underline{V}(0, x) = \underline{u}_{i,\varepsilon}(x) & x \in \Omega. \end{cases}$$

Then by classical regularizing effects of the heat equation (Brezis [5], Theorem X.1), we already know that

$$\underline{V} \in C^\infty((0, +\infty) \times \overline{\Omega}).$$

Concerning \underline{U} : using (4. 7), we see that the function $f : [0, +\infty) \times \overline{\Omega}$ defined by

$$\underline{f}(t, x) := \underline{u}(t, x)(-M - \gamma^+ \underline{u}(t, x))$$

satisfies

$$\underline{f} \in C^0([0, +\infty) \times \overline{\Omega}),$$

and

$$\underline{f}(0, x) = 0 \quad \text{on } \partial\Omega.$$

1 Then we are in position to apply classical regularity results for linear parabolic
 2 equations (see [15], used also in [12] (Theorem 3.4)). We denote $C_{j,\delta'}^{i,\delta}([0, T] \times \bar{\Omega})$
 3 the space of functions on $[0, T] \times \bar{\Omega}$ whose derivatives up to the order i in x and the
 4 order j in t are Hölder continuous with orders δ and δ' respectively, and we recall
 5 the following

6 **Theorem 4.1.** ([15], Theorem 4 page 191) Assume that f is continuous on $[0, T] \times$
 7 $\bar{\Omega}$ with $f(0, \cdot) = 0$ on $\partial\Omega$. Then for any $\alpha \in (0, 1)$, there exists a constant C_α ,
 8 independent of f , such that any solution of

$$(4.13) \quad \begin{cases} y_t - D\Delta y = f(t, x) & (t, x) \in (0, T) \times \Omega, \\ y(t, x) = 0 & (t, x) \in (0, T) \times \partial\Omega, \\ y(0, x) = 0 & x \in \Omega, \end{cases}$$

9 satisfies

$$(4.14) \quad \|y\|_{C_{0,\alpha/2}^{1,\alpha}([0,T] \times \bar{\Omega})} \leq C_\alpha \sup_{[0,T] \times \bar{\Omega}} |f|.$$

Here, we choose any $\alpha \in (0, 1)$, and we apply Theorem 4.1 with $f = \underline{f}$, and we obtain that

$$\forall T > 0, \quad \underline{U} \in C_{0,\alpha/2}^{1,\alpha}([0, T] \times \bar{\Omega}).$$

Combining with the regularity of \underline{V} , we obtain that for all $T > 0$

$$\underline{u} = \underline{U} + \underline{V} \in C_{0,\alpha/2}^{1,\alpha}((0, T] \times \bar{\Omega}),$$

10 which implies (4. 10). □

11 *End of the proof of Lemma 4.3.* First we derive from Lemma 4.4 that the function
 12 $-\partial_\nu \underline{u}$ is continuous on the compact set $[t_0, t_1] \times \partial\Omega$. Then we derive from (4. 9)
 13 that $-\partial_\nu \underline{u}$ is bounded from below by a positive constant on $[t_0, t_1] \times \partial\Omega$.

Now, given $\eta > 0$, denote

$$\Omega_\eta := \{x \in \Omega, d(x, \partial\Omega) < \eta\},$$

and

$$p_{\partial\Omega} : \bar{\Omega}_\eta \rightarrow \partial\Omega$$

the projection onto the boundary: given $x \in \bar{\Omega}_\eta$, $p_{\partial\Omega}(x)$ is the unique point of $\partial\Omega$ that satisfies

$$\|x - p_{\partial\Omega}(x)\| = \inf_{y \in \partial\Omega} \|x - y\|.$$

Since Ω is sufficiently smooth, the projection $p_{\partial\Omega}$ is well-defined and continuous provided η is small enough. Then, using (4. 10), we have that

$$\nabla \underline{u} \in C^0((0, +\infty), C^0(\bar{\Omega})),$$

hence the function

$$(t, x) \in (0, +\infty) \times \bar{\Omega}_\eta \mapsto -\nabla \underline{u}(t, x) \cdot \nu(p_{\partial\Omega}(x))$$

is continuous and takes positive and bounded from below values on $[t_0, t_1] \times \partial\Omega$. Finally, using that $\underline{u} = 0$ on $[t_0, t_1] \times \partial\Omega$, we obtain that there exists $m_0 > 0$ such that

$$(t, x) \in [t_0, t_1] \times \bar{\Omega}_\eta \implies \underline{u}(t, x) \geq m_0 d(x, \partial\Omega).$$

We conclude with the positivity of \underline{u} on the compact subset $[t_0, t_1] \times (\Omega \setminus \Omega_\eta)$ (Lemma 4.2) that there is some $m_{0,0}^{loc-glob} > 0$ such that

$$\forall (t, x) \in [t_0, t_1] \times \Omega, \quad \underline{u}(t, x) \geq m_{0,0}^{loc-glob} d(x, \partial\Omega).$$

14 Since $u \geq \underline{u}$, we obtain (4. 8). This concludes the proof of Lemma 4.3. □

1 4.1.d. *The corresponding local in time - global in space upper estimate.*

2 We prove the following

3 **Lemma 4.5.** *There is some positive uniform constant $\tilde{M}_{0,0}^{loc-glob}$ depending only*
 4 *on D , α , M , γ^- , γ^+ , \underline{u}_i , \overline{u}_i , B_ε , t_0 , and t_1 such that, for all $\mu \in \mathcal{M}_{\alpha,M}$, $\gamma \in$*
 5 *$\mathcal{C}_{\alpha,M,\gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \overline{u}_i}$, the corresponding solution u of (1. 1) satisfies*

$$(4. 15) \quad \forall t \in [t_0, t_1], \forall x \in \Omega, \quad 0 \leq u(t, x) \leq \tilde{M}_{0,0}^{loc-glob} d(x, \partial\Omega).$$

Proof of Lemma 4.5. We proceed as in the previous section. First we decompose u as follows:

$$u = U + V,$$

6 where U is the solution of

$$(4. 16) \quad \begin{cases} U_t - D\Delta U = u(\mu(x) - \gamma(x)u) & (t, x) \in (0, +\infty) \times \Omega, \\ U(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ U(0, x) = 0 & x \in \Omega, \end{cases}$$

7 and V is the solution of

$$(4. 17) \quad \begin{cases} V_t - D\Delta V = 0 & (t, x) \in (0, +\infty) \times \Omega, \\ V(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ V(0, x) = u_i(x) & x \in \Omega. \end{cases}$$

First, concerning the problem (4. 16): we recall (Pao [25], Theorem 5.2, page 66) that the solution u of $(P_{\mu,\gamma})$ satisfies

$$\forall T > 0, \quad u \in C^0([0, T] \times \overline{\Omega}).$$

8 Then we introduce

$$(4. 18) \quad F(t, x) := u(t, x)(\mu(x) - \gamma(x)u(t, x)).$$

9 Since F is continuous on $[0, T] \times \overline{\Omega}$ with $F(0, \cdot) = 0$ on $\partial\Omega$, we are in position to
 10 apply Theorem 4.1, and we obtain that $U \in C_{0,\alpha/2}^{1,\alpha}([0, T] \times \overline{\Omega})$ and that

$$(4. 19) \quad \|U\|_{C_{0,\alpha/2}^{1,\alpha}([0,T] \times \overline{\Omega})} \leq C_0 \sup_{[0,T] \times \overline{\Omega}} |F|.$$

11 The last quantity is uniformly bounded with respect to $\mu \in \mathcal{M}_{\alpha,M}$, $\gamma \in \mathcal{C}_{\alpha,M,\gamma^-, \gamma^+}$
 12 and $u_i \in \mathcal{D}_{\underline{u}_i, \overline{u}_i}$.

Now we turn to problem (4. 17). Once again (Brezis [5], Theorem X.1), we already know that

$$V \in C^\infty((0, +\infty) \times \overline{\Omega}).$$

13 Moreover we can prove that there exists $C(D, t_0, \underline{u}_i, \overline{u}_i)$ such that, for all $u_i \in \mathcal{D}_{\underline{u}_i, \overline{u}_i}$,
 14 we have

$$(4. 20) \quad \forall t \in [\frac{t_0}{2}, T], \quad \|V(t)\|_{C^3(\overline{\Omega})} + \|V_t(t)\|_{C^1(\overline{\Omega})} \leq C(D, t_0, \underline{u}_i, \overline{u}_i),$$

15 see the proof in Appendix (section 7). Then (4. 20) implies that

$$(4. 21) \quad \|V\|_{C_{0,\alpha/2}^{1,\alpha}([0,T] \times \overline{\Omega})} \leq C(D, t_0, \underline{u}_i, \overline{u}_i).$$

Finally, since $u = U + V$, we also obtain that for all $T > t_1$, we have

$$u \in C_{0,\alpha/2}^{1,\alpha}([\frac{t_0}{2}, T] \times \overline{\Omega}),$$

with

$$\|u\|_{C_{0,\alpha/2}^{1,\alpha}([\frac{t_0}{2}, T] \times \overline{\Omega})}$$

uniformly bounded with respect to $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$. Hence there is C independent of $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$ such that

$$\|u\|_{C^0([\frac{t_0}{2}, T]; C^1(\bar{\Omega}))} \leq C,$$

1 and adding that $u = 0$ on $\partial\Omega$, we obtain (4. 15). \square

2 4.2. A local in time - global in space upper estimate for u_t .

3 We prove the following regularity result:

4 **Lemma 4.6.** *There is some positive uniform constant $M_{1,0}^{loc, glob}$ depending only*
 5 *on D , α , M , γ^- , γ^+ , \underline{u}_i , \bar{u}_i , B_ε , t_0 , and t_1 such that, for all $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in$*
 6 *$\mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$, the corresponding solution u of (1. 1) satisfies*

$$(4. 22) \quad \forall t \in [t_0, t_1], \forall x \in \Omega, \quad |u_t(t, x)| \leq M_{1,0}^{loc, glob} d(x, \partial\Omega).$$

Proof of Lemma 4.6. The starting point is the fact, proved previously in Lemma 4.5, that

$$u \in C_{0, \alpha/2}^{1, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega}),$$

with

$$\|u\|_{C_{0, \alpha/2}^{1, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})}$$

7 uniformly bounded with respect to $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$.

8 The proof follows in two steps:

Step 1: a uniform bound on u_t . Here we take $\frac{t_0}{2}$ as new initial time, decomposing

$$u = \tilde{U} + \tilde{V},$$

9 where \tilde{U} is the solution of

$$(4. 23) \quad \begin{cases} \tilde{U}_t - D\Delta\tilde{U} = u(\mu(x) - \gamma(x)u) = F(t, x) & (t, x) \in (\frac{t_0}{2}, +\infty) \times \Omega, \\ \tilde{U}(t, x) = 0 & (t, x) \in (\frac{t_0}{2}, +\infty) \times \partial\Omega, \\ \tilde{U}(\frac{t_0}{2}, x) = 0 & x \in \Omega, \end{cases}$$

10 and \tilde{V} is the solution of

$$(4. 24) \quad \begin{cases} \tilde{V}_t - D\Delta\tilde{V} = 0 & (t, x) \in (\frac{t_0}{2}, +\infty) \times \Omega, \\ \tilde{V}(t, x) = 0 & (t, x) \in (\frac{t_0}{2}, +\infty) \times \partial\Omega, \\ \tilde{V}(\frac{t_0}{2}, x) = u(\frac{t_0}{2}, x) & x \in \Omega. \end{cases}$$

Now we recall from (2. 1) and (2. 2) that there is some $\alpha \in (0, 1)$ such that

$$\mu, \gamma \in C^{0, \alpha}(\bar{\Omega}),$$

and this implies that F defined by (4. 18) satisfies

$$F \in C_{0, \alpha/2}^{0, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega}).$$

11 Then we are in position to apply this other classical regularity result ([15], used
 12 also in [12], Theorem 3.5):

Theorem 4.2. ([15], Theorems 7 and 6 (in this order), page 65) *Assume that $g \in C_{0, \alpha/2}^{0, \alpha}([0, T] \times \bar{\Omega})$ and $h \in C_{1, \alpha/2}^{2, \alpha}([0, T] \times \bar{\Omega})$ for some $\alpha \in (0, 1)$, with the compatibility condition*

$$h_t(0, x) - D\Delta h(0, x) = g(0, x) \quad \text{for } x \in \partial\Omega.$$

1 Then there exists a constant C_1 , independent of g and h , such that the problem

$$(4. 25) \quad \begin{cases} z_t - D\Delta z = g(t, x) & (t, x) \in (0, T) \times \Omega, \\ z(t, x) = h(t, x) & (t, x) \in (0, T) \times \partial\Omega, \\ z(0, x) = h(0, x) & x \in \Omega, \end{cases}$$

2 has a unique solution $z \in C_{1, \alpha/2}^{2, \alpha}([0, T] \times \bar{\Omega})$ which satisfies

$$(4. 26) \quad \|z\|_{C_{1, \alpha/2}^{2, \alpha}([0, T] \times \bar{\Omega})} \leq C_1(\|g\|_{C_{0, \alpha/2}^{0, \alpha}([0, T] \times \bar{\Omega})} + \|h\|_{C_{1, \alpha/2}^{2, \alpha}([0, T] \times \bar{\Omega})})$$

We apply Theorem 4.2 with $g = F$ and $h = 0$, on the time interval $[\frac{t_0}{2}, T]$: then we obtain that

$$\tilde{U} \in C_{1, \alpha/2}^{2, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})$$

and that

$$\|\tilde{U}\|_{C_{1, \alpha/2}^{2, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})} \leq C_1 \|F\|_{C_{0, \alpha/2}^{0, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})}.$$

It is easy to see that

$$\begin{aligned} \|F\|_{C_{0, \alpha/2}^{0, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})} &= \|u(\mu - \gamma u)\|_{C_{0, \alpha/2}^{0, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})} \\ &\leq c\|\mu\|_{C^{0, \alpha}(\bar{\Omega})} \|u\|_{C_{0, \alpha/2}^{0, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})} + c\|\gamma\|_{C^{0, \alpha}(\bar{\Omega})} \|u\|_{C_{0, \alpha/2}^{0, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})}^2, \end{aligned}$$

hence

$$\|F\|_{C_{0, \alpha/2}^{0, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})}$$

is uniformly bounded with respect to $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$, and the same holds for

$$\|\tilde{U}\|_{C_{1, \alpha/2}^{2, \alpha}([\frac{t_0}{2}, T] \times \bar{\Omega})}.$$

On the other hand, since $\tilde{V} \in C^\infty((\frac{t_0}{2}, +\infty) \times \bar{\Omega})$, we have that

$$\tilde{V} \in C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})$$

and

$$\|\tilde{V}\|_{C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})}$$

3 is uniformly bounded with respect to $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$ (and of course also to $\mu \in \mathcal{M}_{\alpha, M}$,
4 $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$), as a consequence of the classical regularity estimates recalled in
5 section 7 (and as in (4. 20), after a shift of $\frac{t_0}{2}$ in time).

6 And thus $u \in C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})$, with a uniform bound with respect to $\mu \in \mathcal{M}_{\alpha, M}$,
7 $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$. This implies that there is some uniform $C_{1, 0}^{loc, glob}$
8 such that

$$(4. 27) \quad \|u_t\|_{C^0([t_0, T] \times \bar{\Omega})} \leq C_{1, 0}^{loc, glob}.$$

9 Step 2: a uniform bound on ∇u_t . We are going to prove an analogous bound on
10 ∇u_t : there is some uniform $C_{1, 1}^{loc, glob}$ such that

$$(4. 28) \quad \|\nabla u_t\|_{C^0([t_0, T] \times \bar{\Omega})} \leq C_{1, 1}^{loc, glob}.$$

11 Then the mean value theorem will imply (4. 22).

12 To prove (4. 28), we introduce $w := u_t$, that is solution of

$$(4. 29) \quad \begin{cases} w_t - D\Delta w = \mu(x)u_t - 2\gamma(x)uu_t & (t, x) \in (\frac{t_0}{2}, +\infty) \times \Omega, \\ w(t, x) = 0 & (t, x) \in (\frac{t_0}{2}, +\infty) \times \partial\Omega, \\ w(\frac{t_0}{2}, x) = u_t(\frac{t_0}{2}, x) & x \in \Omega. \end{cases}$$

13 Here once again we take $\frac{t_0}{2}$ as initial time, and we decompose w as follows:

$$(4. 30) \quad w = \bar{U} + \bar{V},$$

1 where

$$(4.31) \quad \begin{cases} \bar{U}_t - D\Delta\bar{U} = \mu(x)u_t - 2\gamma(x)uu_t & (t, x) \in (\frac{t_0}{2}, +\infty) \times \Omega, \\ \bar{U}(t, x) = 0 & (t, x) \in (\frac{t_0}{2}, +\infty) \times \partial\Omega, \\ \bar{U}(\frac{t_0}{2}, x) = 0 & x \in \Omega, \end{cases}$$

2 and

$$(4.32) \quad \begin{cases} \bar{V}_t - D\Delta\bar{V} = 0 & (t, x) \in (\frac{t_0}{2}, +\infty) \times \Omega, \\ \bar{V}(t, x) = 0 & (t, x) \in (\frac{t_0}{2}, +\infty) \times \partial\Omega, \\ \bar{V}(\frac{t_0}{2}, x) = u_t(\frac{t_0}{2}, x) & x \in \Omega. \end{cases}$$

3 Concerning \bar{V} , we proceed as in section 7: we derive similarly to (7.6) and (7.7)
4 that there exists $C(D, t_0)$ such that

$$(4.33) \quad \forall k \in \{0, 1, \dots, K\}, \forall t \geq \frac{3t_0}{4}, \quad \|\frac{\partial^k \bar{V}}{\partial t^k}(t)\|_{L^2(\Omega)} \leq C(D, t_0) \|u_t(\frac{t_0}{2})\|_{L^2(\Omega)},$$

5 and

$$(4.34) \quad \forall k \in \{0, 1, \dots, K\}, \forall t \geq \frac{3t_0}{4}, \quad \|\Delta^k \bar{V}(t)\|_{L^2(\Omega)} \leq C(D, t_0) \|u_t(\frac{t_0}{2})\|_{L^2(\Omega)}.$$

Since $u_t(\frac{t_0}{2})$ is also uniformly bounded with respect to $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$
and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$, we obtain that

$$\bar{V} \in C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})$$

6 and there is some uniform constant such that

$$(4.35) \quad \|\bar{V}\|_{C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})} \leq C(D, t_0, \underline{u}_i, \bar{u}_i).$$

7 In particular $\nabla \bar{V} \in C^0([t_0, T] \times \bar{\Omega})$ and is uniformly bounded with respect to
8 $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{\underline{u}_i, \bar{u}_i}$.

9 Concerning \bar{U} , we consider the right hand side of (4.31):

$$(4.36) \quad \bar{F}(t, x) := \mu(x)u_t(t, x) - 2\gamma(x)u(t, x)u_t(t, x),$$

and we check that $\bar{F} \in C_{0, \alpha/2}^{0, \alpha}([t_0, T] \times \bar{\Omega})$ and that there exists a universal $C > 0$
such that

$$(4.37) \quad \|\bar{F}\|_{C_{0, \alpha/2}^{0, \alpha}([t_0, T] \times \bar{\Omega})} \leq C \left(\|\mu\|_{C^{0, \alpha}(\bar{\Omega})} \|u\|_{C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})} \right. \\ \left. + \|\gamma\|_{C^{0, \alpha}(\bar{\Omega})} \|u\|_{C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})}^2 \right).$$

Indeed, first

$$\forall t \in [t_0, T], \forall x \in \bar{\Omega}, \quad |\mu(x)u_t(t, x)| \leq \|\mu\|_{L^\infty(\bar{\Omega})} \|u_t\|_{L^\infty([t_0, T] \times \bar{\Omega})},$$

and given $t \in [t_0, T]$, $x \in \bar{\Omega}$, choosing $x' \in \partial\Omega$, we have $u_t(t, x') = 0$, hence

$$|u_t(t, x)| = |u_t(t, x) - u_t(t, x')| \leq \|u\|_{C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})} |x - x'|^\alpha,$$

hence

$$\|\mu u_t\|_{L^\infty([t_0, T] \times \bar{\Omega})} \leq C \|\mu\|_{L^\infty(\bar{\Omega})} \|u\|_{C_{1, \alpha/2}^{2, \alpha}([t_0, T] \times \bar{\Omega})}.$$

Next, given $x, x' \in \bar{\Omega}$ and $t, t' \in [t_0, T]$, we have

$$\begin{aligned} & |\mu(x)u_t(t, x) - \mu(x')u_t(t', x')| \\ &= |(\mu(x) - \mu(x'))u_t(t, x) + \mu(x')(u_t(t, x) - u_t(t', x'))| \\ &\leq \|\mu\|_{C^{0,\alpha}(\bar{\Omega})}|x-x'|^\alpha \|u_t\|_{L^\infty([t_0, T] \times \bar{\Omega})} + \|\mu\|_{L^\infty(\bar{\Omega})} \|u\|_{C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})} \left(|x-x'|^2 + |t-t'|\right)^{\alpha/2} \\ &\leq C \left(|x-x'|^2 + |t-t'|\right)^{\alpha/2} \|\mu\|_{C^{0,\alpha}(\bar{\Omega})} \|u\|_{C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})}. \end{aligned}$$

Hence $\mu u_t \in C_{0,\alpha/2}^{0,\alpha}([t_0, T] \times \bar{\Omega})$, and

$$\|\mu u_t\|_{C_{0,\alpha/2}^{0,\alpha}([t_0, T] \times \bar{\Omega})} \leq C \|\mu\|_{C^{0,\alpha}(\bar{\Omega})} \|u\|_{C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})}.$$

1 We proceed in the same way for $\gamma u u_t$, and we obtain (4. 37).

Then we are in position to apply once again Theorem 4.2, and we obtain that $\bar{U} \in C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})$, and satisfies

$$\|\bar{U}\|_{C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})} \leq C \|\bar{F}\|_{C_{0,\alpha/2}^{0,\alpha}([t_0, T] \times \bar{\Omega})},$$

hence

$$(4. 38) \quad \|\bar{U}\|_{C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})} \leq C' \left(\|\mu\|_{C^{0,\alpha}(\bar{\Omega})} \|u\|_{C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})} + \|\gamma\|_{C^{0,\alpha}(\bar{\Omega})} \|u\|_{C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})}^2 \right).$$

With (4. 35) and (4. 38), we obtain that $u_t = w = \bar{U} + \bar{V}$ belongs to $C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})$ and is uniformly bounded with respect to $\mu \in \mathcal{M}_{\alpha, M}$, $\gamma \in \mathcal{C}_{\alpha, M, \gamma^-, \gamma^+}$ and $u_i \in \mathcal{D}_{u_i, \bar{u}_i}$. Hence ∇w is continuous on $[t_0, T] \times \bar{\Omega}$, and hence bounded on this compact set. Moreover, since

$$\|\nabla u_t\|_{C^0([t_0, T] \times \bar{\Omega})} \leq C \|u_t\|_{C_{1,\alpha/2}^{2,\alpha}([t_0, T] \times \bar{\Omega})},$$

2 we obtain (4. 28). Applying the mean value theorem and the fact that $u_t = 0$ on
3 $\partial\Omega$, we obtain (4. 22). This concludes the proof of Lemma 4.6 . \square

4 5. LIPSCHITZ STABILITY : PROOF OF THEOREM 2.1

5 **Step 1: reduction to some linear inverse problem.** We consider u the solution
6 of $(P_{\mu, \gamma})$ and \tilde{u} solution of $(P_{\tilde{\mu}, \gamma})$. Then we define $w := u - \tilde{u}$ and compute

$$(5. 1) \quad w_t = u_t - \tilde{u}_t = D\Delta w + h_1 + h_2 + h_3,$$

where

$$h_1 := (\mu - \tilde{\mu})u, \quad h_2 := \tilde{\mu}(u - \tilde{u}), \quad h_3 := -\gamma(u^2 - \tilde{u}^2).$$

7 In the following, our aim is to estimate $h_1(T')$ in the problem

$$(5. 2) \quad \begin{cases} w_t - D\Delta w = h_1 + h_2 + h_3 & (t, x) \in (0, +\infty) \times \Omega, \\ w(t, x) = 0 & (t, x) \in (0, +\infty) \times \partial\Omega, \\ w(0, x) = 0 & x \in \Omega. \end{cases}$$

8 Roughly speaking, this first step reduces our nonlinear inverse problem into some
9 non standard linear inverse source problem, namely the determination of the part
10 h_1 of the source term $h_1 + h_2 + h_3$. (And of course the bound from below for u
11 obtained in Lemma 4.3 will help us estimate $\mu - \tilde{\mu}$.)

12 **Step 2: condition satisfied by h_1 .** We recall that in the standard inverse source
13 problem (studied in [19]), that consists in recovering h in (3. 1), the source term h

1 is assumed to satisfy some condition like the following one (otherwise uniqueness
2 may be false) :

$$(5.3) \quad \left| \frac{\partial h}{\partial t}(t, x) \right| \leq C_0 |h(T', x)| \text{ for } (t, x) \in (0, T) \times \Omega$$

3 for some constant $C_0 > 0$.

4 In the present step, we prove that the part h_1 (which is the part that we wish
5 to identify) of the source term $h_1 + h_2 + h_3$ satisfies a similar condition:

6 **Lemma 5.1.** *The quantity $h_1 = (\mu - \tilde{\mu})u$ satisfies*

$$(5.4) \quad \left| \frac{\partial h_1}{\partial t}(t, x) \right| \leq \frac{M_{1,0}^{loc, glob}}{m_{0,0}^{loc-glob}} |h_1(T', x)| \text{ for } (t, x) \in (t_0, t_1) \times \Omega$$

7 where $M_{1,0}^\infty$ and $m_{0,0}^{loc-glob}$ are given by Lemmas 4.3 and 4.6.

Proof of Lemma 5.1. First, using Lemma 4.6, we have

$$\forall t \in [t_0, t_1], \forall x \in \Omega, \quad \left| \frac{\partial h_1}{\partial t}(t, x) \right| = |\mu(x) - \tilde{\mu}(x)| |u_t(t, x)| \leq M_{1,0}^{loc, glob} d(x, \partial\Omega) |\mu(x) - \tilde{\mu}(x)|.$$

Next, using Lemma 4.3, we have

$$\forall x \in \Omega, \quad |h_1(T', x)| = |\mu(x) - \tilde{\mu}(x)| |u(T', x)| \geq m_{0,0}^{loc-glob} d(x, \partial\Omega) |\mu(x) - \tilde{\mu}(x)|.$$

Hence

$$\forall t \in [t_0, t_1], \forall x \in \Omega, \quad \left| \frac{\partial h_1}{\partial t}(t, x) \right| \leq \frac{M_{1,0}^{loc, glob}}{m_{0,0}^{loc-glob}} |h_1(T', x)|. \quad \square$$

8 **Recall:** Before going further in the reasoning, we recall some global Carleman
9 estimate with locally distributed observation in ω . First we recall the weights that
10 will appear in this Carleman estimate:

- the weight in the space variable: let ω_0 be an arbitrary fixed sub-domain of Ω such that $\overline{\omega_0} \subset \omega$; then using [16] or [19, Lemma 2.1], there exists a function $\Psi \in \mathcal{C}^2(\overline{\Omega})$ such that

$$\forall x \in \Omega, \quad \Psi(x) > 0, \quad \Psi|_{\partial\Omega} = 0 \quad \text{and} \quad \forall x \in \overline{\Omega} \setminus \omega_0, \quad |\nabla \Psi(x)| > 0;$$

- the weight in the time variable: it is practical to assume that

$$T' = \frac{t_0 + t_1}{2},$$

and then we will consider as usually

$$\forall t \in (t_0, t_1), \quad \ell(t) := (t - t_0)(t_1 - t)$$

and, for $\lambda > 0$,

$$\forall t \in (t_0, t_1), \forall x \in \Omega, \quad \varphi(t, x) := \frac{e^{\lambda \Psi(x)}}{(t - t_0)(t_1 - t)} = \frac{e^{\lambda \Psi(x)}}{\ell(t)},$$

$$\forall t \in (t_0, t_1), \forall x \in \Omega, \quad \eta(t, x) := \frac{e^{2\lambda \|\Psi\|_{L^\infty(\Omega)}} - e^{\lambda \Psi(x)}}{(t - t_0)(t_1 - t)} = \frac{e^{2\lambda \|\Psi\|_{L^\infty(\Omega)}} - e^{\lambda \Psi(x)}}{\ell(t)}.$$

(In order to generalize to the case $T' \neq \frac{t_0 + t_1}{2}$, one has to choose $\kappa \in (0, 1)$ such that

$$t \in (t_0, t_1) \mapsto (t - t_0)^\kappa (t_1 - t)^{1-\kappa}$$

achieves its maximum at T' , and then

$$\ell(t) = \left((t - t_0)^\kappa (t_1 - t)^{1-\kappa} \right)^{\kappa'},$$

with κ' such that

$$\kappa' \geq \frac{1}{1 - \kappa},$$

which implies that

$$\forall t \in (t_0, t_1), \quad \left(\frac{1}{\ell(t)} \right)' \leq C \left(\frac{1}{\ell(t)} \right)^2.$$

1 When $T = \frac{t_0+t_1}{2}$, $\kappa = \frac{1}{2}$ and $\kappa' = 2$ are suitable choices.)

We also define

$$Q_{t_0, t_1} := (t_0, t_1) \times \Omega \quad \text{and} \quad \omega_{t_0, t_1} := (t_0, t_1) \times \omega.$$

2 Let q be a solution of the parabolic problem

$$(5.5) \quad \begin{cases} q_t - D\Delta q = h(x, t) & (t, x) \in (t_0, t_1) \times \Omega, \\ q(t, x) = 0 & (t, x) \in (t_0, t_1) \times \partial\Omega, \end{cases}$$

for some $h \in L^2(Q_{t_0, t_1})$. Then there exists a number $\hat{\lambda} > 0$ such that for any $\lambda \geq \hat{\lambda}$, we can choose $s_0(\lambda) > 0$ satisfying : there exists a constant $C_0 > 0$ such that for all $s \geq s_0(\lambda)$, the next inequality holds (see [16] or [19, Lemma 2.2]):

$$(5.6) \quad \begin{aligned} & \iint_{Q_{t_0, t_1}} \frac{1}{s\varphi} (|q_t|^2 + |\Delta q|^2) e^{-2s\eta} \\ & + \iint_{Q_{t_0, t_1}} s\varphi |\nabla q|^2 e^{-2s\eta} + \iint_{Q_{t_0, t_1}} s^3 \varphi^3 q^2 e^{-2s\eta} \\ & \leq C_0 \left[\iint_{Q_{t_0, t_1}} h^2 e^{-2s\eta} + \iint_{\omega_{t_0, t_1}} s^3 \varphi^3 q^2 e^{-2s\eta} \right]. \end{aligned}$$

3 **Step 3: application of global Carleman estimates and link with some**
 4 **more standard inverse source problem.** Next we introduce $z := w_t = u_t - \tilde{u}_t$.
 5 Since w is solution of (5. 2), z satisfies

$$(5.7) \quad \begin{cases} z_t - D\Delta z = h_{1,t} + h_{2,t} + h_{3,t} & (t, x) \in (t_0, t_1) \times \Omega = Q_{t_0, t_1}, \\ z(t, x) = 0 & (t, x) \in (t_0, t_1) \times \partial\Omega. \end{cases}$$

Applying (5. 6) to problem (5. 7), we get in particular: for all $\lambda \geq \hat{\lambda}$ and $s \geq s_0(\lambda)$,

$$(5.8) \quad \begin{aligned} & \iint_{Q_{t_0, t_1}} \left(s^3 \varphi^3 z^2 + \frac{1}{s\varphi} z_t^2 \right) e^{-2s\eta} \\ & \leq C_0 \iint_{Q_{t_0, t_1}} (h_{1,t} + h_{2,t} + h_{3,t})^2 e^{-2s\eta} + C_0 \iint_{\omega_{t_0, t_1}} s^3 \varphi^3 z^2 e^{-2s\eta} \\ & \leq 3C_0 \iint_{Q_{t_0, t_1}} h_{1,t}^2 e^{-2s\eta} + 3C_0 \iint_{Q_{t_0, t_1}} (h_{2,t}^2 + h_{3,t}^2) e^{-2s\eta} \\ & \quad + C_0 \iint_{\omega_{t_0, t_1}} s^3 \varphi^3 z^2 e^{-2s\eta}. \end{aligned}$$

6 Inequality (5. 8) is the first step when dealing with the standard inverse source
 7 problem studied in [19]. Here our goal consists in retrieving only the part h_1 of
 8 the source term $h_1 + h_2 + h_3$. Hence our next step is now to absorb the term
 9 $\iint_{Q_{t_0, t_1}} (h_{2,t}^2 + h_{3,t}^2) e^{-2s\eta}$ into the left-hand side of (5. 8). In that purpose, we
 10 state the following fundamental lemma:

1 **Lemma 5.2.** *There exists a positive constant $C > 0$ such that*

$$(5.9) \quad \iint_{Q_{t_0, t_1}} (h_{2,t}^2 + h_{3,t}^2) e^{-2s\eta} \leq C \left(\iint_{Q_{t_0, t_1}} z^2 e^{-2s\eta} + \int_{\Omega} w(T', x)^2 dx \right).$$

Proof of Lemma 5.2. Since $h_2 = \tilde{\mu}(u - \tilde{u}) = \tilde{\mu}w$, we have $h_{2,t} = \tilde{\mu}w_t = \tilde{\mu}z$. Hence $|h_{2,t}| \leq M|z|$. On the other hand, $h_3 = \gamma\tilde{u}^2 - \gamma u^2$. It implies

$$h_{3,t} = 2\gamma\tilde{u}\tilde{u}_t - 2\gamma uu_t = 2\gamma\tilde{u}(\tilde{u}_t - u_t) + 2\gamma(\tilde{u} - u)u_t = -2\gamma\tilde{u}z - 2\gamma wu_t.$$

Therefore, using (4.4), and Lemma 4.6, we have

$$|h_{3,t}| \leq 2\gamma^+ M_{0,0}^\infty |z| + c\gamma^+ M_{1,0}^\infty |w|.$$

2 Hence, there exists some positive constant $C > 0$ such that

$$(5.10) \quad \iint_{Q_{t_0, t_1}} (h_{2,t}^2 + h_{3,t}^2) e^{-2s\eta} \leq C \iint_{Q_{t_0, t_1}} z^2 e^{-2s\eta} + C \iint_{Q_{t_0, t_1}} w^2 e^{-2s\eta}.$$

3 In order to conclude, it remains to estimate the second term in the right-hand side
4 of (5.10). In this purpose, we recall the following classical result:

Lemma 5.3. *There exists a positive constant $C > 0$ such that, for all $z \in L^2(Q_{t_0, t_1})$,*

$$\iint_{Q_{t_0, t_1}} \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 e^{-2s\eta(t,x)} dx dt \leq C \iint_{Q_{t_0, t_1}} z(t, x)^2 e^{-2s\eta(t,x)} dx dt.$$

Proof of Lemma 5.3. First we decompose

$$\begin{aligned} & \iint_{Q_{t_0, t_1}} \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 e^{-2s\eta(t,x)} dx dt \\ &= \int_{\Omega} \int_{t_0}^{T'} \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 e^{-2s\eta(t,x)} dt dx + \int_{\Omega} \int_{T'}^{t_1} \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 e^{-2s\eta(t,x)} dt dx \end{aligned}$$

Let us study the last integral, the other being similar: first we note that

$$\forall t \in (T', t_1), \quad \int_{T'}^t z(\tau, x) d\tau = \int_{T'}^t z(\tau, x) e^{-s\eta(\tau,x)} e^{s\eta(\tau,x)} d\tau,$$

hence, using the Cauchy-Schwarz inequality, we have

$$\forall t \in (T', t_1), \quad \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 \leq \left(\int_{T'}^t z(\tau, x)^2 e^{-2s\eta(\tau,x)} d\tau \right) \left(\int_{T'}^t e^{2s\eta(\tau,x)} d\tau \right);$$

now we note that

$$\tau \in (T', t_1) \mapsto e^{2s\eta(\tau,x)}$$

is nondecreasing, hence

$$\int_{T'}^t e^{2s\eta(\tau,x)} d\tau \leq \int_{T'}^t e^{2s\eta(t,x)} d\tau = (t - T') e^{2s\eta(t,x)};$$

therefore

$$\forall t \in (T', t_1), \quad \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 \leq \left(\int_{T'}^t z(\tau, x)^2 e^{-2s\eta(\tau,x)} d\tau \right) \left((t - T') e^{2s\eta(t,x)} \right),$$

and then

$$\begin{aligned} \forall t \in (T', t_1), \quad \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 e^{-2s\eta(t,x)} &\leq (t - T') \left(\int_{T'}^t z(\tau, x)^2 e^{-2s\eta(\tau,x)} d\tau \right) \\ &\leq (t - T') \left(\int_{T'}^{t_1} z(\tau, x)^2 e^{-2s\eta(\tau,x)} d\tau \right); \end{aligned}$$

integrating with respect to $t \in (T', t_1)$, we obtain

$$\int_{T'}^{t_1} \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 e^{-2s\eta(t,x)} dt \leq \frac{(t_1 - T')^2}{2} \int_{T'}^{t_1} z(\tau, x)^2 e^{-2s\eta(\tau,x)} d\tau,$$

and integrating with respect to $x \in \Omega$, we obtain

$$\int_{\Omega} \int_{T'}^{t_1} \left| \int_{T'}^t z(\tau, x) d\tau \right|^2 e^{-2s\eta(t,x)} dt dx \leq \frac{(t_1 - T')^2}{2} \int_{\Omega} \int_{T'}^{t_1} z(\tau, x)^2 e^{-2s\eta(\tau,x)} d\tau dx;$$

we can repeat the same argument when $t \in (t_0, T')$, using now that

$$\tau \in (t_0, T') \mapsto e^{2s\eta(\tau,x)}$$

- 1 is nonincreasing, and this concludes the proof of Lemma 5.3. \square

So, in order to estimate

$$\iint_{Q_{t_0, t_1}} w^2 e^{-2s\eta},$$

we first write

$$w(t, x) = w(T', x) + \int_{T'}^t w_t(\tau, x) d\tau = w(T', x) + \int_{T'}^t z(\tau, x) d\tau.$$

- 2 Next we apply Lemma 5.3. And coming back to (5. 10), we easily achieve the proof
3 of (5. 9). \square

Let us now come back to (5. 8). Combining (5. 8) and (5. 9), we get:

$$(5. 11) \quad I_0 := \iint_{Q_{t_0, t_1}} \left(s^3 \varphi^3 z^2 + \frac{1}{s\varphi} z_t^2 \right) e^{-2s\eta} \leq C \iint_{Q_{t_0, t_1}} h_{1,t}^2 e^{-2s\eta} \\ + C \iint_{Q_{t_0, t_1}} z^2 e^{-2s\eta} + C \int_{\Omega} w(T', x)^2 dx + C \iint_{\omega_{t_0, t_1}} s^3 \varphi^3 z^2 e^{-2s\eta}.$$

- 4 As a consequence, there exist $s_1(\lambda) > 0$ and a new constant $C > 0$, depending on
5 the various constants and data such that $\forall s \geq s_1(\lambda)$, the following estimate holds:

$$(5. 12) \quad I_0 \leq C \underbrace{\left(\iint_{Q_{t_0, t_1}} h_{1,t}^2 e^{-2s\eta} + \int_{\Omega} w(T', x)^2 dx + \iint_{\omega_{t_0, t_1}} s^3 \varphi^3 z^2 e^{-2s\eta} \right)}_{:= I_1}.$$

- 6 **Step 4: estimate from above of I_1 .** In this step, our purpose is to show that
7 there exists some constant $C > 0$ such that, for all $s \geq s_1(\lambda)$,

$$(5. 13) \quad I_1 \leq C \left[\frac{1}{\sqrt{s}} \int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T', x)} + \|w(T', \cdot)\|_{L^2(\Omega)}^2 + \|w_t\|_{L^2(\omega_{t_0, t_1})}^2 \right].$$

- 8 Since $z = w_t$ and $s^3 \varphi^3 e^{-2s\eta}$ is bounded independently of s large enough, the result
9 directly follows from the definition of I_1 and the following lemma, directly adapted
10 from [19]:

- 11 **Lemma 5.4.** *There exists some constant $C > 0$ (depending on $D, \alpha, M, \gamma^-, \gamma^+$,
12 $u_i, \bar{u}_i, B_\varepsilon, t_0$, and t_1) such that*

$$(5. 14) \quad \iint_{Q_{t_0, t_1}} h_{1,t}^2 e^{-2s\eta} \leq C \frac{1}{\sqrt{s}} \int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T', x)} dx.$$

Proof of Lemma 5.4. First we use (5. 4) to get that

$$\begin{aligned} \iint_{Q_{t_0, t_1}} h_{1,t}^2 e^{-2s\eta} &\leq \left(\frac{M_{1,0}^{loc, glob}}{m_{0,0}^{loc-glob}} \right)^2 \iint_{Q_{t_0, t_1}} h_1(T')^2 e^{-2s\eta} \\ &= \left(\frac{M_{1,0}^{loc, glob}}{m_{0,0}^{loc-glob}} \right)^2 \int_{\Omega} h_1(T', x)^2 \left(\int_{t_0}^{t_1} e^{-2s\eta(t,x)} dt \right) dx. \end{aligned}$$

Now consider

$$\theta(t) = \frac{1}{\ell(t)} = \frac{1}{(t-t_0)(t_1-t)} :$$

θ satisfies

$$\begin{aligned} \theta'(t) &= 2(t-T')\theta^2, \quad \theta''(t) = 2\theta^2 + 8(t-T')^2\theta^3, \\ \theta'''(t) &= 24(t-T')\theta^3 + 48(t-T')^3\theta^4. \end{aligned}$$

We use a Taylor-Lagrange expansion: for all $t \in (t_0, T']$, there exists some $\tau \in (t, T')$ such that

$$\theta(t) = \theta(T') + (t-T')\theta'(T') + \frac{(t-T')^2}{2}\theta''(T') + \frac{(t-T')^3}{6}\theta'''(\tau).$$

Observe that $(t-T')^3 \leq 0$ and $\theta'''(\tau) \leq 0$. We deduce

$$\theta(t) \geq \theta(T') + (t-T')\theta'(T') + \frac{(t-T')^2}{2}\theta''(T') = \theta(T') + \frac{(t-T')^2}{2}\theta''(T').$$

In the same way, when $t \in [T', t_1)$, we get the same estimate. Hence

$$\forall t \in (t_0, t_1), \quad -\theta(t) \leq -\theta(T') - \frac{(t-T')^2}{2}\theta''(T').$$

Then, using the notation

$$\chi(x) = e^{2\lambda\|\Psi\|_{L^\infty(\Omega)}} - e^{\lambda\Psi(x)},$$

so that $\eta(t, x) = \theta(t)\chi(x)$, we have

$$\begin{aligned} \int_{t_0}^{t_1} e^{-2s\eta(t,x)} dt &\leq e^{-2s\chi(x)\theta(T')} \int_{t_0}^{t_1} e^{-\chi(x)\theta''(T')s(t-T')^2} dt \\ &= e^{-2s\eta(T',x)} \int_{t_0}^{t_1} e^{-\chi(x)\theta''(T')s(t-T')^2} dt \leq e^{-2s\eta(T',x)} \int_{-\infty}^{+\infty} e^{-\chi(x)\theta''(T')s(t-T')^2} dt \\ &= e^{-2s\eta(T',x)} \frac{1}{\sqrt{\chi(x)\theta''(T')s}} \int_{-\infty}^{+\infty} e^{-\sigma^2} d\sigma = e^{-2s\eta(T',x)} \frac{\sqrt{\pi}}{\sqrt{\chi(x)\theta''(T')s}}. \end{aligned}$$

Using also the fact that $\theta''(T') = 2\theta(T')^2$, we conclude that

$$\iint_{Q_{t_0, t_1}} h_{1,t}^2 e^{-2s\eta} \leq \left(\frac{M_{1,0}^{loc, glob}}{m_{0,0}^{loc-glob}} \right)^2 \frac{\sqrt{\pi}}{\theta(T')\sqrt{2s}} \int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T',x)} \frac{1}{\sqrt{\chi(x)}} dx,$$

1 which gives (5. 14). □

2 **Step 5: estimate from below of I_0 .** The purpose of the step is to provide the
3 following estimate:

4 **Lemma 5.5.** *There exists a constant $C = C(t_0, t_1) > 0$ such that*

$$(5. 15) \quad \int_{\Omega} z(T', x)^2 e^{-2s\eta(T',x)} \leq \frac{C}{s} I_0 = \frac{C}{s} \iint_{Q_{t_0, t_1}} (s^3 \varphi^3 z^2 + \frac{1}{s\varphi} z_t^2) e^{-2s\eta}.$$

Proof of Lemma 5.5. Since $z(t, x)^2 e^{-2s\eta(t, x)} \rightarrow 0$ as $t \rightarrow t_0$ for a.a. $x \in \Omega$, we can write

$$(5.16) \quad \begin{aligned} \int_{\Omega} z(T', x)^2 e^{-2s\eta(T', x)} dx &= \int_{t_0}^{T'} \frac{\partial}{\partial t} \left(\int_{\Omega} z(t, x)^2 e^{-2s\eta(t, x)} dx \right) dt \\ &= \int_{t_0}^{T'} \int_{\Omega} \left[2z z_t - 2s\eta_t z^2 \right] e^{-2s\eta} dx dt. \end{aligned}$$

First we estimate

$$(5.17) \quad \begin{aligned} s \int_{t_0}^{T'} \int_{\Omega} 2z z_t e^{-2s\eta} &= s \int_{t_0}^{T'} \int_{\Omega} 2s\sqrt{\varphi} z e^{-s\eta} \frac{1}{s\sqrt{\varphi}} z_t e^{-s\eta} \\ &\leq s \int_{t_0}^{T'} \int_{\Omega} \left(s^2 \varphi z^2 e^{-2s\eta} + \frac{1}{s^2 \varphi} z_t^2 e^{-2s\eta} \right) \leq \int_{t_0}^{T'} \int_{\Omega} s^3 \varphi z^2 e^{-2s\eta} + \frac{z_t^2}{s\varphi} e^{-2s\eta} \leq CI_0. \end{aligned}$$

Next we estimate

$$(5.18) \quad \begin{aligned} s \int_{t_0}^{T'} \int_{\Omega} 2s|\eta_t| z^2 e^{-2s\eta} &= \int_{t_0}^{T'} \int_{\Omega} 2s^2 |\theta_t| (e^{2\lambda\|\Psi\|_{\infty}} - e^{\lambda\Psi}) z^2 e^{-2s\eta} \\ &\leq C \int_{t_0}^{T'} \int_{\Omega} s^2 \theta^2 (e^{2\lambda\|\Psi\|_{\infty}} - e^{\lambda\Psi}) z^2 e^{-2s\eta} \leq CI_0, \end{aligned}$$

1 where we recall that $\theta(t) = 1/\ell(t)$ satisfies $|\theta_t(t)| \leq C\theta(t)^2$.

2 Finally, (5.16) associated to (5.17) and (5.18) gives (5.15). \square

Step 6: conclusion. Using (5.15), (5.12) and next (5.13), there exists some constant $C > 0$ such that

$$(5.19) \quad \begin{aligned} \int_{\Omega} z(T', x)^2 e^{-2s\eta(T', x)} &\leq \frac{C}{s^{3/2}} \int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T', x)} dx \\ &\quad + \frac{C}{s} \|w_t\|_{L^2(\omega_{t_0, t_1})}^2 + \frac{C}{s} \|w(T', \cdot)\|_{L^2(\Omega)}^2. \end{aligned}$$

3 On the other hand, let us recall that

$$z(T', x) = w_t(T', x) = D\Delta w(T', x) + h_1(T', x) + h_2(T', x) + h_3(T', x).$$

Therefore,

$$\begin{aligned} &\int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T', x)} dx \\ &\leq C \int_{\Omega} z(T', x)^2 e^{-2s\eta(T', x)} dx + C \int_{\Omega} |D\Delta w(T', x)|^2 e^{-2s\eta(T', x)} dx \\ &\quad + C \int_{\Omega} h_2(T', x)^2 e^{-2s\eta(T', x)} dx + C \int_{\Omega} h_3(T', x)^2 e^{-2s\eta(T', x)} dx. \end{aligned}$$

Applying (5.19) to estimate the term $\int_{\Omega} z(T', x)^2 e^{-2s\eta(T', x)} dx$, we get

$$\begin{aligned} &\int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T', x)} dx \\ &\leq C \frac{1}{s^{3/2}} \int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T', x)} dx + \frac{C}{s} \|w_t\|_{L^2(\omega_{t_0, t_1})}^2 + \frac{C}{s} \|w(T', \cdot)\|_{L^2(\Omega)}^2 \\ &+ C \|\Delta w(T', \cdot)\|_{L^2(\Omega)}^2 + C \|h_2(T', \cdot) e^{-s\eta(T', \cdot)}\|_{L^2(\Omega)}^2 + C \|h_3(T', \cdot) e^{-s\eta(T', \cdot)}\|_{L^2(\Omega)}^2. \end{aligned}$$

Choosing s large enough such that $C/s^{3/2} \leq 1/2$, we get

$$(5.20) \quad \frac{1}{2} \int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T', x)} dx \leq \frac{C}{s} \|w_t\|_{L^2(\omega_{t_0, t_1})}^2 + \frac{C}{s} \|w(T', \cdot)\|_{L^2(\Omega)}^2 \\ + C \|\Delta w(T', \cdot)\|_{L^2(\Omega)}^2 + C \left\| h_2(T', \cdot) e^{-s\eta(T', \cdot)} \right\|_{L^2(\Omega)}^2 + C \left\| h_3(T', \cdot) e^{-s\eta(T', \cdot)} \right\|_{L^2(\Omega)}^2.$$

- 1 Let us now estimate the two last terms of the right hand side of (5.20). First we
2 recall that $|h_2| = |\tilde{\mu}(u - \tilde{u})| \leq M|u - \tilde{u}| = M|w|$ since $\tilde{\mu} \in \mathcal{M}_{\alpha, M}$. Therefore

$$(5.21) \quad \left\| h_2(T', \cdot) e^{-s\eta(T', \cdot)} \right\|_{L^2(\Omega)}^2 \leq M^2 \|w(T', \cdot) e^{-s\eta(T', \cdot)}\|_{L^2(\Omega)}^2.$$

Next we write

$$|h_3| = |\gamma(\tilde{u} - u)(\tilde{u} + u)| = |\gamma(\tilde{u} + u)w| \leq 2\gamma M_{0,0}^{\infty} |w|,$$

- 3 where we used (4.4). This leads to an inequality similar to (5.21).
Finally, (5.20) becomes

$$(5.22) \quad \int_{\Omega} h_1(T', x)^2 e^{-2s\eta(T', x)} dx \\ \leq \frac{C}{s} \|w_t\|_{L^2(\omega_{t_0, t_1})}^2 + C \|w(T', \cdot)\|_{L^2(\Omega)}^2 + C \|\Delta w(T', \cdot)\|_{L^2(\Omega)}^2.$$

And to conclude, using Lemma 4.3, we get

$$h_1(T', x)^2 \geq (m_{0,0}^{loc-glob})^2 d(x, \partial\Omega)^2 (\mu(x) - \tilde{\mu}(x))^2,$$

- 4 which gives (2.4). □

5

6. LIPSCHITZ STABILITY : PROOF OF THEOREM 2.2

The strategy is the same. We observe, that in this second case, if u is solution of $(P_{\mu, \gamma})$ and u^* is solution of (P_{μ, γ^*}) , then $w = u - u^*$ is solution of

$$w_t - D\Delta w = h_2 + h_3 + h_4$$

where

$$h_2 := \mu(u - u^*), \quad h_3 := -\gamma(u - u^*)(u + u^*), \quad h_4 := -(\gamma - \gamma^*)(u^*)^2.$$

The question here is to estimate the part h_4 of the source term $h_2 + h_3 + h_4$. As in the proof of Theorem 2.1, the part of the source term that we want to estimate satisfies the usual estimate: indeed, using first (4.15) and (4.22), and then (4.8), we have

$$\forall t \in (t_0, t_1), \forall x \in \Omega, \quad \left| \frac{\partial h_4}{\partial t}(t, x) \right| = 2u^*(t, x) |u_t^*(t, x)| |\gamma(x) - \gamma^*(x)| \\ \leq 2\tilde{M}_{0,0}^{loc, glob} M_{1,0}^{loc, glob} d(x, \partial\Omega)^2 |\gamma(x) - \gamma^*(x)| \\ \leq \frac{2\tilde{M}_{0,0}^{loc, glob} M_{1,0}^{loc, glob}}{(m_{0,0}^{loc, glob})^2} u^*(T', x)^2 |\gamma(x) - \gamma^*(x)| \\ = \frac{2\tilde{M}_{0,0}^{loc, glob} M_{1,0}^{loc, glob}}{(m_{0,0}^{loc, glob})^2} |h_4(T', x)|.$$

- 6 Then, reasoning as in the proof of Theorem 2.1, we prove that (2.5) holds. □

7. APPENDIX: PROOF OF (4. 20)

1
2 Let V be the solution of (4. 17). As a consequence of the Hille-Yosida theorem
3 (see, e.g., Brezis [5], Theorem VII.7), it is well-known that

$$(7. 1) \quad \forall t > 0, \quad \begin{cases} \|V(t)\|_{L^2(\Omega)} \leq \|u_i\|_{L^2(\Omega)}, \\ \|V_t(t)\|_{L^2(\Omega)} = \|D\Delta(t)\|_{L^2(\Omega)} \leq \frac{1}{t}\|u_i\|_{L^2(\Omega)}. \end{cases}$$

Now consider $K \in \mathbb{N}$, that will be chosen later (large enough), and construct a subdivision of $[0, \frac{t_0}{2}]$:

$$0 < \eta < 2\eta < \dots < K\eta = \frac{t_0}{2}, \quad \text{with} \quad \eta = \frac{t_0}{2K}.$$

4 From (7. 1) we have

$$(7. 2) \quad \forall t \geq \eta, \quad \|V_t(t)\|_{L^2(\Omega)} \leq \frac{1}{\eta}\|u_i\|_{L^2(\Omega)}.$$

5 Since $V \in C^\infty((0, +\infty) \times \bar{\Omega})$, we deduce from (4. 17) that V_t satisfies

$$(7. 3) \quad \begin{cases} (V_t)_t - D\Delta V_t = 0 & (t, x) \in (\eta, +\infty) \times \Omega, \\ V_t(t, x) = 0 & (t, x) \in (\eta, +\infty) \times \partial\Omega. \end{cases}$$

6 Applying again Brezis [5], Theorem VII.7, we get

$$(7. 4) \quad \forall t > \eta, \quad \|(V_t)_t(t)\|_{L^2(\Omega)} \leq \frac{1}{t-\eta}\|V_t(\eta)\|_{L^2(\Omega)}.$$

Hence

$$\forall t > \eta, \quad \|V_{tt}(t)\|_{L^2(\Omega)} \leq \frac{1}{t-\eta} \frac{1}{\eta} \|u_i\|_{L^2(\Omega)},$$

7 which gives

$$(7. 5) \quad \forall t \geq 2\eta, \quad \|V_{tt}(t)\|_{L^2(\Omega)} \leq \frac{1}{\eta^2} \|u_i\|_{L^2(\Omega)}.$$

By iteration, we prove that, for any $j \in \{0, 1, \dots, K-1\}$, $\frac{\partial^j V}{\partial t^j}$ satisfies the problem

$$\begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial^j V}{\partial t^j} \right) - D\Delta \left(\frac{\partial^j V}{\partial t^j} \right) = 0 & (t, x) \in (j\eta, +\infty) \times \Omega, \\ \frac{\partial^j V}{\partial t^j}(t, x) = 0 & (t, x) \in (j\eta, +\infty) \times \partial\Omega, \end{cases}$$

and we have the following estimate

$$\forall t \geq (j+1)\eta, \quad \left\| \frac{\partial^{j+1} V}{\partial t^{j+1}}(t) \right\|_{L^2(\Omega)} \leq \frac{1}{\eta^{j+1}} \|u_i\|_{L^2(\Omega)}.$$

In particular for $j = K-1$, we have

$$\forall t \geq K\eta = \frac{t_0}{2}, \quad \left\| \frac{\partial^K V}{\partial t^K}(t) \right\|_{L^2(\Omega)} \leq \frac{1}{\eta^K} \|u_i\|_{L^2(\Omega)},$$

8 and for any $k \in \{0, 1, \dots, K\}$, we have

$$(7. 6) \quad \forall t \geq \frac{t_0}{2}, \quad \left\| \frac{\partial^k V}{\partial t^k}(t) \right\|_{L^2(\Omega)} \leq \frac{1}{\eta^k} \|u_i\|_{L^2(\Omega)}.$$

From (4. 17) and using the regularity of V , we compute by iteration:

$$\forall k \in \{0, 1, \dots, K\}, \quad \begin{cases} \Delta^k V = \frac{1}{D^k} \frac{\partial^k V}{\partial t^k} & \text{in } \Omega, \\ \Delta^{k-1} V = 0 & \text{on } \partial\Omega. \end{cases}$$

9 So we obtain from (7. 6) that, for any $k \in \{0, 1, \dots, K\}$, we have

$$(7. 7) \quad \forall t \geq \frac{t_0}{2}, \quad \|\Delta^k V(t)\|_{L^2(\Omega)} \leq \frac{1}{D^k \eta^k} \|u_i\|_{L^2(\Omega)}.$$

If we define the operator $A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ in the following classical way:

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ \forall u \in D(A), \quad Au = -\Delta u, \end{cases}$$

then we have

$$\forall k \geq 1, \quad D(A^k) = \{u \in H^{2k}(\Omega), u = \Delta u = \dots = \Delta^{k-1}u = 0 \text{ on } \partial\Omega\},$$

1 and (7. 7) gives that there is $C(t_0, D, \bar{u}_i)$ such that, for all $u_i \in \mathcal{D}_{\bar{u}_i, \bar{u}_i}$, we have

$$(7. 8) \quad \forall t \geq \frac{t_0}{2}, \quad \|V(t)\|_{D(A^k)} \leq C(t_0, D, \bar{u}_i).$$

2 Using Brezis [5] Theorem IX.25, we deduce that

$$(7. 9) \quad \forall t \geq \frac{t_0}{2}, \quad \|V(t)\|_{H^{2k}(\Omega)} \leq C(t_0, D, \bar{u}_i).$$

To conclude, we recall that, since $\Omega \subset \mathbb{R}^d$, then, if $m > \frac{d}{2}$, we have

$$H^m(\Omega) \subset C^k(\bar{\Omega}) \quad \text{for } k = [m - \frac{d}{2}],$$

with continuous injection, see for example Brezis [5] Corollary IX.15 with $p = 2$. Hence it is sufficient to choose K such that $[2K - \frac{d}{2}] = 3$, in order to have

$$H^{2K}(\Omega) \subset C^3(\bar{\Omega}).$$

3 It follows from (7. 9) that

$$(7. 10) \quad \forall t \geq \frac{t_0}{2}, \quad \|V(t)\|_{C^3(\bar{\Omega})} \leq C(t_0, D, \bar{u}_i).$$

This proves the first part of (4. 20). For the second part, we use $V_t = D\Delta V$ in Ω and (7. 10), and we obtain

$$\forall t \geq \frac{t_0}{2}, \quad \|V_t(t)\|_{C^1(\bar{\Omega})} \leq D\|\Delta V(t)\|_{C^1(\bar{\Omega})} \leq D\|V(t)\|_{C^3(\bar{\Omega})} \leq C(t_0, D, \bar{u}_i).$$

4 This concludes the proof of (4. 20). \square

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8

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