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Finite-Time Stabilization of Evolution Equations with Maximal Monotone Maps in Hilbert Space

Moussa Labbadi and Christophe Roman

Abstract—Building on the concept of exact solutions combined with the Cauchy problem and Lyapunov functions, we investigate a robust finite-time and fixed-time stability of abstract systems under nonlinear feedback control laws. First, we design a stabilizing feedback control based on a set-valued map, which precisely rejects matched perturbations and ensures finite-time convergence. Subsequently, an additional nonlinear power term is incorporated to achieve fixed-time convergence. For both cases, we analyze the well-posedness of the closed-loop system using the theory of maximal monotone operators. Furthermore, we extend the analysis to study partial finite-time stability for the same class of abstract systems. The theoretical findings are applied to derive robust partial finite-time stabilization results for the heat equation.

Index Terms—Finite-/ Fixed-Time Stability, Nonlinear Feedback Control, Maximal Monotone Operators, Robust Stabilization.

I. INTRODUCTION

In recent years, the stabilization of Partial Differential Equations (PDEs) has garnered significant attention due to their role in modeling the spatiotemporal dynamics of complex systems, including heat transfer, traffic flow, fluid mechanics, chemical reactions, string vibrations, and electromagnetic phenomena. Their infinite-dimensional nature makes PDE stabilization both critical and challenging. While most existing techniques focus on achieving asymptotic or exponential convergence, finite-time stability has emerged as a key requirement for linear, nonlinear, and infinite-dimensional systems. Finite-time convergence ensures equilibrium within a finite-time, offering enhanced robustness and performance for time-sensitive applications. This can be achieved through homogeneity-based techniques with a negative homogeneity degree [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

Expanding the scope to abstract infinite-dimensional systems, significant contributions have been made to this field, notably in the works on linear and bilinear systems [18], [19], [15]. In the case of the bilinear reaction-diffusion equation, finite-time stability was established directly, with the settling time depending on the system's initial state [20]. Moreover, when time-varying stabilizing feedback control is allowed, finite-time stability can be achieved at any specified time. Further exploration into output and global finite-time stability for abstract bilinear and linear systems was conducted [20]. Global finite-time stabilization for bilinear control systems,

under homogeneous state feedback laws and coercive control operators, has also been studied [21]. Several works have delved into the finite/fixed-time stability analysis of parabolic and hyperbolic PDEs [22], [23], [24], [18], [15].

Another technique related to finite/fixed-time controllability is the sliding mode control (SMC) approach, particularly in the context of evolution systems [25], [26], [27], [28], [29], [30], [31], [32], [33]. In distributed control, SMC has been successfully applied to truncated finite-dimensional models, including applications to parabolic PDE systems [34], [35], heat equations [36], wave equations [32], [37], and Schrödinger equations [31]. Notably, in [38], the authors discuss tracking and control for heat and wave equations using continuous SMC approaches with fractional power techniques to modify the signum function. Robust tracking for a diffusion equation using sliding mode boundaries is addressed in [39], while [40] achieves exponential stabilization of the wave equation with matched perturbations. Additionally, [41] proposes boundary control for a heat process with unbounded matched perturbations using second-order SMC. In a model of earthquake phenomena [42], a cascade system combining a 1D wave equation for fault slip and wave propagation with a 1D diffusion equation for actuator dynamics is utilized.

In the presence of matched perturbations [43], [44], [45], [46], the problem cannot be effectively addressed by merely relying on the power of the state [41]. Instead, the application of set-valued boundary dynamics [47] is required to precisely counteract these perturbations. However, the classical Filippov well-posedness theory [48] cannot be directly applied to prove the well-posedness of the closed-loop system, as its framework does not fully capture the complexities inherent in this scenario.

This paper investigates the robust finite-time and fixed-time stability of a class of abstract systems subjected to matched perturbations. For the proof of robust finite-time stability, we employ exact solutions in conjunction with the Cauchy problem. In the case of robust fixed-time stability, the Lyapunov method is utilized. Subsequently, leveraging the theory of maximal monotone operators, we establish the well-posedness of the closed-loop system in both scenarios and we provide an estimation of the settling time. In the finite-time stability case, the settling time is dependent on the initial condition, whereas in the fixed-time stability case, it is independent of initial conditions. Additionally, we present results on partial finite-time stability. To illustrate the practical application of our results, we provide an example demonstrating the partial finite-time stabilization and disturbance rejection of the heat equation.

NOTATION

For any interval $I \subseteq \mathbb{R}^+$, let $C^k(I, X)$, with $k \in \mathbb{Z}^+$, denote the space of all k -times continuously differentiable functions defined on I and taking values in a space X .

The space $L^2(I, \mathbb{R})$ denotes the set of square-integrable functions $f : I \rightarrow \mathbb{R}$, equipped with the L^2 -norm

$$\|f\| = \sqrt{\int_I f^2(x) dx}.$$

Define $Q(a, b) = (a, b) \times (0, 1)$ for $(a, b) \subset \mathbb{R}^+$ (or Q_t for $(a, b) = (0, t)$, where $t \in \mathbb{R}^+ \cup \{\infty\}$). We use $L^\infty(Q(a, b)) := L^\infty(Q(a, b), \mathbb{R})$ to denote the space of essentially bounded functions $d : Q(a, b) \rightarrow \mathbb{R}$, with the norm

$$\|d\|_{L^\infty(Q(a,b))} = \text{ess sup}_{(s,x) \in (a,b) \times (0,1)} |d(s,x)| < \infty,$$

abusively denoted by $\|\cdot\|_\infty$ in the following. Let us defined the multivoque map $\text{sgn} : \mathcal{X} \rightarrow 2^{\mathcal{X}}$, where \mathcal{X} is a Hilbert space,

$$\text{sgn } y = \begin{cases} \frac{y}{\|y\|} & \text{if } y \neq 0, \\ \{z; \|z\| \leq 1\} & \text{if } y = 0. \end{cases} \quad (1)$$

II. CONCEPTS AND PRELIMINARY RESULTS

A. Preliminaries on Finite-Time Concepts for PDEs

In this section, we recall some fundamental definitions of stability for infinite-dimensional systems.

Consider the following evolution system:

$$\begin{cases} \frac{d}{dt} X(t) = -AX(t), & (2a) \\ X(0) = X_0, & (2b) \end{cases}$$

where $t \geq t_0 \geq 0$, X_0 denotes the initial condition, and $A : \mathcal{D}(A) \subset L^2 \rightarrow L^2$ is a (possibly unbounded) operator.

Definition 1. *The origin of the system (2) is characterized as follows:*

- **Stable** if for every $\varepsilon > 0$, there exists a $\theta > 0$ such that, for any $t_0 \geq 0$ and $X_0 \in L^2$, the condition

$$\|X_0\|_{L^2} \leq \theta \implies \|X(t)\|_{L^2} \leq \varepsilon, \quad \forall t \geq t_0,$$

holds.

- **Asymptotically Stable (AS)** if the system is stable, and in addition,

$$\lim_{t \rightarrow +\infty} \|X(t)\|_{L^2} = 0 \quad \text{for all } X_0 \in L^2.$$

- **Finite-Time Stable (FnTS)** if the system is stable, and for every $X_0 \in L^2$, there exists a time $T(X_0) \geq 0$ such that

$$\|X(t)\|_{L^2} = 0 \quad \text{for all } t \geq T(X_0).$$

The function $T(X_0) = \inf\{T(X_0) \geq 0 : \|X(t)\|_{L^2} = 0, \forall t \geq T(X_0)\}$. $T(X_0)$ represents the settling time of the system (2).

- **Fixed-Time Stable (FxTS)** if it is FnTS and

$$\sup_{X_0 \in L^2} T(X_0) < +\infty.$$

Based on Definition 1 and the results in [49], [50], [15], the following lemma provides some sufficient conditions for the stability of the origin.

Lemma 1. *Let $V^1 : \Omega \subset \mathcal{D}(A) \rightarrow \mathbb{R}_+$ be a continuous functional on Ω , continuously differentiable on $\Omega \setminus \{0\}$, and satisfy the coercivity condition, i.e., there exist two class- \mathcal{K}_∞ functions β_1 and β_2 such that*

$$\beta_1(\|X(t)\|_{L^2}) \leq V(t) \leq \beta_2(\|X(t)\|_{L^2}) \quad \forall t \geq t_0.$$

Then, the following results hold:

- If the time derivative of V along the solutions of (2) (denoted by $\frac{d}{dt}V(t)$ for simplicity in the rest of the paper) satisfies

$$\frac{d}{dt}V(t) \leq 0 \quad \text{in } \Omega \text{ for all } t \geq t_0,$$

then the origin of the system (2) is stable.

- In addition, if there exists a class- \mathcal{K}_∞ function such that

$$\frac{d}{dt}V(t) \leq -\beta_3(\|X(t)\|_{L^2}) \quad \text{in } \Omega \text{ for all } t \geq t_0,$$

then the origin of the system (2) is asymptotically stable [51, Proposition 3.2].

- Alternatively, if there exists $0 \leq T(V(0)) < +\infty$ such that $V(t) = 0$ for all $t \geq T(V(0))$, then the origin of the system (2) is FnTS, with the settling time defined by

$$T(V(0)) = \inf \{T(V(0)) \geq 0 : V(t) = 0, \forall t \geq T(V(0))\}.$$

In particular, if

$$\sup_{X_0 \in \Omega} T(V(0)) < +\infty,$$

then the origin of the system (2) is FxTS.

III. PROBLEM STATEMENT AND PRELIMINARIES

We define a Hilbert space \mathbb{H} equipped with a scalar product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and the induced norm $\|\cdot\|_{\mathbb{H}}$. We focus on the abstract control framework for the following autonomous problems of the form:

$$\begin{cases} \frac{d}{dt} X(t) = -AX(t) + f(u(t), d(t)), & (3a) \\ X(0) = X_0 \in \overline{\mathcal{D}(A)} \subset \mathbb{H}, & (3b) \end{cases}$$

where $-A : \mathcal{D}(A) \subset X \rightarrow X$ is a linear operator which is an infinitesimal generator of a strongly continuous semigroup denoted $(e^{tA})_{t \geq 0}$. This property ensures well-defined time evolution for solutions to the homogeneous equation for each $X_0 \in \overline{\mathcal{D}(A)}$.

We analyze this framework within a Hilbert space, leveraging its inner product and norm properties. Our investigation will encompass the existence, uniqueness, and regularity of solutions, as well as their controllability. The interplay between the linear operator A and the nonlinear term $f(t, \cdot)$ is crucial for designing effective control laws to achieve desired state trajectories.

¹Note that if V is continuously differentiable on $\Omega \setminus \{0\}$, then $\frac{d}{dt}V(t) = \left\langle \frac{\partial V(t)}{\partial X}, \frac{d}{dt}X(t) \right\rangle_{L^2}$.

The primary aim of this paper is to formulate a feedback law that ensures the well-posedness of the system (3) and achieves global finite-time or fixed-time stability. Notably, the system can be interpreted as a cascade system, comprising both an infinite-dimensional component, which may be represented by a partial differential equation, and an ordinary differential equation (ODE). This approach seeks to bridge the complexities inherent in the interaction between the PDE and ODE, ultimately leading to robust stability characteristics.

A common context in which such a system structure emerges is within closed-loop systems featuring control laws that include saturation, integral action, or components that are both time-varying and state-dependent. For example, the saturation case has been studied in [52]. However, we do not make the assumption that the operator A generates an exponentially stable semigroup. Consequently, when the control input is set to zero, the X component of the state may not necessarily converge to zero. Additionally, while the integral case has been addressed in [53], it is crucial to note that their analysis presumes the operator A to be invertible and univoques, which is a strong assumption that contrasts with established results.

In the following, the function $f(t, \cdot)$ can be divided into both the control input $u(X)$ and the matched perturbations $d(t)$. Thus, the system (3) becomes:

$$\begin{cases} \frac{d}{dt}X = -AX + u(X) + d(t), \\ X(0) = X_0, \end{cases} \quad (4)$$

where the perturbation satisfies the following assumptions:

Assumption 1. *The disturbances are bounded, and $\|d\|_\infty < +\infty$.*

Our goal is to design a robust state feedback control uu that ensures the system is input-to-state stable (ISS) [54], with a finite rate of convergence. Additionally, we aim to achieve bounded settling time convergence, meaning that the settling time remains constant.

IV. MAIN RESULTS

In this section, we present the main results of our paper. We begin with two results on well-posedness, followed by two results on the convergence and robustness of the posed systems.

A. Well-posedness

Consider the following inclusion system:

$$\begin{cases} \frac{d}{dt}X(t) + AX(t) - d(t) \in -K \operatorname{sgn}(X(t)), \\ X(0) = X_0. \end{cases} \quad (5a)$$

$$(5b)$$

The following result holds.

Theorem 1. *The abstract evolution problem (5) is well-posed in the case where A is maximal monotone and K is a positive constant. In details the operator $-(A + K \operatorname{sgn})$ is associated with a C_0 semigroup $e^{-(A+K \operatorname{sgn})}(t)$ on the Hilbert \mathbb{H} , and it holds, for all $T > 0$,*

- $\forall X_0 \in \mathcal{D}(A)$, $d \in W^{1,1}(0, T; \mathbb{H})$,

$$X \in C^0([0, T]; \mathcal{D}(A)), \quad \dot{X} \in L^\infty([0, T]; \mathbb{H}). \quad (6)$$

- $\forall X_0 \in \overline{\mathcal{D}(A)}$, $d \in L^1(0, T; \mathbb{H})$, then

$$X \in C^0([0, T]; \overline{\mathcal{D}(A)}). \quad (7)$$

Now, we can enhance the dynamics by incorporating a power term into the control law. Consider the following inclusion system:

$$\begin{cases} \frac{d}{dt}X(t) + AX(t) + d(t) + K_2 \|X(t)\|^\alpha \operatorname{sgn}(X(t)) \\ \qquad \qquad \qquad \in -K \operatorname{sgn}(X(t)), \\ X(0) = X_0, \end{cases} \quad (8)$$

We present the second result on well-posedness, which incorporates the inclusion system with the added power term:

Theorem 2. *The abstract evolution problem (8) is well-posed in the case where A is maximal monotone and K and K_2 are positive constant. In details the operator $-(A + K \operatorname{sgn} + K_2 \|\cdot\|^\alpha \operatorname{sgn})$ is associated with a C_0 semigroup $e^{-(A+K \operatorname{sgn} + K_2 \|\cdot\|^\alpha \operatorname{sgn})}(t)$ on the Hilbert \mathbb{H} , and it holds, for all $T > 0$,*

- $\forall X_0 \in \mathcal{D}(A)$, $d \in W^{1,1}(0, T; \mathbb{H})$,

$$X \in C^0([0, T]; \mathcal{D}(A)), \quad \dot{X} \in L^\infty([0, T]; \mathbb{H}) \quad (9)$$

- $\forall X_0 \in \overline{\mathcal{D}(A)}$, $d \in L^1(0, T; \mathbb{H})$, then

$$X \in C^0([0, T]; \overline{\mathcal{D}(A)}). \quad (10)$$

The proof of Theorem 1 is a particular case of the proof of Theorem 2. The latter is provide in the following

Proof. First the operator sgn is maximal monotone on the Hilbert space \mathbb{H} , this can be draw by comments on page 251 in [55]. The domain of $\|\cdot\|^\alpha \operatorname{sgn}$ is all \mathbb{H} , $\mathcal{D}(\operatorname{sgn}) = \mathbb{H}$, $\forall x, y \in \mathbb{H}$, one gets

$$\langle x, \operatorname{sgn}(x) \rangle = \frac{\langle x, x \rangle}{\|x\|} = \|x\| \quad (11)$$

therefore

$$\begin{aligned} \langle x - y, \|x\|^\alpha \operatorname{sgn}(x) - \|y\|^\alpha \operatorname{sgn}(y) \rangle = \\ \|x\|^{\alpha+1} + \|y\|^{\alpha+1} - \langle x, y \rangle (\|x\|^{\alpha-1} + \|y\|^{\alpha-1}) = \\ (\|x\|^2 + \|y\|^2 - \langle x, y \rangle) (\|x\|^{\alpha-1} + \|y\|^{\alpha-1}) \geq 0 \end{aligned} \quad (12)$$

The operator $\|\cdot\|^\alpha \operatorname{sgn}$ is monotone, now let us check the maximality of the operator in the monotone operator set. First, let us introduce the following result

Theorem 3 (Theorem 1.3 [55] on page 45). *Let \mathcal{X} be a reflexive Banach space and let $B : \mathcal{X} \rightarrow \mathcal{X}^*$ be a monotone hemicontinuous operator. Then B is maximal monotone in $\mathcal{X} \times \mathcal{X}^*$.*

Hence is we just need to show the hemicontinuousness of $\|\cdot\|^\alpha \operatorname{sgn}$.

Definition 2 (Definition 11.2 in [56] on page 111). *$F : D \rightarrow \mathcal{X}$ is called hemicontinuous if $\forall x, y \in D$ and $\forall z \in \mathcal{X}$ then it holds $\langle F(x + ty), z \rangle \rightarrow \langle Fx, z \rangle$ as $t \rightarrow 0^+$.*

One gets $\forall x, y, z \in \mathbb{H}$

$$\langle \|x + ty\|^\alpha \operatorname{sgn}(x + ty), z \rangle = \|x + ty\|^{\alpha-1} \langle (x + ty), z \rangle$$

$$\xrightarrow{t \rightarrow 0^+} \|x\|^{\alpha-1} \langle x, z \rangle \quad (13)$$

$$= \langle \|x\|^\alpha \operatorname{sgn}(x), z \rangle. \quad (14)$$

By applying Theorem 3, it follows that $\|\cdot\|^\alpha \operatorname{sgn}$ is maximal monotone in \mathbb{H} . Utilizing Rockafellar's result (Theorem 9 in Appendix A), we establish that $A + K \operatorname{sgn} + K_2 \|\cdot\|^\alpha \operatorname{sgn}$ is also maximal monotone. Finally, Brezis' Theorem 10 (presented in Appendix A) completes the proof. \square

B. Robust state feedback controls

A robust controller is essential to handle matched perturbations while ensuring finite-time convergence of the system. The following control law provides an effective solution to this problem:

$$u(t) = -K \operatorname{sgn}(X(t)), \quad (15)$$

where K is a positive constant satisfying $K > \|d\|_\infty$, with d representing the perturbation.

We now state the first result on convergence and robustness.

Theorem 4. *Assume that $0 \in A(0)$, and there exists a positive constant η , which depends on the upper bound of the perturbations and the parameter K , such that the operator $A + \eta \operatorname{sgn}$ is m -accretive in $\mathcal{X} \times \mathcal{X}$. Then, the control law (15) steers $X(0) = X_0 \in \overline{D(A)}$ to the origin in finite time, denoted by $T(X_0)$, with an upper bound given by:*

$$T(X_0) \leq \eta^{-1} \|X_0\|. \quad (16)$$

For admissible perturbations satisfying Assumption 1, the trajectory X of the closed-loop system as support on the interval $[0, T(X_0)]$. Specifically, we have $X(t) = 0$ for all $t \geq T(X_0)$.

Proof. The theorem extends [55, Proposition 2.3], which considers the case of no disturbance. We generalize the analysis to account for the disturbance term $d(t)$ under the given assumptions.

If $A + K \operatorname{sgn}$ is m -accretive, then the Cauchy problem (5) admits a unique mild solution $X \in C(\mathbb{R}_+; \mathcal{X})$. This solution is obtained as the uniform limit on compact intervals of the sequence $\{X_\varepsilon(t)\}_{\varepsilon > 0}$, defined by:

$$X(t) = \lim_{\varepsilon \rightarrow 0} X_\varepsilon(t), \quad \forall t > 0, \quad (17)$$

where $X_\varepsilon(t)$ satisfies the following difference inclusion for every $t > \varepsilon$:

$$X_\varepsilon(t) + \varepsilon A X_\varepsilon(t) + \varepsilon K \operatorname{sgn}(X_\varepsilon(t)) - \varepsilon d(t) \ni X_\varepsilon(t - \varepsilon), \quad (18)$$

with the initial condition given as:

$$X_\varepsilon(t) = X_0, \quad \forall t \leq 0.$$

To analyze the behavior of $X_\varepsilon(t)$, multiply both sides of the difference inclusion by a test function $\psi_\varepsilon(t) \in \operatorname{sgn}(X_\varepsilon(t))$. Using the properties of the multivalued operator sgn , which

ensures that $\psi_\varepsilon(t) \cdot X_\varepsilon(t) \geq 0$, and the m -accretivity of $A + K \operatorname{sgn}$, we obtain:

$$\|X_\varepsilon(t)\| + \varepsilon(K - \|d\|_\infty) \|\psi_\varepsilon(t)\| \leq \|X_\varepsilon(t - \varepsilon)\|, \quad \forall t \geq \varepsilon,$$

where $\|d\|_\infty$ denotes the supremum norm of the disturbance $d(t)$ over $t \in \mathbb{R}_+$. Define $\eta = (K - \|d\|_\infty) > 0$, which measures the effective robustness of the system against the disturbance. From the inequality above, it follows by induction:

$$\|X_\varepsilon(j\varepsilon)\| + j\varepsilon\eta \leq \|X_0\|, \quad j = 0, 1, \dots$$

This implies that the trajectory $X_\varepsilon(t)$ diminishes over time, and for sufficiently large j :

$$\|X_\varepsilon(j\varepsilon)\| = 0, \quad \text{for } j\varepsilon > \eta^{-1} \|X_0\|.$$

Taking the limit as $\varepsilon \rightarrow 0$, the uniform convergence of $X_\varepsilon(t)$ to $X(t)$ on compact intervals ensures:

$$X(t) = 0, \quad \text{for } t > \eta^{-1} \|X_0\|.$$

Thus, the solution $X(t)$ vanishes in finite time presented in (16), where $T(X_0)$ depends explicitly on the initial condition and the robustness parameter η . This completes the proof of the finite-time convergence of the solution to zero under the stated conditions. \square

Remark 1. *By using the Lyapunov function $V(t) = \|X(t)\|^2$, we can readily demonstrate the finite-time stability of the system (5).*

It can be observed that the settling time depends on the initial conditions, meaning that it is unbounded. In addition to finite-time convergence, this subsection presents an alternative solution for handling matched perturbations while ensuring fixed-time convergence of the system. Specifically, the design of an appropriate control law is crucial in achieving the desired robustness and convergence properties. Below, we present the following controller.

$$u(t) = -K \operatorname{sgn}(X(t)) - K_2 \|X(t)\|^\alpha \operatorname{sgn}(X(t)), \quad (19)$$

where $K > \|d\|_\infty$ is a positive constant ensuring robustness against the perturbation $d(t)$, which represents external matched disturbances. K_2 and $\alpha > 1$ are additional tuning parameters in $u(t)$ for enhancing convergence rates and addressing matched perturbations.

We now present the second result.

Theorem 5. *If $K_2 > 0$, $K - \|d\|_\infty > 0$, and $\alpha > 1$, assume that $0 \in A(0)$, and that the operator $A + (K - \|d\|_\infty) \operatorname{sgn}$ is m -accretive in $\mathbb{H} \times \mathbb{H}$. Under these conditions, the control law (19) guarantees that the system state $X(t)$, initialized as $X(0) = X_0 \in \overline{D(A)}$, is steered to the origin in fixed time, independent of the initial conditions. The settling time satisfies the following bound:*

$$T(X_0) \leq T_{\max},$$

where:

$$T_{\max} = \frac{1}{K_2(\alpha - 1)} + \frac{1}{K - \|d\|_\infty}. \quad (20)$$

Proof. Consider the following Lyapunov function:

$$2V(t) = \|X(t)\|^2 = \langle X(t), X(t) \rangle, \quad (21)$$

whose time derivative is computed as:

$$\frac{d}{dt}V = \langle X(t), \frac{d}{dt}X(t) \rangle. \quad (22)$$

Substituting

$$\begin{aligned} \frac{d}{dt}X(t) \in & -AX(t) + d(t) - K_2\|X(t)\|^\alpha \operatorname{sgn}(X(t)) \\ & - K \operatorname{sgn}(X(t)). \end{aligned} \quad (23)$$

into $\frac{d}{dt}V$, we obtain:

$$\begin{aligned} \frac{d}{dt}V \in & \langle X(t), -AX(t) + d(t) - K_2\|X(t)\|^\alpha \operatorname{sgn}(X(t)) \\ & - K \operatorname{sgn}(X(t)) \rangle, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{d}{dt}V = & -\langle X(t), AX(t) \rangle + \langle X(t), d(t) \rangle - K_2\|X(t)\|^{\alpha+1} \\ & - K\|X(t)\|. \end{aligned} \quad (25)$$

The following inequalities hold:

$$-\langle X(t), AX(t) \rangle \leq 0, \quad (26)$$

since A is m -accretive and $0 \in A(0)$. Using Cauchy-Schwartz and Holder inequalities it holds

$$\langle X(t), d(t) \rangle \leq \|X(t)\| \|d(t)\| \leq \|X(t)\| \|d\|_\infty, \quad (27)$$

where $\|d\|_\infty$ represents the upper bound of $d(t)$.

Using these inequalities, the time derivative of $V(t)$ can be estimated as:

$$\frac{d}{dt}V \leq -K_2\|X(t)\|^{\alpha+1} - (K - \|d\|_\infty)\|X(t)\| \quad (28)$$

$$= -K_2V(t)^{\frac{\alpha+1}{2}} - (K - \|d\|_\infty)V(t)^{\frac{1}{2}}. \quad (29)$$

Applying Lemma 1 under the conditions $K_2 > 0$, $K - \|d\|_\infty > 0$, and $\frac{\alpha+1}{2} > 1$, the closed-loop system is fixed-time stable and the settling time satisfies (20). \square

V. PARTIAL STABILITY EXTENSIONS

In this section, we provide several conditions for achieving partial finite-time stability. First, we analyze the system under feed-forward compensation to mitigate disturbances. Next, we address the scenario with a bounded uncontrolled part, offering solutions to ensure stability within a finite time.

Let us consider A a maximal monotone operator and B a closed linear relation, such that

$$\begin{cases} \frac{d}{dt}X(t) + AX(t) \in -Bd(t) + BU(t), & (30a) \\ X(0) = X_0 \in \overline{D(A)} \subset \mathbb{H}, & (30b) \end{cases}$$

We consider two cases. In the first, we directly compensate for the influence of the uncontrolled part on the controlled part using a feedforward term. In the second, the result relies on the fact that the influence of the uncontrolled part can be bounded by the controlled part, making direct compensation avoidable under the bounded hypothesis.

A. Feed-forward compensation

Let us consider (30) together with the following feedback law

$$\begin{aligned} U(t) = & -Bk_2\|BX(t)\|^\alpha \operatorname{sgn}(BX(t)) - Bk \operatorname{sgn}(BX(t)) \\ & - BAX(t). \end{aligned} \quad (31)$$

Here, the use of B highlights that the control is associated with the sgn operator, and the disturbance affects only part of the state. The computations performed above remain applicable, ensuring finite-time stabilization of the partial state defined by $B(0)$, which corresponds to the kernel (null space) of B^{-1} . This extension is of practical interest. In other words, we look at the equation of the quotient space $\mathbb{H}/N(B)$, which we established that this is a Hilbert space. Our feedforward term is $BAX(t)$ but we can also take

$$BX(t) \frac{\langle AX(t), BX(t) \rangle}{\|X\|_{\mathbb{H}/N(B)}^2} \quad (32)$$

as feedforward term similarly like it is done in [57]. We have the following result

Theorem 6. *Considering the abstract problem (30) together with the feedback law (31). If $K_2 > 0$, $K - \|d\|_\infty > 0$, and $\alpha > 1$, assume that A is m -accretive with $0 \in A(0)$ and B a linear closed relation, which satisfy*

$$\forall x, y \in X, \langle x, Bx \rangle \geq 0, \quad (33)$$

$$\langle x, By \rangle = \langle Bx, y \rangle, \quad (34)$$

$$BB = B \quad (35)$$

$$\forall x \in \mathcal{D}(A), \langle Ax, Bx \rangle \geq -c\|x\|^2 \quad (36)$$

and $BA - cI$ be maximal monotone then under these conditions, solution of the equation exist in the sens of Theorem 2, and

$$\forall X_0 \in \overline{D(A)}, \|X(t)\|_{X/N(B)} \xrightarrow{t \rightarrow T} 0. \quad (37)$$

The settling time T satisfies the following bound:

$$T(X_0) \leq T_{\max} = \frac{1}{k_2(\alpha - 1)} + \frac{1}{k_1 - \|d\|_\infty}, \quad (38)$$

Definition 3. *A linear relation T is a map from $X \rightarrow 2^X$, (or a set valued map) which has the following property*

$$\forall x, y \in X, \alpha \in \mathbb{R} \quad T(x + \alpha y) = T(x) + \alpha T(y). \quad (39)$$

A linear relation is positive if

$$\forall x \in X, \langle x, Tx \rangle \geq 0. \quad (40)$$

A more detail exposition of linear relation is given in [58].

Lemma 2. *Consider that B is a closed linear relation, which is also positive and auto-adjoint. The bilinear product*

$$\langle y, x \rangle_{\mathbb{H}/N(B)} = \langle y, Bx \rangle, \quad (41)$$

is a scalar product of the Hilbert space $\mathbb{H}/N(B)$, which is \mathbb{H} quotient by the kernel of B , i.e., $N(B)$. The associated semi-norm in \mathbb{H}

$$\|x\|_{\mathbb{H}/N(B)}^2 = \langle x, x \rangle_{\mathbb{H}/N(B)}, \quad (42)$$

is a norm of the quotient space $\mathbb{H}/N(B)$.

Proof. First as B is a closed linear relation in $\mathbb{H} \times \mathbb{H}$ we get that $N(B)$ is closed linear subspace of $\mathbb{H} \times \mathbb{H}$ ([58] Corollary I.2.4 on page 7 and comments on page 43). Using Theorem 8 in appendix one gets that the quotient space $\mathbb{H}/N(B)$ is a Banach space. Now

$$\forall x \in \mathbb{H}, \quad \langle x, Bx \rangle = 0 \Leftrightarrow x = 0 \text{ or } x = N(B), \quad (43)$$

therefore

$$\forall x \in \mathbb{H}/N(B), \quad \langle x, Bx \rangle = 0 \Leftrightarrow x = 0. \quad (44)$$

The remaining property positivity, bilinearity, and symmetry of the bilinear product are directly checked which completes the proof. \square

Lemma 3. Consider B is a linear closed relation where $B^2 = B$ and $B = B^*$. The operators $\mathbb{H} \ni x \rightarrow B \operatorname{sgn}(Bx)$ and $\mathbb{H} \ni x \rightarrow B \|Bx\| \operatorname{sgn}(Bx)$ are maximal monotone in the Hilbert space \mathbb{H} .

Proof. One gets that

$$\langle x - y, B \operatorname{sgn}(Bx) - B \operatorname{sgn}(By) \rangle \quad (45)$$

$$= \langle B(x - y), B \operatorname{sgn}(Bx) - B \operatorname{sgn}(By) \rangle \quad (46)$$

$$+ \langle (1 - B)(x - y), B(\operatorname{sgn}(Bx) - \operatorname{sgn}(By)) \rangle, \quad (47)$$

it holds

$$\langle B(x - y), B \operatorname{sgn}(Bx) - B \operatorname{sgn}(By) \rangle \quad (48)$$

$$= \langle Bx, \operatorname{sgn}(Bx) \rangle + \langle By, \operatorname{sgn}(By) \rangle \quad (49)$$

$$- \langle Bx, \operatorname{sgn}(By) \rangle - \langle By, \operatorname{sgn}(Bx) \rangle \quad (50)$$

$$\geq \|Bx\| + \|By\| - \|Bx\| - \|By\| \quad (51)$$

$$\geq 0. \quad (52)$$

Using $\langle Bx, y \rangle = \langle x, By \rangle$ and the fact that $(B - 1)B = B^2 - B = 0$ one gets

$$\langle x - y, B \operatorname{sgn}(Bx) - B \operatorname{sgn}(By) \rangle \geq 0, \quad (53)$$

and so $\mathbb{H} \ni x \rightarrow B \operatorname{sgn}(Bx)$ is monotone. Similarly one can obtain the monotonicity of $\mathbb{H} \ni x \rightarrow B \|Bx\| \operatorname{sgn}(Bx)$. For establishing that $\mathbb{H} \ni x \rightarrow B \operatorname{sgn}(Bx)$ is maximal in the set of monotone operator we check the range condition, $\forall y, f \in \mathbb{H}$

$$y + B \operatorname{sgn}(By) \ni f, \quad (54)$$

we can still decomposed the problem into two orthogonal part

$$By + B \operatorname{sgn}(By) \ni Bf, \quad (55)$$

$$(1 - B)y = (1 - B)f. \quad (56)$$

Following comments in page 251 in [55], we can show that the solution

$$y_f = \begin{cases} \frac{Bf(\|Bf\| - 1)}{\|Bf\|} + (1 - B)f & \text{if } \|f\| > 1, \\ (1 - B)f & \text{otherwise,} \end{cases} \quad (57)$$

exist and is unique

$$\|y_f - y_g\| \leq \|f - g\|, \quad \forall f, g \in \mathbb{H}. \quad (58)$$

This concludes the proof. \square

Now, the proof of Theorem 6 can be stated as follows:

Proof. The proof follows the same step of the proof of Theorem 5, replacing in the Lyapunov analysis the norm and the scalar product by the one defined in Lemma 2. The key point is using $B^2 = B$ on (24) where for system (6) with feedback (78) one gets

$$2V(t) = \langle X(t), BX(t) \rangle. \quad (59)$$

The derivative of V along the solution is

$$\begin{aligned} \frac{d}{dt} V &\in \langle BX(t), -AX(t) + Bd(t) \\ &\quad - BK_2 \|BX(t)\|^\alpha \operatorname{sgn}(BX(t)) \\ &\quad - BK \operatorname{sgn}(BX(t)) + BAX(t) \rangle, \end{aligned} \quad (60)$$

$$\begin{aligned} \frac{d}{dt} V &\in \langle X(t), B(-AX(t) + d(t)) \\ &\quad - K_2 \|BX(t)\|^\alpha \operatorname{sgn}(BX(t)) \\ &\quad - K \operatorname{sgn}(BX(t)) + BAX(t) \rangle, \end{aligned} \quad (61)$$

$$\begin{aligned} \frac{d}{dt} V &\in \langle BX(t), d(t) - K_2 \|BX(t)\|^\alpha \operatorname{sgn}(BX(t)) \\ &\quad - K \operatorname{sgn}(BX(t)) \rangle. \end{aligned} \quad (62)$$

The last equation is similar to (24), and the remaining follow.

$$\frac{d}{dt} V(t) \leq -k_2 \|X(t)\|^{\alpha+1} - (k_1 - \|d\|_\infty) \|X(t)\| \quad (63)$$

$$= -k_2 V(t)^{\frac{\alpha+1}{2}} - (k_1 - \|d\|_\infty) \sqrt{V(t)}. \quad (64)$$

This concludes the proof for the finite-time convergence component. It is worth noting that (36) is not utilized in the convergence analysis, but it serves as a sufficient condition to establish well-posedness. The reasoning is based on the fact that if $A - \omega I$ is monotone, then A is the generator of a C^0 -semigroup, as discussed in Chapter 4 of [55].

Primarily, (36) gives to the monotonicity of $BA + cI$. Furthermore, by Zorn's lemma, any monotone operator can always be extended to a maximal monotone operator (see Corollary 2.1 in [59]). Noting that $\mathcal{D}(BA) = \mathcal{D}(A)$, we apply Theorem 9 to establish that the solution is well-posed. \square

B. Bounded uncontrolled part

In this section, we no longer rely on the boundedness of the uncontrolled part to establish finite-time convergence and perturbation rejection. Instead, we consider the feedback law

$$U(t) = -k_2 \|BX(t)\|^\alpha \operatorname{sgn}(BX(t)) - k \operatorname{sgn}(BX(t)), \quad (65)$$

we have the following result

Theorem 7. Considering the abstract problem (30) together with the feedback law (31). If $k_2 > \omega_2$, $k_1 - \omega_1 - \|d\|_\infty > 0$, and $\alpha > 1$, assume that A is m -accretive with $0 \in A(0)$ and B a linear closed relation, which satisfy

$$\forall x, y \in X, \quad \langle x, Bx \rangle \geq 0, \quad (66)$$

$$\langle x, By \rangle = \langle Bx, y \rangle, \quad (67)$$

$$BB = B. \quad (68)$$

$$\forall x \in \mathcal{D}(A), \quad \langle -Ax, Bx \rangle \leq \omega_1 \|Bx\|^{\alpha+1} + \omega_2 \|Bx\|, \quad (69)$$

then under these conditions, solution of the equation exist in the sens of Theorem 2, and

$$\forall X_0 \in \overline{D(A)}, \quad \|X(t)\|_{X/N(B)} \xrightarrow{t \rightarrow T} 0. \quad (70)$$

The settling time T satisfies the following bound:

$$T(X_0) \leq T_{\max} = \frac{1}{(k_2 - \omega_2)(\alpha - 1)} + \frac{1}{k_1 - \omega_1 - \|d\|_{\infty}}. \quad (71)$$

Proof. The proof follows the same step of the proof of Theorem 5, replacing in the Lyapunov analysis the norm and the scalar product by the one defined in Lemma 2. The key point is using $B^2 = B$ on (24) where for system (6) with feedback (78) one gets

$$2V(t) = \langle X(t), BX(t) \rangle. \quad (72)$$

The derivative of V along the solution is

$$\begin{aligned} \frac{d}{dt}V(t) \in & \langle BX(t), -AX(t) + Bd(t) \\ & - Bk_2 \|BX(t)\|^{\alpha} \operatorname{sgn}(BX(t)) \\ & - Bk_1 \operatorname{sgn}(BX(t)) \rangle, \end{aligned} \quad (73)$$

$$\begin{aligned} \frac{d}{dt}V(t) \in & \langle BX(t), -AX(t) \rangle + \langle BX(t), Bd(t) \\ & - Bk_2 \|BX(t)\|^{\alpha} \operatorname{sgn}(BX(t)) \\ & - Bk_1 \operatorname{sgn}(BX(t)) \rangle \end{aligned} \quad (74)$$

$$\begin{aligned} & \leq \omega_1 \|BX(t)\|^{\alpha+1} + \omega_2 \|BX(t)\| \\ & - k_2 \|BX(t)\|^{\alpha+1} - (k_1 - \|d\|_{\infty}) \|BX(t)\| \quad (75) \\ & = - (k_2 - \omega_2) V(t)^{\frac{\alpha+1}{2}} \\ & - (k_1 - \omega_1 - \|d\|_{\infty}) \sqrt{V(t)}. \end{aligned} \quad (76)$$

The finite-time convergence is thus established, and the well-posedness follows directly from Theorem 2 and Lemma 2. \square

The requirement in Theorem 6 for $BA - cI$ to be maximal monotone imposes a significant constraint on the feedforward control law. In [57], with the use of the feedforward term (32), the well-posedness was established by leveraging the fact that this term is Lipschitz and associated with $BA - cI$ being monotone. In Theorem 7, the condition shifts to requiring that the uncontrolled part can be bounded by the controlled part, which, while also a stringent assumption, represents a different type of constraint.

VI. TOY EXAMPLE

To illustrate the method, we apply the results to a heat equation.

A. Heat equation with Neumann's boundary

$$\begin{cases} u_t(t, x) - (a(x)u_x(t, x))_x + d(t) \in U(t), & (77a) \\ u_x(t, 0) = 0, & (77b) \\ u_x(t, 1) = 0. & (77c) \end{cases}$$

The input is defined as

$$U(t) = -k_1 \operatorname{sgn}(X(t)) - k_2 \|X(t)\|^{\alpha} \operatorname{sgn}(X(t)), \quad (78)$$

in which

$$X(t) = [u(t, \cdot)], \quad (79)$$

and

$$\|X(t)\|^2 = \int_0^1 u^2 dx. \quad (80)$$

The operator A is defined as

$$\forall z \in \mathcal{D}(A), Az = (az')', \quad (81)$$

with

$$D(A) = \{H^2, z'(0) = 0, z'(1) = 0\}. \quad (82)$$

Obtaining that A is a maximal monotone operator in H^1 is quite direct and can be found in classical book [60]. Therefore Theorem 2 applied and we get that

$$\forall X_0 \in \overline{D(A)}, \quad \|X(t)\|_{X/N(B)} \xrightarrow{t \rightarrow T} 0. \quad (83)$$

The settling time T satisfies the following bound:

$$T(X_0) \leq T_{\max} = \frac{1}{k_2(\alpha - 1)} + \frac{1}{k_1 - \|d\|_{\infty}}. \quad (84)$$

B. Numerical scheme

The operator $Az = z''$ is discretized in space by

$$A_d = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \frac{1}{\Delta x^2}. \quad (85)$$

The space discretization model is

$$\dot{u}_d(t) = A_d u_d(t) + U_d(t) + d(t), \quad (86)$$

where

$$U_d(t) = -k_1 \operatorname{sgn}(u_d(t)) - k_2 \|u_d(t)\|^{\alpha} \operatorname{sgn}(u_d(t)), \quad (87)$$

$$\|u_d(t)\|^2 = \langle u_d(t), u_d(t) \rangle. \quad (88)$$

We then employed the Crank-Nicolson method for time discretization.

$$u_d[k+1] - u_d[k] = (A_d u_d[k] + U[k] + d[k])\Delta t, \quad (89)$$

$$u_d[k+1] - u_d[k] = (A_d u_d[k+1] + U[k] + d[k])\Delta t, \quad (90)$$

$$u_d[k+1] = (2 - \Delta t A)^{-1} [(2 + \Delta t A)u_d[k] + 2U[k]\Delta t + 2d[k]\Delta t]. \quad (91)$$

The following numerical results are presented under various parameters listed in Table I. Figures 1, 2, and 3 illustrate, respectively, the distributed state $u(t, \cdot)$, the distributed control law $U(t, \cdot)$, and the control objective $\log(\|u(t, \cdot)\|)$ for the finite-time case. Specifically, when $k_2 = 0$, the state converges to zero in finite time, which depends on the initial condition. Once the state has converged, the control law matches the disturbance, as shown in Figure 2. Furthermore, the norm of the distributed state decreases to 10^{-10} , which can be interpreted as a numerical zero or explained by the approximation of the sign function used in the numerical simulation. Indeed, the set-valued map has been reformulated as a discontinuous single-valued map.

Symbol	value 1	value 2	value 3
N	100	.	.
t_{end}	10	.	.
Δt	10^{-3}	.	.
$a(x)$	1	.	.
$d(t)$	$\sin(t)$.	.
k_1	3	.	.
k_2	0	$\frac{20}{9}$	$\frac{20}{9}$
α	1.1	.	.
$u(0, \cdot)$	$[1, \dots, 1]$	$[1, \dots, 1]$	$[100, \dots, 100]$
T_{max}	$< +\infty$	5	5

Table I
PARAMETER VALUES FOR THE SIMULATION.

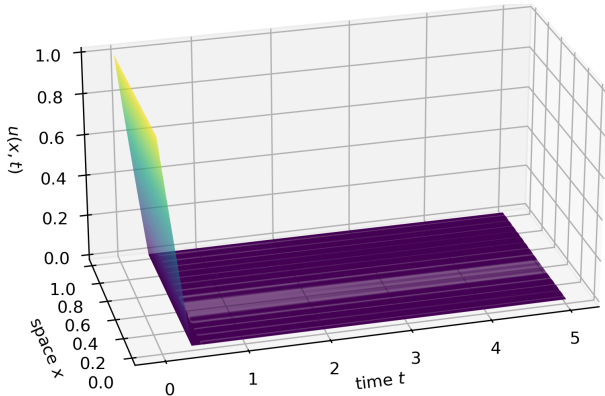


Figure 1. Distributed state $u(t, \cdot)$ for values 1

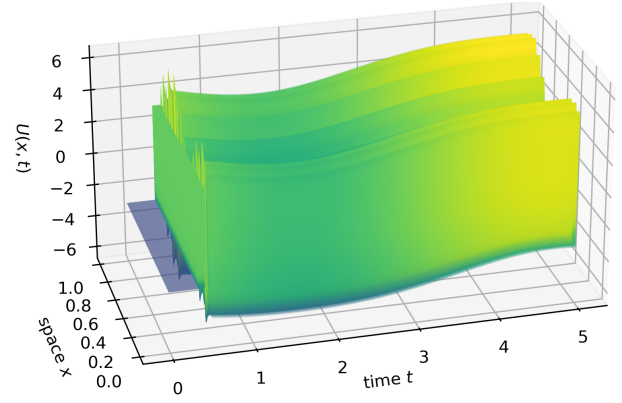


Figure 2. Distributed control $U(t, \cdot)$ for values 1

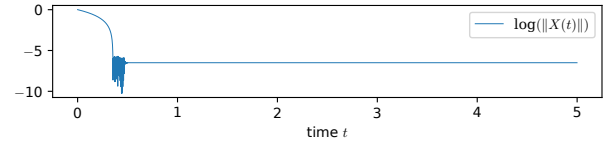


Figure 3. Objective $\log(\|X(t)\|)$ for values 1

Figures 4, 5, and 6 depict, respectively, the distributed state $u(t, \cdot)$, the distributed control law $U(t, \cdot)$, and the control objective $\log(\|u(t, \cdot)\|)$ for the fixed-time case, where the fixed time is set to $T_{max} = 5$. Specifically, when $k_1 = 3$, $k_2 = 20/9$, and $\alpha = 1.1$, the state converges to zero within a fixed time that is independent of the initial conditions. After convergence, the control law aligns with the disturbance, as illustrated in Figure 5. Moreover, the norm of the distributed state decreases to 10^{-10} more rapidly compared to the previous case, primarily due to the additional dissipation introduced into the dynamics. This decrease can still be interpreted as numerical zero or attributed to the approximation of the sign function employed in the numerical simulation.

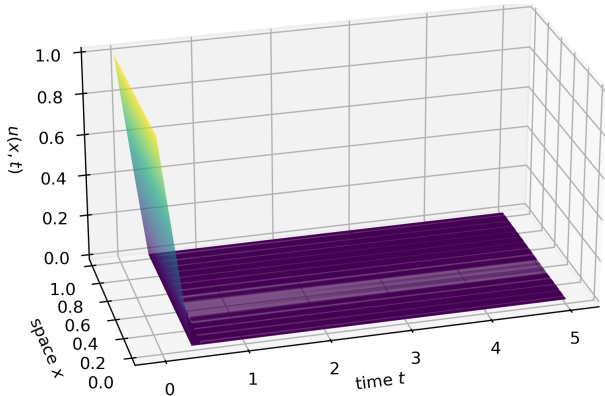
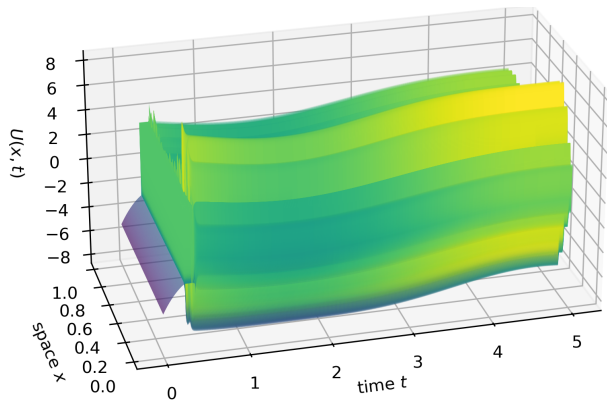
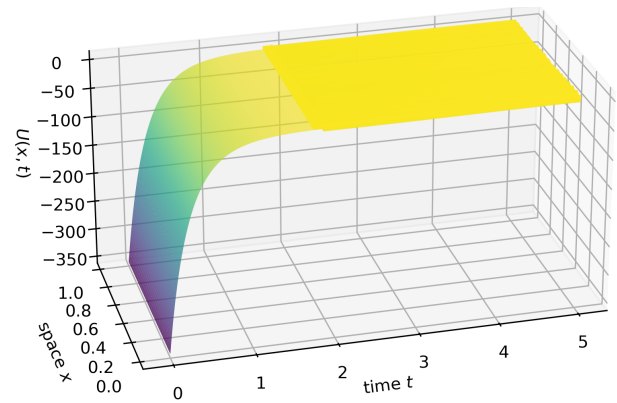
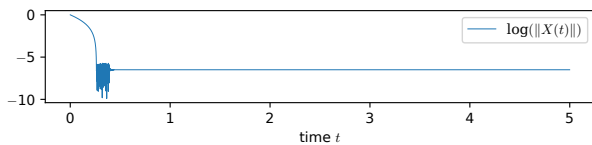
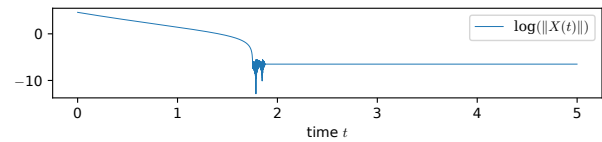
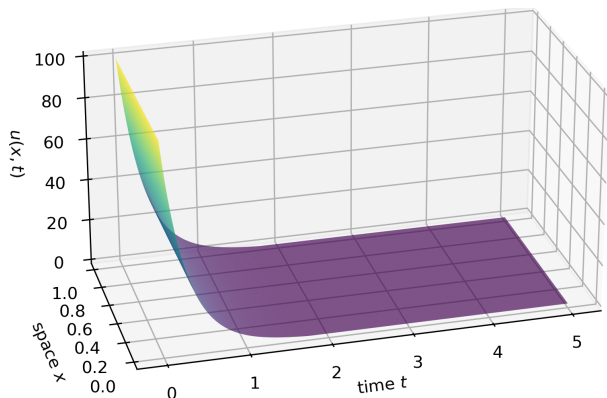


Figure 4. Distributed state $u(t, \cdot)$ for values 2

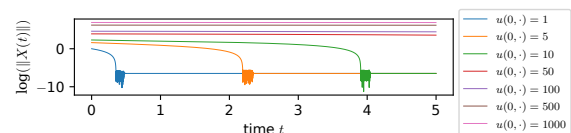
Figure 5. Distributed control $U(t, \cdot)$ for values 2Figure 8. Distributed control $U(t, \cdot)$ for values 3Figure 6. Objective $\log(\|X\|(t))$ for values 2Figure 9. Objective $\log(\|X\|(t))$ for values 3

In the final case presented in Table I, the differences between Figures 4, 5, 6 and Figures 7, 8, 9 lie in the initial conditions. Although the system converges more slowly in this scenario, it still achieves convergence within the fixed time.

Figure 7. Distributed state $u(t, \cdot)$ for values 3

C. Comparison of fixed-time and finite-time stabilization for the objective function

In this section, we present numerical results to highlight the distinction between finite-time and fixed-time stabilization. Figure 10 illustrates the evolution of the norm $\|X(t)\|$ over time on a logarithmic scale for a range of initial conditions, $X_0 \in \{1, 5, 10, 50, 100, 500, 1000\}$. The results reveal that the settling time is dependent on the magnitude of the initial condition. Specifically, larger initial values of X_0 lead to a longer convergence time, which is indicative of finite-time stabilization, where the convergence rate is influenced by the initial perturbation. In contrast, when applying fixed-time control, as depicted in Figure 11, we observe that the settling time is upper-bounded by a constant. This constant settling time is independent of the initial conditions, thereby demonstrating that the convergence time is invariant with respect to the initial state. This behavior is characteristic of fixed-time stabilization, where the convergence time is designed to be constant and independent of the initial conditions.

Figure 10. Finite time with $k_1 = 3, k_2 = 0$ and several initials conditions

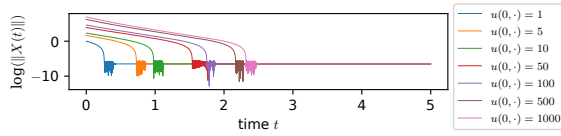


Figure 11. Fixed time $T_{max} = 5$ with $k_1 = 3, k_2 = 20/9, \alpha = 1.1$ and several initials conditions

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we investigate the finite-time stabilization of a class of abstract systems. We extend our results to fixed-time stabilization, where the settling time is bounded. Both analyses are conducted in the presence of matched persistence (L_∞) perturbations. For finite-time stabilization, we employ exact solutions in conjunction with the abstract Cauchy problem, while for fixed-time stabilization, we apply a Lyapunov approach, analogous to methods used for finite-dimensional systems associated with norm in functional space. We establish the well-posedness of the closed-loop system in both cases using the theory of maximal monotone operators. A theoretical and numerical example based on the heat equation is provided to illustrate the results.

For future work, we plan to explore more complex abstractions, including the incorporation of nonlinearities in boundary control, and specific case of partial stabilization.

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APPENDIX

Theorem 8 ([61, Theorem 1.41 (d)]). *Let S be a closed subspace of a topological vector space X . If X is a Banach*

space, then the quotient space X/S is also a Banach space.

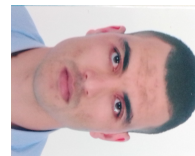
Theorem 9 (Theorem 1.5 in [55]). *Let H be a reflexive Banach space, and let A and B be maximal monotone subset of $H \times H^*$. If it holds*

$$(\text{int } \mathcal{D}(A)) \cap \mathcal{D}(B) \neq \emptyset, \quad (92)$$

then $A + B$ is maximal monotone.

Theorem 10 (Theorem 21 and following Remark in [62]). *Let us consider H being a Hilbert space, A a maximal monotone operator, and $f \in W^{1,1}(0, T; H)$, then for all $X_0 \in \mathcal{D}(A)$, there exists a unique $X(t) : \mathbb{R}^+ \rightarrow H$ such that $\forall t \geq 0$*

- $X(t) \in \mathcal{D}(A)$.
- $\frac{d^+}{dt} X(t) \in L^\infty(0, T; H)$.
- $\frac{d}{dt} X(t) + AX(t) \ni f$ and $X(0) = X_0$.
- X is differentiable from the right $\forall t \in [0, T)$ and $\frac{d^+}{dt} X(t) + (AX(t) - f(t))^0 = 0$. Moreover, $|\frac{d^+}{dt} X(t)| \leq |\frac{d^+}{dt} X(0)| + \int_0^t |\frac{df}{ds}(s)| ds$.
- the mapping $(X_0, f) \mapsto X$ can be extended by continuity from $\mathcal{D}(A) \times L^1(0, T; H)$ into $C(0, T; \overline{\mathcal{D}(A)})$. Moreover, in the case where $f = 0$, the mapping $X_0 \mapsto X(t)$ defines a nonlinear contraction C_0 -semigroup.



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