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# Computation of harmonic functions on higher genus surfaces

Mickaël Nahon\*, Édouard Oudet†

## Abstract

We introduce a method to compute efficiently and with arbitrary precision a basis of harmonic functions with prescribed singularities on a general compact surface of genus two and more. This basis is obtained as a composition of theta functions and the Abel-Jacobi map, which is approximated at spectral speed by complex polynomials. We then implement this method to compute harmonic extensions on genus 2 surfaces with boundary, that are described by their Fenchel-Nielsen coordinates and a smooth parametrization of the boundary. Finally, we prove the spectral convergence of the method for the harmonic extension.

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# 1 Introduction

Harmonic functions are a ubiquitous class of functions that appear in potential theory, fluid dynamics, heat conduction, and more generally in solutions of various optimization problems. We are concerned with the following question: given a compact Riemannian surface  $S$  with boundary  $\partial S$ , and a function  $g : \partial S \rightarrow \mathbb{R}$ , how can we compute precisely and efficiently the harmonic extension

$$u : S \rightarrow \mathbb{R} \text{ such that } \begin{cases} \Delta u = 0 & \text{in } S \\ u = g & \text{in } \partial S \end{cases} . \quad (1.1)$$

When  $S$  is a bounded smooth subset of  $\mathbb{C}$ , one possible approach is through the use of Runge's theorem: choosing one point (which we will call *pole*)  $(w_j)_{j=1,\dots,N}$  in each bounded component of  $\mathbb{C} \setminus S$ , then any harmonic function on  $S$  may be approached by functions of the form

$$\sum_{j=1}^N a_{j,0} \log |z - w_j| + \Re \sum_{j=1}^N \sum_{m=1}^n a_{j,m} (z - w_j)^{-m} + \Re \sum_{m=0}^n b_m z^m \quad (1.2)$$

for some real coefficients  $(a_{j,0})$  and complex coefficients  $(a_{j,m})_{1 \leq m \leq n}$ ,  $(b_m)_{0 \leq m \leq n}$  for some sufficiently large  $n$ . By matching the boundary data (for instance through a least square method on a sampling of the boundary) this gives an efficient method of approximation: this is discussed in the paper [12] with the approximation of Green functions. For a general analytic data one may expect a spectral speed of convergence, in the sense that there is a function  $u_n$  of the form (1.2) that approaches the solution  $u$  of (1.1) with

$$\limsup_{n \rightarrow \infty} \|u - u_n\|_{L^\infty(S)}^{1/n} = q$$

where  $q (< 1)$  depends on how far the solution  $u$  extends harmonically beyond  $S$ . The full characterization of  $q$  is given by the Bernstein-Walsh theorem (see [14, Ch. VII] or [11, Th 6.3.1]).

Moreover this kind of approximation have been adapted to domains with less regularity, for instance in the series of papers [6, 7] in domains with angular points, where moving poles appropriately close to the corners give  $\mathcal{O}(e^{-c\sqrt{n}})$  approximation error, or [2] where this was applied to two-dimensional Stokes equation. See also the more recent paper [13] that explores the rate of convergence in smooth non-convex domains through a careful positioning of the poles.

This approach was generalized in [10] to the case where  $S$  is a subset of a torus, where the authors constructed a similar basis of solutions using the Weierstrass elliptic functions, which may be computed efficiently with high accuracy.

The goal of this paper is to generalize this approach to any compact surface: suppose that  $S$  is a smooth subset of some closed compact surface  $X$ , can we compute efficiently and precisely a basis of harmonic functions on  $X$  (with some poles) that will approximate any harmonic function on  $S$  ?

As we will see, we need to compute two types of functions:

- Given two distinct points  $v, w \in X$ , we need a harmonic function  $z \in X \setminus \{v, w\} \rightarrow \mathbb{R}$  such that

$$G(z) = \log \text{dist}(z - v) + \mathcal{O}_{z \rightarrow v}(1), \quad G(z) = -\log \text{dist}(z - w) + \mathcal{O}_{z \rightarrow w}(1),$$

where  $\text{dist}$  is the distance induced by the metric. When  $X$  is the Riemann sphere, this is the role played by  $G(z) = \log \left| \frac{z-v}{z-w} \right|$ .

- Given a point  $w \in X$ , and an integer  $n \in \mathbb{N}_{\geq 1}$ , we need two harmonic functions defined on  $X \setminus \{w\}$  with a pole of order  $n$  at  $w$ . This is the role played by

$$\Re((z - w)^{-n}), \Im((z - w)^{-n})$$

when  $X$  is the Riemann sphere. Note that these are obtained by taking the real and imaginary part of a fixed holomorphic function, but there is an obstruction to this property already when  $X$  is a torus instead: there is no meromorphic function with a unique pole of order 1.

The case where  $X$  is a subset of the Riemann sphere or of a torus is already known from [10], and the case where  $X$  is non-oriented may be deduced from the oriented case through an orientable double cover. Thus, we focus on the case where  $X$  is a compact oriented surface of genus  $g \geq 2$ .

We will suppose the surface  $X$  is given as a gluing of hyperbolic polygons, and in practice it will be given by the gluing of  $4g - 4$  right-angled hyperbolic hexagon (see the next section for more detail on Fenchel-Nielsen coordinates): while this may seem restrictive, the uniformization theorem states that any higher genus surface is conformal to a surface of constant negative curvature, which may be decomposed as hexagons in this way. Since harmonic functions are preserved by conformal change of metric, we lose no generality with this assumption. Notice however that it is not trivial to compute this hexagon decomposition from a general non-uniformized surface.

In terms of efficiency, our goal is that for a fixed surface  $X$ , we may build once and for all the building blocks to compute directly a basis of harmonic function for any location of the poles: as we will see, our method requires the computation of  $g$  holomorphic functions that act as the coordinates of the Abel-Jacobi map.

## 1.1 Gluing procedure and Fenchel-Nielsen coordinates

In this section we fix a constructions of the surface  $X$ . Consider a finite number of hyperbolic polygons (meaning that their edges are hyperbolic geodesic) in the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  denoted as the closed sets  $(H_p)_{p=1, \dots, m}$ . We call  $\gamma_{p,1}, \dots, \gamma_{p,c(p)}$  the successive sides of  $H_p$ . We then consider a gluing rule for these polygons: each side  $\gamma_{p,i}$  is glued to a unique side  $\gamma_{q,j}$  (we write  $(p, i) \rightarrow (q, j)$ , and this is a symmetric relation) in the sense that there exists a disk automorphism  $g_{p,i} \in PSU(1, 1)$  where

$$PSU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in M_2(\mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1 \right\} / \{\pm I_2\},$$

such that  $g_{p,i}(\gamma_{p,i}) = \gamma_{q,j}$  and the sets  $g_{p,i}(H_p)$  and  $H_q$  have disjoint interior. This implies in particular that glued sides have the same hyperbolic length, and the necessary and sufficient condition for this gluing rule to define a smooth surface is that around each vertex, the sum of the angular opening of all the vertices it is identified with is  $2\pi$ : we will suppose this condition is verified.

Moreover, we will see the embeddings  $H_p \rightarrow X$  ( $p = 1, 2, \dots, m$ ) as a system of charts, since these embedding may be extended to a small neighbourhood of the polygons.

A particular case of this decomposition is when each  $H_p$  is a right-angled hexagon. We remind that any three length  $(l_1, l_2, l_3) \in \mathbb{R}_{>0}$  define (up to isometry) a unique right-angled hexagon where the first, third and fifth sides have respective hyperbolic lengths  $l_1/2, l_2/2, l_3/2$ .

Then any compact oriented surface of genus  $g \geq 2$  is conformally equivalent to a hyperbolic surface obtained as the gluing of  $4g - 4$  right-angled hexagons with the following gluing rules: each hexagon associated to  $(l_1, l_2, l_3)$  is paired with a mirrored hexagon (of lengths  $(l_1, l_3, l_2)$ ) by gluing the alternate side (and obtaining a pair of pants i.e. a hyperbolic compact surface with geodesic boundary that is homeomorphic to a sphere with three disks removed). The  $2g - 2$  pants are then associated by gluing boundary geodesics of same length, with a possible rotation in the gluing of the

geodesics: this leaves a total of  $6g - 6$  independent parameters with  $3g - 3$  lengths (denoted  $l_i > 0$ ) and  $3g - 3$  angles (denoted  $t_i \in \mathbb{R}/\mathbb{Z}$ ), that constitute the Fenchel-Nielsen coordinates of the surface. This construction is illustrated in the genus 2 case in figure 1.

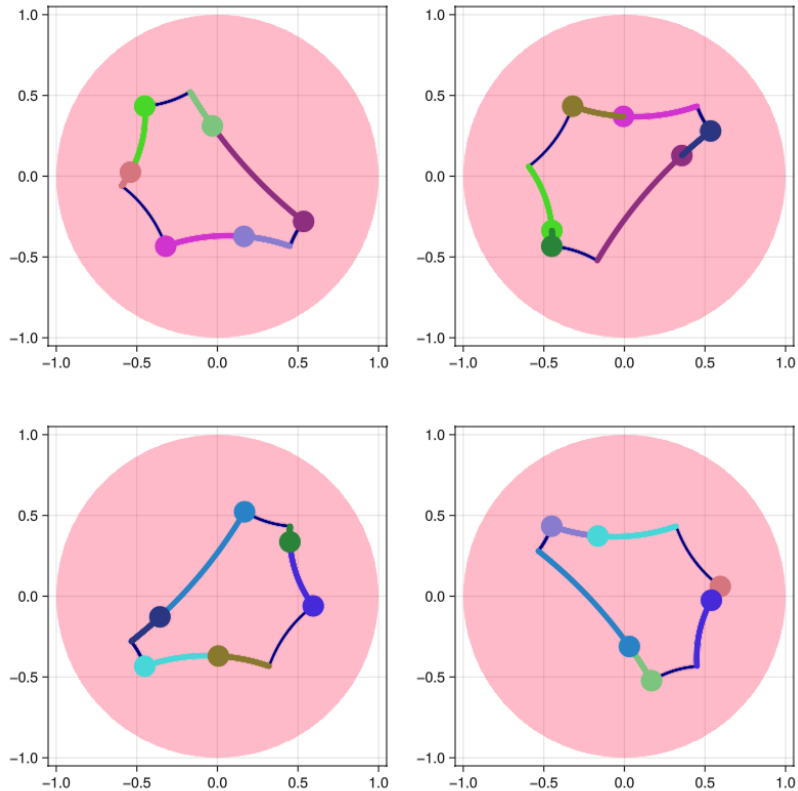


Figure 1: Gluing conditions associated to Fenchel-Nielsen coordinates: every geodesic is divided in four pieces. In this example  $(l_1, t_1; l_2, t_2; l_3, t_3) = (\frac{3}{2}, \frac{1}{10}; 2, \frac{2}{10}; \frac{5}{2}, \frac{1}{8})$ .

## 1.2 Main result

The principal contributions of this paper are both the explicit and computable construction of a basis of harmonic function with prescribed poles, and a complete convergence analysis of the associated discrete scheme: we establish a spectral convergence result for the approximation of harmonic functions which generalize the theorem of Bernstein-Walsch (see [14, Ch. VII] or [11, Th 6.3.1]) to general compact surfaces.

The construction of the basis is as follows:

- 1) We construct the Abel-Jacobi map  $X \rightarrow \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$ , which is an holomorphic immersion of the surface in the complex torus  $\mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$  (here  $\tau$  is a symmetric matrix with positive imaginary part). While there seem to be no explicit way of constructing these functions, we give an elementary least-square method that provides us with a polynomial approximation of the Abel-Jacobi map on each hexagon of the decomposition. This step is independent of the location of the pole, meaning that for a given surface we only need to compute this function once to obtain the basis associated to any pole.
- 2) We construct the Green function associated to any two poles  $v, w$ , meaning a harmonic function  $\log |\hat{\sigma}_{v,w}| : X \setminus \{v, w\} \rightarrow \mathbb{R}$  such that

$$\log |\hat{\sigma}_{v,w}(z)| = \log |z - v| + \mathcal{O}_{z \rightarrow v}(1), \quad \log |\hat{\sigma}_{v,w}(z)| = -\log |z - w| + \mathcal{O}_{z \rightarrow w}(1).$$

This function is obtained as a composition of theta functions and Abel-Jacobi functions, as is detailed in subsection 3.2.

- 3) Finally, for a given generic pole  $w$  (in the sense that it is not a Weierstrass point, as is explained in subsection 2.4), we construct a sequence of functions

$$(\widehat{\wp}_n(z, w))_{1 \leq n \leq g}, (\check{\wp}_n(z, w))_{g+1 \leq n \leq 2g+1}, (\widetilde{\wp}_n(z, w))_{n \geq 2g+2}$$

that may respectively be found in the subsections 3.4, 3.5, 3.6, such that for fixed  $w$ , the  $n$ -th function of the sequence above is harmonic with respect to  $z$  with a unique pole at  $w$  of order  $n$ . More precisely up to a multiplicative constant they have the following asymptotic form near  $w$ :

$$\underbrace{\frac{1 + \mathcal{O}_{z \rightarrow w}(z - w)}{(z - w)^k}}_{\text{meromorphic}} + \underbrace{\mathcal{O}_{z \rightarrow w}(1)}_{\text{harmonic}}.$$

- 4) The case where  $w$  is a Weierstrass point - something which is only possible for a finite number of points of the surface - needs only a small tweaking in the construction of the functions to work: the basis is instead

$$(\widehat{\wp}_n)_{1 \leq n \leq 2g-1}, (\check{\wp}_n)_{2g \leq n \leq 4g-1}, (\widetilde{\wp}_n)_{n \geq 4g},$$

where the functions are constructed in a similar way. We refer to the subsection 3.7 for the modifications.

For any finite set  $S \subset X$ ,  $n \geq 1$ , we denote by  $\mathcal{H}_n(S)$  the set of harmonic function  $h : X \setminus S \rightarrow \mathbb{R}$  such that  $h(z) = \mathcal{O}_{z \rightarrow w}(d_X(z, w)^{-n})$  for each  $w \in S$ , where  $d_X$  is the hyperbolic distance in  $X$ .

**Theorem 1.** *Let  $w_1, \dots, w_N$  be non-Weierstrass points of  $X$ , then  $\mathcal{H}_n(\{w_1, \dots, w_N\})$  admits the following basis: the constant function, the functions*

$$\log |\widehat{\sigma}_{w_j, w_{j+1}}|, \quad j = 1, \dots, N - 1$$

and the real and imaginary parts of

$$(\widehat{\wp}_k(\cdot, w_j))_{1 \leq k \leq g \wedge n}, (\check{\wp}_k(\cdot, w_j))_{g+1 \leq k \leq (2g+1) \wedge n}, (\widetilde{\wp}_k(\cdot, w_j))_{2g+2 \leq k \leq n}, \quad j = 1, \dots, N.$$

In the case where one of the  $(w_j)_{j=1, \dots, N}$  is a Weierstrass point, the result still holds with the modified basis ; in fact we could use the modified basis in every case, but it is computationally more costly to do so.

For the next theorem, we introduce the Green function of a smooth open set  $A \subset X$ , denoted

$$\mathcal{G}_A : \begin{cases} \overline{A}^2 \rightarrow \mathbb{R} \cup \{+\infty\} \\ (z, w) \mapsto \mathcal{G}_A(z, w) \end{cases}, \quad (1.3)$$

the function that verifies

$$-\Delta_z \mathcal{G}_A(\cdot, w) = 2\pi \delta_w \text{ in } A, \quad \mathcal{G}_A(z, w) = 0 \text{ for } (z, w) \in \partial A \times A. \quad (1.4)$$

We remind that  $\mathcal{G}_A$  is symmetric in  $(z, w)$ . By classical potential theory,  $\mathcal{G}_A$  admits an extension to general open sets of  $X$  under the hypothesis that  $X \setminus A$  has positive capacity. In this case it is in general no longer continuous up to the boundary, however in the case where no connected component of  $\partial A$  is reduced to a point, then it is known that every point of  $\partial A$  is regular i.e.  $\lim_{z \rightarrow z_0} \mathcal{G}_A(z, w) = 0$  for every  $z_0 \in \partial A$ ,  $w \in A$  (see for instance [11, Th 4.2.2, 4.4.9]).

**Theorem 2.** *Let  $X$  be a compact oriented surface of genus  $g \geq 1$ , let  $K$  be a compact subset of  $X$  such that none of its connected component is reduced to a point and  $X \setminus K$  has a finite number of connected components, denoted  $D_1, \dots, D_N$ . Let  $w_1, \dots, w_N$  be a point in each  $D_j$ . Let  $h$  be a harmonic function defined on a neighbourhood of  $K$ , and  $h_n$  be the best  $L^\infty(K)$  approximation of  $h$  by an element of  $\mathcal{H}_n(\{w_1, \dots, w_N\})$ . Then for any  $t > 0$  the two following properties are equivalent:*

(a)  *$h$  extends harmonically to  $K \sqcup_{j=1}^N \{\mathcal{G}_{D_j}(\cdot, w_j) < t\}$ .*

(b)  $\limsup_{n \rightarrow \infty} \|h - h_n\|_{L^\infty(K)}^{1/n} \leq e^{-t}$ .

This spectral speed of approximation is observed in solving the harmonic extension problem (see figure 6). We include the genus  $g = 1$  case in the result and this is the only instance in the paper where genus 1 is involved: in this case the basis may be replaced with the (analogous) basis of [10, Th 1.2]. We will use the notations we introduce for genus  $g \geq 2$  and will point out the difference between genus 1 and 2 in the proof.

### 1.3 Organisation of the paper

In the second section we introduce several well-known concept in compact Riemann surfaces: the notion of canonical basis of homology, the space of 1-form and the Abel-Jacobi map, the function  $\Theta$  and its application to the construction of quasi-periodic meromorphic functions, and finally we remind several consequences of the Riemann-Roch theorem and some fact on Weierstrass points.

In the third section, we give a method to effectively construct the Abel-Jacobi map from polynomial approximation, and then the different elements of the basis of harmonic functions for fixed poles. We then prove the main results 1 and 2.

In the fourth section we detail the computational methods used to compute effectively the previously defined basis of harmonic functions with arbitrary precision, and we apply this in the fifth section to the harmonic extension problem in a surface.

In the last section, we give the technical details of the proof of convergence of the least-square method in the approximation of Abel-Jacobi map.

## 2 Some facts and notations on compact Riemann surfaces

We remind in this section some results of the space of 1-forms on a surface, on the Abel-Jacobi map and on  $\Theta$  functions that will be used later to define our basis of harmonic functions.

### 2.1 Canonical basis of homology

Let us go back to the general construction of  $X$  by the gluing of the polygons  $H_1, H_2, \dots, H_m$  introduced in subsection 1.1 (given for instance by the hexagon decomposition). Let  $\Gamma$  be the group of  $PSU(1, 1)$  that is generated from the compositions

$$g_{p_n, i_n} \circ g_{p_{n-1}, i_{n-1}} \circ \dots \circ g_{p_1, i_1}$$

where

- $p_1 = 1$ .
- For all  $k = 1, \dots, n - 1$  there exists  $j \in \{1, \dots, c(p_{k+1})\}$  such that  $(p_k, i_k) \rightarrow (p_{k+1}, j)$ .
- There exists  $j \in \{1, \dots, c(1)\}$  such that  $(p_n, i_n) \rightarrow (1, j)$ .

$X$  may then be identified with the quotient  $\mathbb{D}/\Gamma$  through the mapping  $z \in H_1 \mapsto z \bmod \Gamma$ , which extends uniquely to a conformal bijection  $X \rightarrow \mathbb{D}/\Gamma$ .

We fix in all that follows a canonical basis of homology of  $X$ , meaning a set of  $2g$  closed curves  $(a_i)_{i=1,\dots,g}$  and  $(b_i)_{i=1,\dots,g}$  (which is, in our case, given by a sequence of hyperbolic segment in each polygon  $H_p$ ; the curves  $(a_i)$  and  $(b_i)$  need not be geodesic), such that up to homotopy  $(a_i, a_j), (b_i, b_j), (a_i, b_j)$  do not intersect when  $i \neq j$ , and  $(a_i, b_i)$  intersect once with positive orientation. Each  $a_i$  (resp  $b_i$ ) is homotopic to a unique simple geodesic loop and is associated (in the disk model) to an element  $A_i$  (resp  $B_i$ ) of  $\Gamma$  that preserves a lifting of this geodesic. Up to reordering the curves appropriately then  $A_1, B_1, A_2, B_2, \dots, A_g, B_g$  is a generating family of  $\Gamma$  such that

$$A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} = I_2,$$

and this relation is sufficient to give a presentation of  $\Gamma$ . We refer to [4, I.2.5.] for the construction of a canonical  $4g$ -gon from a general polygonal gluing.

## 2.2 1-forms and Abel-Jacobi maps

For a subset  $E$  of  $\mathbb{C}$  we denote by  $\mathcal{O}(E)$  (resp  $M\mathcal{O}(E)$ ) the set of holomorphic (resp meromorphic) functions defined on a neighbourhood of  $E$ .

We denote by  $\Omega^k(X)$  (resp  $\Omega^k(\mathbb{D}/\Gamma)$ ) the space of holomorphic  $k$ -forms on  $X$  (resp  $\mathbb{D}/\Gamma$ ), and by  $M\Omega^k(X)$  (resp  $M\Omega^k(\mathbb{D}/\Gamma)$ ) the space of meromorphic  $k$ -forms on  $X$  (resp  $\mathbb{D}/\Gamma$ ).

To be more precise, these are defined as follows:  $\Omega^k(X)$  is the subset of holomorphic functions

$$f = (f_{|1}, \dots, f_{|m}) \in \mathcal{O}(H_1) \times \dots \times \mathcal{O}(H_m)$$

such that for every gluing  $(p, i) \rightarrow (q, j)$  we have

$$f_{|p}(z) = (g'_{p,i}(z))^k f_{|q} \circ g_{p,i}(z) \text{ for } z \in \gamma_{p,i}.$$

Similarly,  $\Omega^k(\mathbb{D}/\Gamma)$  is the subset of holomorphic functions  $f \in \mathcal{O}(\mathbb{D})$  such that

$$\forall z \in \mathbb{D}, \forall \gamma \in \Gamma, f(z) = \gamma'(z)^k f(\gamma(z)). \quad (2.1)$$

With our two constructions, the bijection between the spaces  $\Omega^k(X)$  and  $\Omega^k(\mathbb{D}/\Gamma)$  is as follows: for any  $(f_{|1}, \dots, f_{|m}) \in \Omega^k(X)$ , each  $f_{|p}$  extends holomorphically to some function  $\tilde{f}_{|p}$  defined on the disk  $\mathbb{D}$ , and the map

$$\begin{cases} \Omega^k(X) \rightarrow \Omega^k(\mathbb{D}/\Gamma) \\ (f_{|1}, \dots, f_{|m}) \mapsto \tilde{f}_{|1} \end{cases}$$

is a bijection.

We remind that  $\Omega^1(\mathbb{D}/\Gamma)$  is a finite dimensional space of dimension  $g$ , and we fix from now on a basis  $(\omega_1, \omega_2, \dots, \omega_g)$  of  $\Omega^1(\mathbb{D}/\Gamma)$  such that for any  $j, k$ :

$$\int_{a_j} \omega_k = \delta_{j,k}. \quad (2.2)$$

This way we may define the period matrix  $\tau \in \text{Sym}_g(\mathbb{R})$ :

$$\tau_{j,k} = \int_{b_j} \omega_k.$$

$\tau$  is a symmetric complex matrix, and  $\Im(\tau)$  is positive definite.

We now define the **Abel-Jacobi** coordinate maps  $u_j : \mathbb{D} \rightarrow \mathbb{C}$  by

$$u_j(z) = \int_0^z \omega_j.$$



$u_j$  is well-defined on  $\mathbb{D}$ , but does not pass to the quotient on  $\mathbb{D}/\Gamma$  because it verifies the partial periodicity relations

$$u_j(A_k(z)) = u_j(z) + \delta_{j,k}, \quad u_j(B_k(z)) = u_j(z) + \tau_{j,k}.$$

The Abel-Jacobi map based at the point 0 is then

$$u : \begin{cases} \mathbb{D} \rightarrow \mathbb{C}^g \\ z \mapsto (u_1(z), u_2(z), \dots, u_g(z)) \end{cases}$$

It verifies the periodicity relations

$$u(A_j(z)) = u_j(z) + e_j, \quad u(B_j(z)) = u(z) + \tau e_j \quad (2.3)$$

for all  $z$ . In particular, if  $z' = \gamma(z)$  for some  $\gamma \in \Gamma$ , then

$$u(z') - u(z) \in \mathbb{Z}^g + \tau \mathbb{Z}^g.$$

meaning that  $u$  defines a map from  $X$  to the  $2g$ -dimensional torus  $\mathbb{C}^{2g}/(\mathbb{Z}^g + \tau \mathbb{Z}^g)$ . We also introduce the notation

$$u^k : \begin{cases} \mathbb{D}^k \rightarrow \mathbb{C} \\ (z_1, \dots, z_k) \mapsto \sum_{j=1}^k u(z_j) \end{cases}.$$

We denote  $\mathcal{C} \subset \mathbb{C}^g$  the critical values of  $u^g$ : it is a set with (complex) dimension at most  $g-2$ , and it is characterized as the set of points  $\sum_{j=1}^g u(z_j)$  where

$$\det \left( (\omega_j(z_k))_{1 \leq j, k \leq g} \right) = 0.$$

## 2.3 Theta functions and quasi-periodic functions with prescribed zeroes

Let us now define the function

$$\Theta(Z) := \sum_{n \in \mathbb{Z}^g} e^{i\pi(n \cdot \tau n + 2n \cdot Z)}, \quad \forall Z \in \mathbb{C}^g.$$

In many references  $\Theta$  is given (by the same formula) as a function of  $Z$  and  $\tau$ , but since the matrix  $\tau$  is fixed here we leave it out of the arguments. The function  $\Theta$  verifies the following periodicity relations, for any  $Z \in \mathbb{C}^g$ ,  $n \in \mathbb{Z}^g$ :

$$\Theta(Z + n) = \Theta(Z), \quad \Theta(Z + \tau n) = \Theta(Z) e^{-i\pi(n \cdot \tau n + 2n \cdot Z)}, \quad \Theta(-Z) = \Theta(Z). \quad (2.4)$$

Let us define the Riemann constant  $\mathcal{K} \in \mathbb{C}^g$  by its coordinates

$$\mathcal{K}_j = \frac{1}{2} \tau_{jj} - \sum_{k=1}^g \int_0^{A_k(0)} u_j(z) u'_k(z) dz.$$

**Theorem 3.**  $\{Z \in \mathbb{C}^g : \Theta(Z) = 0\} = \mathcal{K} + u^{g-1}(\mathbb{D}^{g-1})$ . Moreover, for any  $\xi \in \mathcal{K} + (u^g(\mathbb{D}^g) \setminus \mathcal{C})$  written as

$$\xi = \mathcal{K} + \sum_{j=1}^g u(w_j),$$

the function

$$z \mapsto \Theta(u(z) - \xi)$$

vanishes with order 1 at every element of  $\{\gamma(w_j), \gamma \in \Gamma, j = 1, \dots, g\}$ , and is non-zero elsewhere.

This is contained in [4, VI.3.2, VI.3.3]. Note that the condition on  $\xi$  imposes that the  $(w_j)$ s must be distincts (modulo  $\Gamma$ ). The periodicity conditions verified by this function are, for any  $j$ ,

$$\Theta(u(A_j(z)) - \xi) = \Theta(u(z) - \xi), \quad \Theta(u(B_j(z)) - \xi) = \Theta(u(z) - \xi) e^{2\pi i(\xi_j - u_j(z) - \frac{1}{2} \tau_{j,j})}. \quad (2.5)$$

## 2.4 Weierstrass points

Let  $D$  be a divisor on  $X$ , represented as formal sum  $D = \sum_{p \in X} n_p [p]$  where  $n_p \in \mathbb{Z}$  is zero for all but a finite number of point, we denote by

$$\ell(D) = \dim(\{f \in M\Omega^0(X) : f = 0 \text{ or } (f) + D \geq 0\}),$$

where  $(f) = \sum_{p \in X} \text{ord}(f, p)[p]$  is the divisor associated to  $f$ . Let  $K$  be the divisor of some holomorphic 1-form, we remind that by the Riemann-Roch theorem:

$$\ell(D) - \ell(K - D) = \deg(D) - g + 1.$$

We summarize the result we will use as follows:

**Proposition 4.** *Let  $p \in X$ , then  $\ell(n[p])$  verifies*

$$\ell(0[p]) = 1, \ell((n+1)[p]) - \ell(n[p]) \in \{0, 1\}, \ell(n[p]) = n - g + 1 \text{ for any } n \geq 2g - 1.$$

Moreover, there is a finite set of points  $W \subset X$  called **Weierstrass points** such that for any  $p \in X \setminus W$  we have

$$\ell(n[p]) = 1 + (n - g)_+.$$

$p$  is a Weierstrass point if and only if

$$\det \left( \left( \omega_j^{(k-1)}(p) \right)_{1 \leq j, k \leq g} \right) = 0,$$

where this quantity is defined up to a (non-zero) multiplicative factor in  $\mathbb{D}/\Gamma$ .

By [4, Cor III.5.11], the cardinal of  $W$  is in the interval  $[2g + 2, g^3 - g]$ , and when  $g = 2$  then  $W$  has exactly 6 element.

Applying the Riemann-Roch theorem to  $D = n[p]$ , this may be reformulated as follows:  $\ell(K - n[p])$  is the dimension of the space of 1-forms  $\omega \in \Omega^1(X)$  that admits a zero of order at least  $n$  at  $p$ . So for any  $p \in X$ ,

$$\ell(K - (2g - 1)[p]) = 0,$$

meaning that any 1-form that has a zero of order larger than  $2g - 1$  is zero everywhere, and if  $p \in X \setminus W$  then

$$\ell(K - g[n]) = 0,$$

meaning that any 1-form that has a zero of order at least  $g$  at a non-Weierstrass point is zero everywhere. As a consequence:

**Lemma 5.** *Let  $z \in X$  (in some polygon  $H_p$ ), the linear map  $\omega \in \Omega^1(X) \mapsto (\omega|_p(z), \omega'_p(z), \dots, \omega^{(2g-2)}_p(z)) \in \mathbb{C}^{2g-1}$  is injective, and if  $z \in X \setminus W$  then the map  $\omega \in \Omega^1(X) \mapsto (\omega|_p(0), \omega'_p(z), \dots, \omega^{(g-1)}_p(z))$  is bijective.*

## 3 Construction of a basis

We split this construction in several steps: first there is the “pre-treatment” that is independent of the location of the pole, where we compute a polynomial that approximates with high accuracy the Abel-Jacobi coordinate maps  $u$ . This allows us to compute the Riemann constant  $\mathcal{K}$  as well.

Next we build, for two fixed points  $v, w$ , the harmonic function with opposing logarithmic poles at  $v, w$ .

We then build for a fixed pole  $w$  the complex harmonic function with a single pole of order  $n$  at  $w$ : we start by constructing the pole of order 1 using the composition of the logarithmic derivative of the  $\Theta$  function with a suitable translation of the Abel-Jacobi map. We then compute formally the successive derivative of this function along the location  $w$  of the pole.

This step is numerically costly, however once we've built sufficiently many meromorphic function with a pole at  $w$ , then the rest of the basis may be built by exponentiation of these functions: depending on whether the pole  $w$  is a Weierstrass point or not, we need to build by successive differentiation either  $2g + 1$  or  $4g - 1$  functions before all the rest may be built by exponentiation.

We will explain in detail the case where the pole  $w$  is **not** a Weierstrass point. We then explain afterward how to adapt it to this case.

### 3.1 Computation of 1-forms

This subsection relies heavily of the representation of  $X$  as a gluing of  $m$  hyperbolic polygons  $(H_p)_{p=1,\dots,m}$ , and we use the notations of section 1.1. We will suppose additionally that the first polygon  $H_1$  contains the center of the disk 0, and that 0 is not a Weierstrass point.

For any  $\omega \in \Omega^1(X)$ , let us define the approximation space

$$\mathcal{A}^{N,\omega} = \left\{ P = (P_{|1}, \dots, P_{|m}) \in \mathbb{C}_N[X]^m : (P_{|1}(0), P'_{|1}(0), \dots, P_{|1}^{(g-1)}(0)) = (\omega_{|1}(0), \omega'_{|1}(0), \dots, \omega_{|1}^{(g-1)}(0)) \right\}.$$

Let  $S(N) = aN$  for some  $a \in \mathbb{N}_{\geq 1}$  that will be taken large enough, and for any side  $\gamma_{p,i}$  we denote by  $\mathcal{S}_{p,i}$  a sampling of  $\gamma_{p,i}$  by  $S(N)$  regularly spaced points.

For any  $P = (P_{|1}, \dots, P_{|m}) \in \mathbb{C}_N[X]^m$ , we let

$$E^N(P) = \sum_{(p,i) \rightarrow (q,j)} \frac{1}{S(N)} \sum_{z \in \mathcal{S}_{p,i}} |P_{|p}(z) - g'_{p,i}(z)P_{|q}(g_{p,i}(z))|^2.$$

Finally, for any compact set  $K \subset \mathbb{C}$  we let  $\mathcal{G}_{\widehat{\mathbb{C}} \setminus K}(z, w)$  be its Green function defined as in equation (1.3),(1.4) (where the ambient manifold is  $\widehat{\mathbb{C}}$  in this case).

The main result of this section is the following:

**Theorem 6.** *Let  $X$  be a surface defined as above, such that  $0 \in H_1$  is not a Weierstrass point. Let  $\rho \in ]0, 1[$  be such that*

$$\rho > \exp \left( - \min_{p=1,\dots,m} \inf_{z \in \partial D} \mathcal{G}_{\widehat{\mathbb{C}} \setminus H_p}(z, \infty) \right),$$

*Then there exists a large enough integer  $a > 0$  and a constant  $C > 0$  such that for any  $\omega \in \Omega^1(X)$ , any  $N \in \mathbb{N}_{\geq g}$ , we denote by  $Q^{N,\omega}$  the unique minimizer of  $E^N$  in  $\mathcal{A}^{N,\omega}$  and we have:*

$$\sum_{p=1}^m \|Q_{|p}^{N,\omega} - \omega_{|p}\|_{L^\infty(H_p)} \leq C \rho^N \sum_{j=0}^{g-1} |\omega_{|1}^{(j)}(0)|.$$

We prove in fact a more precise conclusion: there is a constant  $C_a > 0$  (depending on the geometry and on  $a$ ) such that for any  $Q \in \mathcal{A}^{N,\omega}$ :

$$\sum_{p=1}^m \|Q_{|p} - \omega_{|p}\|_{L^\infty(H_p)}^2 \leq C_a N^3 e^{\frac{C}{\sqrt{a}} N} E^N(Q).$$

Then the result follows from the fact that  $\inf_{\mathcal{A}^{N,\omega}} \sqrt{E^N} = \mathcal{O} \left( \rho^N \sum_{j=0}^{g-1} |\omega_{|1}^{(j)}(0)| \right)$  for any such  $\rho$ , and taking  $a$  large enough: nowhere do we use the fact that  $Q^{N,\omega}$  is a minimizer.

For now we relegate the proof of this theorem to the last section. Let us give some remarks on the optimality and the limitations of this result.

- 1) We may give a more explicit upper bound of  $\rho$  as follows: any  $\rho$  such that the open disk  $\mathbb{D}_\rho$  contains all the closed polygons  $(H_p)_p$  verifies the hypothesis of the theorem. Indeed by maximum principle we would have  $\mathcal{G}_{\widehat{\mathbb{C}} \setminus H_p}(z, \infty) \geq \log(z/\rho)_+$ , so  $\inf_{z \in \partial \mathbb{D}} \mathcal{G}_{\widehat{\mathbb{C}} \setminus H_p} \geq \log(1/\rho)$  (and the inequality would be strict by strong maximum principle).

- 2) The speed of convergence given here is optimal, in the sense that any  $\rho$  lower than the bound given in the theorem would lead to a contradiction. This is discussed at the end of the section 6.
- 3) Each  $Q_{|p}^{N,\omega}$  converges beyond the polygon  $H_p$ : in fact we have  $\|Q_{|p}^{N,\omega} - \omega\|_{L^\infty(H_p \sqcup \{G_{\hat{C}\setminus H_p} < t\})} = \mathcal{O}_{N \rightarrow \infty}((e^t \rho)^N)$  for any  $\rho$  as in the statement and any  $t$ . However this does not mean that  $Q_{|p}^{N,\omega}$  converges to  $\omega_{|p}$  in all of  $\mathbb{D}$ .
- 4) Our proof of the theorem does not give an explicit constant  $C$ , because one of the bounds (see lemma 13) includes an argument by contradiction. It is unclear whether this could be made explicit easily.
- 5)  $0 \in H_1$  may happen to be a Weierstrass point. This happens for instance when  $X$  is obtained by gluing the opposite sides of a single regular octogon centered at 0 with angles  $\frac{\pi}{4}$  (this is the Bolza surface): in this case the Weierstrass points are exactly the origin, the vertices and the middle of the edges (which after identifications correspond to a total of 6 different points). In particular there is a basis of 1-forms  $\alpha, \beta \in \Omega^1(X)$  such that

$$\alpha(z) = 1 + \mathcal{O}_{z \rightarrow 0}(z), \quad \beta(z) = z^2 + \mathcal{O}_{z \rightarrow 0}(z^3).$$

In practice this may be detected by the fact that for some  $l = 1, \dots, g$ , the quantity

$$\inf \left\{ E^N(P), P \in \mathbb{C}_N[X]^m : \forall j = 1, \dots, g, P_{|1}^{(j-1)}(0) = \delta_{j,l} \right\}$$

does not converge to 0. To solve this issue we proceed by perturbation: we choose some random point  $w_{rand} \in H_1$  and impose successive derivative conditions at  $w_{rand}$  instead of 0. We may also change  $H_1$  into  $\varphi(H_1)$  for some small perturbation of the identity  $\varphi \in PSU(1, 1)$ , which amounts to changing  $\gamma_{1,i}$  into  $\varphi(\gamma_{1,i})$  and composing the transition functions  $g_{p,i}$  accordingly.

Overall this least square method allows us to compute a good approximation of the canonical basis of 1-forms (verifying (2.2)). Moreover polynomial expression can be explicitly integrated, so this also give us an approximate period matrix  $\tau$ , Abel-Jacobi map  $u$  and Riemann constant  $\mathcal{K}$ . The details of these computations are explained in sections 4.1 and 4.2.

In the next subsections we give a construction of the harmonic basis with explicit formula given by compositions of the Abel-Jacobi map  $u$  and  $\Theta$  functions (and its derivatives), the period matrix  $\tau$  and the Riemann constant  $\mathcal{K}$ : in practice these will be computed using our approximations.

### 3.2 Logarithmic singularities

In this subsection and all the following we work on the surface  $\mathbb{D}/\Gamma$ , instead of the (equivalent) surface  $X$  obtained by gluing, for the convenience of notations. How these two points of view reconcile in practice is explained in section 4.

Let  $v, w$  be two distinct points and

$$\xi \in (\mathcal{K} + u^{g-1}(\mathbb{D}^{g-1})) \setminus [(\mathcal{C} + \mathcal{K} - u(v)) \cup (\mathcal{C} + \mathcal{K} - u(w))],$$

where  $\mathcal{C}$  is the set defined in subsection 2.4 as the image of the critical points of  $u^g$ .

In other words  $\xi$  is chosen as

$$\xi = \mathcal{K} + \sum_{j=1}^{g-1} u(w_j)$$

where  $w_1, \dots, w_{g-1}$  are generic: since  $\mathcal{C}$  has dimension at most  $g-2$ , generic points will give a suitable  $\xi$ . We let

$$\sigma_{v,w}(z) = \frac{\Theta(u(z) - u(v) - \xi)}{\Theta(u(z) - u(w) - \xi)}. \quad (3.1)$$

Then  $\sigma_{v,w}$  vanishes exactly (modulo  $\Gamma$ ) with order 1 at  $v$ , and has one pole (modulo  $\Gamma$ ) of order 1 at  $w$ . Moreover  $\sigma_{v,w}$  verifies  $\sigma_{v,w}(A_j(z)) = \sigma_{v,w}(z)$  and

$$\sigma_{v,w}(B_j(z)) = \sigma_{v,w}(z)e^{2\pi i(u_j(v)-u_j(w))}.$$

Let  $(c_j)_j \in \mathbb{C}^g$  to be fixed later, and

$$\widehat{\sigma}_{v,w}(z) = \sigma_{v,w}(z)e^{2\pi i \sum_{k=1}^g c_k \Im(u_k(z))}.$$

Then  $\widehat{\sigma}_{w_1, w_2}$  is  $A_j$ -periodic since  $\sigma_{v,w}$  is too and  $\Im(u_k(A_j z)) = \Im(u_k(z) + \delta_{jk}) = \Im(u_k(z))$ . Next,

$$\frac{\widehat{\sigma}_{v,w}(B_j z)}{\widehat{\sigma}_{v,w}(z)} = \exp \left( 2\pi i \left[ u_j(v) - u_j(w) + \sum_{k=1}^g c_k \Im(\tau)_{jk} \right] \right).$$

Since  $\Im(\tau)$  is invertible, we may choose the coefficients  $(c_j)_j$  such that  $\widehat{\sigma}_{v,w}$  is  $\Gamma$ -periodic: it is sufficient to check that for every  $j$  we have

$$\sum_{k=1}^g c_k \Im(\tau)_{jk} = u_j(w) - u_j(v),$$

which amounts to choosing

$$\widehat{\sigma}_{v,w}(z) = \sigma_{v,w}(z) \exp \left( -2\pi i \sum_{1 \leq k, l \leq g} (\Im(\tau)^{-1})_{k,l} \Im(u_k(z))(u_l(v) - u_l(w)) \right). \quad (3.2)$$

The function

$$\log |\widehat{\sigma}_{v,w}(z)| = \log |\sigma_{v,w}(z)| + 2\pi \sum_{1 \leq k, l \leq g} (\Im(\tau)^{-1})_{k,l} \Im(u_k(z)) \Im(u_l(v) - u_l(w)) \quad (3.3)$$

is thus a periodic harmonic function with singularities equal to  $\log(|z - v|) + \mathcal{O}_{z \rightarrow v}(1)$  near  $v$ , and to  $-\log(|z - w|) + \mathcal{O}_{z \rightarrow w}(1)$  near  $w$ .

Note that the function  $\widehat{\sigma}_{v,w}(z)$  might look like it depends on the choice of the generic point  $\xi \in u^{g-1}(\mathbb{D}^{g-1})$ . However, changing  $\xi$  only changes  $\widehat{\sigma}_{v,w}(z)$  by a non-zero constant factor.

### 3.3 Pole of order 1

We use the previously defined functions, where  $w$  is where we aim to build a pole of order 1 and  $v$  is seen as a variable. Let

$$\begin{aligned} \wp_1(z, w) &= \partial_v|_{v=w} \log |\sigma_{v,w}(z)|, \\ \widehat{\wp}_1(z, w) &= \partial_v|_{v=w} \log |\widehat{\sigma}_{v,w}(z)|, \end{aligned}$$

where  $\partial_v|_{v=w}$  is the conformal Wirtinger derivative along the variable  $v$ , taken at the value  $v = w$ . In other words

$$\wp_1(z, w) = \Theta(u(z) - u(w) - \xi)^{-1} \partial_v|_{v=w} \Theta(u(z) - u(w) - \xi) \quad (3.4)$$

$$= -\frac{u'(w) \cdot \nabla \Theta(u(z) - u(w) - \xi)}{\Theta(u(z) - u(w) - \xi)} \quad (3.5)$$

$$\widehat{\wp}_1(z, w) = \wp_1(z, w) - 2\pi i \sum_{1 \leq k, l \leq g} (\Im(\tau)^{-1})_{k,l} \Im(u_k(z)) u_l'(w). \quad (3.6)$$

The function  $\widehat{\wp}_1(z, w)$  is still  $\Gamma$ -periodic along the variable  $z$ : by applying the Wirtinger derivative in  $v$  at  $v = w$  to the periodicity relation

$$\forall \gamma \in \Gamma, \log |\widehat{\sigma}_{v,w}(z)| = \log |\widehat{\sigma}_{v,w}(\gamma(z))|$$

we obtain

$$\forall \gamma \in \Gamma, \widehat{\wp}_1(z, w) = \widehat{\wp}_1(\gamma(z), w).$$

### 3.4 Poles of order 2 to $g$

We define by induction

$$\begin{aligned}\wp_n(z, w) &= \partial_w^{(n-1)} \wp_1(z, w) \\ \widehat{\wp}_n(z, w) &= \partial_w^{(n-1)} \widehat{\wp}_1(z, w).\end{aligned}$$

More precisely,

$$\widehat{\wp}_n(z, w) = \wp_n(z, w) - 2\pi i \sum_{1 \leq k, l \leq g} (\Im(\tau)^{-1})_{k,l} \Im(u_k(z)) u_l^{(n)}(w). \quad (3.7)$$

As before,  $\widehat{\wp}_n(\gamma(z), w) = \widehat{\wp}_n(z, w)$  for any  $\gamma \in \Gamma$ . While the computation of  $u_l^{(n)}(w)$  is straightforward using its polynomial approximation, the function  $\wp_n$  will be computed by an analytic finite difference method explained in section 4.

Note that the function  $\widehat{\wp}_n(z, w)$  by itself is holomorphic with respect to  $z$  but only verifies the periodicity relation

$$\widehat{\wp}_n(A_j(z), w) = \widehat{\wp}_n(z, w), \quad \widehat{\wp}_n(B_j(z), w) = \widehat{\wp}_n(z, w) + 2\pi i u_j^{(m)}(w) \quad (3.8)$$

### 3.5 Poles of order $g + 1$ to $2g + 1$

The fact that  $w$  is not a Weierstrass point appear in this construction. We first define  $(\wp_n)_{g+1 \leq n \leq 2g+1}$  and  $\widehat{\wp}_n(z, w)$  in the same fashion as earlier, and we let  $C(w) \in M_{g \times g}(\mathbb{C})$  to be defined as

$$C(w)_{k,n} = u_k^{(n)}(w), \quad 1 \leq k \leq g, \quad 1 \leq n \leq g$$

such that for any  $n \in \{1, \dots, g\}$ :

$$\widehat{\wp}_n(z, w) = \wp_n(z, w) - 2\pi i \sum_{k=1}^g (\Im(\tau)^{-1} C(w))_{k,n} \Im(u_k(z)).$$

Since  $w$  is not a Weierstrass point, then  $C(w)$  is invertible (as a consequence of lemma 5).

For any  $n = g + 1, \dots, 2g + 1$ , we then let

$$d_n(w) = C(w)^{-1} u^{(n)}(w) \in \mathbb{C}^g$$

be the unique vector that verifies for any  $k = 1, \dots, g$ :

$$u_k^{(n)}(w) = \sum_{p=1}^g d_{n,p}(w) u_k^{(p)}(w).$$

Letting

$$\check{\wp}_n(z, w) = \wp_n(z, w) - \sum_{p=1}^g d_{n,p}(w) \wp_p(z, w),$$

this is a meromorphic function with a pole of order  $n$  at  $z = w$ , and it is  $\Gamma$ -periodic. Indeed, it is equal by construction to

$$\widehat{\wp}_n(z, w) - \sum_{p=1}^g d_{n,p}(w) \widehat{\wp}_p(z, w)$$

which is periodic.

### 3.6 Poles of order $2g + 2$ and more

We now build a periodic meromorphic function with a pole of order  $n$  at  $w$  for  $n \geq 2g + 2$ . For any such  $n$  there exists  $m \in \mathbb{N}^*$ ,  $q \in \{g + 1, g + 2, \dots, 2g + 1\}$  such that

$$n = (g + 1)m + q.$$

We then define

$$\tilde{\wp}_n(z, w) = \check{\wp}_q(z, w)\check{\wp}_{g+1}(z, w)^m.$$

Note that we *could* follow the previous construction: the functions  $\check{\wp}_n$  and  $\tilde{\wp}_n$  differ (up to a nonzero multiplicative constant) by a combination of  $\check{\wp}_m$  (or  $\tilde{\wp}_m$ ) for  $m < n$ . The same can be said between  $\check{\wp}_n$  and  $\hat{\wp}_n$ .

### 3.7 Adaptation to Weierstrass points

Let us now sketch how each of the previous results apply to Weierstrass points. First, as in section 3.1 we compute (using the least square method from theorem 6) an approximation of a basis of  $\Omega^1(X)$  and of the Abel-Jacobi map.

For the construction of the basis of harmonic functions with prescribed poles, for a general Weierstrass point the matrix  $C(w)$  defined in subsection 3.5 is not invertible in general: we need to compute more of the functions  $(\wp_n)_n$  to get a full-rank matrix, as follows:

- 1) We define  $\log |\hat{\sigma}_{v,w}|$  as previously.
- 2) We define similarly  $\wp_1(z, w)$ ,  $\hat{\wp}_1(z, w)$ .
- 3) By differentiation in  $w$  we build the functions  $\hat{\wp}_n$  as previously, this time for the values  $n = 2, 3, \dots, 2g - 1$ .
- 4) We define the non-square matrix  $\tilde{C}(w) \in M_{g \times (2g-1)}(\mathbb{C})$  by:

$$\tilde{C}(w)_{k,n} = u_k^{(n)}(w), \quad 1 \leq k \leq g, \quad 1 \leq n \leq 2g - 1.$$

By lemma 5,  $\tilde{C}(w)$  has rank  $g$  so we find  $d_n(w) \in \mathbb{C}^{2g-1}$  (which may be defined uniquely, up to restricting  $\tilde{C}(w)$  to the right subspace of  $\mathbb{C}^{2g-1}$ ) such that

$$\tilde{C}(w)d_n(w) = c^{(n-1)}(w).$$

and we may then define  $\check{\wp}_n(z, w)$  as previously, for  $n = 2g, 2g + 1, \dots, 4g - 1$ .

- 5) For  $n \geq 4g$ , we make the same Euclidean division  $n = 2gm + q$  where  $2g \leq q < 4g$ , and we let

$$\tilde{\wp}_n(z, w) = \check{\wp}_q(z, w)\check{\wp}_{2g}(z, w)^m.$$

### 3.8 Proof of the main results

Here we give the proof of the two main results 1 and 2.

While the first result is a simple consequence of our construction and of the maximum principle, the second one is more involved: we have to adapt to general surfaces the classical proof of Bernstein-Walsch theorem in  $\mathbb{C}$  (see [14, Ch. VII] or [11, Th 6.3.1] for a more recent reference, to which we will refer for intermediate results).

*Proof of 1.* Let  $h$  be a function as in the statement and let  $w_j$  be one of the poles. We may then develop  $h$  near this point (here we have fixed the charts that is naturally induced by the gluing of polygons)

$$h(z) = a_{j,0} \log(|z - w_j|) + \sum_{k=1}^n \Re [a_{j,k} (z - w_j)^{-k}] + \mathcal{O}_{z \rightarrow w_j}(1),$$

where  $a_{j,0}$  is real and  $a_{j,k}$  is complex. We remind that this is obtained by integrating the Laurent series decomposition of  $\partial_z h(z)$ , which is meromorphic with a pole at  $w_j$ . By construction, there exists a constant  $c \neq 0$  such that

$$\widehat{\rho}_k(z, w_j) = c(-1)^k (k-1)! (z - w_j)^{-k} + \mathcal{O}_{z \rightarrow w_j}((z - w_j)^{-(k-1)}).$$

So there exists  $b_{j,0}, b_{j,1}, b_{j,2}, \dots, b_{j,n}$  such that the function

$$h(z) - \sum_{j=1}^{N-1} b_{j,0} \log |\sigma_{w_j, w_{j+1}}(z)| - \Re \sum_{j=1}^N \left( \sum_{k=1}^g b_{j,n} \widehat{\rho}_k(z, w_j) + \sum_{k=g+1}^{2g+1} b_{j,k} \check{\rho}_k(z, w_j) + \sum_{k=2g+1}^n b_{j,k} \tilde{\rho}_k(z, w_j) \right)$$

is harmonic in  $X \setminus \{w_j, j = 1, \dots, N\}$  and bounded near each pole, thus it is a constant function, from which we get the theorem.  $\square$

For the proof of 2, we first prove a more particular case from which the general case is a direct consequence.

**Lemma 7.** *Let  $X$  be a compact oriented surface of genus  $g \geq 1$ , let  $K$  be a compact subset of  $X$  such that none of its connected component is reduced to a point and  $D := X \setminus K$  is connected, and let  $w \in D$ . Let  $f$  be a holomorphic function defined on a neighbourhood of  $K$ , and  $f_n$  be the best  $L^\infty(K)$  approximation of  $f$  by a meromorphic function of  $X$  with only a pole of order at most  $n$  at  $w$ . Then for any  $t > 0$  the two following properties are equivalent:*

- (a)  $f$  extends holomorphically to  $K \sqcup \{\mathcal{G}_D(\cdot, w_j) < t\}$ .
- (b)  $\limsup_{n \rightarrow \infty} \|f - f_n\|_{L^\infty(K)}^{1/n} \leq e^{-t}$ .

We let  $\mathcal{H} : X^2 \rightarrow \mathbb{R} \cup \{-\infty\}$  be the *other* Green function on  $X$ , defined by

$$\Delta_z \mathcal{H}(\cdot, w) = 2\pi \left( \delta_w - \frac{1}{|X|} \right), \quad \int_X \mathcal{H}(z, w) d\mu_X(z) = 0.$$

where  $\mu_X$  is the area measure induced by the hyperbolic metric. We remind that  $\mathcal{H}(x, y) = \mathcal{H}(y, x)$  with this normalization.

The idea of proof of  $(a \Rightarrow b)$  is as follows.

- 1) If we extend continuously  $\mathcal{G}_D(z, w)$  by 0 in  $K$ , then  $\Delta_z \mathcal{G}_D(\cdot, w) = 2\pi(\nu_K - \delta_w)$  where  $\nu_K$  is a capacitary measure of mass 1 that is supported on  $\partial K$ , called the equilibrium measure of  $K$ . Moreover,  $\nu_K$  may be defined as the probability measure supported on  $K$  that maximizes the quantity

$$I(\nu) = \int_{K^2} (\mathcal{H}(x, y) - \mathcal{H}(x, w) - \mathcal{H}(y, w)) d\nu(x) d\nu(y),$$

and we write  $c(K) = e^{I(\nu_K)}$  the logarithmic capacity of  $K$  with respect to the pole  $w$ . We remind that for any  $z \in D$ :

$$\mathcal{G}_D(z, w) = \int_{\partial K} (\mathcal{H}(z, \zeta) - \mathcal{H}(z, w) - \mathcal{H}(\zeta, w) - I(\nu_K)) d\nu_K(\zeta). \quad (3.9)$$



2) We let  $(z_{1,n}, \dots, z_{n,n})$  be a maximal argument of

$$(z_1, \dots, z_n) \in K^n \mapsto \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} (\mathcal{H}(z_i, z_j) - \mathcal{H}(z_i, w) - \mathcal{H}(z_j, w)) \quad (3.10)$$

and denote  $\log(\delta_n(K))$  its maximal value. Then  $\delta_n(K)$  converges to  $c(K)$

3) We define  $F_n$  to be a meromorphic function, with a pole of order  $n$  at  $w$  and zeroes at the  $\{z_{j,n}, j = 1, \dots, n\}$ : when  $X$  is the Riemann sphere  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  this is always possible by taking the so-called Fekete polynomial  $F_n(X) = \prod_{j=1}^n (X - z_{j,n})$ , however in a general surface this necessitates a correction: we find some  $q \in \mathbb{N}$  (only depending on the geometry of  $K$  and  $X$ ) such that  $F_n$  has  $q$  additional zeroes in  $K$ , and a pole of order  $n + q$  at  $w$ .

We also prove that  $(|F_n|/\|F_n\|_{L^\infty(K)})^{1/n}$  converges locally in  $D$  to  $e^{\mathcal{G}_D(\cdot, w)}$ .

4) Finally, we define  $f_n$  as (through an integral involving  $F_n$ ) as a meromorphic function with a pole of order  $n + \mathcal{O}(1)$  at  $w$  that coincides with  $f$  at the points  $(z_{j,n})_j$ .

*Proof of 7.* We start with  $(b \Rightarrow a)$ . Denote  $g_n = f_n - f_{n-1}$ , such that

$$\limsup_{n \rightarrow \infty} \|g_n\|_{L^\infty(K)}^{1/n} \leq e^{-t}.$$

$\log |g_n|$  is subharmonic in  $X \setminus \{w\}$  and more precisely

$$\Delta \log |g_n| \geq -2\pi n \delta_w,$$

so the function  $z \mapsto \mathcal{G}_D(z, w) - \log \frac{|g_n(z)|}{\|g_n\|_{L^\infty(K)}}$  is superharmonic in  $D$  and nonnegative on  $\partial D$ : as a consequence it is nonnegative in  $D$ . This implies that for any  $s \in (0, t)$ , and any  $z \in K \sqcup \{\mathcal{G}_D(\cdot, w) < s\}$ :

$$|g_n(z)| \leq \|g_n\|_{L^\infty(K)} e^{ns}.$$

So  $\limsup_{n \rightarrow \infty} \|g_n\|_{L^\infty(K \sqcup \{\mathcal{G}_D(\cdot, w) < s\})}^{1/n} \leq e^{s-t} (< 1)$ , meaning  $\|g_n\|_{K \sqcup \{\mathcal{G}_D(\cdot, w) < s\}}$  is summable for all  $s < t$ , and  $f$  extends to  $K \sqcup \{\mathcal{G}_D(\cdot, w) < t\}$  as  $f_0 + \sum_{n \geq 1} g_n$ .

We now prove to the other implication  $(b \Rightarrow a)$ . First note that since  $\mathcal{G}_D(\cdot, w)$  is continuous at  $\partial D$ , without loss of generality we may replace  $K$  with  $K^\epsilon = K \sqcup \{\mathcal{G}_D \leq \epsilon\}$  and  $D$  with  $D^\epsilon = X \setminus K^\epsilon$  for arbitrarily small  $\epsilon \rightarrow 0$ . This way  $\mathcal{G}_{D^\epsilon} = \mathcal{G}_D - \epsilon$  on  $D^\epsilon$ . This is useful in the third step because  $K^\epsilon$  now contains an open set. Thus in the rest of the proof we suppose  $\partial K$  is a finite union of smooth curves.

We follow the four steps evoked earlier: since the first two follow the same steps as the planar case, only the third and fourth are given in full detail.

- 1) The first part is direct (note that since  $\partial K$  is smooth, there is no regularity issue at the boundary). The second part is a consequence of Frostman's theorem (see for instance [11, Th. 3.3.4] with no significant change from the proof of the planar case).
- 2) This is the Fekete-Szego theorem adapted to surface (see for instance [11, Th. 5.5.2] where the same proof adapts).
- 3) We denote by  $\bar{u} : X \rightarrow \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$  the quotiented Abel-Jacobi map. The condition at which there exists a meromorphic function with exactly a pole of order  $n$  at  $w$  and zeroes at the points  $(z_{j,n})_{j=1, \dots, n}$  is given by Abel's theorem as follows:

$$\sum_{j=1}^n (\bar{u}(z_{j,n}) - \bar{u}(w)) = 0.$$

This is not always verified, but we claim that for a large enough  $q \in \mathbb{N}$  we may always find some  $z_{n+1,n}, z_{n+2,n}, \dots, z_{n+q,n} \in K$  such that

$$\sum_{j=1}^{n+q} (\bar{u}(z_{j,n}) - \bar{u}(w)) = 0.$$

Indeed, write  $\bar{u}^k(z_1, \dots, z_k) = \sum_{j=1}^k \bar{u}(z_j)$  for any  $k \in \mathbb{N}^*$ . We remind that  $\bar{u}^g$  is a diffeomorphism outside of a codimension 1 set, and  $K$  has non-empty interior (since we supposed  $K$  to be smooth in the beginning) so  $\bar{u}^g(K^g)$  contains an open set of  $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ . As a consequence, for some large enough  $m \in \mathbb{N}$  we have

$$\bar{u}^{mg}(K^{mg}) = \underbrace{\bar{u}^g(K) + \dots + \bar{u}^g(K)}_{m \text{ times}} = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g).$$

Let  $q = mg$  and we choose  $z_{n+1,n}, \dots, z_{n+q,n} \in K$  such that

$$\sum_{j=n+1}^{n+q} \bar{u}(z_{j,n}) = q\bar{u}(w) - \sum_{j=1}^n (\bar{u}(z_{j,n}) - \bar{u}(w)).$$

This proves our claim: there exists a meromorphic function  $F_n$  with divisor  $\sum_{j=1}^{n+q} ([z_{j,n}] - [w])$ . Up to multiplying  $F_n$  by a scalar constant, we have

$$\log |F_n(z)| = \sum_{j=1}^{n+q} \mathcal{H}(z, z_{j,n}) - \mathcal{H}(z, w) - \mathcal{H}(z_{j,n}, w) \quad (3.11)$$

We now prove the convergence of  $|F_n|^{1/n}$  on  $D$ . It is more convenient (and equivalent) to prove the convergence of  $|F_n|^{\frac{1}{n+q}}$ . Since all zeroes of  $F_n$  are in  $K$ , then  $\frac{1}{n+q} \log \frac{|F_n|}{\|F_n\|_{L^\infty(K)}}$  is harmonic in  $D \setminus \{w\}$  with a logarithmic singularity at  $w$ , and more precisely the function

$$z \mapsto h_n(z) = \mathcal{G}_D(z, w) - \frac{1}{n+q} \log \frac{|F_n(z)|}{\|F_n\|_{L^\infty(K)}}$$

is harmonic with an erasable singularity in  $w$ . Moreover, it is nonnegative at the boundary, as a consequence for any  $z \in D$  we get

$$0 \leq h_n(z) \leq \tau_D(z, w) h_n(w),$$

where  $\tau_D(z, w)$  is the Harnack constant associated to  $(z, w) \in D^2$ . The value of  $h_n$  at  $w$  is  $\log \frac{\|F_n\|_{L^\infty(K)}^{\frac{1}{n+q}}}{c(K)}$  (this is a consequence of equation (3.11) and (3.9) where we remind  $I(\nu_k) = \log(c(K))$ ). As a consequence, the above inequality gives for any  $z \in D$ :

$$e^{\mathcal{G}_D(z, w)} \left( \frac{c(K)}{\|F_n\|_{L^\infty(K)}^{\frac{1}{n+q}}} \right)^{\tau_D(z, w)} \leq \left( \frac{|F_n(z)|}{\|F_n\|_{L^\infty(K)}} \right)^{\frac{1}{n+q}} \leq e^{\mathcal{G}_D(z, w)}. \quad (3.12)$$

Finally, following [11, Th. 5.5.4], for any  $z \in K$  we consider  $(z, z_{1,n}, \dots, z_{n,n})$  as a competitor of  $(z_{1,n+1}, \dots, z_{n+1,n+1})$  in (3.10) and this gives exactly

$$\delta_n(K)^{\frac{n(n-1)}{2}} |F_n(z)| \leq c^q \delta_{n+1}(K)^{\frac{n(n+1)}{2}}.$$

where  $c = \sup_{(z, \zeta) \in K^2} e^{\mathcal{H}(z, \zeta) - \mathcal{H}(z, w) - \mathcal{H}(\zeta, w)}$ . Since  $\delta_n(K)$  is decreasing we get  $\|F_n\|_{L^\infty(K)}^{\frac{1}{n+q}} \leq c^{\frac{q}{n+q}} \delta_n(K)^{\frac{n}{n+q}} \xrightarrow{n \rightarrow \infty} c(K)$ . From this and equation (3.12) we obtain

$$\left( \frac{|F_n(z)|}{\|F_n\|_{L^\infty(K)}} \right)^{\frac{1}{n+q}} \xrightarrow{n \rightarrow \infty} e^{\mathcal{G}_D(z, w)} \text{ locally uniformly in } z \in D. \quad (3.13)$$

4) For the end of the proof, we need the following construction:

**Lemma 8.** *Let  $w \in X$  be fixed, then for any  $z \in X \setminus \{w\}$  there exists a meromorphic 1-form  $\omega_z$  with exactly a pole of order 1 at  $z$  and  $w$ , with residues 1,  $-1$  respectively, such that  $z \mapsto \omega_z$  is itself meromorphic with respect to  $z$ , with a pole at  $z = w$  of order at most  $2g - 1$ .*

*Proof.* We start by the genus 1 case. We may suppose  $X$  is of the form  $\mathbb{C}/L$  for some lattice  $L = \mathbb{Z} + \tau\mathbb{Z}$  where  $\Im(\tau) > 0$ . We then consider the zeta function

$$\mathcal{Z}(z) = \frac{1}{z} + \sum_{\ell \in L \setminus \{0\}} \left\{ \frac{1}{z - \ell} + \frac{1}{\ell} + \frac{z}{\ell^2} \right\}.$$

We remind the periodicity of  $\mathcal{Z}$ , due to Eisenstein:

$$z \mapsto \widehat{\mathcal{Z}}(z) := \mathcal{Z}(z) - \gamma_2 z - \frac{\pi}{\Im(\tau)} \bar{z} \text{ is } L\text{-periodic,}$$

where  $\gamma_2 = \sum_{\ell \in L \setminus \{0\}} \frac{1}{\ell^4}$ . Then we let

$$\omega_z(\zeta) = (\mathcal{Z}(\zeta - z) + \mathcal{Z}(z - w) - \mathcal{Z}(\zeta - w)) d\zeta.$$

$\omega_z$  is meromorphic with respect to  $\zeta, z$  with the appropriate poles. Moreover, since  $\mathcal{Z}(\zeta - z) + \mathcal{Z}(z - w) - \mathcal{Z}(\zeta - w) = \widehat{\mathcal{Z}}(\zeta - z) + \widehat{\mathcal{Z}}(z - w) - \widehat{\mathcal{Z}}(\zeta - w)$  then it is also periodic with respect to  $\zeta, z$ .

We now suppose  $g \geq 2$ . To build this 1-form we work from the disk model  $\mathbb{D}/\Gamma$ . As in section 3.2, we let  $\xi$  be a generic point of  $\mathcal{K} + u^{g-1}(\mathbb{D}^{g-1})$ , and

$$\sigma_{z,w}(\zeta) = \frac{\Theta(u(\zeta) - u(z) - \xi)}{\Theta(u(\zeta) - u(w) - \xi)}.$$

We let

$$\alpha_z(\zeta) = \partial_\zeta \log |\sigma_{z,w}(\zeta)| = \frac{\sigma'_{z,w}(\zeta)}{\sigma_{z,w}(\zeta)}.$$

Then  $\alpha_z \in \Omega^1(\mathbb{D}/\Gamma)$  for every  $z$ , with the poles exactly as in the statement of the lemma (and since it is expressed as a ratio, it does not depend on  $\xi$ ). Moreover it is holomorphic with respect to  $z$  (in particular  $\alpha_w = 1$ ). However it is not periodic with respect to  $z$ , indeed a direct computation gives

$$\alpha_{A_j(z)}(\zeta) = \alpha_z(\zeta), \quad \alpha_{B_j(z)}(\zeta) = \alpha_z(\zeta) + 2\pi i \omega_j(\zeta).$$

Thus it is enough to define

$$\omega_z(\zeta) = \alpha_z(\zeta) - \sum_{k=1}^g c_k(z) \omega_k(\zeta),$$

where  $c_j \in \mathcal{O}(\mathbb{D})$  has poles at  $\Gamma \cdot w$  and verifies the periodicity relations

$$\forall j, k \in \{1, \dots, g\}, \quad c_k(A_j(z)) = c_k(z), \quad c_k(B_j(z)) = c_k(z) + 2\pi i \delta_{j,k}.$$

We built  $c_k$  as a combination of  $(\wp_m)_{m=1,2,\dots,2g-1}$  (which could be replaced by  $m = 1, 2, \dots, g$  when  $w$  is not a Weierstrass point) built in subsection (3.3,3.4). We remind the periodicity of  $\wp_m$ :

$$\wp_m(A_j(z), w) = \wp_m(z, w), \quad \wp_m(B_j(z), w) = \wp_m(z, w) + 2\pi i \omega_j^{(m-1)}(w). \quad (3.14)$$

Letting (as in subsection 3.7)  $\tilde{C}(w) = (\omega_j^{(m-1)}(w))_{1 \leq j \leq g, 1 \leq m \leq 2g-1}$ , then  $\tilde{C}(w)$  has rank  $g$  according to lemma 5 so for each  $k$  there exists some  $b^k = (b_m^k)_{0 \leq m \leq 2g-2} \in \mathbb{C}^{2g-1}$  such that  $\tilde{C}(w)b^k = e_k$ . We then define

$$c_k(z) = \sum_{m=0}^{2g-2} b_m^k \varrho_m(z).$$

This way  $\omega_z(w)$  is periodic with respect to  $z$ . □

Let  $K_t = K \sqcup \{\mathcal{G}_D > t\}$ . Up to a small perturbation of  $t$  we may suppose that  $\partial K_t$  is smooth and does not contain zeroes of  $f$ . Let then

$$f_n(z) = \frac{1}{2\pi i} \oint_{\partial K_t} \frac{F_n(\zeta) - F_n(z)}{F_n(\zeta)} f(\zeta) \omega_z(d\zeta), \quad \forall z \in K_t.$$

$f_n$  is meromorphic with a pole of order at most  $n + q + 2g - 1$  at  $w$ . Indeed by residue theorem

$$f_n(z) = - \sum_{j=1}^n f(z_{j,n}) F_n(z) \operatorname{res} \left( \zeta \mapsto \frac{\omega_z(d\zeta)}{F_n(\zeta)}, z_{j,n} \right),$$

and  $F_n$  has order  $n + q$  at  $w$ ,  $\omega_z$  has order at most  $2g - 1$ .

Then by residue theorem  $f(z) = \frac{1}{2\pi i} \oint_{\partial K_t} f(\zeta) \omega_z(d\zeta)$  for any  $z \in K_t$ , so

$$f(z) - f_n(z) = \frac{1}{2\pi i} \oint_{\partial K_t} \frac{F_n(z)}{F_n(\zeta)} f(\zeta) \omega_z(d\zeta).$$

For some constant  $C > 0$ , and  $z \in K$ :

$$|f(z) - f_n(z)|^{\frac{1}{n+q}} \leq \left( C \frac{\|F_n\|_{L^\infty(K)}}{\inf_{\partial K_t} |F_n|} \right)^{\frac{1}{n+q}} \xrightarrow{n \rightarrow \infty} e^{-t}.$$

where we used the convergence result (3.13). □

*Proof of 2.* Let  $h$  be a harmonic function as in the statement of the theorem, defined in some neighbourhood of  $K$  denoted  $K_t = K \sqcup_{j=1}^N \{\mathcal{G}_{D_j}(\cdot, w_j) < t\}$ . Up to a small perturbation of  $t$  we suppose  $\partial K_t$  is smooth. Write  $G(w) = \{n \in \mathbb{N}^* : \ell(n[p]) = \ell((n-1)[p])\}$  the set of gaps at  $w$ :  $G(w) = \{1, \dots, g\}$  when  $w$  is not a Weierstrass point, and in general it is a subset of  $\{1, \dots, 2g-1\}$  of cardinal  $g$  such that the matrix

$$M(w) = (\omega_j^{(m-1)}(w))_{(j,m) \in \{1, \dots, g\} \times G(w)}$$

is invertible.

We claim we may split  $h$  as follows:

$$h(z) = h^{res}(z) + h^{per}(z) + h^{red}(z),$$

where

- The *residue* part  $h^{res}$  is a combination of the functions  $\log |\widehat{\sigma}_{w_l, w_{l+1}}(z)|$  for  $l = 1, \dots, N-1$ .
- The *period* part  $h^{per}$  is a combination of  $(\Re \widehat{\rho}_m(z, w_1), \Im \widehat{\rho}_m(z, w_1))_{m \in G(w_1)}$ .
- The *reduced* part  $h^{red}$  is the real part of some element  $f \in \mathcal{O}(K_t)$ , which is uniquely defined up to fixing  $f = h^{red}$  at some reference point in  $K$ .

Moreover  $h^{res}$  and  $h^{per}$  (which belong to a finite dimensional space) depend smoothly on  $h$ . We prove this claim in three steps:

- 1) Write  $D_{j,t} = D_j \setminus K_t$  and  $r_l(h) = \sum_{j=1}^l \frac{1}{2\pi i} \int_{\partial D_{j,t}} \partial_z h$ , such that denoting

$$h^{res} = \sum_{l=1}^{N-1} r_l(h) \log |\widehat{\sigma}_{w_l, w_{l+1}}(z)|$$

then for any  $j = 1, \dots, N$  we have

$$\int_{\partial D_{j,t}} \partial_z (h - h^{res}) = 0.$$

- 2) The periods of  $\partial_z (h - h^{res})$  are now well-defined: indeed for any smooth open set  $A \subset X$  such that  $\partial A \subset K_t$ , we have

$$\int_{\partial A} \partial_z (h - h^{res}) = \sum_{l=1}^N \int_{\partial D_{j,t} \subset A} \partial_z (h - h^{res}) = 0.$$

We show that coefficients  $(c_m(h), d_m(h))_{m \in G(w_1)}$  may be chosen such that, denoting

$$h^{per}(z) = \Re \sum_{m \in G(w_1)} c_m(h) \Re \widehat{\wp}_m(z, w_1) + d_m(h) \Im \widehat{\wp}_m(z, w_1),$$

then  $\int_{\mathcal{C}} \partial_z (h - h^{res} - h^{per}) = 0$  for any closed loop  $\mathcal{C}$  in  $K_t$ . To prove this, it is sufficient to prove that any set of periods may be reached with these forms, meaning:

$$\text{Span}_{\mathbb{C}} \left\{ \left( \begin{array}{c} \left( \int_{a_j} \partial_z \Re \widehat{\wp}_m(z, w_1) \right)_j \\ \left( \int_{b_j} \partial_z \Re \widehat{\wp}_m(z, w_1) \right)_j \end{array} \right), \left( \begin{array}{c} \left( \int_{a_j} \partial_z \Im \widehat{\wp}_m(z, w_1) \right)_j \\ \left( \int_{b_j} \partial_z \Im \widehat{\wp}_m(z, w_1) \right)_j \end{array} \right), m = 1, \dots, 2g - 1 \right\} = \mathbb{C}^{2g}.$$

Suppose it is not the case, meaning there exists some combination

$$\omega(z) = \sum_{m \in P(w_1)} c_m \partial_z \widehat{\wp}_m(z, w_1) + d_m \overline{\partial_z \widehat{\wp}_m(z, w_1)}$$

such that  $\int_{a_j} \omega = \int_{b_j} \omega = 0$  for all  $j$ : since  $\omega$  is meromorphic this means there is a meromorphic function  $f$  defined on  $X$ , with at most a pole at  $w_1$ , such that  $\partial_z f = \omega$ .

Since  $\overline{\partial_z \widehat{\wp}_m(z, w_1)}$  is in  $\Omega^1(X)$  for every  $m$ , we directly see that all coefficients  $c_m$  must vanish: if  $c_m \neq 0$  where  $m$  is supposed to be the largest possible index, then  $f$  has a pole of order exactly  $m$  at  $w_1$ , which is impossible since  $m \in G(p)$ . Then, by the expression of  $\widehat{\wp}_m$  (see equation (3.7)) we have

$$\overline{\partial_z \widehat{\wp}_m(z, w_1)} = \pi \sum_{j=1}^g (\Im(\tau)^{-1} \overline{M_{\cdot, m}(w_1)})_j \omega_j(z).$$

and since  $M(w_1)$  has full rank, it is not possible that all periods of  $\sum_{m \in G(w_1)} d_m \overline{\partial_z \widehat{\wp}_m(z, w_1)}$  vanish, unless all coefficients  $d_m$  vanish. This concludes the claim of the existence of  $c_m(h)$ ,  $d_m(h)$ .

- 3)  $h - h^{res} - h^{per}$  verifies  $\int_{\mathcal{C}} \partial_z (h - h^{res} - h^{per}) = 0$  for any closed loop  $\mathcal{C}$ , so we may find an antiderivative  $f \in \mathcal{O}(K_t)$ , which is uniquely defined up to fixing its value at some reference points of  $K_t$ .

We now prove the two implications.

( $a \Rightarrow b$ ) Suppose  $\limsup_{n \rightarrow \infty} \|h_n - h\|_{L^\infty(K)}^{1/n} \leq e^{-t}$ . Write  $f, f_n$  the holomorphic functions such that  $\partial_z f = \partial_z h^{red}$ ,  $\partial_z f_n = \partial_z h_n^{red}$  (we may suppose that  $f = f_n = 0$  at some fixed reference point of  $K$ ). Then since  $h^{res}$ ,  $h^{per}$  depend smoothly on  $h$ , we get also

$$\limsup_{n \rightarrow \infty} \|f_n - f\|_{L^\infty(K)}^{1/n}.$$

The first implication of lemma 7 applies.

( $b \Rightarrow a$ ) We start by decomposing  $h$  as above. Then our goal is to decompose  $f$  as

$$f = f_1 + \dots + f_N + r,$$

where each  $f_j$  is in  $\mathcal{O}(X \setminus (D_j \setminus K_t))$  (and  $r \in \mathcal{H}_{2g-1}(\{w_1, \dots, w_N\})$ ), so that we may apply lemma 7 to each  $f_j$ .

Let  $\omega_z$  be the parametrization of 1-forms built from lemma 8 with the basis point  $w_1$ . By residue theorem:

$$f(z) = \frac{1}{2\pi i} \oint_{\partial K_t} f(\zeta) \omega_z(d\zeta) = \sum_{j=1}^N \underbrace{\frac{1}{2\pi i} \oint_{D_j \cap \partial K_t} f(\zeta) \omega_z(d\zeta)}_{\tilde{f}_j}.$$

Then  $\tilde{f}_1 \in \mathcal{O}(X \setminus (D_1 \setminus K_t))$  and each  $\tilde{f}_j$  for  $j \geq 2$  is meromorphic in  $X \setminus (D_j \setminus K_t)$  with a single pole of order at most  $2g - 1$  at  $w_1$ . By the main theorem 1, there exists a harmonic function  $f_j^{corr}$  with a single pole (of order at most  $2g - 1$ ) in  $w_1$  such that  $\tilde{f}_j - f_j^{corr}$  has an erasable singularity at  $w_1$ . We then decompose

$$f = \underbrace{\tilde{f}_1 + \sum_{j=2}^N f_j^{corr}}_{g_1} + \sum_{j=2}^N \underbrace{\tilde{f}_j - f_j^{corr}}_{g_j}.$$

Here each function  $g_j$  is harmonic in  $X \setminus (D_j \setminus K_t)$ : by the same reduction as earlier, it may be decomposed as  $f_j + h_j$  where  $h_j \in \mathcal{H}_{2g-1}(\{w_j\})$  and  $f_j \in \mathcal{O}(X \setminus (D_j \setminus K_t))$ . We may then conclude by applying the lemma 7 to each  $f_j$ .

□

## 4 Computational approach

We describe in this section the several steps required to compute a numerical approximation of the harmonic basis introduced in section 3. Following [5], we illustrate our method on three distinct surfaces of genus 2 described by their Fenchel-Nielsen coordinates (see also section 1.1):

- **The surface with symmetry group  $D_6 \times \mathbb{Z}_2$**  for which

$$(l_1, t_1; l_2, t_2; l_3, t_3) = (2 \operatorname{arccosh}(2), 0; 2 \operatorname{arccosh}(2), 0; 2 \operatorname{arccosh}(2), 0)$$

which coordinates correspond to four regular orthogonal hexagons with no twists.

- **The Bolza surface** which can either be parametrized by Fenchel-Nielsen coordinates with one single nonzero twist

$$(l_1, t_1; l_2, t_2; l_3, t_3) = (2 \operatorname{arccosh}(3 + 2\sqrt{2}), \frac{1}{2}; 2 \operatorname{arccosh}(1 + \sqrt{2}), 0; 2 \operatorname{arccosh}(1 + \sqrt{2}), 0)$$

or equivalently by the uniform representation

$$(l_1, t_1; l_2, t_2; l_3, t_3) = (l, t; l, t; l, t)$$

where  $l = \operatorname{arccosh}(1 + \sqrt{2})$  and  $t = \frac{1}{l} \operatorname{arccosh}\left(\frac{2\sqrt{3+\sqrt{2}}}{7}\right)$ .

- **The Gutzwiller octagon** of coordinates

$$(l_1, t_1; l_2, t_2; l_3, t_3) = (2 \operatorname{arccosh}\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right), \frac{1}{2}; 4 \operatorname{arccosh}\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right), \frac{1}{4}; 2 \operatorname{arccosh}\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right), \frac{1}{2}).$$

Every triple of alternate edge lengths  $(l_1, l_2, l_3)$  determines a unique orthogonal hexagon in Poincaré's disk up to automorphisms of the disk. The practical construction of these hexagons for three alternate lengths is described in [8] chapter 3. We plot the tilings of the hyperbolic disk obtained by the four orthogonal hexagons of every surface in figure 2.

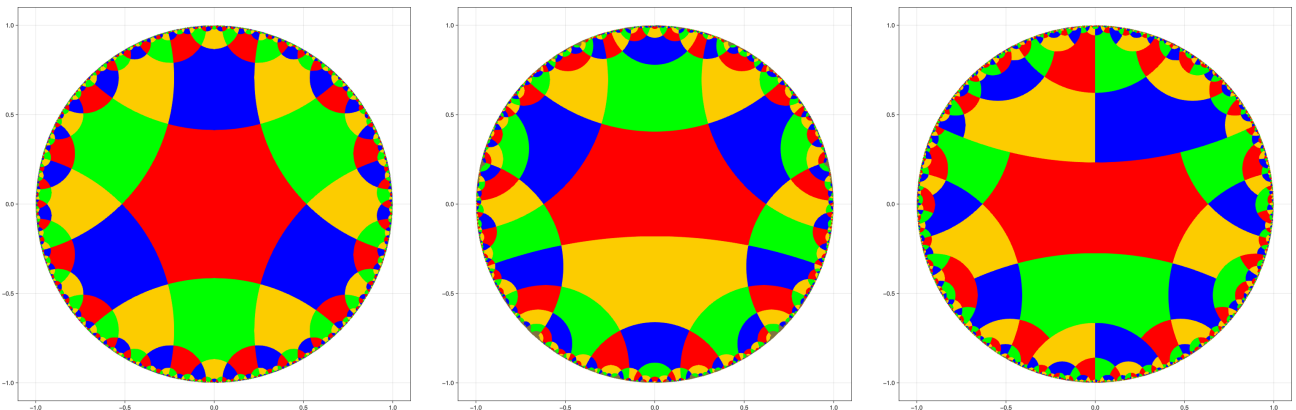


Figure 2:  $D_6 \times \mathbb{Z}_2$ , Bolza and Gutzwiller hexagonal tilings (left to right).

## 4.1 Approximation of 1-forms

Our construction of a basis of harmonic functions strongly relies on the approximation of a basis of one 1-forms. We already described in the introduction the least square approximation procedure that we implemented to obtain a polynomial approximation of this basis. We prove in section 6 the spectral convergence in infinity norm of the approximation with respect to the degree of the polynomials.

Assuming that the basis point is not a Weierstrass point (see remark 4 below theorem 6 for a discussion of the detection and correction when the center is a Weierstrass point), we compute for some large  $N$  the optimal polynomials  $Q_l^N = (Q_{l,p}^N)_{p=1,\dots,m} \in \mathbb{C}_N[X]^m$  defined as

$$\operatorname{argmin} \left\{ E^N(P), P \in \mathbb{C}_N[X]^m : \forall j = 1, \dots, g, P_{|1}^{(j-1)}(0) = \delta_{j,l} \right\}.$$

These act as an approximate basis of the 1-forms  $\Omega^1(X)$ . To illustrate this convergence, we list in table 1 the a posteriori errors obtained with several degrees  $N$  of approximation. The number of

sampling points  $S(N)$  to enforce the periodicity conditions on every circular edge is fixed to be  $3N$ . To avoid roundoff errors, we implemented the method using arbitrary precision with a number of 256 bits which correspond to a maximal floating point precision of 154 digits. The computation in the required precision of the polynomial approximations required from several seconds to a few minutes (for  $N = 10$  to  $N = 200$ ). Observe that these computations have to be considered as pretreatment: to evaluate our full basis of harmonic functions we only need to perform these computations once. The error values given in table 1 correspond to the maximal error on periodicity conditions

$$P|_p(z) = g'_{p,i}(z)P|_q(g_{p,i}(z))$$

numerically evaluated on  $6N$  random sampling points  $z$  on every circular edge. As expected, we recover a spectral precision with respect to  $N$  for all three test cases.

	$N = 10$	$N = 20$	$N = 50$	$N = 100$	$N = 160$	$N = 200$
$D_6 \times \mathbb{Z}_2$	6.93e-04	1.58e-06	6.89e-16	1.20e-31	2.01e-48	6.29e-59
Bolza surface	6.15e-03	1.52e-05	9.24e-14	6.06e-27	3.00e-43	1.79e-52
Gutzwiller octagon	1.95e-02	2.61e-05	1.04e-13	7.76e-27	7.51e-42	1.15e-51

Table 1: Spectral convergence of numerical a posteriori periodicity errors

## 4.2 Approximation of the period matrix and the Abel-Jacobi map

The period matrix of a surface is a fundamental data required in our construction to build meromorphic functions of prescribed orders. We explained in previous section how to compute an approximation of a basis of holomorphic 1-forms. The computation of a period matrix reduces to the evaluation of path integrals of these forms along the canonical basis  $(a_j, b_j)_{j=1, \dots, g}$  introduced in 2.2. Starting from the polynomials  $Q_l^N = (Q_{l|p}^N)_{p=1, \dots, m} \in \mathbb{C}_N[X]^m$  defined previously, we first let

$$A^N = \left( \left( \int_{a_k} Q_l^N \right)_{1 \leq k, l \leq g} \right),$$

where the integral of a element  $Q = (Q_{|1}, \dots, Q_{|m}) \in \mathbb{C}_N[X]^m$  along a smooth (say analytic by part) loop  $c : [0, 1] \rightarrow X$  is defined as follows:  $[0, 1]$  may be partitioned with intervals

$$0 = t_0 < t_1 < t_2 < \dots < t_r = 1,$$

such that for any  $i \in \{0, \dots, r-1\}$ ,  $t \in [t_i, t_{i+1}] \mapsto c(t)$  lies in a fixed polygon  $H_{p_i}$ , and  $c(t_i)$  is in the boundary of  $H_{p_i}$  for  $0 < i < r$ . We then let

$$\int_c Q = \sum_{i=0}^{r-1} \int_{t_i}^{t_{i+1}} Q_{|p_i}(c(t))c'(t)dt.$$

While this definition may seem standard, we remind that in this case the functions  $(Q_{|p})_{p=1, \dots, m}$  do not verify exactly the periodicity condition.

The matrix  $A^N$  is invertible at least for a large enough  $N$  and we make the change of variable

$$\tilde{Q}_k^N = \sum_{l=1}^g ((A^N)^{-1})_{l,k} Q_l^N$$

such that  $\int_{a_k} \tilde{Q}_l = \delta_{k,l}$ . This way  $\tilde{Q}_l^N$  is an approximation of the  $l$ -th canonical 1-form  $\omega_l$  verifying condition (2.2). We then let

$$\tau^N = \left( \left( \frac{1}{2} \int_{b_k} \tilde{Q}_l^N + \frac{1}{2} \int_{b_l} \tilde{Q}_k^N \right)_{1 \leq k, l \leq g} \right)$$



the approximate period matrix. Note that we force its symmetry by construction: we could define  $\tau_{k,l}^N = \int_{b_k} \tilde{Q}_l^N$ , and since  $\tilde{Q}_l^N$  is close to  $\omega^l$ ,  $\tau^N$  would be close to being symmetric. In our simulations, we used the very efficient *Arb* C library [9, 3] which has been now incorporated in the *Flint* library. The functions available in the library may provide results at a precision depending on the precision of the input. In particular, the call of the multidimensional theta function requires the input period matrix to be symmetric, which motivates this forced symmetry.

We evaluate the numerical precision of our approximations of period matrices on the well known case of Bolza surface. O. Bolza computed analytically the associated period matrix in [1]:

$$\tau = \begin{pmatrix} \frac{-1+i\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1+i\sqrt{2}}{2} \end{pmatrix}.$$

Moreover, the Siegel reduction of  $\tau$  is given by

$$\tau_{Siegel} = (A\tau + B)(C\tau + D)^{-1},$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using previous formula we obtain

$$\tau_{Siegel} = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + i \begin{pmatrix} \frac{2\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2\sqrt{2}}{3} \end{pmatrix}.$$

We reproduce below in table 2 the infinity norm error between  $\tau_{Siegel}$  and the approximations  $\tau_{Siegel}^N$ . As expected, we recover the same order of convergence as the one associated to the periodicity errors of previous section. We observed the same qualitative behavior when approximating the period matrices of the  $D_6 \times \mathbb{Z}_2$  surface and Gutzwiller octagon.

	$N = 10$	$N = 20$	$N = 50$	$N = 100$	$N = 160$	$N = 200$
$\ \tau_{Siegel} - \tau_{Siegel}^N\ _\infty$	8.95e-05	5.69e-08	4.51e-17	6.33e-30	8.30e-46	1.53e-55

Table 2: Spectral convergence of the approximation of Bolza period matrix.

Finally, the approximation of the Abel-Jacobi map  $u = (u_1, \dots, u_g)$  is defined from taking a primitive of  $\tilde{Q}^N$  on each polygon and adjusting it with a constant: we fix a path  $c : [0, 1] \rightarrow X$  that meet every polygon exactly once and define on each polygon  $H_p$  the vector-valued polynomial  $U_{|p}^N = (U_{1|p}^N, \dots, U_{g|p}^N) \in \mathbb{C}_{N+1}[X]^g$  such that for every  $l = 1, \dots, g$

$$(U_{l|p}^N)' = \tilde{Q}_{l|p}^N \text{ and } t \mapsto U_l^N(c(t)) \text{ is continuous.}$$

While this choice of path may seem arbitrary, we remind that any different choice would only differ by an element of  $\mathbb{Z}^g + \tau\mathbb{Z}^g$  on each polygon, which will be transparent after a composition of  $\Theta$ .

Finally, we compute the approximate Riemann constant

$$\mathcal{K}_j^N = \frac{1}{2} \tau_{jj}^N - \sum_{k=1}^g \int_{a_k} U_j^N(z) (U_k^N)'(z) dz,$$

where the last term may be computed exactly since  $U_j^N(z) (U_k^N)'(z) \in \mathbb{C}_{2N+1}[X]^m$ .

### 4.3 Approximation of meromorphic function of logarithmic order and of order 1

We described in sections 3.2 and 3.3, that the evaluation of functions  $\log |\hat{\sigma}_{v,w}|$  and  $\hat{\varphi}_1(z, w)$  requires the knowledge of the period matrix  $\tau$  of the surface (to evaluate the  $\Theta$  function) and a basis of 1-forms. We already discussed the accurate approximation of these data in the two previous sections.

Several libraries provide implementations of the vectorial Riemann theta function for a given  $\tau$  matrix in the Siegel upper half-space  $\mathbb{H}_g$  (the space of symmetric complex matrices with positive definite imaginary part).

All  $\Theta$  function evaluations of this section were performed with a precision of 512 using formula 3.3 and 3.4. We a posteriori evaluated the numerical precision computing the periodicity errors of our approximations of the  $\log |\hat{\sigma}|$  and  $\hat{\varphi}_1$  function. Using the same criteria as in section 4.1, we observed periodicity errors of the same order with respect to  $N$  as the one given in table 1. We plot in figures 3, 4 and 5 the graphs of the  $\log |\hat{\sigma}|$  function for every three surfaces. The left four graphs represent the restriction of the function to the Fenchel hexagons (with colored edges periodicity). The right plots represent the full graph of the associated function on Poincare disk.

As an example, we also provide plots of the real and imaginary part of  $\hat{\varphi}_1$  of the Bolza surface in the appendix (see figures 8 and 9).

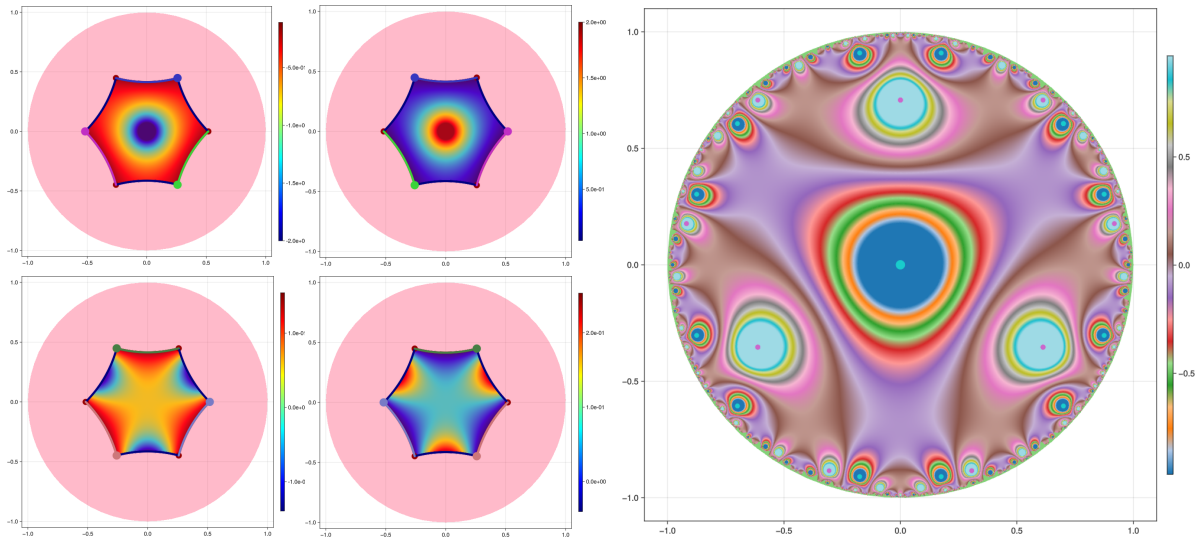


Figure 3: Logarithmic poles on  $D_6 \times \mathbb{Z}_2$ 's surface.

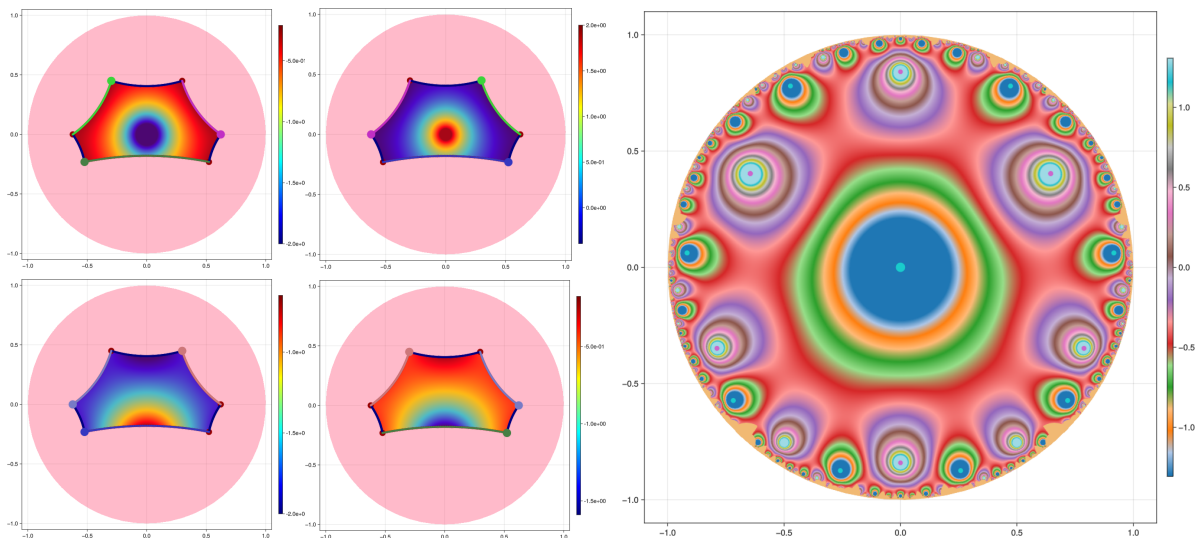


Figure 4: Logarithmic poles on Bolza's surface.

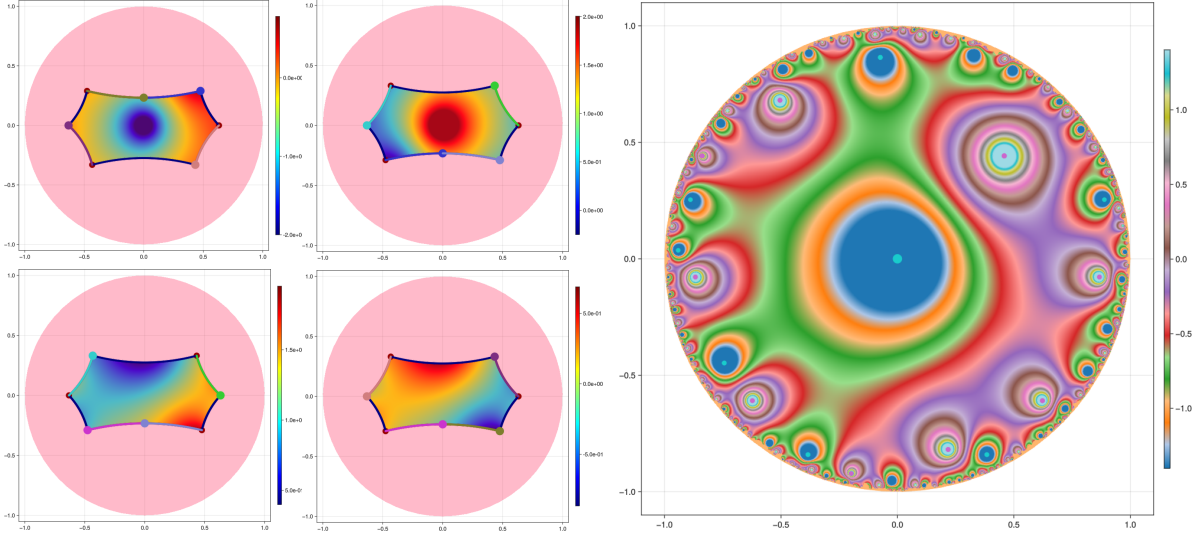


Figure 5: Logarithmic poles on Gutzwiller's surface.

#### 4.4 Approximation of meromorphic function of higher order by complex differentiation

The definitions of harmonic functions of order  $2$  to  $2g + 1$  introduced in sections 3.6 and 3.5 involve a new difficulty from a computational point of view: The derivatives of a rational function of  $\Theta$  (composed with anti-derivatives of pre-computed 1-forms) have to be evaluated. For this task, we implemented a finite difference procedure well adapted to analytic functions. In this context, finite differences avoid round off errors coming from high order algebraic derivatives minimizing the number of function evaluations. We recall a simple process to approximate the derivative of order  $m \in \mathbb{N}^*$  of a function  $f$  of a single complex variable  $z$ . The generalization of the method to complex vectorial variable is straightforward.

Let  $\varepsilon > 0$  and  $\zeta = e^{\frac{2i\pi}{m}}$ . Assume  $f(z + h) = \sum_{k \in \mathbb{N}} a_k h^k$  for a small complex perturbation  $h \in \mathbb{C}$ . For any fixed  $1 \leq p < m$ , we have using algebraic simplifications

$$\sum_{n=0}^{m-1} \zeta^{-pn} f(z + h_n) = m \sum_{k \in \mathbb{N}, k \equiv p \pmod{m}} a_k \varepsilon^k$$

where  $h_n = \varepsilon \zeta^n$ . Thus, previous equality gives an approximation of  $a_p$  with an error term of order  $\varepsilon^m$ . Consequently,  $m$  evaluations of the function  $f$  lead to an approximation of all its derivatives (deduced from the  $a_k$ ) up to order  $m - 1$ . In order to avoid round off errors in the finite difference process, we systematically call the *Flint* library (to evaluate Riemann  $\Theta$  function) with a (doubled) accuracy of 512 bits.

Finally, meromorphic functions of order greater than  $2g + 2$  can be easily obtained as powers and product of previous functions as detailed in section 3.6. Similarly to the approximation of the  $\log |\hat{\sigma}_{v,w}|$  and  $\hat{\varphi}_1(z, w)$  function of previous section, we observe no loss of accuracy with respect to the periodicity. We obtained once again the same order of convergence as the one obtained in the approximation of 1-forms. As an illustration, we plot in the appendix the real and imaginary part of a meromorphic function of the surface with symmetry group  $D_6 \times \mathbb{Z}_2$  of order 3 (see figures 10 and 11) and of order 6 (see figures 12 and 13).

## 5 Numerical harmonic extension in genus 2

In the same spirit as the numerical experiments of [12, 10] we illustrate in this section the use of our basis of harmonic functions to approximate harmonic extension by the method of particular

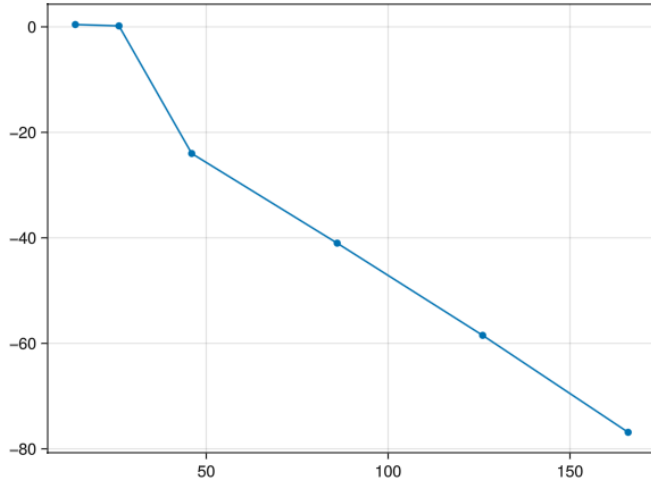


Figure 6: Spectral convergence with respect to the number of degrees of freedom: plot of the logarithm of the error in the boundary condition of (5.1) with respect to the number of degrees of freedom.

solutions. Namely, we consider the Gutzwiller surface  $X$  to solve for a given  $g$  the Laplace equation

$$u : S \rightarrow \mathbb{R} \text{ such that } \begin{cases} \Delta u = 0 & \text{in } X \setminus (B_1 \cup B_2) \\ u = g & \text{on } \partial B_1 \cup \partial B_2 \end{cases}, \quad (5.1)$$

where  $B_1$  and  $B_2$  are two closed geodesic balls.

Theorem 1 is the key ingredient of the method of particular solution to solve (5.1) for a surface of genus greater than 1. For a finite set of harmonic functions, the idea is to identify the linear combination which minimizes the boundary error in the least square sense. To simplify the notations, we denote by  $(\phi_i)_{1 \leq i \leq M}$  a finite set of harmonic functions in  $X \setminus (B_1 \cup B_2)$  provided by theorem 1 which is dense in the space of harmonic functions when  $M$  tends to infinity. We thus look for the unknown coefficients  $(v_i)_{1 \leq i \leq M}$  such that the function

$$u(z) = \sum_{i=1}^M v_i \phi_i(z) \quad (5.2)$$

is close to satisfy the boundary condition imposed by  $g$ .

To identify the optimal coefficients, we sample uniformly the boundary  $\partial B_1 \cup \partial B_2$  with respect to arclength and denote the collection of all sampled points by  $(p_\ell)_{\ell \in S}$ . We define the matrix  $B \in \mathbb{R}^{S \times M}$  to be

$$B_{\ell,i} = \phi_i(p_\ell).$$

The least-squares solution is found by solving the normal equations

$$B^t B v = B^t b \quad (5.3)$$

where the vector  $b = (g(p_\ell))_{\ell \in S}$  is the evaluation of the boundary condition at the sampling points.

We implemented the numerical method in Julia using arbitrary precision provided by the packages *GenericLinearAlgebra.jl* and *ArbNumerics.jl* (a wrapper of the *Arb* C library we already introduced). All computational experiments were performed with a precision of 512 bits which corresponds to a machine epsilon approximately equal to  $10^{-154}$ .

In our example, we choose the two disks  $B_1$  and  $B_2$  to be centered circular holes of radii  $r = 0.1$  of the first and third hexagons representing Gutzwiller surface. We systematically fixed the number of sampling points per circular edge to be equal to  $3M$ . We define the boundary condition as  $g(\theta) = \sin(3\theta)$  and  $g(\theta) = \sin(7\theta)$  respectively on  $\partial B_1$  and  $\partial B_2$ , where  $\theta$  are the polar angles with

respect to the centers of the holes. By the maximum principle, the accuracy of the solution can be computed by looking at the error on the boundary that is  $\sup_{x \in \partial B_1 \cup \partial B_2} |u(x) - f(x)|$ .

We plot in figure 6 the graph of the logarithm of the boundary error a posteriori evaluated on  $3S$  random points with respect to the number  $M$  of basis elements. As expected we recover a spectral convergence as the results obtained on similar test cases in [12, 10]. Finally, we plot in figure 7 the full graph of the harmonic extension of  $g$  on Poincare disk. To visualize the small variations of the function  $u$ , we apply a threshold to  $u$  and plot the function  $\max(\min(u, t), -t)$  where  $t = 0.1$ .

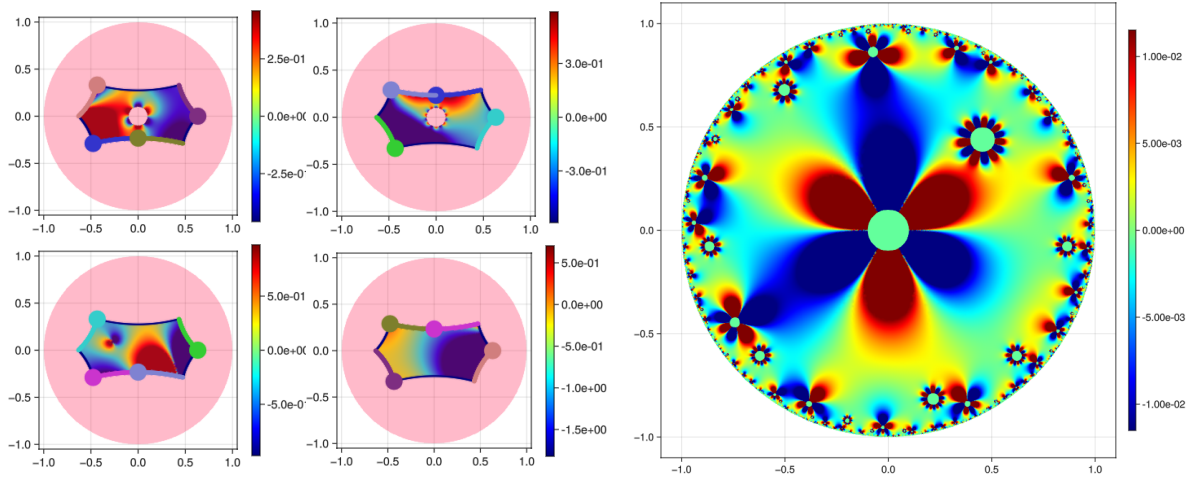


Figure 7: Harmonic extension on Gutzwiller surface by the method of particular solution.

## 6 Convergence of the least square approximation

In this section we prove the theorem 6. We fix  $X$  as in the statement of the result (in particular  $0 \in H_1$  is not a Weierstrass point), and we divide the proof in several intermediate results. We will use the following abuse of notation: the edge  $\gamma_{p,i}$  is seen as a parametrizing function

$$\gamma_{p,i} : [0, 1] \rightarrow \mathbb{D}$$

with constant speed (equal to the length of  $\gamma_{p,i}$  denoted  $L(\gamma_{p,i})$ ). This way the sampling set is simply

$$\mathcal{S}_{p,i} = \{\gamma_{p,i}(k/aN), k = 0, 1, 2, \dots, aN\}.$$

We start by introducing the notation

$$\gamma_{p,i}^a = \gamma_{p,i}([1/a, 1 - 1/a]).$$

For any  $K \subset \mathbb{C}$ , we write  $\mathcal{N}^\epsilon(K) = \{z \in \mathbb{C} : \text{dist}(z, K) \leq \epsilon\}$ . We introduce the auxiliary functional:

$$F(P) = \sum_{(p,i) \rightarrow (q,j)} \|P|_p - g'_{p,i} P|_q \circ g_{p,i}\|_{L^\infty(\gamma_{p,i}^a)}^2$$

and

$$G(P) = \sum_{(p,i) \rightarrow (q,j)} \|P|_p - g'_{p,i} P|_q \circ g_{p,i}\|_{L^\infty\left(\mathcal{N}^{\frac{2L(\gamma_{p,i})}{a}}(\gamma_{p,i}^a)\right)}^2.$$

Note that  $\mathcal{N}^{\frac{2L(\gamma_{p,i})}{a}}(\gamma_{p,i}^a)$  contains  $\mathcal{N}^{\frac{L(\gamma_{p,i})}{a}}(\gamma_{p,i}([0, 1]))$ .

**Lemma 9.** *Let  $\rho$  be as in theorem 6, then for any large enough  $N$  and any  $\omega \in \Omega^1(X)$ :*

$$\inf_{\mathcal{A}^{N,\omega}} \sqrt{E^N} < \rho^N \sum_{j=0}^{g-1} |\omega|_1^{(j)}(0).$$

*Proof.* This is a consequence of the Bernstein-Walsch theorem (see [14, Ch. VII] or [11, Th 6.3.1] for a more recent reference): since  $\omega|_p$  extends as a holomorphic function on  $\mathbb{D}$  (for any  $p$ ) and no further, then

$$\limsup_{N \rightarrow \infty} \inf_{Q \in \mathbb{C}^N[X]} \|Q - \omega|_p\|_{L^\infty(H_p)}^{1/N} = r,$$

where  $r \in ]0, 1[$  is the smallest value such that  $\omega|_p$  extends holomorphically to  $H_p \sqcup \{z \in \mathbb{C} : \mathcal{G}_{\widehat{\mathbb{C}} \setminus H_p}(z, \infty) < \log(1/r)\}$ , meaning the smallest value such that  $\{z \in \mathbb{C} : \mathcal{G}_{\widehat{\mathbb{C}} \setminus H_p}(z, \infty) < \log(1/r)\}$  is included in the disk  $\mathbb{D}$ . By definition of  $\rho$  we have, for any large enough  $N$  and any  $p$ :

$$\inf_{Q \in \mathbb{C}^N[X]} \|Q - \omega|_p\|_{L^\infty(H_p)} < \rho^N.$$

The result is then obtained from the fact that  $\Omega^1(X)$  has finite dimension,  $E^N$  is a quadratic form and  $\omega \in \Omega^1(X) \mapsto \sum_{j=0}^{g-1} |\omega|_1^{(j)}(0)|$  is a norm (since the origin of  $H_1$  is supposed to be generic).  $\square$

**Lemma 10.** *There exists a geometric constant  $C > 0$  such that for any  $P = (P_1, \dots, P_m) \in \mathbb{C}_N[X]^m$  we have*

$$F(P) \lesssim C a N^3 e^{NC/a} E^N(P).$$

*Proof.* Let  $P = (P_1, \dots, P_m) \in \mathbb{C}_N[X]^m$ , let  $(p, i)$  and  $(q, j)$  be such that  $(p, i) \rightarrow (q, j)$  and the gluing is made by some  $g_{p,i} \in PSU(1, 1)$  of the form

$$g_{p,i}(z) = \frac{az + b}{z - \omega}$$

where  $\omega := g_{p,i}^{-1}(\infty) \in \mathbb{C} \setminus \overline{\mathbb{D}}$ . We define

$$f(z) = P|_p(z) - g'_{p,i}(z) P|_q(g_{p,i}(z)) \in \frac{\mathbb{C}_{2N+2}[z]}{(z - \omega)^{N+2}}.$$

Our goal is to give a bound of  $\|f\|_{L^\infty(\gamma_{p,i}^a)}$  by  $\frac{1}{aN} \sum_{k=0}^{aN-1} |f(\gamma_{p,i}(k/aN))|^2$ . Let  $z = \gamma_{p,i}(t)$  for some  $t \in [\frac{1}{a}, 1 - \frac{1}{a}]$ . There exists some integer interval  $\{s, s+1, \dots, s+2N+2\} \subset \{0, 1, 2, \dots, aN\}$  such that, denoting  $t_k = \frac{s+k}{aN}$ , we have

$$t_N \leq t \leq t_{N+1}.$$

Denote also  $z_k = \gamma_{p,i}(t_k)$ . For some constant  $C > 0$  that only depend on the geometry of  $\gamma_{p,i}$  such that for any  $s \neq t$  in  $[t_0, t_{2N+2}]$  we have

$$e^{-C/a} \leq \frac{|\gamma_{p,i}(t) - \gamma_{p,i}(s)|}{L(\gamma_{p,i})|t - s|} \leq e^{C/a},$$

where  $L(\gamma_{p,i})$  is the length of  $\gamma_{p,i}$ .

Since  $(z - \omega)^{N+2} f(z)$  is a polynomial of degree  $2N+2$  at most, then by classical Lagrange interpolation we have:

$$f(z) = \sum_{k=0}^{2N+2} \left( \frac{z_k - \omega}{z - \omega} \right)^{N+2} f(z_k) \prod_{l=0, l \neq k}^{2N+2} \frac{z - z_l}{z_k - z_l},$$

and so

$$|f(z)|^2 \leq \left( \frac{1}{aN} \sum_{k=0}^{aN} |f(\gamma_{p,i}(k/aN))|^2 \right) \left( aN \sum_{k=0}^{2N+2} \left| \frac{z_k - \omega}{z - \omega} \right|^{2N+4} \prod_{l=0, l \neq k}^{2N+2} \left| \frac{z - z_l}{z_k - z_l} \right|^2 \right).$$

We now bound each term separately:

- $\left|\frac{z_k - \omega}{z - \omega}\right|^{2N+4} \leq e^{NC/a}$  for some geometric constant  $C > 0$  (since  $|z_k - \omega| \leq (1 + C/a)|z - \omega|$  for some  $C > 0$ ).
- $\prod_{l=0, l \neq k}^{2N+2} |z_k - z_l| \geq e^{-(2N+2)C/a} L(\gamma_{p,i})^{2N+2} \prod_{l=0, l \neq k}^{2N+2} \left|\frac{k-l}{aN}\right|$ , since  $|z_k - z_l| > e^{-C/a} L(\gamma_{p,i})$  for some  $C > 0$ . The product  $\prod_{l=0, l \neq k}^{2N+2} |k-l| = (k-1)!(2N+2-k)!$  is minimal at  $k = N+1$  we have

$$\prod_{l=0, l \neq k}^{2N+2} |z_k - z_l| \geq e^{-(2N+2)C/a} L(\gamma_{p,i})^{2N+2} (aN)^{-2N-2} N!(N+1)!.$$

- $\prod_{l=0, l \neq k}^{2N+2} |z - z_l| \leq e^{(2N+2)C/a} L(\gamma_{p,i})^{2N+2} (aN)^{-2N-2} \prod_{l=0, l \neq k}^{2N+2} |r-k|$  where  $r := aNt - s$  is in the interval  $[N, N+1]$ . As a consequence, we find that

$$\prod_{l=0, l \neq k}^{2N+2} |z - z_l| \leq C e^{(2N+2)C/a} L(\gamma_{p,i})^{2N+2} (aN)^{-2N-2} (N+1)!^2$$

for some constant  $C > 0$

From this we get (up to a possibly different geometric constant  $C > 0$ ):

$$|f(z)|^2 \leq C a N^3 e^{NC/a} \left( \frac{1}{aN} \sum_{k=0}^{aN} |f(\gamma_{p,i}(k/aN))|^2 \right),$$

which implies the result.  $\square$

**Lemma 11.** *Let  $\omega \in \widehat{\mathbb{C}} \setminus \mathbb{D}$  be fixed, the map  $t \in [0, 1/4] \mapsto \mathcal{G}_{\widehat{\mathbb{C}} \setminus \gamma_{p,i}([t, 1-t])}(\cdot, \omega) \in \mathcal{C}^{\frac{1}{2}}(\mathbb{D}, \mathbb{R})$  is continuous.*

*Proof.* There is a continuously-defined homography  $h_t$  that maps the circle arc  $\gamma_{p,i}([t, 1-t])$  to the segment  $[-1, 1]$ , then we have

$$\mathcal{G}_{\widehat{\mathbb{C}} \setminus \gamma_{p,i}([t, 1-t])}(z, \omega) = \mathcal{G}_{\widehat{\mathbb{C}} \setminus [-1, 1]}(h_t(z), h_t(\omega)).$$

and  $\mathcal{G}_{\widehat{\mathbb{C}} \setminus [-1, 1]}$  is  $\frac{1}{2}$ -Hölder.  $\square$

**Lemma 12.** *There exists  $\beta > 0$  such that for any  $P = (P_1, \dots, P_m) \in \mathbb{C}_N[X]^m$ , we have the inequality*

$$G(P) \leq e^{N\beta/\sqrt{a}} F(P).$$

*Proof.* Let

$$\beta = \max_{(p,i)} \max_{0 \leq t \leq 1/4} \sqrt{L(\gamma_{p,i})} \left[ \|\mathcal{G}_{\widehat{\mathbb{C}} \setminus \gamma_{p,i}([t, 1-t])}(\cdot, \infty)\|_{\mathcal{C}^{\frac{1}{2}}(\mathbb{D}, \mathbb{C})} + \|\mathcal{G}_{\widehat{\mathbb{C}} \setminus \gamma_{p,i}([t, 1-t])}(\cdot, g_{p,i}^{-1}(\infty))\|_{\mathcal{C}^{\frac{1}{2}}(\mathbb{D}, \mathbb{C})} \right],$$

which is finite by lemma 11 (note that  $g_{p,i}^{-1}(\infty) \in \mathbb{C} \setminus \overline{\mathbb{D}}$ ). We let

$$\mathcal{G}(z) = \mathcal{G}_{\widehat{\mathbb{C}} \setminus \gamma_{p,i}([1/a, 1-1/a])}(z, \infty) + \mathcal{G}_{\widehat{\mathbb{C}} \setminus \gamma_{p,i}^a}(z, g_{p,i}^{-1}(\infty)).$$

As previously, we let

$$f(z) = P|_p(z) - g'_{p,i}(z) P|_q(g_{p,i}(z)) \in \frac{\mathbb{C}_{2N+2}[z]}{(z-\omega)^{N+2}}$$

and

$$h(z) = \frac{1}{N} \log \frac{|f(z)|}{\|f\|_{L^\infty(\gamma_{p,i}^a)}} - \mathcal{G}(z),$$

that we see as a function of  $\widehat{\mathbb{C}} \setminus \gamma_{p,i}^a$  with a (finite) number of logarithmic singularities. Without loss of generality we may slightly perturb  $f$  so that it does not vanish on  $\gamma_{p,i}$ . Notice then that:

- 1)  $h \leq 0$  in  $\gamma_{p,i}^a$ .
- 2)  $h$  is subharmonic in  $\widehat{\mathbb{C}} \setminus \gamma_{p,i}^a$ : in fact  $h$  is harmonic outside a finite number of logarithmic singularities at the zeroes of  $f$  (where  $h$  is then subharmonic) at and  $\infty, g_{p,i}^{-1}(\infty)$  (where  $h$  is also subharmonic due to the degree condition of  $f$ ).

Thus by maximum principle,  $h \leq 0$  everywhere, so

$$|f(z)| \leq e^{NG(z)} \|f\|_{L^\infty(\gamma_{p,i}^a)}.$$

Since every point of  $\mathcal{N}^{\frac{2L(\gamma_{p,i})}{a}}(\gamma_{p,i}^a)$  has distance  $\frac{2L(\gamma_{p,i})}{a}$  to some point of  $\gamma_{p,i}^a$ , then by the definition of the constant  $\beta$  we have for any  $z \in \mathcal{N}^{\frac{2L(\gamma_{p,i})}{a}}(\gamma_{p,i}^a)$

$$\mathcal{G}(z) \leq \beta/\sqrt{a},$$

which gives is the expected result. □

**Lemma 13.** *There exists a constant  $c_a > 0$  such that, for any*

$$f = (f_{|1}, \dots, f_{|m}) \in \mathcal{O}(\mathcal{N}^{\frac{L(\gamma_{p,i})}{a}}(H_1)) \times \mathcal{O}(\mathcal{N}^{\frac{L(\gamma_{p,i})}{a}}(H_2)) \times \dots \times \mathcal{O}(\mathcal{N}^{\frac{L(\gamma_{p,i})}{a}}(H_m)),$$

we have

$$c_a \sum_{p=1}^m \|f_{|p}\|_{L^\infty(H_p)} \leq G(f)^{1/2} + \sum_{j=0}^{g-1} |f_{|1}^{(j)}(0)|.$$

The constant  $c_a$  that we obtain here is not explicit due to the fact that our proof is done by contradiction and compactness: it depends on the geometry and on  $a$  in a non-trivial way.

*Proof.* Suppose that this statement is false. Then there exists a sequence  $f^k = (f_{|1}^k, \dots, f_{|m}^k)$  verifying

$$G(f^k) + \sum_{j=0}^{g-1} |f_{|1}^{k(j)}(0)| \xrightarrow{k \rightarrow \infty} 0$$

and for any  $k$ :

$$\sum_{p=1}^m \|f_{|p}^k\|_{L^\infty(H_p)} = 1.$$

**Claim:** there exists a constant  $C > 0$  such that

$$\sum_{p=1}^m \|f_{|p}^k\|_{L^\infty(\tilde{H}_p)} \leq C,$$

where  $\tilde{H}_p$  is a neighbourhood of  $H_p$ .

This is obtained by induction. Let us first remind that, denoting  $L$  the minimal length among all sides, we have

$$G(f^k) \geq \sum_{(p,i) \rightarrow (q,j)} \|f_{|p}^k - g'_{p,i} f_{|q}^k \circ g_{p,i}\|_{L^\infty(\mathcal{N}^{\frac{L}{a}}(\gamma_{p,i}))}^2.$$

- For every  $p = 1, \dots, m$ , we have

$$\|f_{|p}^k\|_{L^\infty(H_p)} \leq 1.$$

- For every  $(p, i)$ , denote  $(q_{p,i}, j_{p,i})$  the corresponding edge. Then for every  $z \in \mathcal{N}^{\frac{L}{a}}(\gamma_{p,i}) \cap g_{p,i}^{-1}(H_{q_{p,i}})$ , we have

$$\begin{aligned} |f_{|p}^k(z)| &\leq |f_{|p}^k(z) - g'_{p,i}(z) f_{|q_{p,i}}^k \circ g_{q_{p,i}, j_{p,i}}(z)| + |g'_{p,i}(z) f_{|q_{p,i}}^k \circ g_{p,i}(z)| \\ &\leq G(f^k)^{\frac{1}{2}} + \|g_{p,i}\|_{L^\infty(\mathbb{D})}. \end{aligned}$$



- For every  $p$ , and every  $n \in \mathbb{N}$ , let  $H_p^{(n)}$  be the set defined inductively by

$$\begin{cases} H_p^{(0)} = H_p \\ H_p^{(n)} = H_p^{(n-1)} \cup \bigcup_{i=1}^{c(p)} \left( \mathcal{N}_{\frac{L}{a}}^c(\gamma_{p,i}) \cap g_{p,i}^{-1}(H_{q_{p,i}}^{(n-1)}) \right). \end{cases}$$

Then by the same computation as above, each  $f_p$  is bounded independently of  $k$  in  $L^\infty(H_p^{(n)})$  by

$$\sup_{p=1,\dots,m} \|f_p^k\|_{L^\infty(H_p^{(n)})} \leq G(f^k)^{\frac{1}{2}} + \left( \sup_{(p,i)} \|g'_{p,i}\|_{L^\infty(\mathbb{D})} \right) \left( \sup_{p=1,\dots,m} \|f_p^k\|_{H_p^{(n-1)}} \right).$$

- Taking  $n$  to be larger than the maximal number of vertices of the polygons ( $H_p$ ) that are identified to a single point (we may take the worst case with  $n > c(1) + c(2) + \dots + c(m)$ ), then  $H_p^{(n)}$  is a neighbourhood of  $H_p$  in  $\mathbb{D}$ , that is denoted  $\tilde{H}_p$  in the claim.

In particular, up to extraction we may suppose that each  $f_p^k$  converges locally uniformly in a (possibly smaller) neighbourhood of  $H_p$  to some limit  $f_p$  when  $k \rightarrow \infty$ , and the limits verify the periodicity relation

$$f_p = g'_{p,i} f_q \circ g_{p,i}$$

when  $(p, i) \rightarrow (q, j)$ . As a consequence,  $f = (f_1, \dots, f_m)$  is an element of  $\Omega^1(X)$ , that is non-zero since  $\sum_{p=1}^m \|f_p\|_{L^\infty(H_p)} = 1$  (because this is verified by  $f_p^k$ , which converges to  $f_p$  up to the boundary), and such that  $\sum_{j=0}^{g-1} |f_1^{(j)}(0)| = 0$ : this is in contradiction with the fact that 0 is not a Weierstrass point, meaning that  $\omega \in \Omega^1(X) \mapsto (\omega_1(0), \dots, \omega_1^{(g-1)}(0))$  must be injective.  $\square$

We may now prove the theorem 6:

*Proof.* By the previous results we have

$$\begin{aligned} G(P^{N,\omega}) &\leq e^{\frac{\beta}{\sqrt{a}}N} F(P^{N,\omega}) && \text{by lemma 12} \\ &\leq C a N^3 e^{\left[\frac{c}{a} + \frac{\beta}{\sqrt{a}}\right]N} E^N(P^{N,\omega}) && \text{by lemma 10.} \end{aligned}$$

Let now  $\rho$  be as in the statement of theorem 6, let  $\rho' < \rho$  still verifying the same lower bound. We may choose  $a$  large enough such that

$$\rho' e^{\frac{c}{a} + \frac{\beta}{\sqrt{a}}} < \rho.$$

Then by lemma 9 we have, for large enough  $N$ :

$$G(P^{N,\omega}) \leq \rho^{2N}.$$

We apply now lemma 13 to  $P^{N,\omega} - \omega$ ; note that  $G(P^{N,\omega} - \omega) = G(P^{N,\omega})$  by the periodicity of  $\omega$ , and the first  $g - 1$  derivatives of  $P_1^{N,\omega} - \omega_1$  at  $0 \in H_1$  are zero, so:

$$c_a \sum_{p=1}^m \|P_p^{N,\omega} - \omega_p\|_{L^\infty(H_p)} \leq G(P^{N,\omega} - \omega)^{\frac{1}{2}} \leq \rho^N.$$

$\square$

Let us briefly comment on the fact that the lower bound on  $\rho$  given in theorem 6 is optimal: suppose that the convergence is still valid for some  $\rho$  strictly lower than this bound, meaning that for some  $p$  the set

$$\{z \in \mathbb{C} : \mathcal{G}_{\hat{\mathbb{C}} \setminus H_p}(z, \infty) < \log(1/\rho)\}$$

is not included in the disk  $\mathbb{D}$ . Then by Bernstein-Walsh theorem this gives us an extension of  $\omega_p$  to the set  $\{z \in \mathbb{C} : \mathcal{G}_{\hat{\mathbb{C}} \setminus H_p}(z, \infty) < \log(1/\rho)\}$  which contains a point of  $\partial\mathbb{D}$ : this is a contradiction, as the functions  $\omega_p$  cannot be extended beyond the boundary  $\partial\mathbb{D}$ .

# A Some meromorphic functions on genus 2 surfaces

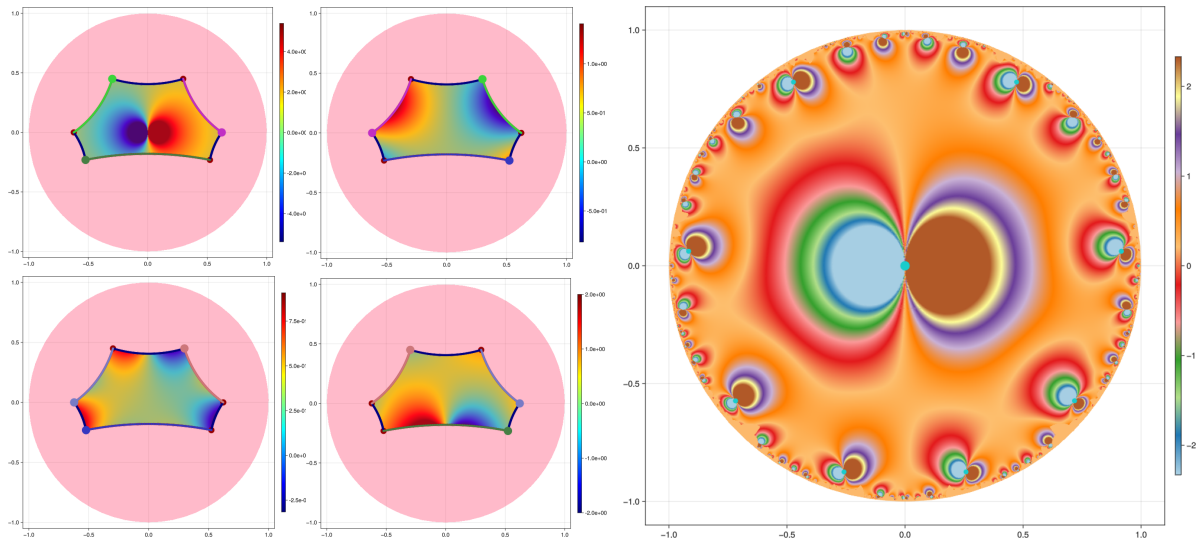


Figure 8: Pole of order 1 on Bolza's surface. The Real part.

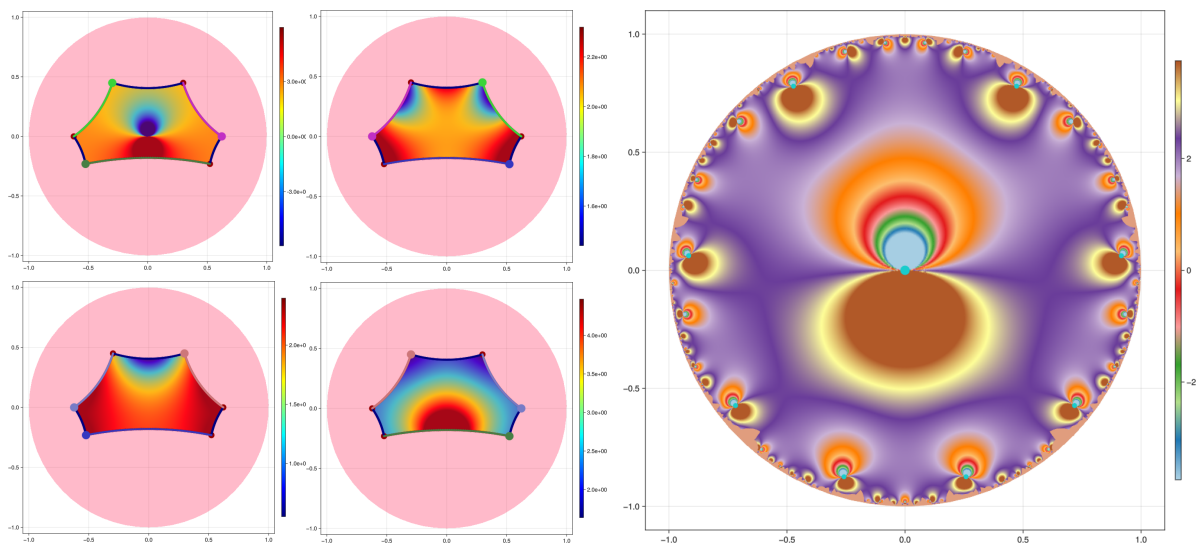


Figure 9: Pole of order 1 on Bolza's surface. The Imaginary part.

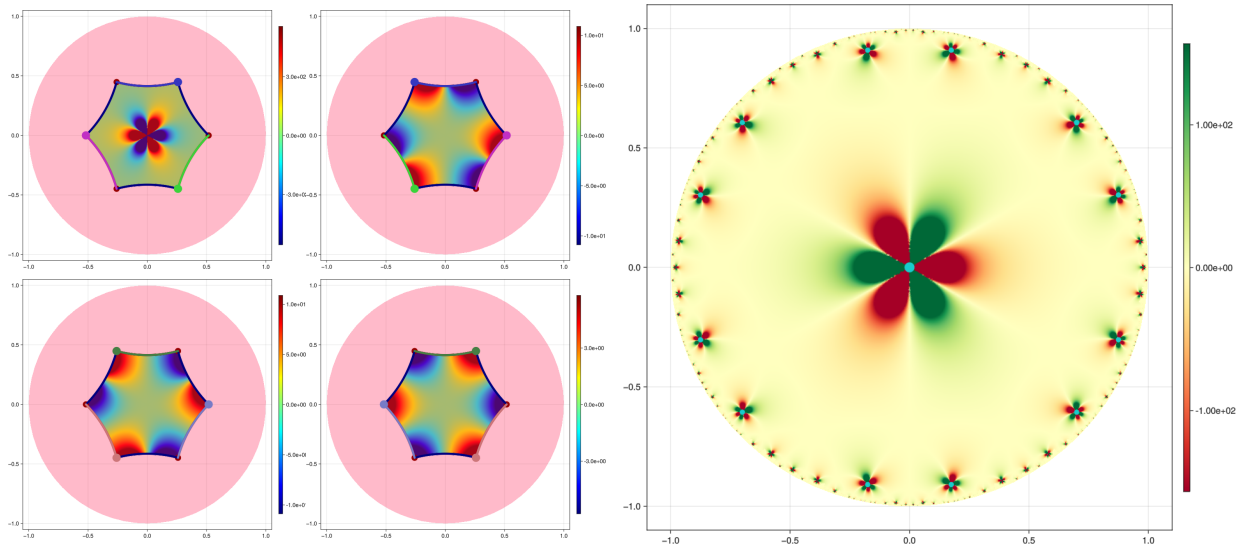


Figure 10: Pole of order 3 on  $D_6 \times \mathbb{Z}_2$  surface. The Real part.

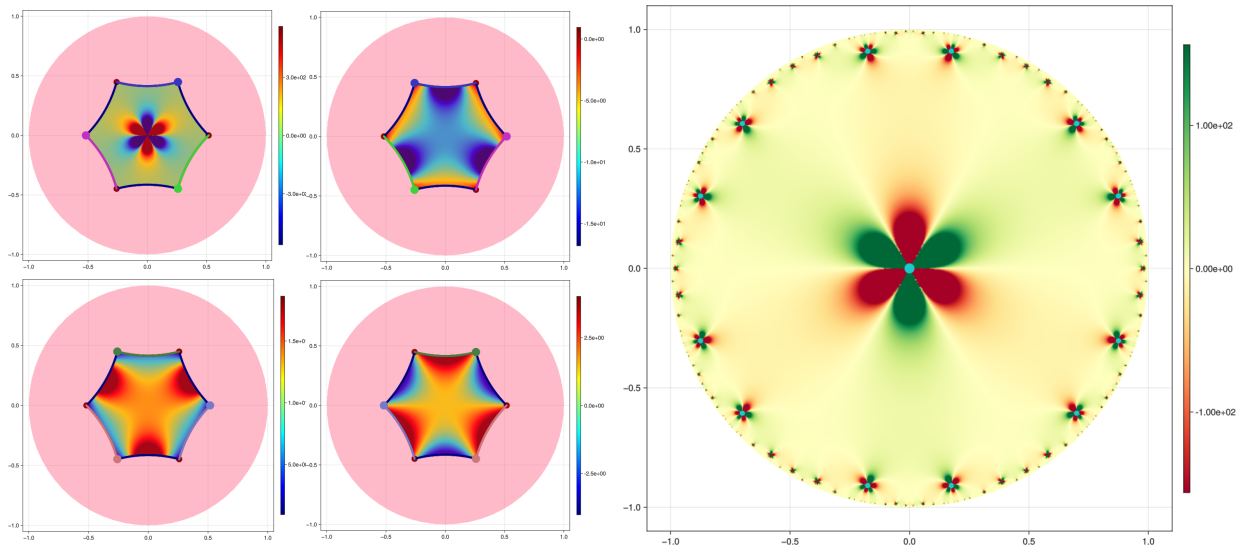


Figure 11: Pole of order 3 on  $D_6 \times \mathbb{Z}_2$  surface. The Imaginary part.

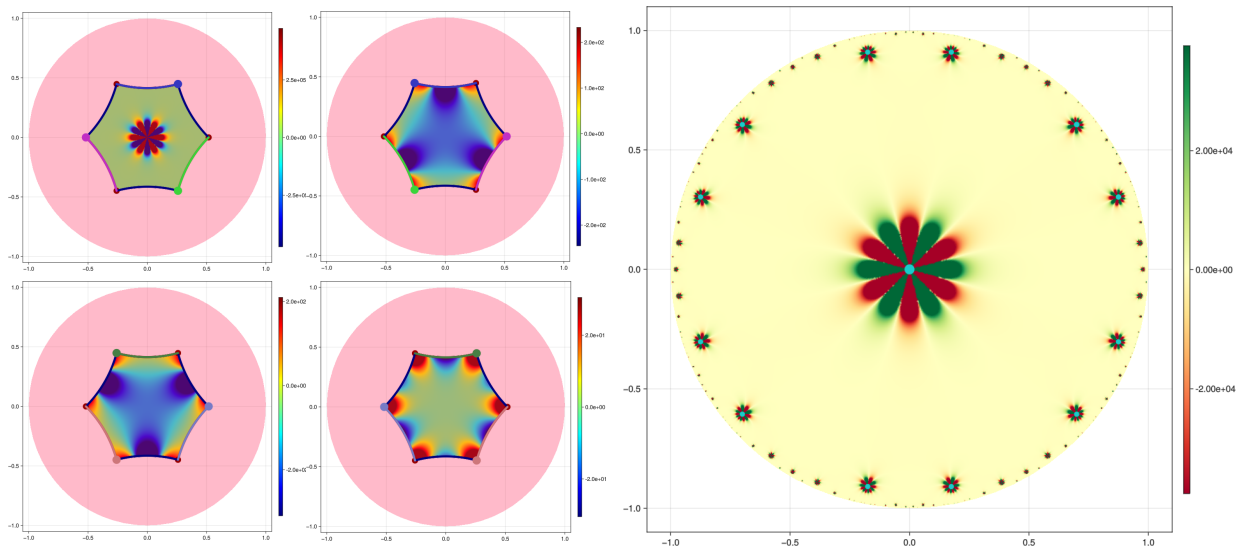


Figure 12: Pole of order 6 on  $D_6 \times \mathbb{Z}_2$  surface. The Real part.

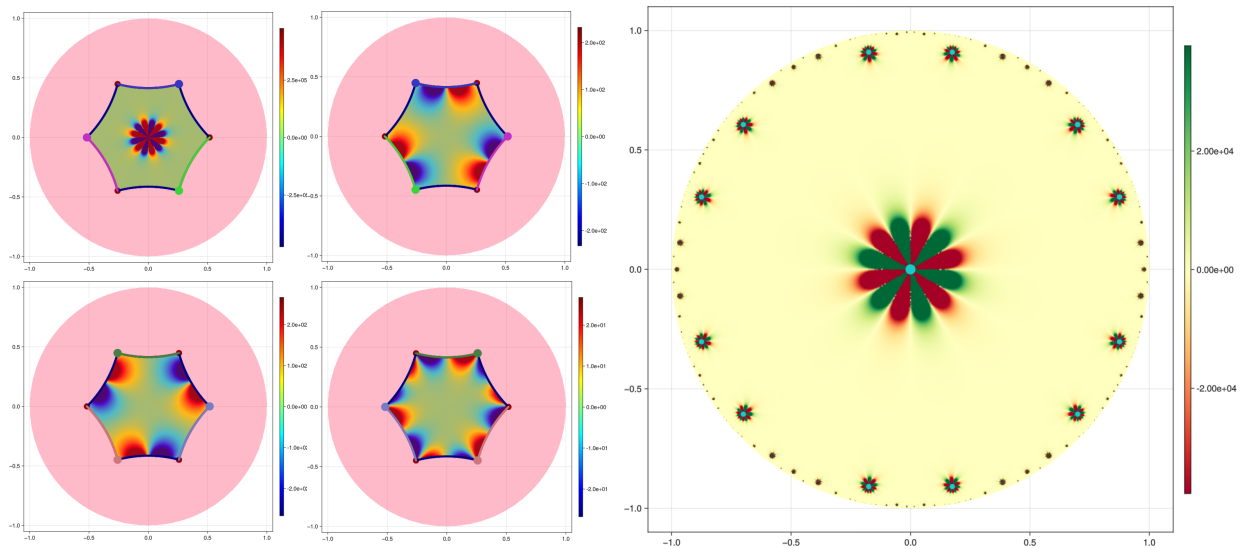


Figure 13: Pole of order 6 on  $D_6 \times \mathbb{Z}_2$  surface. The Imaginary part.

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