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Matteo Bonini[1], Martino Borello[2,3], and Eimear Byrne[4]

[1] Aalborg University, Department of Mathematical Sciences, Aalborg, Denmark mabo@math.aau.dk
[2] Université Paris 8, Laboratoire de Géométrie, Analyse et Applications, LAGA, Université Sorbonne Paris Nord, CNRS, UMR 7539, France martino.borello@univ-paris8.fr
[3] INRIA Saclay & LIX, CNRS UMR 7161, École Polytechnique, France
[4] School of Mathematics and Statistics, University College Dublin, Ireland ebyrne@ucd.ie

Abstract. We introduce the concept of a sum-rank saturating system and outline its correspondence to a covering properties of a sum-rank metric code. We consider the problem of determining the shortest sum-rank-ρ-saturating systems of a fixed dimension, which is equivalent to the covering problem in the sum-rank metric. We obtain upper and lower bounds on this quantity. We also give constructions of saturating systems arising from geometrical structures.

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Introduction

Researchers have extensively explored the connections between linear codes and sets of points in finite geometries, as evidenced by previous works such as [2,11,15,16,18,20]. The construction of a generator matrix or parity check matrix for a linear code can be accomplished through a multiset of projective points, with the supports of codewords corresponding to complements of hyperplanes in a fixed projective set. The interconnection between these two domains facilitates the application of methods from one field to the other. Notably, this approach has been employed in constructing codes with a bounded *covering radius*, associated with *saturating sets* in projective space. Recent investigations into the geometry of rank-metric codes codes [3,33] reveal their correspondence to q-systems and linear sets. A similar correspondence holds for sum-rank metric codes [29,34].

The sum-rank metric may be viewed as a hybrid of the rank and Hamming metric, extending both notions. Sum-rank metric codes have found applications in network coding, space-time coding, and distributed storage systems [14,26–28]. They have been studied in terms of decoding and code-optimality (see, e.g. [1,12,32]). One of the key reasons this metric has gained attention in recent years is that sum-rank metric codes beat traditional codes in terms of the field size needed to construct codes that meet the Singleton-like bound, due to the existence of linearized Reed-Solomon codes [25]. In this paper, we focus on sum-rank metric codes that are $\mathbb{F}_{q^m}$-linear subspaces. While there exists a more general description of sum-rank metric codes simply as linear spaces of matrices over $\mathbb{F}_q$, this restriction has immediate connections to geometric approaches [7,29].

This paper focus on the covering radius. The covering radius of a code is the smallest positive integer ρ such that the union of the spheres of radius ρ about each codeword equals the entire ambient space. The covering radius serves as an indicator of combinatorial properties, such as *maximality*, and is an invariant of code equivalence. It also provides insight into error-correcting capabilities by determining the maximal weight of a correctable error. This essential coding theoretical parameter has been extensively studied for codes in the context of the Hamming metric [8,13,15–19]. However, only a few papers in the literature on rank-metric codes and sum-rank metric ones address this parameter [6,10,22,30]. Recently, in [6], a purely geometrical approach based on saturating systems was proposed to study the covering radius in the rank metric. This approach led to new bounds and interesting examples of covering codes in the rank metric, see [4,6].

In this paper, we extend these ideas to the sum-rank metric by introducing the concept of a sum-rank saturating system, aligning it with a sum-rank metric covering code. We also provide new bounds for covering codes in the sum-rank metric, as well as examples arising from partitions and cutting systems.

After recalling some main definitions and results in Section 1, we introduce the main object of the paper in Section 2: we introduce the notion of a sum-rank saturating system, give equivalent characterizations of such systems, and outline the connection to the rank covering radius of a sum-rank

metric code. In Section 3 we give upper and lower bounds on the minimum $\mathbb{F}_q$-dimension of a sum-rank saturating system. Finally, in 4 we provide explicit constructions of sum-rank saturating systems from partitions of projective spaces and cutting designs.

1 Preliminaries

1.1 Vector sum-rank metric codes

Throughout this paper, we will denote by $\mathbf{n} = (n_1, \dots, n_t) \in \mathbb{N}^t$ an ordered tuple with $n_1 \geq n_2 \geq \dots \geq n_t$ and $N = n_1 + \dots + n_t$. We use the notation $\mathbb{F}_{q^m}^{\mathbf{n}} = \bigoplus_{i=1}^t \mathbb{F}_{q^m}^{n_i}$ for the direct sum of vector spaces.

We recall that the *rank* of a vector $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^m}^n$ is defined as $\mathrm{rk}(v) = \dim_{\mathbb{F}_q}(\langle v_1, \dots, v_n \rangle_{\mathbb{F}_q})$ and the *sum-rank weight* of an element $x = (x_1, \dots, x_t) \in \mathbb{F}_{q^m}^{\mathbf{n}}$ is $w(x) = \sum_{i=1}^t \mathrm{rk}(x_i)$.

Definition 1. *A sum-rank metric code $\mathcal{C}$ is an $\mathbb{F}_{q^m}$-subspace of $\mathbb{F}_{q^m}^{\mathbf{n}}$ endowed with the sum-rank distance, which is defined to be:*

$$d(x, y) = w(x - y) = \sum_{i=1}^{t} \mathrm{rk}(x_i - y_i),$$

for all $x = (x_1, \dots, x_t), y = (y_1, \dots, y_t) \in \mathbb{F}_{q^m}^{\mathbf{n}}$.

Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^{\mathbf{n}}$ be a sum-rank metric code. We say that $\mathcal{C}$ is an $[\mathbf{n}, k, d]_{q^m/q}$ code (or an $[\mathbf{n}, k]_{q^m/q}$ code) if k is the $\mathbb{F}_{q^m}$-dimension of $\mathcal{C}$ and d is its minimum distance, i.e.

$$d = d(\mathcal{C}) = \min\{d(x, y) \colon x, y \in \mathcal{C}, x \neq y\}.$$

Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^{\mathbf{n}}$ be a linear sum-rank metric code. A *generator matrix* G of $\mathcal{C}$ is a matrix whose lines generate $\mathcal{C}$, i.e. $G = (G_1 | \dots | G_t)$ with $G_1, \dots, G_t \in \mathbb{F}_{q^m}^{k \times n_i}$. We define $\mathcal{C}$ to be *nondegenerate* if the columns of G_i are $\mathbb{F}_q$-linearly independent for $i \in \{1, \dots, t\}$. Without loss of generality, in this paper we will only consider nondegenerate codes.

1.2 q-systems

We introduce some results regarding the connections of sum-rank metric codes and sets of subspaces, see [29] for further details.

Definition 2. *For each $i \in \{1, \dots, t\}$, let $\mathcal{U}_i$ be an $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^m}^k$ of dimension n_i. If the ordered t-tuple $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_t)$ satisfies $\langle \mathcal{U}_1, \dots, \mathcal{U}_t \rangle_{\mathbb{F}_{q^m}} = \mathbb{F}_{q^m}^k$ then $\mathcal{U}$ is called an $[\mathbf{n}, k]_{q^m/q}$* system. *We say that $\mathcal{U}$ has dimension (or rank) $\mathbf{n}$. A* generator matrix *for $\mathcal{U}$ is a $k \times \sum_{j=1}^t n_j$ matrix over $\mathbb{F}_{q^m}$ the form $G = [G_1 | \cdots | G_t]$, where for each i, G_i is a generator matrix for the $[n_i, k]_{q^m/q}$ system $\mathcal{U}_i$, i.e., such that the $\mathbb{F}_q$-span of the columns of each G_i is $\mathcal{U}_i$.*

For any $\mathbb{F}_q$-subspace $\mathcal{V}$ of $\mathbb{F}_{q^m}^k$ and $a \in \mathbb{F}_{q^m}$ we write $a\mathcal{V} := \{av : v \in \mathcal{V}\}$.

Definition 3. *Two sum-rank systems $(\mathcal{U}_1, \dots, \mathcal{U}_t)$ and $(\mathcal{V}_1, \dots, \mathcal{V}_t)$ are equivalent if there exists an isomorphism $\varphi \in \mathrm{GL}(k, \mathbb{F}_{q^m})$, an element $\mathbf{a} = (a_1, \dots, a_t) \in (\mathbb{F}_{q^m}^*)^t$ and a permutation $\sigma \in \mathcal{S}_t$, such that for every $i \in \{1, \dots, t\}$*

$$\varphi(\mathcal{U}_i) = a_i \mathcal{V}_{\sigma(i)}.$$

The following result allows us to establish a connection between systems and codes that will be very useful later on for our scopes, see [29] for further details.

Theorem 1 ([29, Theorem 3.1]). *Let $\mathcal{C}$ be an $[\mathbf{n}, k, d]_{q^m/q}$. Let $G = (G_1 | \dots | G_t)$ be a generator matrix of $\mathcal{C}$. Let $\mathcal{U}_i \subseteq \mathbb{F}_{q^m}^k$ be the $\mathbb{F}_q$-span of the columns of G_i, for $i \in \{1, \dots, t\}$. The sum-rank weight of an element $xG \in \mathcal{C}$, with $x = (x_1, \dots, x_k) \in \mathbb{F}_{q^m}^k$ is*

$$w(xG) = N - \sum_{i=1}^{t} \dim_{\mathbb{F}_q}(\mathcal{U}_i \cap x^\perp), \tag{1}$$

where $x^\perp = \{y = (y_1, \ldots, y_k) \in \mathbb{F}_{q^m}^k \colon \sum_{i=1}^k x_i y_i = 0\}$. *In particular, the minimum distance of* $\mathcal{C}$ *reads as follows*

$$d = N - \max\left\{\sum_{i=1}^{t} \dim_{\mathbb{F}_q}(\mathcal{U}_i \cap H) \colon H \text{ is an } \mathbb{F}_{q^m}\text{-hyperplane of } \mathbb{F}_{q^m}^k\right\}. \tag{2}$$

So $(\mathcal{U}_1, \ldots, \mathcal{U}_t)$ *in an* $[\mathbf{n}, k, d]_{q^m/q}$*-system.*

Moreover, it is possible to prove that there is a one-to-one correspondence between equivalence classes of sum-rank nondegenerate $[\mathbf{n}, k, d]_{q^m/q}$ code and equivalence classes of $[\mathbf{n}, k, d]_{q^m/q}$-systems, see [29].

1.3 Linear sets

Let us define linear sets, which were introduced by Lunardon in [24] for the construction of blocking sets and have become a topic of significant research in recent years. A thorough discussion of linear sets is available in [31].

Definition 4. *Let* V *be a* k*-dimensional vector space over* $\mathbb{F}_{q^m}$ *and consider* $\Lambda = \mathrm{PG}(V, \mathbb{F}_{q^m}) = \mathrm{PG}(k-1, q^m)$*. Let* $\mathcal{U}$ *be an* $\mathbb{F}_q$*-subspace of* V *of dimension* n*. Then the point-set*

$$L_{\mathcal{U}} = \{\langle u \rangle_{\mathbb{F}_{q^m}} : u \in \mathcal{U} \setminus \{0\}\} \subseteq \Lambda$$

is called an $\mathbb{F}_q$*-linear set of rank* n*.*

Definition 5. *Let* $P = \langle v \rangle_{\mathbb{F}_{q^m}}$ *be a point in* Λ*. The weight of* P *in* $L_{\mathcal{U}}$ *is defined as*

$$w_{L_{\mathcal{U}}}(P) = \dim_{\mathbb{F}_q}(\mathcal{U} \cap \langle v \rangle_{\mathbb{F}_{q^m}}).$$

An basic upper bound on the number of points that a linear set contains is

$$|L_{\mathcal{U}}| \leq \frac{q^n - 1}{q - 1}. \tag{3}$$

We say that, $L_{\mathcal{U}}$ is *scattered* if it meets this number of points, or equivalently, if all points of $L_{\mathcal{U}}$ have weight one.

2 Sum-rank saturating systems

In this section we will discuss the main object of this paper, namely, sum-rank saturating systems.

We start by recalling the definition of a ρ-saturating set.

Definition 6. *Let* $\mathcal{S} \subseteq \mathrm{PG}(k-1, q^m)$*.*

(a) *A point* $Q \in \mathrm{PG}(k-1, q^m)$ *is said to be* ρ-saturated *by* $\mathcal{S}$ *if there exist* $\rho+1$ *points* $P_1, \ldots, P_{\rho+1} \in \mathcal{S}$ *such that* $Q \in \langle P_1, \ldots, P_{\rho+1} \rangle_{\mathbb{F}_{q^m}}$*. We also say that* $\mathcal{S}$ ρ-saturates Q*.*

(b) *The set* $\mathcal{S}$ *is called a* ρ-saturating *set of* $\mathrm{PG}(k-1, q^m)$ *if every point* $Q \in \mathrm{PG}(k-1, q^m)$ *is* ρ*-saturated by* $\mathcal{S}$ *and* ρ *is the smallest value with this property.*

The following is the main object of this paper.

Definition 7. *An* $[\mathbf{n}, k]_{q^m/q}$ *system* $\mathcal{U}$ *is sum-rank* ρ*-saturating if* $L_{\mathcal{U}_1} \cup \cdots \cup L_{\mathcal{U}_t}$ *is* $(\rho-1)$*-saturating.*

As in the rank-metric case, we may get a characterisation of sum-rank saturating systems.

Theorem 2. *Let* $\mathcal{U}$ *be an* $[\mathbf{n}, k]_{q^m/q}$ *system and let* G *be any generator matrix of* $\mathcal{U}$*. The following are equivalent:*

(a) $\mathcal{U}$ *is sum-rank* ρ*-saturating.*

(b) *For each vector* $v \in \mathbb{F}_{q^m}^k$ *there exists* $\lambda = (\lambda_1, \ldots, \lambda_t) \in \mathbb{F}_{q^m}^{1\times n_1} \times \ldots \times \mathbb{F}_{q^m}^{1\times n_t}$ *with* $\mathrm{wt}_{\mathrm{srk}}(\lambda) \leq \rho$ *such that*

$$v = G(\lambda_1, \ldots, \lambda_t)^T,$$

and ρ *is the least integer with this property.*

(c) *We have*

$$\mathbb{F}_{q^m}^k = \bigcup_{\substack{(\mathcal{S}_i : i\in[t]):\, \mathcal{S}_i \leq_{\mathbb{F}_q} \mathcal{U}_i, \\ \sum_{i=1}^t \dim_{\mathbb{F}_q} \mathcal{S}_i \leq \rho}} \left(\bigcup_{i=1}^t \langle \mathcal{S}_i \rangle_{\mathbb{F}_{q^m}} \right)$$

and ρ is the least integer with this property.

Proof. (a) $\implies$ (b) Let $Q = \langle v \rangle \in PG(k-1, q^m)$. Since $\mathcal{U}$ is sum-rank ρ-saturating, there exist $P_1, \ldots, P_\rho \in L_{\mathcal{U}_1} \cup \cdots \cup L_{\mathcal{U}_t}$ such that $Q \in \langle P_1, \ldots, P_\rho \rangle_{\mathbb{F}_{q^m}}$. For each i, let $P_i = \langle w_i \rangle$ for some $w_i \in \mathcal{U}_{w(i)}$ and $w(i) \in [t]$. Then $v = \sum_{j=1}^{\rho} \gamma_j w_j$ for some $\gamma_j \in \mathbb{F}_{q^m}$. For each $\ell \in [t]$, let the set $B_\ell = \{u_{\ell,1}, \ldots, u_{\ell,n_\ell}\}$ be an $\mathbb{F}_q$-basis of $\mathcal{U}_\ell$. For each i, there exist $\tau_{\ell,i,r} \in \mathbb{F}_q$ such that $w_i = \sum_{\ell=1}^{t} \sum_{r=1}^{n_\ell} \tau_{\ell,i,r} u_{\ell,r}$, where $\tau_{\ell,i,r} = 0$ whenever $\ell \neq w(i)$.

We may now express v as follows:

$$v = \sum_{\ell=1}^{t} \sum_{r=1}^{n_\ell} u_{\ell,r} \sum_{i=1}^{\rho} \gamma_i \tau_{\ell,i,r}.$$

Define the matrices $\tau^{(\ell)} = (\tau_{\ell,i,r}) \in \mathbb{F}_q^{\rho \times n_\ell}$, and hence define:

$$\lambda = (\lambda_1, \ldots, \lambda_t) := (\gamma_1, \ldots, \gamma_\rho)[\tau^{(1)} | \cdots | \tau^{(t)}] \in \mathbb{F}_{q^m}^{1\times n_1} \times \cdots \times \mathbb{F}_{q^m}^{1 \times n_t}$$

Each matrix $\tau^{(\ell)}$ has every j-th row all-zeroes if $w(j) \neq \ell$ and, furthermore, there are at most ρ distinct non-zero matrices $\tau^{(\ell)}$. We have $\lambda_j = \gamma_i \tau_i^{(j)}$, for i, j satisfying $w(i) = j$. It follows that $\mathrm{wt}_{\mathrm{srk}}(\lambda) \leq \rho$.

The proofs that (b) implies (c) and (c) implies (a) are very similar to those of [6, Theorem 2.3].

Definition 8. *Let $\mathcal{U}$ be an $[\mathbf{n}, k]_{q^m/q}$ system. For each positive integer ρ, we define*

$$\mathbb{S}_\rho(\mathcal{U}) := \bigcup_{\substack{(\mathcal{S}_i : i\in[t]):\, \mathcal{S}_i \leq_{\mathbb{F}_q} \mathcal{U}_i, \\ \sum_{i=1}^t \dim_{\mathbb{F}_q} \mathcal{S}_i \leq \rho}} \left(\bigcup_{i=1}^t (\mathcal{S}_i \otimes \mathbb{F}_{q^m}) \right).$$

It is immediate from Theorem 2 that $\mathcal{U}$ is sum-rank ρ-saturating if ρ is the least integer satisfying $\mathbb{F}_{q^m}^k = \mathbb{S}_\rho(\mathcal{U})$.

The following statement is the sum-rank analogue of [6, Theorem 2.5]. The proof is very similar and hence is omitted.

Theorem 3. *Let $\mathcal{U}$ be an $[\mathbf{n}, k]_{q^m/q}$ system associated to a code $\mathcal{C}$. The following are equivalent.*

(a) *$\mathcal{U}$ is sum-rank ρ-saturating.*
(b) *$\rho_{srk}(\mathcal{C}^\perp) = \rho$.*

Definition 9. *For $i = 1, 2$, let $\mathcal{U}_i$ be a sum-rank ρ_i-saturating $[\mathbf{n}_i, k_i]_{q^m/q}$ system associated with a code $\mathcal{C}_i$ that has generator matrix G_i. We define the* direct sum *of $\mathcal{U}_1$ and $\mathcal{U}_2$, which we denote by $\mathcal{U}_1 \oplus \mathcal{U}_2$, to be the $[(\mathbf{n}_1, \mathbf{n}_2), k_1 + k_2]_{q^m/q}$ system associated with the direct sum of $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e. the code whose generator matrix is*

$$G_1 \oplus G_2 := \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}.$$

It is straightforward to establish the following (c.f. [6]).

Theorem 4. *For $i \in [t]$, let $\mathcal{U}_i$ be a sum-rank ρ_i-saturating $[\mathbf{n}_i, k_i]_{q^m/q}$ system. Then $\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_t$ is an $[(\mathbf{n}_1, \ldots, \mathbf{n}_t), k_1 + \cdots + k_t]_{q^m/q}$ system and is sum-rank ρ-saturating, for some $\rho \leq \rho_1 + \cdots + \rho_t$.*

Definition 10. *A sum-rank ρ-saturating system $\mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_t$ is called* reducible *if there exists $i \in \{1, \ldots, t\}$ such that the system $\mathcal{U}_1 \oplus \cdots \mathcal{U}_{i-1} \oplus \mathcal{U}_{i+1} \cdots \oplus \mathcal{U}_t$ is sum-rank ρ-saturating. Otherwise, the system is called* irreducible.

3 Bounds on the dimension of sum-rank saturating systems

As in the case of Hamming-metric and rank-metric codes, it is interesting to know the shortest length of any sum-rank metric code of a given dimension and covering radius ρ, or equivalently, the least rank of any sum-rank ρ-saturating system in a given vector space.

We start with a bound which follows from the geometric characterisation of our systems. In the proof, we will use the following well-known estimates:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q < f(q)\, q^{b(a-b)}, \qquad \text{for } a, b \in \mathbb{N}, \tag{4}$$

$$q^{e_1} + \ldots + q^{e_r} < \frac{q}{q-1} q^{e_r}, \qquad \text{for } e_i \in \mathbb{Z},\ 0 \le e_1 < \ldots < e_r. \tag{5}$$

where $f(q) = \prod_{i=1}^{+\infty} (1 - q^{-i})^{-1}$.

Theorem 5. *Let $\mathcal{U}$ be a sum-rank ρ-saturating $[\mathbf{n}, k]_{q^m/q}$ system. Then*

$$q^{m\rho} \sum_{\mathbf{s} \in \mathcal{N}, |\mathbf{s}| = \rho} \begin{bmatrix} \mathbf{n} \\ \mathbf{s} \end{bmatrix}_q \ge q^{mk}.$$

In particular,

$$\frac{1}{4t} \cdot \sum_{1 \le i < j \le t} (n_j - n_i)^2 + \frac{\rho(|n| - \rho)}{t} + 2t \ge m(k - \rho). \tag{6}$$

Proof. Since $\mathcal{U}$ is a sum-rank ρ-saturating $[\mathbf{n}, k]_{q^m/q}$ system, we have that $\mathbb{S}_\rho(\mathcal{U}) = \mathbb{F}_{q^m}^k$. Therefore

$$q^{m\rho} \sum_{\mathbf{s} \in \mathcal{N}, |\mathbf{s}| = \rho} \begin{bmatrix} \mathbf{n} \\ \mathbf{s} \end{bmatrix}_q \ge |\mathbb{S}_\rho(\mathcal{U})| = |\mathbb{F}_{q^m}^k| = q^{mk}.$$

By (4), we have:

$$\sum_{\mathbf{s} \in \mathcal{N}, |\mathbf{s}| = \rho} \begin{bmatrix} \mathbf{n} \\ \mathbf{s} \end{bmatrix}_q < \sum_{\mathbf{s} \in \mathcal{N}, |\mathbf{s}| = \rho} \prod_{i=1}^{t} f(q) q^{s_i(n_i - s_i)} = f(q)^t \sum_{\mathbf{s} \in \mathcal{N}, |\mathbf{s}| = \rho} q^{\sum_{i=1}^t s_i(n_i - s_i)}$$

We go on studying the following quantity

$$p(s_1, \ldots, s_{t-1}) := \sum_{i=1}^{t-1} s_i(n_i - s_i) + \left(\rho - \sum_{i=1}^{t-1} s_i \right) \left(n_t - \rho + \sum_{i=1}^{t-1} s_i \right).$$

Notice that

$$\frac{\partial p}{\partial s_i}(s_1, \ldots, s_{t-1}) = n_i - n_t + 2\rho - 4s_i - 2 \sum_{j \ne i} s_j.$$

Since,

$$(t-1) \frac{\partial p}{\partial s_i}(s_1, \ldots, s_{t-1}) - \sum_{j \ne i} \frac{\partial p}{\partial s_j}(s_1, \ldots, s_{t-1}) = (t-1)n_i - \sum_{j \ne i} n_j + 2\rho - 2ts_i,$$

and since we easily see that the maximum of $p(s_1, \ldots, s_{t-1})$ is achieved for

$$s_i = \frac{1}{2t} \left((t-1)n_i - \sum_{j \ne i} n_j + 2\rho \right),$$

we get:

$$p(s_1, \ldots, s_{t-1}) = \frac{1}{4t} \cdot \sum_{1 \le i < j \le t} (n_j - n_i)^2 + \frac{\rho(|n| - \rho)}{t}.$$

Therefore, by (5), we get

$$\frac{q \cdot f(q)^t}{q-1} \cdot q^{\left(\frac{1}{4t} \cdot \sum_{1 \le i < j \le t} (n_j - n_i)^2 + \frac{\rho(|n| - \rho)}{t} \right)} > q^{m(k-\rho)}.$$

When $q > 2$, since $\frac{q \cdot f(q)^t}{q-1} \leq q^t$, we get

$$\frac{1}{4t} \cdot \sum_{1 \leq i < j \leq t} (n_j - n_i)^2 + \frac{\rho(|n| - \rho)}{t} + t \geq m(k - \rho),$$

while if $q = 2$ we obtain

$$\frac{1}{4t} \cdot \sum_{1 \leq i < j \leq t} (n_j - n_i)^2 + \frac{\rho(|n| - \rho)}{t} + 2t \geq m(k - \rho).$$

Remark 1. We have that $f(q) \longrightarrow 1$ as $q \longrightarrow \infty$, and so asymptotically $\frac{qf(q)^t}{q-1} \longrightarrow 1$ as $q \longrightarrow \infty$. For this reason, as q grows, we may replace (6) with

$$\frac{1}{4t} \cdot \sum_{1 \leq i < j \leq t} (n_j - n_i)^2 + \frac{\rho(|n| - \rho)}{t} \geq m(k - \rho),$$

for sufficiently large q. Indeed, even for relatively small values of q, $\frac{qf(q)^t}{q-1}$ takes values much smaller than q, for t not exceeding q. For example, for $q = 211, t = 20$, we have $\frac{qf(q)^t}{q-1} \approx 1.105407$; for $q = 111, t = 111$ we have $\frac{qf(q)^t}{q-1} \approx 2.780617$.

Remark 2. For $t = 1$ (the rank-metric case), the bound coincides asymptotically with the one obtained in [6] (while for small q, in [6] the rough estimate could be avoided). For $n_1 = \cdots = n_t = n$, we have:

$$N = tn \geq \frac{tm}{\rho}(k - \rho) + \rho - \frac{2t^2}{\rho}. \tag{7}$$

Lemma 1. *Fix t and N. Let $[n_1, \ldots, n_t]$ and $[n'_1, \ldots, n'_t]$ be such that $n_1 \geq \ldots \geq n_t$, $n'_1 \geq \ldots \geq n'_t$ and $N = n_1 + \ldots + n_t = n'_1 + \ldots + n'_t$. Then*

$$\sum_{1 \leq i < j \leq t} (n_j - n_i)^2 \leq \sum_{1 \leq i < j \leq t} (n'_j - n'_i)^2$$

if and only if $[n_1, \ldots, n_t] \preceq [n'_1, \ldots, n'_t]$ in the lexicographic ordering.

Proof. Let $[n_1, \ldots, n_t]$ be as stated in the hypothesis, define $n_0 = +\infty$ and $n_{t+1} = 0$ and take $1 \leq x < y \leq t$ such that $n_{x-1} \geq n_x + 1$, $n_y - 1 \geq n_{y+1}$. Let $n'_i = n_i$ for $i \notin \{x, y\}$, $n'_x = n_x + 1$ and $n'_y = n_y - 1$. Since

$$\sum_{1 \leq i \leq t, i \notin \{x,y\}} (n'_x - n'_i)^2 = \sum_{1 \leq i \leq t, i \notin \{x,y\}} (n_x - n_i + 1)^2 = \sum_{1 \leq i \leq t, i \notin \{x,y\}} ((n_x - n_i)^2 + 2(n_x - n_i) + 1),$$

$$\sum_{1 \leq i \leq t, i \notin \{x,y\}} (n'_y - n'_i)^2 = \sum_{1 \leq i \leq t, i \notin \{x,y\}} (n_y - n_i - 1)^2 = \sum_{1 \leq i \leq t, i \notin \{x,y\}} ((n_y - n_i)^2 - 2(n_y - n_i) + 1),$$

$$(n'_x - n'_y)^2 = (n_x - n_y + 2)^2 = (n_x - n_y)^2 + 4(n_x - n_y) + 4$$

and the rest of terms are equal, we get that

$$\sum_{1 \leq i < j \leq t} (n'_j - n'_i)^2 - \sum_{1 \leq i < j \leq t} (n_j - n_i)^2 = t(2(n_x - n_y) + 2) \geq 2t.$$

The general case follows iterating the process.

This means that, for fixed ρ, t, N, the left hand-side of (6) takes its minimum and maximum values for $n_1 = \ldots = n_t$ and for $n_2 = \ldots = n_t = 1$, respectively. This gives sense to the following definition.

Definition 11. *Let t be a positive integer. We define the* shortest length *to be*

$$s_{q^m/q}(k,\rho,t) := \min\left\{\sum_{i=1}^{t}\dim(\mathcal{U}_i) : \mathcal{U}_i \leq_{\mathbb{F}_q} \mathbb{F}_{q^m}^k, (\mathcal{U}_1,\dots,\mathcal{U}_t) \text{ is sum-rank-}\rho \text{ saturating}\right\},$$

i.e. it is the minimal sum of the $\mathbb{F}_q$-dimensions of the $\mathcal{U}_i$, $i \in \{1,\dots,t\}$, of a sum-rank ρ-saturating system $\mathcal{U} = (\mathcal{U}_1,\dots,\mathcal{U}_t)$ in $\mathbb{F}_{q^m}^k$.
We define the homogeneous shortest length *to be*

$$s^{\hom}_{q^m/q}(k,\rho,t) := \min\left\{tn : \mathcal{U}_i \leq_{\mathbb{F}_q} \mathbb{F}_{q^m}^k, \dim(\mathcal{U}_i) = n, (\mathcal{U}_1,\dots,\mathcal{U}_t) \text{ is sum-rank-}\rho \text{ saturating}\right\},$$

i.e. it is the minimal sum of the $\mathbb{F}_q$-dimensions of the $\mathcal{U}_i$, $i \in \{1,\dots,t\}$, of a sum-rank ρ-saturating system $\mathcal{U} = (\mathcal{U}_1,\dots,\mathcal{U}_t)$ in $\mathbb{F}_{q^m}^k$, with the additional hypothesis that each $\mathcal{U}_i$ has the same dimension n.

Let $\mathcal{U} = (\mathcal{U}_1,\dots,\mathcal{U}_t)$ be a sum-rank saturating system with generator matrix $G = [G_1|\cdots|G_t]$. Consider the system $\mathcal{U}' = (\mathcal{U}_1,\dots,\mathcal{U}_{t-2},\mathcal{U}'_{t-1})$, which has generator matrix $G = [G_1|\cdots|G_{t-2}|G'_{t-1}]$, where G'_{t-1} is a matrix whose columns are a union of $\mathbb{F}_q$-bases of $\mathcal{U}_{t-1}$ and $\mathcal{U}_t$. Since $\mathcal{U}_{t-1}+\mathcal{U}_t = \mathcal{U}'_{t-1}$ we have $\dim_{\mathbb{F}_q}(\mathcal{U}'_{t-1}) \leq \dim_{\mathbb{F}_q}(\mathcal{U}_{t-1}) + \dim_{\mathbb{F}_q}(\mathcal{U}_t)$, while $\rho(\mathcal{U}') \leq \rho(\mathcal{U})$.

For this reason we have the following proposition.

Proposition 1 (Monotonicity in t). *We have that $s_{q^m/q}(k,\rho,t) \leq s_{q^m/q}(k,\rho,t+1)$.*

Lemma 2. *Let $\mathcal{U} = (\mathcal{U}_1,\dots,\mathcal{U}_t)$ be a sum-rank ρ-saturating $[\mathbf{n},k]_{q^m/q}$ system. Suppose for some $i \in [t]$, $L_{\mathcal{U}_i}$ is not scattered. Let $\mathcal{U}'_i = \langle u_{i,1},\dots,u_{i,n_i-1}\rangle_{\mathbb{F}_q}$ for some $\mathbb{F}_q$-basis $\{u_{i,1},\dots,u_{i,n_i}\}$ of $\mathcal{U}_i$ such that $u_{i,n_i} \in \lambda\langle u_{i,1},\dots,u_{i,n_i-1}\rangle_{\mathbb{F}_q}$ for some $\lambda \in \mathbb{F}_{q^m}$. Then $\mathcal{U}' = (\mathcal{U}_1,\dots,\mathcal{U}'_i,\dots,\mathcal{U}_t)$ is a sum-rank-ρ'-saturating $[\mathbf{n}',k]_{q^m/q}$ system satisfying $\rho' \leq \rho+1$ and $\mathbf{n}' = (n_1,\dots,n_i-1,\dots,n_t)$.*

Proof. The statement follows as a direct consequence of [6, Lemma 4.5], which gives that if $L_{\mathcal{U}_i}$ is scattered then $\mathcal{U}'_i$ is an $[n_i-1,k_i]_{q^m/q}$ rank-ρ'_i-saturating system satisfying $\rho'_i \leq \rho_i+1$.

More generally, we have the following.

Lemma 3. *Let $\mathcal{U} = (\mathcal{U}_1,\dots,\mathcal{U}_t)$ be a sum-rank ρ-saturating $[\mathbf{n},k]_{q^m/q}$ system. Suppose that for each $i \in [t]$, $\mathcal{U}_i$ has an $\mathbb{F}_q$-basis $\{u_1^{(i)},\dots,u_{n_i}^{(i)}\}$ such that*

$$u_{n_t}^{(t)} = \lambda\sum_{i\in S}\sum_{\substack{j=1,\\ j\neq n_t}}^{n_i} a_j^{(i)}u_j^{(i)},$$

for some $a_j^{(i)} \in \mathbb{F}_q$ and $S \subseteq [t]$. Then $\mathcal{U}' = (\mathcal{U}_1,\dots,\mathcal{U}_{t-1},\mathcal{U}'_t)$ is a sum-rank-ρ'-saturating $[\mathbf{n}',k]_{q^m/q}$ system satisfying $\rho' \leq \rho+|S|$ and $\mathbf{n}' = (n_1,\dots,n_{t-1},n_t-1)$.

Proof. Let $v \in \mathbb{F}_{q^m}^k$. As in the proof of Theorem 2, there exist $\gamma_1,\dots\gamma_\rho \in \mathbb{F}_{q^m}$ such that

$$v = \sum_{\ell=1}^{t}\sum_{r=1}^{n_\ell} u_r^{(\ell)}\sum_{i=1}^{\rho}\gamma_i\tau_{i,r}^{(\ell)}.$$

for some $\tau_{i,r}^{(\ell)} \in \mathbb{F}_q$ with $\tau_{i,r}^{(\ell)}$ non-zero for at most one pair (ℓ, i) and at most ρ of the matrices $\tau^{(\ell)} := (\tau_{i,r}^{(\ell)}) \in \mathbb{F}_q^{\rho \times n_\ell}$ are non-zero. Define $a_r^{(\ell)} := 0$ for all $r \in [n_\ell]$ whenever $\ell \notin S$. This yields that

$$\begin{aligned}
v &= \sum_{\ell=1}^{t} \sum_{r=1}^{n_\ell} u_r^{(\ell)} \sum_{i=1}^{\rho} \gamma_i \tau_{i,r}^{(\ell)} \\
&= \sum_{\ell=1}^{t-1} \sum_{r=1}^{n_\ell} u_r^{(\ell)} \sum_{i=1}^{\rho} \gamma_i \tau_{i,r}^{(\ell)} + \sum_{r=1}^{n_t-1} u_r^{(t)} \sum_{i=1}^{\rho} \gamma_i \tau_{i,r}^{(t)} + \lambda \sum_{\ell=1}^{t} \sum_{\substack{r=1, \\ r \neq n_t}}^{n_\ell} a_r^{(\ell)} u_r^{(\ell)} \sum_{i=1}^{\rho} \gamma_i \tau_{i,n_t}^{(t)} \\
&= \sum_{\ell=1}^{t-1} \sum_{r=1}^{n_\ell} u_r^{(\ell)} \left(\sum_{i=1}^{\rho} \gamma_i \tau_{i,r}^{(\ell)} + a_r^{(\ell)} \lambda \sum_{i=1}^{\rho} \gamma_i \tau_{i,n_t}^{(t)} \right) + \sum_{r=1}^{n_t-1} u_r^{(t)} \left(\sum_{i=1}^{\rho} \gamma_i \tau_{i,r}^{(t)} + a_r^{(t)} \lambda \sum_{i=1}^{\rho} \gamma_i \tau_{i,n_t}^{(t)} \right) \\
&= \sum_{\ell=1}^{t-1} \sum_{r=1}^{n_\ell} u_r^{(\ell)} \left(\sum_{i=1}^{\rho} \gamma_i \tau_{i,r}^{(\ell)} + a_r^{(\ell)} \gamma_{\rho+1} \right) + \sum_{r=1}^{n_t-1} u_r^{(t)} \left(\sum_{i=1}^{\rho} \gamma_i \tau_{i,r}^{(t)} + a_r^{(t)} \gamma_{\rho+1} \right) \\
&= \sum_{\ell=1}^{t-1} \sum_{r=1}^{n_\ell} u_r^{(\ell)} \sum_{i=1}^{\rho+1} \gamma_i \tau_{i,r}^{(\ell)} + \sum_{r=1}^{n_t-1} u_r^{(t)} \sum_{i=1}^{\rho+1} \gamma_i \tau_{i,r}^{(t)}, \\
&= \sum_{\ell=1}^{t} \sum_{r=1}^{n'_\ell} u_r^{(\ell)} \sum_{i=1}^{\rho+1} \gamma_i \tau_{i,r}^{(\ell)}
\end{aligned}$$

where $n'_t = n_t - 1$, $n'_\ell = n_\ell$ if $\ell < t$; $\gamma_{\rho+1} = \lambda \sum_{i=1}^{\rho} \gamma_i \tau_{i,n_t}^{(t)}$, and $\tau_{\rho+1,r}^{(\ell)} = a_r^{(\ell)}$ for each $\ell \in [t]$. Then

$$v = \sum_{\ell=1}^{t} G^{(\ell)} \lambda_\ell^T = G \lambda^T,$$

where

$$G = [G^{(1)} | \cdots | G^{(t)}], G^{(\ell)} = [u_1^{(\ell)}, \ldots, u_{n'_\ell}^{(\ell)}],$$

$$(\lambda_1, \ldots, \lambda_t) = (\gamma_1, \ldots, \gamma_{\rho+1})[\tau^{(1)} | \cdots | \tau^{(t)}] = [\gamma\tau^{(1)} | \cdots | \gamma\tau^{(t)}],$$

and $\hat{\tau}^{(\ell)} = (\tau_{i,r}^{(\ell)})$ for each $\ell \in [t]$. Now consider the value of $\mathrm{w}_{\mathrm{srk}}(\lambda) = \sum_{\ell=1}^{t} \mathrm{rk}(\lambda_\ell) = \sum_{\ell=1}^{t} \mathrm{w}_{\mathrm{rk}}(\gamma\hat{\tau}^{(\ell)})$. Let $\ell \in S$. If $\tau^{(\ell)}$ is not the all-zero matrix, then $\hat{\tau}^{(\ell)}$ has at most 2 non-zero rows and so

$$\mathrm{w}_{\mathrm{rk}}(\lambda_\ell) = \mathrm{w}_{\mathrm{rk}}(\gamma\hat{\tau}^{(\ell)}) \leq 2.$$

Otherwise $\lambda_\ell = \gamma_{\rho+1}(\tau_{\rho+1,1}^{(\ell)}, \ldots, \tau_{\rho+1,n_\ell}^{(\ell)})$ has rank weight at most 1. If $\ell \notin S$ then $\mathrm{w}_{\mathrm{rk}}(\lambda_\ell) \leq 1$. It follows that $\mathrm{w}_{\mathrm{srk}}(\lambda) \leq \rho + |S|$.

In particular, Lemma 2 follows as a special case of Lemma 3.

We have the following observations on the monotonicity of $s_{q^m/q}(k, \rho, t)$. The proofs are similar to those of [6, Theorem 4.6].

Theorem 6 (Monotonicity in ρ). *Let $|\mathbf{n}| > k$. The following hold.*

1. $s_{q^m/q}(k, \rho, t) \leq s_{q^m/q}(k, \rho + 1, t)$.
2. $s_{q^m/q}(k, \rho, t) \leq s_{q^m/q}(k + 1, \rho, t) - 1$.
3. $s_{q^m/q}(k + 1, \rho + 1, t) \leq s_{q^m/q}(k, \rho + 1, t) + 1$.

Proof. Let $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_t)$ be an $[\mathbf{n}, k]_{q^m/q}$ sum-rank-ρ-saturating system such that $|\mathbf{n}| = s_{q^m/q}(k, \rho, t)$. Without loss of generality, we may assume that a generator matrix of a code associated to $\mathcal{U}$ has the form:

$$G = [G_1, \ldots, G_t] = \left[\begin{array}{cc|cc|c|cc} I_{k_1} & A_{1,1} & 0 & A_{1,2} & \cdots & 0 & A_{1,t} \\ 0 & 0 & I_{k_2} & A_{2,2} & \cdots & 0 & A_{2,t} \\ 0 & 0 & 0 & 0 & \cdots & 0 & A_{3,t} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I_{k_t} & A_{t,t} \end{array}\right] \tag{8}$$

We may assume that $n_1 > k_1 \geq 1$, since otherwise we can permute the submatrices G_i and apply arbitrary elementary row operations to the block-permuted matrix and perform $\mathbb{F}_q$-linear column operations within each G_i to find a matrix of the required form. Over all such choices of $\mathcal{U}$ and G, let G be one such that the rightmost column of $A_{1,1}$, is a vector $y \in \mathbb{F}_{q^m}^{k_1}$ that has minimal rank weight. If $\mathrm{w_{rk}}(y) = 1$, then by Lemma 2, there exists an $[\mathbf{n}', k]_{q^m/q}$ sum-rank-ρ'-saturating system with $|\mathbf{n}'| = |\mathbf{n}| - 1$ and $\rho' \leq \rho + 1$; indeed we then have $\rho' = \rho + 1$ by the minimality of $|\mathbf{n}|$. In this case we have $s_{q^m/q}(k, \rho + 1, t) \leq s_{q^m/q}(k, \rho, t) - 1$.

If $\mathrm{w_{rk}}(y) = \ell \geq 2$, then $y = \sum_{i=1}^{\ell} \alpha_i y^{(i)}$ for an $\mathbb{F}_q$-basis $\{\alpha_1, \ldots, \alpha_\ell\} \subseteq \mathbb{F}_{q^m}$ of the $\mathbb{F}_q$-span of the coefficients of y and $y^{(i)} \in \mathbb{F}_q^{k_1}$. A straightforward computation verifies that the matrix $G^{(1)}$ found by replacing y in $A_{1,1}$ with $x^{(1)} := \sum_{i=1}^{\ell-1} \alpha_i y^{(i)}$ yields an $[\mathbf{n}, k]_{q^m/q}$ sum-rank-$\rho^{(1)}$-saturating system with $\rho^{(1)} \leq \rho + 1$. If $\rho^{(1)} = \rho + 1$ then the statement of the theorem holds, so suppose otherwise. If $\rho^{(1)} = \rho$ then we arrive at a contradiction by the minimality of $\mathrm{w_{rk}}(y) > \mathrm{w_{rk}}(x^{(1)})$, so suppose $\rho^{(1)} \leq \rho - 1$. The matrix $G^{(2)}$ found by replacing y with $x^{(2)} := \sum_{i=1}^{\ell-2} \alpha_i y^{(i)}$ yields an $[\mathbf{n}, k]_{q^m/q}$ sum-rank-$\rho^{(2)}$-saturating system with $\rho^{(2)} \leq \rho^{(1)} + 1 \leq \rho$. As before, by the minimality of $\mathrm{w_{rk}}(y) > \mathrm{w_{rk}}(x^{(2)})$, we have $\rho^{(2)} \neq \rho$ and so $\rho^{(2)} \leq \rho - 1$. Repeated applications of the above argument lead to a sequence of $[\mathbf{n}, k]_{q^m/q}$ sum-rank-$\rho^{(i)}$-saturating systems with $\rho^{(i)} \leq \rho - 1$ for each $i \in [\ell - 1]$. The final matrix $G^{(\ell-1)}$ in this sequence has rightmost column of $A_{1,1}$ equal to $x^{(\ell-1)} = \alpha_1 y^{(1)}$, so we may apply Lemma 2 to see that deleting this column from $G^{(\ell-1)}$ results in an $[\mathbf{n}', k]_{q^m/q}$ sum-rank-ρ-saturating system with $|\mathbf{n}'| = |\mathbf{n}| - 1$, giving a contradiction. We deduce that $\rho^{(1)} = \rho + 1$, in which we have $s_{q^m/q}(k, \rho + 1, t) \leq s_{q^m/q}(k, \rho, t)$. This proves 1.

The proofs that 2 and 3 hold are very similar to the rank-metric case and are omitted (see [6, Theorem 4.6]).

Definition 12. *For each $i \in \{1, 2\}$, let $\mathcal{U}^{(i)}$ be an $[\mathbf{n^{(i)}}, k_i]_{q^m/q}$ system, associated with an $[\mathbf{n^{(i)}}, k_i]_{q^m/q}$ sum-rank metric code $\mathcal{C}_i$. Let $f : \mathbb{F}_{q^m}^{\mathbf{n^{(1)}}} \longrightarrow \mathbb{F}_{q^m}^{\mathbf{n^{(2)}}}$ be an $\mathbb{F}_{q^m}$-linear map. The code*

$$\mathcal{C} := \{(u, f(u) + v) : u \in \mathcal{C}_1, v \in \mathcal{C}_2\}$$

is an $[(\mathbf{n^{(1)}}, \mathbf{n^{(2)}}), k_1 + k_2]_{q^m/q}$, which we call the f-sum of $\mathcal{C}_1$ and $\mathcal{C}_2$. Its associated $[(\mathbf{n^{(1)}}, \mathbf{n^{(2)}}), k_1 + k_2]_{q^m/q}$ system is called the f-sum of $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$, which we denote by $\mathcal{U}^{(1)} \oplus_f \mathcal{U}^{(2)}$. If f is the zero map, we write $\mathcal{U}^{(1)} \oplus \mathcal{U}^{(2)}$, and call it the direct sum; if f is the identity map, we we write $\mathcal{U}^{(1)} \oplus_\iota \mathcal{U}^{(2)}$ and call it the Plotkin-sum of $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$.

Proposition 2. *For each $i \in \{1, 2\}$, let $\mathbf{n}^{(i)} = (\mathbf{n}_1^{(i)}, \ldots, \mathbf{n}_{t_i}^{(i)})$, and let $\mathcal{U}^{(i)}$ be an $[\mathbf{n^{(i)}}, k_i]_{q^m/q}$ sum-rank-ρ_i-saturating system, associated with an $[n_i, k_i]_{q^m/q}$ code $\mathcal{C}_i$. Let $f : \mathbb{F}_{q^m}^{\mathbf{n^{(1)}}} \longrightarrow \mathbb{F}_{q^m}^{\mathbf{n^{(2)}}}$ be an $\mathbb{F}_{q^m}$-linear map. Then $\mathcal{U}^{(1)} \oplus_f \mathcal{U}^{(2)}$ is an $[(\mathbf{n^{(1)}}, \mathbf{n^{(2)}}), k_1 + k_2]_{q^m/q}$ system that is sum-rank-ρ-saturating, where $\rho \leq \rho_1 + \rho_2$. In particular, if $\rho_1 + \rho_2 \leq \min\{k_1 + k_2, m\}$, then*

$$s_{q^m/q}(k_1 + k_2, \rho_1 + \rho_2, t_1 + t_2) \leq s_{q^m/q}(k_1, \rho_1, t_1) + s_{q^m/q}(k_2, \rho_2, t_2).$$

Proof. Let $\mathcal{U}'$ be the $[\mathbf{n}^{(2)}, k_1]_{q^m/q}$ system associated with $f(\mathcal{U})$. Then $\mathcal{U}_1 + \mathcal{U}'$ is a sum-rank-ρ'-saturating $[(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}), k_1]_{q^m/q}$ system, satisfying $\rho' \leq \rho_1$. Therefore,

$$\mathbb{F}_{q^m}^{k_1+k_2} = \mathbb{S}_{\rho'}((\mathcal{U}_1 + \mathcal{U}') \oplus \mathbf{0}_{k_2}) \cup \mathbb{S}_{\rho_2}(\mathbf{0}_{k_1} \oplus \mathcal{U}_2) = \mathbb{S}_{\rho'+\rho_2}(\mathcal{U}_1 \oplus_f \mathcal{U}_2),$$

and so $\mathcal{U}_1 \oplus_f \mathcal{U}_2$ is an $[(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}), k_1 + k_2]_{q^m/q}$ system that is sum-rank-ρ-saturating for $\rho \leq \rho_1 + \rho_2$. The rest now follows by choosing each $\mathcal{U}_i$ to have length $s_{q^m/q}(k_i, \rho_i, t_i)$ and applying Theorem 6.

Theorem 7. *Let $\mathbb{F}_{q^m} = \mathbb{F}_q[\alpha]$, $r \geq 1$, $h \geq r$ and*

$$A_{h,r} := \left[\begin{array}{c|c|c|c|c} I_r & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & I_{h-r} & \alpha I_{h-r} & \cdots & \alpha^{m-1} I_{h-r} \end{array}\right]$$

Then

$$G_t := \underbrace{\left[\begin{array}{c|c|c|c} A_{h,r} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & A_{h,r} & \cdots & \mathbf{0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & \cdots & A_{h,r} \end{array}\right]}_{t \text{ times}}$$

generates an homogeneous sum-rank rt-saturating system. So

$$s^{\mathrm{hom}}_{q^m/q}(th, tr, t) \leq t(m(h-r)+r).$$

Proof. From [6, Theorem 4.4], we have that $A_{h,r}$ is the generator matrix of a code associated to an r-rank saturating $[m(h-r)+r, h]_{q^m/q}$ system $\mathcal{U}_{h,r}$. The matrix G_t is the generator matrix of a code associated to the direct sum of t copies of $\mathcal{U}_{h,r}$, which from Proposition 2 is a sum-rank ρ-saturating $[t(m(h-r)+r), th]_{q^m/q}$ system, with $\rho \leq tr$. It is not hard to see that $\rho = tr$. Let $v = (v^{(1)}, \ldots, v^{(t)}) \in \mathbb{F}_{q^m}^{th}$ such that each $v^{(i)} \in \mathbb{F}_{q^m}^{h}$ has its first r coefficients non-zero. Then any expression of v as an $\mathbb{F}_{q^m}$-linear combination of the columns of G_t requires the use of all its tr columns.

Remark 3. Since

$$t\left(\frac{m}{r}(h-r)+r\right) \leq s^{\mathrm{hom}}_{q^m/q}(th, tr, t) \leq t(m(h-r)+r),$$

we see immediately that when $r = 1$ the lower and the upper bounds coincide, so that

$$s^{\mathrm{hom}}_{q^m/q}(th, t, t) = t(m(h-1)+1).$$

4 Constructions of sum-rank saturating systems

In this final section we present some constructions of sum-rank-ρ saturating systems of small $\mathbb{F}_q$-dimension.

4.1 Sum-rank saturating systems from partitions of the projective space

We construct sum-rank saturating systems form partitions of the projective space. First observe that if $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_t)$ is such that $L_{\mathcal{U}_1} \cup \cdots \cup L_{\mathcal{U}_t} = \mathrm{PG}(k-1, q^m)$, than $\mathcal{U}$ is sum-rank 1-saturating.

Example 1. In [23, Theorem 4.28] we get that, if $(m, k) = 1$, there exists a partition of $\mathrm{PG}(k-1, q^m)$ into

$$t = \frac{(q^{mk}-1)(q-1)}{(q^m-1)(q^k-1)}$$

subgeometries $\mathrm{PG}(k-1, q)$. This gives us a sum-rank 1-saturating system of total length

$$k \cdot \frac{(q^{mk}-1)(q-1)}{(q^m-1)(q^k-1)}.$$

We mention another construction of a sum-rank ρ-saturating system based on a partition of $PG(k, q^m)$ into subgeometries.

Proposition 3. *Let $\mathcal{P} = \{\mathcal{P}_i\}_{i\in\{1,\ldots,t\}}$ a partition of $\mathrm{PG}(k-1, q^m)$ into subspaces. Let k_i be a positive integer such that $\mathcal{P}_i \simeq \mathrm{PG}(k_i-1, q^m)$. If $\mathcal{U}$ is such that each $\mathcal{U}_i$ is rank ρ-saturating in $\mathcal{P}_i$, then $\mathcal{U}$ is sum-rank ρ'-saturating with $\rho' \leq \rho$.*

Proof. Let $P \in \mathrm{PG}(k-1, q^m)$. Then $P \in \mathcal{P}_i$ for some i and $\mathcal{U}_i$ is a ρ-rank saturating system. In particular, P is in the span of at most ρ elements of $L_{\mathcal{U}_i}$. Therefore, $\mathcal{U}$ is sum-rank ρ'-saturating with $\rho' \leq \rho$.

A partition of the vector space $\mathbb{F}_{q^m}^k$ yields a partition of $\mathrm{PG}(k-1, q^m)$ into subspaces. In [9], some necessary conditions and constructions of partitions are presented. Thanks to Proposition 3, every such partition may be combined with other constructions.

4.2 Sum-rank $(k-1)$-saturating systems from cutting designs

In this section, we introduce the concept of sum-rank metric minimal codes and we examine their parameters. The geometry of minimal codes has been significant in constructing and establishing bounds in both the Hamming and rank metric, through the so-called *strong blocking sets*. These sets, first introduced in [16] in order to get small saturating sets, are collections of points in the projective space such that the intersection with every hyperplane spans the hyperplane. In [21], strong blocking sets are referred to as generator sets and are formed by unions of disjoint lines. They have recently garnered

renewed interest in coding theory, especially since [5], where they are called *cutting blocking sets* and are utilized to construct minimal codes. Quite surprisingly, they have been demonstrated to be the geometric counterparts of minimal codes [2, 35].

Minimal codes and their geometric counterparts may be introduced also in the context of the sum-rank metric.

Definition 13. *Let $\mathcal{C}$ be an $[\mathbf{n}, k]_{q^m/q}$ sum-rank metric code. A nonzero codeword $c \in \mathcal{C}$ is said* minimal *if for every $c' \in \mathcal{C}$ such that $\mathrm{supp}_{\mathbf{n}}(c') \subseteq \mathrm{supp}_{\mathbf{n}}(c)$ then $c' = \lambda c$ for some $\lambda \in \mathbb{F}_{q^m}$. We say that $\mathcal{C}$ is* minimal *if all of its nonzero codewords are minimal.*

Definition 14. *A system $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_t) \subset \mathbb{F}_{q^m}^k$ is* cutting *if $L_{\mathcal{U}_1} \cup \ldots \cup L_{\mathcal{U}_t}$ is a* strong blocking set *in $\mathrm{PG}(k-1, q^m)$, that is if*

$$\langle (L_{\mathcal{U}_1} \cup \ldots \cup L_{\mathcal{U}_t}) \cap \mathcal{H} \rangle_{\mathbb{F}_{q^m}} = \mathcal{H},$$

for every hyperplane $\mathcal{H}$ in $\mathrm{PG}(k-1, q^m)$.

The following is a generalization of the geometric characterization of minimal codes in the Hamming and in the rank metric.

Theorem 8 ([34, Corollary 10.25]). *A sum-rank metric code is minimal if and only if an associated system is cutting.*

As in the other metrics, also in the sum-rank one cutting systems give rise to saturating systems.

Theorem 9. *If $\mathcal{U}$ is a cutting system in $\mathbb{F}_{q^m}^k$, then $\mathcal{U}$ is a sum-rank $(k-1)$-saturating system in $\mathbb{F}_{q^{m(k-1)}}^k$.*

Proof. The system $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_t)$ is cutting in $\mathbb{F}_{q^m}^k$, so that the associated code $\mathcal{C}$ is minimal (by Theorem 9). Then the associated Hamming-metric code to $\mathcal{C}$ defined in [34, Definition 9.26.] is minimal, by [34, Corollary 9.27.]. Hence $L_{\mathcal{U}_1} \cup \ldots \cup L_{\mathcal{U}_t}$ is a strong blocking set in $\mathrm{PG}(k-1, q^m)$. Then $L_{\mathcal{U}_1} \cup \ldots \cup L_{\mathcal{U}_t}$ is a $(k-2)$-saturating set in $\mathrm{PG}(k-1, q^{m(k-1)})$ by [16]. By definition, this means that $\mathcal{U}$ is a sum-rank $(k-1)$-saturating system in $\mathbb{F}_{q^{m(k-1)}}^k$.

Example 2. In [7, Section 4], the authors provide bounds on the parameters and constructions of minimal sum-rank codes. These last have either one or two nonzero weights. Thanks to Theorem 9, these constructions provide more examples of saturating systems in the sum-rank metric.

The doubly extended linearized Reed-Solomon code (see [29]) with parameters

$$[((\underbrace{m, \ldots, m}_{q-1 \text{ times}}, 1, 1), 2]_{q^m/q}$$

and their geometric dual with parameters

$$[(\underbrace{m, \ldots, m}_{q-1 \text{ times}}, 2m-1, 2m-1), 2]_{q^m/q}$$

are both minimal sum-rank codes (see [7, Remark 4.6.]). The first ones have length $(q-1)m+2$, which meets the lower bound for the length of minimal sum-rank codes, for any m and the second ones have length $(q-1)m+4m-2$, which is minimal for $m=2$. Note the parameters of these codes do not meet our lower bound.

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