



**HAL**  
open science

## Dihedral Quantum Codes

Nadja Willenborg, Martino Borello, Anna-Lena Horlemann, Habibul Islam

► **To cite this version:**

Nadja Willenborg, Martino Borello, Anna-Lena Horlemann, Habibul Islam. Dihedral Quantum Codes. 2025. hal-04897228

**HAL Id: hal-04897228**

**<https://hal.science/hal-04897228v1>**

Preprint submitted on 20 Jan 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Dihedral Quantum Codes

Nadja Willenborg<sup>2</sup>, Martino Borello<sup>1,3</sup>, Anna-Lena Horlemann<sup>2</sup>, and Habibul Islam<sup>2</sup>

<sup>1</sup>Université Paris 8, Laboratoire de Géométrie, Analyse et Applications, LAGA,  
Université Sorbonne Paris Nord, CNRS, UMR 7539, France

<sup>2</sup>University of St.Gallen, Switzerland

<sup>3</sup>INRIA Saclay & LIX, CNRS UMR 7161, École Polytechnique, France

## Abstract

We establish dihedral quantum codes of short block length, a class of CSS codes obtained by the lifted product construction. We present the code construction and give a formula for the code dimension, depending on the two classical codes that the CSS code is based on. We also give a lower bound on the code distance and construct an example of short dihedral quantum codes.

## 1 Introduction

In recent years, significant progress has been made in the theory of quantum codes, particularly through the development of lifted product constructions [14, 15]. These methods have led to the discovery of families of quantum codes with both linearly growing distance and dimension. However, most existing constructions have focused on abelian group algebras, specifically the cyclic group algebras.

The aim of this article is to explore the construction of quantum codes using a nonabelian group algebra, specifically the dihedral group algebra  $\mathbb{F}_q[D_{2n}]$ . This approach is novel in the context of quantum information, as nonabelian group algebras have not been extensively used in quantum code constructions. Our focus on the dihedral group algebra allows us to introduce new methods and derive useful bounds on code parameters, opening a pathway for further research on nonabelian structures in quantum coding.

The construction of quantum codes can be transformed into a problem of constructing classical linear codes with certain self-orthogonal properties. From [3] it is known that asymptotically good binary dihedral codes exist (in the classical case). However, it is not possible to obtain asymptotically good dihedral quantum codes with the lifted product construction of [15], since finite solvable groups of bounded derived length never give a family of expanders [11].

In our work, we present a novel nonabelian quantum CSS (Calderbank-Shor-Steane) code construction using dihedral groups. Some of these codes are identified as moderate-density parity-check (MDPC) codes, which offer advantages for quantum error correction. In addition, their potential in the development of code-based public key encryption schemes with compact public keys is discussed, highlighting their relevance to future cryptographic protocols.

The paper is structured as follows: We start by introducing the necessary mathematical background. After defining lifted product codes, we start constructing dihedral lifted product codes from these in Section 3. In that section we also present the dimension formula and a distance bound for these codes. Afterwards, we use the dihedral group of order 180 and are able to compute a concrete code example. In Section 4 we summarize our results and their potential future applications.

## 2 Preliminaries

In this paper, we adhere to classical notations for matrix spaces. Specifically,  $\text{Mat}_{m \times n}(\mathcal{H})$  denotes the space of  $m \times n$ -matrices with elements in an algebra  $\mathcal{H}$  and we use  $\text{Mat}_m(\mathcal{H})$  when referring to square  $m \times m$ -matrices over  $\mathcal{H}$ . Furthermore, since we restrict to finite groups, our algebra will be denoted as  $\mathbb{F}_q[G]$ , i.e., the group algebra of  $G$  over the finite field  $\mathbb{F}_q$ .

### 2.1 Classical Codes

A classical linear code  $\mathcal{C}$  with parameters  $[n, k]_{\mathbb{F}_q}$  is a  $k$ -dimensional vector space in  $\mathbb{F}_q^n$ . The Hamming distance  $d(v, v')$  between  $v, v' \in \mathbb{F}_q^n$  is the number of positions, where  $v, v'$  differ. The parameter

$$d(\mathcal{C}) = \min\{d(v, v') : v \neq v', v, v' \in \mathcal{C}\}$$

is called the *minimum (Hamming) distance* of  $\mathcal{C}$ . A linear  $[n, k]_{\mathbb{F}_q}$  code  $\mathcal{C}$  with  $d(\mathcal{C}) = d$  is called an  $[n, k, d]_{\mathbb{F}_q}$  code. A linear  $[n, k]_{\mathbb{F}_q}$  code can be defined as the kernel of a matrix  $H \in \mathbb{F}_q^{(n-k) \times n}$  with  $\text{rk}H = n - k$ , called a *parity-check matrix* of the code. The rows of  $H$  are orthogonal to any vector in  $\mathcal{C}$ . The code defined by a parity-check matrix  $H$  is denoted by  $\mathcal{C}(H)$ .

### 2.2 Quantum CSS codes

We consider the complex Hilbert space  $\mathbb{C}^q$  of dimension  $q$  and its  $N$ -fold tensor product  $(\mathbb{C}^q)^{\otimes N}$ , also known as  *$N$ -qudit space*, where each component corresponds to one qudit. A quantum error-correcting code of length  $N$  and dimension  $K$  is a  $q^K$ -dimensional subspace of  $(\mathbb{C}^q)^{\otimes N}$ ; if it can correct up to  $\lfloor (D-1)/2 \rfloor$  quantum errors we denote it by  $[[N, K, D]]_q$ . Quantum *Calderbank-Shor-Steane* (CSS) codes form an important subclass of quantum error-correcting codes [5, 16]. A CSS code is defined by a pair of classical linear codes  $\mathcal{C}_X, \mathcal{C}_Z \subseteq \mathbb{F}_q^N$ , such that the CSS code is isomorphic to the following direct sum of two quotient spaces,

$$Q(\mathcal{C}_X, \mathcal{C}_Z) := \mathcal{C}_Z / \mathcal{C}_X^\perp \oplus \mathcal{C}_X / \mathcal{C}_Z^\perp. \quad (2.1)$$

The dimension of the CSS code corresponding to  $Q(\mathcal{C}_X, \mathcal{C}_Z)$  is  $K = \dim \mathcal{C}_X / \mathcal{C}_Z^\perp$ , which can be reformulated as follows

$$K = \dim \mathcal{C}_X - \dim \mathcal{C}_Z^\perp = \dim \ker H_Z - \text{rk}H_X = N - \text{rk}H_Z - \text{rk}H_X. \quad (2.2)$$

On the other hand, its minimum distance is given by  $D = \min\{D_X, D_Z\}$ , where

$$D_Z := \min_{c \in \mathcal{C}_Z \setminus \mathcal{C}_X^\perp} |c|, \quad D_X := \min_{c \in \mathcal{C}_X \setminus \mathcal{C}_Z^\perp} |c|.$$

Note that by considering two classical error-correcting codes in the CSS construction, this definition enables the code to handle the two primary types of quantum errors: bit-flip and phase-flip errors, whose capabilities are quantified by  $D_X$ , respectively  $D_Z$ .

To guarantee that the CSS code in (2.1) is well-defined, we need  $\mathcal{C}_X^\perp \subseteq \mathcal{C}_Z$  (or equivalently  $\mathcal{C}_Z^\perp \subseteq \mathcal{C}_X$ ). Let  $H_X$  be a parity-check matrix of  $\mathcal{C}_X$  and  $H_Z$  be a parity-check matrix of  $\mathcal{C}_Z$ , then we can express this via the following orthogonality condition

$$H_X H_Z^\top = 0. \quad (2.3)$$

Indeed, since the parity-check matrix  $H_X$  is the generator matrix of  $\mathcal{C}_X^\perp$  and  $H_Z$  is a parity-check matrix of  $\mathcal{C}_Z$ , we have that the row space of  $H_X$  is contained in  $\mathcal{C}_Z$ , i.e.,  $\mathcal{C}_X^\perp \subseteq \mathcal{C}_Z$ .

### 2.3 Group codes

Let  $G$  be a finite group with neutral element  $e$  and  $\mathbb{F}_q$  be a field. The group algebra  $\mathbb{F}_q[G]$  over  $\mathbb{F}_q$  is the set

$$\mathbb{F}_q[G] := \left\{ \sum_{g \in G} a_g g \mid a_g \in \mathbb{F}_q \right\},$$

with the following operations for  $a_g, b_g \in \mathbb{F}_q$ :

$$\begin{aligned} \sum_{g \in G} a_g g + \sum_{g \in G} b_g g &:= \sum_{g \in G} (a_g + b_g)g, \\ c \cdot \left( \sum_{g \in G} a_g g \right) &:= \sum_{g \in G} ca_g g, \\ \left( \sum_{g \in G} a_g g \right) \cdot \left( \sum_{g \in G} b_g g \right) &:= \sum_{g \in G} \left( \sum_{\mu\nu=g} a_\mu b_\nu \right) g. \end{aligned}$$

**Definition 2.1.** Let  $\sigma : G \times \mathbb{F}_q[G] \rightarrow \mathbb{F}_q[G]$  be a free action of a group  $G$  on  $\mathbb{F}_q[G]$ , i.e.,  $\forall g \in G \setminus \{e\} \forall x \in \mathbb{F}_q[G] : \sigma(g, x) \neq x$ . Let  $\mathcal{C}$  be a  $\mathbb{F}_q$ -submodule  $\mathcal{C} \subseteq \mathbb{F}_q[G]^\ell$  for some  $\ell \in \mathbb{N}$ . If  $\mathcal{C}$  is invariant under  $\sigma$ , i.e.,

$$\forall g \in G \forall c \in \mathcal{C} : \sigma(g, c) \in \mathcal{C},$$

where  $\sigma$  operates componentwise on  $\mathcal{C}$ , then  $\mathcal{C}$  is called a *generalized quasi-group code of index  $\ell$* . If  $\ell = 1$  we call  $\mathcal{C}$  a *generalized group code*.

**Remark 2.2.** Note that the invariance of the quasi-group code  $\mathcal{C}$  under the group action  $\sigma$  as defined implies invariance under specific types of group actions. If  $\sigma$  acts in a manner similar to a left-group action (i.e.,  $\sigma(g, x) = gx, \forall g \in G, \forall x \in \mathbb{F}_q[G]$ ), then invariance under  $\sigma$  implies invariance under the left-group action  $\lambda_g : x \mapsto gx$ . On the other hand, if  $\sigma$  behaves like the corresponding right-group action (i.e.,  $\sigma(g, x) = g^{-1}x$ ), then invariance under  $\sigma$  corresponds to invariance under the right-group action  $\rho_g : x \mapsto g^{-1}x$ . Consequently, every quasi-group code  $\mathcal{C}$  can be regarded either as a left module (i.e., invariant under  $\lambda_g$ ) or right module (i.e., invariant under  $\rho_g$ ), depending on the specific behaviour of  $\sigma$  and which group action— $\lambda_g$  or  $\rho_g$ —is more relevant for the context considered.

To represent group codes as codes over  $\mathbb{F}_q$  and use them in the CSS construction, we will use the following representations:

**Definition 2.3.** Let  $a \in \mathbb{F}_q[G]$ . The *right* (resp. *left*) *regular matrix representation* with respect to a fixed basis of  $\mathbb{F}_q^{|G|}$  is defined as the  $|G| \times |G|$ -matrix of the linear operator  $\rho_a : \mathbb{F}_q^{|G|} \rightarrow \mathbb{F}_q^{|G|}, x \mapsto xa$  (resp.  $\lambda_a : \mathbb{F}_q^{|G|} \rightarrow \mathbb{F}_q^{|G|}, x \mapsto ax$ ). We denote the right regular matrix representation of  $a$  by  $R(a)$  and its left regular matrix representation by  $L(a)$ . Clearly, when  $\mathbb{F}_q[G]$  is commutative we do not need to distinguish between left and right regular representations. In this case we simply denote the corresponding matrix over  $\mathbb{F}_q$  by  $\mathbf{a}$ , and for  $A \in \text{Mat}_m(\mathbb{F}_q[G])$  we denote its corresponding matrix over  $\mathbb{F}_q$  by  $\mathbf{A}$ .

The following will be useful in our code construction:

**Proposition 2.4.** For any  $a, b \in \mathbb{F}_q[G]$

$$L(a)^\top R(b) = R(b)L(a)^\top.$$

*Proof.* Since multiplication in  $\mathbb{F}_q[G]$  is associative we have for any  $x \in \mathbb{F}_q[G]$  that  $a(xb) = (ax)b$  holds. This shows for any  $x \in \mathbb{F}_q^{|G|}$  that applying  $L(a)^\top$  after  $R(b)$  on  $x$  gives the same result as first applying  $L(a)^\top$  on  $x$  and then  $R(b)$ .  $\square$

It is well-known (see for example [8, Chapter 16]) that if  $\text{char}(\mathbb{F}_q) \nmid |G|$ , all group codes over  $\mathbb{F}_q[G]$  are *principal*, i.e., that there exists  $c \in \mathbb{F}_q[G]$  with  $\mathcal{C} = \mathbb{F}_q[G]c$ .

## 2.4 Lifted product construction

The lifted product was introduced in [14] and formalizes many known constructions of quantum codes. The idea is to lift the elements in matrices over  $\mathbb{F}_q$  up to some ring  $R$  that is also a finite dimensional  $\mathbb{F}_q$ -algebra. To define these codes we need the Kronecker product over  $\mathbb{F}_q[G]$  and the conjugated transpose of a matrix  $H \in \text{Mat}_{m \times n}(\mathbb{F}_q[G])$ . We start by defining these concepts in the context of group algebras.

**Definition 2.5.** Let  $a \in \mathbb{F}_q[G]$  such that  $a = \sum_{g \in G} a_g g$ . Then its *reciprocal*  $a^*$  is defined as

$$a^* := \sum_{g \in G} a_g^{-1} g. \quad (2.4)$$

If  $H = (h_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$  is a matrix over  $\mathbb{F}_q[G]$  we define its *conjugated transpose* as  $H^* := (h_{j,i}^*)_{1 \leq j \leq n, 1 \leq i \leq m}$ , where  $h_{j,i}^*$  is the reciprocal of  $h_{i,j} \in \mathbb{F}_q[G]$ .

**Remark 2.6.** Note that the Hamming weight is invariant under taking the reciprocal. Moreover, relabeling each group element in (2.4) with its inverse gives for any  $a, b \in \mathbb{F}_q[G]$  :

$$(a + b)^* = \sum_{g \in G} (a_g + b_g) g^{-1} = a^* + b^*.$$

We also have the following relation:

**Lemma 2.7.**

$$R(a^*) = R(a)^\top \text{ and } L(a^*) = L(a)^\top.$$

*Proof.* In fact, since  $g_i g = g_j$  implies  $g = g_i^{-1} g_j$ , the coefficient of  $g_j$  in the product  $g_i a$  is  $a_{g_i^{-1} g_j}$ . On the other hand, the coefficient of  $g_i$  in  $g_j a^*$  is  $a_{(g_j^{-1} g_i)^{-1}}$ . Hence the two coefficients are equal, which shows that the  $(j, i)$ -th entry of  $L(a)$  is equal to the  $(i, j)$ -th entry of  $L(a^*)$ .  $\square$

The lifted product construction is based on the well-known Kronecker product:

**Definition 2.8.** Let  $A \in \text{Mat}_{m_A \times n_A}(\mathbb{F}_q[G])$  and  $B \in \text{Mat}_{m_B \times n_B}(\mathbb{F}_q[G])$ . Then the *Kronecker product* of  $A$  and  $B$  is the  $m_A m_B \times n_A n_B$  block matrix  $A \otimes B$  given by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

**Proposition 2.9.** Let  $A \in \text{Mat}_{m_A \times n_A}(\mathbb{F}_q[G])$ ,  $B \in \text{Mat}_{m_B \times n_B}(\mathbb{F}_q[G])$  and define

$$A^\natural := [L(a_{ij})^\top]_{1 \leq i \leq m_A, 1 \leq j \leq n_A} \in \text{Mat}_{m_A \times n_A}(\mathbb{F}_q^{|G| \times |G|}),$$

$$B^\natural := [R(b_{ij})]_{1 \leq i \leq m_B, 1 \leq j \leq n_B} \in \text{Mat}_{m_B \times n_B}(\mathbb{F}_q^{|G| \times |G|}).$$

Moreover let

$$H_X^\natural := [A^\natural \otimes I_{m_B}, -I_{m_A} \otimes B^\natural], \quad H_Z^\natural := [I_{n_A} \otimes B^{\natural \top}, A^{\natural \top} \otimes I_{n_B}]. \quad (2.5)$$

Then we have

$$H_X^\natural H_Z^{\natural \top} = 0.$$

*Proof.* We obtain

$$\begin{aligned}
H_X^{\natural} H_Z^{\natural \top} &= (A^{\natural} \otimes I_{m_B})(I_{n_A} \otimes B^{\natural}) + (-I_{m_A} \otimes B^{\natural})(A^{\natural} \otimes I_{n_B}) \\
&= \begin{pmatrix} a_{11}^{\natural} & \cdots & \mathbf{0} & & a_{1n_A}^{\natural} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & a_{11}^{\natural} & & \mathbf{0} & \cdots & a_{1n_A}^{\natural} \\ \hline & \vdots & & \ddots & & \vdots & \\ a_{m_A 1}^{\natural} & \cdots & \mathbf{0} & & a_{m_A n_A}^{\natural} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & a_{m_A 1}^{\natural} & & \mathbf{0} & \cdots & a_{m_A n_A}^{\natural} \end{pmatrix} \begin{pmatrix} b_{11}^{\natural} & \cdots & b_{1n_B}^{\natural} & & & & \\ \vdots & \ddots & \vdots & \cdots & & & \mathbf{0} \\ b_{m_B 1}^{\natural} & \cdots & b_{m_B n_B}^{\natural} & & & & \\ \hline & \vdots & & \ddots & & & \\ \mathbf{0} & & & \cdots & b_{11}^{\natural} & \cdots & b_{1n_B}^{\natural} \\ & & & & \vdots & \ddots & \vdots \\ & & & & b_{m_B 1}^{\natural} & \cdots & b_{m_B n_B}^{\natural} \end{pmatrix} \\
&- \begin{pmatrix} b_{11}^{\natural} & \cdots & b_{1n_B}^{\natural} & & & & \\ \vdots & \ddots & \vdots & \cdots & & & \mathbf{0} \\ b_{m_B 1}^{\natural} & \cdots & b_{m_B n_B}^{\natural} & & & & \\ \hline & \vdots & & \ddots & & & \\ \mathbf{0} & & & \cdots & b_{11}^{\natural} & \cdots & b_{1n_B}^{\natural} \\ & & & & \vdots & \ddots & \vdots \\ & & & & b_{m_B 1}^{\natural} & \cdots & b_{m_B n_B}^{\natural} \end{pmatrix} \begin{pmatrix} a_{11}^{\natural} & \cdots & \mathbf{0} & & a_{1n_A}^{\natural} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & a_{11}^{\natural} & & \mathbf{0} & \cdots & a_{1n_A}^{\natural} \\ \hline & \vdots & & \ddots & & \vdots & \\ a_{m_A 1}^{\natural} & \cdots & \mathbf{0} & & a_{m_A n_A}^{\natural} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & a_{m_A 1}^{\natural} & & \mathbf{0} & \cdots & a_{m_A n_A}^{\natural} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}^{\natural} b_{11}^{\natural} & \cdots & a_{11}^{\natural} b_{1n_B}^{\natural} & & a_{12}^{\natural} b_{11}^{\natural} & \cdots & a_{12}^{\natural} b_{1n_B}^{\natural} & & a_{1n_A}^{\natural} b_{11}^{\natural} & \cdots & a_{1n_A}^{\natural} b_{1n_B}^{\natural} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{11}^{\natural} b_{m_B 1}^{\natural} & \cdots & a_{11}^{\natural} b_{m_B n_B}^{\natural} & & a_{12}^{\natural} b_{m_B 1}^{\natural} & \cdots & a_{12}^{\natural} b_{m_B n_B}^{\natural} & & a_{1n_A}^{\natural} b_{m_B 1}^{\natural} & \cdots & a_{1n_A}^{\natural} b_{m_B n_B}^{\natural} \\ \hline a_{21}^{\natural} b_{11}^{\natural} & \cdots & a_{21}^{\natural} b_{1n_B}^{\natural} & & & & & & & & \\ \vdots & \ddots & \vdots & & & & & & & & \\ a_{21}^{\natural} b_{m_B 1}^{\natural} & \cdots & a_{21}^{\natural} b_{m_B n_B}^{\natural} & & & & & & & & \\ \hline & \vdots & & & & & & & & & \\ a_{m_A 1}^{\natural} b_{11}^{\natural} & \cdots & a_{m_A 1}^{\natural} b_{1n_B}^{\natural} & & & & & & a_{m_A n_A}^{\natural} b_{11}^{\natural} & \cdots & a_{m_A n_A}^{\natural} b_{1n_B}^{\natural} \\ \vdots & \ddots & \vdots & & & & & & \vdots & \ddots & \vdots \\ a_{m_A 1}^{\natural} b_{m_B 1}^{\natural} & \cdots & a_{m_A 1}^{\natural} b_{m_B n_B}^{\natural} & & & & & & a_{m_A n_A}^{\natural} b_{m_B 1}^{\natural} & \cdots & a_{m_A n_A}^{\natural} b_{m_B n_B}^{\natural} \end{pmatrix} \\
&- \begin{pmatrix} b_{11}^{\natural} a_{11}^{\natural} & \cdots & b_{1n_B}^{\natural} a_{11}^{\natural} & & b_{11}^{\natural} a_{12}^{\natural} & \cdots & b_{1n_B}^{\natural} a_{12}^{\natural} & & b_{11}^{\natural} a_{1n_A}^{\natural} & \cdots & b_{1n_B}^{\natural} a_{1n_A}^{\natural} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ b_{m_B 1}^{\natural} a_{11}^{\natural} & \cdots & b_{m_B n_B}^{\natural} a_{11}^{\natural} & & b_{m_B 1}^{\natural} a_{12}^{\natural} & \cdots & b_{m_B n_B}^{\natural} a_{12}^{\natural} & & b_{m_B 1}^{\natural} a_{1n_A}^{\natural} & \cdots & b_{m_B n_B}^{\natural} a_{1n_A}^{\natural} \\ \hline b_{11}^{\natural} a_{21}^{\natural} & \cdots & b_{1n_B}^{\natural} a_{21}^{\natural} & & & & & & & & \\ \vdots & \ddots & \vdots & & & & & & & & \\ b_{m_B 1}^{\natural} a_{21}^{\natural} & \cdots & b_{m_B n_B}^{\natural} a_{21}^{\natural} & & & & & & & & \\ \hline & \vdots & & & & & & & & & \\ b_{11}^{\natural} a_{m_A 1}^{\natural} & \cdots & b_{1n_B}^{\natural} a_{m_A 1}^{\natural} & & & & & & b_{11}^{\natural} a_{m_A n_A}^{\natural} & \cdots & b_{1n_B}^{\natural} a_{m_A n_A}^{\natural} \\ \vdots & \ddots & \vdots & & & & & & \vdots & \ddots & \vdots \\ b_{m_B 1}^{\natural} a_{m_A 1}^{\natural} & \cdots & b_{m_B n_B}^{\natural} a_{m_A 1}^{\natural} & & & & & & b_{m_B 1}^{\natural} a_{m_A n_A}^{\natural} & \cdots & b_{m_B n_B}^{\natural} a_{m_A n_A}^{\natural} \end{pmatrix} \\
&= 0,
\end{aligned}$$

where the last equality follows from Proposition 2.4, using that  $a_{ij}^{\natural} = L(a_{i,j})^{\top}$ ,  $b_{ij}^{\natural} = R(b_{i,j})$ .  $\square$

**Definition 2.10.** (Lifted Product Construction, see [14])

Let  $A \in \text{Mat}_{m_A \times n_A}(\mathbb{F}_q[G])$  and  $B \in \text{Mat}_{m_B \times n_B}(\mathbb{F}_q[G])$ , and define

$$H_X = [A \otimes I_{m_B}, -I_{m_A} \otimes B], \quad H_Z = [I_{n_A} \otimes B^*, A^* \otimes I_{n_B}]. \quad (2.6)$$

A  $LP(A, B)$  code, defined by its parity-check matrices  $H_X, H_Z$ , is a well-defined quantum CSS code of length  $N = |G|(n_A m_B + n_B m_A)$  over  $\mathbb{F}_q$  by identifying it with the matrices  $H_X^\natural, H_Z^\natural$  from Proposition 2.9.

One interesting fact about lifted product codes as defined above is that they are moderate-density parity-check (MDPC) codes<sup>1</sup> (i.e., codes who have a parity-check matrix whose rows have Hamming weights in  $O(\sqrt{N})$ , see e.g. [2]), if  $m_B$  and  $m_A$  are in the same order of magnitude as  $n_A$  and  $n_B$ , respectively. We will prove the statement for the case  $m_B = n_A, m_A = n_B$ , but the analog holds for  $m_B \in \mathcal{O}(n_A), m_A \in \mathcal{O}(n_B)$ .

**Proposition 2.11.** Let  $A \in \text{Mat}_{n_B \times n_A}(\mathbb{F}_q[G])$  and  $B \in \text{Mat}_{n_A \times n_B}(\mathbb{F}_q[G])$ . Then a  $LP(A, B)$  code with parity-check matrices  $H_X, H_Z$ , as defined in (2.6), is a moderate-density parity-check (MDPC) code.

*Proof.* Again we create the matrices

$$A^\natural \in \text{Mat}_{n_B \times n_A}(\mathbb{F}_q^{|G| \times |G|}), \quad B^\natural \in \text{Mat}_{n_A \times n_B}(\mathbb{F}_q^{|G| \times |G|}),$$

by replacing the elements  $a_{ij}$  of  $A$ , (respectively  $b_{ij}$  of  $B$ ) by  $L(a_{ij})^\top$ , (respectively  $R(b_{ij})$ ) and the analogs of  $H_X$  and  $H_Z$  in the following way

$$H_X^\natural = [A^\natural \otimes I_{n_A}, -I_{n_B} \otimes B^\natural], \quad H_Z^\natural = [I_{n_A} \otimes B^{\natural\top}, A^{\natural\top} \otimes I_{n_B}].$$

Let  $\omega_X, \omega_Z$  be the maximal row weights of  $H_X^\natural, H_Z^\natural$  and let  $N$  be the length of the  $LP(A, B)$  code, i.e.,  $N = (n_A^2 + n_B^2)|G|$ . To show that the  $LP(A, B)$  code is MDPC we use Definition 2.2 from [2] and show that  $\omega_X, \omega_Z \in O(\sqrt{N})$ , as  $N \rightarrow \infty$ . We easily see that the row weights of the parity-check matrices are upper bounded by  $(n_A + n_B)|G|$ . Since

$$\limsup_{n_\bullet \rightarrow \infty} \frac{|G|^2(n_A^2 + 2n_A n_B + n_B^2)}{|G|(n_A^2 + n_B^2)} < \infty \quad \text{for } \bullet \in \{A, B\},$$

we have  $\omega_X^2, \omega_Z^2 \in O(N)$  and the statement follows.  $\square$

### 3 Dihedral lifted product codes

We now present our main results: the parameters of dihedral lifted product codes. Throughout this section we assume  $\text{char}(\mathbb{F}_q) \nmid |G|$ . We let  $n \geq 3$  and briefly introduce the cyclic group and the dihedral group. The cyclic group of order  $n$ , containing  $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ , denoted by  $C_n$ , is defined as  $C_n := \langle \alpha \rangle$ , where  $\alpha^n = 1$  and  $\alpha^m \neq 1$  for  $m < n$ . The dihedral group of order  $2n$ , containing  $1, \alpha, \alpha^2, \dots, \alpha^{n-1}, \beta, \alpha\beta, \alpha^2\beta, \dots, \alpha^{n-1}\beta$ , denoted by  $D_{2n}$ , is defined as  $D_{2n} := \langle \alpha, \beta \rangle$ , where  $\alpha^n = \beta^2 = 1$  and  $\beta\alpha = \alpha^{n-1}\beta, \alpha^m \neq 1$ .

Let

$$x^n - 1 = \prod_{i=1}^r f_i \prod_{i=r+1}^{r+s} f_i^* f_i$$

be the factorization of  $x^n - 1 \in \mathbb{F}_q[x]$  into irreducible factors, where  $r$  is the number of self-

---

<sup>1</sup>These codes are particularly interesting in the area of code-based cryptography.

reciprocal factors, i.e.,  $f_i^* = f_i$  and  $2s$  the number of non-self-reciprocal factors. Let

$$\theta(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

and let  $\mathbb{F}_q \subseteq F_i$  be extension fields of  $\mathbb{F}_q$  such that  $[F_i : \mathbb{F}_q] = \deg f_i/2$  if  $\theta(n) + 1 \leq i \leq r$  and  $[F_i : \mathbb{F}_q] = \deg f_i$  in all other cases. In [18, 12] explicit decompositions of  $\mathbb{F}_q[D_{2n}]$  were obtained. For our construction we use a more generic framework and decompose  $\mathbb{F}_q[D_{2n}]$  as

$$\mathbb{F}_q[D_{2n}] \cong \bigoplus_{i=1}^{r+s} R_i, \quad (3.1)$$

where

$$R_i = \begin{cases} \mathbb{F}_q[C_2] & \text{if } 1 \leq i \leq \theta(n) \\ \text{Mat}_2(F_i) & \text{if } \theta(n) + 1 \leq i \leq r + s \end{cases}.$$

As presented in [18] an arbitrary code  $\mathcal{C} \subseteq \mathbb{F}_q[D_{2n}]$  can be decomposed as

$$\mathcal{C} \cong \bigoplus_{i=1}^{r+s} C_i, \quad C_i = \begin{cases} R_i & \text{if } i \in J_1 \\ I_i & \text{if } i \in J_2 \\ 0 & \text{if } i \notin J_1 \cup J_2 \end{cases},$$

where  $I_i \subsetneq R_i$  is a proper ideal and  $J_1, J_2 \subseteq [r + s]$  are two index sets which we call the *corresponding sets* of the code.

### 3.1 Dimension formula

We start with some preliminary results that facilitate the exploration of our main result of this section in Theorem 3.3. To derive this dimension formula we analyze the matrices  $A, B$  defined over  $\mathbb{F}_q$ , stemming from the decomposition outlined in Equation (3.1). Specifically we restrict to  $\text{Mat}_m(\mathbb{F}_q)$  to obtain good distance properties as illustrated in the subsequent sections and such that singular value decomposition (SVD) can be used. Note that  $\mathbb{F}_q$  can also be seen as the group algebra over the trivial group.

**Lemma 3.1.** Let  $A, B \in \text{Mat}_m(\mathbb{F}_q)$ ,  $k_A := \dim \ker A$ ,  $k_B := \dim \ker B$  and let  $H_X, H_Z$  be defined as in (2.6). Then we have

$$\text{rk} H_X = m^2 - k_A k_B, \quad \text{rk} H_Z = m^2 - k_A k_B.$$

*Proof.* Using singular value decomposition (SVD) we can express  $A \otimes I_m$  as

$$A \otimes I_m = U \Sigma V^*,$$

where  $U, V \in \text{Mat}_{m^2}(\mathbb{F}_q)$  are unitary matrices, i.e.  $U^*U = UU^* = I_{m^2}$ , and  $\Sigma = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_{m^2}})$ , where  $\lambda_1, \dots, \lambda_{m^2}$  are the eigenvalues of  $(A \otimes I_m)^\top (A \otimes I_m)$ . We can now construct idempotent matrices  $E_A \in \text{Mat}_{m^2}(\mathbb{F}_q)$ ,  $F_A \in \text{Mat}_{m^2}(\mathbb{F}_q)$  both of the same rank as  $A$ , such that

$$E_A^2 = E_A, F_A^2 = F_A, E_A(A \otimes I_m) = (A \otimes I_m)F_A = (A \otimes I_m),$$

by

$$E_A = U \Sigma_{m^2} U^*, \quad F_A = V \Sigma_{m^2} V^*,$$

where  $\Sigma_{m^2} = \text{diag}(1, \dots, 1, 0, \dots, 0) \in \text{Mat}_{m^2}(\mathbb{F}_q)$  has exactly  $\text{rk} A$  many ones along the diagonal. Since  $E_A E_A^{-1} E_A = E_A = E_A^3$ ,  $F_A F_A^{-1} F_A = F_A = F_A^3$ , i.e., any idempotent matrix is a generalized



inverse of itself, [13, Theorem 5] gives

$$\begin{aligned}\mathrm{rk}H_X &= \mathrm{rk}[A \otimes I_m, -I_m \otimes B] = \mathrm{rk}(A \otimes I_m) + \mathrm{rk}((I_m - E_A)(-I_m \otimes B)) \\ &= m \cdot \mathrm{rk}A + m \cdot \mathrm{rk}B - \mathrm{rk}(E_A(-I_m \otimes B)).\end{aligned}$$

The last equality follows from  $\mathrm{rk}(A \otimes B) = \mathrm{rk}A \cdot \mathrm{rk}B$  and the decomposition

$$\mathrm{rk}(-I_m \otimes B) = \mathrm{rk}E_A(-I_m \otimes B) + \mathrm{rk}(I_{m^2} - E_A)(-I_m \otimes B),$$

where  $E_A(I_m \otimes B)$  and  $(I_{m^2} - E_A)(-I_m \otimes B)$  represent projections of the columns of  $-I_m \otimes B$  onto two orthogonal subspaces. More precisely  $E_A(-I_m \otimes B)$  is the projection of  $-I_m \otimes B$  onto the column space of  $A \otimes I_m$  and  $(I_{m^2} - E_A)(-I_m \otimes B)$  is the projection of  $-I_m \otimes B$  onto the orthogonal complement of the column space of  $A \otimes I_m$ . We have

$$\mathrm{rk}(E_A(-I_m \otimes B)) = \mathrm{rk}(E_A(-I_m \otimes B)(A \otimes I_m)) = \mathrm{rk}((A \otimes I_m)(-I_m \otimes B))$$

and hence

$$\begin{aligned}\mathrm{rk}H_X &= m \cdot \mathrm{rk}B + m \cdot \mathrm{rk}A - \mathrm{rk}A \cdot \mathrm{rk}B \\ &= (m - \mathrm{rk}A)\mathrm{rk}B + (m - \mathrm{rk}B)\mathrm{rk}A + \mathrm{rk}A \cdot \mathrm{rk}B \\ &= k_A \mathrm{rk}B + k_B \mathrm{rk}A + \mathrm{rk}A \cdot \mathrm{rk}B \\ &= (k_A + \mathrm{rk}A)(k_B + \mathrm{rk}B) - k_A k_B \\ &= m^2 - k_A k_B.\end{aligned}$$

The equation for  $\mathrm{rk}H_Z$  follows similarly by using  $F_A$  instead of  $E_A$ . □

**Proposition 3.2.** Let  $A, B \in \mathrm{Mat}_m(\mathbb{F}_q)$  and  $k_A := \dim \ker A, k_B := \dim \ker B$ . Then

$$\dim LP(A, B) = 2k_A k_B.$$

*Proof.* From Lemma 3.1 we have  $\mathrm{rk}H_X = m^2 - k_A k_B$  and  $\mathrm{rk}H_Z = m^2 - k_A k_B$ . Hence using formula (2.2) for the quantum dimension we obtain

$$K = 2m^2 - \mathrm{rk}H_X - \mathrm{rk}H_Z = 2k_A k_B. \quad \square$$

**Theorem 3.3.** Let  $e_A, e_B \in \mathbb{F}_q[D_{2n}]$ , such that  $(e_A), (e_B)$  have corresponding sets  $J_i^A, J_i^B$ , for  $i = 1, 2$ . Let  $\mathbf{a} = (a_1 e_A, \dots, a_m e_A), \mathbf{b} = (b_1 e_B, \dots, b_m e_B)$  and  $a_1, \dots, a_m, b_1, \dots, b_m$  such that  $(a_i) + (e_A) = (1)$ , respectively  $(b_i) + (e_B) = (1)$ . Let  $A, B \in \mathrm{Mat}_m(\mathbb{F}_q[D_{2n}])$  such that all rows lie in  $\mathbf{a}\mathbb{F}_q[D_{2n}]$ , respectively  $\mathbf{b}\mathbb{F}_q[D_{2n}]$ . Then

$$\begin{aligned}\dim LP(A, B) &= \sum_{j=1}^r \deg f_j(\mathbf{1}_{I_1} \omega_1 + \mathbf{1}_{I_2} \omega_2 + \mathbf{1}_{I_3} \omega_3 + \mathbf{1}_{I_4} \omega_4 + \mathbf{1}_{I_5} \omega_5 + \mathbf{1}_{I_6} \omega_6) \\ &\quad + \sum_{j=r+1}^{r+s} 2 \deg f_j(\mathbf{1}_{I_1} \omega_1 + \mathbf{1}_{I_2} \omega_2 + \mathbf{1}_{I_3} \omega_3 + \mathbf{1}_{I_4} \omega_4 + \mathbf{1}_{I_5} \omega_5 + \mathbf{1}_{I_6} \omega_6),\end{aligned}$$

where

$$\begin{aligned}I_1 &:= J_2^A \cap J_2^B \\ I_2 &:= (J_2^A \cap J_1^B) \cup (J_2^B \cap J_1^A)\end{aligned}$$

$$\begin{aligned}
I_3 &:= J_1^A \cap J_1^B \\
I_4 &:= [r+s] \setminus (J_1^A \cup J_2^A) \cap [r+s] \setminus (J_1^B \cup J_2^B) \\
I_5 &:= \left( J_1^A \cap [r+s] \setminus (J_1^B \cup J_2^B) \right) \cup \left( J_1^B \cap [r+s] \setminus (J_1^A \cup J_2^A) \right) \\
I_6 &:= ([r+s] \setminus (J_1^A \cup J_2^A) \cap (J_2^B) \cup ([r+s] \setminus (J_1^B \cup J_2^B) \cap (J_2^A) \\
\omega_1 &:= (2m-1)^2 \\
\omega_2 &:= 2(m-1)(2m-1) \\
\omega_3 &:= 2^2(m-1)^2 \\
\omega_4 &:= 2^2m^2 \\
\omega_5 &:= 2^2m(m-1) \\
\omega_6 &:= 2m(2m-1).
\end{aligned}$$

*Proof.* Let

$$(e_A) = \bigoplus_{i=1}^{r+s} C_i^A, \quad (e_B) = \bigoplus_{i=1}^{r+s} C_i^B$$

be the decompositions of the codes  $(e_A), (e_B)$ . Using the decomposition of  $\mathbb{F}_q[D_{2n}]$  any matrix  $A \in \text{Mat}_m(\mathbb{F}_q[D_{2n}])$  can be uniquely represented by the collection of matrices  $(A_i)_{i \in [r+s]}$ , where  $A_i$  is the corresponding matrix over  $R_i$ , see [17] for more details. This gives

$$\begin{aligned}
\dim LP(A, B) &= \sum_{i=1}^{\theta(n)} \dim_{F_i} LP(A_i, B_i) + \sum_{i=\theta(n)+1}^r \dim_{F_i} LP(A_i, B_i) \cdot \frac{\deg f_i}{2} \\
&\quad + \sum_{i=r+1}^{r+s} \dim_{F_i} LP(A_i, B_i) \cdot \deg f_i. \tag{3.2}
\end{aligned}$$

We have  $\dim_{F_i} \mathcal{C}(A_i) = 2m - \text{rk}_{F_i} A_i$  and  $\dim_{F_i} \mathcal{C}(B_i) = 2m - \text{rk}_{F_i} B_i$ . Let  $\Theta \in \{A, B\}$  then

$$\text{rk}_{F_i} \Theta_i = \begin{cases} 0 & \text{if } i \in [r+s] \setminus J_1^\Theta \cup J_2^\Theta \\ 1 & \text{if } i \in J_2^\Theta \\ 2 & \text{if } i \in J_1^\Theta \end{cases}$$

and hence by Proposition 3.2

$$\dim_{F_i} LP(A_i, B_i) = \begin{cases} 2(2m-1)^2 & \text{if } i \in I_1 \\ 2^2(m-1)(2m-1) & \text{if } i \in I_2 \\ 2^3(m-1)^2 & \text{if } i \in I_3 \\ 2^3m^2 & \text{if } i \in I_4 \\ 2^3m(m-1) & \text{if } i \in I_5 \\ 2^2m(2m-1) & \text{if } i \in I_6. \end{cases} \tag{3.3}$$

Now the result follows by combining (3.2) and (3.3).  $\square$

### 3.2 Induced codes

Let  $\ell$  be an arbitrary divisor of  $n$  and  $t < \ell$ . Consider the two proper normal subgroups  $D_{2(n/\ell)} = \langle a^\ell, ba^t \rangle$  and  $C_\ell = \langle a^{n/\ell} \rangle$  of the dihedral group  $D_{2n}$ . Let  $I$  be a left ideal of  $\mathbb{F}_q[C_\ell]$ , then the code  $\mathcal{C} := (\mathbb{F}_q[D_{2n}])I$  is called  $C_\ell$ -induced. In [20] it was shown that if  $I$  is a  $[\ell, k, d]$  code, then  $\mathcal{C}$  is a  $[2n, |\Gamma|k, d]$  code, where  $\Gamma$  is the right transversal for  $C_\ell$  in  $D_{2n}$ . To distinguish

between the different decompositions and the corresponding auxiliary code constructions, when referring to the algebra  $\mathbb{F}_q[C_\ell]$ , we add  $\hat{\cdot}$  to the notation.

**Theorem 3.4.** (cf. [18, Theorem 6]) Let  $\hat{x}^\ell - 1 = (\prod_{i=1}^{\hat{r}} \hat{f}_i(\hat{x}))(\prod_{i=\hat{r}+1}^{\hat{r}+\hat{s}} \hat{f}_i(\hat{x})\hat{f}_i^*(\hat{x}))$  be the factorization of  $\hat{x}^\ell - 1$  into irreducible factors,  $\hat{g} \mid \hat{x}^\ell - 1$  and  $\hat{\mathcal{C}}_{\hat{g}} := (\hat{g})$ . Let  $\Omega : \mathbb{F}_q[C_\ell] \hookrightarrow \mathbb{F}_q[D_{2n}]$  be the embedding into  $\mathbb{F}_q[D_{2n}]$  and let  $\mathcal{C} = (\mathbb{F}_q[D_{2n}])\Omega(\hat{\mathcal{C}}_{\hat{g}})$  be the induced code. Then  $\mathcal{C}$  is a  $[2n, \frac{2kn}{\ell}, d]$  code, where  $k$  is the dimension of  $\hat{\mathcal{C}}_{\hat{g}}$  and  $d$  its minimum distance. Moreover,

$$\mathcal{C} \cong \bigoplus_{j=1}^{r+s} B_j, \quad B_j = \begin{cases} R_j & \text{if } j \in J_1 \\ I_j & \text{if } j \in J_2 \\ 0 & \text{if } j \notin J_1 \cup J_2 \end{cases},$$

where

$$J_1 = \{j \in [r+s] : (f_j(x) \mid \hat{g}(x^{n/\ell}) \wedge f_j^*(x) \mid \hat{g}(x^{n/\ell}))\},$$

$$J_2 = \{j \in [r+s] \setminus [\theta(n)] : \neg(f_j(x) \mid \hat{g}(x^{n/\ell}) \wedge f_j^*(x) \mid \hat{g}(x^{n/\ell}))\}.$$

**Theorem 3.5.** (cf. [18, Theorem 7]) Let  $\hat{x}^{n/\ell} - 1 = (\prod_{i=1}^{\hat{r}} \hat{f}_i(\hat{x}))(\prod_{i=\hat{r}+1}^{\hat{r}+\hat{s}} \hat{f}_i(\hat{x})\hat{f}_i^*(\hat{x}))$  be the factorization of  $\hat{x}^{n/\ell} - 1$  into irreducible factors, let  $\Omega : \mathbb{F}_q[D_{2(n/\ell)}] \hookrightarrow \mathbb{F}_q[D_{2n}]$  be the embedding into  $\mathbb{F}_q[D_{2n}]$  and let  $\hat{\mathcal{C}} \subseteq \mathbb{F}_q[D_{2(n/\ell)}]$  be a code such that

$$\hat{\mathcal{C}} \cong \bigoplus_{i=1}^{\hat{r}+\hat{s}} \hat{B}_i,$$

where, for  $1 \leq i \leq \hat{r}$ ,  $\hat{B}_i = 0$  or  $\hat{B}_i = \hat{A}_i$ . Let  $\mathcal{C} = (\mathbb{F}_q[D_{2n}])\Omega(\hat{\mathcal{C}})$  be the induced code, i.e., a  $[2n, \ell k, d]$  code, where  $k$  is the dimension of  $\hat{\mathcal{C}}_{\hat{g}}$  and  $d$  its minimum distance. Moreover, suppose that

$$\mathcal{C} \cong \bigoplus_{j=1}^{r+s} B_j.$$

Then, for all  $1 \leq i \leq \hat{r} + \hat{s}$ ,

$$B_j = \begin{cases} R_j & \text{if } f_j(x) \mid \hat{f}_i(x^\ell) \text{ and } \hat{B}_i = \hat{A}_i, \\ 0 & \text{if } f_j(x) \mid \hat{f}_i(x^\ell) \text{ and } \hat{B}_i = 0, \\ I_j & \text{else.} \end{cases}$$

### 3.3 Distance bound

We now determine the last missing parameter of our codes: the minimum distance. We will derive a lower bound on the minimum distance of the lifted product code, depending on the minimum distance of the codes related to the matrices used in the construction.

**Notation 3.6.** For  $c \in [\mathbb{F}_q[G]]^m$  we consider the block vector

$$\mathbf{c} := [\mathbf{c}_1, \dots, \mathbf{c}_m] \in \mathbb{F}_q^{|G|^m},$$

where  $\mathbf{c}_i \in \mathbb{F}_q^{|G|}$  contains the coefficients of  $c_i \in \mathbb{F}_q[G]$ .

We need the following lemma in the proof of our main result in Theorem 3.8.

**Lemma 3.7** (See [10], Lemma 16). Let  $G_A, G_B \subseteq D_{2n}$  be two proper normal subgroups such that  $[G_A : N] \cdot [G_B : N] \cdot |N| = 2n$  and  $N = G_A \cap G_B$  is abelian and normal in both  $G_A, G_B$ . Let  $A \in \text{Mat}_{m_A}(\mathbb{F}_q[N]), B \in \text{Mat}_{m_B}(\mathbb{F}_q[N])$  with  $\dim \mathcal{C}(A) = \dim \mathcal{C}(B) = 0$ . Then the quasi-abelian code  $LP(A, B)$  has zero dimension, i.e.

$$\dim LP(A, B) = 0.$$

The following theorem is a variant of [9, Theorem 5], respectively [10, Statement 12]. For completeness we include a proof for our case.

**Theorem 3.8.** Let  $G_A, G_B \subseteq D_{2n}$  be two proper normal subgroups such that  $[G_A : N] \cdot [G_B : N] \cdot |N| = 2n$  and  $N := G_A \cap G_B$  is abelian and normal in both  $G_A, G_B$ . Let  $A \in \text{Mat}_{m_A}(\mathbb{F}_q[G_A]), B \in \text{Mat}_{m_B}(\mathbb{F}_q[G_B])$  and define

$$d_0 := \min\{d(\mathcal{C}(A)), d(\mathcal{C}(B)), d(\mathcal{C}(A^\top)), d(\mathcal{C}(B^\top))\}.$$

Then the minimum distance of the lifted product code  $LP(A, B)$  satisfies

$$D \geq \left\lfloor \frac{d_0}{|N|} \right\rfloor.$$

*Proof.* Let  $\ell_A := [G_A : N], \ell_B := [G_B : N]$ . Moreover we replace the elements  $a_{i,j}, b_{i,j}$  of  $A, B$  by some square matrices  $a_{ij}^\# \in \text{Mat}_{\ell_A}(\mathbb{F}_q[N]), b_{ij}^\# \in \text{Mat}_{\ell_B}(\mathbb{F}_q[N])$ . More precisely we consider the left (respectively right) regular matrix representation with respect to a fixed basis of  $\mathbb{F}_q[N]$  and define  $A^\# := [L_N(a_{i,j}^\top)]_{1 \leq i,j \leq m_A}, B^\# := [R_N(b_{i,j}^\#)]_{1 \leq i,j \leq m_B}$ . Note that since the algebra  $\mathbb{F}_q[N]$  is commutative, we do not need to distinguish between the left and right representations of  $\mathbb{F}_q[N]$ . Thus we use the bold notation from Definition 2.3 and define

$$\mathbf{A} = \mathbf{A}^\# \otimes I_{\ell_B}, \quad \mathbf{B} = I_{\ell_A} \otimes \mathbf{B}^\#$$

and the parity-check matrices of  $LP(A, B)$  by

$$\mathbf{H}_X = [(\mathbf{A}^\# \otimes I_{\ell_B}) \otimes I_{m_B}, -I_{m_A} \otimes (I_{\ell_A} \otimes \mathbf{B}^\#)], \quad \mathbf{H}_Z = [I_{n_A} \otimes (I_{\ell_A} \otimes \mathbf{B}^\#), -(\mathbf{A}^\# \otimes I_{\ell_B}) \otimes I_{n_B}].$$

Let  $c \in \mathcal{C}(H_X)$  such that  $w_H(c) < \lfloor d_0/|N| \rfloor$ . We define reduced matrices

$$A^\#[I_A] := (a_{i,j}^\#)_{[m_A \ell_A] \times I_A}, \quad B^\#[I_B] := (b_{i,j}^\#)_{[m_B \ell_B] \times I_B},$$

where  $I_A \subseteq [m_A \ell_A], I_B \subseteq [m_B \ell_B]$  label the columns of  $A^\#, B^\#$  incident to nonzero elements of  $\mathbf{c}$  in  $\mathbf{H}_X \mathbf{c} = 0$ . Let  $\mathcal{I}_A, \mathcal{I}_B$  be the index sets of all columns in the corresponding  $\mathbf{A}^\#, \mathbf{B}^\#$  and let  $\mathcal{I} = \mathcal{I}_A \times [m_B \ell_B] \sqcup \mathcal{I}_B \times [m_A \ell_A]$  be the labeling of all such columns in  $\mathbf{H}_X$ . Each element of  $\mathbb{F}_q[N]$  corresponds to a block of size  $|N|$ . Thus

$$|\mathcal{I}_\mu| = |N| |I_\mu| \leq |N| w_H(c), \quad \mu \in \{A, B\}.$$

Hence  $\mathbf{A}^\#[I_A], \mathbf{B}^\#[I_B]$  have at most  $d_0 - 1$  columns which implies that all columns in the parity-check matrices are linearly independent. This gives

$$\dim \mathcal{C}(A^\#[I_A]) = \dim \mathcal{C}(B^\#[I_B]) = 0.$$

Considering  $LP(A^\#[I_A], B^\#[I_B])$  with

$$\begin{aligned} H_X[\mathcal{I}] &= [A^\#[I_A] \otimes I_{m_B}, -I_{m_A} \otimes B^\#[I_B]] \\ H_Z[\mathcal{I}] &= [I_{|I_A|} \otimes B^{\#*}[I_B], A^{\#*}[I_A] \otimes I_{|I_B|}]. \end{aligned}$$

Lemma 3.7 gives  $\dim LP(A^\sharp[I_A], B^\sharp[I_B]) = 0$ . But this implies  $\mathcal{C}(H_X[\mathcal{S}]) = \mathcal{C}(H_Z[\mathcal{S}])^\perp$ . Clearly, the reduced vector  $c[\mathcal{S}]$  belongs to  $\mathcal{C}(H_X[\mathcal{S}])$  by construction. Hence  $c[\mathcal{S}] \in \mathcal{C}(H_Z[\mathcal{S}])^\perp$ . Since  $c$  can be obtained from  $c[\mathcal{S}]$  by extending it with zeroes on the positions  $[2m_{AMB}] \setminus \mathcal{S}$  we have  $c \in \mathcal{C}(H_Z)^\perp$ . Similar arguments show that  $c \in \mathcal{C}(H_Z)$  with  $w_H(c) < \lfloor d_0/|N| \rfloor$  belongs to  $\mathcal{C}(H_X)^\perp$ .  $\square$

**Remark 3.9.** Note that with the theorem above, the best distance statements are achieved by taking two proper normal subgroups of  $D_{2n}$  in the construction, which have a trivial intersection. Therefore, particularly suitable are  $C_\ell, D_{2(n/\ell)}$ , where  $\ell|n$ . We illustrate the construction with these groups in the following example.

**Example 3.10.** We consider the dihedral group  $D_{180}$  and its two proper normal subgroups  $D_{20} = \langle a^9, ba^t \rangle$  ( $t < 9$ ) and  $C_9 = \langle a^{10} \rangle$ . In particular we consider the code  $\hat{\mathcal{C}}_A \subseteq \mathbb{F}_{11}[C_9]$  with generator

$$\hat{g}_A = (x^2 + x + 1)(x^6 + x^3 + 1).$$

and let  $e_A = \frac{(x^9-1)}{\hat{g}_A}$ . Moreover we define  $\mathbf{a} = (a_1 e_A, \dots, a_m e_A)$  with  $a_i \in \mathbb{F}_{11}[D_{180}]$  such that  $(b_i) + (e_B) = (1)$ . Using MAGMA we obtain that the minimum distance of  $\hat{\mathcal{C}}_A$  is 8 and

$$\hat{g}(x^{10}) = \prod_{i=1}^{20} \hat{g}_i,$$

where

$$\begin{aligned} \hat{g}_1 &= x^2 + x + 1, & \hat{g}_2 &= x^2 + 2x + 4, \\ \hat{g}_3 &= x^2 + 3x + 9, & \hat{g}_4 &= x^2 + 4x + 5, \\ \hat{g}_5 &= x^2 + 5x + 3, & \hat{g}_6 &= x^2 + 6x + 3, \\ \hat{g}_7 &= x^2 + 7x + 5, & \hat{g}_8 &= x^2 + 8x + 9, \\ \hat{g}_9 &= x^2 + 9x + 4, & \hat{g}_{10} &= x^2 + 10x + 1, \\ \hat{g}_{11} &= x^6 + x^3 + 1, & \hat{g}_{12} &= x^6 + 2x^3 + 4, \\ \hat{g}_{13} &= x^6 + 3x^3 + 9, & \hat{g}_{14} &= x^6 + 4x^3 + 5, \\ \hat{g}_{15} &= x^6 + 5x^3 + 3, & \hat{g}_{16} &= x^6 + 6x^3 + 3, \\ \hat{g}_{17} &= x^6 + 7x^3 + 5, & \hat{g}_{18} &= x^6 + 8x^3 + 9, \\ \hat{g}_{19} &= x^6 + 9x^3 + 4, & \hat{g}_{20} &= x^6 + 10x^3 + 1. \end{aligned}$$

For the dihedral code we use the  $\hat{\mathcal{C}} = [20, 8, 8]_{11}$  code as obtained in [17, Section 3]. More precisely, we consider  $\hat{\mathcal{C}} \subseteq \mathbb{F}_{11}[D_{20}]$  with decomposition

$$\hat{\rho}(\hat{\mathcal{C}}) = \bigoplus_{i=1}^6 \hat{B}_i,$$

where

$$\hat{B}_1 = \mathbb{F}_{11} \oplus \mathbb{F}_{11}, \hat{B}_2 = 0 \oplus 0, \hat{B}_3 = I_3(1, -1), \hat{B}_4 = M_2(\mathbb{F}_{11}[3]), \hat{B}_5 = 0, \hat{B}_6 = 0.$$

This code has a generator  $\hat{g}_B$ . The check element is the generator  $\hat{g}_B^\perp$  of the dual  $\hat{\mathcal{C}}^\perp$ . We let  $e_B = \hat{g}_B^\perp$  and define  $\mathbf{b} = (b_1 e_B, \dots, b_m e_B)$  with  $b_i \in \mathbb{F}_{11}[D_{180}]$  such that  $(b_i) + (e_B) = (1)$ . Moreover we consider  $A, B \in \text{Mat}_m(\mathbb{F}_{11}[D_{180}])$  such that all rows lie in  $\mathbf{a}\mathbb{F}_{11}[D_{180}]$ , respectively  $\mathbf{b}\mathbb{F}_{11}[D_{180}]$  and we consider the following factorization

$$(x^{90} - 1) = \prod_{i=1}^6 f_i \prod_{i=7}^{18} f_i^* f_i,$$

where

$$\begin{aligned}
f_1 &= x + 10, & f_2 &= x + 1, \\
f_3 &= x^2 + x + 1, & f_4 &= x^2 + 10x + 1, \\
f_5 &= x^6 + x^3 + 1, & f_6 &= x^6 + 10x^3 + 1, \\
f_7 &= x + 9, & f_7^* &= x + 5, \\
f_8 &= x + 7, & f_8^* &= x + 8, \\
f_9 &= x + 3, & f_9^* &= x + 4, \\
f_{10} &= x + 2, & f_{10}^* &= x + 6, \\
f_{11} &= x^2 + 2x + 4, & f_{11}^* &= x^2 + 6x + 3, \\
f_{12} &= x^2 + 3x + 9, & f_{12}^* &= x^2 + 4x + 5, \\
f_{13} &= x^2 + 7x + 5, & f_{13}^* &= x^2 + 8x + 9, \\
f_{14} &= x^2 + 5x + 3, & f_{14}^* &= x^2 + 9x + 4, \\
f_{15} &= x^6 + 2x^3 + 4, & f_{15}^* &= x^6 + 6x^3 + 3, \\
f_{16} &= x^6 + 3x^3 + 9, & f_{16}^* &= x^6 + 4x^3 + 5, \\
f_{17} &= x^6 + 5x^3 + 3, & f_{17}^* &= x^6 + 9x^3 + 4, \\
f_{18} &= x^6 + 7x^3 + 5, & f_{18}^* &= x^6 + 8x^3 + 9.
\end{aligned}$$

Note that  $r = 6$  and  $s = 12$ . Using the notation of Theorem 3.4 we obtain

$$J_1^A = \{1, 2, 7, 8, 9, 10\}, \quad J_2^A = \emptyset.$$

Note that the decomposition of  $x^{10} - 1$  is

$$x^{10} - 1 = \hat{f}_1 \hat{f}_2 [\hat{f}_3 \hat{f}_3^* \hat{f}_4 \hat{f}_4^* \hat{f}_5 \hat{f}_5^* \hat{f}_6 \hat{f}_6^*],$$

where

$$\begin{aligned}
\hat{f}_1 &= (x - 1), & \hat{f}_3 &= (x - 2), & \hat{f}_4 &= (x - 3), & \hat{f}_5 &= (x - 7), & \hat{f}_6 &= (x - 9), \\
\hat{f}_2 &= (x + 1), & \hat{f}_3^* &= (x - 6), & \hat{f}_4^* &= (x - 4), & \hat{f}_5^* &= (x - 8), & \hat{f}_6^* &= (x - 5).
\end{aligned}$$

We have

$$\begin{aligned}
\hat{f}_1(x^9) &= (x + 10)(x^2 + x + 1)(x^6 + x^3 + 1) \\
\hat{f}_3(x^9) &= (x + 5)(x^2 + 6x + 3)(x^6 + 7x^3 + 5) \\
\hat{f}_4(x^9) &= (x + 7)(x^2 + 4x + 5)(x^6 + 9x^3 + 4).
\end{aligned}$$

Applying Theorem 3.5 we have for the induced code  $\mathcal{C} = (\mathbb{F}_{11}[D_{2,90}])\Omega(\hat{\mathcal{C}})$  that

$$\rho(\mathcal{C}) = \bigoplus_{i=1}^{18} B_i,$$

where

$$B_i = \begin{cases} \mathbb{F}_{11} \oplus \mathbb{F}_{11} & \text{if } i \in \{1, 3, 5\}, \\ I_3(1, -1) & \text{if } i \in \{7, 11, 18\}, \\ \text{Mat}_2(\mathbb{F}_{11}[3]) & \text{if } i \in \{8, 12, 17\}, \\ 0 & \text{else.} \end{cases}$$

and hence

$$J_1^B = \{1, 3, 5, 8, 12, 17\}, \quad J_2^B = \{7, 11, 18\}.$$

Theorem 3.3 gives

$$\begin{aligned}
I_1 &= \emptyset, \\
I_2 &= \{7\}, \\
I_3 &= \{1, 8\}, \\
I_4 &= \{4, 6, 13, 14, 15, 16\}, \\
I_5 &= \{2, 3, 5, 9, 10, 12, 17\}, \\
I_6 &= \{11, 18\}.
\end{aligned}$$

We have

$$\deg f_i = \begin{cases} 1 & \text{if } i \in \{1, 2, 7, 8, 9, 10\}, \\ 2 & \text{if } i \in \{3, 4, 11, 12, 13, 14\}, \\ 6 & \text{if } i \in \{5, 6, 15, 16, 17, 18\} \end{cases}$$

and since  $r = 6$ ,  $s = 12$  and  $\theta(n) = 2$  we obtain from Theorem 3.3 that

$$\dim_{\mathbb{F}_{11}} LP(A, B) = 4(m-1)(2m-1) + 12(m-1)^2 + 160m^2 + 116m(m-1) + 32m(2m-1).$$

We show the possible parameter sets (for  $m = 1, \dots, 5$ ) of these codes in Table 1.

$m$	$[[N, K, D]]_{11}$ code	$K/N$
1	$[[360, 192, 8]]_{11}$	0.53
2	$[[1440, 1088, 8]]_{11}$	0.76
3	$[[3240, 2704, 8]]_{11}$	0.83
4	$[[5760, 5040, 8]]_{11}$	0.88
5	$[[9000, 8096, 8]]_{11}$	0.9

Table 1: Dihedral lifted product codes obtained from a  $[9, 1, 8]_{11}$ -cyclic code and a  $[20, 8, 8]_{11}$ -dihedral code. The third column describes the rate of the code.

## 4 Final remarks

In this paper we concentrated on nonabelian group code constructions. Although the existence and construction of good quantum CSS codes over various fields, including  $\mathbb{F}_{11}$ , have been extensively explored and demonstrated in [7], our codes offer distinct advantages due to their MDPC structure. One significant advantage of MDPC codes is their decodability via graph-based decoders, which are known for their efficiency in classical code-based cryptography. These characteristics potentially enhance decoding efficiency in quantum computing applications. In particular, one could develop and analyze a quantum McEliece public-key cryptosystem as proposed in [6] using our dihedral quantum codes.

Notably, among the 2-block codes proposed in [19], our codes appear to be the first non-abelian quantum MDPC codes, paving the way for future research, particularly in exploring their superiority in specific quantum error correction scenarios.

For future work it would be interesting to consider and analyze decoding algorithms for these dihedral codes; for example, via the generalized discrete Fourier transforms, or the Morita correspondence between  $\mathbb{F}_q[x]/(x^m - 1)$  submodules and left ideals in  $\text{Mat}_2(\mathbb{F}_q)[x]/(x^m - 1)$  as discussed in [1, 4].

## 5 Acknowledgements

We would like to thank Pavel Panteleev for helping us to refine Definition 2.1 in a more elegant way. We also thank Markus Grassl for providing us with the latest quantum code parameters and Anthony Leverrier for enlightening discussions on quantum codes.

## References

- [1] Morgan Barbier, Christophe Chabot, and Guillaume Quintin, *On quasi-cyclic codes as a generalization of cyclic codes*, Finite Fields and Their Applications **18** (2012), no. 5, 904–919.
- [2] Jessica Bariffi, Sam Mattheus, Alessandro Neri, and Joachim Rosenthal, *Moderate-density parity-check codes from projective bundles*, Designs, Codes and Cryptography **90** (2022), no. 12, 2943–2966.
- [3] Louay MJ Bazzi and Sanjoy K Mitter, *Some randomized code constructions from group actions*, IEEE Transactions on Information Theory **52** (2006), no. 7, 3210–3219.
- [4] Martino Borello and Abdelillah Jamous, *Dihedral codes with prescribed minimum distance*, Arithmetic of Finite Fields: 8th International Workshop, WAIFI 2020, Rennes, France, July 6–8, 2020, Revised Selected and Invited Papers 8, Springer, 2021, pp. 147–159.
- [5] A Robert Calderbank and Peter W Shor, *Good quantum error-correcting codes exist*, Physical Review A **54** (1996), no. 2, 1098.
- [6] Hachiro Fujita, *Quantum mceliece public-key cryptosystem*, Quantum information & computation **12** (2012), no. 3-4, 181–202.
- [7] Markus Grassl and Martin Rötteler, *Quantum mds codes over small fields*, 2015 IEEE International Symposium on Information Theory (ISIT), IEEE, 2015, pp. 1104–1108.
- [8] W Cary Huffman, Jon-Lark Kim, and Patrick Solé, *Concise encyclopedia of coding theory*, CRC Press, 2021.
- [9] Alexey A Kovalev and Leonid P Pryadko, *Quantum Kronecker sum-product low-density parity-check codes with finite rate*, Physical Review A **88** (2013), no. 1, 012311.
- [10] Hsiang-Ku Lin and Leonid P Pryadko, *Quantum two-block group algebra codes*, arXiv preprint arXiv:2306.16400 (2023).
- [11] Alexander Lubotzky and Benjamin Weiss, *Groups and expanders.*, Expanding graphs, Princeton, NJ, 1992, pp. 95–109.
- [12] FE Brochero Martínez, *Structure of finite dihedral group algebra*, Finite Fields and Their Applications **35** (2015), 204–214.
- [13] George Matsaglia and George PH Styan, *Equalities and inequalities for ranks of matrices*, Linear and multilinear Algebra **2** (1974), no. 3, 269–292.
- [14] Pavel Panteleev and Gleb Kalachev, *Quantum LDPC codes with almost linear minimum distance*, IEEE Transactions on Information Theory **68** (2021), no. 1, 213–229.
- [15] ———, *Asymptotically good quantum and locally testable classical LDPC codes*, Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing, 2022, pp. 375–388.



- [16] Andrew M Steane, *Simple quantum error-correcting codes*, Physical Review A **54** (1996), no. 6, 4741.
- [17] Kirill V Vedenev and Vladimir M Deundyak, *Codes in a dihedral group algebra*, Automatic Control and Computer Sciences **53** (2019), 745–754.
- [18] ———, *Relationship between codes and idempotents in a dihedral group algebra*, Mathematical Notes **107** (2020), no. 1-2, 201–216.
- [19] Renyu Wang, Hsiang-Ku Lin, and Leonid P Pryadko, *Abelian and non-abelian quantum two-block codes*, arXiv preprint arXiv:2305.06890 (2023).
- [20] Karl-Heinz Zimmermann, *Beiträge zur algebraischen codierungstheorie mittels modularer darstellungstheorie*, Lehrstuhl II für Mathematik, Universität Bayreuth, 1994.