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► **To cite this version:**

Valentina Astore, Martino Borello, Marco Calderini, Flavio Salizzoni. A geometric invariant of linear rank-metric codes. 2025. hal-04897043

HAL Id: hal-04897043

<https://hal.science/hal-04897043v1>

Preprint submitted on 20 Jan 2025

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A GEOMETRIC INVARIANT OF LINEAR RANK-METRIC CODES

VALENTINA ASTORE^{1,2}, MARTINO BORELLO^{3,1}, MARCO CALDERINI⁴, AND FLAVIO SALIZZONI⁵

ABSTRACT. Rank-metric codes have been a central topic in coding theory due to their theoretical and practical significance, with applications in network coding, distributed storage, crisscross error correction, and post-quantum cryptography. Recent research has focused on constructing new families of rank-metric codes with distinct algebraic structures, emphasizing the importance of invariants for distinguishing these codes from known families and from random ones. In this paper, we introduce a novel geometric invariant for linear rank-metric codes, inspired by the Schur product used in the Hamming metric. By examining the sequence of dimensions of Schur powers of the extended Hamming code associated with a linear code, we demonstrate its ability to differentiate Gabidulin codes from random ones. From a geometric perspective, this approach investigates the vanishing ideal of the linear set corresponding to the rank-metric code.

MSC Classification Codes: 11T71, 51E20, 94B27.

Key words: Rank-metric codes, Schur product, Overbeck’s distinguisher, generalized Gabidulin codes, Castelnuovo-Mumford regularity.

INTRODUCTION

Rank-metric codes have been a topical subject in coding theory since their introduction in 1978. While their initial development was motivated by theoretical reasons [12,15], the interest in these codes stems from practical applications such as network coding [40], distributed data storage [39], crisscross error correction [33], and code-based cryptography. Interestingly, the first code-based cryptosystem utilizing Gabidulin codes was proposed as early as 1991 [16]. However, it wasn’t until the recent push for post-quantum cryptography that Gabidulin codes gained significant prominence in this area. Although no cryptographic system based on rank-metric codes has made it past the second round of the NIST Post-Quantum Cryptography Standardization process, NIST believes that rank-based cryptography should continue to be researched, as rank-metric cryptosystems offer a nice alternative to traditional Hamming metric codes with comparable bandwidth [1].

Driven by these applications, in recent years there has been a growing interest in building new families of optimal codes with different algebraic structures, especially since it has been shown that most MRD codes are not Gabidulin codes when sufficiently large field extension degrees are considered [25]. A first explicit construction for different MRD codes, called *twisted Gabidulin codes*,

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V. A. is funded by AID (French *Agence de l’innovation de défense*).

M. B. is partially supported by the ANR-21-CE39-0009 - BARRACUDA (French *Agence Nationale de la Recherche*).

The research of M. C. was partially supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM).

F. S. is supported by the P500PT_222344 SNSF project and by the 2023-28 Germaine de Staël project.

was presented in [35]. Other constructions can be found, among the others, in [11,19,23,36]. For a more complete list of results we refer to [37].

A key step in constructing new codes is to prove that they are not equivalent to already known families. Although determining whether two linear rank-metric codes are equivalent or not can be done in polynomial time [8], there is much interest in constructing invariants that allow to complete this task in an easy way. Moreover, finding an invariant that discriminates the code used in a cryptographic system from a random code results in mining any security proof of the scheme, and potentially lead to an attack. However, the literature on this topic is currently still limited and only few useful invariants are available. For instance, in [19] the authors introduced an invariant for generalized Gabidulin codes, which is based on the dimension of the intersection of the code with itself under some field automorphism. This was investigated further in [17] and generalized in [26].

Even in the scenario of the Hamming metric, similar problems arise, and in this case, the Schur product plays a fundamental role. Inspired by [14], in [28] it was proven that the dimension of the Schur product of a linear code (in the Hamming metric) with itself allows us to differentiate between a generalized Reed Solomon code from a random one. This invariant was then used successfully in [9] to construct a key-recovery attack on a code-based cryptographic scheme.

In this paper, we introduce a geometric invariant for linear rank-metric codes which is inspired and similar to the dimension of the Schur products. In the case of rank-metric codes, just considering the Schur powers is not enough and, moreover, is not rank-equivalence-invariant. To overcome this problem, it is necessary to consider the extended Hamming code, introduced in [2], and examine its Schur products. In particular, we show that the sequence of the dimensions of the Schur powers of the extended code associated with a linear code distinguishes a Gabidulin code from a random one. From a geometric point of view, this corresponds to investigate the vanishing ideal of a linear set corresponding to the rank-metric code. In particular, the behavior of forms of a certain degree distinguishes (generalized) Gabidulin codes from random ones. In the last part of the paper we inquired which properties of the code can be determined from the dimension sequence. This leads also to an investigation of zeros of forms on linear sets.

Outline: The structure of this paper is as follows. Section 1 presents the preliminaries on rank-metric codes, including an overview of known invariants in the rank-metric context, an introduction to the Schur product in the Hamming metric, and the geometric interpretation of rank-metric codes. In Section 2, we define and analyse the dimension sequence of codes in the rank metric. In particular, we study the behavior of Gabidulin codes and compare it with the general case of random codes. Section 3 explores the geometric properties that support the results presented in the previous section. Specifically, we demonstrate how the findings from Section 2 relate to the evaluation of forms on linear sets. Finally, Section 4 is dedicated to study the zero locus of the forms introduced in Section 3.

1. BACKGROUND

Throughout this paper, let q be a prime power, \mathbb{F}_q be the finite field of order q , and m, n be two positive integers. A (linear) *rank-metric code* \mathcal{C} is an \mathbb{F}_{q^m} -linear subspace of $\mathbb{F}_{q^m}^n$ endowed with the rank metric (for which the distance between two vectors (v_1, \dots, v_n) and (w_1, \dots, w_n) is defined as the $\dim_{\mathbb{F}_q} \langle v_1 - w_1, \dots, v_n - w_n \rangle_{\mathbb{F}_q}$). If k is its dimension and d is its minimum rank distance, we say that \mathcal{C} is an $[n, k, d]_{q^m/q}$ code. A *generator matrix* G for \mathcal{C} is a matrix whose rows form a basis of \mathcal{C} . If the \mathbb{F}_q -dimension of the columns of G is equal to n , then we say that \mathcal{C} is *nondegenerate* (see [2, Definition 3.1. and Proposition 3.2.] for equivalent definitions). Two rank-metric codes \mathcal{C} and \mathcal{C}' are *equivalent* if there exists an \mathbb{F}_{q^m} -linear *isometry* (that is a map $\varphi_{\alpha, A} : v \mapsto \alpha v A$, with $\alpha \in \mathbb{F}_{q^m}^*$ and $A \in \text{GL}_n(\mathbb{F}_q)$) sending \mathcal{C} to \mathcal{C}' . We refer to [18] for a complete introduction on this topic.

1.1. Maximum rank distance codes. We introduce now the main objects of the paper.

Theorem 1.1 ([15, Section 2]). Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be an $[n, k, d]_{q^m/q}$ code. Then

$$k \leq n - d + 1.$$

Codes attaining the previous bound are called *maximum rank distance (MRD) codes*. Any linear MRD code $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ of dimension k has a generator matrix $G \in \mathbb{F}_{q^m}^{k \times n}$ in systematic form, i.e.,

$$G = [I_k \mid X],$$

where all entries of X are elements of $\mathbb{F}_{q^m} \setminus \mathbb{F}_q$ [19, Lemma 5.3]. It was shown in [25, Theorem 4.6] that when large field extension degrees are considered, then a randomly chosen systematic generator matrix defines an MRD code with high probability.

In this paper, we focus on a specific family of MRD codes, originally introduced independently by Delsarte in [12] and by Gabidulin in [15], and later generalized in [20].

Definition 1.2. Let $\alpha_1, \dots, \alpha_n \in \mathbb{F}_{q^m}$ be linearly independent elements over \mathbb{F}_q and $s \in \mathbb{N}$ such that $\gcd(s, m) = 1$. A *generalized Gabidulin code* $\mathcal{G}_{s,k}(\alpha_1, \dots, \alpha_n)$ with parameter s and of dimension k over \mathbb{F}_{q^m} is the rank-metric code whose generator matrix is

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^{[s]} & \alpha_2^{[s]} & \cdots & \alpha_n^{[s]} \\ \vdots & \vdots & & \vdots \\ \alpha_1^{[s(k-1)]} & \alpha_2^{[s(k-1)]} & \cdots & \alpha_n^{[s(k-1)]} \end{pmatrix},$$

where, for a nonnegative integer i , we write $[i]$ to mean q^i . In particular, when $s = 1$, we will simply say that $\mathcal{G}_k = \mathcal{G}_{1,k}(\alpha_1, \dots, \alpha_n)$ is a *Gabidulin code of dimension k* .

In the following, we will frequently compare structured MRD codes with random ones. Thus, we conclude this section by giving a formal definition of this notion. For us, a *random code* is a random variable that selects with uniform probability a matrix in systematic form, without zero columns.

1.2. Known invariants in the rank metric. Hereafter, for a positive integer s and a given matrix (or vector) X , we denote by $X^{[s]}$ the matrix (or vector) obtained by raising all entries of X to the power q^s .

The following results present useful characterizations of generalized Gabidulin codes.

Theorem 1.3 ([19, Theorem 4.8]). Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a linear MRD code of dimension $k < n$ and let $\mathcal{C}^{[s]} := \{c^{[s]} : c \in \mathcal{C}\}$ for some $s \in \{1, \dots, m-1\}$ with $\gcd(s, m) = 1$. Then, $\dim(\mathcal{C} \cap \mathcal{C}^{[s]}) = k - 1$ if and only if \mathcal{C} is a generalized Gabidulin code with parameter s .

Lemma 1.4 ([25, Lemma 3.3]). Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a linear MRD code of dimension k with generator matrix $[I_k \mid X]$, and $s \in \{1, \dots, m-1\}$ with $\gcd(s, m) = 1$. Then, \mathcal{C} is a generalized Gabidulin code with parameter s if and only if $\text{rk}(X^{[s]} - X) = 1$.

Following the notation introduced in [10], for a positive integer i , we will denote the i -th q -sum of a linear rank-metric code $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ by

$$\Lambda_i(\mathcal{C}) := \mathcal{C} + \mathcal{C}^{[1]} + \dots + \mathcal{C}^{[i]}.$$

As firstly introduced by Overbeck in his attack for the Gabidulin version of McEliece cryptosystem (GPT) [27], when the degree of the extension field m is large enough, then the first q -sum distinguishes a Gabidulin code from a random one.

Theorem 1.5 ([9, Proposition 3],[7, Proposition 1]). Let $\mathcal{G}_k, \mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ respectively be a Gabidulin and a random k -dimensional code. Then, for any positive integer $i < n - k$, it holds that

$$\dim(\Lambda_i(\mathcal{G}_k)) = k + i < k(i + 1) = \dim(\Lambda_i(\mathcal{C})),$$

where for any nonnegative integer a , $\Pr(\dim(\Lambda_i(\mathcal{C})) \leq k(i + 1) - a) = \mathcal{O}(q^{-ma})$.

Further investigations and generalizations of these results can be found, e.g., in [9,10,17,19,25].

1.3. Schur product in the Hamming metric. We denote by $*$ the standard component-wise product in $\mathbb{F}_{q^m}^n$, i.e., for $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n) \in \mathbb{F}_{q^m}^n$,

$$v * w = (v_1 w_1, \dots, v_n w_n).$$

Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{F}_{q^m}^n$ be two linear codes. The *Schur product* \star of \mathcal{C}_1 and \mathcal{C}_2 is defined as

$$\mathcal{C}_1 \star \mathcal{C}_2 = \langle c_1 * c_2 : c_1 \in \mathcal{C}_1, c_2 \in \mathcal{C}_2 \rangle_{\mathbb{F}_{q^m}}.$$

It is not hard to see that the set of linear codes contained in $\mathbb{F}_{q^m}^n$, together with the operations $+$ and \star , is a commutative semiring ordered under inclusion.

Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a linear code and let $\mathcal{C}^{(0)} = \langle (1, \dots, 1) \rangle_{\mathbb{F}_{q^m}}$. For $i \geq 1$, the i -th *Schur power* of \mathcal{C} is

$$\mathcal{C}^{(i)} = \mathcal{C} \star \mathcal{C}^{(i-1)} = \underbrace{\mathcal{C} \star \dots \star \mathcal{C}}_{i \text{ times}}.$$

The Schur product between linear codes has been largely studied, for instance in [6,9,24,30–32]. In particular, it is well-known that the dimension of the Schur square distinguishes algebraic structured linear codes, such as Reed-Solomon codes, from random ones (see [24, Corollary 27]).

In this context, we are particularly interested in the following definition.

Definition 1.6. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a linear code. The sequence of integers

$$\dim(\mathcal{C}^{(i)}) \text{ for } i \geq 0$$

is called the *dimension sequence*, or the *Hilbert sequence*, of \mathcal{C} . The *Castelnuovo-Mumford regularity* of \mathcal{C} is the smallest integer $r = r(\mathcal{C}) \geq 0$ such that, for every $t \geq r$,

$$\dim(\mathcal{C}^{(t)}) = \dim(\mathcal{C}^{(r)}).$$

Notice that the Castelnuovo-Mumford regularity is well-defined. It is easy to show that the dimension sequence of a nonzero linear code is nondecreasing, therefore it stabilizes after a finite number of steps [30, Proposition 11]. The terms *Hilbert sequence* and *Castelnuovo-Mumford regularity* are borrowed from commutative algebra, where analogous concepts are defined, see for instance [13]. The close connection between these objects is clarified in the following proposition.

Proposition 1.7 ([32, Proposition 1.28]). Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a linear code with generator matrix $G = [g_1 \mid g_2 \mid \dots \mid g_n]$, where each column g_i is nonzero for all $i \in \{1, \dots, n\}$. Let

$$\Pi_G = \{\langle g_1 \rangle_{q^m}, \dots, \langle g_n \rangle_{q^m}\} \subseteq \text{PG}(k - 1, q^m),$$

where $\langle g_i \rangle_{q^m}$ is the projective point associated to g_i . Then,

- (1) the dimension sequence of \mathcal{C} is equal to the Hilbert function of Π_G ,
- (2) the regularity $r(\mathcal{C})$ of \mathcal{C} is equal to the Castelnuovo-Mumford regularity of Π_G ,
- (3) $\dim(\mathcal{C}^{(t)}) = |\Pi_G|$, for all $t \geq r(\mathcal{C})$,

where we define the Hilbert function and the Castelnuovo-Mumford regularity of Π_G as those of its homogeneous coordinate ring.

Remark 1.8. The dimension sequence is well-defined. Indeed, the Hilbert function of an ideal is invariant under projective automorphisms. This implies that the Hilbert function of the homogeneous coordinate ring of Π_G does not depend on the choice of the generator matrix G . Furthermore, the dimension of $\mathcal{C}^{(i)}$ is invariant under monomial equivalence: if \mathcal{C}_1 and \mathcal{C}_2 are two linear codes with generator matrices G_1 and G_2 , where $G_1 = G_2M$ for a monomial matrix M , it is trivial to conclude that $\Pi_{G_1} = \Pi_{G_2}$.

1.4. The associated Hamming-metric code. In order to generalize the previous concept to rank-metric codes, we need to introduce the notion of Hamming-metric code associated to a rank-metric one, as well as the geometric interpretation of rank-metric codes.

Let \mathcal{C} be a nondegenerate $[n, k, d]_{q^m/q}$ rank-metric code. Let G be a generator matrix of \mathcal{C} and $g_1, \dots, g_n \in \mathbb{F}_{q^m}^k$ be the columns of G . The \mathbb{F}_q -vector space

$$\mathcal{U}_G = \langle g_1, \dots, g_n \rangle_{\mathbb{F}_q}$$

has \mathbb{F}_q -dimension n and $\langle \mathcal{U}_G \rangle_{\mathbb{F}_{q^m}} = \mathbb{F}_{q^m}^k$. Thus, \mathcal{U}_G is naturally called an $[n, k]_{q^m/q}$ -system associated to the code \mathcal{C} . Two $[n, k]_{q^m/q}$ -systems \mathcal{U} and \mathcal{U}' are said to be *equivalent* if there is an \mathbb{F}_{q^m} -isomorphism $\varphi : \mathbb{F}_{q^m}^k \rightarrow \mathbb{F}_{q^m}^k$ such that $\varphi(\mathcal{U}) = \mathcal{U}'$. Clearly, if G and G' are two generator matrices of the same code, \mathcal{U}_G and $\mathcal{U}_{G'}$ are equivalent. As a consequence, with a little abuse of notation, we may drop the index and simply talk about the $[n, k]_{q^m/q}$ -system \mathcal{U} associated to \mathcal{C} . It is also straightforward to see that equivalent codes are associated to equivalent systems (for more details, see the appendix of [2]).

Let now $\sim_{\mathbb{F}_q}$ be the proportionality relation over \mathbb{F}_q^k such that, for $u, v \in \mathbb{F}_q^k$, $u \sim_{\mathbb{F}_q} v$ if and only if $u = \lambda v$ for some $\lambda \in \mathbb{F}_q^*$.

Definition 1.9. Let \mathcal{U} be the $[n, k]_{q^m/q}$ -system associated to the nondegenerate $[n, k, d]_{q^m/q}$ rank-metric code \mathcal{C} . Let $G^H(\mathcal{U}) \in \mathbb{F}_{q^m}^{k \times N}$ be a matrix whose columns form a set of representatives for the set of equivalence classes $(\mathcal{U} \setminus \{0\}) / \sim_{\mathbb{F}_q}$. We will denote $G^H(\mathcal{U})$ as an *extended (generator) matrix* of \mathcal{C} . Moreover, let \mathcal{C}^H be the $[N, k]_{q^m}$ code generated by $G^H(\mathcal{U})$, where $N = (q^n - 1)/(q - 1)$. Then, we will denote \mathcal{C}^H as a *Hamming-metric code associated to \mathcal{C}* .

Proposition 1.10. Let \mathcal{C} be a nondegenerate $[n, k, d]_{q^m/q}$ rank-metric code. Then, its associated Hamming-metric code \mathcal{C}^H is unique up to columns permutation and right multiplication by a diagonal matrix with entries in \mathbb{F}_q^* .

Proof. Let \mathcal{U} be the $[n, k]_{q^m/q}$ -system associated to \mathcal{C} . Consider two matrices $G_1^H(\mathcal{U}), G_2^H(\mathcal{U}) \in \mathbb{F}_{q^m}^{k \times N}$ whose columns form two sets of representatives $\{g_{1,1}, \dots, g_{1,n}\}$ and $\{g_{2,1}, \dots, g_{2,n}\}$ for $(\mathcal{U} \setminus \{0\}) / \sim_{\mathbb{F}_q}$. By construction, these two sets are such that $g_{1,1} = \lambda_1 g_{2,i_1}, \dots, g_{1,n} = \lambda_n g_{2,i_n}$ for some indices permutation $\pi(1, \dots, n) = (i_1, \dots, i_n)$ and some $\lambda_1, \dots, \lambda_n \in \mathbb{F}_q^*$. Hence,

$$G_1^H(\mathcal{U}) = (g_{1,1} | \dots | g_{1,n}) = (g_{2,i_1} | \dots | g_{2,i_n}) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

In conclusion, the two Hamming-metric codes $\mathcal{C}_1^H = \text{rowsp}(G_1^H(\mathcal{U}))$ and $\mathcal{C}_2^H = \text{rowsp}(G_2^H(\mathcal{U}))$ associated to \mathcal{C} satisfy the desired uniqueness condition. \square

As a consequence, it makes sense again, with a little abuse of notation, to talk about the Hamming-metric code associated to \mathcal{C} . For all the details about this object, see [2, Section 4.2].

1.5. The geometry of rank-metric codes. The associated Hamming-metric code is closely related to some geometric objects called *linear sets*. These objects were introduced by Lunardon in [22] to construct blocking sets and have been the subject of intense research in recent years. A detailed treatment of linear sets can be found in [29].

Definition 1.11. Let \mathcal{U} be an $[n, k]_{q^m/q}$ -system. The \mathbb{F}_q -linear set of rank n associated with \mathcal{U} is

$$L_{\mathcal{U}} := \{ \langle u \rangle_{\mathbb{F}_q} : u \in \mathcal{U} \setminus \{0\} \} \subseteq \text{PG}(k-1, q^m).$$

Two linear sets are said to be *equivalent* if their systems are equivalent.

Remark 1.12. The original definition of a linear set does not require that $\langle \mathcal{U} \rangle_{\mathbb{F}_q} = \mathbb{F}_q^k$. If $\dim_{\mathbb{F}_q}(\langle \mathcal{U} \rangle_{\mathbb{F}_q}) = h < k$, we can assume, without loss of generality, that $\mathcal{U} \subseteq \mathbb{F}_q^h$. In this case, we can then study $L_{\mathcal{U}}$ within $\text{PG}(h-1, q^m)$.

Then, the *geometric object associated* to a rank-metric code \mathcal{C} with generator matrix G is $L_{\mathcal{U}_G}$. Notice that changing the generator matrix trivially results in equivalent linear sets. Moreover, equivalent rank-metric codes correspond to equivalent associated linear sets. The rank metric can be deduced from linear sets by examining their intersections with hyperplanes (taking into account weights of points, see for example [2]). In our context, the following remark is particularly important.

Remark 1.13. Let \mathcal{C} be a $[n, k]_{q^m/q}$ code, G be a generator matrix of \mathcal{C} and \mathcal{C}^H be the Hamming-metric code associated to \mathcal{C} , with generator matrix G^H . Then,

$$L_{\mathcal{U}_G} = \Pi_{G^H}.$$

In the next section we will explore some geometric invariants of these objects.

2. \mathbb{F}_q -DIMENSION SEQUENCE OF RANK-METRIC CODES

As we mentioned earlier, the dimension sequence is an important geometric invariant of Hamming-metric codes, which may be used to differentiate between algebraic structured codes and random ones. It might seem natural to define the dimension sequence similarly for rank-metric codes. However, this approach presents two relevant drawbacks:

- the dimension sequence often converges too quickly, so there is not enough “space” to discriminate between families of MRD codes and random ones;
- even though the dimension sequence is invariant under monomial equivalences, as observed in Remark 1.8, it is not invariant under rank-metric equivalences, as highlighted in the following example.

Example 2.1. Let $\mathcal{C}_1 = \langle (1, 0, 1), (1, 1, 0) \rangle_{\mathbb{F}_q}$ and $\mathcal{C}_2 = \langle (1, 0, 0), (0, 1, 0) \rangle_{\mathbb{F}_q}$ be two $[3, 2, 1]_{q^m/q}$ codes. It is easy to see that, although the two codes are equivalent, we have

$$\dim(\mathcal{C}_1^{(2)}) = 3 \quad \text{and} \quad \dim(\mathcal{C}_2^{(2)}) = 2.$$

To fix these problems, in this section we propose the following definition of \mathbb{F}_q -dimension sequence for rank-metric codes.

Definition 2.2. The \mathbb{F}_q -dimension sequence, or \mathbb{F}_q -Hilbert sequence, $\{h_i(\mathcal{C})\}_{i \geq 0}$ of a nondegenerate rank-metric code \mathcal{C} over \mathbb{F}_q is the Hilbert sequence of the associate Hamming-metric code \mathcal{C}^H , that is

$$h_i(\mathcal{C}) = \dim_{\mathbb{F}_q}(\mathcal{C}^{H(i)}).$$

Moreover, the \mathbb{F}_q -Castelnuovo-Mumford regularity $r(\mathcal{C})$ of \mathcal{C} is the Castelnuovo-Mumford regularity of \mathcal{C}^H .

Remark 2.3. Notice that the \mathbb{F}_q -dimension sequence and the \mathbb{F}_q -Castelnuovo-Mumford regularity do not depend on the choice of \mathcal{C}^H , hence they are well-defined. Consider two equivalent codes \mathcal{C}_1 and \mathcal{C}_2 and let G_1^H and G_2^H be two extended matrices of \mathcal{C}_1 and \mathcal{C}_2 respectively. We have already noticed in Proposition 1.10 that extended codes are unique up to monomial equivalences. Then, the columns of G_1^H must be equal to those of G_2^H up to permutation and right scalar multiplication. This implies that $\Pi_{G_1^H} = \Pi_{G_2^H}$, and therefore $h_i(\mathcal{C}_1) = h_i(\mathcal{C}_2)$, for all $i \geq 0$. Looking at Remark 1.13, we realize that we are merely considering the dimension sequence of the linear set $L_{\mathcal{U}_G}$ associated with the code \mathcal{C} via a generator matrix G .

Theorem 2.4. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a code of dimension k and generator matrix G . Let also $\mathcal{I}(L_{\mathcal{U}_G})$ be the vanishing ideal of $L_{\mathcal{U}_G}$ in $\mathbb{F}_{q^m}[x_1, \dots, x_k]$. Then, for every positive integer i ,

$$h_i(\mathcal{C}) = \binom{k+i-1}{i} - \dim_{\mathbb{F}_{q^m}}(\mathcal{I}(L_{\mathcal{U}_G})_i),$$

where $\mathcal{I}(L_{\mathcal{U}_G})_i$ is the set of all homogeneous polynomials in $\mathcal{I}(L_{\mathcal{U}_G})$ of degree i . In particular, for $i \in \{1, \dots, q\}$,

$$h_i(\mathcal{C}) = \binom{k+i-1}{i}.$$

Proof. By Proposition 1.7 and Remark 1.13, we know that the \mathbb{F}_q -dimension sequence $\{h_i(\mathcal{C})\}_{i \geq 0}$ is equal to the dimension sequence of $L_{\mathcal{U}_G}$. Hence,

$$h_i(\mathcal{C}) = \dim_{\mathbb{F}_{q^m}} \mathbb{F}_{q^m}[x_1, \dots, x_k]_i - \dim_{\mathbb{F}_{q^m}}(\mathcal{I}(L_{\mathcal{U}_G})_i),$$

where $\mathbb{F}_{q^m}[x_1, \dots, x_k]_i$ is the set of homogeneous polynomials of degree i , so that

$$h_i(\mathcal{C}) = \binom{k+i-1}{i} - \dim_{\mathbb{F}_{q^m}}(\mathcal{I}(L_{\mathcal{U}_G})_i).$$

As we remarked above, the dimension sequence does not depend on the choice of G . Hence, we can assume that $\text{PG}(k-1, q) \subseteq L_{\mathcal{U}_G}$ (it is sufficient to consider a generator matrix G in systematic form). This implies that

$$\mathcal{I}(L_{\mathcal{U}_G}) \subseteq \mathcal{I}(\text{PG}(k-1, q)) \subseteq \mathbb{F}_{q^m}[x_1, \dots, x_k].$$

Therefore, knowing the ideal $\mathcal{I}(\text{PG}(k-1, q))$ implies knowing what kind of polynomials we can expect to find in $\mathcal{I}(L_{\mathcal{U}_G})$. Let

$$\mathcal{F}_s := \langle x_i^{[s]} x_j - x_i x_j^{[s]} : 1 \leq i < j \leq k \rangle_{\mathbb{F}_{q^m}}.$$

By [4, Theorems 2.5 and 2.8], we know that the vanishing ideal over \mathbb{F}_{q^m} of $\text{PG}(k-1, q)$ is the ideal of $\mathbb{F}_{q^m}[x_1, \dots, x_k]$ generated by \mathcal{F}_1 . Hence, if we restrict ourselves to the case $i \in \{1, \dots, q\}$, we find that $\mathcal{I}(L_{\mathcal{U}_G})_i = \emptyset$, hence the second part of the thesis holds. \square

As a consequence, the first significant element in the \mathbb{F}_q -dimension sequence is the $(q+1)$ -th. As we will see later, this is also sufficient to discriminate between Gabidulin and random codes.

The next theorem provides a simple way to compute $h_{q+1}(\mathcal{C})$. We will postpone its proof to the next section, where we will study the link between Overbeck's invariant (Theorem 1.5), $h_{q+1}(\mathcal{C})$, and the \mathcal{F}_s spaces.

Theorem 2.5. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a code of dimension k with generator matrix $[I_k | X]$, where $X \in \mathbb{F}_{q^m}^{k \times (n-k)}$, and $r = \text{rk}(X^{[1]} - X)$. Then,

$$h_{q+1}(\mathcal{C}) = \binom{k+q}{q+1} - \binom{k-r}{2}.$$

Proof. See next section. \square

The previous theorem allows us to compute the exact value of $h_{q+1}(\mathcal{G}_k)$ for a Gabidulin code \mathcal{G}_k .

Corollary 2.6. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be an MRD code of dimension $k < n$. Then, \mathcal{C} is a Gabidulin code if and only if

$$h_{q+1}(\mathcal{C}) = \binom{k+q}{q+1} - \binom{k-1}{2}.$$

Proof. Without loss of generality, we can assume that \mathcal{C} has a generator matrix in systematic form, that is $[I_k | X]$. Since \mathcal{C} is MRD, by Lemma 1.4, we know that $\text{rk}(X^{[1]} - X) = 1$ if and only if \mathcal{C} is a Gabidulin code. Therefore, we immediately get the thesis via Theorem 2.5. \square

Now that the Gabidulin case has been fully studied up to the $(q+1)$ -th term of the \mathbb{F}_q -dimension sequence, it is natural to investigate the general behavior of a random code. To address this question, we rely on the analysis made in [25], where the following result is specifically presented in the same scenario that fits our framework.

Lemma 2.7 (Schwartz-Zippel, Corollary 1 in [34]). Let $p \in \mathbb{F}_q[x_1, \dots, x_k]$ be a nonzero polynomial of degree $d \geq 0$. Let \mathbb{F}_{q^m} be an extension field and $S \subseteq \mathbb{F}_{q^m}$ be a finite set. If s_1, \dots, s_k are selected independently and uniformly at random from S , then

$$\Pr(p(s_1, \dots, s_k) = 0) \leq \frac{d}{|S|}.$$

Hence, we obtain the following.

Lemma 2.8. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a random code of dimension k , with generator matrix $[I_k | X]$. Let also $r = \min\{k, n-k\}$ and $s \in \{1, \dots, m-1\}$ with $\gcd(s, m) = 1$. Then,

$$\Pr(\text{rk}(X^{[s]} - X) = r) \geq 1 - \frac{r}{q^{m-1}}.$$

Proof. Since \mathcal{C} is random, the entries of X have been chosen independently and uniformly at random from \mathbb{F}_{q^m} . When considering the matrix $X^{[s]} - X$ (instead of X), this translates into having selected independently and uniformly at random elements in $S = \{\alpha \in \mathbb{F}_{q^m} : \text{Tr}_{q^m/q}(\alpha) = 0\}$. It is well-known that $|S| = q^{m-1}$, see [21, Section 2.3] for the details. Let now $r = \min\{k, n-k\}$ and consider any $r \times r$ block B of $X^{[s]} - X$. Then,

$$\Pr(\text{rk}(X^{[s]} - X) = r) \geq \Pr(\det(B) \neq 0) = 1 - \Pr(\det(B) = 0).$$

If we regard the entries of B as the variables x_1, \dots, x_{r^2} , we can look at $\det(B)$ as a polynomial in $\mathbb{F}_q[x_1, \dots, x_{r^2}]$ of degree r . Then, by Schwartz-Zippel's lemma, we get that $\Pr(\det(B) = 0) \leq \frac{r}{|S|}$, which directly implies the thesis. \square

Theorem 2.9. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a random code of dimension k and $r = \min\{k, n-k\}$. Then,

$$\Pr\left(h_{q+1}(\mathcal{C}) = \binom{k+q}{q+1} - \binom{k-r}{2}\right) \geq 1 - \frac{r}{q^{m-1}}.$$

Proof. Given that $[I_k | X]$ is a generator matrix of \mathcal{C} , by Theorem 2.5 it follows that

$$\Pr(\text{rk}(X^{[1]} - X) = r) = \Pr\left(h_{q+1}(\mathcal{C}) = \binom{k+q}{q+1} - \binom{k-r}{2}\right),$$

where $\binom{k-r}{2}$ is meant to be equal to 0 when $r \geq k-1$. Hence, the thesis is a direct consequence of Lemma 2.8. \square

Remark 2.10. In this paper we focus on linear rank-metric codes. However, the dimension sequence can be also used to differentiate between a linear set and a random set of points. In particular, we expect the Castelnuovo-Mumford regularity of a random set of points to be strictly lower than that of a linear set.

In order to support our theoretical findings, we conducted a series of tests to study and compare the \mathbb{F}_q -dimension sequences of Gabidulin and random codes. In this analysis, we extended our study beyond the $(q + 1)$ -th Schur power of codes, exploring higher powers as well. These investigations were carried out using MAGMA [5], see [3] for more details about the implementation, and the results are presented in Figure 1. As can be observed, the experimental results document the above remarks regarding the first $(q + 1)$ -th terms of the \mathbb{F}_q -dimension sequences for both Gabidulin and random codes. Furthermore, the figure shows that the two \mathbb{F}_q -dimension sequences have different \mathbb{F}_q -Castelnuovo-Mumford regularities, with random codes having it smaller than the Gabidulin ones.

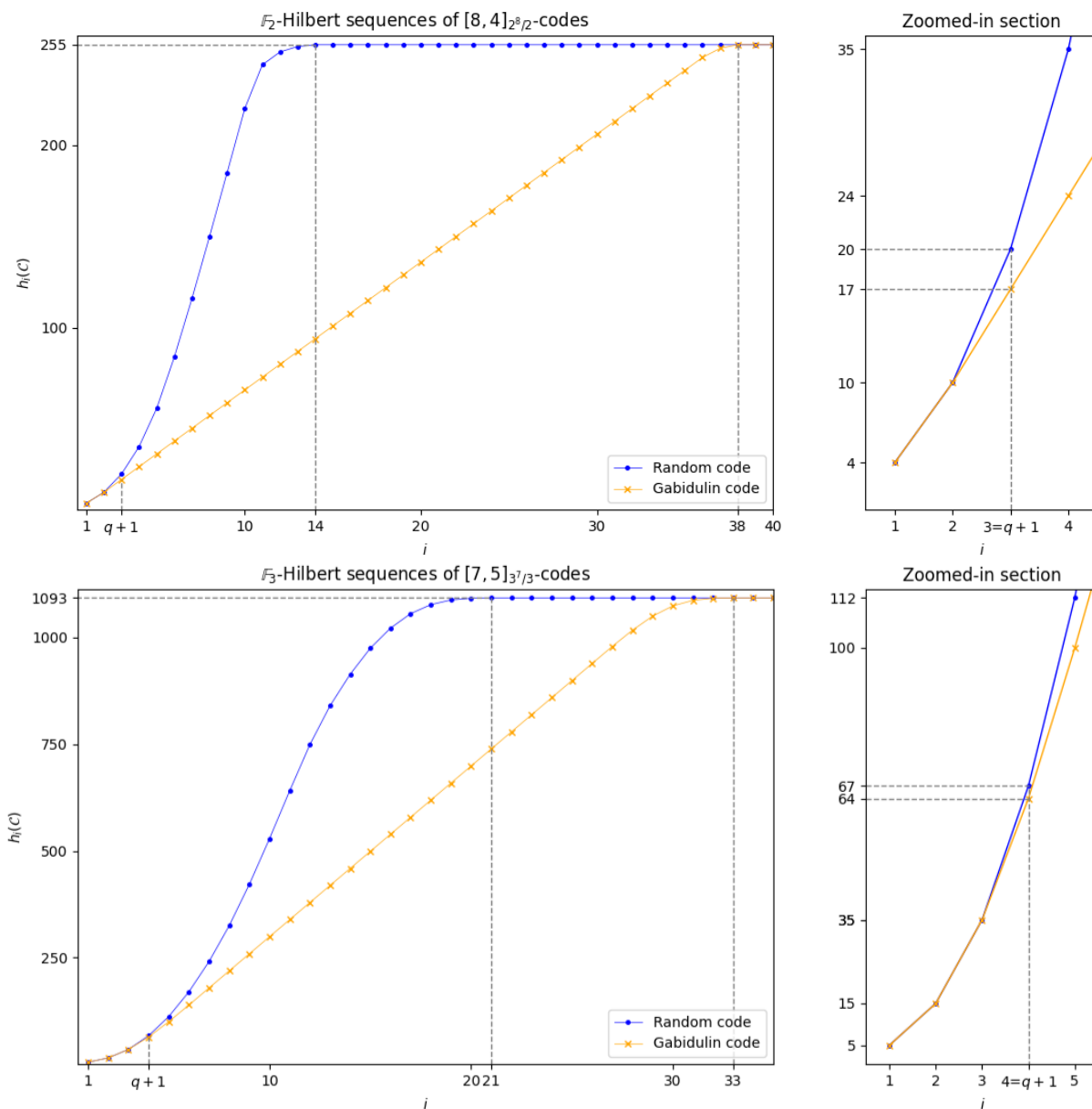


FIGURE 1. Experimental investigations

Remark 2.11. In this section, we have shown that the $(q + 1)$ -th term of the \mathbb{F}_q -dimension sequence characterizes Gabidulin codes. A natural question that follows is whether the \mathbb{F}_q -Castelnuovo-Mumford regularity could serve a similar purpose. So far, the Castelnuovo-Mumford regularity

of codes has been studied by Randriambololona in the context of Reed-Solomon codes [32]. The Gabidulin case is more complex, and obtaining the desired results in this context requires further investigations.

3. EVALUATION OF $(q^s + 1)$ -FORMS ON LINEAR SETS

As we have seen in Theorem 1.5, the dimension of the first q -sum of a given code allows to decide whether it is a Gabidulin or a random one. In particular, to compute the dimension of the first q -sum of a Gabidulin code \mathcal{G}_k with generator matrix $[I_k | X]$, it is sufficient to notice that

$$\dim(\mathcal{G}_k + \mathcal{G}_k^{[1]}) = k + \text{rk}(X^{[1]} - X),$$

and then apply Lemma 1.4. This gives $\dim(\mathcal{G}_k + \mathcal{G}_k^{[1]}) = k + 1$, whereas the dimension of the first q -sum is typically $2k$ in most other cases. A careful reader could have already noticed that the condition $\text{rk}(X^{[1]} - X) = 1$ appearing in Lemma 1.4 is the same condition that is stated in Theorem 2.5. In this section, we will show that $\text{rk}(X^{[s]} - X)$ is related to the dimension of the subspaces of \mathcal{F}_s (where we recall that $\mathcal{F}_s := \langle x_i^{[s]}x_j - x_ix_j^{[s]} : 1 \leq i < j \leq k \rangle_{\mathbb{F}_{q^m}}$) vanishing on the points of the linear set associated to the code generated by $[I_k | X]$. In the particular case of $s = 1$, this will result in a proof of Theorem 2.5.

We begin by studying the set of forms in \mathcal{F}_s vanishing on a coset of \mathbb{F}_q^k . In the following, we align to the convention that an empty sum is equal to zero.

Lemma 3.1. A $(q^s + 1)$ -form

$$p(x) := \sum_{1 \leq i < j \leq k} A_{i,j}(x_i^{[s]}x_j - x_ix_j^{[s]}) \in \mathcal{F}_s$$

vanishes on $\mathcal{V}_\alpha = \alpha + \mathbb{F}_q^k$ with $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{F}_{q^m}^k$ if and only if $p(\alpha) = 0$ and

$$\left\{ \sum_{i=1}^{t-1} A_{i,t}(\alpha_i^{[s]} - \alpha_i) - \sum_{j=t+1}^k A_{t,j}(\alpha_j^{[s]} - \alpha_j) = 0 \quad t \in \{1, \dots, k\} \right\}.$$

Proof. Consider a $(q^s + 1)$ -form

$$p(x) := \sum_{1 \leq i < j \leq k} A_{i,j}(x_i^{[s]}x_j - x_ix_j^{[s]}) \in \mathcal{F}_s$$

vanishing on $\mathcal{V}_\alpha := \alpha + \mathbb{F}_q^k$. For $a \in \mathbb{F}_q^k$, by evaluating $p(x)$ at $\alpha + a$, we obtain

$$0 = p(\alpha + a) = p(\alpha) + \sum_{1 \leq i < j \leq k} A_{i,j}((\alpha_i^{[s]} - \alpha_i)a_j - (\alpha_j^{[s]} - \alpha_j)a_i).$$

Since $p(\alpha) = 0$ by hypothesis, we have

$$\sum_{1 \leq i < j \leq k} \left(A_{i,j}(\alpha_i^{[s]} - \alpha_i)a_j - A_{i,j}(\alpha_j^{[s]} - \alpha_j)a_i \right) = 0,$$

implying that

$$\sum_{t=1}^k \left(\sum_{i=1}^{t-1} A_{i,t}(\alpha_i^{[s]} - \alpha_i) - \sum_{j=t+1}^k A_{t,j}(\alpha_j^{[s]} - \alpha_j) \right) a_t = 0.$$

Since this is true for every $a \in \mathbb{F}_q^k$ and $p(\alpha) = 0$, we get the following linear system

$$\begin{cases} \sum_{i=1}^{t-1} A_{i,t}(\alpha_i^{[s]} - \alpha_i) - \sum_{j=t+1}^k A_{t,j}(\alpha_j^{[s]} - \alpha_j) = 0 & t \in \{1, \dots, k\} \\ \sum_{1 \leq i < j \leq k} A_{i,j}(\alpha_i^{[s]} \alpha_j - \alpha_i \alpha_j^{[s]}) = 0 \end{cases}.$$

To conclude, it is sufficient to observe that the last equation is a linear combination of the first k ones. Indeed, one can easily check that

$$\sum_{1 \leq i < j \leq k} A_{i,j}(\alpha_i^{[s]} \alpha_j - \alpha_i \alpha_j^{[s]}) = \sum_{t=1}^k \left(\sum_{i=1}^{t-1} A_{i,t}(\alpha_i^{[s]} - \alpha_i) - \sum_{j=t+1}^k A_{t,j}(\alpha_j^{[s]} - \alpha_j) \right) \alpha_t.$$

To prove the converse it is sufficient to compute $p(\alpha + a)$ and to check that it is a linear combination of the equations appearing in the system. \square

The next lemma shows that if a form in \mathcal{F}_s vanishes at two cosets \mathcal{V}_{α_1} and \mathcal{V}_{α_2} , then it vanishes at the coset of any linear combination of α_1 and α_2 over \mathbb{F}_q . Equivalently, the set of cosets at which a form in \mathcal{F}_s vanishes is an \mathbb{F}_q -vector space.

Lemma 3.2. Let $\alpha_1 = (\alpha_{1,1}, \dots, \alpha_{1,k})$ and $\alpha_2 = (\alpha_{2,1}, \dots, \alpha_{2,k})$ be vectors in \mathbb{F}_q^k . A $(q^s + 1)$ -form $p(x) \in \mathcal{F}_s$ vanishes on \mathcal{V}_{α_1} and on \mathcal{V}_{α_2} if and only if it vanishes on $\mathcal{V}_{\lambda_1 \alpha_1 + \lambda_2 \alpha_2}$ for any $\lambda_1, \lambda_2 \in \mathbb{F}_q$. Equivalently, a form $p(x) := \sum_{1 \leq i < j \leq k} A_{i,j}(x_i^{[s]} x_j - x_i x_j^{[s]}) \in \mathcal{F}_s$ vanishes on $\mathcal{V}_{\lambda_1 \alpha_1 + \lambda_2 \alpha_2}$ for any $\lambda_1, \lambda_2 \in \mathbb{F}_q$, if and only if it satisfies the following system

$$\begin{cases} \sum_{i=1}^{t-1} A_{i,t}(\alpha_{1,i}^{[s]} - \alpha_{1,i}) - \sum_{j=t+1}^k A_{t,j}(\alpha_{1,j}^{[s]} - \alpha_{1,j}) = 0 & t \in \{1, \dots, k\} \\ \sum_{i=1}^{t-1} A_{i,t}(\alpha_{2,i}^{[s]} - \alpha_{2,i}) - \sum_{j=t+1}^k A_{t,j}(\alpha_{2,j}^{[s]} - \alpha_{2,j}) = 0 & t \in \{1, \dots, k\} \end{cases}.$$

Proof. First of all, we observe that if $p(x)$ vanishes on \mathcal{V}_{α} , then for any $\lambda \in \mathbb{F}_q$, it also vanishes on $\mathcal{V}_{\lambda \alpha}$. Indeed, let $\lambda \in \mathbb{F}_q^*$ ($\lambda = 0$ is obvious), since $p(x)$ is homogeneous we have $p(\lambda \alpha + a) = \lambda^2 p(\alpha + a/\lambda)$, for any $a \in \mathbb{F}_q^k$.

Assume that $p(x)$ vanishes on \mathcal{V}_{α_1} and \mathcal{V}_{α_2} . Then, by Lemma 3.1, we have

$$\begin{cases} \sum_{i=1}^{t-1} A_{i,t} \bar{\alpha}_{1,i} - \sum_{j=t+1}^k A_{t,j} \bar{\alpha}_{1,j} = 0 & t \in \{1, \dots, k\} \\ \sum_{i=1}^{t-1} A_{i,t} \bar{\alpha}_{2,i} - \sum_{j=t+1}^k A_{t,j} \bar{\alpha}_{2,j} = 0 & t \in \{1, \dots, k\} \end{cases},$$

where $\bar{y} := y^{[s]} - y$. Since $\overline{\lambda_1 \alpha_{1,i} + \lambda_2 \alpha_{2,i}} = \lambda_1 \bar{\alpha}_{1,i} + \lambda_2 \bar{\alpha}_{2,i}$ for any $\lambda_1, \lambda_2 \in \mathbb{F}_q$, by appropriately adding the previous equations, we obtain

$$\left\{ \sum_{i=1}^{t-1} A_{i,t} \overline{\lambda_1 \alpha_{1,i} + \lambda_2 \alpha_{2,i}} - \sum_{j=t+1}^k A_{t,j} \overline{\lambda_1 \alpha_{1,j} + \lambda_2 \alpha_{2,j}} = 0 \quad t \in \{1, \dots, k\} \right\}.$$

By Lemma 3.1, we conclude that if $p(x)$ vanishes on \mathcal{V}_{α_1} and \mathcal{V}_{α_2} , then it vanishes on $\mathcal{V}_{\lambda_1 \alpha_1 + \lambda_2 \alpha_2}$ for any $\lambda_1, \lambda_2 \in \mathbb{F}_q$. \square

We now proceed to prove the main result of this section.

Theorem 3.3. Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a code of dimension k with generator matrix $G = [I_k | X]$, where $X \in \mathbb{F}_q^{k \times (n-k)}$, and let $s \in \{1, \dots, m-1\}$ with $\gcd(s, m) = 1$. Let also $\mathcal{I}(L_{\mathcal{U}_G})$ be the vanishing ideal of the linear set $L_{\mathcal{U}_G}$ (which contains $\text{PG}(k-1, q)$). Then, for $r \in \{0, \dots, k\}$,

$$\text{rk}(X^{[s]} - X) = r \text{ if and only if } \dim \mathcal{F}_s \cap \mathcal{I}(L_{\mathcal{U}_G}) = \binom{k-r}{2},$$

where $\binom{k-r}{2} = 0$ for $r = k-1$ or $r = k$.

Proof. Suppose that $\text{rk}(X^{[s]} - X) = r$ and let $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,k})^T$ be the i -th column of X . Up to permuting the columns of X , we can assume that $\bar{\alpha}_1, \dots, \bar{\alpha}_r$, where $\bar{y} := y^{[s]} - y$, are linearly independent. By Lemma 3.2, a polynomial $p(x) = \sum_{1 \leq i < j \leq k} A_{i,j}(x_i^{[s]}x_j - x_i x_j^{[s]}) \in \mathcal{F}_s$ vanishes on $L_{\mathcal{U}_G}$ if and only if the following system is satisfied

$$(1) \quad \left\{ \sum_{i=1}^{t-1} A_{i,t} \bar{\alpha}_{\ell,i} - \sum_{j=t+1}^k A_{t,j} \bar{\alpha}_{\ell,j} = 0, \quad \ell \in \{1, \dots, r\}, t \in \{1, \dots, k\} \right\}.$$

The desired conclusion follows by determining the dimension of the solution space of the previous linear system. Since the solution space is invariant under row operations, up to a permutation of the indices, we can suppose that $\bar{\alpha}_1 = (\bar{\alpha}_{1,1}, \dots, \bar{\alpha}_{1,k})^T$, $\bar{\alpha}_2 = (0, \bar{\alpha}_{2,2}, \dots, \bar{\alpha}_{2,k})^T, \dots, \bar{\alpha}_r = (0, \dots, 0, \bar{\alpha}_{r,r}, \dots, \bar{\alpha}_{r,k})^T$, where $\bar{\alpha}_{i,i} \neq 0$ $i \in \{1, \dots, r\}$. In this way, we obtain r subsystems of (1) with associated matrices

$$M_1 = [M_{1,1} | M_{1,2} | \dots | M_{1,k-1}], \dots, M_r = [M_{r,1} | M_{r,2} | \dots | M_{r,k-1}],$$

where $M_{t,l}$ is a $k \times (k-l)$ matrix such that

$$M_{t,l} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 0 & -\bar{\alpha}_{t,t} & \dots & -\bar{\alpha}_{t,k} \\ 0 & & \ddots & & & \\ \vdots & & & \ddots & & \\ 0 & \dots & & & & 0 \end{pmatrix} \text{ or } M_{t,l} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ -\bar{\alpha}_{t,l+1} & \dots & -\bar{\alpha}_{t,k} \\ \bar{\alpha}_{t,l} & & \\ & \ddots & \\ & & \bar{\alpha}_{t,l} \end{pmatrix},$$

for $l < t$ and $l \geq t$ respectively.

Let us now consider the matrix

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_r \end{pmatrix}$$

associated to the whole system (1). The rank of M is at least $k-1 + k-2 + \dots + k-r$ since the pivot elements $\bar{\alpha}_{i,i}$ are different from zero in the last $k-i$ rows of the block matrices $M_{i,i}$. We now prove that the rank of M is at most $k-1 + k-2 + \dots + k-r$. First of all, note that in the matrix M_i the l -th row, for $l \in \{1, \dots, i\}$, depends on the last $k-i+1$ rows of M_l and on the last $k-l$ rows of M_i . In particular, the left kernel of M contains the vectors of the form

$$(0, \dots, 0 | \dots | \underbrace{0, \dots, 0, \bar{\alpha}_{i,i}, \dots, \bar{\alpha}_{i,k}}_{i-1} | 0, \dots, 0 | \underbrace{0, \dots, 0, \bar{\alpha}_{l,l}, \dots, \bar{\alpha}_{l,k}}_{l-1} | \dots | 0, \dots, 0) \in \mathbb{F}_q^{rk},$$

for $1 \leq l \leq i \leq r$, that is

$$\begin{aligned}
& (\bar{\alpha}_{1,1}, \dots, \bar{\alpha}_{1,k} | 0_k | \cdots | 0_k) \\
& (0, \bar{\alpha}_{2,2}, \dots, \bar{\alpha}_{2,k} | \bar{\alpha}_{1,1}, \dots, \bar{\alpha}_{1,k} | 0_k | \cdots | 0_k) \\
& \quad \vdots \\
& (0_{r-1}, \bar{\alpha}_{r,r}, \dots, \bar{\alpha}_{r,k} | 0_k | \cdots | \bar{\alpha}_{1,1}, \dots, \bar{\alpha}_{1,k}) \\
& (0_k | 0, \bar{\alpha}_{2,2}, \dots, \bar{\alpha}_{2,k} | 0_k | \cdots | 0_k) \\
& (0_k | 0, 0, \bar{\alpha}_{3,3}, \dots, \bar{\alpha}_{3,k} | 0, \bar{\alpha}_{2,2}, \dots, \bar{\alpha}_{2,k} | \cdots | 0_k) \\
& \quad \vdots \\
& (0_k | 0_{r-1}, \bar{\alpha}_{r,r}, \dots, \bar{\alpha}_{r,k} | 0_k | \cdots | 0, \bar{\alpha}_{2,2}, \dots, \bar{\alpha}_{2,k}) \\
& \quad \vdots \\
& (0_k | \cdots | 0_{r-2}, \bar{\alpha}_{r-1,r-1}, \dots, \bar{\alpha}_{r-1,k} | 0_k) \\
& (0_k | \cdots | 0_{r-1}, \bar{\alpha}_{r,r}, \dots, \bar{\alpha}_{r,k} | 0_{r-2}, \bar{\alpha}_{r-1,r-1}, \dots, \bar{\alpha}_{r-1,k}) \\
& (0_k | \cdots | 0_k | 0_{r-1}, \bar{\alpha}_{r,r}, \dots, \bar{\alpha}_{r,k}),
\end{aligned}$$

where 0_i denotes a string of 0's of length i .

To show that these vectors are contained in the left kernel of M , let us just consider

$$(0_k | \cdots | 0_l, \bar{\alpha}_{l+1,l+1}, \dots, \bar{\alpha}_{l+1,k} | 0_{l-1}, \bar{\alpha}_{l,l}, \dots, \bar{\alpha}_{l,k} | \cdots | 0_k)$$

and focus on the blocks $M_{l,l}, M_{l+1,l}, M_{l,l+1}$ and $M_{l+1,l+1}$. The other cases can be similarly treated. Then, we have

$$\begin{aligned}
M_{l,l} &= \begin{pmatrix} 0_{l-1 \times k-l} & & \\ -\bar{\alpha}_{l,l+1} & \cdots & -\bar{\alpha}_{l,k} \\ \bar{\alpha}_{l,l} & & \\ & \ddots & \\ & & \bar{\alpha}_{l,l} \end{pmatrix}, & M_{l+1,l} &= \begin{pmatrix} 0_{l-1 \times k-l} & & \\ -\bar{\alpha}_{l+1,l+1} & \cdots & -\bar{\alpha}_{l+1,k} \\ 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}, \\
M_{l,l+1} &= \begin{pmatrix} 0_{l \times k-l-1} & & \\ -\bar{\alpha}_{l,l+2} & \cdots & -\bar{\alpha}_{l,k} \\ \bar{\alpha}_{l,l+1} & & \\ & \ddots & \\ & & \bar{\alpha}_{l,l+1} \end{pmatrix}, & M_{l+1,l+1} &= \begin{pmatrix} 0_{l \times k-l-1} & & \\ -\bar{\alpha}_{l+1,l+2} & \cdots & -\bar{\alpha}_{l+1,k} \\ \bar{\alpha}_{l+1,l+1} & & \\ & \ddots & \\ & & \bar{\alpha}_{l+1,l+1} \end{pmatrix}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
(0_l, \bar{\alpha}_{l+1,l+1}, \dots, \bar{\alpha}_{l+1,k}) \cdot M_{l,l} &= (\bar{\alpha}_{l,l} \bar{\alpha}_{l+1,l+1}, \dots, \bar{\alpha}_{l,l} \bar{\alpha}_{l+1,k}), \\
(0_{l-1}, \bar{\alpha}_{l,l}, \dots, \bar{\alpha}_{l,k}) \cdot M_{l+1,l} &= (-\bar{\alpha}_{l,l} \bar{\alpha}_{l+1,l+1}, \dots, -\bar{\alpha}_{l,l} \bar{\alpha}_{l+1,k})
\end{aligned}$$

and

$$\begin{aligned}
(0_l, \bar{\alpha}_{l+1,l+1}, \dots, \bar{\alpha}_{l+1,k}) \cdot M_{l,l+1} &= (-\bar{\alpha}_{l,l+2} \bar{\alpha}_{l+1,l+1} + \bar{\alpha}_{l,l+1} \bar{\alpha}_{l+1,l+2}, \dots, -\bar{\alpha}_{l,k} \bar{\alpha}_{l+1,l+1} + \bar{\alpha}_{l,l+1} \bar{\alpha}_{l+1,k}), \\
(0_{l-1}, \bar{\alpha}_{l,l}, \dots, \bar{\alpha}_{l,k}) \cdot M_{l+1,l+1} &= (\bar{\alpha}_{l,l+2} \bar{\alpha}_{l+1,l+1} - \bar{\alpha}_{l,l+1} \bar{\alpha}_{l+1,l+2}, \dots, \bar{\alpha}_{l,k} \bar{\alpha}_{l+1,l+1} - \bar{\alpha}_{l,l+1} \bar{\alpha}_{l+1,k}).
\end{aligned}$$

Hence, these parts erase each other. All these $\binom{r+1}{2}$ vectors are linearly independent, therefore $rk - \binom{r+1}{2} = k - 1 + \cdots + k - r$, implying that $\text{rk}(M) = k - 1 + k - 2 + \cdots + k - r$. In conclusion, we have shown that the space of solutions of (1) has dimension $\binom{k-r}{2}$.

Note that when $r = k - 1$ or $r = k$, we have that M has maximal rank and therefore the only solution is the zero solution. \square

Corollary 3.4. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be a code of dimension k with generator matrix $G = [I_k | X]$, where $X \in \mathbb{F}_{q^m}^{k \times (n-k)}$, and let $s \in \{1, \dots, m-1\}$ with $\gcd(s, m) = 1$. Then,

$$h_{q^s+1}(\mathcal{C}) \leq \binom{k+q^s}{q^s+1} - \binom{k-r}{2}.$$

Proof. Let $\mathcal{I}(L_{\mathcal{U}_G})$ be the vanishing ideal of the linear set $L_{\mathcal{U}_G}$. It suffices to notice that

$$\dim_{\mathbb{F}_{q^m}}(\mathcal{I}_{q^s+1}(L_{\mathcal{U}_G})) \geq \dim_{\mathbb{F}_{q^m}}(\mathcal{I}(L_{\mathcal{U}_G}) \cap \mathcal{F}_s).$$

Then, by Theorem 3.3 we obtain

$$h_{q^s+1}(\mathcal{C}) = \binom{k+q^s}{q^s+1} - \dim(\mathcal{I}_{q^s+1}(L_{\mathcal{U}_G})) \leq \binom{k+q^s}{q^s+1} - \binom{k-r}{2}. \quad \square$$

We are now in position to prove Theorem 2.5.

Proof of Theorem 2.5. Since $\text{PG}(k-1, q)$ is contained in $L_{\mathcal{U}_G}$, we have that $\mathcal{I}_{q+1}(L_{\mathcal{U}_G}) \subseteq \mathcal{F}_1$, and therefore $\mathcal{I}_{q+1}(L_{\mathcal{U}_G}) = \mathcal{I}(L_{\mathcal{U}_G}) \cap \mathcal{F}_1$. The thesis follows by applying Theorem 3.3. \square

From Theorem 3.3 and Lemma 1.4, we immediately obtain the following characterization of generalized Gabidulin codes.

Corollary 3.5. Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ be an MRD code of dimension k with generator matrix $G = [I_k | X]$, where $X \in \mathbb{F}_{q^m}^{k \times (n-k)}$, and let $s \in \{1, \dots, m-1\}$ with $\gcd(s, m) = 1$. Let also $\mathcal{I}(L_{\mathcal{U}_G})$ be the vanishing ideal of the linear set $L_{\mathcal{U}_G}$. Then, \mathcal{C} is a generalized Gabidulin code with parameter s if and only if

$$\dim \mathcal{F}_s \cap \mathcal{I}(L_{\mathcal{U}_G}) = \binom{k-1}{2}.$$

4. SOME CONSIDERATIONS ON THE NUMBER OF “ZEROS” OF A $(q^s + 1)$ -FORM

In the previous sections, we have shown that two equivalent linear rank-metric codes have the same \mathbb{F}_q -dimension sequence, which allows us to distinguish a Gabidulin code from a random one. At this point, it is natural to ask which properties of a linear code can be derived from its \mathbb{F}_q -dimension sequence. For instance, given a linear rank-metric code \mathcal{C} with \mathbb{F}_q -dimension sequence $\{h_i(\mathcal{C})\}_{i \geq 0}$, we can easily find its dimension as $\dim_{\mathbb{F}_{q^m}}(\mathcal{C}) = h_1(\mathcal{C})$. However, we are not able to determine the length of the code, as one can see in the following example.

Example 4.1. Let \mathcal{C}_1 and \mathcal{C}_2 be two \mathbb{F}_4 -linear rank-metric codes generated respectively by

$$G_1 = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \alpha \end{pmatrix},$$

where α is a primitive element of \mathbb{F}_4 . Even though $\mathcal{U}_{G_1} \neq \mathcal{U}_{G_2}$, we have that $\text{PG}(1, 4) = L_{\mathcal{U}_{G_1}} = L_{\mathcal{U}_{G_2}}$. Therefore, $h_i(\mathcal{C}_1) = h_i(\mathcal{C}_2)$ for all $i \geq 0$. However, \mathcal{C}_1 has length 3, while \mathcal{C}_2 has length 4.

Even though the length of a code is not determined by its \mathbb{F}_q -dimension sequence, we can still recover some partial information about it.

Lemma 4.2. Let \mathcal{C} be an $[n, k, d]_{q^m/q}$ code with generator matrix G . Then, for all $i \geq 0$,

$$n \geq \log_q((q-1)h_i(\mathcal{C}) + 1).$$

Proof. By Proposition 1.7, $|L_{\mathcal{U}_G}| = h_r(\mathcal{C}) \geq h_i(\mathcal{C})$. Hence, since $q^n \geq |\mathcal{U}_G| \geq (q-1)|L_{\mathcal{U}_G}| + 1$, we immediately get the thesis. \square

When the length of the code \mathcal{C} is given, its \mathbb{F}_q -dimension sequence allows to recover further information about it.

Remark 4.3. When $n \leq m$, a code is MRD if and only if the linear set associated is scattered with respect to hyperplanes [38, Corollary 5.7]. Clearly this implies that the linear set is scattered, hence its cardinality is equal to $\frac{q^n-1}{q-1}$. Therefore, if $h_{r(\mathcal{C})}(\mathcal{C}) < \frac{q^n-1}{q-1}$, the code \mathcal{C} can not be MRD. Moreover, if $k = 2$, hyperplanes are points, implying that being scattered with respect to hyperplanes coincides with being scattered. In this case, $h_{r(\mathcal{C})}(\mathcal{C}) = \frac{q^n-1}{q-1}$ if and only if \mathcal{C} is MRD.

Even when only a subsequence of the \mathbb{F}_q -dimension sequence is known, it is still possible to obtain information on $h_{r(\mathcal{C})}(\mathcal{C})$, or equivalently on $|L_{\mathcal{U}_G}|$. Since the \mathbb{F}_q -dimension sequence is nondecreasing, it immediately provides a lower bound on $h_{r(\mathcal{C})}(\mathcal{C})$. On the other side, $h_i(\mathcal{C})$ allows us to determine $\dim(\mathcal{I}(L_{\mathcal{U}_G})_i)$, and any upper bound on the variety described by the ideal generated by $\mathcal{I}(L_{\mathcal{U}_G})_i$ gives an upper bound on $|L_{\mathcal{U}_G}|$. In particular, for $i = q^s + 1$, we have that

$$|L_{\mathcal{U}_G}| \leq |\mathbf{V}(\mathcal{F}_s \cap \mathcal{I}(L_{\mathcal{U}_G}))|,$$

where $\mathbf{V}(\mathcal{F}_s \cap \mathcal{I}(L_{\mathcal{U}_G}))$ is the variety of the ideal generated by $\mathcal{F}_s \cap \mathcal{I}(L_{\mathcal{U}_G})$. Since for each point in $L_{\mathcal{U}_G}$ there exists an $\alpha \in \mathbb{F}_{q^m}^k$ such that $p(\mathcal{V}_\alpha) = 0$ for all $p \in \mathcal{F}_s \cap \mathcal{I}(L_{\mathcal{U}_G})$, we can consider only the zeros of this type. Therefore, we obtain

$$|L_{\mathcal{U}_G}| \leq \frac{|\{\mathcal{V}_\alpha : \alpha \in \mathbb{F}_{q^m}^k \text{ and } p(\mathcal{V}_\alpha) = 0 \text{ for all } p \in \mathcal{F}_s \cap \mathcal{I}(L_{\mathcal{U}_G})\}|q^k - 1}{q - 1}.$$

Hence, to derive an upper bound on $h_{r(\mathcal{C})}(\mathcal{C})$ it is sufficient to upper bound the number of \mathcal{V}_α 's vanishing on a subset \mathcal{S} of \mathcal{F}_s . In the last part of the section, we focus on the case where $|\mathcal{S}| = 1$. Note that the question about the behavior of forms over linear sets has its own mathematical interest.

We recall that $s \in \{1, \dots, m-1\}$ and $\gcd(s, m) = 1$.

Lemma 4.4. Let $\alpha, \beta \in \mathbb{F}_{q^m}^k$. Then, $\mathcal{V}_\alpha = \mathcal{V}_\beta$ if and only if $\alpha^{[s]} - \alpha = \beta^{[s]} - \beta$.

Proof. $\mathcal{V}_\alpha = \mathcal{V}_\beta$ directly implies that $\beta = \alpha + v$ for some $v \in \mathbb{F}_q^k$. Therefore, we immediately get that $\beta^{[s]} - \beta = (\alpha + v)^{[s]} - (\alpha + v) = \alpha^{[s]} - \alpha$.

On the other side, $\alpha^{[s]} - \alpha = \beta^{[s]} - \beta$ implies that $\alpha^{[s]} - \beta^{[s]} = \alpha - \beta$. Since $\gcd(s, m) = 1$, we have that $\alpha - \beta \in \mathbb{F}_q^k$, hence $\mathcal{V}_\alpha = \mathcal{V}_\beta$. \square

The previous lemma implies that the number of sets \mathcal{V}_α on which a form p vanishes is given by the cardinality of

$$\bar{\mathcal{Z}}_p := \{\alpha^{[s]} - \alpha : \alpha \in \mathbb{F}_{q^m}^k, p(\mathcal{V}_\alpha) = 0\}.$$

In addition, by Theorem 3.3, we can conclude the following.

Lemma 4.5. Let $\alpha_1, \dots, \alpha_{k-1} \in \mathbb{F}_{q^m}^k$ be such that $\bar{\alpha}_i = \alpha_i^{[s]} - \alpha_i$, for $i \in \{1, \dots, k-1\}$, are linearly independent. If $p \in \mathcal{F}_s$ vanishes on all the sets \mathcal{V}_{α_i} , then $p \equiv 0$.

Proof. Take $\alpha_1, \dots, \alpha_{k-1} \in \mathbb{F}_{q^m}^k$ to be the column of a $(k-1) \times k$ matrix X over \mathbb{F}_{q^m} . Now, $\text{rk}(X^{[s]} - X) = k-1$. Hence, $\dim \mathcal{F}_s \cap \mathcal{I}(L_{\mathcal{U}_G}) = 0$, where $G = [I_k | X]$, by Theorem 3.3. Then p should be the zero form. \square

Theorem 4.6. Let $p \in \mathcal{F}_s$ be a nonzero $(q^s + 1)$ -form. If $\dim_{\mathbb{F}_{q^m}}(\langle \bar{\mathcal{Z}}_p \rangle_{\mathbb{F}_{q^m}}) = r$, then there exist $\alpha_1, \dots, \alpha_r$ such that

$$q^{\sum_{i=1}^r \dim_{\mathbb{F}_q}(\mathcal{H}_{\alpha_i})} \leq |\bar{\mathcal{Z}}_p| \leq q^{r(m-1)},$$

where $\mathcal{H}_{\alpha_i} = \{v \in \mathbb{F}_{q^m}^k : \text{Tr}((\alpha_{i,j}^{[s]} - \alpha_{i,j})v) = 0 \text{ for all } j \in \{1, \dots, k\}\}$. In particular, the maximum number of sets \mathcal{V}_α on which a $(q^s + 1)$ -form in $\mathcal{F}_s \setminus \{0\}$ can vanish is $q^{(k-2)(m-1)}$.

Proof. Let $\alpha_1, \dots, \alpha_r \in \mathbb{F}_{q^m}^k$ be such that $\bar{\mathcal{Z}}_p \subseteq \langle \bar{\alpha}_1, \dots, \bar{\alpha}_r \rangle_{\mathbb{F}_{q^m}}$, where $\bar{\alpha}_i = \alpha_i^{[s]} - \alpha_i$. Let $\mathcal{T}_0 := \{\bar{\beta} : \beta \in \mathbb{F}_{q^m}^k\}$ be the set of vectors in $\mathbb{F}_{q^m}^k$ with entries having trace equal to 0 over \mathbb{F}_q . Then, from Lemma 3.1, we obtain that

$$\bar{\mathcal{Z}}_p = \langle \bar{\alpha}_1, \dots, \bar{\alpha}_r \rangle_{\mathbb{F}_{q^m}} \cap \mathcal{T}_0.$$

Since $\bar{\alpha}_1, \dots, \bar{\alpha}_r$ are linearly independent over \mathbb{F}_{q^m} , there exist r entries such that the projection of $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_r \rangle_{\mathbb{F}_{q^m}}$ over these entries is a bijection between $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_r \rangle_{\mathbb{F}_{q^m}}$ and $\mathbb{F}_{q^m}^r$. Hence, the image of $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_r \rangle_{\mathbb{F}_{q^m}} \cap \mathcal{T}_0$ under this projection is contained in the set of vectors in $\mathbb{F}_{q^m}^r$ whose entries have trace equal to zero, that has cardinality equal to $q^{r(m-1)}$. On the other side, since $\bar{\mathcal{Z}}_p$ is an \mathbb{F}_q -linear space, every element of the form $\sum_{i=1}^r v_i \bar{\alpha}_i$, for $v_i \in \mathcal{H}_{\alpha_i}$, belongs to \mathcal{T}_0 , hence to $\bar{\mathcal{Z}}_p$. Thus, we obtain the lower bound on the cardinality of $\bar{\mathcal{Z}}_p$. Finally, by Lemma 4.5, we get that the cardinality of $\bar{\mathcal{Z}}_p$ can be at most $q^{(k-2)(m-1)}$. \square

We conclude noting that the upper bound in Theorem 4.6 is tight. Indeed, we can meet it for instance by considering $\gamma e_1, \dots, \gamma e_r$, where $\gamma \in \mathbb{F}_{q^m} \setminus \mathbb{F}_q$ and e_i is the i -th vector of the canonical base of $\mathbb{F}_{q^m}^k$.

The situation when $|\mathcal{S}| > 1$ is way more complicated and needs further investigations.

ACKNOWLEDGMENTS

The authors of this paper would like to thank Alain Couvreur, Elisa Gorla, and Ferdinando Zullo for their insightful feedback and suggestions.

REFERENCES

- [1] G. Alagic, J. Alperin-Sheriff, D. Apon, D. Cooper, Q. Dang, J. Kelsey, Y.-K. Liu, C. Miller, D. Moody, R. Peralta, et al. Status report on the second round of the NIST post-quantum cryptography standardization process. *US Department of Commerce, NIST*, 2:69, 2020.
- [2] G. Alfarano, M. Borello, A. Neri, and A. Ravagnani. Linear cutting blocking sets and minimal codes in the rank metric. *Journal of Combinatorial Theory, Series A*, 192:105658, 2022.
- [3] V. Astore, 2024. <https://github.com/Valentina-Astore/HilbertSequencesOfCodes>.
- [4] P. Beelen, M. Datta, and S. R. Ghorpade. Vanishing ideals of projective spaces over finite fields and a projective footprint bound. *Acta Mathematica Sinica, English Series*, 35(1):47–63, 2019.
- [5] W. Bosma, J. Cannon, and C. Playoust. The magma algebra system i: The user language. *Journal of Symbolic Computation*, 24(3-4):235–265, 1997.
- [6] I. Cascudo, R. Cramer, D. Mirandola, and G. Zémor. Squares of Random Linear Codes. *IEEE Transactions on Information Theory*, 61(3):1159–1173, 03 2015.
- [7] D. Coggia and A. Couvreur. On the security of a Loidreau rank metric code based encryption scheme. *Designs, Codes and Cryptography*, 88:1941–1957, 2020.
- [8] A. Couvreur, T. Debris-Alazard, and P. Gaborit. On the hardness of code equivalence problems in rank metric. 2021.
- [9] A. Couvreur, P. Gaborit, V. Gauthier-Umaña, A. Otmani, and J. Tillich. Distinguisher-based attacks on public-key cryptosystems using Reed–Solomon codes. *Designs, Codes and Cryptography*, 73:641–666, 2014.
- [10] A. Couvreur and I. Zappatore. An extension of Overbeck’s attack with an application to cryptanalysis of twisted Gabidulin-based schemes. In *Post-Quantum Cryptography*, pages 3–37. Springer Nature Switzerland, 2023.
- [11] B. Csajbók, G. Marino, O. Polverino, and C. Zanella. A new family of MRD-codes. *Linear Algebra and its applications*, 548:203–220, 2018.
- [12] P. Delsarte. Bilinear forms over a finite field, with applications to coding theory. *Journal of Combinatorial Theory, Series A*, 25(3):226–241, 1978.
- [13] D. Eisenbud. A second course in commutative algebra and algebraic geometry. *The Geometry of Syzygies. Graduate Texts in Mathematics*, 229, 2005.
- [14] J.-C. Faugere, V. Gauthier-Umana, A. Otmani, L. Perret, and J.-P. Tillich. A distinguisher for high-rate McEliece cryptosystems. *IEEE Transactions on Information Theory*, 59(10):6830–6844, 2013.
- [15] E. M. Gabidulin. Theory of codes with maximum rank distance (translation). *Problemy peredachi informatsii*, 21(1):3–16, 1985.

- [16] E. M. Gabidulin, A. Paramonov, and O. Tretjakov. Ideals over a non-commutative ring and their application in cryptology. In *Advances in Cryptology—EUROCRYPT’91: Workshop on the Theory and Application of Cryptographic Techniques Brighton, UK, April 8–11, 1991 Proceedings 10*, pages 482–489. Springer, 1991.
- [17] L. Giuzzi and F. Zullo. Identifiers for MRD-codes. *Linear Algebra and its Applications*, 575:66–86, 2019.
- [18] E. Gorla. Rank-metric codes. In *Concise Encyclopedia of Coding Theory*, pages 227–250. Chapman and Hall/CRC, 2021.
- [19] A.-L. Horlemann-Trautmann and K. Marshall. New criteria for MRD and Gabidulin codes and some rank-metric code constructions. *Advances in Mathematics of Communications*, 11(3):533–548, 2017.
- [20] A. Kshevetskiy and E. Gabidulin. The new construction of rank codes. In *Proceedings. International Symposium on Information Theory, 2005. ISIT 2005.*, pages 2105–2108. IEEE, 2005.
- [21] R. Lidl. *Introduction to finite fields and their applications*. Cambridge University Press, 1994.
- [22] G. Lunardon. Normal spreads. *Geometriae Dedicata*, 75:245–261, 1999.
- [23] G. Lunardon, R. Trombetti, and Y. Zhou. Generalized twisted Gabidulin codes. *Journal of Combinatorial Theory, Series A*, 159:79–106, 2018.
- [24] D. Mirandola and G. Zémor. Critical pairs for the product singleton bound. *IEEE Transactions on Information Theory*, 61(9):4928–4937, 2015.
- [25] A. Neri, A.-L. Horlemann-Trautmann, T. Randrianarisoa, and J. Rosenthal. On the genericity of maximum rank distance and Gabidulin codes. *Designs, Codes and Cryptography*, 86:341–363, 2018.
- [26] A. Neri, S. Puchinger, and A.-L. Horlemann-Trautmann. Equivalence and characterizations of linear rank-metric codes based on invariants. *Linear Algebra and its Applications*, 603:418–469, 2020.
- [27] R. Overbeck. Structural attacks for public key cryptosystems based on Gabidulin codes. *Journal of Cryptology*, 21:280–301, 2008.
- [28] R. Pellikaan and I. Márquez-Corbella. Error-correcting pairs for a public-key cryptosystem. In *Journal of Physics: Conference Series*, volume 855, page 012032. IOP Publishing, 2017.
- [29] O. Polverino. Linear sets in finite projective spaces. *Discrete mathematics*, 310(22):3096–3107, 2010.
- [30] H. Randriambololona. Asymptotically good binary linear codes with asymptotically good self-intersection spans. *IEEE Transactions on Information Theory*, 59(5):3038–3045, 2013.
- [31] H. Randriambololona. An upper bound of singleton type for componentwise products of linear codes. *IEEE Transactions on Information Theory*, 59(12):7936–7939, 2013.
- [32] H. Randriambololona. On products and powers of linear codes under componentwise multiplication. *Algorithmic arithmetic, geometry, and coding theory*, 637:3–78, 2015.
- [33] R. M. Roth. Maximum-rank array codes and their application to crisscross error correction. *IEEE Transactions on Information Theory*, 37(2):328–336, 1991.
- [34] J. T. Schwartz. Fast probabilistic algorithms for verification of polynomial identities. *Journal of the ACM (JACM)*, 27(4):701–717, 1980.
- [35] J. Sheekey. A new family of linear maximum rank distance codes. *Advances in Mathematics of Communications*, 10(3):475–488, 2016.
- [36] J. Sheekey. New semifields and new MRD codes from skew polynomial rings. *Journal of the London Mathematical Society*, 101(1):432–456, 2020.
- [37] J. Sheekey, K. Schmidt, and A. Winterhof. 13. MRD codes: constructions and connections. In *Combinatorics and finite fields: Difference sets, polynomials, pseudorandomness and applications*, volume 23, pages 255–286. de Gruyter, 2019.
- [38] J. Sheekey and G. Van de Voorde. Rank-metric codes, linear sets, and their duality. *Designs, Codes and Cryptography*, 88(4):655–675, 2020.
- [39] N. Silberstein, A. S. Rawat, and S. Vishwanath. Error resilience in distributed storage via rank-metric codes. In *2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1150–1157. IEEE, 2012.
- [40] D. Silva, F. R. Kschischang, and R. Koetter. A rank-metric approach to error control in random network coding. *IEEE Transactions on Information Theory*, 54(9):3951–3967, 2008.