



HAL
open science

Brochette first-passage percolation

Maxime Marivain

► **To cite this version:**

| Maxime Marivain. Brochette first-passage percolation. 2025. hal-04896984

HAL Id: hal-04896984

<https://hal.science/hal-04896984v1>

Preprint submitted on 20 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

BROCHETTE FIRST-PASSAGE PERCOLATION

Maxime Marivain

CMAP, École Polytechnique, CNRS, Route de Saclay, 91128 Palaiseau Cedex, France
maxime.marivain@polytechnique.edu

January 20, 2025

Abstract

We investigate a novel first-passage percolation model, referred to as the Brochette first-passage percolation model, where the passage times associated with edges lying on the same line are equal. First, we establish a point-to-point convergence theorem, identifying the time constant. In particular, we explore the case where the time constant vanishes and demonstrate the existence of a wide range of possible behaviours. Next, we prove a shape theorem, showing that the limiting shape is the L^1 diamond. Finally, we extend the analysis by proving a point-to-point convergence theorem in the setting where passage times are allowed to be infinite.

Keywords : First-passage percolation, time constant, shape theorem.

AMS classification : 60K35, 60C05, 60Fxx.

1 Introduction

1.1 First-passage percolation in the independent setting

First-passage percolation is a probabilistic model introduced in 1965 by Hammersley and Welsh in [HW65] to study the flow of fluids through porous media. We refer to [ADH17] for a survey on the subject. In the classical model, we take the set of vertices to be \mathbb{Z}^d . We consider the set of edges \mathbb{E}^d with endpoints $x, y \in \mathbb{Z}^d$ such that $\|x - y\|_1 = 1$. For each edge $e \in \mathbb{E}^d$, we associate a non-negative random variable τ_e with common cumulative distribution function F , called the passage time of edge e . A path is a finite or infinite sequence of edges in \mathbb{E}^d such that consecutive edges share a unique endpoint. For such a path Γ , we define:

$$T(\Gamma) = \sum_{e \in \Gamma} \tau_e,$$

which we call the passage time of Γ . To define the passage time between $x, y \in \mathbb{R}^d$, we first note x' as the unique vertex in \mathbb{Z}^d such that $x \in x' + [0; 1)^d$. Now, if $x, y \in \mathbb{R}^d$, we define the passage time between x and y as:

$$T(x, y) = \inf_{\Gamma} T(\Gamma), \tag{1}$$

where this infimum is taken over the set of paths Γ with endpoints x' and y' . We focus on the random balls for this pseudo-distance, fundamental objects of our study. For $t \geq 0$ let

$$B_t = \{x \in \mathbb{R}^d : T(0, x) \leq t\} \tag{2}$$

be the set of points reachable from the origin in a time less than t . This model has been extensively studied since its introduction in 1965 in the case where $(\tau_e)_{e \in \mathbb{E}^d}$ are chosen to be independent and identically distributed. We now recall the relevant results in the classical case within the context of this article. A first result of this model is that $T(0, nx)$ grows linearly. More precisely we have:

Theorem 1.1 ([Kin73], see Theorem 2.1 [ADH17]). *Assume that $(\tau_e)_{e \in \mathbb{E}^d}$ are i.i.d. random variables such that $\mathbb{E}[\min(\tau_1, \dots, \tau_{2d})] < +\infty$, then for all $x \in \mathbb{Z}^d$ there exists a constant $\mu(x) \geq 0$ (called the time constant) such that:*

$$\lim_{n \rightarrow +\infty} \frac{T(0, nx)}{n} = \mu(x) \text{ a.s. and in } L^1.$$

However, in this model, no non-trivial distribution is known for which the time constant μ can be computed. In 1981, Cox and Durrett improved this result with their shape theorem:

Theorem 1.2 ([CD81]). *Assume now that $\mathbb{E}[\min(\tau_1^d, \dots, \tau_{2d}^d)] < +\infty$ and that $F(0) < p_c(d)$, where $p_c(d)$ is the critical parameter for bond percolation in \mathbb{Z}^d . Then, there exists a deterministic, compact and convex subset B_* of \mathbb{R}^d such that for all $\epsilon > 0$ we have:*

$$\mathbb{P}((1 - \epsilon)B_* \subset \frac{B_t}{t} \subset (1 + \epsilon)B_* \text{ for all large } t) = 1.$$

Furthermore, B_* is symmetric with respect to the axes of \mathbb{R}^d and has a non-empty interior.

Several slightly different models have been introduced, we mention in particular

- The case where no integrability condition holds for τ , in particular where $\mathbb{P}(\tau = +\infty) > 0$ has also been studied, notably by Garet and Marchand in [GM04] and by Cerf and Th  ret in [CT16]. They also proved an analogue of Theorem 1.1 with convergence in probability.
- The case where passage times are non-independent. For example, the stationary ergodic case has been studied in [Boi90]. This model, still rich, has led to many analogous results to the classical ones.

In contrast to this, we consider in this article a long-range dependence model, which, to our knowledge, has not yet been studied. The name of this model is inspired by the one studied by Duminil-Copin, Hilario, Kozma and Sidoravicius in [DCHKS18] in the context of bond percolation. As they do in their article we introduce a strong dependence along the lines.

1.2 Brochette first-passage percolation

As in the classical case we consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$. In this model, the passage times associated with edges on the same line are equal. However, the passage times associated with edges on distinct lines still are independent. Let us now define our model more formally.

Definition 1.3. *An integer line is the set of points in \mathbb{Z}^d that lie on a same line in \mathbb{R}^d parallel to one of the axes. We denote by Δ the set of these lines.*

Definition 1.4 (Brochette first-passage percolation). *Let $(\tau_\delta)_{\delta \in \Delta}$ be a family of non-negative i.i.d random variables with common cumulative distribution function F and indexed by the set of integer lines. In the Brochette first-passage percolation we associate the same passage time to all edges whose endpoints lie on the same integer line $\delta \in \Delta$. In other words, for an edge $e \in \mathbb{E}^d$, the passage time τ_e of e is equal to τ_δ where δ is the unique integer line to which e belongs.*

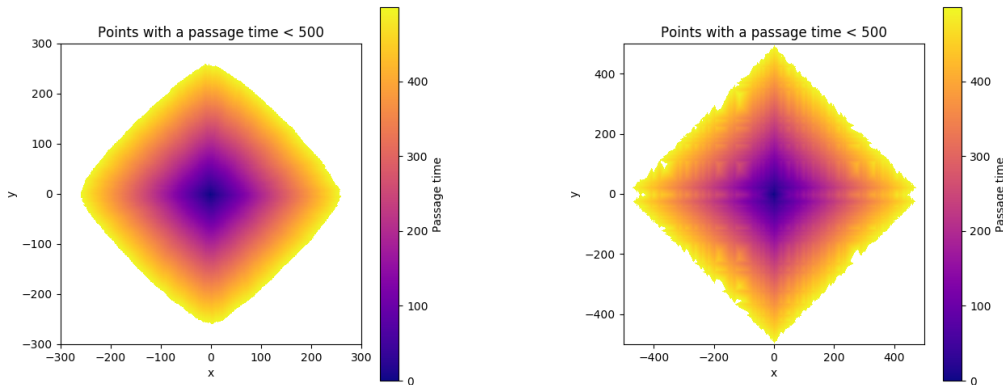


Figure 1: The random balls B_t for $t = 500$ for the distribution $1 + \text{Bernoulli}(0.95)$ for the classical model (on the left) and the Brochette model (on the right).

From now on T and B_t denote passage times and random balls for Brochette first-passage percolation. Throughout the text, we assume that the support of our passage times is included in an interval of the form $[a; +\infty)$, where $a \geq 0$ and a equals the infimum of this support. In particular we will use that :

$$\forall \delta > 0, F(a + \delta) > 0 \text{ and } F(a - \delta) = 0. \quad (3)$$

In the Brochette model, $T(0, n)$ is more sensitive to small values of passage times. Small passage times generate kind of *highways* that geodesics tend to go through. Thus, the random balls do not look alike for the classical and the Brochette models. (See a simulation on Figure 1.)

1.3 The results

Our first result is that we can identify the time constant in the Brochette model.

Theorem 1.5 (Point to point convergence theorem). *For Brochette first-passage percolation, if we assume that*

$$\mathbb{E}[\min(\tau_1, \dots, \tau_d)] < +\infty, \quad (4)$$

where τ_1, \dots, τ_d are d independent copies with distribution F , then we have:

$$\forall x \in \mathbb{Z}^d, \quad \lim_{n \rightarrow +\infty} \frac{T(0, nx)}{n} = a \|x\|_1 \quad \text{a.s. and in } L^1,$$

where we recall that a is the infimum of the support of τ .

Observe that by the definition of a , $T(0, x) \geq a \|x\|_1$ a.s. so the non trivial part of Theorem 1.5 is the upper bound.

Remark 1.6. *Let us note that Assumption (4) is not the same as in the i.i.d. case, as there are d independent integer lines passing through each vertex instead of $2d$ edges as in the classical model. It follows that (4) is optimal. Indeed, if Assumption (4) is not satisfied $T(0, e_1)$ is not integrable.*

Our Theorem 1.5 states that in the case $a = 0$, $T(0, n)$ is sublinear, with the exact order of magnitude yet to be determined. In the classical model, it is known that $\mathbb{E}[T(0, n)]$ is sublinear as soon as $F(0) \geq p_c$ (see Theorem 1.15 in [Kes86]). Several works tried to better understand the behaviour of $\mathbb{E}[T(0, n)]$ in this case. First Zhang [Zha95] established that $\mathbb{E}[T(0, n)]$ remains bounded if $F(0) > p_c$. The case $F(0) = p_c$ is more subtle. We mention a few results in dimension $d = 2$. Chayes, Chayes and Durrett proved in [CCD86] that if F is a Bernoulli distribution of parameter p_c , then there exist $C_1, C_2 > 0$ such that $C_1 \log(n) \leq \mathbb{E}[T(0, n)] \leq C_2 \log(n)$ for all n large enough. Besides Zhang also showed in [Zha99] that there are some distributions F such that $F(0) = p_c$ for which $\mathbb{E}[T(0, n)]$ converges to a non-negative constant (see also [DLW17] for more related results).

We will see that, for the Brochette model, depending on the behaviour of the distribution function F of the passage times near 0, a wide variety of behaviours can emerge even in dimension $d = 2$.

Theorem 1.7. *Consider Brochette first-passage percolation with $a = 0$ in dimension $d = 2$. If there exist $C > 0, \beta \geq 0$ such that $F(t) \sim_{t \rightarrow 0^+} Ct^\beta$, and if the passage times are bounded and atomless, then for all $x \in \mathbb{Z}^2$ there exist $A_1, A_2 > 0$ such that for all $n \geq 2$:*

- If $\beta < 1$, $A_1 \leq \mathbb{E}[T(0, nx)] \leq A_2$.
- If $\beta = 1$, $A_1 \log(n) \leq \mathbb{E}[T(0, nx)] \leq A_2 \log(n)$.
- If $\beta > 1$, $A_1 n^{1-\frac{1}{\beta}} \leq \mathbb{E}[T(0, nx)] \leq A_2 n^{1-\frac{1}{\beta}}$.

Remark 1.8. *We could not obtain comparable results in higher dimensions, which is why we focus on the case $d = 2$.*

The third result is the shape theorem. We denote by \diamond the closed unit ball for the L^1 norm.

Theorem 1.9. *For Brochette first-passage percolation, if we assume that*

$$\mathbb{E}[\min(\tau_1^d, \dots, \tau_d^d)] < +\infty \tag{5}$$

and that $a > 0$ verifies (3), then for all $\epsilon > 0$ we have:

$$\mathbb{P}\left(\frac{1-\epsilon}{a} \diamond \subset \frac{B_t}{t} \subset \frac{1}{a} \diamond \text{ for all large } t\right) = 1.$$

Furthermore, if $a = 0$ we have for all $M > 0$:

$$\mathbb{P}\left(M \diamond \subset \frac{B_t}{t} \text{ for all large } t\right) = 1.$$

Remark 1.10. *It will be seen in Proposition 4.5 that Condition (5) is optimal: there is no shape theorem if (5) does not hold.*

Finally, we consider one last case: when $\mathbb{P}(\tau = +\infty) > 0$. In this case, it amounts to studying first-passage percolation on the cluster \mathcal{C}_* composed of lines whose passage times are finite. Moreover, as soon as $\mathbb{P}(\tau < +\infty) > 0$ the obtained subgraph is both connected and infinite whereas it is not the case in the classical model. Indeed, if an edge has a finite passage time, then all the edges on the same line have the same finite passage time. Thus, it is easy to see that the subgraph is a union of integer lines. In order to state Theorem 1.12, we need to introduce a generalized notion of passage time.

Definition 1.11. If $x, y \in \mathbb{Z}^d$, let $T^*(x, y) = T(x^*, y^*)$ where x^* is the closest vertex in L^1 norm to x in the cluster of edges with finite passage times with an arbitrary rule to break ties.

Theorem 1.12. For Brochette first-passage percolation, if $\mathbb{P}(\tau < +\infty) > 0$, we have:

$$\forall x \in \mathbb{Z}^d \quad \lim_{n \rightarrow +\infty} \frac{T^*(0, nx)}{n} = a\|x\|_1 \quad \text{in probability.}$$

This is the analogue of Theorem 4 in [CT16], we follow the same structure of proof. An important difference is that [CT16] relies on non-trivial estimates in classical percolation. In our case, the infinite cluster has a rather simple shape, which allows for self-contained proofs. To conclude, we outline how the different sections of this article will be structured.

- In Section 2 we focus on point to point convergence Theorem. We prove Theorem 1.5.
- In Section 3 we prove Theorem 1.7 about the variety of behaviours for the expectation of the passage time in dimension $d = 2$.
- In Section 4 we obtain a shape theorem by proving Theorem 1.9.
- In Section 5 we prove Theorem 1.12 about the convergence in probability of $\frac{T^*(0, nx)}{n}$.

2 Point to point almost sure convergence

This section is devoted to the proof of Theorem 1.5 : the *a.s.* convergence of $\frac{T(0, nx)}{n}$. In addition to prove the linear growth of the passage time we are able to determine the limit constant which is equal to $a\|x\|_1$. This is a first step towards understanding the behaviour of $T(0, x)$ in the context of the Brochette first-passage percolation. This result will be improved in Section 4.

Proof of Theorem 1.5. To prove this theorem we will use the following version of Kingman's theorem:

Theorem 2.1 (Subadditive Ergodic Theorem [Lig85]). Let $(X_{m,n})_{0 \leq m \leq n}$ be a family of random variables that satisfy the following assumptions:

- a) $X_{0,n} \leq X_{0,m} + X_{m,n}$ for all $0 < m < n$.
- b) The sequences $(X_{m,m+k})_{k \geq 1}$ and $(X_{m+1,m+k+1})_{k \geq 1}$ have the same distribution for all $m \geq 0$.
- c) For all $k \geq 1$, the sequence $(X_{nk, (n+1)k})_{n \geq 0}$ is stationary.
- d) $\mathbb{E}[X_{0,1}] < +\infty$ and $\mathbb{E}[X_{0,n}] > -cn$ for some finite constant c .

Then there exists a random variable X such that:

$$\frac{X_{0,n}}{n} \xrightarrow{n \rightarrow +\infty} X \quad \text{a.s. and in } L^1.$$

Fix $x \in \mathbb{Z}^d$. We aim to apply Theorem 2.1 to $X_{m,n} = T(mx, nx)$. We verify briefly that a), b), c), d) hold. a) is immediate. As in the i.i.d case the environment is invariant under shifts of \mathbb{Z}^d so assumptions b) and c) hold. Moreover, $\mathbb{E}[X_{0,n}] > -cn$ because passage times are non-negative. It remains to prove $\mathbb{E}[X_{0,1}] < +\infty$. There exist d paths $\Gamma_1, \dots, \Gamma_d$ from 0 to x whose

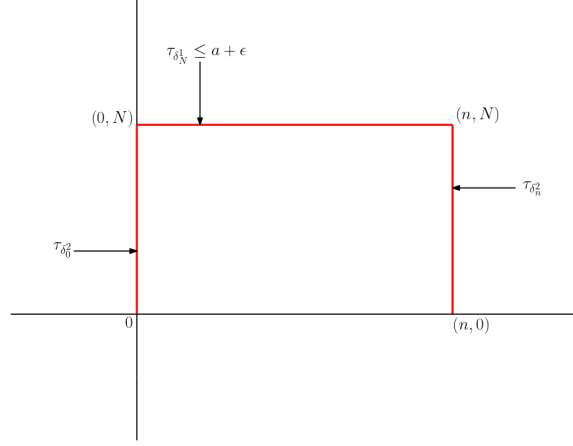


Figure 2: The path going from 0 to ne_1 following the construction.

passage times are independent. Thus, $T(0, x) \leq \min(T(\Gamma_1), \dots, T(\Gamma_d))$ and if we let L the length of the longest of these paths (say Γ_1) we have:

$$\begin{aligned} \mathbb{P}(T(0, x) > s) &\leq \mathbb{P}\left(\bigcap_{i=1}^d \{T(\Gamma_i) > s\}\right) \\ &= \prod_{i=1}^d \mathbb{P}(T(\Gamma_i) > s) \\ &\leq \mathbb{P}(T(\Gamma_1) > s)^d \\ &= (L\mathbb{P}(\tau_e > \frac{s}{L}))^d. \end{aligned}$$

Therefore, considering $Y := \min(\tau_1, \dots, \tau_d)$ where τ_1, \dots, τ_d are i.i.d. with the same distribution F , we have:

$$\mathbb{P}(T(0, x) > s) \leq L^d \mathbb{P}(Y > \frac{s}{L}),$$

from which we deduce, thanks to (4), that $\mathbb{E}[X_{0,1}] < +\infty$. Kingman's theorem therefore implies that there exists a random variable X_x such that:

$$\lim_{n \rightarrow +\infty} \frac{T(0, nx)}{n} = X_x \text{ a.s. and in } L^1.$$

We first prove that for $x = e_1$, it suffices to show that:

$$\forall \epsilon > 0, \liminf \frac{T(0, ne_1)}{n} \leq a + \epsilon \text{ a.s.}$$

Let $\epsilon > 0$. We denote by δ_k^1 the line $ke_2 + \mathbb{Z}e_1$ with $e_2 := (0, 1, 0, \dots, 0)$. Let $N = \inf\{k \geq 1, \tau_{\delta_k^1} \leq a + \epsilon\}$. Since $(\delta_k^1)_{k \in \mathbb{N}}$ are i.i.d. and by definition of a , $N < +\infty$ a.s.. Let δ_k^2 be the line $ke_1 + \mathbb{Z}e_2$ (See Figure 2). Then:

$$T(0, ne_1) \leq N\tau_{\delta_0^2} + n(a + \epsilon) + N\tau_{\delta_n^2}.$$

Thus

$$\frac{T(0, ne_1)}{n} \leq \frac{N}{n}\tau_{\delta_0^2} + (a + \epsilon) + \frac{N}{n}\tau_{\delta_n^2}.$$

Since $N < +\infty$ almost surely, doesn't depend on n and $\tau_{\delta_0^2} < +\infty$ almost surely too, we have $\lim_{n \rightarrow +\infty} \frac{N}{n} \tau_{\delta_0^2} = 0$. Moreover, since $(\tau_{\delta_n^2})_n$ are i.i.d. with distribution F , they are infinitely often smaller than $a + 1$ and we have $\liminf_{n \rightarrow +\infty} \frac{N}{n} \tau_{\delta_n^2} = 0$. We conclude that

$$\liminf_{n \rightarrow +\infty} \frac{T(0, ne_1)}{n} \leq a + \epsilon \text{ a.s.}$$

Therefore we have

$$\lim_{n \rightarrow +\infty} \frac{T(0, ne_1)}{n} = a \text{ a.s. and in } L^1.$$

Now, let $x = (x_1, \dots, x_d) \in \mathbb{N}^d$ (the proof for $x \in \mathbb{Z}^d$ is the same). Let $y_i = (x_1, \dots, x_i, 0, \dots, 0)$. Then:

$$T(0, nx) \leq \sum_{i=0}^{d-1} T(ny_i, ny_{i+1}).$$

Since $T(ny_i, ny_{i+1}) \stackrel{(d)}{=} T(0, nx_{i+1}e_{i+1})$ we have $\frac{T(ny_i, ny_{i+1})}{n} \xrightarrow[n \rightarrow +\infty]{} ax_{i+1}$ in probability. Thus, we get:

$$\lim_{n \rightarrow +\infty} \frac{T(0, nx)}{n} \leq a \sum_{i=0}^{d-1} x_{i+1} = a \|x\|_1.$$

□

3 The case $a = 0$

In this section, we study the case where $a = 0$ and $d = 2$. We will see that, depending on F , a diverse range of behaviours for $T(0, n)$ can arise. For this purpose we define the following random variable.

Definition 3.1. If τ_1, \dots, τ_n are independent passage times we denote:

$$M_n := \min(\tau_1, \dots, \tau_n).$$

To prove Theorem 1.7, we will see it as a corollary of the following theorem.

Theorem 3.2. For the Brochette first-passage percolation when $d = 2$, if $a = 0$ with a defined by (3), if the distribution of τ is atomless and if we assume that $\mathbb{E}[\min(\tau_1, \tau_2)] < +\infty$ then we have:

$$\forall x \in \mathbb{Z}^2, \quad \sum_{k=0}^{\|nx\|_1-1} \mathbb{E}[M_{4k+2}] \leq \mathbb{E}[T(0, nx)] \leq 8 \sum_{k=0}^{\|nx\|_1-1} \mathbb{E}[M_k].$$

Proof of Theorem 3.2. Let us denote X_n the random variable which is equal to the abscissa of the vertical edges with the smallest passage time in $\llbracket 0; n \rrbracket^2$ and Y_n the random variable which is equal to the ordinate of the horizontal edges with the smallest passage time in $\llbracket 0; n \rrbracket^2$. We denote $I_n := T(0, (X_n, Y_n))$, with $I_0 := 0$. Then by sub-additivity, we have:

$$I_{n+1} - I_n \leq T((X_n, Y_n), (X_{n+1}, Y_{n+1})),$$

which gives:

$$\mathbb{E}[I_{n+1} - I_n] \leq \mathbb{E}[T((X_n, Y_n), (X_{n+1}, Y_{n+1}))].$$

Now if $(X_{n+1}, Y_{n+1}) \neq (X_n, Y_n)$, we have:

$$(X_{n+1}, Y_{n+1}) \in \{(X_n, n+1); (n+1, Y_n); (n+1, n+1)\}.$$

Moreover, X_n and Y_n are independent and since the passage times are atomless we have:

$$\mathbb{P}(X_{n+1} = n+1) = \mathbb{P}(Y_{n+1} = n+1) = \frac{1}{n+2}.$$

Let $T_{n,n+1} := T((X_n, Y_n), (X_{n+1}, Y_{n+1}))$. If $n \in \mathbb{N}$ we denote by $\tau_{v,n}$ the passage time associated with the vertical integer line of abscissa n and by $\tau_{h,n}$ the passage time associated with the horizontal integer line of ordinate n . We have:

1. If $(X_{n+1}, Y_{n+1}) = (X_n, n+1)$, $T_{n,n+1} \leq (n+1)\tau_{v,X_n}$.
2. If $(X_{n+1}, Y_{n+1}) = (n+1, Y_n)$, $T_{n,n+1} \leq (n+1)\tau_{h,Y_n}$.
3. If $(X_{n+1}, Y_{n+1}) = (n+1, n+1)$, $T_{n,n+1} \leq (n+1)\tau_{v,X_n} + (n+1)\tau_{h,n+1}$.

Finally we have, since X_n is independent of the horizontal passage times and Y_n is independent of the vertical passage times:

$$\begin{aligned} \mathbb{E}[I_{n+1} - I_n] &\leq \mathbb{E}[T_{n,n+1}] \\ &= \mathbb{E}[T_{n,n+1} \mathbb{1}_{(X_{n+1}, Y_{n+1})=(X_n, n+1)}] + \mathbb{E}[T_{n,n+1} \mathbb{1}_{(X_{n+1}, Y_{n+1})=(n+1, Y_n)}] \\ &\quad + \mathbb{E}[T_{n,n+1} \mathbb{1}_{(X_{n+1}, Y_{n+1})=(n+1, n+1)}] \\ &\leq \mathbb{E}[(n+1)\tau_{v,X_n} \mathbb{1}_{(X_{n+1}, Y_{n+1})=(X_n, n+1)}] + \mathbb{E}[(n+1)\tau_{h,Y_n} \mathbb{1}_{(X_{n+1}, Y_{n+1})=(n+1, Y_n)}] \\ &\quad + \mathbb{E}[((n+1)\tau_{v,X_n} + (n+1)\tau_{h,n+1}) \mathbb{1}_{(X_{n+1}, Y_{n+1})=(n+1, n+1)}] \\ &\leq \mathbb{E}[\tau_{v,X_n} \mathbb{1}_{X_{n+1}=X_n}] + \mathbb{E}[\tau_{h,Y_n} \mathbb{1}_{Y_{n+1}=Y_n}] + \mathbb{E}[\tau_{v,X_n} \mathbb{1}_{X_{n+1}=n+1}] + \mathbb{E}[\tau_{h,n+1} \mathbb{1}_{Y_{n+1}=n+1}] \\ &\leq 2\mathbb{E}[M_n]. \end{aligned}$$

Hence, by summing, we obtain:

$$\mathbb{E}[I_n] \leq 2 \sum_{k=0}^{n-1} \mathbb{E}[M_k].$$

Moreover, to go from 0 to $(n, 0)$, we can construct with the same method a path from $(n, 0)$ to (X_n, Y_n) . We can do the same computation and obtain that:

$$\mathbb{E}[T(0, (n, 0))] \leq 2\mathbb{E}[I_n].$$

Now, to go from 0 to $x = (x_1, x_2)$, we can use the same method twice to go from 0 to $(x_1, 0)$ and then from $(x_1, 0)$ to (x_1, x_2) . It follows that by telescoping we have:

$$\mathbb{E}[T(0, nx)] \leq 4\mathbb{E}[I_{\|nx\|_1}] \leq 8 \sum_{k=0}^{\|nx\|_1 - 1} \mathbb{E}[M_k].$$

For the lower bound we proceed differently. We note that to go from 0 to nx , a path must cross all the deterministic L^1 balls of radius k for k between 0 and $\|nx\|_1 - 1$. Thus, the passage time from 0 to nx is greater than the sum of the minimum passage times among the edges of

the sphere of radius k for k ranging from 0 to $\|nx\|_1 - 1$. Moreover, in dimension 2, $4k + 2$ independent edges originate from a vertex on the sphere of radius k . Thus, we have:

$$\sum_{k=0}^{\|nx\|_1-1} \mathbb{E}[M_{4k+2}] \leq \mathbb{E}[T(0, nx)].$$

□

Proof of Theorem 1.7. To prove Theorem 1.7, we apply the estimate of Lemma A.1 to Theorem 3.2. □

4 Shape Theorem

This section is devoted to the proof of Theorem 1.9. First, by the definition of our model, $\frac{B_t}{t} \subset \frac{1}{a} \diamond$ for all t . Only the first inclusion remains to be proven. To improve a pointwise convergence into a shape theorem, all usual proofs rely at some point on a uniform control of $T(x, y)$. To obtain such control, we prove here the analogue of the Difference-Estimate Lemma (Lemma 2.20, [ADH17]). This step presents a particular difficulty in the "Brochette" case. This step follows the same strategy as the one developed in [CD81] but presents a particular difficulty in the "Brochette" case. As we will see, the dependence makes the construction of certain paths more delicate.

Lemma 4.1. *Consider Brochette first-passage percolation and assume that we have the integrability condition (5). Then, there exists a constant $\kappa > 0$ such that, for every $z \in \mathbb{Z}^d$,*

$$\mathbb{P}\left(\sup_{\substack{x \in \mathbb{Z}^d \\ x \neq z}} \frac{T(z, x)}{\|x - z\|_1} < \kappa\right) > 0.$$

Proof. By shift invariance, it is sufficient to prove the result for $z = 0$. We provide all the details of the proof in the case $d = 2$ and we will briefly explain at the end how to generalize it to an arbitrary d . Set $x = (x_1, x_2)$. Without loss of generality, we assume that $x_1 \geq x_2 \geq 0$. Similarly to classical proofs, the estimates will rely on the construction of paths from 0 to x .

- We first construct $(d + 1) = 3$ paths $\Gamma_1, \Gamma_2, \Gamma_3$ that primarily use edges whose passage times are independent of most of the edges in the other paths.
- Then for $1 \leq i \leq 3$, we will bound $\text{Var}(T(\Gamma_i))$. To do this, we will decompose each Γ_i into several segments, which we will call steps, and most of these are independent.
- In the last step of the proof, we exploit the fact that Γ_1, Γ_2 and Γ_3 are mostly independent in order to control $T(0, x)$.

Construction of paths $\Gamma_1, \Gamma_2, \Gamma_3$. (see Figure 3). We begin by defining a deterministic path S_1 from 0 to x , which we will refer to as the skeleton. It is composed of three staircases: it descends and then ascends to $(x_1 - x_2, 0)$ before ending at (x_1, x_2) . Formally, if $p = \lfloor \frac{x_1 - x_2}{15} \rfloor$ and

$p' = \lfloor \frac{x_2}{15} \rfloor$, S_1 is the path that passes through the following points:

$$\begin{aligned} x_0^1 &= (0, 0) \rightarrow x_1^1 = (0, -15) \rightarrow x_2^1 = (15, -15) \rightarrow \cdots \rightarrow x_{2\lceil \frac{p}{2} \rceil}^1 = (\lceil \frac{p}{2} \rceil 15, -\lceil \frac{p}{2} \rceil 15) \\ &\rightarrow x_{2\lceil \frac{p}{2} \rceil + 1}^1 = (\lceil \frac{p}{2} \rceil 15, -(\lceil \frac{p}{2} \rceil - 1)15) \rightarrow x_{2\lceil \frac{p}{2} \rceil + 2}^1 = ((\lceil \frac{p}{2} \rceil + 1)15, -(\lceil \frac{p}{2} \rceil - 1)15) \\ &\rightarrow \cdots \rightarrow x_{m_1}^1 = (x_1 - x_2, 0) \rightarrow x_{m_1+1}^1 = (x_1 - x_2, 15) \\ &\rightarrow x_{m_1+2}^1 = (x_1 - x_2 + 15, 15) \rightarrow \cdots \rightarrow x_{m_1+n_1}^1 = (x_1, x_2), \end{aligned}$$

and such that between two consecutive points x_i^1 , it follows the straight line. Note that x_i^1 and x_{i+1}^1 are almost always at most 15 edges apart. Depending on the value of p and p' , $m_1 \in \{4\lceil \frac{p}{2} \rceil - 1; 4\lceil \frac{p}{2} \rceil\}$ and $n_1 \in \{2p'; 2p' + 2\}$. Similarly, we construct two other skeletons with the same shape but slightly offset from S_1 for independence reasons that we will see later in the proof.

- S_2 goes through:
 $x_0^2 = (0, 0) \rightarrow x_1^2 = (0, -10) \rightarrow x_2^2 = (10, -10) \rightarrow x_3^2 = (10, -25) \rightarrow \cdots \rightarrow x_{2\lceil \frac{p}{2} \rceil}^2 = (10 + (\lceil \frac{p}{2} \rceil - 1)15, -10 - (\lceil \frac{p}{2} \rceil - 1)15) \rightarrow x_{2\lceil \frac{p}{2} \rceil + 1}^2 = (10 + (\lceil \frac{p}{2} \rceil - 1)15, -10 - (\lceil \frac{p}{2} \rceil - 2)15) \rightarrow \cdots \rightarrow x_{4\lceil \frac{p}{2} \rceil - 1}^2 = ((2\lceil \frac{p}{2} \rceil - 1)15 - 5, 0) \rightarrow x_{4\lceil \frac{p}{2} \rceil =: m_2}^2 = (x_1 - x_2, 0) \rightarrow x_{m_2+1}^2 = (x_1 - x_2, 10) \rightarrow x_{m_2+2}^2 = (x_1 - x_2 + 10, 10) \rightarrow x_{m_2+3}^2 = (x_1 - x_2 + 10, 10 + 15) \rightarrow x_{m_2+4}^2 = (x_1 - x_2 + 10 + 15, 10 + 15) \rightarrow \cdots \rightarrow x_{m_2+n_2}^2 = (x_1, x_2)$,
with $n_2 \in \{2p' + 2; 2p' + 4\}$.
- S_3 goes through:
 $x_0^3 = (0, 0) \rightarrow x_1^3 = (0, -5) \rightarrow x_2^3 = (5, -5) \rightarrow x_3^3 = (5, -20) \rightarrow \cdots \rightarrow x_{2\lceil \frac{p}{2} \rceil}^3 = (5 + (\lceil \frac{p}{2} \rceil - 1)15, -5 - (\lceil \frac{p}{2} \rceil - 1)15) \rightarrow x_{2\lceil \frac{p}{2} \rceil + 1}^3 = (5 + (\lceil \frac{p}{2} \rceil - 1)15, -5 - (\lceil \frac{p}{2} \rceil - 2)15) \rightarrow \cdots \rightarrow x_{4\lceil \frac{p}{2} \rceil - 1}^3 = ((2\lceil \frac{p}{2} \rceil - 1)15 - 10, 0) \rightarrow x_{4\lceil \frac{p}{2} \rceil =: m_3}^3 = (x_1 - x_2, 0) \rightarrow x_{m_3+1}^3 = (x_1 - x_2, 5) \rightarrow x_{m_3+2}^3 = (x_1 - x_2 + 5, 5) \rightarrow x_{m_3+3}^3 = (x_1 - x_2 + 5, 5 + 15) \rightarrow x_{m_3+4}^3 = (x_1 - x_2 + 5 + 15, 5 + 15) \rightarrow \cdots \rightarrow x_{m_3+n_3}^3 = (x_1, x_2)$,
with $n_3 \in \{2p' + 2; 2p' + 4\}$.

A maximal sequence of consecutive edges in the same direction within S_ℓ is called a *step*. (We see from the construction that steps consist of at most 39 edges but most often contain only 15.)

Definition 4.2. For $1 \leq \ell \leq 3$ let s be a step of S_ℓ . Assume s is a horizontal (resp. vertical) step. Let $a = (a_1, a_2)$ and $b = (b_1, a_2)$ (resp. $a = (a_1, a_2)$ and $b = (a_1, b_2)$) be the endpoints of s . Then the envelope of s is defined by the set of edges forming the borders of the rectangle with vertices

$$(a_1, a_2), (b_1, a_2), (b_1, a_2 + 1), (a_1, a_2 + 1)$$

(resp.

$$(a_1, a_2), (a_1 + 1, a_2), (a_1 + 1, b_2), (a_1, b_2)).$$

In the above definition, if a and b are the vertices x_i^ℓ and x_{i+1}^ℓ in the traversal of S_ℓ , the envelope of the step a, b is referred to as the i -th envelope.

Lemma 4.3. Let S_ℓ be a skeleton and E the envelope of a step of S_ℓ . Then $\{\tau_e, e \in E\}$ is independent of $\{\tau_e, e \in E'\}$ for all envelope E' of S_ℓ but at most 10 other envelopes E' .

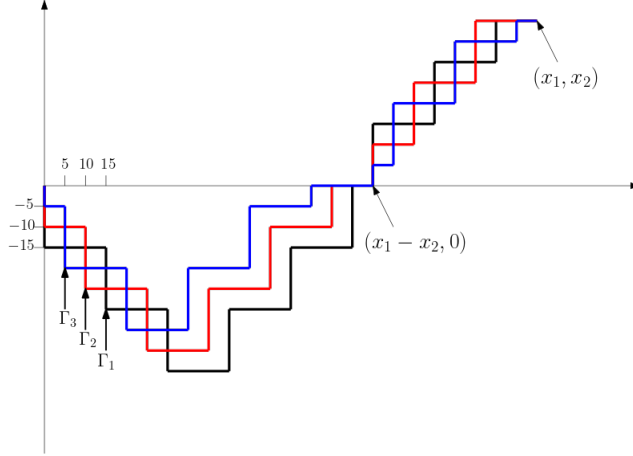


Figure 3: Paths $\Gamma_1, \Gamma_2, \Gamma_3$.

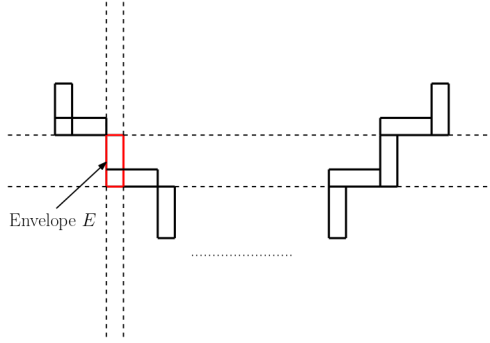


Figure 4: Envelopes which are dependent of the red one.

Proof of lemma 4.3. If the passage times of the edges in an envelope depend on those of E , then this envelope must necessarily touch one of the two strips defined by the sides of E (see Figure 4). Furthermore, based on the construction of our paths and by looking at Figure 4, we can see that there are at most ten such envelopes. \square

We are now ready to build Γ_ℓ . It is a random path connecting 0 to x and contained in the set of envelopes of the steps of S_ℓ . Let γ_i^ℓ be the path with the shortest passage time among the two paths connecting x_i^ℓ and x_{i+1}^ℓ in the i -th envelope of S_ℓ . Then we define Γ_ℓ as the concatenation¹ of $\gamma_1^\ell, \gamma_2^\ell, \dots, \gamma_{k_\ell}^\ell$ where $k_\ell = m_\ell + n_\ell$ is the number of steps of S_ℓ . Notice that the distribution of $T(\gamma_i^\ell)$ only depends on the number of edges of the i -th step of S_ℓ . We can then choose \tilde{T} such that $\tilde{T} \stackrel{(d)}{=} \min(39\tau_1, \tau_2 + 39\tau_3 + \tau_4)$. Thus, for all i and ℓ , \tilde{T} stochastically dominates $T(\gamma_i^\ell)$.

Lemma 4.4. *For $1 \leq \ell \leq 3$ and $1 \leq i \leq k_\ell$ set $T_i^\ell = T(\gamma_i^\ell)$. Then for all $1 \leq k \leq k_\ell$ there exists $C > 0$ such that we have:*

$$\mathbb{P}\left(\sum_{i=1}^k T_i^\ell > k(\mathbb{E}[\tilde{T}] + 1)\right) \leq \frac{C}{k}.$$

¹It is possible that the path Γ_ℓ passes through the same vertex several times.

Proof. Using straightforward calculation (see eq (2.3) in [ADH17]) and thanks to Assumption (5), for all i and l , $\text{Var}(T_i^\ell) < C'$ for some C' which does not depend on k, i and ℓ . Using Lemma 4.3, we have:

$$\text{Var}\left(\sum_{i=1}^k T_i^\ell\right) = \sum_{1 \leq i, j \leq k} \text{Cov}(T_i^\ell, T_j^\ell) \leq \sum_{i=1}^k 10 \max_j \text{Cov}(T_i^\ell, T_j^\ell).$$

Moreover, if $\text{Cov}(T_i^\ell, T_j^\ell) \neq 0$, then we have, according to the Cauchy-Schwarz inequality:

$$|\text{Cov}(T_i^\ell, T_j^\ell)| \leq \sqrt{\text{Var}(T_i^\ell)\text{Var}(T_j^\ell)} \leq C'.$$

Consequently, we have:

$$\text{Var}\left(\sum_{i=1}^k T_i^\ell\right) \leq 10kC'.$$

Thus, since by definition $\mathbb{E}[\tilde{T}] \geq \mathbb{E}[T_i^\ell]$ for all $1 \leq i \leq k$ we have thanks to Tchebychev inequality:

$$\mathbb{P}\left(\sum_{i=1}^k T_i^\ell > k(\mathbb{E}[\tilde{T}] + 1)\right) \leq \mathbb{P}\left(\sum_{i=1}^k T_i^\ell > \sum_{i=1}^k \mathbb{E}[T_i^\ell] + k\right) \leq \frac{10}{k}C'.$$

Denoting $C = 10C'$, the result follows. \square

In order to conclude the proof, we need to control $T(0, x)$ using the independence of Γ_1, Γ_2 and Γ_3 . By construction, Γ_1, Γ_2 and Γ_3 involve independent envelopes, except in the neighborhoods of $0, (x_1 - x_2, 0)$ and x . In order to exploit this rigorously we introduce a few notation:

- $U_\ell = \sum_{i=3}^{m_\ell-2} T_i^\ell + \sum_{i=m_\ell+3}^{k_\ell-2} T_i^\ell$
- $R = \max_{1 \leq \ell \leq 3} (T_1^\ell + T_2^\ell + \sum_{i=m_\ell-1}^{m_\ell+2} T_i^\ell + T_{k-1}^\ell + T_k^\ell).$

We note that in this way, the U_ℓ are independent. Indeed, by construction, the envelopes of S_ℓ on which U_ℓ depends consist of edges located on integer lines distinct from those forming the envelopes of $S_{\ell'}$ on which $U_{\ell'}$ depends, for all distinct $\ell, \ell' \in \{1; 2; 3\}$. Moreover, we have thanks to the union bound:

$$\mathbb{P}(R > \|x\|_1) \leq 24\mathbb{P}(\tilde{T} > \frac{\|x\|_1}{8}).$$

Additionally, we notice that $T(0, x) \leq R + \min_{1 \leq \ell \leq 3} U_\ell$. Therefore, we have with the union bound:

$$\begin{aligned} \mathbb{P}(T(0, x) > (\mathbb{E}[\tilde{T}] + 2)\|x\|_1) &\leq \mathbb{P}(R > \|x\|_1) + \mathbb{P}\left(\min_{1 \leq \ell \leq 3} U_\ell > (\mathbb{E}[\tilde{T}] + 1)\|x\|_1\right) \\ &\leq 24\mathbb{P}(\tilde{T} > \frac{\|x\|_1}{8}) + \mathbb{P}\left(\min_{1 \leq \ell \leq 3} U_\ell > (\mathbb{E}[\tilde{T}] + 1)\|x\|_1\right). \end{aligned}$$

Now according to Lemma 4.4 $\mathbb{P}(U_\ell > k_\ell(\mathbb{E}[\tilde{T}] + 1)) \leq \frac{C}{k_\ell}$. Thus, by the independence of the U_ℓ , we have, since each path is composed of less than $\|x\|_1$ steps but more than $\frac{\|x\|_1}{15}$:

$$\mathbb{P}(T(0, x) > (\mathbb{E}[\tilde{T}] + 2)\|x\|_1) \leq 24\mathbb{P}(\tilde{T} > \frac{\|x\|_1}{8}) + (15C)^3 \|x\|_1^{-3}.$$

Thus, if $K = \mathbb{E}[\tilde{T}] + 2$,

$$\sum_{x \in \mathbb{Z}^2} \mathbb{P}\left(\frac{T(0, x)}{\|x\|_1} > K\right) \leq 24 \sum_{x \in \mathbb{Z}^2} \mathbb{P}(\tilde{T} > \frac{\|x\|_1}{8}) + (15C)^3 \sum_{x \in \mathbb{Z}^2} \|x\|_1^{-3}.$$

Thanks to Assumption (5), $\mathbb{E}[\tilde{T}^d] = \mathbb{E}[\tilde{T}^2] < +\infty$ and so $\sum_{x \in \mathbb{Z}^2} \mathbb{P}(\tilde{T} > \frac{\|x\|_1}{8}) < +\infty$. Therefore, since the series of $\|x\|_1^{-3}$ over \mathbb{Z}^2 converges, according to the Borel-Cantelli Lemma there exists $\kappa > 0$ such that for all $z \in \mathbb{Z}^2$

$$\mathbb{P}(\forall x \in \mathbb{Z}^2 \setminus \{z\}, \frac{T(z, x)}{\|z - x\|_1} < \kappa) > 0.$$

□

Everything we have done here can be generalized in dimension $d \geq 3$. Indeed the biggest difference is that we need to construct $d + 1$ paths. This construction is achievable by simply spacing out the paths and moving along orthogonal planes to maintain as much independence as possible. Moreover, we then consider d -dimensional envelopes and denote γ_i^ℓ the path with the shortest passage time among the d paths connecting x_i^ℓ and x_{i+1}^ℓ in the i -th d -dimensional envelope of S_ℓ . Thanks to Assumption (5), we have in particular $\mathbb{E}[\min(\tau_1, \dots, \tau_d)^2] < +\infty$, so $\text{Var}(T_i^\ell)$ are still bounded and with obvious notations, we get $\mathbb{P}(\min_{1 \leq \ell \leq d+1} U_\ell > (\mathbb{E}[\tilde{T}] + 1)\|x\|_1) \leq C'\|x\|_1^{-(d+1)}$. Finally, using that $\mathbb{E}[\min(\tau_1, \dots, \tau_d)^d] < +\infty$, we also get that $\sum_{x \in \mathbb{Z}^d} \mathbb{P}(\tilde{T} > \frac{\|x\|_1}{8}) < +\infty$.

Proof of Theorem 1.9. We can now use Lemma 4.1 to prove Theorem 1.9 by following the proof in [ADH17]. The translation f with vector $\zeta \in \mathbb{Z}^d$ is not ergodic when ζ is on one of the axes. However, ergodicity outside the axes is sufficient to prove Theorem 1.9. Indeed, by carefully reading the proof of Theorem 2.16 in [ADH17], just after formula (2.11), we see that ergodicity along directions arbitrarily close to the axes is sufficient. The result follows. □

Proposition 4.5. *Assumption (5) is necessary for B_t to have a limiting shape. More precisely if $\mathbb{E}[\min(\tau_1^d, \dots, \tau_d^d)] = +\infty$ then for any $K > 0$,*

$$\frac{T(0, x)}{\|x\|_1} > K \text{ for infinitely many } x \in \mathbb{Z}^d, \text{ a.s.}$$

Proof. Recall that a is the infimum of the support of the distribution of the passage times. Let $K > a$. Let $Q = \{z = (z_1, \dots, z_d) \in \mathbb{N}^d \text{ such that } z_2, \dots, z_d \leq z_1\}$. Let for all $z \in Q$, $A_z := \{\forall i \in \llbracket 1; d \rrbracket, \tau_{z+e_i} \geq dz_1 K\}$. If A_z occurs, $T(0, z) \geq K\|z\|_1$. Then if for any $K > a$ an infinite number of A_z occur, there is no shape theorem. To prove that this is the case with probability 1 if $\mathbb{E}[\min(\tau_1^d, \dots, \tau_d^d)] < +\infty$, we use the following Borel-Cantelli lemma:

Lemma 4.6. [ER59] Lemma C). *Let $(A_n)_{n \geq 1}$ be such that $\sum_{n=1}^{+\infty} \mathbb{P}(A_n) = +\infty$ and such that:*

$$\limsup_{n \rightarrow +\infty} \frac{(\sum_{i=1}^n \mathbb{P}(A_i))^2}{\sum_{1 \leq i, j \leq n} \mathbb{P}(A_i \cap A_j)} = 1.$$

Then $\mathbb{P}(\limsup A_n) = 1$.

In our case, we want to prove that $\limsup_{n \rightarrow +\infty} \frac{(\sum_{z \in Q \cap \llbracket 1; n \rrbracket^d} \mathbb{P}(A_z))^2}{\sum_{z, z' \in Q \cap \llbracket 1; n \rrbracket^d} \mathbb{P}(A_z \cap A_{z'})} = 1$. We have $\mathbb{P}(A_z) = p_{z_1}^d$, where we put $p_i := \mathbb{P}(\tau \geq dKi)$. Let

$$V_n := \left(\sum_{z \in Q \cap \llbracket 1; n \rrbracket^d} \mathbb{P}(A_z) \right)^2 = \left(\sum_{i=1}^n i^{d-1} p_i^d \right)^2.$$

Now by assumption $\mathbb{E}[\min(\tau_1, \dots, \tau_d)^d] = +\infty$ and by series-integral comparison there exists a constant $R > 0$ such that $\sum_{i=1}^{+\infty} i^{d-1} p_i^d \geq R \mathbb{E}[\min(\tau_1, \dots, \tau_d)^d]$. So

$$\sum_{i=1}^{+\infty} i^{d-1} p_i^d = +\infty.$$

Let $z = (z_1, \dots, z_d), z' = (z'_1, \dots, z'_d) \in Q$ and let us denote $I(z, z') := \{j \in \llbracket 1; d \rrbracket : z_j = z'_j\}$. We have:

$$\mathbb{P}(A_z \cap A_{z'}) = \begin{cases} \mathbb{P}(A_z) \mathbb{P}(A_{z'}) & \text{if } |I| = 0 \\ p_{z_1}^{2d-k} & \text{if } |I| = k, k > 0, 1 \in I \\ p_{\min(z_1, z'_1)}^{d-k} p_{\max(z_1, z'_1)}^d & \text{if } |I| = k, k > 0, 1 \notin I. \end{cases}$$

Then:

$$\begin{aligned} \sum_{z, z' \in Q \cap \llbracket 1; n \rrbracket^d} \mathbb{P}(A_z \cap A_{z'}) &\leq \sum_{\substack{z, z' \in Q \cap \llbracket 1; n \rrbracket^d \\ |I|=0}} \mathbb{P}(A_z) \mathbb{P}(A_{z'}) + \sum_{k=1}^d \sum_{\substack{z, z' \\ |I(z, z')|=k \\ 1 \in I(z, z')}} p_{z_1}^{2d-k} + \sum_{k=1}^{d-1} \sum_{\substack{z, z' \\ |I(z, z')|=k \\ 1 \notin I(z, z')}} p_{\max(z_1, z'_1)}^d p_{\min(z_1, z'_1)}^{d-k} \\ &\leq \underbrace{\sum_{\substack{z, z' \in Q \cap \llbracket 1; n \rrbracket^d \\ |I|=0}} \mathbb{P}(A_z) \mathbb{P}(A_{z'})}_{=: S_{1,n}} + \sum_{k=1}^d 2 \binom{d-1}{k-1} \underbrace{\sum_{z_1=1}^n z_1^{2d-k-1} p_{z_1}^{2d-k}}_{=: S_{2,k,n}} \\ &\quad + \sum_{k=1}^{d-1} 2 \binom{d-1}{k} \underbrace{\sum_{z_1=1}^n z_1^{d-1} p_{z_1}^{d-k} \sum_{z'_1=z_1+1}^n z'_1{}^{d-k-1} p_{z'_1}^d}_{=: S_{3,k,n}}. \end{aligned}$$

We will now estimate each part of the right-hand term. First, we notice that $S_{1,n} \leq V_n$. Let us first estimate $S_{2,k,n}$ for $k \in \llbracket 1; d \rrbracket$. First we write:

$$S_{2,k,n} = \sum_{i=1}^n i^{2d-k-1} p_i^{2d-k} = \sum_{i=1}^n i^{d-1} p_i^d (ip_i)^{d-k}.$$

Let $k \in \llbracket 0; d-1 \rrbracket$ and $i \in \mathbb{N}^*$ there exists a constant $C_k > 0$ such that:

$$\begin{aligned} i(ip_i)^k &= i^{k+1} p_i^k \\ &\leq \left(\sup_{i_0 \in \mathbb{N}} \frac{i_0^{k+1}}{\sum_{j=1}^{i_0} j^k} \right) \sum_{j=1}^i j^k p_j^k \text{ because } (p_i) \text{ is a decreasing sequence} \\ &= C_k \sum_{j=1}^i j^{\frac{k}{d}} j^{\frac{k(d-1)}{d}} p_j^k \\ &\leq C_k \left(\sum_{j=1}^i j^{\frac{k}{d-k}} \right)^{\frac{d-k}{d}} \left(\sum_{j=1}^i j^{d-1} p_j^d \right)^{\frac{k}{d}} \text{ according to Hölder's inequality} \\ &\leq C_k i^{\left(\frac{k}{d-k} + 1\right) \frac{d-k}{d}} \left(\sum_{j=1}^i j^{d-1} p_j^d \right)^{\frac{k}{d}} \\ &= C_k i \left(\sum_{j=1}^i j^{d-1} p_j^d \right)^{\frac{k}{d}}. \end{aligned}$$

We then have:

$$\forall k \in \llbracket 0; d-1 \rrbracket, (ip_i)^k \stackrel{i \rightarrow +\infty}{\underset{=}{\sim}} o\left(\sum_{j=1}^i j^{d-1} p_j^d\right).$$

It follows that we have:

$$\forall k \in \llbracket 1; d \rrbracket, (ip_i)^{d-k} \stackrel{i \rightarrow +\infty}{\underset{=}{\sim}} o\left(\sum_{j=1}^i j^{d-1} p_j^d\right).$$

Since $\sum_{j=1}^i j^{d-1} p_j^d \leq \sum_{j=1}^n j^{d-1} p_j^d$, we conclude that:

$$\forall k \in \llbracket 1; d \rrbracket, S_{2,k,n} \stackrel{n \rightarrow +\infty}{\underset{=}{\sim}} o(V_n).$$

Let us now estimate $S_{3,k,n}$. We see that :

$$S_{3,k,n} = \sum_{i=1}^n i^{d-1} p_i^{d-k} \sum_{j=i+1}^n j^{d-k-1} p_j^d = \sum_{j=2}^n j^{d-k-1} p_j^d \sum_{i=1}^{j-1} i^{d-1} p_i^{d-k}.$$

Now we have:

$$\begin{aligned} \sum_{i=1}^{j-1} i^{d-1} p_i^{d-k} &= \sum_{i=1}^{j-1} i^{\frac{(d-1)k}{d}} p_i^{d-k} i^{\frac{(d-1)(d-k)}{d}} \\ &\leq \left(\sum_{i=1}^{j-1} i^{\frac{d}{k} \frac{(d-1)k}{d}}\right)^{\frac{k}{d}} \left(\sum_{i=1}^{j-1} p_i^{\frac{(d-k)d}{d-k}} i^{\frac{(d-1)(d-k)}{d} \frac{d}{d-k}}\right)^{\frac{d-k}{d}} \text{ thanks to Hölder's inequality} \\ &\leq C_k j^{\frac{(d-1)k}{d} + \frac{k}{d}} \left(\sum_{i=1}^{j-1} i^{d-1} p_i^d\right)^{\frac{d-k}{d}} \\ &\leq C_k j^k \left(\sum_{i=1}^{j-1} i^{d-1} p_i^d\right)^{\frac{d-k}{d}}. \end{aligned}$$

Then by summing we finally have that:

$$\forall k \in \llbracket 1; d \rrbracket, S_{3,k,n} \stackrel{n \rightarrow +\infty}{\underset{=}{\sim}} o(V_n).$$

To conclude, since $S_{1,n} \leq V_n$ and $S_{2,k,n} + S_{3,k,n} \stackrel{n \rightarrow +\infty}{\underset{=}{\sim}} o(V_n)$, we have:

$$\limsup_{n \rightarrow +\infty} \frac{V_n}{S_{1,n} + S_{2,k,n} + S_{3,k,n}} \geq 1.$$

Thanks to Lemma 4.6, Proposition 4.5 is proven. \square

5 Brochette First-Passage Percolation with infinite passage times

In this section, we allow our passage times to be infinite:

$$\mathbb{P}(\tau = +\infty) > 0.$$

The proof used here follows the same outline as [CT16] by Cerf and Th er et. They proved similar results in the context of classical first-passage percolation. Their idea is to introduce a regularized notion of passage times. For any $M > a$, we consider the set of edges whose passage time is strictly less than M . The cluster formed by these edges and their endpoints is denoted by \mathcal{C}_M . It is easy to see that \mathcal{C}_M is infinite, connected, and composed of integer lines. Moreover, we can then define, for all $x, y \in \mathbb{Z}^d$, $\tilde{T}_M(x, y) = T(\tilde{x}_M, \tilde{y}_M)$ where \tilde{x}_M is the point in the cluster \mathcal{C}_M closest to x in L^1 norm with an arbitrary rule to break ties. In the same way, we define \mathcal{C}_* the cluster \mathcal{C}_M with $M = +\infty$. The proof of Theorem 1.12 is intertwined with that of the following proposition, which shows that \tilde{T}_M does not depend on M asymptotically.

Proposition 5.1. *For the Brochette first passage percolation, if $\mathbb{P}(\tau < +\infty) > 0$, we have for all $M > a$:*

$$\forall x \in \mathbb{Z}^d, \quad \lim_{n \rightarrow +\infty} \frac{\tilde{T}_M(0, nx)}{n} = a \|x\|_1 \quad \text{a.s. and in } L^1.$$

First to understand and then to prove this theorem we now define several objects. If $M > a$, we define the following objects:

- $p_M := \mathbb{P}(\tau \leq M)$.
- If $x, y \in \mathbb{Z}^d$, $D_M(x, y) = \inf\{|\gamma| : \gamma \text{ path from } x \text{ to } y \text{ in } \mathcal{C}_M\}$ is the graph distance in \mathcal{C}_M , often called the chemical distance between x and y in \mathcal{C}_M in the context of first-passage percolation.
- $D(x, y) = \inf\{|\gamma| : \gamma \text{ path from } x \text{ to } y \text{ in } \mathcal{C}_*\}$ is the chemical distance between x and y in \mathcal{C}_* .

To prove Proposition 5.1, we will once again use Kingman's subadditive theorem. The only assumption of Kingman's Theorem which is not immediate is the integrability of the process $(\tilde{T}_M(mx, nx))_{m < n}$. This is the purpose of our first lemma.

Lemma 5.2. *Consider Brochette first-passage percolation, then for all $M > a$ there exist $A_1(M), A_2(M) > 0$ such that for all $x \in \mathbb{Z}^d$ and for all $\ell \geq (2d + 1)\|x\|_1$*

$$\mathbb{P}(0 \leftrightarrow x, D_M(0, x) \geq \ell) \leq A_1(M) \exp(-A_2(M)\ell).$$

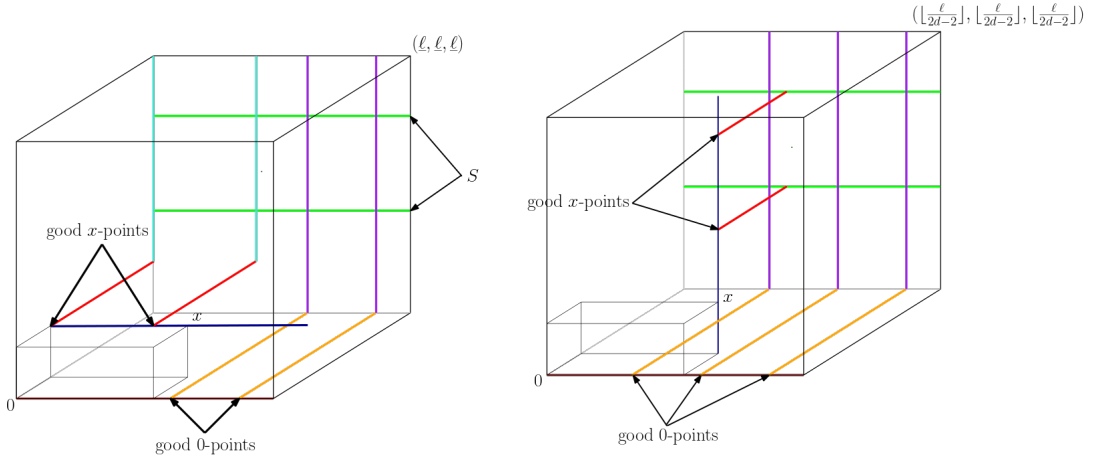
Proof of Lemma 5.2. We assume that $x = (x_1, \dots, x_d)$ has all its coordinates positive (the other cases are treated in the same way). Let $\ell \geq (2d + 1)\|x\|_1$. We consider the situation where only one edge emanating from 0 and from x are present, and they are parallel (this is the worst case; the proof is similar and sometimes shorter in other cases). Moreover to simplify the notations we assume that these edges are $\{0; e_1\}$ and $\{x; x + e_1\}$. Let us define $\underline{\ell} := \lfloor \frac{\ell}{2d+1} \rfloor$. If x and y belong to the same integer line δ , we denote $x \xrightarrow[M]{} y$ if the passage time of δ is less than M , i.e., if $x, y, \delta \in \mathcal{C}_M$. Let $1 \leq c \leq \underline{\ell}$. We say that $(c, 0, \dots, 0)$ is a good 0-point if the event

$$(c, 0, \dots, 0) \xrightarrow[M]{} (c, \underline{\ell}, 0, \dots, 0) \xrightarrow[M]{} (c, \underline{\ell}, \underline{\ell}, 0, \dots, 0) \xrightarrow[M]{} (c, \underline{\ell}, \dots, \underline{\ell})$$

occurs. Similarly, if $1 \leq c' \leq \underline{\ell}$ we say that $(c', 0, \dots, 0)$ is a good x -point if the event:

$$(c', x_1, \dots, x_d) \xrightarrow[M]{} (c', \underline{\ell}, x_3, \dots, x_d) \xrightarrow[M]{} (c, \underline{\ell}, \underline{\ell}, x_4, \dots, x_d) \xrightarrow[M]{} (c', \underline{\ell}, \dots, \underline{\ell})$$

occurs. The number B of good 0-points and the number B' of good x -points follow a binomial distribution with parameters $\underline{\ell}$ and p_M^{d-1} . Lastly, if we denote S the set of integers $1 \leq e \leq \underline{\ell}$



(a) Illustration of Lemma 5.2 in the case where the edges emanating from 0 and x are parallel.

(b) Illustration of Lemma 5.2 in the case where the edges emanating from 0 and x are not parallel.

Figure 5

such that $(0, \underline{\ell}, \dots, \underline{\ell}, e) \xrightarrow{M} (\underline{\ell}, \dots, \underline{\ell}, e)$ then its cardinal \tilde{B} follows a binomial distribution with parameters $\underline{\ell}$ and p_M . Moreover, if $B, B', \tilde{B} > 0$, then there exists a path connecting 0 to x in \mathcal{C}_M with a length at most $(2d+1)\underline{\ell} \leq \ell$. We have illustrated the construction in the case of dimension 3 in Figure 5. We leave it to the readers to convince themselves that the construction is similar when the edges emanating from 0 and x are not parallel (see Figure 5b). Let us show that the non-existence of such a path has an exponentially small probability to occur. We have:

$$\begin{aligned}
\mathbb{P}(0 \leftrightarrow x, D_M(0, x) \geq \ell) &\leq \mathbb{P}(0 \leftrightarrow x, D_M(0, x) \geq (2d+1)\underline{\ell}) \\
&\leq \mathbb{P}(B = 0) + \mathbb{P}(B' = 0) + \mathbb{P}(\tilde{B} = 0) \\
&\leq 3\mathbb{P}(\tilde{B} = 0) \\
&\leq 3 \exp(\underline{\ell} \log(1 - p_M)) \\
&\leq A_1(M) \exp(-A_2(M)\ell).
\end{aligned}$$

□

Proposition 5.3. *There exist three positive constants C_1, C_2 and C_3 which depend on M such that for all $x \in \mathbb{Z}^d$ and for all $\ell \geq C_3\|x\|_1$*

$$\mathbb{P}(\tilde{T}_M(0, x) > \ell) \leq C_1 e^{-C_2 \ell}.$$

Proof of Proposition 5.3 .

$$\begin{aligned}
\mathbb{P}(\tilde{T}_M(0, x) > \ell) &\leq \mathbb{P}(\tilde{0} \notin B_1(0, \frac{\epsilon\ell}{2})) + \mathbb{P}(\tilde{x} \notin B_1(x, \frac{\epsilon\ell}{2})) \\
&+ \sum_{\substack{a \in B_1(0, \frac{\epsilon\ell}{2}) \cap \mathbb{Z}^d \\ b \in B_1(x, \frac{\epsilon\ell}{2}) \cap \mathbb{Z}^d}} \mathbb{P}(\tilde{0} = a, \tilde{x} = b, D_M(a, b) \geq \frac{\ell}{M}) \\
&\leq \mathbb{P}(\mathcal{C}_M \cap B_1(0, \frac{\epsilon\ell}{2}) = \emptyset) + \mathbb{P}(\mathcal{C}_M \cap B_1(x, \frac{\epsilon\ell}{2}) = \emptyset) \\
&+ \sum_{\substack{a \in B_1(0, \frac{\epsilon\ell}{2}) \cap \mathbb{Z}^d \\ b \in B_1(x, \frac{\epsilon\ell}{2}) \cap \mathbb{Z}^d}} \mathbb{P}(a \overset{M}{\longleftrightarrow} b, D_M(a, b) \geq \frac{\ell}{M}).
\end{aligned}$$

To bound the first two terms, we use the following rough estimate:

$$\mathbb{P}(\mathcal{C}_M \cap B_1(0, r) = \emptyset) \leq (1 - p_M)^r.$$

Moreover, if $a \in B_1(0, \frac{\epsilon\ell}{2}) \cap \mathbb{Z}^d$ and $b \in B_1(x, \frac{\epsilon\ell}{2}) \cap \mathbb{Z}^d$ then $\|a - b\|_1 \leq \|x\|_1 + \epsilon\ell$. Thus, for $\epsilon = \frac{1}{2(2d+1)M}$ and $\ell \geq 2M(2d+1)\|x\|_1$ we can use Lemma 5.2 and we then have:

$$\mathbb{P}(\tilde{T}_M(0, x) > \ell) \leq 2(1 - p_M)^{\frac{\epsilon\ell}{2}} + |B_1(0, \frac{\epsilon\ell}{2}) \cap \mathbb{Z}^d|^2 A_1 \exp(-A_2\ell).$$

Hence the result. \square

Proof of Proposition 5.1. We first fix $M > a$. All the assumptions of Kingman's Theorem are satisfied, and $\tilde{T}_M(0, x)$ is integrable thanks to Proposition 5.3. Then we know that:

$$\frac{\tilde{T}_M(0, nx)}{n} \xrightarrow{n \rightarrow +\infty} X_M \text{ a.s. and in } L^1, \quad (6)$$

with X_M a random variable. However, as in the case where passage times are finite almost surely, the lack of ergodicity implies that, for now, we do not know that the limiting random variable is almost surely equal to a constant. To see that we will study the convergence of a subsequence of $(\frac{\tilde{T}_M(0, nx)}{n})_{n \geq 1}$ in the following lemma.

Lemma 5.4. *For all $\epsilon > 0$, $x \in \mathbb{Z}^d$ and $M > a$, we have:*

$$a\|x\|_1 \leq X_M \leq M\|x\|_1 \text{ a.s..}$$

Proof of Lemma 5.4. Let $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$. By symmetry we can assume that $x_1, \dots, x_d \geq 0$. By definition, there exists at least one integer line with a passage time less than M passing through $\tilde{0}$. Without loss of generality, assume that the line $\tilde{0} + \mathbb{Z}e_1$ has a passage time less than M . Let $J = \{n \in \mathbb{N}, nx \in \mathcal{C}_M, D_M(\tilde{0}, nx) \leq \|\tilde{0} - nx\|_1 + 2\sqrt{n}\}$. Let us show that J is infinite almost surely. If $nx \in \mathcal{C}_M$ is such that $nx + \mathbb{Z}e_1$ has a passage time less than M we consider $K(n)$ to be the abscissa of the point in $x_1e_1 + \mathbb{N}e_1$ closest to x_1 such that:

$$\tilde{0} + K(n)e_1 \xrightarrow{M} (\tilde{0} + K(n), x_2, \tilde{0}_3, \dots, \tilde{0}_d) \xrightarrow{M} \dots \xrightarrow{M} (\tilde{0} + K(n), x_2, \dots, x_d) \xrightarrow{M} (x_1, x_2, \dots, x_d) = x.$$

We then have that $D_M(\tilde{0}, nx) \leq \|\tilde{0} - nx\|_1 + 2K(n)$. $K(n)$ follows a geometric distribution with parameter $p := p_M^{d-1}$, so we have:

$$\mathbb{P}(K(n) > \sqrt{n}) = (1 - p)^{\lfloor \sqrt{n} \rfloor}.$$

Consequently by the Borel-Cantelli lemma $K(n) \leq \sqrt{n}$ infinitely often, almost surely. It follows that J is infinite almost surely. Thus we have:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\tilde{T}_M(0, nx)}{n} &= \lim_{\substack{n \rightarrow +\infty \\ n \in J}} \frac{\tilde{T}_M(0, nx)}{n} \\ &\leq M \liminf_{\substack{n \rightarrow +\infty \\ n \in J}} \frac{D_M(\tilde{0}, nx)}{n} \\ &\leq M \liminf_{\substack{n \rightarrow +\infty \\ n \in J}} \left(\frac{\|\tilde{0} - 0\|_1}{n} + \frac{\|0 - nx\|_1}{n} \right) \\ &\leq M \|x\|_1. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\tilde{T}_M(0, nx)}{n} &\geq \lim_{\substack{n \rightarrow +\infty \\ n \in J}} a \frac{D_M(\tilde{0}, nx)}{n} \\ &\geq a \lim_{\substack{n \rightarrow +\infty \\ n \in J}} \frac{\|\tilde{0} - nx\|_1}{n} = a \|x\|_1. \end{aligned}$$

Thus, according to (6), if $M > a$ we have:

$$a \|x\|_1 \leq X_M \leq M \|x\|_1 \text{ a.s.} \quad (7)$$

□

So, we need to show that, in fact, X_M does not depend on M . To do this, let us fix $x \in \mathbb{Z}^d$. Recall that $T^*(x, y) = T(x^*, y^*)$ where x^* is the closest vertex in L^1 norm to x in \mathcal{C}_* with an arbitrary rule to break ties. According to the triangular inequality on T ,

$$|T^*(0, nx) - \tilde{T}_M(0, nx)| \leq T(0^*, \tilde{0}) + T(nx^*, \tilde{nx}).$$

Since $T(0^*, \tilde{0})$ and $T(nx^*, \tilde{nx})$ have same distribution :

$$\mathbb{P}(|T^*(0, nx) - \tilde{T}_M(0, nx)| \geq \epsilon n) \leq \mathbb{P}(T(0^*, \tilde{0}) + T(nx^*, \tilde{nx}) \geq \epsilon n) \leq 2\mathbb{P}(T(0^*, \tilde{0}) \geq \frac{\epsilon n}{2}).$$

Since 0^* and $\tilde{0}$ are both in \mathcal{C}_* , $T(0^*, \tilde{0}) < +\infty$ and the above inequality implies:

$$\forall x \in \mathbb{Z}^d, \frac{T^*(0, nx)}{n} - \frac{\tilde{T}_M(0, nx)}{n} \xrightarrow{\mathbb{P}} 0. \quad (8)$$

We can conclude that $\frac{T^*(0, nx)}{n}$ and $\frac{\tilde{T}_M(0, nx)}{n}$ have the same limit which can not depend on M since $\frac{T^*(0, nx)}{n}$ does not depend on M . By letting M tend to a in (7), we deduce that:

$$\frac{\tilde{T}_M(0, nx)}{n} \xrightarrow{n \rightarrow +\infty} a \|x\|_1 \text{ almost surely,} \quad (9)$$

which concludes the proof of Theorem 5.1. □

Proof of Theorem 1.12. Combining (8) and (9) we obtain:

$$\frac{T^*(0, nx)}{n} \xrightarrow{n \rightarrow +\infty} a \|x\|_1.$$

□

A Appendix

The asymptotic of M_n is studied in numerous references, but we have not found any reference regarding the study of $\mathbb{E}[M_n]$. For the sake of self-containedness we provide an estimate of $\mathbb{E}[M_n]$.

Lemma A.1. *Assume that the passage times are bounded. Let $(\tau_i)_{i \in \mathbb{N}}$ be i.i.d. random variables with cumulative distribution function F such that:*

$$F(t) \sim_{t \rightarrow 0^+} Ct^\beta, \quad \text{for a constant } C > 0 \text{ and } \beta > 0.$$

Then, if $M_n = \min(\tau_1, \dots, \tau_n)$ we have:

$$\mathbb{E}[M_n] \sim_{n \rightarrow +\infty} \frac{1}{n^{\frac{1}{\beta}}} \frac{1}{\beta C^{\frac{1}{\beta}}} \Gamma\left(\frac{1}{\beta}\right).$$

Proof of Lemma A.1. Let $K > 0$ be such that $\tau \leq K$ almost surely. Let us denote $X_n := M_n n^{\frac{1}{\beta}}$. We have:

$$\mathbb{E}[X_n] = \int_0^{+\infty} \mathbb{P}(X_n \geq t) dt = \int_0^{Kn^{\frac{1}{\beta}}} \left(1 - F\left(\frac{t}{n^{\frac{1}{\beta}}}\right)\right)^n dt.$$

Now, for a fixed t , we have:

$$\left(1 - F\left(\frac{t}{n^{\frac{1}{\beta}}}\right)\right)^n = \left(1 - C\frac{t^\beta}{n} + o\left(\frac{1}{n}\right)\right)^n \xrightarrow{n \rightarrow +\infty} \exp(-Ct^\beta).$$

Let $C' > 0$ such that $F(u) \geq C'u^\beta$ for $u \in [0; K]$. Using that $(1 - \frac{x}{n})^n \leq e^{-x}$, we have that for all $t \geq 0$, $\left(1 - F\left(\frac{t}{n^{\frac{1}{\beta}}}\right)\right)^n \leq \exp(-C't^\beta)$. According to the Dominated Convergence Theorem and with the changes of variables $u = Ct^\beta$ we have:

$$\mathbb{E}[X_n] \xrightarrow{n \rightarrow +\infty} \int_0^{+\infty} \exp(-Ct^\beta) dt = \Gamma\left(\frac{1}{\beta}\right) \times \frac{1}{\beta C^{\frac{1}{\beta}}}.$$

So we finally have:

$$\mathbb{E}[M_n] \sim_{n \rightarrow +\infty} \frac{1}{n^{\frac{1}{\beta}}} \frac{1}{\beta C^{\frac{1}{\beta}}} \Gamma\left(\frac{1}{\beta}\right).$$

□

Acknowledgments

I would like to warmly thank my two PhD advisors, Anne-Laure Basdevant and Lucas Gerin, first for suggesting this topic and for their support throughout the writing of this article. This work was partially supported by ANR LOUCCOUM.

References

- [ADH17] Antonio Auffinger, Michael Damron, and Jack Hanson. *50 years of first-passage percolation*, volume 68. American Mathematical Soc., 2017.

- [Boi90] Daniel Boivin. First passage percolation: the stationary case. *Probability theory and related fields*, 86(4):491–499, 1990.
- [CCD86] Jennifer Tour Chayes, Lincoln Chayes, and Richard Durrett. Critical behavior of the two-dimensional first passage time. *Journal of statistical physics*, 45:933–951, 1986.
- [CD81] John Theodore Cox and Richard Durrett. Some limit theorems for percolation processes with necessary and sufficient conditions. *The Annals of Probability*, pages 583–603, 1981.
- [CT16] Raphaël Cerf and Marie Thérét. Weak shape theorem in first passage percolation with infinite passage times. In *Annales de l’Institut Henri Poincaré-Probabilités et Statistiques*, volume 52, pages 1351–1381, 2016.
- [DCHKS18] Hugo Duminil-Copin, Marcelo Richard Hilario, Gady Kozma, and Vladas Sidoravicius. Brochette percolation. *Israel Journal of Mathematics*, 225:479–501, 2018.
- [DLW17] Michael Damron, Wai-Kit Lam, and Xuan Wang. Asymptotics for 2d critical first passage percolation. *The Annals of Probability*, 45(5):2941–2970, 2017.
- [ER59] Paul Erdős and Alfréd Rényi. On Cantor’s series with convergent $\sum \frac{1}{q_n}$. *Ann. Univ. Sci. Budapest. Eötvös. Sect. Math.*, 2:93–109, 1959.
- [GM04] Olivier Garet and Régine Marchand. Asymptotic shape for the chemical distance and first-passage percolation on the infinite bernoulli cluster. *ESAIM: Probability and Statistics*, 8:169–199, 2004.
- [HW65] John Michael Hammersley and Dominic Welsh. First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Bernoulli 1713, Bayes 1763, Laplace 1813: Anniversary Volume. Proceedings of an International Research Seminar Statistical Laboratory University of California, Berkeley 1963*, pages 61–110. Springer, 1965.
- [Kes86] Harry Kesten. Aspect of first passage percolation. *Lecture Notes in Math.*, 1180:126–263, 1986.
- [Kin73] John Frank Charles Kingman. Subadditive ergodic theory. *The Annals of Probability*, pages 883–899, 1973.
- [Lig85] Thomas Milton Liggett. An improved subadditive ergodic theorem. *The Annals of Probability*, 13(4):1279–1285, 1985.
- [Zha95] Yu Zhang. Supercritical behaviors in first-passage percolation. *Stochastic processes and their applications*, 59(2):251–266, 1995.
- [Zha99] Yu Zhang. Double behavior of critical first-passage percolation. *Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten*, pages 143–158, 1999.