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# College Admissions in France: Affirmative Action, Overlapping Reserves, and Housing Quotas

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## Abstract

This paper analyzes the current college admissions system in France, known as Parcoursup. The mechanism is based on the iterative version of College-Proposing Deferred Acceptance algorithm (CDA), which matches candidates – who may possess multiple traits – to regular educational institutions and boarding schools (i.e., institutions with private dormitories). First, I identify a flaw in Parcoursup that leads to instability, even when a single regular institution is considered. Subsequently, I propose two alternatives to Parcoursup. The first is a stable CDA (SCDA), constructed with stable choice rules satisfying substitutes condition for regular institutions and boarding schools. However, such choice rules exist only under the restricted model. Therefore, I also design a second alternative under the general model: a modified CDA (MCDA) that utilizes stable choice rules constructed under a social choice framework.

**JEL Classification:** C61, C62, C78, D47, D82

**Keywords:** Matching theory; Social choice; College admissions; Stability; Affirmative action; Housing quotas; Overlapping reserves

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# 1 Introduction

In 2018, the college admissions procedure in France, known as APB (Admission Post-Bac), was replaced by a newly designed mechanism called Parcoursup. There were three primary reasons for this reform. First, although both mechanisms use College-Proposing Deferred Acceptance (CDA), APB directly asked for candidates' strict preferences and then performed a centralized CDA, which felt more like a black box for candidates, while Parcoursup implements CDA in a step-by-step, iterative way, asking only for candidates' not ordered lists of acceptable institutions. Second, Parcoursup does not ask for candidates' preferences until all institutions submit their rankings to ensure that the former does not affect the latter, which was not the case with APB.<sup>1</sup> Third, APB randomly selected students for oversubscribed non-selective courses, while Parcoursup allows such courses to implement criteria such as motivation letters or specific academic records to better match students to available spots. Additionally, APB allowed for 24 wishes (applications) per candidate, generating too many portfolios for institutions to manually order, whereas Parcoursup reduces the maximum number of wishes per candidate to ten, with an option for sub-wishes for institutions offering similar courses to the primary wish. Regarding the transparency of information, Parcoursup provides detailed information about each course, including success rates and profiles of successful candidates, which was not as accessible under APB. This change helps students make more informed decisions about their educational paths.

Parcoursup operates as the primary online platform for managing applications to higher education in France. It consists of four subsequent phases.<sup>2</sup>

## 1. Candidates formulate their wishes (middle of January – beginning of April):

First, candidates register on the platform and provide personal details, academic records, and other relevant information such as their National Student Identifier (INE). Then, candidates indicate their **not ranked** preferences by listing them as “wishes” (*voeux*) on their profile. They can list up to ten primary wishes and multiple sub-wishes (*sous-voeux*) under each primary wish, depending on the program specifics. They are choosing from over 23,000 courses offered by various higher education institutions across France. These courses include university degree programs, preparatory classes, and vocational training.<sup>3</sup>

## 2. Institutions rank received wishes (beginning of April – end of May):

Each educational institution reviews the applications based on specific criteria they publish on the platform and submits a strict ranking over candidates who “wished” to study at this institution.

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<sup>1</sup>In the context of APB, assignment decisions in overbooked non-selective courses could take into account students' preferences. According to the [Guide du Candidat 2017](#) (page 9): “...a so-called ‘tensive’ course (i.e., one which receives more applications than it offers places) is only offered if it is classified in the first wishes of the candidate...”.

<sup>2</sup>The full up to date description of Parcoursup (in french) can be found at [https://services.dgesip.fr/T454/S764/algorithmes\\_nationales\\_de\\_parcoursup](https://services.dgesip.fr/T454/S764/algorithmes_nationales_de_parcoursup).

<sup>3</sup>For certain selective programs, students may need to submit additional documents, such as motivation letters, project descriptions, or portfolios. These help institutions assess the suitability of candidates beyond their academic scores.

3. **Institutions continuously send admission offers and candidates respond (end of May – middle of July):** After reviewing applications, institutions continuously send admission offers to candidates. Students must respond to these offers within a set timeframe (usually, two days): by either tentatively or definitely accepting one offer and declining the others (if a student received at least one offer), or exiting the procedure.<sup>4</sup>
4. **Additional admission phase (middle of June – middle of September):** If during main admission phase a student does not receive an offer or wish to reconsider their options, Parcoursup provides an additional admission phase where students can modify their wishes, add new ones, or accept pending offers. This phase aims to ensure that as many students as possible secure a placement.

Since the introduction of the classical college admissions problem by Gale and Shapley (1962) the main desideratum of the final allocation has always been its stability. In their seminal paper, the authors introduce two versions of a stable mechanism: College-Proposing and Student-Proposing Deferred Acceptance (SDA). The admission phases of Parcoursup are designed as iterative CDA, where all steps of the algorithm are performed sequentially. The brief description presented above suggests at least three causes for the instability of the final allocation: first, College-Proposing Deferred Acceptance is not strategy-proof for candidates (Balinski and Sönmez, 1999); second, number of wishes per candidate is limited; and third, the time for the iterative Deferred Acceptance is also limited.<sup>5</sup> The first cause alone may lead to an unstable allocation due to strategic behavior of candidates. The second cause alone also forces candidates to construct their wishes strategically (Beyhaghi et al., 2017) which is a difficult task for them to accomplish, thus leading to mistakes due to being over or under cautious, which in turn may result in an unstable allocation. The third cause alone simply may not allow the mechanism to finish the Deferred Acceptance procedure, therefore forcing it to produce an “under-cooked” unstable result. While each of these three features of Parcoursup contains an obvious threat to its stability, I am not addressing any of them in this paper for the following reasons.

- **Why iterative Deferred Acceptance?** As stated above, one of the core reasons why the previous mechanism APB was replaced is that it explicitly asked for candidates’ strict preferences over wishes. The only way to still perform a DA without such stage is to make it iterative: now at each stage each candidate should just choose the best offer out of all received ones. Unfortunately, sometimes one and a half month may not be enough for the iterative CDA to properly end.
- **Why College-Proposing DA?** Student-Proposing DA, first, Pareto dominates any other stable mechanism; and second, is the only mechanism that is stable and strategy-

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<sup>4</sup>In the middle of June response times to admission offers are suspended to allow high school students to focus on their written *baccalaureate exams*. Failure to pass these exams (candidates have an option to retake them) will partially disqualify candidates from further participation in Parcoursup: only vocational training admissions do not depend on the results of baccalaureate exams.

<sup>5</sup>A mechanism is strategy-proof for candidates if no candidate ever benefits by misreporting their preferences.

proof for candidates (Balinski and Sönmez, 1999).<sup>6</sup> However, it lacks a very desirable feature of CDA once we consider their iterative versions: with the iterative CDA, the “quality” of a candidate’s tentative match cannot deteriorate over time, while for the iterative SDA it is the other way around. In other words, for the iterative CDA, once a candidate is tentatively matched to an institution it is guaranteed that this candidate will end up studying in at least that institution (or maybe in even better one), while for the iterative SDA a candidate cannot be sure that they will study even somewhere until the very end of the procedure.

- **Why to limit the number of wishes per candidates?** During the two month of the second phase of Parcoursup each institution should construct its strict ranking over all received applications, that may be tens of thousands each containing creative elements (e.g., a portfolio or motivational letter). A smaller number of choices helps reduce the administrative burden on both the students and the institutions. It simplifies the decision-making process for institutions by focusing on candidates who are genuinely interested in their offerings. Without short-lists popular institutions simply will not be able to qualitatively rank all received applications within a reasonable timeframe.

The hidden cause of instability of Parcoursup that I focus on in this paper takes place right before the admission phase and is caused by the undergoing affirmative action policy. In France an institution may have a target percentage of the total quota (soft bound) for each of the two kinds of candidates: scholarship-holders (candidates who are eligible for a scholarship once admitted) and residents (candidates who live nearby this institution). Moreover, there may be candidates who are at the same time scholarship-holders and residents for some institution, i.e., these two traits may overlap. Once all institutions have submitted their quotas, target percentages, and merit lists, the mechanism reorganizes each merit list into the corresponding *call order* by boosting the best (according to the merit list) candidates with traits, such that, regardless of how many best (according to the call order) candidates are picked, the percentage of candidates possessing some trait is as close as possible to the corresponding target percentage. Proposition 1 shows that only this part may cause instability of the final allocation even if there is only one institution.

Moreover, some institutions are, in fact, boarding schools, i.e., they have a limited number of dormitory rooms to distribute among admitted candidates. Additionally, during the second phase, each boarding school constructs a second strict ranking among candidates based solely on their room eligibility. Note that this room-eligibility ranking may differ from the merit ranking and the call order of a boarding school. Parcoursup allows candidates to explicitly state in their not ranked preferences whether it is acceptable or not to be admitted to a boarding school with

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<sup>6</sup>Roth and Peranson (1999) indicate that in a scenario where participants are provided with randomly generated preference lists of limited length, only a small portion of them can gain an advantage through deceit. Within the context of the probabilistic marriage model, Immorlica and Mahdian (2005) establish that the proportion of individuals benefiting from deception tends to zero as the size of the market tends to infinity.

or without obtaining a room on campus.<sup>7</sup> However, Parcoursup integrates room distribution into the admission phase using controversial call orders of boarding schools, thus opening up possibilities for improvement. To my knowledge, this is the first paper that simultaneously addresses affirmative action and room distribution in college admissions.

Unfortunately, Proposition 6 shows that there may not be a stable matching under the general model. Therefore, this paper proposes two approaches to improving Parcoursup. The first is based on the social choice agenda and focuses on finding a(II) stable chosen subset(s) under a given institution and a proposed set of contracts. Meanwhile, the second approach strives to achieve stability of the final matching by restricting the general model.

Under the social choice approach, for a regular institution that cares about both traits I construct the choice correspondence that finds all stable subsets of a proposed set of candidates. This correspondence is based on the novel *anchor* approach that is more elegant than a “brute-force” method of Sönmez and Yenmez (2019) and suggests a way to order all stable subsets from the one with the best possible high-ranked candidates and some low-ranked candidates to the one with the highest number of middle-ranked candidates. In order to obtain the next stable subset, a high-ranked candidate with one trait and a low-ranked candidate with another trait are substituted with middle-ranked candidates: one with no traits and one with both traits (the latter is called *the anchor* of the new stable subset, since this candidate pins down this subset by being the lowest-ranked chosen candidate with both traits).

For boarding schools, Example 3 shows that to take into account room-eligibility ranking under stability, we should consider only already chosen candidates with weak demand for a room, i.e., ones who can study without a room but still desire a room. Then, under the social choice approach, I design a stable choice rule based on the binary linear programming technique that produces a specific stable subset if a boarding school cares about both traits. Furthermore, using the anchor approach, I construct the choice correspondence that yields all stable subsets for a boarding school that cares about only one trait. It again suggests a way to order all stable subsets. However, now it goes from the one with the highest number of middle-ranked candidates to the one with the best possible high-ranked candidates and some low-ranked candidates. In order to obtain the next stable subset, a middle-ranked candidate with a trait who can study without a room and another middle-ranked candidate without a trait who cannot study without a room are substituted with a higher-ranked and a lower-ranked candidates: one without a trait who can study without a room and one with a trait who cannot study without a room (the latter is called *the anchor* of the new stable subset, since this candidate pins down this subset by being the lowest-ranked chosen candidate with a trait who cannot study without a room).

Under the stable matching approach, I adopt the method of Hatfield and Milgrom (2005) and try to construct choice rules that satisfy substitutes condition for a regular institution and for a boarding school. Sönmez and Yenmez (2019) show that in order to accomplish this for a regular institution we should switch from the *one-to-all convention*, where an admitted

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<sup>7</sup>Similar to how cadets report their preferences over combinations of a branch and (an extended) service duration in ROTC branching setting of Sönmez (2013) and Sönmez and Switzer (2013).

candidate accommodates one slot for each of their traits, to the *one-to-one convention*, where an admitted candidate accommodates one slot for only one of their traits. Therefore, for a regular institution I use the designed by the authors unique stable choice rule that satisfies substitutes condition under the one-to-one convention, the horizontal envelope choice rule.

Proposition 6 shows that there may not be a stable matching if boarding schools care about at least one trait. Therefore, I propose a stable choice rule for boarding schools that satisfies substitutes condition under no affirmative action, and show that by embedding this choice rule for boarding schools together with the horizontal envelope choice rule for regular institutions into the altered version of iterative CDA we indeed obtain a stable mechanism.

The rest of the paper is organized as follows. Section 2 overviews related literature. Section 3 introduces the model without boarding schools. Section 4 describes the Parcoursup algorithm without boarding schools and shows that it is not stable even for one institution. Section 5 constructs the choice correspondence that contains all stable choice rules for a regular institution. Section 6 introduces boarding schools to the model by allowing institutions to have housing quotas and house-eligibility rankings. It shows that there may not be a stable matching if boarding schools care about traits. Then it constructs a stable choice rule under both traits with use of binary linear programming. Next, it designs the choice correspondence that contains all stable choice rules under only one trait. Section 7 presents a stable College-Proposing DA under restricted model. Section 8 designs a modified College-Proposing DA under general model that incorporates stable choice rules from Sections 5 and 6. Section 9 summarizes the main findings of the paper and proposes future avenues for research. Appendix A contains omitted proofs.

## 2 Related Literature

In broad sense, this paper contributes to studies of college admissions introduced in Gale and Shapley (1962). In particular, it considers an affirmative action agenda with soft bounds and overlapping reserves for at most two traits. Abdulkadiroglu and Sönmez (2003) propose and briefly discuss diversity concerns in context of school choice using the Deferred Acceptance mechanism. Ehlers et al. (2014) compare hard and soft bounds while assuming that each agent has exactly one trait, and propose soft bounds as more promising. Echenique and Yenmez (2015) highlight the importance of diversity-oriented choice rules that satisfy the substitutes condition as a tool in transferring general results from the theory of matching with contracts to the domain of matching under diversity goals. One of the first papers considering overlapping reserves is Goto et al. (2017). The authors allow for any number of overlapping traits and prove that there may not be a stable matching under one-to-all convention and three overlapping traits. Then they switch to one-to-one convention and develop a stable mechanism called Deferred Acceptance for Overlapping Types assuming that individuals have strict preferences on reserved seats of different types. Sönmez and Yenmez (2019) are the first to construct the unique one-to-one stable choice rule without any assumptions on individuals preferences over

types or tie-breaking rules. Moreover, the authors prove that the constructed rule satisfies the substitutes condition, which allows applying the results of matching with contracts to the case of the one-to-one convention with overlapping reserves. In addition, they consider the case of one-to-all convention with at most two overlapping types and construct the choice correspondence that contains all stable choice rules. However, to achieve this they use a “brute-force” approach, while I propose a more elegant approach that I also use as a tool for constructing stable choice rules for boarding schools under one trait. Dur and Zhang (2023) build on Sönmez and Yenmez (2019) and propose the unique choice rule that maximises cutoff score under one-to-all reserve matching with any number of overlapping types. However, they assume that the total number of the minimum requirements for all types for an institutions may not exceed the total quota of this institution, while my model does not impose such constraint. Aygün and Turhan (2020) studies affirmative action policies under one-to-one approach where students have preferences over traits. Kamada and Kojima (2024) consider matching problems with general upper-bound constraints that also hold for our model, however, they allow for wasteful solutions, while this paper does not.

Also, this article studies a fair distribution of available student housing. Abdulkadiroglu and Sönmez (1999) and Sönmez and Ünver (2008) develop and explore the well-behaved *You Request My House-I Get Your Turn* mechanism for allocating dormitory rooms to students on college campuses.<sup>8</sup> However, the procedure is assumed to start after the admissions process. Sokolov (2023) investigates student housing allocation during the admissions phase under distributional housing constraints. Afacan (2024) is able to embed the room-eligibility ranking into stable version of DA by considering candidates *without strong demand for a house*, i.e., for whom, first, both contracts with and without a room are acceptable; and second, these two contracts stand back-to-back in candidates preferences. Note that in order to use this approach we need to know preferences of candidates prior to iterative CDA, which is not allowed by Parcoursup.

This study also contributes to a growing literature analysing real-life school and college admissions in various cities and countries including (but not limited to) Boston (Abdulkadiroglu and Sönmez, 2003; Abdulkadiroglu et al., 2005b; Ergin and Sönmez, 2006; Pathak and Sönmez, 2008), New York (Abdulkadiroglu et al., 2005a), Chicago (Pathak and Sönmez, 2013), the United Kingdom (Pathak and Sönmez, 2013), Germany (Westkamp, 2013), Hungary (Biró, 2008, 2012), Turkey (Balinski and Sönmez, 1999; Yuret and Dogan, 2011), Brazil (Aygün and Bó, 2021; Bó and Hakimov, 2021), Chile (Correa et al., 2022), China (Pu, 2021), Taiwan (Dur et al., 2022), and Russia (Sokolov, 2023).

### 3 Model

There exist a finite set of candidates  $\mathcal{C}$ , a finite set of institutions  $\mathcal{F}$ , and a set of traits  $\mathcal{T} = \{b\} \cup (\cup_{f \in \mathcal{F}} \{r^f\})$ .<sup>9</sup> Each candidate  $c \in \mathcal{C}$  has a subset of traits  $\tau(c) = \tau_b(c) \cup \tau_f(c)$ , where

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<sup>8</sup>Their setting is a generalization of *housing markets* model by Shapley and Scarf (1974), and house allocation problems by Hylland and Zeckhauser (1979).

<sup>9</sup>In french an institution is *la formation*.



$\tau_b(c) \subseteq \{b\}$ , and  $\tau_f(c) \subseteq \cup_{f \in \mathcal{F}} \{r^f\}$ . If a candidate has a trait  $b$  or a trait  $r^f$ , it means that this candidate is either a scholarship-holder, or a resident near institution  $f$ , respectively.<sup>10</sup> Each institution  $f \in \mathcal{F}$  has  $q^f$  identical positions to allocate (its total quota), and two target numbers of positions  $q_b^f$  and  $q_r^f$  out of total quota  $q^f$  that are reserved for candidates with corresponding traits (soft bounds). Each target number of reserved positions is less than the total quota, however the sum of two numbers of reserved positions may exceed the total quota of an institution, i.e.,  $q_b^f \leq q^f$ ,  $q_r^f \leq q^f$ , and  $(q_b^f + q_r^f)$  may be greater than  $q^f$  for any institution  $f \in \mathcal{F}$ .

Each institution  $f \in \mathcal{F}$  has a strict ranking  $\pi^f$  over the set of all acceptable candidates  $C^f \subseteq \mathcal{C}$ . Therefore, for any  $c, \bar{c} \in C^f$ , notation  $c \pi^f \bar{c}$  means that for an institution  $f$  a candidate  $c$  has a strictly higher priority than a candidate  $\bar{c}$ . A list of candidates ranked from the most preferred to the least preferred is called a ranked order list (ROL). Moreover, according to the Law №2018-166 of March 8, 2018, each institution prefers the scholarship holders' trait  $b$  over the residents' trait  $r$ , i.e., for each institution it is more important to accommodate target positions for the trait  $b$  than for the trait  $r$ .

Each candidate  $c \in \mathcal{C}$  has a strict preference order  $\succ^c$  over the set of all institutions  $\mathcal{F}$ .

For any set of candidates  $C \subseteq \mathcal{C}$  and a trait  $t \in \mathcal{T}$ , let  $C_t$  denote the set of all candidates in  $C$  with the trait  $t$ , i.e.,  $C_t = \{c \in C | t \in \tau(c)\}$ .

A **college admission market** is  $\langle \mathcal{C}, \mathcal{F}, \mathcal{T}, \tau, \{q^f, q_b^f, q_r^f, \pi^f, C^f\}_{f \in \mathcal{F}}, \{\succ^c\}_{c \in \mathcal{C}} \rangle$ .

Note that we are in the world of the **one-to-all reserve matching**, i.e., upon admission, a candidate with multiple traits accommodates each of the reserves they qualify for. For now, we assume that there are no boarding schools among institutions, thus there is no need to care about their housing quotas and house-eligibility rankings.

### 3.1 Matching and Stability

**Definition 1.** (*Matching*). A set of pairs  $\mu \subseteq \mathcal{C} \times \mathcal{F}$  is a matching if any candidate appears in at most one pair, and any institution  $f$  appears in at most  $q^f$  pairs.

So, a matching is any allocation where any candidate is admitted to at most one institution, and any institution admits at most its total quota amount of candidates.

**Definition 2.** (*Individual rationality*). A matching  $\mu$  is individually rational if it does not contain a pair  $(c, f)$ , where  $c$  is unacceptable for  $f$ , i.e.,  $c \notin C^f$ .

A matching is individually rational if no institution admits an unacceptable candidate.

For any matching  $\mu$  and a candidate  $c \in \mathcal{C}$  denote their match under  $\mu$  by  $\mu_c \in \mathcal{F} \cup \{c\}$ , where  $\mu_c = c$  means that the candidate  $c$  is alone under  $\mu$ , i.e., there is no pair with  $c$  in  $\mu$ . Furthermore, for any matching  $\mu$  and an institution  $f \in \mathcal{F}$  denote the set of all its matches under  $\mu$  by  $\mu_f \subseteq \mathcal{C} \cup \{\emptyset\}$ , where  $\mu_f = \emptyset$  means that the institution  $f$  does not admit anyone under  $\mu$ , i.e., there is no pair with  $f$  in  $\mu$ .

<sup>10</sup>In french a scholarship holder is *la boursier*.

**Definition 3.** (*Non-wastefulness*). A matching  $\mu$  is non-wasteful if there is no institution  $f$  and a candidate  $c \in C^f$ , such that, first,  $|\mu_f| < q^f$ ; and second,  $f \succ^c \mu_c$ .

A matching is non-wasteful if no candidate prefers an empty seat of some institution to their current match.

Denote the number of unfilled reserved positions of a trait  $t \in \{b, r^f\}$  for an institution  $f$  that admits a set of candidates  $C \subseteq C^f$  by  $\Delta_f^C(t) = \max\{q_t^f - |C_t|, 0\}$ .

**Definition 4.** (*Justified envy*). A candidate  $\bar{c}$  has a justified envy towards a candidate  $c$  under a matching  $\mu$  if, first,  $\mu_c = f \in \mathcal{F}$ ; second,  $\mu_{\bar{c}} \neq f$ ; third,  $c, \bar{c} \in C^f$ ; fourth,  $f \succ^{\bar{c}} \mu_{\bar{c}}$ ; and one of the following holds, denote  $\bar{\mu} = (\mu \setminus \{(\bar{c}, \mu_{\bar{c}}), (c, f)\}) \cup (\bar{c}, f)$ ,

1.  $\Delta_{\bar{\mu}}^f(b) < \Delta_{\mu}^f(b)$ ;
2.  $\Delta_{\bar{\mu}}^f(b) = \Delta_{\mu}^f(b)$  and  $\Delta_{\bar{\mu}}^f(r) < \Delta_{\mu}^f(r)$ ;
3.  $\Delta_{\bar{\mu}}^f(b) = \Delta_{\mu}^f(b)$ ,  $\Delta_{\bar{\mu}}^f(r) = \Delta_{\mu}^f(r)$ , and  $\bar{c} \pi^f c$ .

Under a given matching, a candidate  $\bar{c}$  that prefers to study at some institution  $f$  to their current match has a justified envy towards another candidate  $c$  admitted to this institution  $f$  if, by rejecting  $c$  and admitting  $\bar{c}$ ,  $f$  either accommodates more reserve positions for the trait  $b$ ; or number of accommodated reserve positions for the trait  $b$  does not change, while  $f$  accommodates more reserve positions for the trait  $r$ ; or both numbers of accommodated reserve positions for traits  $b$  and  $r$  do not change, while  $f$  prefers candidate  $\bar{c}$  to  $c$ .

A matching **eliminates justified envy** if there is no such pair of candidates, i.e., there is no justified envy under this matching.

**Definition 5.** (*Stable matching*). A matching is stable if it is individually rational, non-wasteful, and eliminates justified envy.

Therefore, under a stable matching, no candidate-institution pair has incentives to break it.

## 3.2 Choice Rule

A choice rule  $Ch$  is a function that, for a given institution  $f \in \mathcal{F}$  and a proposed set of acceptable candidates  $C \subseteq C^f$ , produces a subset  $Ch(C) \subseteq C$ , such that  $|Ch(C)| \leq q^f$ .

I naturally extend non-wastefulness, elimination of justified envy, and stability to subsets and choice rules.

**Definition 6.** (*Subset, Choice rule: Non-wastefulness*). Fix an institution  $f \in \mathcal{F}$  and a set of candidates  $C \subseteq C^f$ . A subset  $C' \subseteq C$  is non-wasteful if  $|C'| = \min\{q^f, |C|\}$ .

A choice rule  $Ch$  is non-wasteful if, for any institution  $f \in \mathcal{F}$  and any set of candidates  $C \subseteq C^f$ , it produces a non-wasteful subset of  $C$ .

**Definition 7.** (*Subset, Choice rule: Elimination of justified envy*). Fix an institution  $f \in \mathcal{F}$  and a set of candidates  $C \subseteq C^f$ . A subset  $C' \subseteq C$  eliminates justified envy if there is no pair of candidates  $c, \bar{c} \in C$ , such that, first,  $c \in C'$ ; second,  $\bar{c} \in C \setminus C'$ ; and one of the following holds, denote  $\bar{C} = (C' \setminus c) \cup \bar{c}$ ,

1.  $\Delta_f^{\bar{C}}(b) < \Delta_f^{C'}(b)$ ;
2.  $\Delta_f^{\bar{C}}(b) = \Delta_f^{C'}(b)$  and  $\Delta_f^{\bar{C}}(r) < \Delta_f^{C'}(r)$ ;
3.  $\Delta_f^{\bar{C}}(b) = \Delta_f^{C'}(b)$ ,  $\Delta_f^{\bar{C}}(r) = \Delta_f^{Ch(C)}(r)$ , and  $\bar{c} \pi^f c$ .

A choice rule  $Ch$  eliminates justified envy if for any institution  $f \in \mathcal{F}$  and any set of candidates  $C \subseteq C^f$  it produces a subset of  $C$  that eliminates justified envy.

**Definition 8.** (*Subset, Choice rule: Stability*). Fix an institution  $f \in \mathcal{F}$  and a set of candidates  $C \subseteq C^f$ . A subset  $C' \subseteq C$  is stable if it is non-wasteful, and eliminates justified envy.

A choice rule is stable if it is non-wasteful, and eliminates justified envy.

## 4 Parcoursup Algorithm

The college admission mechanism in France consists of four phases. The following is known prior to the first phase: the set of candidates with their traits, the set of institutions, all quotas and all numbers of reserved positions, i.e.,  $\langle \mathcal{C}, \mathcal{F}, \mathcal{T}, \tau, \{q^f, q_b^f, q_r^f\}_{f \in \mathcal{F}} \rangle$  is known.

During the first phase, each candidate  $c \in \mathcal{C}$  formulates and submits to the central planner the set of their *wishes*: a set of institutions  $W^c \subseteq \mathcal{F}$  that they will apply to. There is an upper bound on the number of wishes for each candidate.

During the second phase, each institution  $f \in \mathcal{F}$  identifies a set of acceptable candidates  $C^f = \{c \in \mathcal{C} | f \in W^c\}$ , i.e., all the candidates that wished to apply to  $f$ , ranks them from the best to the worst, and submits this strict ranking  $\pi^f$  to the central planner.

At the beginning of the third phase, for each institution  $f$  the central planner produces another strict ranking  $\pi_{call}^f$  on the set of acceptable candidates  $C^f$ . Then the **iterative College-Proposing Deferred Acceptance** mechanism with a fixed number of steps is performed, where the *call orders*  $\{\pi_{call}^f\}_{f \in \mathcal{F}}$  are treated as the preferences of the institutions.

At the beginning of the fourth (complementary) phase, all candidates who have not received an admission offer, or who are not entirely satisfied with their current offer, may formulate ten new wishes for institutions with vacant places. Then, the protocol of the second and third phases is repeated, excluding candidates who have been finally admitted and the seats at institutions that have been filled.

For simplicity, for the rest of this paper, I assume that, first, there is no capacity constraint on the set of wishes during the first phase, so the sets of acceptable candidates  $\{C^f\}_{f \in \mathcal{F}}$  fully describe where candidates want to apply; and second, that the duration of the third phase is long enough to ensure that the iterative College-Proposing Deferred Acceptance mechanism does not end prematurely, thereby eliminating the need for a fourth phase. It is easy to demonstrate that both capacity constraints and a premature end to the Deferred Acceptance (even with a complementary phase) may lead to an unstable outcome. Therefore, the main focus of this paper is to highlight more fundamental features of Parcoursup that may also contribute to an unfair solution.

## 4.1 Call Order

Here is how the Parcoursup platform calculates a call order for a fixed institution at the beginning of a third phase. Fix an institution  $f \in \mathcal{F}$ , consider its quota  $q$ , its numbers of reserved positions  $q_b$  and  $q_r$ , the set of acceptable candidates  $C$  along with their traits, and the priority ranking  $\pi$  over  $C$ .<sup>11</sup>

- **Step 1:**

- If  $q_b = q_r = 0$ , then set  $\pi_{call} = \pi$  and finish the procedure, so the resulting *call order list* (COL)  $L_{call}$  is exactly the ROL of the institution.
- If  $q_b > 0$  and  $q_r = 0$ , then pick the best candidate from  $C$  with the trait  $b$ , or if no such candidate exists, pick the best candidate from  $C$ , let it be  $c_1$ , and set  $L_{call} = \{c_1\}$ .
- If  $q_b = 0$  and  $q_r > 0$ , then pick the best candidate from  $C$  with the trait  $r$ , or if no such candidate exists, pick the best candidate from  $C$ , let it be  $c_1$ , and set  $L_{call} = \{c_1\}$ .
- If  $q_b > 0$  and  $q_r > 0$ , then pick the best candidate from  $C$  with both traits, or if no such candidate exists, pick the best candidate from  $C$  with the trait  $b$ , or if no such candidate exists, pick the best candidate from  $C$  with the trait  $r$ , or if no such candidate exists, pick the best candidate from  $C$ , let it be  $c_1$ , and set  $L_{call} = \{c_1\}$ .

- **Step  $k$  ( $2 \leq k \leq |C|$ ):**

- If  $\frac{|(L_{call})_b|}{k} \geq \frac{q_b}{q}$  and  $\frac{|(L_{call})_r|}{k} \geq \frac{q_r}{q}$ , then pick the best candidate from  $C \setminus L_{call}$ , let it be  $c_k$ , and add him to the bottom of  $L_{call}$ , i.e.,  $L_{call} = \{c_1, \dots, c_k\}$ .
- If  $\frac{|(L_{call})_b|}{k} < \frac{q_b}{q}$  and  $\frac{|(L_{call})_r|}{k} \geq \frac{q_r}{q}$ , then pick the best candidate from  $C \setminus L_{call}$  with the trait  $b$ , or if no such candidate exists, pick the best candidate from  $C \setminus L_{call}$ , let it be  $c_k$ , and add him to the bottom of  $L_{call}$ , i.e.,  $L_{call} = \{c_1, \dots, c_k\}$ .
- If  $\frac{|(L_{call})_b|}{k} \geq \frac{q_b}{q}$  and  $\frac{|(L_{call})_r|}{k} < \frac{q_r}{q}$ , then pick the best candidate from  $C \setminus L_{call}$  with the trait  $r$ , or if no such candidate exists, pick the best candidate from  $C \setminus L_{call}$ , let it be  $c_k$ , and add him to the bottom of  $L_{call}$ , i.e.,  $L_{call} = \{c_1, \dots, c_k\}$ .
- If  $\frac{|(L_{call})_b|}{k} < \frac{q_b}{q}$  and  $\frac{|(L_{call})_r|}{k} < \frac{q_r}{q}$ , then pick the best candidate from  $C \setminus L_{call}$  with both traits, or if no such candidate exists, pick the best candidate from  $C \setminus L_{call}$  with the trait  $b$ , or if no such candidate exists, pick the best candidate from  $C \setminus L_{call}$  with the trait  $r$ , or if no such candidate exists, pick the best candidate from  $C \setminus L_{call}$ , let it be  $c_k$ , and add him to the bottom of  $L_{call}$ , i.e.,  $L_{call} = \{c_1, \dots, c_k\}$ .

As a result, we obtain a COL  $L_{call}$  that gives us a strict call order  $\pi_{call}$  on  $C$  for  $f$ . We denote it by  $L_{call}(\pi, C)$ .

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<sup>11</sup>I omit superscripts  $f$  for clarity.

**Example 1.** Suppose that we have an institution  $f$  with quota  $q = 3$ , numbers of reserved positions  $q_b = q_r = 1$ , and the set  $C$  of four acceptable candidates with the following traits (from the best to the worst according to  $\pi$ )

$c_1$	$c_2$	$c_3$	$c_4$
$b$			$b$
	$r$		$r$

- **Step 1:** Since  $q_b = 1 > 0$  and  $q_r = 1 > 0$ , then we pick the best candidate with both traits, i.e.,  $L_{call} = \{c_4\}$ .
- **Step 2:** Since  $\frac{|(L_{call})_b|}{2} = \frac{1}{2} \geq \frac{1}{3} = \frac{q_b}{q}$  and  $\frac{|(L_{call})_r|}{2} = \frac{1}{2} \geq \frac{1}{3} = \frac{q_r}{q}$ , then we pick the best candidate from  $C \setminus L_{call}$ , i.e.,  $L_{call} = \{c_4, c_1\}$ .
- **Step 3:** Since  $\frac{|(L_{call})_b|}{3} = \frac{2}{3} \geq \frac{1}{3} = \frac{q_b}{q}$  and  $\frac{|(L_{call})_r|}{3} = \frac{1}{3} \geq \frac{1}{3} = \frac{q_r}{q}$ , then we pick the best candidate from  $C \setminus L_{call}$ , i.e.,  $L_{call} = \{c_4, c_1, c_2\}$ .
- **Step 4:** We pick the only candidate from  $C \setminus L_{call}$ , i.e.,  $L_{call} = \{c_4, c_1, c_2, c_3\}$ .

As a result,  $L_{call}(\pi, C) = \{c_4, c_1, c_2, c_3\}$ . Therefore, the COL with corresponding traits for  $f$  is (from the first to the last according to obtained  $\pi_{call}$ )

$c_4$	$c_1$	$c_2$	$c_3$
$b$	$b$		
$r$		$r$	

△

## 4.2 Iterative College-Proposing Deferred Acceptance

Before the start of the procedure the following is known: the set of candidates with their traits, the set of institutions with their strict rankings, all quotas, all numbers of reserved positions, and all sets of acceptable candidates, i.e.,  $\langle \mathcal{C}, \mathcal{F}, \mathcal{T}, \tau, \{q^f, q_b^f, q_r^f, \pi^f, C^f\}_{f \in \mathcal{F}} \rangle$  is known.

- **Step 0:** Call order  $\pi_{call}^f$  is calculated for each institution  $f \in \mathcal{F}$ .
- **Step 1:** Each institution  $f \in \mathcal{F}$  proposes to the first  $\min\{q^f, |C^f|\}$  candidates according to the call order  $\pi_{call}^f$ .

Each candidate  $c \in \mathcal{C}$  receives a set of offers. If this set is non-empty, they pick one offer that they tentatively or definitely accept and reject all the others.

All candidates that definitely accepted an offer leave the market together with occupied seats and accommodated reserved positions.

If there were no rejections, the procedure is terminated and all tentative acceptances become final assignments.

- **Step  $k$  ( $k \geq 2$ ):** Each institution  $f \in \mathcal{F}$  proposes to the next bunch of candidates according to the call order  $\pi_{call}^f$  who have not rejected it before in order to exactly fill the rest of the quota  $q^f$ .

Each candidate  $c \in \mathcal{C}$  receives a set of new offers and combines it with previously tentatively accepted one (if any). If the resulting set is not empty, they pick one offer that they tentatively or definitely accept and reject all the others.

All candidates that definitely accepted an offer leave the market together with occupied seats and accommodated reserved positions.

If there were no rejections, the procedure is terminated and all tentative acceptances become final assignments.

Since both the sets of candidates and institutions are finite, the described mechanism is certain to complete in a finite number of steps. Furthermore, it possesses the following advantageous feature: once a candidate accepts an offer, it is guaranteed that their final match will be at least as preferable for him as the currently accepted offer. I believe that this exact property led policymakers to choose the College-Proposing version of Deferred Acceptance instead of the Student-Proposing version (where the quality of tentatively matched institutions decay over time), and, as a direct consequence, to certainly sacrifice the strategy-proofness of the entire mechanism.

A matching mechanism is a function that for any college admissions market produces a matching. A mechanism is stable if it always produces a stable matching. Unfortunately, the described above mechanism may not produce a stable outcome even if there is only one institution.

**Proposition 1.** *Parcoursup mechanism is not stable even if  $|\mathcal{F}| = 1$ .*

**Proof.** Consider Example 1. Since there is only one institution, the iterative College-Proposing Deferred Acceptance mechanism stops after Step 1 and produces the following matching  $\mu = \{(c_4, f), (c_1, f), (c_2, f)\}$ . Note that the only unmatched candidate  $c_3$  has justified envy towards matched  $c_4$ , since, first,  $c_1$  and  $c_2$  together accommodate all reserved positions of  $f$ ; and second,  $c_3 \pi^f c_4$ . Thus, the resulting  $\mu$  is not stable.  $\square$

This result makes it impossible to fix the Parcoursup mechanism by recalculating the call list at the beginning of each step, i.e., getting rid of Step 0 and embedding the following choice rule. Consider an institution  $f$  and a proposed set of candidates  $C \subseteq C^f$ . Parcoursup choice rule  $Ch^{PCS}$  picks the bunch of first candidates from  $C$  according to  $\pi_{call}$  up to the quota.

**Corollary 1.**  *$Ch^{PCS}$  is not stable.*

Furthermore, Parcoursup mechanism fails to be stable if there are at least two institutions and only one trait.

**Proposition 2.** *Parcoursup mechanism is not stable even if  $|\mathcal{F}| = 2$  and  $|\mathcal{T}| = 1$ .*

**Proof.** Consider the following market with two institutions  $f_1$  and  $f_2$  with quotas  $q^1 = 4$  and  $q^2 = 2$ , and target numbers  $q_b^1 = 2$  and  $q_b^2 = 0$ ; and seven candidates, where only  $c_1, c_3$  and  $c_7$  have trait  $b$ . Preferences of institutions are  $\pi^1 = \pi^2 = c_1 \pi c_2 \dots c_6 \pi c_7$ .

Preferences of candidates are as follows

$$\begin{aligned} c_1, c_3, c_5, c_6, c_7 &: f_1 \succ f_2 \\ c_2, c_4 &: f_2 \succ f_1 \end{aligned}$$

Call order for the first institution is  $\pi_{call} = c_1 \pi_{call} c_2 \pi_{call} c_3 \pi_{call} c_4 \pi_{call} c_7 \pi_{call} c_5 \pi_{call} c_6$ . Thus, the Parcoursup mechanism will result in the following matching:  $\{c_1, c_3, c_5, c_7\}$  are matched to  $f_1$ ,  $\{c_2, c_4\}$  are matched to  $f_2$ , while  $c_6$  is unmatched. Note that  $c_6$  has a justified envy towards  $c_7$ , since the target number of  $f_1$  is fully satisfied by  $c_1$  and  $c_3$ . So, the resulting matching is not stable.  $\square$

Therefore, we need to use an approach different from the call list construction.

## 5 Stable Choice Rules under Overlapping Reserves

In this section, I solve a social choice problem and construct the stable choice correspondence that contains all stable choice rules, and also present a natural ordering on the set of all stable choice rules.

### 5.1 No Overlapping Reserves

Suppose that each candidate has at most than one trait, i.e., we are in an economy without overlapping reserves. Consider the following choice rule  $Ch^{NOI}$ .

Fix an institution  $f$  and a proposed set of candidates  $C \subseteq C^f$ .

- **Step 1:** Set  $Ch = \{\}$ . Pick the best  $\min\{q_b, |C_b|\}$  candidates with the trait  $b$  from  $C$  and add them to  $Ch$ . Set  $\bar{q} = q - \min\{q_b, |C_b|\}$ .
- **Step 2:** Pick the best  $\min\{\bar{q}, q_r, |C_r|\}$  candidates with the trait  $r$  from  $C$  and add them to  $Ch$ . Set  $\bar{q} = \bar{q} - \min\{\bar{q}, q_r, |C_r|\}$ .
- **Step 3:** Pick the best  $\bar{q}$  candidates from  $C \setminus Ch$  and add them to  $Ch$ . Set  $Ch^{NOI}(C) = Ch$ .

**Proposition 3.** *In the economy without overlapping reserves the choice rule is stable if, and only if, it is  $Ch^{NOI}$ .*

Thus, there is the unique choice rule if a candidate may have at most one trait.

## 5.2 With Overlapping Reserves

In this section we allow for candidates with two traits. Consider an institution  $f$  and a proposed set of candidates  $C \subseteq C^f$ . Suppose that the set  $C$  has exactly  $m$  candidates with two traits, i.e.,  $|(C_b)_r| = m$ . We say that there are exactly  $(\min\{q, m\} + 1)$  **anchors**  $A(C) = (\cup_{k=1}^{\min\{q, m\}} a_k) \cup \{\emptyset\}$ , where  $a_k \in (C_b)_r$  for  $k \in \{1, \dots, \min\{q, m\}\}$  denotes the  $k$ -th best candidate from  $(C_b)_r$  according to  $\pi$ , and  $a_0$  denotes an empty anchor  $\emptyset$ . I assume that  $|A(C)| > 1$ , i.e., there is at least one candidate with two traits. For any candidate  $c \in C$  denote its lower contour set by  $lc(C, c) = \{c' \in C | c \pi c'\} \cup \{c\}$ .

We use an anchor in order to pin down a choice set with the following function  $Ch^{An}(\cdot | \cdot)$ .<sup>12</sup> Fix an anchor  $a_k$ , where  $k \in \{0, 1, \dots, \min\{q, m\}\}$ .

- **Step 1:** Set  $Ch = \{\}$  if  $k = 0$ , or  $Ch = \cup_{j=1}^k a_j$  if  $k > 0$ . Then set  $C' = C \setminus Ch$ . If  $k < \min\{q, m\}$ , set  $C'' = C' \setminus lc(C, a_{k+1})$ .
- **Step 2:** Calculate  $Ch^{NOI} = Ch^{NOI}(C')$  under quota  $q' = q - k$  and numbers of reserved positions  $q'_b = \max\{0, q_b - k\}$ ,  $q'_r = \max\{0, q_r - k\}$ . Set  $Ch^{An}(C | a_k) = Ch^{NOI} \cup Ch$ .

For any subset of anchors  $A' \subseteq A(C)$  denote by  $\mathbf{Ch}^{An}(C | A')$  the following set of subsets of  $C$ :  $\mathbf{Ch}^{An}(C | A') = \cup_{a \in A'} \{Ch^{An}(C | a)\}$ . Therefore  $\mathbf{Ch}^{An}(\cdot | \cdot)$  is a correspondence that for a pair of a subset  $C \subseteq C^f$  and a subset of anchors  $A \subseteq A(C)$  produces a set of subsets of  $C$  (one per each anchor).

**Proposition 4.** *Fix an institution  $f$  and a proposed set  $C \subseteq C^f$ . There exists two numbers  $k_{left} \leq k_{right}$  both from  $\{0, 1, \dots, \min\{q, m\}\}$ , such that the correspondence  $\mathbf{Ch}^{An}(C | \cup_{k=k_{left}}^{k_{right}} \{a_k\})$  produces exactly all stable subsets of  $C$ .*

**Proof.** I assume that, first,  $|C| > q$  (otherwise we just pick the whole set  $C$ ); and second,  $q_b > 0$ ,  $q_r > 0$ , there is at least one candidate with a trait  $b$ , at least one candidate with a trait  $r$ , and at least one candidate with both traits (otherwise we just pick  $Ch^{NOI}(C)$ ).

**Lemma 1.** *If  $Ch^{An}(C | a_k)$  is not stable subset, then there is no stable subsets among subsets, where  $a_k$  is the worst chosen anchor.*

**Lemma 2.** *Consider two subsequent anchors  $a_k$  and  $a_{k+1}$  from  $A(C)$ . If  $\Delta^{Ch^{An}(C | a_k)}(b) > 0$ , then, first,  $Ch^{An}(C | a_k)$  is not a stable subset of  $C$ ; and second,  $\Delta^{Ch^{An}(C | a_k)}(b) > \Delta^{Ch^{An}(C | a_{k+1})}(b) \geq 0$ .*

**Lemma 3.** *Consider two subsequent anchors  $a_k$  and  $a_{k+1}$  from  $A(C)$ . If  $\Delta^{Ch^{An}(C | a_k)}(b) = 0$  and  $\Delta^{Ch^{An}(C | a_k)}(r) > 0$ , then, first,  $Ch^{An}(C | a_k)$  is not a stable subset of  $C$ ; and second,*

- $\Delta^{Ch^{An}(C | a_{k+1})}(b) = 0$ , and
- $\Delta^{Ch^{An}(C | a_k)}(r) > \Delta^{Ch^{An}(C | a_{k+1})}(r) \geq 0$ .

<sup>12</sup>Note that this is not a choice rule since it may produce more than one choice set for the same proposed set  $C$  by considering different anchors.



**Lemma 4.** Consider two subsequent anchors  $a_k$  and  $a_{k+1}$  from  $A(C)$ . If  $\Delta^{Ch^{An}(C|a_k)}(b) = \Delta^{Ch^{An}(C|a_k)}(r) = 0$  and  $|Ch^{An}(C|a_k)| < q$ , then, first,  $Ch^{An}(C|a_k)$  is not a stable subset of  $C$ ; and second,

- $\Delta^{Ch^{An}(C|a_{k+1})}(b) = \Delta^{Ch^{An}(C|a_{k+1})}(r) = 0$ , and
- $|Ch^{An}(C|a_k)| < |Ch^{An}(C|a_{k+1})| \leq q$ .

**Lemma 5.** Consider two subsequent anchors  $a_k$  and  $a_{k+1}$  from  $A(C)$ . If one of the following holds:

- either  $\Delta^{Ch^{An}(C|a_k)}(b) > 0$ ,
- or  $\Delta^{Ch^{An}(C|a_k)}(b) = 0$  and  $\Delta^{Ch^{An}(C|a_k)}(r) > 0$ ,
- or  $\Delta^{Ch^{An}(C|a_k)}(b) = \Delta^{Ch^{An}(C|a_k)}(r) = 0$  and  $|Ch^{An}(C|a_k)| < q$ ,

and also holds  $\Delta^{Ch^{An}(C|a_{k+1})}(b) = \Delta^{Ch^{An}(C|a_{k+1})}(r) = 0$  and  $|Ch^{An}(C|a_{k+1})| = q$ , then  $Ch^{An}(C|a_{k+1})$  is a stable subset of  $C$ .

**Lemma 6.** Consider the last anchor  $a_{\min\{q,m\}}$ . If one of the following holds:

- either  $\Delta^{Ch^{An}(C|a_{\min\{q,m\}})}(b) > 0$ ,
- or  $\Delta^{Ch^{An}(C|a_{\min\{q,m\}})}(b) = 0$  and  $\Delta^{Ch^{An}(C|a_{\min\{q,m\}})}(r) > 0$ ,
- or  $\Delta^{Ch^{An}(C|a_{\min\{q,m\}})}(b) = \Delta^{Ch^{An}(C|a_{\min\{q,m\}})}(r) = 0$  and  $|Ch^{An}(C|a_{\min\{q,m\}})| < q$ ,

then  $Ch^{An}(C|a_{\min\{q,m\}})$  is the only stable subset of  $C$ .

**Lemma 7.** Consider two subsequent anchors  $a_k$  and  $a_{k+1}$  from  $A(C)$ . If  $\Delta^{Ch^{An}(C|a_k)}(b) = \Delta^{Ch^{An}(C|a_k)}(r) = 0$  and  $|Ch^{An}(C|a_k)| = q$ , and  $Ch^{An}(C|a_k)$  is a stable subset of  $C$ , then  $Ch^{An}(C|a_{k+1})$  may be both: stable and not stable subset of  $C$ .

For  $Ch^{An}(C|a_k)$  pick the best not chosen candidate  $c_{best}^\times$  from  $C$ , the worst chosen candidate with only trait  $b$ ,  $c_{worst}^{\checkmark,b}$ , and the worst chosen candidate with only trait  $r$ ,  $c_{worst}^{\checkmark,r}$ .

If all three such candidates exist, and  $c_{best}^\times \pi c_{worst}^{\checkmark,b}$  and  $c_{best}^\times \pi c_{worst}^{\checkmark,r}$ , then  $Ch^{An}(C|a_{k+1})$  is a stable subset of  $C$ . Otherwise,  $Ch^{An}(C|a_{k+1})$  is not a stable subset of  $C$ .

**Lemma 8.** Consider not the last anchor  $a_k$  from  $A(C)$ . If  $\Delta^{Ch^{An}(C|a_k)}(b) = \Delta^{Ch^{An}(C|a_k)}(r) = 0$  and  $|Ch^{An}(C|a_k)| = q$ , and  $Ch^{An}(C|a_k)$  is not a stable subset of  $C$ , then  $Ch^{An}(C|a_j)$  is also not a stable subset of  $C$  for any  $j \geq k + 1$ .

**Lemma 9.** If  $\Delta^{Ch^{An}(C|a_0)}(b) = \Delta^{Ch^{An}(C|a_0)}(r) = 0$  and  $|Ch^{An}(C|a_0)| = q$ , then  $Ch^{An}(C|a_0)$  is a stable subset of  $C$ .

Combining all the results above, here is the algorithm that calculates  $k_{left}$  and  $k_{right}$ .

- **Step 1:** If one of the following holds:

- either  $\Delta^{Ch^{An}(C|a_{\min\{q,m\}})}(b) > 0$ ,

- or  $\Delta^{Ch^{An}(C|_{a_{\min\{q,m\}}})}(b) = 0$  and  $\Delta^{Ch^{An}(C|_{a_{\min\{q,m\}}})}(r) > 0$ ,
- or  $\Delta^{Ch^{An}(C|_{a_{\min\{q,m\}}})}(b) = \Delta^{Ch^{An}(C|_{a_{\min\{q,m\}}})}(r) = 0$  and  $|Ch^{An}(C|_{a_{\min\{q,m\}}})| < q$ ,

then  $k_{left} = k_{right} = \min\{q, m\}$  by Lemma 6. Otherwise, go to the Step 2.

- **Step 2:** Find the lowest  $k$ , such that  $\Delta^{Ch^{An}(C|_{a_k})}(b) = \Delta^{Ch^{An}(C|_{a_k})}(r) = 0$  and  $|Ch^{An}(C|_{a_k})| = q$ . By Lemmas 2-5 and 9 set  $k_{left} = k$ .
- **Step 3:** Find the highest  $k \geq k_{left}$ , such that for  $Ch^{An}(C|_{a_k})$ :  $c_{best}^\times \pi c_{worst}^{\checkmark,b}$  and  $c_{best}^\times \pi c_{worst}^{\checkmark,r}$ . If there is no such  $k$ , set  $k_{right} = k_{left}$ , otherwise, set  $k_{right} = k + 1$ , by Lemmas 7 and 8.

Also, by Lemma 1 we considered all potential candidates for stable subsets. This concludes the proof of the proposition.  $\square$

A choice correspondence  $\mathbf{Ch}(\cdot)$  is a correspondence that for a given institution  $f \in \mathcal{F}$  and a set of candidates  $C \subseteq C^f$  produces a set of subsets  $\mathbf{Ch}(C) \subseteq 2^C$ , such that each subset has at most  $q$  candidates.

Consider the following choice correspondence  $\mathbf{Ch}^{Ol}$ . Fix an institution  $f$  and a proposed set  $C \subseteq C^f$ .

- **Step 1:** Calculate  $k_{left}$  and  $k_{right}$ .
- **Step 2:** Set  $\mathbf{Ch}^{Ol}(C) = \mathbf{Ch}^{An}(C|_{\cup_{k=k_{left}}^{k_{right}} a_k})$ .

**Theorem 1.** *A choice rule is stable if, and only if, it is a selection from the choice correspondence  $\mathbf{Ch}^{Ol}$ .*<sup>13</sup>

Next result describes how stable sets change when we move from the first anchor to the last.

**Proposition 5.** *Fix an institution  $f$  and a proposed set  $C \subseteq C^f$ . Consider two subsequent anchors  $a_k$  and  $a_{k+1}$  from  $A(C)$ , such that  $k_{left} \leq k < k + 1 \leq k_{right}$ .*

*For  $Ch^{An}(C|_{a_k})$  pick the best not chosen candidate  $c_{best}^\times$  from  $C$ , the worst chosen candidate  $c_{worst}^{\checkmark}$ , the worst chosen candidate with only trait  $b$ ,  $c_{worst}^{\checkmark,b}$ , and the worst chosen candidate with only trait  $r$ ,  $c_{worst}^{\checkmark,r}$ .*

*For  $Ch^{An}(C|_{a_{k+1}})$  pick the best not chosen candidate  $\bar{c}_{best}^\times$  from  $C$  and the worst chosen candidate  $\bar{c}_{worst}^{\checkmark}$ .*

*The following holds:*

- *first,  $c_{worst}^{\checkmark} \pi^f \bar{c}_{worst}^{\checkmark}$ ;*
- *second,  $c_{best}^\times \pi^f \bar{c}_{best}^\times$ ;*
- *third,  $Ch^{An}(C|_{a_{k+1}}) = (Ch^{An}(C|_{a_k}) \setminus \{c_{worst}^{\checkmark,b}, c_{worst}^{\checkmark,r}\}) \cup \{c_{best}^\times, a_{k+1}\}$ ;*

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<sup>13</sup>Sönmez and Yenmez (2019) propose a similar solution to the problem without preferences over traits. However, my approach is simpler and more intuitive, as it is not based on a “brute-force” case-by-case analysis.

- fourth, if  $k_{left} < k_{right}$ , then  $q$  is exhausted for  $Ch^{An}(C|a_k)$  and  $Ch^{An}(C|a_{k+1})$ , and  $q_b - |(Ch^{An}(C|a_k))_b| = q_r - |(Ch^{An}(C|a_k))_r| = 0$  for any  $k \in \{k_{left}, \dots, k_{right}\}$ .

Thus, by considering a lower-ranked anchor for producing a stable subset, we pick more very high ranked candidates at the cost of also picking more very low ranked candidates by reducing amount of picked candidates with middle ranks.<sup>14</sup>

Example 2 illustrates the design of the stable choice correspondence  $\mathbf{Ch}^{Ol}$ .

**Example 2.** Consider an institution  $f$  with quota  $q = 5$ , numbers of reserved positions  $q_b = 4$ ,  $q_r = 3$ , and the proposed set  $C$  of twelve acceptable candidates with the following traits (from the best to the worst according to  $\pi$ )

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$	$c_9$	$c_{10}$	$c_{11}$	$c_{12}$
		$b$		$b$	$b$		$b$		$b$		$b$
	$r$	$r$			$r$	$r$			$r$	$r$	$r$

There are five anchors (non-empty ones are framed by vertical lines):  $a_0 = \emptyset$ ,  $a_1 = c_3$ ,  $a_2 = c_6$ ,  $a_3 = c_{10}$ , and  $a_4 = c_{12}$ .

We consider all anchors one by one, starting with the empty anchor  $a_0$ :

- $Ch^{An}(C|a_0) = Ch^{NOI}(\{c_1, c_2\}) = \{c_1, c_2\}$  – not a stable subset, since it is wasteful;
- $Ch^{An}(C|a_1) = \{c_3\} \cup Ch^{NOI}(\{c_1, c_2, c_4, c_5\}) = \{c_1, \dots, c_5\}$  – not a stable subset, since  $c_6$  has a justified envy towards  $c_1$ ;
- $Ch^{An}(C|a_2) = \{c_3, c_6\} \cup Ch^{NOI}(\{c_1, c_2, c_4, c_5, c_7, c_8, c_9\}) = \{c_2, c_3, c_5, c_6, c_8\}$  – stable subset;
- $Ch^{An}(C|a_3) = \{c_3, c_6, c_{10}\} \cup Ch^{NOI}(\{c_1, c_2, c_4, c_5, c_7, c_8, c_9, c_{11}\}) = \{c_1, c_3, c_5, c_6, c_{10}\}$  – stable subset;
- $Ch^{An}(C|a_4) = \{c_3, c_6, c_{10}, c_{12}\} \cup Ch^{NOI}(\{c_1, c_2, c_4, c_5, c_7, c_8, c_9, c_{11}\}) = \{c_1, c_3, c_6, c_{10}, c_{12}\}$  – not a stable subset, since  $c_5$  has a justified envy towards  $c_{12}$ .

So,  $\mathbf{Ch}^{Ol}(C) = \{Ch^{An}(C|a_2), Ch^{An}(C|a_3)\}$ .

Therefore, first, there are two stable subsets of  $C$ ; and second, stable subset  $Ch^{An}(C|a_2)$  does not contain  $c_1$  and does not contain anyone worse than  $c_8$ , while another stable subset  $Ch^{An}(C|a_3)$  contains  $c_1$  and also contains  $c_{10}$ .  $\triangle$

## 6 Housing Constraints

Suppose now that there are boarding schools in the economy, i.e., there are institutions with limited capacity for housing. Thus, each institution  $f \in \mathcal{F}$  also has a housing quota

<sup>14</sup> $Ch^{An}(C|a_{k_{left}})$  and  $Ch^{An}(C|a_{k_{right}})$  correspond to  $C^{\max \min}$  and  $C^{\min \max}$  from Sönmez and Yenmez (2019) respectively.

$0 \leq q_H^f \leq q^f$ .<sup>1516</sup> A candidate may be admitted to an institution with a housing ( $H$ ) or without a housing ( $NH$ ). Moreover, some candidates may consider it unacceptable to study at a boarding school without (or with) housing.<sup>17</sup>

Each candidate  $c \in \mathcal{C}$  has a strict preference order  $\succ^c$  over the set  $(\mathcal{F} \times \{H, NH\}) \cup \{c\}$ , such that, if this candidate is acceptable for some institution  $f$ , i.e.,  $c \in C^f$ , then there is a pair  $(f, h) \succ^c c$  for some housing need  $h \in \{H, NH\}$ . A pair  $(f, h)$  is acceptable for  $c$  if  $c$  prefers this pair to being not admitted, i.e.,  $(f, h) \succ^c c$ . A pair  $(f, h)$  is unacceptable for  $c$  if  $c$  prefers to be not admitted to this pair, i.e.,  $c \succ^c (f, h)$ . I also naturally assume that if for some institution  $f$  and a candidate  $c \in C^f$  both pairs  $(f, H)$  and  $(f, NH)$  are acceptable for  $c$ , then  $(f, H) \succ^c (f, NH)$ . Denote by  $P^f$  the set of all acceptable pairs for  $f$ ,  $P^f = \{(c, h) \in C^f \times \{H, NH\} \mid (f, h) \succ^c c\}$ .

Thus, now we are considering contracts (not a pairs of a candidate and an institution as before), where a set of contracts is  $\Xi = \mathcal{C} \times \mathcal{F} \times \{H, NH\}$ , and a contract  $\xi = (c, f, H)$  means that a candidate  $c$  is admitted to an institution  $f$  and has a housing, while a contract  $\xi = (c, f, NH)$  means that a candidate  $c$  is admitted to an institution  $f$  and does not have a housing. For any contract  $\xi = (c, f, h) \in \Xi$  denote  $\xi_C = c$ ,  $\xi_F = f$ , and  $\xi_D = h \in \{H, NH\}$ .

Given a fixed boarding school  $f$ , we call a candidate  $c$  either a **weak-demand-for-a-house-type (WDH)** candidate if  $(f, H) \succ^c (f, NH) \succ^c c$ , or a **strong-demand-for-a-house-type (SDH)** candidate if  $(f, H) \succ^c c \succ^c (f, NH)$ , or a **no-demand-for-a-house-type (NDH)** candidate if  $(f, NH) \succ^c c \succ^c (f, H)$ .

Moreover, together with a merit ranking  $\pi^f$ , each boarding school  $f$  also submits a house-eligibility strict ranking  $\pi_H^f$  on  $C^f$ . Therefore, if for two candidates  $\{c, c'\} \subseteq C^f$  we have  $c \pi_H^f c'$ , that means that  $c$  is more eligible to receive a house from  $f$  than  $c'$ . Note that these two strict rankings  $\pi^f$  and  $\pi_H^f$  may differ.

A **college admission market** is  $\langle \mathcal{C}, \mathcal{F}, \mathcal{T}, \tau, \{q^f, q_b^f, q_r^f, q_H^f, \pi^f, \pi_H^f, C^f\}_{f \in \mathcal{F}}, \{\succ^c\}_{c \in \mathcal{C}} \rangle$ .

## 6.1 Matching and Stability

We included boarding schools into the economy, thus we need to redefine stability properties of a matching, so it deals with housing allocation constraints and house-eligibility rankings.

**Definition 9.** (*Matching*). A set of contracts  $\mu \subseteq \Xi$  is a matching if any candidate appears in at most one contract, and for any institution  $f$  the following holds:

- $f$  appears in at most  $q^f$  contracts, and
- there are at most  $q_H^f$  contracts from the set  $\mathcal{C} \times \{f\} \times \{H\}$ .

<sup>15</sup>For a boarding school  $0 < q_H^f$ , and for a regular institution  $q_H^f = 0$ .

<sup>16</sup>In reality, some boarding schools may have a common dormitory, which leads to one housing quota among several institutions. However, as noted by Sokolov (2023), such feature causes non-existence of stable matchings even if we disregard all traits. Therefore, in this paper I assume that each boarding school has its own set of houses.

<sup>17</sup>For regular institutions (with  $q_H = 0$ ) we admit all candidates without a house ( $NH$ ).

Now, for an allocation to be a matching, we also require that each boarding school distribute no more houses than its housing quota.

**Definition 10.** (*Individual rationality*). A matching  $\mu$  is individually rational if it does not contain a contract  $(c, f, h)$ , such that, either  $c$  is unacceptable for  $f$ , i.e.,  $c \notin C^f$ , or  $(f, h)$  is unacceptable for  $c$ , i.e.,  $c \succ^c (f, h)$ .

A matching is individually rational if no institution is matched to an unacceptable pair.

For any matching  $\mu$  and a candidate  $c \in \mathcal{C}$  denote their match under  $\mu$  by  $\mu_c \in (\mathcal{F} \times \{H, NH\}) \cup \{c\}$ , where  $\mu_c = c$  means that the candidate  $c$  is alone under  $\mu$ , i.e., there is no contract with  $c$  in  $\mu$ . Furthermore, for any matching  $\mu$  and an institution  $f \in \mathcal{F}$  denote the set of all its matches under  $\mu$  by  $\mu_f \subseteq (\mathcal{C} \times \{H, NH\}) \cup \{\emptyset\}$ , where  $\mu_f = \emptyset$  means that the institution  $f$  does not admit anyone under  $\mu$ , i.e., there is no contract with  $f$  in  $\mu$ . Denote the subset of pairs with a housing need of a set  $P \subseteq \mathcal{C} \times \{H, NH\}$  by  $(P)_H$ .

Denote the set of candidates matched to an institution  $f$  under a matching  $\mu$  by  $(\mu_f)_C$ .

**Definition 11.** (*Non-wastefulness*). A matching  $\mu$  is non-wasteful if there is no institution  $f$  and a candidate  $c \in C^f$ , such that,  $|\mu_f| < q^f$ ,  $c \notin (\mu_f)_C$ , and there exists a pair  $(f, h) \succ^c \mu_c$ , such that  $\mu' = (\mu \setminus (c, \mu_c)) \cup (c, f, h)$  is a matching.

We say that a matching is non-wasteful if there is no not admitted to some institution candidate who would like to and can be admitted to this institution with some acceptable contract.

**Definition 12.** (*Justified envy*). A candidate  $\bar{c}$  has a justified envy towards a candidate  $c$  under a matching  $\mu$  if, first,  $\mu_c = (f, h) \in \mathcal{F} \times \{H, NH\}$ ; second,  $\mu_{\bar{c}}$  does not contain  $f$ ; third,  $c, \bar{c} \in C^f$ ; fourth,  $(f, \bar{h}) \succ^{\bar{c}} \mu_{\bar{c}}$  for some  $\bar{h} \in \{H, NH\}$ ; fifth,  $\bar{\mu} = (\mu \setminus \{(\bar{c}, \mu_{\bar{c}}), (c, f, h)\}) \cup (\bar{c}, f, \bar{h})$  is a matching; and one of the following holds

1.  $\Delta_f^{(\bar{\mu}_f)c}(b) < \Delta_f^{(\mu_f)c}(b)$ ;
2.  $\Delta_f^{(\bar{\mu}_f)c}(b) = \Delta_f^{(\mu_f)c}(b)$  and  $\Delta_f^{(\bar{\mu}_f)c}(r) < \Delta_f^{(\mu_f)c}(r)$ ;
3.  $\Delta_f^{(\bar{\mu}_f)c}(b) = \Delta_f^{(\mu_f)c}(b)$ ,  $\Delta_f^{(\bar{\mu}_f)c}(r) = \Delta_f^{(\mu_f)c}(r)$ , and
  - (a) either  $\bar{h} = NH$  and  $\bar{c} \pi^f c$ ,
  - (b) or  $\bar{h} = H$ ,  $h = NH$ ,  $\bar{c} \succ^{\bar{c}} (f, NH)$ , and  $\bar{c} \pi^f c$ ,
  - (c) or  $\bar{h} = h = H$ ,  $\bar{c} \succ^{\bar{c}} (f, NH)$ , and  $\bar{c} \pi^f c$ .

Example 3 shows that we cannot use  $\pi_H^f$  instead of  $\pi^f$  in point 3c of Definition 12.

**Example 3.** Consider an institution with quota  $q = 1$ , housing quota  $q_H = 1$ , and a proposed set with three candidates ordered according to preferences  $\pi: c_1^2 \pi c_2 \pi c_3^1$  and  $\pi_H: c_3^1 \pi_H c_1^2 \pi_H c_2$  with the following housing needs

$$\begin{array}{ccc} c_1^2 & c_2 & c_3^1 \\ \hline H & & H \\ & NH & \end{array}$$

There may be four matchings and none of them is non-wasteful and eliminates justified envy if we use  $\pi_H^f$  instead of  $\pi^f$  in point 3c of Definition 12.:

1.  $\{\}$  is wasteful;
2.  $\{(c_3^1, H)\}$  has a justified envy: is blocked by  $(c_2, NH)$ ;
3.  $\{(c_2, NH)\}$  has a justified envy: is blocked by  $(c_1^2, H)$ ;
4.  $\{(c_1^2, H)\}$  has a justified envy: is blocked by  $(c_3^1, H)$ .

△

Therefore, justified envy under presence of houses is a direct extension of itself from Definition 4 (when we considered only regular institutions).

**Proposition 6.** *There may not exist a non-wasteful and individually rational matching that eliminates justified envy even if there is only one trait and no WDH candidates.*

We prove it with the following example.

**Example 4.** Consider two institutions  $f_1$  and  $f_2$  with quotas  $q^1 = q^2 = 2$ , housing quotas  $q_H^1 = q_H^2 = 1$ , and target numbers for the trait  $b$ :  $q_b^1 = 1$  and  $q_b^2 = 0$ ; and three candidates ordered according to preferences  $\pi^1$ :  $c_1 \pi^1 c_2 \pi^1 c_3$  and  $\pi^2$ :  $c_1 \pi^2 c_2$  ( $c_3 \notin C^2$ ) with the following housing needs

For  $f_1$  :  $q = 2, q_b = q_H = 1$ ,

For  $f_2$  :  $q = 2, q_H = 1$ ,

$$\begin{array}{ccc} c_1 & c_2 & c_3 \\ \hline H & & H \\ & NH & \\ & b & b \end{array}$$

$$\begin{array}{cc} c_1 & c_2 \\ \hline H & H \end{array}$$

Preferences of candidates are as follows

$$\begin{aligned} c_1 &: f_1 \succ f_2 \\ c_2 &: f_2 \succ f_1 \\ c_3 &: f_1 \end{aligned}$$

There are only two non-wasteful and individually rational matchings:

1. If  $c_3 \in f_1 \Rightarrow c_1 \notin f_1 \Rightarrow c_1 \in f_2 \Rightarrow c_2 \in f_1 \Rightarrow$  matching is  $\{(c_1, f_2, H), (c_2, f_1, NH), (c_3, f_1, H)\}$  –  $c_1$  justifiably envies  $c_3$ ;

2. If  $c_3 \notin f_1 \Rightarrow c_1 \in f_1 \Rightarrow c_2 \in f_2 \Rightarrow$  matching is  $\{(c_1, f_1, H), (c_2, f_2, H)\}$  –  $c_3$  justifiably envies  $c_1$ .

Thus, there is no non-wasteful and individually rational matching that eliminates justified envy.  $\triangle$

Proposition 6 suggests that we need to restrict the model to obtain existence of a stable matching. However, we can solve for all stable subsets under social choice perspective for only one boarding school.

## 6.2 Choice Rule

A choice rule  $Ch(\cdot)$  is a function that for a given institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$  produces a subset of pairs  $Ch(P) \subseteq P$ , such that

- first, for each candidate  $c \in C^f$  there is at most one pair in  $Ch(P)$  containing  $c$ ;
- second,  $|Ch(P)| \leq q^f$ ;
- third,  $|(Ch(P))_H| \leq q_H^f$ .

Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . We call a subset of  $P$  a *proper subset* if it can be chosen from  $P$  by a choice rule.

**Definition 13.** (*Subset, Choice rule: Non-wastefulness*). Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . A proper subset  $P' \subseteq P$  is non-wasteful if there is no pair  $(c, h) \in P \setminus P'$ , such that  $P'' = P' \cup (c, h)$  is also proper.

A choice rule  $Ch$  is non-wasteful if, for any institution  $f \in \mathcal{F}$  and any set of pairs  $P \subseteq P^f$ , it produces a non-wasteful subset of  $P$ .

**Definition 14.** (*Subset, Choice rule: Justified envy*). Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . Under a proper subset  $P' \subseteq P$  a candidate  $\bar{c}$  has a justified envy towards a different candidate  $c$  through pairs  $(\bar{c}, \bar{h}) \in P \setminus P'$  and  $(c, h) \in P'$  if, first, subset  $\bar{P} = (P' \setminus (c, h)) \cup (\bar{c}, \bar{h})$  is proper, and, second, one of the following holds

1.  $\Delta_f^{(\bar{P})c}(b) < \Delta_f^{(P')c}(b)$ ;
2.  $\Delta_f^{(\bar{P})c}(b) = \Delta_f^{(P')c}(b)$  and  $\Delta_f^{(\bar{P})c}(r) < \Delta_f^{(P')c}(r)$ ;
3.  $\Delta_f^{(\bar{P})c}(b) = \Delta_f^{(P')c}(b)$ ,  $\Delta_f^{(\bar{P})c}(r) = \Delta_f^{(P')c}(r)$ ,  $\bar{c} \pi^f c$ , and
  - (a) either  $\bar{h} = NH$ ,
  - (b) or  $\bar{h} = H$  and  $(\bar{c}, NH) \notin P$ .

A choice rule  $Ch$  eliminates justified envy if, for any institution  $f \in \mathcal{F}$  and any set of pairs  $P \subseteq P^f$ , it produces a subset of  $P$  that eliminates justified envy.

However, under presence of WDH candidates (with two proposed pairs in  $P$ ) it may be suboptimal to award them with houses before all other candidates have been considered.

**Definition 15.** (*Subset, Choice rule: Non-wastefulness via house*). Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . A proper subset  $P' \subseteq P$  is non-wasteful via house if  $|P'| < q^f$ ,  $|P'_H| = q_H^f$ , and there is no pair of candidates  $c$  and  $\bar{c}$ , such that,  $c$  has no pair in  $P'$ ,  $(\bar{c}, H) \in P'$ , and  $\{(c, H), (\bar{c}, NH)\} \in P \setminus P'$ .

A choice rule  $Ch$  is non-wasteful via house if, for any institution  $f \in \mathcal{F}$  and any set of pairs  $P \subseteq P^f$ , it produces a non-wasteful via house subset of  $P$ .

In words, a proper subset is wasteful via house if the boarding school with an empty seat and no empty houses has given a house to a candidate who can study without it, while some not admitted candidate wants to study at this institution and cannot do so without a house.

**Example 5.** Consider an institution  $f$  with quota  $q = 2$ , housing quota  $q_H = 1$  numbers of reserved positions  $q_b = 1$  and  $q_r = 0$ , and a proposed set with three pairs containing two candidates ordered according to preferences  $\pi: c_1 \pi c_2$  with the following traits and housing needs

$c_1$	$c_2$
$H$	$H$
	$NH$
	$b$

Proper subset  $\{(c_2, H)\}$  eliminates justified envy and is non-wasteful, but is wasteful via house, since we can admit both candidates by taking a house from  $c_2$  and giving it to  $c_1$ . Therefore, there is only one proper subset that satisfies all three properties:  $\{(c_1, H), (c_2, NH)\}$ .  $\triangle$

**Definition 16.** (*Subset, Choice rule: House-justified envy*). Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . Under a proper subset  $P' \subseteq P$  a candidate  $\bar{c}$  has a house-justified envy towards a candidate  $c$  through pairs  $(\bar{c}, H) \in P \setminus P'$  and  $(c, H) \in P'$  if,  $(\bar{c}, NH) \in P'$ ,  $(c, NH) \in P \setminus P'$ , and  $\bar{c} \pi_H^f c$ .

A choice rule  $Ch$  eliminates house-justified envy if, for any institution  $f \in \mathcal{F}$  and any set of pairs  $P \subseteq P^f$ , it produces a subset of  $P$  that eliminates house-justified envy.

In words, if there are two admitted candidates with two proposed pairs each: one is admitted with a house and another – without, then the former should have higher house-eligibility rank. Otherwise, there is a house-justified envy.

**Definition 17.** (*Subset, Choice rule: House-seat-justified envy*). Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . Under a proper subset  $P' \subseteq P$  a candidate  $\bar{c}$  has a house-seat-justified envy towards a pair of candidates  $c$  and  $c'$  through pairs  $(\bar{c}, H) \in P \setminus P'$ ,  $(c, H) \in P'$ , and  $(c', NH) \in P'$  if,



1. quota  $q$  and housing quota  $q_H$  are completely filled, i.e.,  $|P'| = q^f$  and  $|P'_H| = q_H$ ,
2.  $c \pi \bar{c}$ ,
3.  $(\bar{c}, NH) \notin P$ ,
4.  $(c, NH) \in P$ , and
5. set  $\bar{P} = (P' \setminus \{(c, H), (c', NH)\}) \cup \{(\bar{c}, H), (c, NH)\}$ :
  - (a) either  $\Delta_f^{(\bar{P})c}(b) < \Delta_f^{(P')c}(b)$ ,
  - (b) or  $\Delta_f^{(\bar{P})c}(b) = \Delta_f^{(P')c}(b)$  and  $\Delta_f^{(\bar{P})c}(r) < \Delta_f^{(P')c}(r)$ ,
  - (c) or  $\Delta_f^{(\bar{P})c}(b) = \Delta_f^{(P')c}(b)$  and  $\Delta_f^{(\bar{P})c}(r) = \Delta_f^{(P')c}(r)$ , and  $\bar{c} \pi c'$ .

A choice rule  $Ch$  eliminates house-seat-justified envy if, for any institution  $f \in \mathcal{F}$  and any set of pairs  $P \subseteq P^f$ , it produces a subset of  $P$  that eliminates house-seat-justified envy.

**Example 6.** Consider an institution  $f$  with quota  $q = 2$ , housing quota  $q_H = 1$  numbers of reserved positions  $q_b = q_r = 0$ , and a proposed set with four pairs containing three candidates ordered according to preferences  $\pi: c_1 \pi c_2 \pi c_3$  with the following housing needs

$c_1$	$c_2$	$c_3$
$H$	$H$	
$NH$		$NH$

Proper subset  $\{(c_1, H), (c_3, NH)\}$  eliminates justified envy and house-justified envy, is non-wasteful and non-wasteful via house, but does not eliminate house-seat-justified envy, since we can admit  $c_2$  by taking house from  $c_1$  and a seat from  $c_3$ . Thus, there is only one proper subset that satisfies all five properties:  $\{(c_1, NH), (c_2, H)\}$ .  $\triangle$

**Example 7.** Consider an institution  $f$  with quota  $q = 2$ , housing quota  $q_H = 1$  numbers of reserved positions  $q_b = 2$ , and a proposed set with four pairs containing three candidates ordered according to preferences  $\pi: c_1 \pi c_2 \pi c_3$  with the following traits and housing needs

$c_1$	$c_2$	$c_3$
	$H$	$H$
$NH$	$NH$	
	$b$	$b$

It is straightforward to verify that a proper subset  $\{(c_1, NH), (c_2, H)\}$  eliminates justified envy and house-justified envy, is non-wasteful and non-wasteful via house, but does not eliminate house-seat-justified envy. While, another proper subset  $\{(c_2, NH), (c_3, H)\}$  satisfies all five properties and accommodates one more reserve position for  $b$ .  $\triangle$

**Definition 18.** (*Subset, Choice rule: Stability*). Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . A proper subset  $P' \subseteq P$  is stable if it is non-wasteful, non-wasteful via house, and eliminates justified envy, house-justified envy, and house-seat-justified envy.

A choice rule  $Ch$  is stable if, for any institution  $f \in \mathcal{F}$  and any set of pairs  $P \subseteq P^f$ , it produces a stable subset of  $P$ .

Stable proper subset maximally accommodates the total quota seats along with reserve positions for both traits, while controlling that if a candidate who can study without a house gets a house, then all more eligible for housing accepted candidates who can study without a house, and want a house, also get one.

**Definition 19.** (*Subset, Choice rule: Non-house-wastefulness*). Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . A proper subset  $P' \subseteq P$  is non-house-wasteful if either  $|P'_H| = q_H^f$ , or  $|P'_H| < q_H^f$ , and there is no candidate  $c$ , such that,  $(c, NH) \in P'$  and  $(c, H) \in P \setminus P'$ .

Non-house-wastefulness forces a boarding school to maximally distribute its houses across admitted candidates.

**Definition 20.** (*Subset, Choice rule: Strong stability*). Fix an institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$ . A proper subset  $P' \subseteq P$  is strongly stable if it is stable and non-house-wasteful.

A choice rule  $Ch$  is strongly stable if, for any institution  $f \in \mathcal{F}$  and any set of pairs  $P \subseteq P^f$ , it produces a strongly stable subset of  $P$ .

Therefore, for strong stability we distribute empty houses as much as possible among admitted candidates.

Under stability and strong stability, if we consider two candidates  $c, c'$ , such that  $c \pi^f c'$  and  $c' \pi_H^f c$ , we should care about  $\pi_H^f$  if, and only if, both candidates are matched to  $f$ , can study without a house, want a house (both are WDH candidates for  $f$ ), and we decide whom to give a house.

Note that, in absence of houses, stability from Definition 18 and strong stability from Definition 20 both become stability from Definition 8.

Also note that, under presence of houses there are two resources that can be obtained by a candidate in the context of one institution: a seat (from the quota  $q$ ) and a house  $H$  (from the quota  $q_H$ ). Moreover, a candidate may obtain each of these resources either from an institution  $f$ , or from some other candidate  $c$ . Table 1 shows that strong stability covers all possible cases under presence of houses, where the second column indicates the source of a seat and the third column indicates the source of a house for a candidate.

Table 1: Strong stability properties.

	Resource		Property	Definition
	Seat	$H$		
Source	$f$	$f$	non-wastefulness	11
	$f$	$c$	non-wastefulness via house	15
	$c$	$f$	justified envy	12
	$c$	$c$	justified envy	12
	$c$	$c'$	house-seat-justified envy	17
	$c$	-	justified envy	12
	$f$	-	non-wastefulness	11
	-	$c$	house-justified envy	16
	-	$f$	non-house-wastefulness	19

The notion includes the word *wastefulness* if either a seat should be taken from an institution, or no seat should be taken and a house should be taken from an institution. While, the notion includes words *justified envy* if either a seat should be taken from a candidate, or no seat should be taken and a house should be taken from a candidate.

### 6.3 Binary Linear Programming: Strongly Stable Choice

In this section I construct a strongly stable choice rule with use of binary linear programming techniques. Such a solution produces only one strongly stable subset, while there may be others.

**Proposition 7.** *There is no unique strongly stable subset even if we allow for only one trait.*

**Example 8.** Consider an institution with quota  $q = 2$ , housing quota  $q_H = 1$ , numbers of reserved positions  $q_b = 1$  and  $q_r = 0$ , proposed set with four candidates ordered according to preferences  $\pi: c_1 \pi c_2 \pi c_3 \pi c_4$  with the following traits and housing needs

$c_1$	$c_2$	$c_3$	$c_4$
$H$		$H$	
	$NH$		$NH$
		$b$	$b$

Note that both subsets  $\{(c_1, H), (c_4, NH)\}$  and  $\{(c_2, NH), (c_3, H)\}$  are strongly stable.  $\triangle$

Thus, there may be many strongly stable subsets for a given proposed set. Moreover, since a strongly stable subset is also stable, there may be many stable subsets for a given proposed set.

Now, consider a fixed institution  $f$  with quota  $q$ , housing quota  $q_H$  and numbers of reserved positions for both traits  $q_b$  and  $q_r$ , together with a proposed set of candidates  $C = \{c_1, c_2, \dots, c_{|C|}\} \subseteq C^f$  with corresponding traits and proposed pairs  $P$ . For each candidate  $c_i \in C$  denote by  $i$  their rank in  $\pi$ .

For each candidate  $c$  construct a 3-by-1 binary vector  $A_c = (I_H, I_b, I_r)'$ , where  $I_t$  for  $t \in \{b, r\}$  is equal to one if  $c$  possesses a corresponding trait  $t$ , and is equal to zero otherwise; and  $I_H = 1$  if  $(f, H) \in P$  and  $(f, NH) \notin P$ , and  $I_H = 0$  otherwise. Combine all  $|C|$  column-vectors into a 3-by- $|C|$  matrix  $A = [A_{c_1}, A_{c_2}, \dots, A_{c_{|C|}}]$ . Denote all three rows of  $A$  by  $A = [A^H, A^b, A^r]'$ .

We have two binary vector decision variables:

1. a  $|C|$ -by-1 vector  $x$ , where  $x_k = 1$  if a candidate  $c_k$  is chosen, and  $x_k = 0$  otherwise;
2. a  $|C|$ -by-1 vector  $h$ , where  $h_k = 1$  if a candidate  $c_k$  is given a house, and  $h_k = 0$  otherwise.

First, the total quota should be satisfied:

$$\sum_{i=1}^{|C|} x_i \leq q \quad (1)$$

Second, the housing quota should be satisfied:

$$\sum_{i=1}^{|C|} h_i \leq q_H \quad (2)$$

Third, we need to make sure that we give houses only to chosen candidates:

$$h_i \leq x_i, \quad i = 1, \dots, |C| \quad (3)$$

Fourth, we need to make sure that we give houses only to candidates who cannot study without a house:

$$h_i \leq A_i^H, \quad i = 1, \dots, |C| \quad (4)$$

Set  $\alpha = \frac{(|C|+1)|C|}{2}$ . The optimization problem is

$$\begin{aligned} \min_{x, h} \quad & \max\{0; q_b - A^b \cdot x\} \cdot 2^{|C|}(\alpha + 1)(q + 1)(q_r + 1) \\ & + \max\{0; q_r - A^r \cdot x\} \cdot 2^{|C|}(\alpha + 1)(q + 1) \\ & + \left( q - \sum_{i=1}^{|C|} x_i \right) \cdot 2^{|C|}(\alpha + 1) \\ & + \left( \sum_{i=1}^{|C|} i \cdot x_i \right) \cdot 2^{|C|} \\ & + \sum_{i=1}^{|C|} 2^{|C|-i}(1 - x_i) \\ \text{s.t.} \quad & (1)-(4). \end{aligned} \quad (5)$$

Minimization of the objective function from (5) makes sure that

- primarily we accommodate maximum reserves for the trait  $b$ , since it contains the following term

$$\max\{0; q_b - A^b \cdot x\} \cdot 2^{|\mathcal{C}|}(\alpha + 1)(q + 1)(q_r + 1) \quad (6)$$

and the following inequality holds

$$\begin{aligned} & \max\{0; q_r - A^r \cdot x\} \cdot 2^{|\mathcal{C}|}(\alpha + 1)(q + 1) \\ & + \left( q - \sum_{i=1}^{|\mathcal{C}|} x_i \right) \cdot 2^{|\mathcal{C}|}(\alpha + 1) \\ & + \left( \sum_{i=1}^{|\mathcal{C}|} i \cdot x_i \right) \cdot 2^{|\mathcal{C}|} \\ & + \sum_{i=1}^{|\mathcal{C}|} 2^{|\mathcal{C}|-i}(1 - x_i) < \\ & q_r \cdot 2^{|\mathcal{C}|}(\alpha + 1)(q + 1) \\ & + q \cdot 2^{|\mathcal{C}|}(\alpha + 1) \\ & + \alpha \cdot 2^{|\mathcal{C}|} \\ & + 2^{|\mathcal{C}|} = 2^{|\mathcal{C}|}(\alpha + 1)(q + 1)(q_r + 1) \end{aligned} \quad (7)$$

- then we accommodate maximum reserves for the trait  $r$ , since it contains the following term

$$\max\{0; q_r - A^r \cdot x\} \cdot 2^{|\mathcal{C}|}(\alpha + 1)(q + 1) \quad (8)$$

and the following inequality holds

$$\begin{aligned} & \left( q - \sum_{i=1}^{|\mathcal{C}|} x_i \right) \cdot 2^{|\mathcal{C}|}(\alpha + 1) \\ & + \left( \sum_{i=1}^{|\mathcal{C}|} i \cdot x_i \right) \cdot 2^{|\mathcal{C}|} \\ & + \sum_{i=1}^{|\mathcal{C}|} 2^{|\mathcal{C}|-i}(1 - x_i) < \\ & q \cdot 2^{|\mathcal{C}|}(\alpha + 1) \\ & + \alpha \cdot 2^{|\mathcal{C}|} \\ & + 2^{|\mathcal{C}|} = 2^{|\mathcal{C}|}(\alpha + 1)(q + 1) \end{aligned} \quad (9)$$

- then we accommodate maximum seats from the total quota, since it contains the following term

$$\left( q - \sum_{i=1}^{|C|} x_i \right) \cdot 2^{|\mathcal{C}|} (\alpha + 1) \quad (10)$$

and the following inequality holds

$$\begin{aligned} & \left( \sum_{i=1}^{|C|} i \cdot x_i \right) \cdot 2^{|\mathcal{C}|} \\ & + \sum_{i=1}^{|C|} 2^{|\mathcal{C}|-i} (1 - x_i) < \\ & \alpha \cdot 2^{|\mathcal{C}|} \\ & + 2^{|\mathcal{C}|} = 2^{|\mathcal{C}|} (\alpha + 1) \end{aligned} \quad (11)$$

- then, we choose the best possible candidates, since it contains the following term

$$\left( \sum_{i=1}^{|C|} i \cdot x_i \right) \cdot 2^{|\mathcal{C}|} \quad (12)$$

and the following inequality holds

$$\sum_{i=1}^{|C|} 2^{|\mathcal{C}|-i} (1 - x_i) < 2^{|\mathcal{C}|} \quad (13)$$

- finally, we pin down a specific subset by going one-by-one from the best candidate to the worst and trying to pick exactly him, since for any  $k = 1, \dots, |C|$  it contains the following term

$$2^{|\mathcal{C}|-k} (1 - x_k) \quad (14)$$

and the following inequality holds if  $k < |C|$

$$\sum_{i=k+1}^{|C|} 2^{|\mathcal{C}|-i} (1 - x_i) < 2^{|\mathcal{C}|-k} \quad (15)$$

However, in order to use solution techniques for binary linear optimization problems we also need to get rid of  $\max\{\cdot, \cdot\}$  parts of the objective function.

Note that we can rewrite the function  $\max\{a, b\}$ , where there is an upper bound  $M$  for  $a$  and  $b$ , i.e.,  $a \leq M$  and  $b \leq M$ , as follows

$$\begin{aligned}
& X \\
\text{s.t. } & a \leq X \\
& b \leq X \\
& X \leq a + M(1 - y) \\
& X \leq b + My
\end{aligned} \tag{16}$$

where  $y$  is a binary variable, i.e.,  $y \in \{0, 1\}$ .

Note that both numbers of reserved positions  $q_b$  and  $q_r$  have an upper bound  $q$ . Set two binary variables  $y_b$  and  $y_r$ . Therefore, the optimization problem (5) can be rewritten as follows

$$\begin{aligned}
\min_{x, h, y_b, y_r} & \quad \delta_b \cdot 2^{|\mathcal{C}|} (\alpha + 1) (q + 1) (q_r + 1) \\
& + \delta_r \cdot 2^{|\mathcal{C}|} (\alpha + 1) (q + 1) \\
& + \left( q - \sum_{i=1}^{|\mathcal{C}|} x_i \right) \cdot 2^{|\mathcal{C}|} (\alpha + 1) \\
& + \left( \sum_{i=1}^{|\mathcal{C}|} i \cdot x_i \right) \cdot 2^{|\mathcal{C}|} \\
& + \sum_{i=1}^{|\mathcal{C}|} 2^{|\mathcal{C}|-i} (1 - x_i) \\
\text{s.t. } & \quad (1)-(4) \\
& 0 \leq \delta_b \\
& q_b - A^b \cdot x \leq \delta_b \\
& \delta_b \leq q(1 - y_b) \\
& \delta_b \leq (q_b - A^b \cdot x) + q \cdot y_b \\
& 0 \leq \delta_r \\
& q_r - A^r \cdot x \leq \delta_r \\
& \delta_r \leq q(1 - y_r) \\
& \delta_r \leq (q_r - A^r \cdot x) + q \cdot y_r
\end{aligned} \tag{17}$$

Note that, first, all variables  $\{x, h, y_b, y_r\}$  are binary; and second, the objective function and all constraints are linear with respect to all variables. Thus, (17) is indeed a binary linear problem.

Denote by  $Ch^{Opt,1}$  a choice rule that picks a solution of (17). We can use  $Ch^{Opt,1}$  to construct the following choice rule  $Ch^{Opt}$ .

Fix a proposed set  $P \subseteq P^f$ .

- **Step 1:** Solve (17) and take the resulting set  $Ch^{Opt,1}(P) = P' \subseteq P$ .
- **Step 2 (houses distribution):** If all  $q_H$  is exhausted, then set  $Ch^{Opt}(P) = P'$ . Oth-

erwise, give the rest housing slots to the most eligible for houses chosen candidates who desires a house but does not have it, name the new set of pairs  $P''$ . Set  $Ch^{Opt}(P) = P''$ .

**Proposition 8.**  *$Ch^{Opt}$  is strongly stable.*

Note that, by Proposition 7, there is no unique strongly stable choice rule if we care about traits. Also note, that  $Ch^{Opt}$  always gives the same output for the same inputs, i.e., it is a function.

## 6.4 (Strongly) Stable Choice Rules under One Trait

The obvious drawback of the binary linear approach is that it yields only one strongly stable subset. However, there is a way to partially fix it and find all stable subsets by neglecting trait  $r$ , since it is less important, and focusing on the other trait  $b$ . This section constructs the stable choice correspondence for such setting via the anchor approach from Section 5.

Fix a boarding school  $f$ . Suppose that any candidate has at most one pair in  $P^f$ . Therefore, any pair  $(c, h) \in P^f$ , first, contains a unique candidate; and second, can be one of four types:

- **b-pair:**  $b \in \tau(c)$  and  $h = NH$ ;
- **H-pair:**  $b \notin \tau(c)$  and  $h = H$ ;
- **NH-pair:**  $b \notin \tau(c)$  and  $h = NH$ ;
- **bH-pair:**  $b \in \tau(c)$  and  $h = H$ .

**Proposition 9.** *Fix a boarding school  $f$ . Suppose that any candidate has at most one pair in  $P^f$ . Consider a proposed set  $P \subseteq P^f$ .*

*If  $P' \subseteq P$  is stable subset and bH-pair is chosen, then all b-pairs and bH-pairs with higher ranked candidates are chosen.*

*If  $P' \subseteq P$  is stable subset and H-pair is chosen, then all pairs with higher ranked candidates are chosen.*

*If  $P' \subseteq P$  is stable subset and NH-pair is chosen, then all b-pairs and NH-pairs with higher ranked candidates are chosen.*

*If  $P' \subseteq P$  is stable subset and b-pair is chosen, then all b-pairs with higher ranked candidates are chosen.*

### 6.4.1 No bH-pairs

Suppose that there is no bH-pair in  $P^f$ . Consider the following choice rule  $Ch^{bH-NOI}$ . Fix a proposed set  $P \in P^f$ .

- **Step 1:** Pick  $q' = \min\{q_b, |((P)_c)_b|\}$  b-pairs with the best candidates. Update the quota  $q = q - q'$ .
- **Step 2.k** ( $1 \leq k \leq |\mathbf{P}|$ ): If  $q = 0$ , end the procedure. Otherwise, do the following.



- If  $q_H > 0$ , then consider a not considered before pair  $(c, h)$  with the best candidate. If there is such pair,
  - \* if  $h = NH$ , pick this pair and update  $q = q - 1$ ;
  - \* if  $h = H$ , pick this pair and update  $q = q - 1$  and  $q_H = q_H - 1$ .
 If there is no such pair, end the procedure.
- If  $q_H = 0$ , then consider a not considered before pair  $(c, NH)$  with the best candidate. If there is such pair, pick it and update  $q = q - 1$ . If there is no such pair, end the procedure.

**Proposition 10.** *Consider an economy with one trait where, first, each candidate has at most one pair for each boarding school; and second, there is no bH-pair. A choice rule is stable if, and only if, it is  $Ch^{bH-NOI}$ .*

By analogy with regular institutions and no overlapping reserves, there is the unique choice rule if a candidate may have at most one pair for each boarding school.

#### 6.4.2 With bH-pairs

In this section we allow for bH-pairs. Fix a boarding school  $f$ . Suppose that any candidate has at most one pair in  $P^f$ . Also suppose that the proposed set  $P \subseteq P^f$  has exactly  $m$  bH-pairs. We will say that there are exactly  $(\min\{q_H, m\} + 1)$  **anchors**  $A(P) = (\cup_{k=1}^{\min\{q_H, m\}} a_k) \cup \{\emptyset\}$ , where  $a_k$  for  $k \in \{1, \dots, \min\{q_H, m\}\}$  denotes  $k$ -th best bH-pair, and  $a_0$  denotes an empty anchor  $\emptyset$ . I assume that  $|A(P)| > 1$ .

We use an anchor in order to pin down a choice set with the following function  $Ch^{bAn}(\cdot|\cdot)$ .<sup>18</sup> Fix an anchor  $a_k$ , where  $k \in \{0, 1, \dots, \min\{q_H, m\}\}$ .

- **Step 1:** Set  $Ch = \{\}$  if  $k = 0$ , or  $Ch = \cup_{j=1}^k a_j$  if  $k > 0$ . Then set  $P' = P \setminus \cup_{j=1}^{\min\{q_H, m\}} a_j$ .
- **Step 2:** Calculate  $Ch^{bH-NOI} = Ch^{bH-NOI}(P')$  under quota  $q' = q - k$ , housing quota  $q'_H = q_H - k$ , and number of reserved positions  $q'_b = \max\{0, q_b - k\}$ . Set  $Ch^{bAn}(P|a_k) = Ch^{bH-NOI} \cup Ch$ .

For any subset of anchors  $A' \subseteq A(P)$  denote by  $\mathbf{Ch}^{bAn}(P|A')$  the following set of subsets of  $P$ :  $\mathbf{Ch}^{bAn}(P|A') = \cup_{a \in A'} \{Ch^{bAn}(P|a)\}$ . Thus  $\mathbf{Ch}^{bAn}(\cdot|\cdot)$  is a correspondence that for a pair of a subset  $P \subseteq P^f$  and a subset of anchors  $A \subseteq A(P)$  produces a set of subsets of  $P$  (one per each anchor).

**Proposition 11.** *Fix a boarding school  $f$ . Suppose that any candidate possesses at most one trait  $b$ , and has at most one pair in  $P^f$ . Consider a proposed set  $P \subseteq P^f$ , such that  $|A(P)| > 1$ .*

*There exists two numbers  $k_{left} \leq k_{right}$  both from  $\{0, 1, \dots, \min\{q_H, m\}\}$ , such that the correspondence  $\mathbf{Ch}^{bAn}(P|\cup_{k=k_{left}}^{k_{right}} \{a_k\})$  produces exactly all stable subsets of  $P$ .*

<sup>18</sup>Note that this is not a choice rule since it may produce more than one choice set for the same proposed set  $P$  by considering different anchors.

**Proof.** I assume that, first,  $|(P)_H| > q_H > 0$  (otherwise we just neglect housing constraint and return to the case from Section 5); and second,  $q_b > 0$  (case with  $q_b = 0$  is considered in Section 7).

**Lemma 10.** *If  $Ch^{bAn}(P|a_k)$  is not stable subset, then there is no stable subsets among subsets, where  $a_k$  is the worst chosen anchor.*

**Lemma 11.** *Consider a set  $Ch^{bAn}(P|a_k)$  for some  $k < \min\{q_H, m\}$ . When should we switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset (by understanding that  $Ch^{bAn}(P|a_k)$  is not stable)? Each of three quotas  $q$ ,  $q_H$ , and  $q_b$  can be either exhausted (e) or not exhausted (ne) under  $Ch^{bAn}(P|a_k)$ . Therefore, there can be eight cases.*

1. If  $q$ : ne,  $q_H$ : ne,  $q_b$ : ne, then we should switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset.
2. If  $q$ : ne,  $q_H$ : ne,  $q_b$ : e, then we should switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset.
3. If  $q$ : ne,  $q_H$ : e,  $q_b$ : ne, then we should switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset if, and only if, there is a chosen  $H$ -pair in  $Ch^{bAn}(P|a_k)$ .
4. If  $q$ : ne,  $q_H$ : e,  $q_b$ : e, then we should switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset if, and only if, there is a chosen  $H$ -pair in  $Ch^{bAn}(P|a_k)$  that is lower ranked than  $a_{k+1}$ .
5. If  $q$ : e,  $q_H$ : ne,  $q_b$ : ne, then we should switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset if, and only if, there is a chosen by  $Ch^{bAn}(P|a_k)$  pair that is not a  $bH$ -pair.
6. If  $q$ : e,  $q_H$ : ne,  $q_b$ : e, then we should switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset if, and only if, there is a chosen by  $Ch^{bAn}(P|a_k)$  pair that is lower ranked than  $a_{k+1}$ .
7. If  $q$ : e,  $q_H$ : e,  $q_b$ : ne, then we should switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset if, and only if, there is a chosen  $H$ -pair in  $Ch^{bAn}(P|a_k)$ .
8. If  $q$ : e,  $q_H$ : e,  $q_b$ : e, then we should switch to  $Ch^{bAn}(P|a_{k+1})$  as to possibly stable subset if, and only if, there is a chosen  $H$ -pair in  $Ch^{bAn}(P|a_k)$  that is lower ranked than  $a_{k+1}$ .

**Lemma 12.** *Starting from anchor  $a_0$ , switch to the next one if Lemma 11 says so. Consider the anchor  $a_k$ , such that either we do not switch to the next one, or it is the last anchor. Subset  $Ch^{bAn}(P|a_k)$  is stable.*

**Lemma 13.** *Consider two subsequent anchors  $a_k$  and  $a_{k+1}$ . If  $Ch^{bAn}(P|a_k)$  is stable, then  $Ch^{bAn}(P|a_{k+1})$  is stable if, and only if,*

- first, there are the worst chosen by  $Ch^{bAn}(P|a_k)$   $H$ -pair,  $H'_{worst}$ , and  $b$ -pair,  $b'_{worst}$ ;
- second, there is the best not chosen by  $Ch^{bAn}(P|a_k)$   $NH$ -pair,  $NH^\times_{best}$ ;
- third,  $H'_{worst}$  has higher rank than both  $a_{k+1}$  and  $NH^\times_{best}$ ;

- fourth,  $b_{worst}^\vee$  has lower rank than both  $a_{k+1}$  and  $NH_{best}^\times$ .

Also, if  $Ch^{bAn}(P|a_{k+1})$  is stable, then  $Ch^{bAn}(P|a_{k+1}) = (Ch^{bAn}(P|a_k) \setminus \{H_{worst}^\vee, b_{worst}^\vee\}) \cup \{a_{k+1}, NH_{best}^\times\}$ .

**Lemma 14.** Consider two subsequent anchors  $a_k$  and  $a_{k+1}$ . If  $Ch^{bAn}(P|a_k)$  is stable, but  $Ch^{bAn}(P|a_{k+1})$  is not stable, then  $Ch^{bAn}(P|a_{k+l})$  is not stable for any  $l > 1$ .

Combining all the results above, here is the algorithm that calculates  $k_{left}$  and  $k_{right}$ .

- **Step 1:** Starting from anchor  $a_0$ , switch to the next one if Lemma 11 says so. Consider the anchor  $a_k$ , such that either we do not switch to the next one, or it is the last anchor. By Lemma 12, if  $a_k$  is the last anchor, set  $k_{right} = k_{left} = k$ . Otherwise, set  $k_{left} = k$  and go to the Step 2.
- **Step 2:** Starting from anchor  $a_{k_{left}}$ , switch to the next stable one if Lemma 13 says so. Consider the anchor  $a_k$ , such that either we do not switch to the next stable one (since the next one is not stable), or it is the last anchor. By Lemma 14 set  $k_{right} = k$ .

Also, by Lemma 10 we considered all potential candidates for stable subsets. This concludes the proof of the proposition.  $\square$

**Proposition 12.** Fix an institution  $f$  and a proposed set  $P \subseteq P^f$ , where a candidate may have at most one proposed pair. Consider two subsequent anchors  $a_k$  and  $a_{k+1}$  from  $A(C)$ , such that  $k_{left} \leq k < k+1 \leq k_{right}$ .

For  $Ch^{bAn}(P|a_k)$  pick the worst chosen  $H$ -pair,  $H_{worst}^\vee$ , and  $b$ -pair,  $b_{worst}^\vee$ , and the best not chosen  $NH$ -pair from  $P$ ,  $NH_{best}^\times$ . Also pick the best not chosen pair  $p_{best}^\times$  from  $P$  and the worst chosen pair  $p_{worst}^\vee$ .

For  $Ch^{bAn}(P|a_{k+1})$  pick the best not chosen pair  $\bar{p}_{best}^\times$  from  $P$  and the worst chosen pair  $\bar{p}_{worst}^\vee$ .

The following holds:

- first,  $p_{worst}^\vee = b_{worst}^\vee$ , and  $\bar{p}_{worst}^\vee \pi^f p_{worst}^\vee$ ;
- second,  $\bar{p}_{best}^\times = H_{worst}^\vee$ , and  $\bar{p}_{best}^\times \pi^f p_{best}^\times$ ;
- third,  $Ch^{bAn}(P|a_{k+1}) = (Ch^{bAn}(P|a_k) \setminus \{H_{worst}^\vee, b_{worst}^\vee\}) \cup \{a_{k+1}, NH_{best}^\times\}$ ;
- fourth, if  $k_{left} < k_{right}$ , then quotas  $q$  and  $q_H$  are exhausted, and  $Ch^{bAn}(P|a_k)$  has exactly  $q_b$  pairs with candidates possessing trait  $b$  for any  $k \in \{k_{left}, \dots, k_{right}\}$ .

Thus, by considering a lower-ranked anchor for producing a stable subset, we pick less very low ranked candidates at the cost of also picking less very high ranked candidates by increasing amount of picked candidates with middle ranks.

### 6.4.3 Bringing Back Candidates with Two Pairs

Suppose now that there may be candidates with two pairs in  $P^f$ . A choice correspondence  $\mathbf{Ch}(\cdot)$  is a correspondence that for a given institution  $f \in \mathcal{F}$  and a set of pairs  $P \subseteq P^f$  produces a set of subsets  $\mathbf{Ch}(P) \subseteq 2^P$ , such that each subset can be chosen by a choice rule.

Consider the following choice correspondences  $\mathbf{Ch}^{tD-Ol, nH}$  and  $\mathbf{Ch}^{tD-Ol}$ . Fix an institution  $f$  and a proposed set  $P \subseteq P^f$ .

- **Step 1:** Construct  $P' \subseteq P$  by excluding all pairs with houses for candidates who have two proposed pairs in  $P$ . Calculate  $k_{left}$  and  $k_{right}$  for  $P'$ . Set  $\mathbf{Ch}' = \mathbf{Ch}^{bAn}(P' | \bigcup_{k=k_{left}}^{k_{right}} a_k)$ , and  $\mathbf{Ch}^{tD-Ol, nH}(P) = \mathbf{Ch}'$ . If either  $k_{left} < k_{right}$  or  $k_{left} = k_{right}$  and there are no empty houses under the only stable subset, set  $\mathbf{Ch}^{tD-Ol}(P) = \mathbf{Ch}'$ . Otherwise, go to the Step 2.
- **Step 2 (houses distribution):** Pick the only set  $P'$  from  $\mathbf{Ch}'$ , give one house to the most eligible for houses chosen candidate who wants a house and include the result into  $\mathbf{Ch}'$ . Again, pick  $P'$ , give two houses (if there is two empty houses) to the two most eligible for houses chosen candidates who want a house and include the result into  $\mathbf{Ch}'$ . And so on, until there are no more empty houses. Set  $\mathbf{Ch}^{tD-Ol}(P) = \mathbf{Ch}'$ .

**Theorem 2.** *A choice rule is stable if, and only if, it is a selection from the choice correspondence  $\mathbf{Ch}^{tD-Ol}$ .*

As for strong stability, consider the following choice correspondence  $\mathbf{Ch}^{tD-Ol, aH}$ . Fix an institution  $f$  and a proposed set  $P \subseteq P^f$ .

- **Step 1:** Construct  $P' \subseteq P$  by excluding all pairs with houses for candidates who have two proposed pairs in  $P$ . Calculate  $k_{left}$  and  $k_{right}$  for  $P'$ . Set  $\mathbf{Ch}' = \mathbf{Ch}^{bAn}(P' | \bigcup_{k=k_{left}}^{k_{right}} a_k)$ , and  $\mathbf{Ch}^{tD-Ol, nH}(P) = \mathbf{Ch}'$ . If either  $k_{left} < k_{right}$  or  $k_{left} = k_{right}$  and there are no empty houses under the only stable subset, set  $\mathbf{Ch}^{tD-Ol, aH}(P) = \mathbf{Ch}'$ . Otherwise, go to the Step 2.
- **Step 2 (houses distribution):** Pick the only set from  $\mathbf{Ch}'$ , give all houses to the most eligible for houses chosen candidates who want a house and include the result into empty  $\mathbf{Ch}''$ . Set  $\mathbf{Ch}^{tD-Ol, aH}(P) = \mathbf{Ch}''$ .

**Theorem 3.** *A choice rule is strongly stable if, and only if, it is a selection from the choice correspondence  $\mathbf{Ch}^{tD-Ol, aH}$ .*

Moreover, if  $|\mathbf{Ch}^{tD-Ol, aH}(P)| > 1$  for some  $P \subseteq P^f$ , then, first, for any candidate  $c$  with two pairs in  $P$ , pair  $(c, H)$  is never chosen; and second,  $\mathbf{Ch}^{tD-Ol, aH}(P) = \mathbf{Ch}^{tD-Ol}(P)$ .

In words, second part of Theorem 3 states the following: if a strongly stable subset is not unique, then, first, no candidate who can study without a house receives a house under any stable subset; and second, all stable subsets are strongly stable.

Examples 9 and 10 illustrate designs of the stable choice correspondence  $\mathbf{Ch}^{tD-Ol}$  and strongly stable choice correspondence  $\mathbf{Ch}^{tD-Ol, aH}$ .

**Example 9.** Consider an institution  $f$  with quota  $q = 7$ , housing quota  $q_H = 4$ , number of reserved positions  $q_b = 4$ , and the proposed set of fourteen pairs  $P$  with twelve acceptable candidates with the following traits and housing needs (from the best to the worst according to  $\pi$ )

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$	$c_9$	$c_{10}$	$c_{11}$	$c_{12}$
$H$		$H$	$H$	$H$	$H$	$H$		$H$	$H$	$H$	
	$NH$			$NH$			$NH$		$NH$		$NH$
		$b$			$b$		$b$	$b$	$b$	$b$	

There are five anchors in  $A(P)$  (non-empty ones are framed by vertical lines):  $a_0 = \emptyset$ ,  $a_1 = c_3$ ,  $a_2 = c_6$ ,  $a_3 = c_9$ , and  $a_4 = c_{11}$ .

First, we temporarily exclude pairs  $(c_5, H)$  and  $(c_{10}, H)$ , since both candidates have two proposed pairs,  $P' = P \setminus \{(c_5, H), (c_{10}, H)\}$ . Then we consider all anchors one by one, starting with the empty anchor  $a_0$ :

- $Ch^{bAn}(P'|a_0) = Ch^{bH-NOI}(P' \setminus A(P)) = \{(c_1, H), (c_2, NH), (c_4, H), (c_5, NH), (c_7, H), (c_8, NH), (c_{10}, NH)\}$  – not a stable subset, since  $(c_3, H)$  has a justified envy towards  $(c_7, H)$ ;
- $Ch^{bAn}(P'|a_1) = \{(c_3, H)\} \cup Ch^{bH-NOI}(P' \setminus A(P)) = \{(c_1, H), (c_2, NH), (c_3, H), (c_4, H), (c_5, NH), (c_8, NH), (c_{10}, NH)\}$  – not a stable subset, since  $(c_6, H)$  has a justified envy towards  $(c_5, NH)$  (there is an empty house);
- $Ch^{bAn}(P'|a_2) = \{(c_3, H), (c_6, H)\} \cup Ch^{bH-NOI}(P' \setminus A(P)) = \{(c_1, H), (c_2, NH), (c_3, H), (c_4, H), (c_6, NH), (c_8, NH), (c_{10}, NH)\}$  – stable subset without empty houses;
- $Ch^{bAn}(P'|a_3) = \{(c_3, H), (c_6, H), (c_9, H)\} \cup Ch^{bH-NOI}(P' \setminus A(P)) = \{(c_1, H), (c_2, NH), (c_3, H), (c_5, NH), (c_6, NH), (c_8, NH), (c_9, H)\}$  – stable subset without empty houses;
- $Ch^{bAn}(P'|a_4) = \{(c_3, H), (c_6, H), (c_9, H), (c_{11}, H)\} \cup Ch^{bH-NOI}(P' \setminus A(P)) = \{(c_2, NH), (c_3, H), (c_5, NH), (c_6, NH), (c_8, NH), (c_9, H), (c_{11}, H)\}$  – not a stable subset, since  $(c_1, H)$  has a justified envy towards  $(c_{11}, H)$ .

So,  $\mathbf{Ch}^{tD-OI}(P) = \mathbf{Ch}^{tD-OI, aH}(P) = \{Ch^{bAn}(P'|a_2), Ch^{bAn}(P'|a_3)\}$ .

Therefore, first, there are two stable subsets of  $P$ , each of which is also strongly stable; and second,  $Ch^{bAn}(P'|a_3)$  is equal to  $(Ch^{bAn}(P'|a_2) \setminus \{(c_4, H), (c_{10}, NH)\}) \cup \{(c_5, NH), (c_9, NH)\}$ .

△

**Example 10.** Consider an institution  $f$  with quota  $q = 4$ , housing quota  $q_H = 3$ , number of reserved positions  $q_b = 2$ , and the proposed set of ten pairs  $P$  with seven acceptable candidates with the following traits and housing needs (from the best to the worst according to  $\pi$ )

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$
	$H$	$H$		$H$	$H$	$H$
$NH$		$NH$	$NH$	$NH$		$NH$
	$b$			$b$	$b$	$b$

There are three anchors in  $A(P)$  (non-empty ones are framed by vertical lines):  $a_0 = \emptyset$ ,  $a_1 = c_2$ , and  $a_2 = c_6$ .

First, we temporarily exclude pairs  $(c_3, H)$ ,  $(c_5, H)$ , and  $(c_7, H)$ , since these three candidates have two proposed pairs,  $P' = P \setminus \{(c_3, H), (c_5, H), (c_7, H)\}$ . Then we consider all anchors one by one, starting with the empty anchor  $a_0$ :

- $Ch^{bAn}(P'|a_0) = Ch^{bH-NOI}(P' \setminus A(P)) = \{(c_1, NH), (c_3, NH), (c_5, NH), (c_7, NH)\}$  – not a stable subset, since  $(c_2, H)$  has a justified envy towards  $(c_7, NH)$ ;
- $Ch^{bAn}(P'|a_1) = \{(c_2, H)\} \cup Ch^{bH-NOI}(P' \setminus A(P)) = \{(c_1, NH), (c_2, H), (c_3, NH), (c_5, NH)\}$  – stable subset with two empty houses;
- $Ch^{bAn}(P'|a_2) = \{(c_2, H), (c_6, H)\} \cup Ch^{bH-NOI}(P' \setminus A(P)) = \{(c_1, NH), (c_2, H), (c_3, NH), (c_6, H)\}$  – not a stable subset, since  $(c_5, NH)$  has a justified envy towards  $(c_6, H)$ .

So, under proposed set  $P'$  there is only one stable subset  $Ch^{bAn}(P'|a_1) = \{(c_1, NH), (c_2, H), (c_3, NH), (c_5, NH)\}$  with two empty houses. Note that it contains two candidates who would like to get a house:  $c_3$  and  $c_5$ .

Suppose that,  $c_5 \pi_H c_3$ . Thus, there are three stable subsets of  $P$ :  $P_1 = \{(c_1, NH), (c_2, H), (c_3, NH), (c_5, NH)\}$ ,  $P_2 = \{(c_1, NH), (c_2, H), (c_3, NH), (c_5, H)\}$ , and  $P_3 = \{(c_1, NH), (c_2, H), (c_3, H), (c_5, H)\}$ , while only  $P_3$  is strongly stable.

Therefore,  $\mathbf{Ch}^{tD-OI, aH}(P) = \{P_3\} \subset \{P_1, P_2, P_3\} = \mathbf{Ch}^{tD-OI}(P)$ .  $\triangle$

## 7 Stable Matching

This section presents all restrictions that allow to construct a stable (iterative) College-Proposing Deferred Acceptance mechanism.

### 7.1 Regular Institutions

Hatfield and Milgrom (2005) and Aygün and Sönmez (2013) show that a stable mechanism can be obtained through DA by using a stable choice rule, provided that this choice rule satisfies the following substitutes condition.

**Definition 21.** (*Substitutes condition (Kelso and Crawford, 1982)*). A choice rule  $Ch$  satisfies substitutes condition if, for every set  $C \subseteq C^f$  for every  $f$ , and any two candidates  $c \in C$  and  $\bar{c} \in C^f \setminus C$  the following holds

$$c \in Ch(C \cup \{\bar{c}\}) \Rightarrow c \in Ch(C).$$

In words, if a candidate is chosen from a proposed set of candidates, then they should stay chosen once some other chosen candidate rejects the offer, i.e., all candidates are substitutes for an institution.

**Proposition 13.** *There is no stable choice rule that satisfies substitutes condition.*

It is proved with the following example.

**Example 11.** Consider an institution with quota  $q = 2$  and numbers of reserved positions  $q_b = q_r = 1$ . There is a set of four candidates from  $C^f$  ordered according to preferences  $\pi: c_1 \pi c_2 \pi c_3 \pi c_4$  with the following traits and housing needs

$c_1$	$c_2$	$c_3$	$c_4$
-----			
	$b$	$b$	
		$r$	$r$

Suppose that  $Ch$  is a stable choice rule. Then, if the proposed set is  $\{c_1, c_2, c_3, c_4\}$ , the rule should choose  $Ch(\{c_1, c_2, c_3, c_4\}) = \{c_1, c_3\}$ . Note that  $c_1$  is chosen. However, if the proposed set is  $\{c_1, c_2, c_4\}$  (candidate  $c_3$  is excluded), the rule should choose  $Ch(\{c_1, c_2, c_4\}) = \{c_2, c_4\}$ . Now  $c_1$  is not chosen, thus, there is no stable choice rule that satisfies substitutes condition.  $\triangle$

Proposition 13 shows that, under one-to-all reserve convention, no stable choice rule can be embedded into DA in order to get a stable mechanism. Therefore, in order to guarantee existence of a stable matching we should switch from one-to-all to **one-to-one approach** (Sönmez and Yenmez, 2019), i.e., upon admission, a candidate with multiple traits accommodates only one of the reserves they qualify for.

Since institutions prioritize trait  $b$  over trait  $r$ , we should recalculate the number of reserved positions for  $r$ :  $q_r = \min\{q_r, q - q_b\}$ . By this we make sure that all initial reserved positions for trait  $b$  will have a chance to be accommodated.

Now we use the approach from Sönmez and Yenmez (2019). Fix a set of candidates  $C \subseteq \mathcal{C}$ . Let  $n(C)$  denote the maximum amount of reserved positions that can be assigned to individuals from  $C$ .

**Definition 22.** (*Maximal compliance with reservations (Sönmez and Yenmez, 2019)*). A choice rule  $Ch$  maximally complies with reservations if, for every set  $C \subseteq C^f$  for every  $f$ ,  $Ch(C)$  is a subset with maximal  $n(Ch(C))$ , i.e.,  $Ch(C) \in \arg \max_{C' \subseteq C \mid |C'| \leq q} n(C')$ .

**Definition 23.** (*Elimination of one-to-one justified envy (Sönmez and Yenmez, 2019)*). A choice rule  $Ch$  eliminates one-to-one justified envy if, for every set  $C \subseteq C^f$  for every  $f$ ,  $c \in Ch(C)$ , and  $c' \in C \setminus Ch(C)$ ,

$$c' \pi^f c \Rightarrow n((Ch(C) \setminus \{c\}) \cup \{c'\}) < n(Ch(C)).$$

We naturally extend elimination of one-to-one justified envy to matchings.

Consider the following **horizontal envelope choice rule**  $Ch^{\boxtimes}$ :

- **Step 1.1:** Fix an institution  $f$  and a proposed set  $C \subseteq C^f$ . Choose the highest priority candidate with a trait that has a reserved position. Let  $c_1$  be this candidate and  $C_1$  be the set including only this candidate. If there exists no such candidate, go to Step 2.
- **Step 1.k** ( $2 \leq k \leq (q_b + q_r)$ ): Starting from the candidate who has the next highest priority after  $c_{k-1}$ , check one by one if for the next candidate  $c$  the following inequality holds  $n(C_{k-1} \cup \{c\}) > n(C_{k-1})$ . If it does, choose this candidate and denote him by  $c_k$ . In this case, let  $C_k = C_{k-1} \cup \{c_k\}$  be the set of individuals chosen so far. Otherwise, if no such candidate exists, go to Step 2.
- **Step 2:** For unfilled positions, choose remaining individuals with the highest priority until all positions are filled or there are no unchosen individuals remaining.

**Definition 24.** (*Choice rule: one-to-one stability*). A choice rule is one-to-one stable if it maximally complies with reservations, eliminates one-to-one justified envy, and is non-wasteful.

**Definition 25.** (*Matching: one-to-one stability*). A matching is one-to-one stable if it eliminates one-to-one justified envy and is non-wasteful.

**Proposition 14.** (*Sönmez and Yenmez, 2019*) Consider the one-to-one reserve matching convention. A choice rule is one-to-one stable if, and only if, it is the horizontal envelope choice rule.

**Definition 26.** (*Irrelevance of rejected individuals (Aygün and Sönmez, 2013)*). A choice rule  $Ch$  satisfies the irrelevance of rejected individuals condition, if, for every set  $C \subseteq C^f$  for every  $f$ ,

$$c \in C \setminus Ch(C) \Rightarrow Ch(C \setminus \{c\}) = Ch(C).$$

**Proposition 15.** (*Sönmez and Yenmez, 2019*) The horizontal envelope choice rule satisfies the substitutes condition and the irrelevance of rejected individuals condition.

## 7.2 Boarding Schools

Proposition 6 suggests that we need to focus only on housing needs of candidates.

We already redefined individual rationality, justified envy and non-wastefulness for matchings under economy with boarding schools. Now we redefine non-wastefulness via house, house-seat-justified envy, house-justified envy, non-house wastefulness, stability, and strong stability for matchings under economy with boarding schools which neglect all traits.

**Definition 27.** (*Matching: Non-wastefulness via house*). A matching  $\mu$  is non-wasteful via house if there are no boarding school  $f$  and a pair of SDH for  $f$  candidate  $c$  and WDH for  $f$  candidate  $\bar{c}$ , such that,  $|\mu_f| < q^f$ ,  $|(\mu_f)_H| = q_H^f$ ,  $c$  is not admitted to  $f$ , and  $(\bar{c}, H)$  is matched to  $f$ .



**Definition 28.** (*Matching: House-justified envy*). A candidate  $\bar{c}$  has a house-justified envy towards a candidate  $c$  under a matching  $\mu$  if there is a boarding school  $f$ , such that, both  $c$  and  $\bar{c}$  are WDH candidates for  $f$ ,  $(c, H)$  and  $(\bar{c}, NH)$  are matched to  $f$ , and  $\bar{c} \pi_H^f c$ .

**Definition 29.** (*Matching: House-seat-justified envy*). A candidate  $\bar{c}$  has a house-seat-justified envy towards a pair of candidates  $c$  and  $c'$  under a matching  $\mu$  if there is a boarding school  $f$ , such that,  $\bar{c}$  is SDH candidate for  $f$ ,  $c$  is WDH candidate for  $f$ , and the following holds

1. quota  $q^f$  and housing quota  $q_H^f$  are completely filled, i.e.,  $|\mu_f| = q^f$  and  $|(\mu_f)_H| = q_H^f$ ,
2.  $c \pi \bar{c} \pi c'$ ,
3.  $\{(c, f, H), (c', f, NH)\} \subseteq \mu$ , and
4.  $(\bar{c}, f, NH) \notin \mu$ .

**Definition 30.** (*Matching: Stability*). A matching  $\mu$  is stable if it is non-wasteful, non-wasteful via house, and eliminates justified envy, house-justified envy, and house-seat-justified envy.

**Definition 31.** (*Matching: Non-house-wastefulness*). A matching  $\mu$  is non-house-wasteful if there is no pair of a boarding school  $f$  and a WDH for  $f$  candidate  $c$ , such that,  $(c, f, NH) \in \mu$  and  $|(\mu_f)_H| < q_H^f$ .

**Definition 32.** (*Matching: Strong stability*). A matching  $\mu$  is strongly stable if it is stable and non-house-wasteful.

We also need to reformulate the substitutes condition for our matching market with contracts.

**Definition 33.** (*Substitutes condition (Hatfield and Milgrom, 2005)*). A choice rule  $Ch$  satisfies substitutes condition for an institution  $f$  if for any set  $P \subset P^f$  and any two pairs  $(c, h) \in P$  and  $(\bar{c}, \bar{h}) \in P^f \setminus P$  the following holds

$$(c, h) \in Ch(P \cup \{(\bar{c}, \bar{h})\}) \Rightarrow (c, h) \in Ch(P).$$

So, for a choice rule to satisfy substitutes condition there may not be two complementary contracts, i.e., any chosen contract stays chosen even if any other contract is excluded from the proposed set.

Consider the following choice rule  $Ch^D$ . Fix a proposed set  $P \in P^f$ , neglect all traits

- **Step 0:** Set an empty set  $P'$  that will be the chosen set at the end of the procedure. For each candidate  $c$ , who is WDH for  $f$  and has  $(c, H)$  in  $P$ , exclude  $(c, H)$  from  $P$ . Consider two disjoint sets of WDH candidates for  $f$ :  $C^2$  includes only WDH candidates who proposed two pairs in  $P$ , while  $C^1$  includes only WDH candidates who proposed only one pair (with a house) in  $P$ . Go to Step 1.1.
- **Step 1.k:** If  $q = 0$ , end the procedure. Otherwise, do the following.

– If  $q_H > 0$ , then consider a not considered before pair  $(c, h)$  from  $P$  with the best candidate according to  $\pi$ . If there is such pair,

\* if  $h = NH$ , add this pair to  $P'$  and update  $q = q - 1$ , go to Step 1.(k+1);

\* if  $h = H$ , add this pair to  $P'$  and update  $q = q - 1$  and  $q_H = q_H - 1$ .

If there is no such pair, consider a subset of  $C^2$  with candidates who has a pair in  $P'$ , name it  $C^{2'}$ , go to Step 2.1.

– If  $q_H = 0$ , then consider a not considered before pair  $(c, NH)$  from  $P$  with the best candidate. If there is such pair, add it to  $P'$  and update  $q = q - 1$ , go to Step 1.(k+1). If there is no such pair, end the procedure.

- **Step 2.k:** If either  $q = 0$ , or  $q_H = 0$ , or  $C^1$  is empty, end the procedure. Otherwise, pick the highest house-eligibility ranked candidate  $c$  from  $C^1$ , such that the total number of candidates from  $C^{2'}$  with higher house-eligibility ranking than  $c$ , let this number be  $m$ , is strictly smaller than  $q_H$ .<sup>19</sup>

If there is no such candidate, end the procedure. Otherwise, add  $(c, H)$  to  $P'$ , exclude  $c$  from  $C^1$ , exclude those  $m$  candidates from  $C^{2'}$ , and update  $q = q - 1$  and  $q_H = q_H - (m + 1)$ . Go to Step 2.(k+1).

**Proposition 16.** *Neglect all traits.  $Ch^D$  satisfies substitutes condition.*

Therefore,  $Ch^D$  may be incorporated into DA.

**Definition 34.** *(Law of aggregate demand). A choice rule  $Ch$  satisfies the law of aggregate demand (LAD), if, for every two subsets  $P'$  and  $P$  of  $P^f$  for every  $f$ ,*

$$P' \subset P \Rightarrow |Ch(P')| \leq |Ch(P)|.$$

**Definition 35.** *(Irrelevance of rejected pairs). A choice rule  $Ch$  satisfies the irrelevance of rejected pairs condition, if, for every set  $P \subseteq P^f$  for every  $f$ ,*

$$(c, h) \in P \setminus Ch(P) \Rightarrow Ch(P \setminus \{(c, h)\}) = Ch(P).$$

**Proposition 17.**  *$Ch^D$  satisfies the law of aggregate demand and irrelevance of rejected pairs condition.*

### 7.3 Restoring Stability of Iterative College-Proposing DA

Consider the following **stable iterative College-Proposing Deferred Acceptance** mechanism (SCDA).

Before the start of the procedure the following is known: the set of candidates with their traits, the set of institutions with their strict rankings and strict house-eligibility rankings,

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<sup>19</sup>By doing this we are making sure that we will be able to eliminate house-justified envy of the final matching after proposing  $c$  with a pair  $(f, H)$ .

all quotas, all housing quotas, all numbers of reserved positions, and all sets of acceptable pairs (if an institution  $f$  is not a boarding school, i.e.,  $q_H^f = 0$ , then  $P^f = C^f \times \{NH\}$ ), i.e.,  $\langle \mathcal{C}, \mathcal{F}, \mathcal{T}, \tau, \{q^f, q_H^f, q_b^f, q_r^f, \pi^f, \pi_H^f, P^f\}_{f \in \mathcal{F}} \rangle$  is known.

- **Step 1.1:** Each regular institution  $f \in \mathcal{F}$  (with  $q_H^f = 0$ ) proposes to candidates from  $Ch^\boxtimes((P^f)_c)$ .

Each boarding school  $f \in \mathcal{F}$  (with  $q_H^f > 0$ ) proposes to candidates from  $Ch^D(P^f)$  with corresponding pairs.

Each candidate  $c \in \mathcal{C}$  receives a set of offers. If this set is not empty, they pick the one that they tentatively or definitely accept and reject all the others.

All candidates that definitely accepted an offer leave the market together with occupied seats and accommodated reserved positions and houses.

For each institution  $f$ , denote the set of all tentatively accepted together with all not yet rejected pairs from  $P^f$  by  $P_1^f$ .

If there were no rejections, then matching is finalized: all tentative acceptances become final assignments.

- **Step 1.k (k ≥ 2):** Each regular institution  $f \in \mathcal{F}$  (with initial  $q_H^f = 0$ ) proposes to candidates from  $Ch^\boxtimes((P_{k-1}^f)_c)$ .

Each boarding school  $f \in \mathcal{F}$  (with initial  $q_H^f > 0$ ) proposes to candidates from  $Ch^D(P_{k-1}^f)$  with corresponding pairs.

Each candidate  $c \in \mathcal{C}$  receives a set of offers. If this set is not empty, they pick the one that they tentatively or definitely accept and reject all the others.

All candidates that definitely accepted an offer leave the market together with occupied seats and accommodated reserved positions and houses.

For each institution  $f$ , denote the set of all tentatively accepted together with all not yet rejected pairs from  $P_{k-1}^f$  by  $P_k^f$ .

If there were no rejections, then matching is finalized: all tentative acceptances become final assignments.

- **Step 2 (houses distribution):** Each boarding school with empty houses gives them to the most eligible for houses chosen candidates who do not have a house, but want it (they will accept them). Matching is finalized: all tentative acceptances become final assignments.

**Theorem 4.** *If regular institutions use one-to-one convention and boarding schools disregard all traits, then SCDA is one-to-one stable for regular institutions and strongly stable for boarding schools.*

Theorem 4 shows that we can always find a one-to-one stable for regular institutions and strongly stable for boarding schools matching under one-to-one convention for regular institutions and no traits for boarding schools.

## 8 Modified Iterative College-Proposing DA

If policymakers care more about one-to-all convention and affirmative action for boarding schools than about stability of the final matching, then one can incorporate stable choice rules for regular institutions and boarding schools in the following way.<sup>20</sup>

Consider the **modified iterative College-Proposing Deferred Acceptance** mechanism (MCDA). Before the start of the procedure the following is known: the set of candidates with their traits, the set of institutions with their strict rankings, house-eligibility rankings, all quotas, all housing quotas, all numbers of reserved positions, and all sets of acceptable pairs (if an institution  $f$  is not a boarding school, i.e.,  $q_H^f = 0$ , then  $P^f = C^f \times \{NH\}$ ), i.e.,  $\langle \mathcal{C}, \mathcal{F}, \mathcal{T}, \tau, \{q^f, q_H^f, q_b^f, q_r^f, \pi^f, \pi_H^f, P^f\}_{f \in \mathcal{F}} \rangle$  is known. Denote by  $Ch^{Ol}$  any selection from  $\mathbf{Ch}^{Ol}$ .

- **Step 1.1:** Each regular institution  $f \in \mathcal{F}$  (with  $q_H^f = 0$ ) proposes to candidates from some selection  $Ch^{Ol}((P^f)_c)$ .

Each boarding school  $f \in \mathcal{F}$  (with  $q_H^f > 0$ ) sets a boarding bar  $D_1^f$ . Then it creates a new set of pairs  $P^{f,D}$  by excluding from  $P^f$  all pairs  $(c, H)$ , such that rank of  $c$  in house-eligibility ranking  $\pi_H^f$  is below  $D_1^f$ . Then  $f$  proposes to candidates from  $Ch_f^B(P^{f,D})$  with corresponding pairs.

Each candidate  $c \in \mathcal{C}$  receives a set of offers. If this set is not empty, they pick the one that they tentatively or definitely accept and reject all the others.

All candidates that definitely accepted an offer leave the market together with occupied seats, accommodated reserved positions and houses.

For each institution  $f$ , denote the set of all tentatively accepted pairs from  $P^f$  by  $P_1^{f,Acc}$ .

For each institution  $f$ , we update its quotas and a set of pairs  $P_1^f$  that are still on the market.

If there were no rejections, go to Step 2.

- **Step 1.k (k ≥ 2):** Each regular institution  $f \in \mathcal{F}$  (with initial  $q_H^f = 0$ ) proposes to candidates from  $(P_{k-1}^{f,Acc})_c \cup Ch^{Ol}((P_{k-1}^f \setminus P_{k-1}^{f,Acc})_c)$ , thus, while constructing a set of offers, institution  $f$  assumes that all candidates from  $(P_{k-1}^{f,Acc})_c$  should have an offer.

Each boarding school  $f \in \mathcal{F}$  (with initial  $q_H^f > 0$ ) sets a new boarding bar  $D_k^f$ . Then it creates a new set of pairs  $P^{f,D}$  by excluding from  $P_{k-1}^f$  all pairs  $(c, H)$ , such that rank of  $c$  in house-eligibility ranking  $\pi_H^f$  is below  $D_k^f$ . Then  $f$  proposes to candidates from

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<sup>20</sup>The choice of policymakers is not straightforward, since there are many real-life constraints that may break stability of initially stable mechanism, e.g., capacity constraint on amount of wishes per candidate, or fixed amount of steps of the iterative DA algorithm.

$P_{k-1}^{f,Acc} \cup Ch_f^B(P^{f,D} \setminus P_{k-1}^{f,Acc})$  with corresponding pairs, thus, while constructing a set of offers, institution  $f$  assumes that all pairs from  $P_{k-1}^{f,Acc}$  should be offered.

Each candidate  $c \in \mathcal{C}$  receives a set of offers. If this set is not empty, they pick the one that they tentatively or definitely accept and reject all the others.

All candidates that definitely accepted an offer leave the market together with occupied seats, accommodated reserved positions and houses.

For each institution  $f$ , denote the set of all tentatively accepted pairs from  $P^f$  by  $P_k^{f,Acc}$ .

For each institution  $f$ , we update its quotas and a set of pairs  $P_k^f$  that are still on the market.

If there were no rejections, go to Step 2.

- **Step 2 (houses distribution):** Each boarding school with empty houses gives them to the most eligible for houses chosen candidates who do not have a house, but want it (they will accept them). Matching is finalized: all tentative acceptances become final assignments.

Where, either  $Ch_f^B = Ch^{Opt,1}$ , if a boarding school  $f$  cares about two traits, or  $Ch_f^B$  is any selection from  $\mathbf{Ch}^{tD-Ol,nH}$ , if a boarding school  $f$  cares about one trait.

During Steps 1. of MCDA house-eligibility ranks are used in the same fashion as they are used now in Parcoursup: during each step of DA for each boarding school these ranks together with a chosen boarding bar determine which candidates are eligible for a house, and which are not. In addition, Step 2 makes sure that MCDA is non-house-wasteful.

Also note that, by construction of MCDA, the “quality” of a candidate’s tentatively accepted contract cannot decline over time, i.e., we artificially imitate substitutes condition.

The presented MCDA mechanism makes sure that the proposed by institutions sets of offers during any given Step 1.k are stable under the market given at the beginning of this Step 1.k, keeping fixed tentatively accepted offers from before. Proposition 1 shows that it is not the case for Parcoursup.

## 9 Conclusion

To the best of my knowledge, this is the first study to propose various types of choice rules that simultaneously deal with affirmative action with overlapping reserves and student housing allocation. Furthermore, it is the first paper that formally analyses stability properties of Parcoursup mechanism.

Table 2 sums up the social choice results of this research, i.e., (strongly) stable choice rules for regular institutions and boarding schools under affirmative action.<sup>21</sup> The top-left cell of Table 2 contains, first, the choice correspondence  $\mathbf{Ch}^{Ol}$  containing all stable choice rules for regular institutions under one-to-all reserve convention; and second, the unique stable horizontal

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<sup>21</sup>Only  $Ch^\boxtimes$  (Sönmez and Yenmez, 2019) satisfies substitutes condition of Hatfield and Milgrom (2005).

envelop choice rule  $Ch^{\boxtimes}$  for regular institutions under one-to-one reserve convention (Sönmez and Yenmez, 2019). The bottom-left(right) cell of Table 2 contains, first, a (strongly) stable choice rule  $Ch^{Opt,1}$  ( $Ch^{Opt}$ ) for boarding schools based on binary linear programming approach; and second, the choice correspondence  $\mathbf{Ch}^{tD-Ol}$  ( $\mathbf{Ch}^{tD-Ol,aH}$ ) containing all (strongly) stable choice rules for boarding schools that care about only one trait.

Table 2: (Strongly) stable choice rules.

	Stability	Strong stability
Regular institutions	one-to-all: $\mathbf{Ch}^{Ol}$ (!) one-to-one: $Ch^{\boxtimes}$ (!)	–
Boarding schools	two traits: $Ch^{Opt,1}$ one trait: $\mathbf{Ch}^{tD-Ol}$ (!)	two traits: $Ch^{Opt}$ one trait: $\mathbf{Ch}^{tD-Ol,aH}$ (!)

(!) signals that all such choice rules are found.

Under matching agenda, there may be college admission markets without a stable matching if boarding schools care about at least one trait. Based on the horizontal envelop choice rule  $Ch^{\boxtimes}$  (Sönmez and Yenmez, 2019) for regular institutions and brand new  $Ch^D$  for boarding schools we construct a SCDA mechanism that is one-to-one stable for regular institutions and strongly stable for boarding schools under no traits. However, it is an open question whether a stable matching always exists under more general model with one-to-all convention.

This study proposes an improvement over existing college admissions mechanism in France assuming that there are no common dormitories, i.e each boarding school has its own housing quota. However, in reality common dormitories do exist in France and some other countries and may cause non-existence of a stable matching even when boarding schools do not care about affirmative action (Sokolov, 2023). Therefore, finding a way to solve this problem is a very prominent research topic as a continuation of distributional constraints literature (Kamada and Kojima, 2012, 2017, 2018, 2024).

Another venue of research would be the analysis of MCDA’s stability properties under different approaches to setting boarding schools’ bars. For instance, if during some step of MCDA for some boarding school there are  $q_H$  currently empty houses, we may allow this school to propose with a house only to the  $q_H$  most house-eligible candidates who want a house.

Finally, it would be extremely interesting to compare stability metrics of MCDA and Parcoursup final matchings on the real Parcoursup data in order to assess the scale of possible benefits of the proposed alternative mechanism.

## Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

# A Proofs

**Proof of Proposition 3.** First, I show that  $Ch^{NOI}$  is stable.

$Ch^{NOI}$  is non-wasteful, since, by construction, it picks exactly  $\min\{q, |C|\}$  candidates.

Suppose that  $Ch^{NOI}$  does not eliminate justified envy. Hence, there should be a pair of candidates  $c, \bar{c} \in C$ , such that, first,  $c \in Ch^{NOI}(C)$ ; second,  $\bar{c} \in C \setminus Ch^{NOI}(C)$ ; and one of the following holds, denote  $\bar{C} = (Ch^{NOI}(C) \setminus c) \cup \bar{c}$ ,

1.  $\Delta_{\bar{C}}^{\bar{c}}(b) < \Delta_f^{Ch^{NOI}(C)}(b)$ ;
2.  $\Delta_{\bar{C}}^{\bar{c}}(b) = \Delta_f^{Ch^{NOI}(C)}(b)$  and  $\Delta_{\bar{C}}^{\bar{c}}(r) < \Delta_f^{Ch^{NOI}(C)}(r)$ ;
3.  $\Delta_{\bar{C}}^{\bar{c}}(b) = \Delta_f^{Ch^{NOI}(C)}(b)$ ,  $\Delta_{\bar{C}}^{\bar{c}}(r) = \Delta_f^{Ch^{NOI}(C)}(r)$ , and  $\bar{c} \pi c$ .

Suppose that the first one holds. This implies that, first,  $\Delta_f^{Ch^{NOI}(C)}(b) > 0$ ; and second, there exists a not chosen candidate  $\bar{c}$  with trait  $b$ . The former means that the algorithm of  $Ch^{NOI}(C)$  stopped adding candidates during the Step 1. Thus, by construction, we should have chosen all candidates with trait  $b$ , which contradicts the latter.

Suppose that the second one holds. The equality  $\Delta_{\bar{C}}^{\bar{c}}(b) = \Delta_f^{Ch^{NOI}(C)}(b)$  implies that either both sets  $Ch^{NOI}(C)$  and  $\bar{C}$  contain all candidates with trait  $b$ , or they both contain exactly  $q_b$  of them, where  $Ch^{NOI}(C)$  contains the best  $q_b$  candidates with trait  $b$ , by construction. The inequality  $\Delta_{\bar{C}}^{\bar{c}}(r) < \Delta_f^{Ch^{NOI}(C)}(r)$  implies that  $\Delta_f^{Ch^{NOI}(C)}(r) > 0$ . Thus, once  $Ch^{NOI}$  picked the best (or all) candidates with trait  $b$  during Step 1 it started picking the best candidates with trait  $r$  one by one and stopped adding candidates during Step 2. Thus, it picked exactly  $\min\{q, |C|\} - |(Ch^{NOI}(C))_b|$  best candidates with trait  $r$ . Since, first,  $|\bar{C}| = |Ch^{NOI}(C)|$ ; second,  $|(\bar{C})_b| = |(Ch^{NOI}(C))_b|$ ; and third,  $|(Ch^{NOI}(C))_r| = |Ch^{NOI}(C)| - |(Ch^{NOI}(C))_b|$ , we have  $|(\bar{C})_r| = |(Ch^{NOI}(C))_r|$ , which contradicts the inequality  $\Delta_{\bar{C}}^{\bar{c}}(r) < \Delta_f^{Ch^{NOI}(C)}(r)$ .

Suppose that the third one holds. There may be one of the following two cases:

1. The chosen set  $Ch^{NOI}(C)$  contains exactly the best  $\min\{q, |C|\}$  candidates from  $C$ . Thus, there may not be a not chosen candidate  $\bar{c}$ , such that  $\bar{c} \pi c$ . Contradiction.
2. The chosen set  $Ch^{NOI}(C)$  does not contain the best  $\min\{q, |C|\}$  candidates from  $C$ . This implies that  $|C| > q$ , which in turn implies that  $|Ch^{NOI}(C)| = q$ . Moreover, by construction of Steps 1 and 2, a not chosen candidate  $\bar{c}$ , such that  $\bar{c} \pi c$ , may only have no traits, while a chosen candidate  $c$  should have some trait  $t$ , i.e.,  $\tau(c) = t$ . And moreover,  $|(Ch^{NOI}(C))_b| \leq q_b$  and  $|(Ch^{NOI}(C))_r| \leq q_r$ . Thus, we have a contradiction with the equality  $\Delta_{\bar{C}}^{\bar{c}}(t) = \Delta_f^{Ch^{NOI}(C)}(t)$ .

Now I show that a stable choice rule should be  $Ch^{NOI}$ .

Suppose that there is a stable choice rule  $Ch$ , such that  $Ch(C) \neq Ch^{NOI}(C)$ . Non-wastefulness implies that  $|C| > q$ . Thus, the following holds,  $|Ch(C)| = |Ch^{NOI}(C)| = q$ . Moreover, we should have  $\Delta_f^{Ch(C)}(b) = \Delta_f^{Ch^{NOI}(C)}(b)$  and  $\Delta_f^{Ch(C)}(r) = \Delta_f^{Ch^{NOI}(C)}(r)$ .

Denote by  $(Ch^{NOI}(C))^k$  the set of candidates picked by  $Ch^{NOI}(C)$  during Step  $k \in \{1, 2, 3\}$ . There may be one of three cases:



1. There is a candidate  $c \in (Ch^{NOI}(C))^1 \setminus Ch(C)$ . Since, first, during Step 1  $Ch^{NOI}(C)$  picks the best possible candidates with  $b$ ; and second,  $\Delta_f^{Ch(C)}(b) = \Delta_f^{Ch^{NOI}(C)}(b)$ , there should be another candidate  $\bar{c} \in Ch(C)$ , such that  $\tau(\bar{c}) = b$  and  $c \pi \bar{c}$ . Thus, under  $Ch(C)$  candidate  $c$  has a justified envy towards  $\bar{c}$ . Contradiction.
2. There are no candidates in the set  $(Ch^{NOI}(C))^1 \setminus Ch(C)$ , but there is a candidate  $c \in (Ch^{NOI}(C))^2 \setminus Ch(C)$ . This implies that both  $Ch^{NOI}(C)$  and  $Ch(C)$  contain the best  $\min\{|C_b|, q_b\}$  candidates from  $C_b$ . Since, first, during Step 2  $Ch^{NOI}(C)$  picks the best possible candidates with  $r$ ; and second,  $\Delta_f^{Ch(C)}(r) = \Delta_f^{Ch^{NOI}(C)}(r)$ , there should be another candidate  $\bar{c} \in Ch(C)$ , such that  $\tau(\bar{c}) = r$  and  $c \pi \bar{c}$ . Thus, under  $Ch(C)$  candidate  $c$  has a justified envy towards  $\bar{c}$ . Contradiction.
3. There are no candidates in the set  $((Ch^{NOI}(C))^1 \cup (Ch^{NOI}(C))^2) \setminus Ch(C)$ , but there is a candidate  $c \in (Ch^{NOI}(C))^3 \setminus Ch(C)$ . This implies that both  $Ch^{NOI}(C)$  and  $Ch(C)$  contain the best  $\min\{|C_b|, q_b\}$  candidates from  $C_b$  and the best  $\min\{|C_r|, q_r\}$  candidates from  $C_r$ . Since, during Step 3  $Ch^{NOI}(C)$  picks the best possible candidates that were not picked before, there should be another candidate  $\bar{c} \in Ch(C)$ , such that  $c \pi \bar{c}$ . Thus, under  $Ch(C)$  candidate  $c$  has a justified envy towards  $\bar{c}$ . Contradiction.

This concludes the proof. □

**Proof of Lemma 1.** By Proposition 3, any other subset is not stable. □

**Proof of Lemma 2.** By including anchor  $a_{k+1}$  into the chosen set we will for sure accommodate more reserves for the trait  $b$ , since  $q_b \leq q$ : we can either use an empty seat, thus,  $Ch^{An}(C|a_k)$  is wasteful, or take it from a chosen candidate, who does not have the trait  $b$ , thus  $Ch^{An}(C|a_k)$  has a justified envy. Hence,  $Ch^{An}(C|a_k)$  is not stable, and  $\Delta^{Ch^{An}(C|a_k)}(b) > \Delta^{Ch^{An}(C|a_{k+1})}(b)$ . □

Denote by  $(Ch^{An}(C|a))^j$  the set of candidates picked during Step 2 of  $Ch^{An}(C|a)$  by  $Ch^{NOI}$  during its Step  $j \in \{1, 2, 3\}$ .

**Proof of Lemma 3.** There may be two cases:  $|(Ch^{An}(C|a_k))^1| + |(Ch^{An}(C|a_k))^2| < (q - k)$  and  $|(Ch^{An}(C|a_k))^1| + |(Ch^{An}(C|a_k))^2| = (q - k)$ .

- If  $|(Ch^{An}(C|a_k))^1| + |(Ch^{An}(C|a_k))^2| < (q - k)$ , then we have two cases:
  - if there are unfilled seats under  $Ch^{An}(C|a_k)$ , i.e.,  $|C \setminus lc(C, a_{k+1})| < q$ , then, first,  $Ch^{An}(C|a_k)$  is wasteful; and second, we will accommodate more reserves for the trait  $r$  without affecting reserves for  $b$  by giving an empty seat to  $a_{k+1}$ ;
  - if there are no unfilled seats under  $Ch^{An}(C|a_k)$ , i.e.,  $|C \setminus lc(C, a_{k+1})| \geq q$ , then, we will accommodate more reserves for the trait  $r$  without affecting reserves for  $b$  by giving to  $a_{k+1}$  a seat taken from anyone from  $(Ch^{An}(C|a_k))^3$ , thus,  $Ch^{An}(C|a_k)$  has a justified envy.

Hence,  $Ch^{An}(C|a_k)$  is not stable, and the following holds:  $\Delta^{Ch^{An}(C|a_{k+1})}(b) = 0$  and  $\Delta^{Ch^{An}(C|a_k)}(r) > \Delta^{Ch^{An}(C|a_{k+1})}(r)$ .

- If  $|(Ch^{An}(C|a_k))^1| + |(Ch^{An}(C|a_k))^2| = (q - k)$ , then we have two cases:
  - If  $|(Ch^{An}(C|a_k))^1| > 0$ , then we will accommodate more reserves for the trait  $r$  without affecting reserves for  $b$  by giving to  $a_{k+1}$  a seat taken from anyone from  $(Ch^{An}(C|a_k))^1$ , thus,  $Ch^{An}(C|a_k)$  has a justified envy;
  - If  $|(Ch^{An}(C|a_k))^1| = 0$ , then  $|(Ch^{An}(C|a_k))^2| = (q - k)$ , then in  $Ch^{An}(C|a_k)$  all  $q$  candidates have the trait  $r$ , hence,  $\Delta^{Ch^{An}(C|a_k)}(r) = 0$ , contradiction.

Hence,  $Ch^{An}(C|a_k)$  is not stable, and the following holds:  $\Delta^{Ch^{An}(C|a_{k+1})}(b) = 0$  and  $\Delta^{Ch^{An}(C|a_k)}(r) > \Delta^{Ch^{An}(C|a_{k+1})}(r)$ .

This concludes the proof. □

**Proof of Lemma 4.** Note that since  $|Ch^{An}(C|a_k)| < q$ , then  $Ch^{An}(C|a_k) = C \setminus lc(C, a_{k+1})$ . Thus, there is an empty seat and a not chosen anchor  $a_{k+1}$ . So, first,  $Ch^{An}(C|a_k)$  is wasteful; and second, we will not affect reserves for traits  $b$  and  $r$  by giving an empty seat to  $a_{k+1}$ , while the total filled quota will increase. □

**Proof of Lemma 5.**  $Ch^{An}(C|a_{k+1})$  is non-wasteful, since  $|Ch^{An}(C|a_{k+1})| = q$ .

By Lemmas 2-4, the set  $Ch^{An}(C|a_k)$  is not stable. Thus, for any anchor  $a$  prior to  $a_{k+1}$  the only candidate for a stable set,  $Ch^{An}(C|a)$ , appears to be not stable due to inaccessibility of candidates from  $C \setminus lc(C, a)$ . Thus,  $Ch^{An}(C|a_{k+1})$  picks the best  $q$  candidates from  $C$  (from  $C \setminus lc(C, a_{k+2})$ , if  $a_{k+2}$  exists) that fully accommodate reserves for both traits, hence,  $Ch^{An}(C|a_{k+1})$  eliminates justified envy. □

**Proof of Lemma 6.**  $Ch^{An}(C|a_{\min\{q,m\}})$  is non-wasteful by construction.

By Lemmas 2-4, for any anchor  $a$  prior to  $a_{\min\{q,m\}}$  the only candidate for a stable set,  $Ch^{An}(C|a)$ , appears to be not stable due to inaccessibility of candidates from  $C \setminus lc(C, a)$ . Thus,  $Ch^{An}(C|a_{\min\{q,m\}})$  picks the best  $q$  candidates from  $C$  that try to, first, fully accommodate reserves for the trait  $b$ ; and second, fully accommodate reserves for the trait  $r$ , hence,  $Ch^{An}(C|a_{k+1})$  eliminates justified envy.

Since  $a_{\min\{q,m\}}$  is the last anchor, hence  $Ch^{An}(C|a_{\min\{q,m\}})$  is the only stable subset of  $C$ . □

**Proof of Lemma 7.** Suppose that all three such candidates exist, and that  $c_{\text{best}}^\times \pi c_{\text{worst}}^{\checkmark,b}$  and  $c_{\text{best}}^\times \pi c_{\text{worst}}^{\checkmark,r}$ . Thus, by construction the following holds:  $Ch^{An}(C|a_{k+1}) = (Ch^{An}(C|a_k) \setminus \{c_{\text{worst}}^{\checkmark,b}, c_{\text{worst}}^{\checkmark,r}\}) \cup \{c_{\text{best}}^\times, a_{k+1}\}$ , since by including  $a_{k+1}$  with both traits we now need to decrease each  $|(Ch^{An}(C|a_k))^1|$  and  $|(Ch^{An}(C|a_k))^2|$  by one, such that one seat from  $c_{\text{worst}}^{\checkmark,b}$  will go to  $a_{k+1}$ , and another seat from  $c_{\text{worst}}^{\checkmark,r}$  will go to the best not chosen candidate  $c_{\text{best}}^\times$ , who has higher rank than both  $c_{\text{worst}}^{\checkmark,b}$  and  $c_{\text{worst}}^{\checkmark,r}$ . Hence,  $Ch^{An}(C|a_{k+1})$  stays non-wasteful and still eliminates justified envy.

Suppose now that all three such candidates exist, but  $c_{\text{worst}}^{\check{,}b} \pi c_{\text{best}}^{\times} \pi c_{\text{worst}}^{\check{,}r}$ . Thus, by construction the following holds:  $Ch^{An}(C|a_{k+1}) = (Ch^{An}(C|a_k) \setminus \{c_{\text{worst}}^{\check{,}r}\}) \cup \{a_{k+1}\}$ . Hence, under  $Ch^{An}(C|a_{k+1})$  candidate  $c_{\text{worst}}^{\check{,}r}$  has a justified envy towards  $a_{k+1}$ .

Suppose now that all three such candidates exist, but  $c_{\text{worst}}^{\check{,}r} \pi c_{\text{best}}^{\times} \pi c_{\text{worst}}^{\check{,}b}$ . Thus, by construction the following holds:  $Ch^{An}(C|a_{k+1}) = (Ch^{An}(C|a_k) \setminus \{c_{\text{worst}}^{\check{,}b}\}) \cup \{a_{k+1}\}$ . Hence, under  $Ch^{An}(C|a_{k+1})$  candidate  $c_{\text{worst}}^{\check{,}b}$  has a justified envy towards  $a_{k+1}$ .

Suppose now that all three such candidates exist, but  $c_{\text{worst}}^{\check{,}b} \pi c_{\text{best}}^{\times}$  and  $c_{\text{worst}}^{\check{,}r} \pi c_{\text{best}}^{\times}$ . Thus, by construction the following holds:  $Ch^{An}(C|a_{k+1}) = (Ch^{An}(C|a_k) \setminus \{c_{\text{worst}}^{\check{,}r}\}) \cup \{a_{k+1}\}$ , where  $c_{\text{worst}}^{\check{,}r}$  is the worst chosen candidate. Hence, under  $Ch^{An}(C|a_{k+1})$  candidate  $c_{\text{worst}}^{\check{,}r}$  has a justified envy towards  $a_{k+1}$ .

If  $c_{\text{best}}^{\times}$  does not exist, then  $Ch^{An}(C|a_k)$  simply picks the best  $q$  candidates from  $C$ , thus it is the only stable subset of  $C$ .

If  $c_{\text{worst}}^{\check{,}b}$  does not exist, then

- either  $Ch^{An}(C|a_{k+1}) = (Ch^{An}(C|a_k) \setminus \{c_{\text{worst}}^{\check{,}r}\}) \cup \{a_{k+1}\}$ , hence, under  $Ch^{An}(C|a_{k+1})$  candidate  $c_{\text{worst}}^{\check{,}r}$  has a justified envy towards  $a_{k+1}$ ,
- or  $Ch^{An}(C|a_{k+1}) = (Ch^{An}(C|a_k) \setminus \{c_{\text{worst}}^{\check{,}b}\}) \cup \{a_{k+1}\}$ , hence, under  $Ch^{An}(C|a_{k+1})$  candidate  $c_{\text{worst}}^{\check{,}b}$  has a justified envy towards  $a_{k+1}$ .

Finally, if  $c_{\text{worst}}^{\check{,}r}$  does not exist, then

- either  $Ch^{An}(C|a_{k+1}) = (Ch^{An}(C|a_k) \setminus \{c_{\text{worst}}^{\check{,}b}\}) \cup \{a_{k+1}\}$ , hence, under  $Ch^{An}(C|a_{k+1})$  candidate  $c_{\text{worst}}^{\check{,}b}$  has a justified envy towards  $a_{k+1}$ ,
- or  $Ch^{An}(C|a_{k+1}) = (Ch^{An}(C|a_k) \setminus \{c_{\text{worst}}^{\check{,}r}\}) \cup \{a_{k+1}\}$ , hence, under  $Ch^{An}(C|a_{k+1})$  candidate  $c_{\text{worst}}^{\check{,}r}$  has a justified envy towards  $a_{k+1}$ .

This concludes the proof. □

**Proof of Lemma 8.** From Lemma 7 it follows that for the first anchor  $a$  (the one with the highest rank), such that  $\Delta^{Ch^{An}(C|a)}(b) = \Delta^{Ch^{An}(C|a)}(r) = 0$  and  $|Ch^{An}(C|a)| = q$ , and  $Ch^{An}(C|a)$  is not a stable subset of  $C$ , there always is a justified envy from some candidate  $c$  to another candidate  $c'$  under  $Ch^{An}(C|a)$ . Moreover, by construction of  $Ch^{An}(\cdot|\cdot)$ , this justified envy from  $c$  to  $c'$  will stay under  $Ch^{An}(C|a_j)$  for any  $j \geq k + 1$ . Thus,  $Ch^{An}(C|a_j)$  is not a stable subset of  $C$  for any  $j \geq k + 1$ . □

**Proof of Lemma 9.**  $Ch^{An}(C|a_0)$  is non-wasteful by construction.

Also,  $Ch^{An}(C|a_0)$  picks the best  $q$  candidates from  $C \setminus lc(C, a_1)$  that fully accommodate reserves for both traits, hence,  $Ch^{An}(C|a_0)$  eliminates justified envy. □

**Proof of Theorem 1.** By Proposition 4, any selection from the choice correspondence  $\mathbf{Ch}^{Ol}$  is stable.

Also by Proposition 4, if a choice rule is stable, then it should be a selection from the choice correspondence  $\mathbf{Ch}^{Ol}$ , since this correspondence always produces all stable subsets. □

**Proof of Proposition 5.** Follows from the proof of Lemma 7.  $\square$

**Proof of Proposition 8.** Fix an institution  $f$  with proposed set of pairs  $P \subseteq P^f$ . By construction  $Ch^{Opt}(P)$  picks a set from  $\mathbf{P}^b = \arg \min_{P' \subseteq P | P' \text{ is feasible}} \{\Delta^{(P')c}(b)\}$ .<sup>22</sup> Moreover, it picks a set from  $\mathbf{P}^{b,r} = \arg \min_{P' \in \mathbf{P}^b} \{\Delta^{(P')c}(r)\} \subseteq \mathbf{P}^b$ . Moreover, it picks a set from  $\mathbf{P}^{b,r,q} = \arg \max_{(P') \in \mathbf{P}^{b,r}} \{|P'|\} \subseteq \mathbf{P}^{b,r}$ . Furthermore, it picks a set from  $\mathbf{P}^{b,r,q,best} = \arg \min_{(P') \in \mathbf{P}^{b,r,q}} \{\sum_{c_k \in (P')_c} k\} \subseteq \mathbf{P}^{b,r,q}$ , where  $c_k$  is  $k$ -th best candidate in  $C^f$ .

$Ch^{Opt}$  eliminates justified envy, since it picks a set from  $\mathbf{P}^{b,r,q,best}$ .

$Ch^{Opt}$  eliminates house-seat-justified envy, since it picks a set from  $\mathbf{P}^{b,r,q,best}$ .

$Ch^{Opt}$  satisfies non-wastefulness, since it picks a set from  $\mathbf{P}^{b,r,q}$ .

$Ch^{Opt}$  satisfies non-wastefulness via house, since it picks a set from  $\mathbf{P}^{b,r,q}$ .

$Ch^{Opt}$  eliminates house-justified envy, since it picks a set from  $\mathbf{P}^{b,r,q,best}$ , such that houses are given only to the best possible candidates who cannot attend without a house, and during Step 2 houses are given to the most eligible for houses chosen candidates who need a house and do not have it.

$Ch^{Opt}$  satisfies non-house-wastefulness, since during Step 2 it tries to distribute all empty houses.

Thus,  $Ch^{Opt}$  is strongly stable.  $\square$

**Proof of Proposition 9.** First, if bH-pair is chosen and some higher ranked b-pair or bH-pair is not chosen, then there is a justified envy.

Second, if H-pair is chosen and some higher ranked pair is not chosen, then there is a justified envy.

Third, if NH-pair is chosen and some higher ranked b-pair or NH-pair is not chosen, then there is a justified envy.

Fourth, if b-pair is chosen and some higher ranked b-pair is not chosen, then there is a justified envy.  $\square$

**Proof of Proposition 10.** Since each candidate has at most one pair for each boarding school, then the following parts of stability are not applicable: non-wastefulness via house, house-justified envy, and house-seat justified envy.

$Ch^{bH-NOI}$  is non-wasteful, since it stops distributing seats only if there are no more houses and all not chosen candidates need a house.

$Ch^{bH-NOI}$  eliminates justified envy, since, first, it maximizes utilization of  $q_b$ ; and second, once  $q_b$  is maximally filled, it picks  $(c', h')$  and does not pick  $(c, h)$ , such that  $c\pi c'$ , only if there is no empty houses and  $h' = NH$  and  $h = H$ .

Suppose that there is a different stable choice rule  $Ch$ . Thus, there is a proposed set  $P$ , such that there is a pair  $(c, h)$ , such that  $(c, h) \in Ch^{bH-NOI}$  and  $(c, h) \notin Ch$  (otherwise,  $Ch$  can choose not feasible subset). If  $\tau(c) = b$ , then  $h = NH$ , then either  $Ch(P)$  is wasteful or there is a justified envy under  $Ch(P)$ , since it picks the best b-pairs during Step 1. If  $c$  has no traits and  $h = H$ , then either  $Ch(P)$  is wasteful or there is a justified envy under  $Ch(P)$ , since

<sup>22</sup>A subset  $P \subseteq P^f$  is feasible if it can be chosen by a choice rule.

it picks the best H-pairs during Steps 2.. If  $c$  has no traits and  $h = NH$ , then either  $Ch(P)$  is wasteful or there is a justified envy under  $Ch(P)$ , since it picks the best NH-pairs during Steps 2.. Thus,  $Ch$  is not stable.  $\square$

**Proof of Lemma 10.** By Proposition 10, any other subset is not stable.  $\square$

**Proof of Lemma 11.** Consider all cases one by one.

1.  $Ch^{bAn}(P|a_k)$  is wasteful, since by choosing  $a_{k+1}$  we will fill one more seat.
2.  $Ch^{bAn}(P|a_k)$  is wasteful, since by choosing  $a_{k+1}$  we will fill one more seat.
3. The only case when we can include  $a_{k+1}$  (and increase utilization of  $q_b$ ) is when he can take a house from not an anchor.
4. The only case when we can include  $a_{k+1}$  is when he can take a house from some lower ranked candidate.
5. The only case when we can include  $a_{k+1}$  (and increase utilization of  $q_b$ ) is when he can a seat from not an anchor.
6. The only case when we can include  $a_{k+1}$  is when he can take a seat from some lower ranked candidate.
7. The only case when we can include  $a_{k+1}$  (and increase utilization of  $q_b$ ) is when he can take a house from not an anchor.
8. The only case when we can include  $a_{k+1}$  is when he can take a house from some lower ranked candidate.

This concludes the proof of the lemma.  $\square$

**Proof of Lemma 12.** Note that once we switched and fixed an anchor  $a_k$  we calculate a subset  $Ch^{bAn}(P|a_k)$  disregarding only all lower ranked anchors  $\cup_{i=k+1}^{\min\{q_h, m\}} a_i$ . Hence, only these lower ranked anchors may cause not stability of  $Ch^{bAn}(P|a)$ . Moreover, in each case out of cases 3-8 from Lemma 11 there is a unique way of how not stability may be caused. Thus, if, given a case, not stability is not caused through this way, then  $Ch^{bAn}(P|a_k)$  is stable.

If  $a_k$  is the last anchor, then, first, during constructing  $Ch^{bAn}(P|a_k)$  we considered all pairs from  $P$ ; and second, all other anchors yield not stable subsets. Hence,  $Ch^{bAn}(P|a_k)$  is stable.  $\square$

**Proof of Lemma 13.** Suppose that all four requirements hold. By Lemma 11 under stable  $Ch^{bAn}(P|a_k)$  we can be in either case 4, or case 8, i.e., both  $q_H$  and  $q_b$  are exhausted, and only  $q$  may be not exhausted. Moreover, if  $q$  is not exhausted, then  $Ch^{bAn}(P|a_k)$  picks all b-pairs and NH-pairs. Thus, the only possible case is all three quotas,  $q$ ,  $q_H$  and  $q_b$ , are exhausted.

Thus, by construction,  $Ch^{bAn}(P|a_{k+1}) = (Ch^{bAn}(P|a_k) \setminus \{H_{\text{worst}}^{\checkmark}, b_{\text{worst}}^{\checkmark}\}) \cup \{a_{k+1}, NH_{\text{best}}^{\times}\}$ . Thus,  $Ch^{bAn}(P|a_{k+1})$  is stable, since again all three quotas,  $q$ ,  $q_H$  and  $q_b$  are exhausted, and  $Ch^{bAn}(P|a_k)$  was stable.

Suppose that there is no chosen by  $Ch^{bAn}(P|a_k)$  H-pair. Since there is no empty houses, it implies that  $a_k$  is the last anchor. Contradiction.

Suppose that there is the worst chosen by  $Ch^{bAn}(P|a_k)$  H-pair,  $H_{\text{worst}}^\vee$ , that has higher rank than  $a_{k+1}$ , but there is no chosen by  $Ch^{bAn}(P|a_k)$  b-pair. By construction,  $Ch^{bAn}(P|a_{k+1}) = (Ch^{bAn}(P|a_k) \setminus \{H_{\text{worst}}^\vee\}) \cup \{a_{k+1}\}$ . Thus,  $H_{\text{worst}}^\vee$  has a justified envy towards  $a_{k+1}$ .

Suppose that all hold but the existence of  $NH_{\text{best}}^\times$ . By construction,  $Ch^{bAn}(P|a_{k+1}) = (Ch^{bAn}(P|a_k) \setminus \{H_{\text{worst}}^\vee\}) \cup \{a_{k+1}\}$ . Thus,  $H_{\text{worst}}^\vee$  has a justified envy towards  $a_{k+1}$ .

Following the same logic it is straightforward to show that all four requirements should hold for stability of  $Ch^{bAn}(P|a_{k+1})$ .  $\square$

**Proof of Lemma 14.** If  $a_k$  is the first anchor that yields a stable subset, hence we are in one of cases 3-8 from Lemma 11, and the requirement of this case does not hold.

- **Case 3:**  $q$ : ne,  $q_H$ : e,  $q_b$ : ne, and there is no chosen H-pair. Hence,  $a_k$  is the last anchor. Contradiction.
- **Case 4:**  $q$ : ne,  $q_H$ : e,  $q_b$ : e, and there is no chosen H-pair that is lower ranked than  $a_{k+1}$ . By considering any anchor  $a_{k+l}$ , first, we can either stay in the case 4, or switch to 8; and second, there will not be chosen H-pair that is lower ranked than  $a_{k+l+1}$ . Thus,  $Ch^{bAn}(P|a_{k+l})$  is not stable for any  $l > 1$ .
- **Case 5:**  $q$ : e,  $q_H$ : ne,  $q_b$ : ne, and all chosen pairs are bH-pairs. This cannot be since  $q \geq q_H$ . Contradiction.
- **Case 6:**  $q$ : e,  $q_H$ : ne,  $q_b$ : e, and there is no chosen pair that is lower ranked than  $a_{k+1}$ . By considering any anchor  $a_{k+l}$ , we can either stay in the case 6, or switch to 8. If we stay in the case 6, then there will not be chosen pair that is lower ranked than  $a_{k+l+1}$ . If we switch to 8, then there will not be chosen H-pair that is lower ranked than  $a_{k+l+1}$ . Thus,  $Ch^{bAn}(P|a_{k+l})$  is not stable for any  $l > 1$ .
- **Case 7:**  $q$ : e,  $q_H$ : e,  $q_b$ : ne, and there is no chosen H-pair. Hence,  $a_k$  is the last anchor. Contradiction.
- **Case 8:**  $q$ : e,  $q_H$ : e,  $q_b$ : e, and there is no chosen H-pair that is lower ranked than  $a_{k+1}$ . By considering any anchor  $a_{k+l}$ , first, we will stay in the case 8; and second, there will not be chosen H-pair that is lower ranked than  $a_{k+l+1}$ . Thus,  $Ch^{bAn}(P|a_{k+l})$  is not stable for any  $l > 1$ .

If  $a_k$  is not the first anchor that yields a stable subset, then, by Lemma 13, we are in the case 8. Thus, from above,  $Ch^{bAn}(P|a_{k+l})$  is not stable for any  $l > 1$ .  $\square$

**Proof of Proposition 12.** Follows from Proposition 9 and the proof of Lemma 13.  $\square$

**Proof of Theorem 2.** By Proposition 11 any selection from the choice correspondence  $\mathbf{Ch}'$  from Step 1 of  $\mathbf{Ch}^{tD-OI}$  is stable. Moreover, by Proposition 12, if  $k_{\text{left}} < k_{\text{right}}$ , then for any

stable subset there are no empty houses. In Step 2 we consider unique previously found stable subset with empty houses and add new ones by distributing all amounts of empty houses across the most eligible for houses chosen candidates in need. Thus, all subsets in  $\mathbf{Ch}^{tD-Ol}(P)$  are stable.

Also by Proposition 11, if a choice rule is stable when no candidate propose two pairs, then it should be a selection from the choice correspondence  $\mathbf{Ch}'$  from Step 1 of  $\mathbf{Ch}^{tD-Ol}$ , since this correspondence always produces all stable subsets. Thus, by design of Step 2,  $\mathbf{Ch}^{tD-Ol}(P)$  contains all stable subsets of  $P$ .  $\square$

**Proof of Theorem 3.** Follows from the proof of Theorem 2 and Proposition 12.  $\square$

**Proof of Proposition 16.** Suppose that there is a chosen pair  $(c, h) \in Ch^D(P)$  that becomes not chosen once some other pair  $(\bar{c}, \bar{h})$  is excluded from  $P$ . Note that  $c$  and  $\bar{c}$  cannot be the same candidate for this to happen. Candidate  $c$  cannot be SDH or NDH, or WDH with two proposed pairs by construction of Steps 1.k. Moreover, candidate  $c$  cannot be WDH with one proposed pair by construction of Steps 2.k, since taking these steps implies that only WDH with one proposed pair are currently not chosen and after rejection of any pair  $(\bar{c}, \bar{h})$  there will be even more seats or seats and houses to distribute. Thus, there may not be such pair  $(c, h)$ . Contradiction.  $\square$

**Proof of Proposition 17.** It is straightforward to show that  $Ch^D$  satisfies LAD.

Proposition 16 implies that  $Ch^D$  satisfies substitutes condition. Substitutes condition together with LAD imply irrelevance of rejected pairs condition (Aygün and Sönmez, 2013).  $\square$

**Proof of Theorem 4.** Suppose that all candidates have fixed preferences and never definitely accept offers. Since both choice rules, horizontal envelope choice rule and  $Ch^D$ , satisfy substitutes condition, then the following holds. If candidate tentatively accepts a pair  $p$  at Step  $k$  and then accepts different pair  $p'$  at Step  $(k+1)$ , then  $p' \succ^c p$ . Thus, if at the end of some Step  $k$  a candidate  $c$  does not have a pair  $(c, h)$  in  $P_k^f$ , then he is no longer interested in this pair.

The horizontal envelope choice rule is one-to-one stable, thus the final matching is also one-to-one stable for regular institutions.

Fix a market with only one boarding school  $f$  that neglects all traits. We show that  $Ch^D$  produces non-wasteful via house matching that eliminates justified envy and house-seat-justified envy.

$Ch^D$  produces non-wasteful via house matching. Note that for a matching to be wasteful via house there should be a not admitted SDH candidate together with an empty seat, which is not possible by construction of Steps 1.k.

$Ch^D$  produces matching that eliminates house-seat-justified envy. Note that for a matching to have house-seat-justified envy there should be chosen WDH candidate with a house, which indicates that Steps 2.k were used. Also there should be a not chosen SDH candidate, which indicates that Steps 2.k were not used. Contradiction.

$Ch^D$  produces matching that eliminates justified envy. Note that for a matching to have justified envy there should be a not chosen candidate with higher rank and chosen candidate with lower rank, such that

- either former proposed a contract without a house, which is not possible by construction of Steps 1.k;
- or former is SDH (proposed only contract with a house) and either latter has a house or there is an empty house, which is not possible by construction of Steps 1.k.

$Ch^D$  may produce wasteful matching but only for not chosen WDH candidates with one proposed pair. Note that for a matching to be wasteful there should be a not admitted candidate who would like to and can be admitted to the boarding school, i.e. there should be an empty seat for him. Thus, this candidate cannot be SDH or NDH, or WDH with two proposed pairs by construction of Steps 1.k.

Thus the resulting matching prior to Step 2 is non-wasteful via house matching, eliminates justified envy, and house-seat-justified envy for boarding schools. Steps 2.k of  $Ch^D$  make sure that if a WDH candidate receives a house, then all chosen WDH candidates with higher house-eligibility rankings also receive a house in the final matching. Thus, Step 2 makes sure that final matching eliminates house-justified envy. Moreover, Steps 2.k make sure that if there is a not chosen WDH candidates with one proposed pair, empty seat, and an empty house, then there are not enough empty houses for chosen WDH candidates with higher house-eligibility ranks, so this candidate cannot be chosen in order not to cause house-justified envy. Thus, Step 2 makes sure that the final matching is non-wasteful.

Also, Step 2 makes sure that the final matching is non-house-wasteful, since it distributes maximum empty houses among the most eligible already admitted candidates. Thus, the final matching is also strongly stable for boarding schools.  $\square$

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