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► **To cite this version:**

Narayanan Harihar. Applications of the Implicit Inversion and Implicit Duality theorems. 2025. ⟨hal-04895387⟩

**HAL Id: hal-04895387**

**<https://hal.science/hal-04895387v1>**

Preprint submitted on 18 Jan 2025

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# Applications of the Implicit Inversion and Implicit Duality theorems

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## Abstract

This is intended to be the first chapter of a monograph entitled ‘Implicit linear algebra and its Applications’. Proofs are sometimes omitted. In every case, the proofs are available in the accompanying ‘Problems’ paper [24] as part of the solutions.

Linear systems often involve, as a basic building block, solutions of equations of the form

$$\begin{aligned}A_S f_S + A_P f_P &= 0 \\ A'_S f_S &= 0,\end{aligned}$$

where our primary interest might be in the vector variable  $f_P$ . Usually, neither  $f_S$  nor  $f_P$  can be written as a function of the other but they are linked through the linear relationship, that of  $(f_S, f_P)$  belonging to  $\mathcal{V}_{SP}$ , the solution space of the first of the two equations. Often  $\mathcal{V}_{SP}$ , by physical considerations, has a different nature from  $\mathcal{V}_S$ , the solution space of the second equation. For instance in electrical networks,  $\mathcal{V}_{SP}$  has a topological origin so the numbers can be treated more precisely than through floating-point arithmetic which might be required for dealing with  $\mathcal{V}_S$ , which could arise from ‘device- characteristic’. We may regard the final space of solutions  $\mathcal{V}_P$  as derived from the other two spaces by an operation, say, ‘ $\mathcal{V}_P = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S$ ’ ( $\mathcal{V}_P := \{f_P : (f_S, f_P) \in \mathcal{V}_{SP}, f_S \in \mathcal{V}_S\}$ ). This operation, together with subspace addition, intersection and linear relationships, can be used to build a version of linear algebra which we call ‘implicit linear algebra’ where linear transformations do not hold centre stage. There are two basic results - an ‘implicit inversion theorem’ which describes when  $\mathcal{V}_S$  can be obtained from  $\mathcal{V}_P$  and  $\mathcal{V}_{SP}$ , and an ‘implicit duality theorem’ which says  $\mathcal{V}_P^\perp = \mathcal{V}_{SP}^\perp \leftrightarrow \mathcal{V}_S^\perp$ . This paper gives a quick summary of the applications of these results, which should also give an idea of the applications of implicit linear algebra.

*Keywords:* Linear Dynamical Systems, State Space Theory, Behavioural Systems Theory, Controlled and Conditioned Invariant Spaces, Duality in Adjoint System, Implicit Linear Algebra, Implicit Duality Theorem

*2000 MSC:* 15A03, 15A04, 15A18

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## 1. Introduction

The basic ideas for the work described in this paper arose when electrical network topology was studied with the aim of building efficient circuit simulators ([9, 10, 11, 12, 13, 15]). An electrical network is defined by *topological constraints*, namely Kirchhoff’s current and voltage laws and *device- characteristic constraints*. Our approach, towards building circuit simulators, was to get maximum advantage out of manipulating the topological - rather than the device- characteristic, constraints. The topological constraints can be treated by efficient graph theoretic algorithms. Mixing them up, at an early stage, with the device- characteristic constraints brings in the inherently less efficient floating-point algorithms.

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Some fundamental questions here are: how to *connect* subnetworks, how to *decompose* networks into multiports, what is the *minimum number of variables* through which blocks of a partition of edges of the network *interact*, how to make *minimal changes* in the topology of the network for *faster simulation* etc. For solving these problems we have to deal with subspaces rather than with transformations. The spaces, that arise, can be regarded as subspaces of a ‘global’ space with a specified basis - in a vector  $(x_1, \dots, x_n)$ ,  $x_i$  refers to the variable associated with say the  $i^{\text{th}}$  edge or node. The variables of interest are currents and voltages associated with edges of the graph of the network and the injected currents and potentials at nodes. Often we would be interested in some of the variables which would be buried in a much larger set of network variables and we have to deal with the former *implicitly* without trying to *eliminate* the latter, since elimination destroys the topological structure of the constraints which we are trying to exploit.

When one connects subnetworks, taking the device independent point of view, there is no *direction* to the flow of ‘information’, merely *interaction*, which could be regarded as working both ways. To capture this situation, linear transformations are not suitable, rather one needs linear relations. To give an instance, a basic problem associated with ‘multiports’ (networks with certain edges specified as ‘ports’ which do not have any device constraints) is to obtain the voltage-current spaces associated with the ports, given the affine device- characteristic constraints of the multiport. It should be noted that the voltages (currents) of the devices usually will not uniquely determine the port voltages (port currents). Neither would the port quantities uniquely determine the ‘interior’ quantities. Therefore, even when the device- characteristic is linear, linear maps are not suitable for studying such problems.

The first applications of this approach dealt, as described above, with electrical networks. (Work in this area continued and is reported, for instance, in [16, 27, 28].) But soon it became clear that there would be applications wherever there is a duality associated with some form of orthogonality. Multivariable control systems appeared to be of this kind (a preliminary work is reported in [14]). This work was revived after the advent of behavioural system theory in the 90’s ([34, 35, 26]). The motivation was two fold. First, duality did not play as fundamental a role in behavioural system theory as it did in state variable based classical control theory. Use of ‘implicit linear algebra’ in particular, the implicit duality theorem (Theorem 5), can restore its central character ([17, 20, 21]). Secondly, since behavioural system algorithms were based on rings of polynomials, they were not as efficient as real matrix based state variable ones. Here again use of implicit linear algebra gives us algorithms which are as efficient as the latter, with the added advantage that no effort is needed to put the system into an artificial canonical form ([20, 21]). (The representation-free formulation of this paper based on the first derivative of the dynamic variable has been used by some behavioural theorists ([32], [33].)

There is some indication that ideas of implicit linear algebra are relevant to other areas of mathematics, such as matroids and number lattices ([29, 23, 22]).

Our aim in this paper is to give a quick overview of the applications of the implicit inversion and duality theorems. These two results are basic to implicit linear algebra. So this would also constitute a summary of the applications of implicit linear algebra, without first having to set up an elaborate notation. There is an accompanying paper of problems with solutions which would flesh out the sketch in this paper.

We now give a brief description of the notation of implicit linear algebra. In our approach we avoid the use of explicit maps and work rather with vector spaces and the operations sum, intersection, restriction, contraction and the operation  $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$ , of generalized minor. A vector space  $\mathcal{V}_S$  is a collection of functions  $f_S : S \rightarrow \mathfrak{R}$ , closed under addition and scalar multiplication,  $\mathcal{V}_{SP} := \mathcal{V}_{S \cup P}$ , where  $S, P$  are disjoint sets, sum  $\mathcal{V}_A + \mathcal{V}_B := \{(f_{A \setminus B}, f_{A \cap B} + g_{A \cap B}, g_{B \setminus A}) : (f_{A \setminus B}, f_{A \cap B}) \in \mathcal{V}_A, (g_{A \cap B}, g_{B \setminus A}) \in \mathcal{V}_B\}$ , intersection  $\mathcal{V}_A \cap \mathcal{V}_B := \{(f_{A \setminus B}, h_{A \cap B}, g_{B \setminus A}) : (f_{A \setminus B}, h_{A \cap B}) \in \mathcal{V}_A, (h_{A \cap B}, g_{B \setminus A}) \in \mathcal{V}_B\}$ , restriction  $\mathcal{V}_{SP} \circ S := \{f_S : (f_S, f_P) \in \mathcal{V}_{SP}\}$ , contraction  $\mathcal{V}_{SP} \times S := \{f_S : (f_S, 0_P) \in \mathcal{V}_{SP}\}$ ,  $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P := \{f_S : (f_S, f_P) \in \mathcal{V}_{SP}, f_P \in \mathcal{V}_P\}$ . (Note that the sum and intersection operations are mild generalizations of the usual sum and intersection of vector spaces.) These operations are defined in detail in Subsection 2.1.1, and Subsection 2.1.2.

We illustrate the use of the notation below.

- Instead of considering the map defined by  $f_S = Kf_P$ , we would look at the solution space  $\mathcal{V}_{SP}$  of the equation  $If_S - Kf_P = 0$ ; instead of saying  $K$  is symmetric, we would say the space  $\mathcal{V}_{SP}$  is

complementary orthogonal to the solution space  $\mathcal{V}'_{SP}$  of the equation  $Kf_S + If_P = 0$  (see for instance [19]).

- When we need to speak of feedback  $u = Kx$  on the system  $\mathcal{V}_{X\dot{X}U} := \{(\dot{x}, x, u) : \dot{x} = Ax + Bu\}$ , we would instead speak of the intersection  $\mathcal{V}_{XU} \cap \mathcal{V}_{X\dot{X}U}$  of the spaces  $\mathcal{V}_{XU} := \{(x, u) : u = Kx\}$  and  $\mathcal{V}_{X\dot{X}U}$  and its restriction  $(\mathcal{V}_{XU} \cap \mathcal{V}_{X\dot{X}U}) \circ X \uplus \dot{X}$  to the set  $X \uplus \dot{X}$  ( $X, \dot{X}$  being disjoint copies of each other with  $\dot{e} \in \dot{X}, e \in X$ , corresponding to each other); when we need to speak of injection  $\dot{x} = Ly$  into the system  $\mathcal{V}_{X\dot{X}Y} := \{(\dot{x}, x, y) : \dot{x} = Ax, y = Cy\}$ , we would instead speak of the sum  $\mathcal{V}_{\dot{X}Y} + \mathcal{V}_{X\dot{X}Y}$  of the spaces  $\mathcal{V}_{\dot{X}Y} := \{(\dot{x}, y) : \dot{x} = Ly\}$  and  $\mathcal{V}_{X\dot{X}Y}$  and its contraction  $(\mathcal{V}_{\dot{X}Y} + \mathcal{V}_{X\dot{X}Y}) \times X \uplus \dot{X}$  to the set  $X \uplus \dot{X}$  (see Subsection 5.1).
- When we need to speak of a controlled invariant space  $\mathcal{V}_X$  (i.e., a space that satisfies  $A(\mathcal{V}_X) \subseteq \mathcal{V}_X + \text{Im}(B)$ ) of the system  $\mathcal{V}_{X\dot{X}U} := \{(\dot{x}, x, u) : \dot{x} = Ax + Bu\}$ , we would instead say that  $(\mathcal{V}_{X\dot{X}U} \circ X \cup \dot{X}) \leftrightarrow (\mathcal{V}_X)_{\dot{X}} \supseteq \mathcal{V}_X$ ; ( $X, \dot{X}$  being disjoint copies of each other with  $\dot{e} \in \dot{X}, e \in X$ , corresponding to each other, and  $(\mathcal{V}_X)_{\dot{X}} := \{f_{\dot{X}}, f_X \in \mathcal{V}_X\}, f_{\dot{X}}(\dot{e}) = f_X(e), e \in X$ ); when we need to speak of a conditioned invariant space  $\mathcal{V}_X$  (i.e., a space that satisfies  $\mathcal{V}_X \supseteq A(\mathcal{V}_X \cap \text{Ker}(C))$ ) of the system  $\mathcal{V}_{X\dot{X}Y} := \{(\dot{x}, x, y) : \dot{x} = Ax, y = Cy\}$ , we would instead say that  $(\mathcal{V}_{X\dot{X}Y} \times X \cup \dot{X}) \leftrightarrow \mathcal{V}_X \subseteq (\mathcal{V}_X)_{\dot{X}}$  (see Subsection 5.2, [20]).

Our algorithms are all built in terms of the above mentioned vector space operations. They deal with the systems as given and therefore are capable of exploiting desirable properties, such as sparsity, if possessed by the original representation of the system.

In [24] we have shown how to translate results and algorithms with the state-input-output format to the above notation. This automatically shows how to compute the outputs of algorithms based on state-input-output format, starting instead from the way the systems were originally presented.

Section 2 is on preliminary definitions and results. Among others, matched and skewed compositions are defined. An important role is played by the operations of building copies, of sets and of vector spaces on copies of index sets, in increasing the effectiveness of these operations. Implicit inversion theorem (IIT) and implicit duality theorem (IDT) are stated with the proof available in the appendix.

Section 3 is on multiport decomposition and topological transformation of electrical networks.

Section 4 is on how input-output behaviour of linear systems with large number of inputs can be computed by effectively working with the transpose through the use of the implicit duality theorem.

Section 5 is on how dynamical systems can be handled implicitly in the spirit of behavioural system theory, but algorithmically more efficiently, through the use of IIT and IDT.

Section 6 is on how IIT and IDT are directly applicable to behavioural system theory in the computation of controllers for building a desired dynamical system.

Section 7 is on a general framework for implicit duality based on the ‘dot product’.

Section 8 is mainly on IDT for matroids which goes through under appropriate definitions even though there is no ‘dot product’. IIT holds for a special class of matroids.

The sections (3,4,5,6,7,8) depend only on the preliminary section (2) and are largely independent of each other. Whenever there is dependence on a preceding section, there is explicit mention with a star at the beginning of the subsection. Whenever a statement is made without proof, reference is made to the solution of a problem given in [24]. This paper is written tersely, but with the intention of giving the reader a fair idea of the subject without having to read the entire paper. For instance, to get an idea of what is possible through application of the theorems, scanning the preliminaries section and reading either of the sections on electrical networks (Section 3 (4 pages)) or dynamical systems (Section 5 (6 pages)) would be adequate.

## 2. Preliminaries

The preliminary results and the notation used are from [16] and more specifically from [20]. A **vector**  $\mathbf{f}$  on  $X$  over  $\mathbb{F}$  is a function  $f : X \rightarrow \mathbb{F}$  where  $\mathbb{F}$  is a field and would usually be denoted by  $\mathbf{f}_X$ . In this paper, we work primarily with the real field. We use  $X - Y$  in place of  $X \setminus Y$ , for better readability. When  $X, Y$  are disjoint,  $X \uplus Y$  denotes the disjoint union of  $X$  and  $Y$ . A vector  $f_{X \uplus Y}$  on  $X \uplus Y$  would be written as  $f_{XY}$  and would often be written as  $(f_X, f_Y)$  during operations dealing with such vectors. The **sets** on which vectors are defined will be referred to as **index** sets and will always be **finite**. When a vector  $x$  figures in an equation, we will use the convention that  $x$  denotes a column vector and  $x^T$  denotes a row vector such as in  $Ax = b, x^T A = b^T$ .

Let  $X \subseteq Y$ . The **restriction**  $\mathbf{f}_Y|_X$  of  $f_Y$  to  $X$  is defined as follows:  $f_Y|_X \equiv g_X$ , where  $g_X(e) = f_Y(e), e \in X$ .

If  $f_X$  is over  $\mathbb{F}$ ,  $\lambda \in \mathbb{F}$ , then  $\lambda \mathbf{f}_X$  is defined by  $(\lambda f_X)(e) \equiv \lambda[f_X(e)], e \in X$ .

If  $f_X$  and  $g_Y$  both are over  $\mathbb{F}$ , we define  $\mathbf{f}_X + \mathbf{g}_Y$  on  $X \cup Y$  by  $(f_X + g_Y)(e) \equiv f_X(e) + g_Y(e)$ , if  $e \in X \cap Y$ ,  $(f_X + g_Y)(e) \equiv f_X(e)$ , if  $e \in (X - Y)$ ,  $(f_X + g_Y)(e) \equiv g_Y(e)$ , if  $e \in (Y - X)$

When  $X \cap Y = \emptyset$  then  $f_X + g_Y$  is written as  $\mathbf{f}_X \oplus \mathbf{g}_Y$  or as  $(\mathbf{f}_X, \mathbf{g}_Y)$ . When  $f_X, g_X$  are over  $\mathbb{F}$  the **dot product**  $\langle f_X, g_X \rangle$  of  $f_X$  and  $g_X$  is defined by  $\langle f_X, g_X \rangle \equiv \sum_{e \in X} f_X(e)g_X(e)$ . We say  $f_X, g_X$  are **orthogonal** iff  $\langle f_X, g_X \rangle = 0$ . The vector  $0_X$  is defined by  $0_X(e) = 0, e \in X$ .

An **arbitrary collection** of vectors on  $X$  with  $0_X$  as a member would be denoted by  $\mathcal{K}_X$  and referred to as a **zero-member collection**. The zero-member collection  $\mathcal{K}_X$  is a **vector space** iff  $f_X, g_X \in \mathcal{K}$  implies  $\lambda f_X + \sigma g_X \in \mathcal{K}$  for  $\lambda, \sigma \in \mathbb{F}$ . We will use the symbol  $\mathcal{V}_X$  for vector space on  $X$  as opposed to  $\mathcal{K}_X$  for arbitrary collections of vectors on  $X$  with a zero vector as a member. The symbol  $\mathcal{F}_X$  refers to the **collection of all vectors** on  $X$  and  $\mathbf{0}_X$  to the **zero vector space** on  $X$  as well as the **zero vector** on  $X$ . The collection of all vectors orthogonal to vectors in  $\mathcal{K}_X$  is denoted by  $\mathcal{K}_X^\perp$ . When  $\mathcal{K}_X$  is a vector space we have  $(\mathcal{K}_X^\perp)^\perp = \mathcal{K}_X$ . When  $X, Y$  are disjoint we usually write  $\mathcal{K}_{XY}$  in place of  $\mathcal{K}_{X \uplus Y}$ . We do not distinguish between  $\mathcal{K}_{XY}$  and  $\mathcal{K}_{YX}$  except in the definition of copies (see Subsection 2.1.3). The collection  $\{(f_X, \lambda f_Y) : (f_X, f_Y) \in \mathcal{K}_{XY}\}$  is denoted by  $\mathcal{K}_{X(\lambda Y)}$  and when  $\lambda = -1$  we would write more simply  $\mathcal{K}_{X(-Y)}$ .

The **restriction** of  $\mathcal{K}_{SP}$  to  $S$ , denoted  $\mathcal{K}_{SP} \circ S$  is defined to be the collection of vectors  $h_S$  such that for some  $h_P, (h_S, h_P) \in \mathcal{K}_{SP}$ . The **contraction** of a zero-member collection  $\mathcal{K}_{SP}$  to  $S$ , denoted  $\mathcal{K}_{SP} \times S$  is defined to be the collection of vectors  $h_S$  such that  $(h_S, 0_P) \in \mathcal{K}_{SP}$ . Here  $\mathcal{K}_{XYZ} \circ \mathbf{XY}, \mathcal{K}_{XYZ} \times \mathbf{XY}$ , respectively, would denote  $\mathcal{K}_{XYZ} \circ \mathbf{X} \uplus \mathbf{Y}, \mathcal{K}_{XYZ} \times \mathbf{X} \uplus \mathbf{Y}$ . when  $X, Y, Z$  are pairwise disjoint. We simplify  $(\mathcal{K}_{XYZ} \times YZ) \circ Z, (\mathcal{K}_{XYZ} \circ YZ) \times Z$ , respectively to  $\mathcal{K}_{XYZ} \times YZ \circ Z, \mathcal{K}_{XYZ} \circ YZ \times Z$ . For vector spaces  $\mathcal{V}_S$ , we have the basic fact that the maximal linearly independent subset of vectors contained in any set of vectors  $\mathcal{K}_S \subseteq \mathcal{V}_S$  always has the same size. This is the **rank**  $r(\mathcal{K}_S)$  of  $\mathcal{K}_S$ . We have the following result on ranks of contraction and restriction. A proof may be found in [24].

**Theorem 1.**  $r(\mathcal{V}_{SP}) = r(\mathcal{V}_{SP} \circ S) + r(\mathcal{V}_{SP} \times P)$ .

### 2.1. Basic operations

The basic operations we use in this paper are as follows:

#### 2.1.1. Sum and Intersection

Let  $\mathcal{K}_X, \mathcal{K}_Y$  be zero-member collections of vectors on sets  $X, Y$  respectively. The **sum**  $\mathcal{K}_X + \mathcal{K}_Y$  of  $\mathcal{K}_X, \mathcal{K}_Y$  is defined over  $X \cup Y$  as follows:  $\mathcal{K}_X + \mathcal{K}_Y \equiv \{(f_X, 0_{Y-X}) + (0_{X-Y}, g_Y), \text{ where } f_X \in \mathcal{K}_X, g_Y \in \mathcal{K}_Y\}$ . When  $X, Y$  are disjoint,  $\mathcal{K}_X + \mathcal{K}_Y$  is usually written as  $\mathcal{K}_X \oplus \mathcal{K}_Y$  and is called the **direct sum**. Thus,  $\mathcal{K}_X + \mathcal{K}_Y = (\mathcal{K}_X \oplus \mathbf{0}_{Y-X}) + (\mathbf{0}_{X-Y} \oplus \mathcal{K}_Y)$ .

The **intersection**  $\mathcal{K}_X \cap \mathcal{K}_Y$  of  $\mathcal{K}_X, \mathcal{K}_Y$  is defined over  $X \cup Y$  as follows:  $\mathcal{K}_X \cap \mathcal{K}_Y \equiv \{f_{X \cup Y} : f_{X \cup Y} = (f_X, k_{Y-X}), f_{X \cup Y} = (h_{X-Y}, f_Y), \text{ where } f_X \in \mathcal{K}_X, f_Y \in \mathcal{K}_Y, k_{Y-X}, h_{X-Y} \text{ are arbitrary vectors on } Y - X, X - Y \text{ respectively}\}$ . Thus,  $\mathcal{K}_X \cap \mathcal{K}_Y = (\mathcal{K}_X \oplus \mathcal{F}_{Y-X}) \cap (\mathcal{F}_{X-Y} \oplus \mathcal{K}_Y)$ .

### 2.1.2. Matched and Skewed Composition

The **matched composition**  $\mathcal{K}_X \leftrightarrow \mathcal{K}_Y$ , where  $\mathcal{K}_X, \mathcal{K}_Y$ , are zero-member collections, is on  $(X - Y) \uplus (Y - X)$  and is defined as follows:

$$\mathcal{K}_X \leftrightarrow \mathcal{K}_Y := \{f : f = (g|_{X-Y}, h|_{Y-X}), \text{ where } g \in \mathcal{K}_X, h \in \mathcal{K}_Y \text{ \& } g|_{X \cap Y} = h|_{Y \cap X}\}.$$

(The matched composition is referred to as matched sum in [16].)

The **skewed composition**  $\mathcal{K}_X \Rightarrow \mathcal{K}_Y$  is on  $(X - Y) \uplus (Y - X)$  and is defined as follows:

$$\mathcal{K}_X \Rightarrow \mathcal{K}_Y := \{f : f = (g|_{X-Y}, h|_{Y-X}), \text{ where } g \in \mathcal{K}_X, h \in \mathcal{K}_Y \text{ \& } g|_{X \cap Y} = -h|_{Y \cap X}\}.$$

When  $X, Y$  are disjoint, the matched and skewed composition both correspond to direct sum.

The **generalized minor** of  $\mathcal{K}_{XY}$  relative to  $\mathcal{K}_Y$  is  $\mathcal{K}_{XY} \leftrightarrow \mathcal{K}_Y$ . When  $\mathcal{K}_{XY}, \mathcal{K}_Y$  are vector spaces, observe that  $\mathcal{K}_{XY} \leftrightarrow \mathcal{K}_Y = (\mathcal{K}_{XY} \Rightarrow \mathcal{K}_Y)$ . We note that  $\mathcal{K}_{XY} \circ X = \mathcal{K}_{XY} \leftrightarrow \mathcal{F}_Y$  and  $\mathcal{K}_{XY} \times X = \mathcal{K}_{XY} \leftrightarrow \mathbf{0}_Y$ .

When  $X, Y$  and  $Z$  are disjoint sets,  $(\mathcal{K}_{XYZ} \leftrightarrow \mathcal{K}_X) \leftrightarrow \mathcal{K}_Y = (\mathcal{K}_{XYZ} \leftrightarrow \mathcal{K}_Y) \leftrightarrow \mathcal{K}_X = \mathcal{K}_{XYZ} \leftrightarrow (\mathcal{K}_X \oplus \mathcal{K}_Y)$ . When  $\mathcal{K}_X := \mathbf{0}_X, \mathcal{K}_Y := \mathcal{F}_Y$ , the above reduces to  $\mathcal{K}_{XYZ} \times YZ \circ Z = \mathcal{K}_{XYZ} \circ XZ \times Z$ .

The ' $\leftrightarrow$ ' operation is not inherently associative. For instance, consider the expression  $\mathcal{K}_{AB} \leftrightarrow \mathcal{K}_{BC} \leftrightarrow \mathcal{K}_{BD}$ . We have  $(\mathcal{K}_{AB} \leftrightarrow \mathcal{K}_{BC}) \leftrightarrow \mathcal{K}_{BD} = (\mathcal{K}_{AB} \leftrightarrow \mathcal{K}_{BC}) \oplus \mathcal{K}_{BD}$  and  $(\mathcal{K}_{AB} \leftrightarrow \mathcal{K}_{BD}) \leftrightarrow \mathcal{K}_{BC} = (\mathcal{K}_{AB} \leftrightarrow \mathcal{K}_{BD}) \oplus \mathcal{K}_{BC}$ . Clearly, these expressions need not be identical in general. Next, consider the expression  $(\mathcal{K}_{AB} \leftrightarrow \mathcal{K}_{BC}) \leftrightarrow (\mathcal{K}_{CA} \leftrightarrow \mathcal{K}_{PQ})$ . Here, if we remove the brackets, we get a subexpression  $\mathcal{K}_{AB} \leftrightarrow \mathcal{K}_{BC} \leftrightarrow \mathcal{K}_{CA}$ , where the index set becomes null and the ' $\leftrightarrow$ ' operation is not defined for such a situation. We say such an expression contains a 'null' subexpression. Let us only look at expressions which do not contain null subexpressions. For such expressions, there is a special case where the brackets can be got rid of without ambiguity. This is when no index set occurs more than twice in the expression. Consider for instance the expression  $\mathcal{K}_{A_1 B_1} \leftrightarrow \mathcal{K}_{A_2 B_2} \leftrightarrow \mathcal{K}_{A_1 B_2}$ , where all the  $A_i, B_j$  are mutually disjoint. It is clear that this expression has a unique meaning, namely, the space of all  $(f_{A_2}, g_{B_1})$  such that there exist  $h_{A_1}, k_{B_2}$  with  $(h_{A_1}, g_{B_1}) \in \mathcal{K}_{A_1 B_1}, (f_{A_2}, k_{B_2}) \in \mathcal{K}_{A_2 B_2}, (h_{A_1}, k_{B_2}) \in \mathcal{K}_{A_1 B_2}$ . Essentially the terms of the kind  $p_{D_i}$  survive if they occur only once and they get coupled through the terms which occur twice. These latter don't survive in the final expression. Such an interpretation is valid even if we are dealing with signed index sets of the kind  $-A_i$ . (We remind the reader that  $\mathcal{K}_{(-A)_B}$  is the space of all vectors  $(-f_A, g_B)$  where  $(f_A, g_B)$  belongs to  $\mathcal{K}_{AB}$ .) If either  $-A_i$ , or  $A_i$  occurs, it counts as one occurrence of the index set  $A_i$ . We state this result as a theorem but omit the routine proof.

**Theorem 2.** *An expression of the form  $\leftrightarrow_{i,j,k,\dots} (\mathcal{K}_{(\pm A_i)(\pm B_j)(\pm C_k)\dots})$ , where the index sets  $\pm A_i, \pm B_j, \pm C_k, \dots$  are all mutually disjoint has a unique meaning provided it contains no null subexpression and no index set occurs more than twice in it.*

**Remark 1.** *The condition 'contains no null subexpression' is not necessary if we define an object called  $\mathcal{K}_\emptyset$ , with the property that  $\mathcal{K}_\emptyset \leftrightarrow \mathcal{K}_A = \mathcal{K}_A$ , for every  $\mathcal{K}_A$ , including  $\mathcal{K}_\emptyset$ . Whenever we have a term  $(\mathcal{K}_A \leftrightarrow \mathcal{K}_A)$  or  $(\mathcal{K}_A \leftrightarrow \mathcal{K}_{-A})$ , we treat it as  $\mathcal{K}_\emptyset$ .*

### 2.1.3. Building Copies

An operation that is very useful is that of building **copies** of sets and **copies** of collections of vectors on copies of sets. We say set  $A, A'$  are **copies of each other** iff there is a bijection, usually clear from the context, mapping  $e \in A$  to  $e' \in A'$ . We need this operation in order to talk, for instance, of the vector  $(v, i)$  where  $v$  is a voltage vector on the edge set of a graph while  $i$  is a current vector on the same set. We handle this by building a copy  $E'$  of the edge set  $E$  and say  $v$  is on  $E$  and  $i$  is on  $E'$ .

The vectors  $f_A$  and  $f_{A'}$  are said to be copies of each other iff  $f_{A'}(e') = f_A(e)$  when  $A$  and  $A'$  are copies of each other. If the vectors on  $A$  and  $A'$  are not copies of each other, they would be distinguished using notation, such as,  $f_A, \widehat{f_{A'}}$  etc. Vectors without subscripts like,  $x, \dot{x}$  and  $w, \dot{w}$  are not necessarily copies of each other.

When  $A, A'$  are copies of each other  $(\mathcal{K}_A)_{A'} := \{f_{A'} : f_{A'}(e') = f_A(e), f_A \in \mathcal{K}_A\}$ . If  $A', B', C'$  are copies of pairwise disjoint sets  $A, B, C$ , respectively, with  $A', B', C'$ , being pairwise disjoint,  $(\mathcal{K}_{ABC})_{A'B'C'} := \{f_{A'B'C'} : f_{A'B'C'}(e') = f_{ABC}(e), e \in A \uplus B \uplus C, e' \in A' \uplus B' \uplus C'\}$ , being copies of each other}. When  $A$  and  $A'$  are copies of each other, the notation for interchanging the positions of variables on  $A$  and  $A'$

in a collection  $\mathcal{K}_{AA'B}$  is given by  $(\mathcal{K}_{AA'B})_{A'AB}$ , that is  $(\mathcal{K}_{AA'B})_{A'AB} = \{(g_A, f_{A'}, h_B) \mid (f_A, g_{A'}, h_B) \in \mathcal{K}_{AA'B}, g_A \text{ being copy of } g_{A'}, f_{A'} \text{ being copy of } f_A\}$ .

Note that when we are not talking about copies the order of the disjoint index sets does not matter -  $\mathcal{K}_{XY}$  and  $\mathcal{K}_{YX}$  denote the same collection of vectors.

**Remark 2.** Working with copies is often needed when we use the matched composition operation. Consider the expression  $(\mathcal{V}_{AB} \leftrightarrow \mathcal{V}_{BC}) \leftrightarrow \mathcal{V}_{BC}$ . It can be seen that this is the same as  $(\mathcal{V}_{AB})_{AB'} \leftrightarrow (\mathcal{V}_{BC})_{B'C} \leftrightarrow \mathcal{V}_{BC}$ . The latter expression has no index set occurring more than twice. So has no need for brackets.

Suppose we want to write  $y = A(Ax)$ . Let  $X, Y, Z$  be pairwise disjoint copies of each other. Let  $\mathcal{V}_{XZ} := \lambda^T(I|A^T)$ ,  $\lambda$ , a real vector. Now take  $(z^T, y^T) := z^T(I|A^T)$ ,  $(x^T, z^T) := x^T(I|A^T)$ . Therefore we write  $y = A(Ax)$  as  $(x^T|y^T) \in (\mathcal{V}_{XZ})_{ZY} \leftrightarrow \mathcal{V}_{XZ} = \mathcal{V}_{XZ} \leftrightarrow (\mathcal{V}_{XZ})_{ZY}$ .

While dealing with voltage vector  $v_E$  and current vector  $i_E$  simultaneously in equations such as  $(A|B) \begin{pmatrix} v_E \\ i_E \end{pmatrix} = s$ , we might want to index the column set by  $E$  and its copy  $E'$  because we want to treat the row space of  $(A|B)$  as a vector space  $\mathcal{V}_{EE'}$ .

Interchanging the positions of variables is needed, for instance, to describe the adjoint of a dynamical system (see Section 5.4). Let  $\mathcal{V}_{W\dot{W}}$  be described by  $\dot{w} = Aw$ . (Here  $W, \dot{W}$ , are copies of each other.) The adjoint  $\mathcal{V}_{W\dot{W}}^a$  of this system would be described by  $\dot{w} = A^T w$ . In our notation, this becomes  $\mathcal{V}_{W\dot{W}}^a := (\mathcal{V}_{W\dot{W}}^\perp)_{\dot{W}(-W)}$ , noting that the space orthogonal to the one described by  $\dot{w} = Aw$  is  $w = -A^T \dot{w}$ .

## 2.2. Graphs and electrical networks

The motivation for the ideas in this paper was the attempt to understand the topological processes of electrical circuit theory and exploit them efficiently. So we require a few definitions from graph theory to state results from electrical networks. We follow [16].

An **undirected graph** is a triple  $(V, E, f^u)$ , where  $V$  is the set of **vertices**,  $E$  is the set of **edges**, and  $f^u$  is the **incidence function** which associates a pair of vertices with each edge. The **incidence function** defines the end vertices of the edges. Similarly, a **directed graph**  $\mathcal{G}$  (in brief **graph**  $\mathcal{G}$ ) is also a triple  $(V, E, f^d)$ , where  $f^d$  is the **incidence function** which associates with each edge an ‘‘ordered pair’’ of vertices and the sets  $V, E$  define the **vertices**, **edges**, respectively of the graph. When the edge set is  $E$  we say  $\mathcal{G}$  is on  $E$  and denote  $\mathcal{G}$  by  $\mathcal{G}_E$ . The **incidence function** defines the end vertices and also the direction of arrow for the edges.

We use terms such as path, connected component, tree, cotree, forest, coforest, loop, cutset, etc. in the conventional manner (see, for instance, [16]).

Let  $\mathcal{G}_E$  be a directed graph and Let  $T \subseteq E$ . Then  $\mathcal{G}_E \times (\mathbf{E} - \mathbf{T})$  denotes the graph obtained by removing the edges  $T$  from  $\mathcal{G}_E$  and fusing the end vertices of the removed edges.  $\mathcal{G}_E \circ (\mathbf{E} - \mathbf{T})$  denotes the graph obtained by removing the edges  $T$  from  $\mathcal{G}_E$  and removing the isolated vertices. (Note that  $\times, \circ$  are also used to denote vector space operations. However, the context would make clear whether the objects involved are graphs or vector spaces.)

We refer to the space of vectors  $v_E$ , which satisfy Kirchhoff’s Voltage Law (KVL) (sum of signed voltages around a loop equals zero) of the graph  $\mathcal{G}_E$ , by  $\mathcal{V}^v(\mathcal{G}_E)$  and to the space of vectors  $i_E$ , which satisfy Kirchhoff’s Current Law (KCL) (sum of signed currents leaving a node equals zero) of the graph  $\mathcal{G}_E$ , by  $\mathcal{V}^i(\mathcal{G}_E)$ . It is often convenient for us to deal with the pair  $(v_E, i_E)$  as a single vector  $(v_E, i_{E'})$  on  $E \uplus E'$ , where  $E'$  is a disjoint copy of  $E$ .

**Tellegen’s Theorem** ([16]) states that  $\mathcal{V}^v(\mathcal{G}_E) = \mathcal{V}^i(\mathcal{G}_E)^\perp$ . The following result on vector spaces associated with graphs is useful.

$$\begin{aligned} \mathcal{V}^v(\mathcal{G}_E \circ T) &= \mathcal{V}^v(\mathcal{G}_E) \circ T, & \mathcal{V}^v(\mathcal{G}_E \times T) &= \mathcal{V}^v(\mathcal{G}_E) \times T, & T &\subseteq E(\mathcal{G}_E); \\ \mathcal{V}^i(\mathcal{G}_E \circ T) &= \mathcal{V}^i(\mathcal{G}_E) \times T, & \mathcal{V}^i(\mathcal{G}_E \times T) &= \mathcal{V}^i(\mathcal{G}_E) \circ T, & T &\subseteq E(\mathcal{G}_E). \end{aligned}$$

An **electrical network**  $\mathcal{N}$  is a pair  $(\mathcal{G}, \mathcal{D})$ . Here  $\mathcal{G}$  is a directed graph and  $\mathcal{D}$  is a collection of pairs of vector functions of time  $(v(\cdot), i(\cdot))$ , where  $v(t), i(t)$  are vectors on the edge set of the graph.  $\mathcal{D}$  is called

the ‘device- characteristic’ of the network. A **solution** of  $\mathcal{N}$  is a pair  $(v(\cdot), i(\cdot))$ , satisfying

$$v(t) \in \mathcal{V}^v(\mathcal{G}), \quad i(t) \in \mathcal{V}^i(\mathcal{G}) \quad (KVL, KCL) \quad \text{and} \quad (v(\cdot), i(\cdot)) \in \mathcal{D}.$$

It is often convenient for us to deal with the pair  $(v_E, i_E)$  as a single vector  $(v_E, i_{E'})$  on  $E \uplus E'$ , where  $E'$  is a disjoint copy of  $E$  and  $\mathcal{G}_{E'}$  is a copy of  $\mathcal{G}_E$ . In this case  $\mathcal{D}$  would be a collection  $\mathcal{K}_{EE'}$  of vector functions  $(v_E(\cdot), i_{E'}(\cdot))$ , the solution would be a vector function  $(v_E(\cdot), i_{E'}(\cdot))$ , satisfying

$$v_E(t) \in \mathcal{V}^v(\mathcal{G}_E), \quad i_{E'}(t) \in \mathcal{V}^i(\mathcal{G}_{E'}) \quad (KVL, KCL) \quad \text{and} \quad (v_E(\cdot), i_{E'}(\cdot)) \in \mathcal{K}_{EE'}.$$

The network is **static** iff the vector functions are constant functions. An **electrical multiport**  $\mathcal{N}_P$ , is a network  $(\mathcal{G}_{EP}, \mathcal{D}_E \times \mathcal{D}_P)$ , where  $\mathcal{D}_P := \mathcal{F}_{PP'}$ , i.e., there are no device- characteristic constraints on  $P$  and the current and voltage vectors on  $P$  are only constrained by KCL and KVL equations of the graph. The device- characteristic  $\mathcal{D}_E$  is the constraint  $(v_E, i_{E'}) \in \mathcal{K}_{EE'}$ . A vector  $(v_{EP}, i_{E'P'}) \in \mathcal{D}_E \times \mathcal{D}_P$  iff  $(v_{EP}|_E, i_{E'P'}|_{E'}) \in \mathcal{D}_E$  and  $(v_{EP}|_P, i_{E'P'}|_{P'}) \in \mathcal{D}_P$ . We would usually write  $\mathcal{N}_P := (\mathcal{G}_{EP}, \mathcal{D}_E)$  since  $\mathcal{D}_P := \mathcal{F}_{PP'}$ .

### 2.3. Relationship of orthogonality to other vector space operations

If the zero-member collections of vectors, which go to make up the expression, are all closed under addition, it is clear that  $+, \cap, \leftrightarrow, \rightleftharpoons$ , yield collections which are closed under addition. It is also clear that  $+, \cap, \leftrightarrow, \rightleftharpoons$ , all yield vector spaces when they operate on vector spaces.

For any collection of vectors  $\mathcal{K}_X$ , the collection  $\mathcal{K}_X^\perp$  is defined by  $\mathcal{K}_X^\perp = \{g_X : \langle f_X, g_X \rangle = 0, f_X \in \mathcal{K}_X\}$ . Clearly,  $\mathcal{K}_X^\perp$  is a vector space for any collection of vectors  $\mathcal{K}_X$ . When  $\mathcal{V}_X$  is a vector space and  $X$  is finite, it can be shown that  $\mathcal{V}_X$  is **involutionary**, i.e.,  $(\mathcal{V}_X^\perp)^\perp = \mathcal{V}_X$ . This is fundamental for the results of the present work.

The following results are easy to see

$$\begin{aligned} (\mathcal{V}_X + \widehat{\mathcal{V}}_X)^\perp &= \mathcal{V}_X^\perp \cap \widehat{\mathcal{V}}_X^\perp, & X \text{ finite} \\ (\mathcal{V}_X \cap \widehat{\mathcal{V}}_X)^\perp &= \mathcal{V}_X^\perp + \widehat{\mathcal{V}}_X^\perp, & X \text{ finite} . \end{aligned}$$

When  $X, Y$  are disjoint, it is easily verified that  $(\mathcal{V}_X \oplus \mathcal{V}_Y)^\perp = \mathcal{V}_X^\perp \oplus \mathcal{V}_Y^\perp$ . When  $X, Y$  are not disjoint,  $\mathcal{V}_X + \mathcal{V}_Y := (\mathcal{V}_X \oplus \mathbf{0}_{Y-X}) + (\mathcal{V}_Y \oplus \mathbf{0}_{X-Y})$ . So

$$(\mathcal{V}_X + \mathcal{V}_Y)^\perp = (\mathcal{V}_X^\perp \oplus \mathcal{F}_{Y-X}) \cap (\mathcal{V}_Y^\perp \oplus \mathcal{F}_{X-Y}) = \mathcal{V}_X^\perp \cap \mathcal{V}_Y^\perp,$$

by the definition of intersection of vector spaces on two distinct sets.

Using  $(\mathcal{V}^\perp)^\perp = \mathcal{V}$ , we have  $(\mathcal{V}_X \cap \mathcal{V}_Y)^\perp = \mathcal{V}_X^\perp + \mathcal{V}_Y^\perp$ .

The following results can also be easily verified:

$$\begin{aligned} (\mathcal{V}_{XY} \circ X)^\perp &= \mathcal{V}_{XY}^\perp \times X \\ (\mathcal{V}_{XY} \times X)^\perp &= \mathcal{V}_{XY}^\perp \circ X \quad \text{using} \quad (\mathcal{V}^\perp)^\perp = \mathcal{V}. \end{aligned}$$

The above pair of results will be referred to as the **dot-cross duality**.

### 2.4. Implicit inversion and implicit duality

The results in this subsection were originally proved in [13, 14, 15]. A comprehensive account is in [16].

The following result is the **implicit inversion theorem**(IIT).

**Theorem 3.** Consider the equation  $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ} = \mathcal{V}_{SQ}$ . Then

1. given  $\mathcal{V}_{SP}, \mathcal{V}_{SQ}, \exists \mathcal{V}_{PQ}$ , satisfying the equation iff  $\mathcal{V}_{SP} \circ S \supseteq \mathcal{V}_{SQ} \circ S$  and  $\mathcal{V}_{SP} \times S \subseteq \mathcal{V}_{SQ} \times S$ ;
2. given  $\mathcal{V}_{SP}, \mathcal{V}_{PQ}, \mathcal{V}_{SQ}$  satisfying the equation, we have  $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{SQ} = \mathcal{V}_{PQ}$  iff  $\mathcal{V}_{SP} \circ P \supseteq \mathcal{V}_{PQ} \circ P$  and  $\mathcal{V}_{SP} \times P \subseteq \mathcal{V}_{PQ} \times P$ .

It can be shown that the special case where  $Q = \emptyset$  is equivalent to the above result. For convenience we state it below.

**Theorem 4.** *Let  $\mathcal{V}_S, \mathcal{V}_{SP}$  be vector spaces on  $S, S \uplus P$ , respectively.*

1.  $\mathcal{V}_S = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$  for some  $\mathcal{V}_P$ , iff  $\mathcal{V}_{SP} \circ S \supseteq \mathcal{V}_S$  and  $\mathcal{V}_{SP} \times S \subseteq \mathcal{V}_S$ ;
2.  $\mathcal{V}_P = \mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S$  iff  $\mathcal{V}_{SP} \circ P \supseteq \mathcal{V}_P$  and  $\mathcal{V}_{SP} \times P \subseteq \mathcal{V}_P$ .

The following result is the **implicit duality theorem** (IDT).

**Theorem 5.**  $(\mathcal{V}_{XY} \leftrightarrow \mathcal{V}_{YZ})^\perp = (\mathcal{V}_{XY}^\perp \Leftarrow \mathcal{V}_{YZ}^\perp)$ . In particular,  $(\mathcal{V}_{XY} \leftrightarrow \mathcal{V}_Y)^\perp = \mathcal{V}_{XY}^\perp \leftrightarrow \mathcal{V}_Y^\perp$ .

Observe that the dot-cross duality is also a consequence of the implicit duality theorem since

$$(\mathcal{V}_{XY} \times X)^\perp = (\mathcal{V}_{XY} \leftrightarrow 0_X)^\perp = \mathcal{V}_{XY}^\perp \leftrightarrow \mathcal{F}_X = \mathcal{V}_{XY}^\perp \circ X.$$

Proofs of Theorem 3 and Theorem 5 are given in the appendix.

Implicit duality theorem and its applications are dealt with in detail in [13], [15], [14], [16], [20]. There has been more recent interest in the applications of this result in [7], in the context of ‘Pontryagin duality’. Versions for other contexts (such as orthogonality being replaced by polarity, or by the constraint of dot product being an integer) are available in [16], for operators in [17], [18], for matroids in [29]. There is a detailed discussion of a general version of the implicit duality theorem based on the dot product in Section 7 and in [24].

## 2.5. Summary of applications of the two theorems

The main applications are related to dealing with systems as they are without having to put them into some canonical structure. This avoids linear algebraic operations which are computationally expensive for large systems.

### 2.5.1. In the present paper

- Implicit inversion theorem is used to formalize multiport decomposition and topological transformation of electrical networks. It enables us to obtain conditions for minimization of ports between parts of a network and for building a minimum size larger vector space which contains two given vector spaces as generalized minors. These conditions are used to construct algorithms for the relevant purposes (see Section 3).
- Variations of the implicit inversion theorem are used to formalize state feedback, output injection, controlled and conditioned invariant spaces and emulators (generalization of state equations) for dynamical systems not necessarily represented in the state-input-output form (see Section 5).
- Implicit duality theorem allows us to exploit duality inherent in the system. For instance, if the current space of an electrical network is decomposed into multiports, this automatically results in the decomposition of the voltage space. If we build a larger space containing given spaces as generalized minors, its complementary orthogonal space will contain the complementary orthogonal spaces of the given spaces as generalized minors (see Section 3).
- Implicit duality theorem is used to build transposes of implicitly defined systems and adjoints of dynamical systems, not necessarily represented in the state-input-output form. This brings out the inherent duality between state feedback and output injection and between controlled and conditioned invariant spaces. Also, adjoints of emulators of dynamical systems are shown to be emulators of the adjoint dynamical system (see Section 5).
- The two theorems are used to solve the problem of building controllers for behavioural systems (see Section 6).
- Implicit duality theorem holds for matroids under suitable interpretation of the ‘ $\leftrightarrow$ ’ operation. Implicit inversion theorem holds for some classes of matroids (see Section 8).

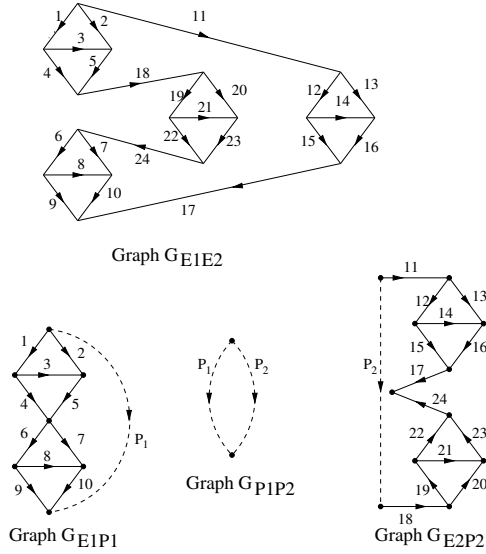


Figure 1: A graph and its multiport decomposition

### 2.5.2. In the accompanying ‘Problems’ paper

The ideas of the previous subsection are greatly expanded in the accompanying ‘Problems’ paper ([24]).

- A version of the implicit inversion theorem for affine spaces is applied to affine electrical multiports to construct port voltage-current behaviour in terms of device- characteristic. The two theorems are used to develop the idea of ‘rigid’ multiports (affine multiports which have solutions for arbitrary internal sources and whose internal voltages and currents uniquely fix the port voltages and currents). Rigid multiports capture affine multiports which can be handled by general purpose circuit simulators after suitable termination. By terminating rigid multiports by their adjoints and duals respectively, general versions of Thevenin-Norton and maximum power transfer theorems are developed, which do not depend upon existence of a hybrid representation for port behaviour.
- The two theorems are used to build a theory of generalized operators (‘genops’) which are defined implicitly. There is a spectral theory for genops which is analogous to the usual spectral theory of operators. Their use allows us to convert pole placement algorithms based on the state-input-output representation to that based on implicit representations for dynamical systems (such as a resistor, capacitor, inductor network).
- The two theorems are used to build a duality theory for behavioural systems which are not necessarily in the state-input-output form.

Applications of the implicit duality theorem for ‘polarity’ (cones) and ‘integrality’ (integral lattices) systems are described in [16, 22].

## 3. Multiport decomposition and topological transformation of electrical networks

There are two useful ideas of electrical network theory, which are essentially notions of implicit linear algebra, but which appear to have no counterparts in classical linear algebra. These are the methods called multiport decomposition and topological transformation (see [13], [15], [16]).

### 3.1. Multiport decomposition

*Multiport decomposition* is a classical method, characteristic of electrical network theory, both in analysis and synthesis (see for instance [4], [30]). Suppose a network  $\mathcal{N}$  has the graph  $\mathcal{G}_{E_1E_2}$  of Figure 1. Here  $E_1 := \{1, \dots, 10\}, E_2 := \{11, \dots, 24\}$ . Suppose also that the device-characteristic of the network has the current and voltage variables of  $E_1$ , independent of the current and voltage variables of  $E_2$  (for instance, it is composed entirely of resistors and sources). Then we can break up the network into two ‘multiports’ with graphs  $\mathcal{G}_{E_1P_1}, \mathcal{G}_{E_2P_2}$  and a ‘port connection graph’  $\mathcal{G}_{P_1P_2}$  as in the figure. Here the edges in  $P_1, P_2$ , have no device-characteristic constraints on their voltages and currents. As far as  $v_{E_1E_2}$  is concerned, the Kirchhoff voltage constraints of the graph  $\mathcal{G}_{E_1E_2}$  would be equivalent to the voltage constraints of  $\mathcal{G}_{E_1P_1}, \mathcal{G}_{E_2P_2}$  and  $\mathcal{G}_{P_1P_2}$  taken together and similarly also the Kirchhoff current constraints, as far as  $i_{E_1E_2}$  is concerned. Since the device-characteristic of  $E_1, E_2$  are independent we can treat the original network as being composed of two multiports whose ports satisfy the matching constraints imposed by the voltage and current constraints of the port connection graph.

We now examine this method from the point of view of the present work.

We are given a vector space  $\mathcal{V}_{E_1E_2}$  (say the solution space of the Kirchhoff voltage constraints of the graph  $\mathcal{G}_{E_1E_2}$ ) and we have to construct spaces  $\mathcal{V}_{E_1P_1}, \mathcal{V}_{E_2P_2}, \mathcal{V}_{P_1P_2}$ , such that  $\mathcal{V}_{E_1E_2} = (\mathcal{V}_{E_1P_1} \oplus \mathcal{V}_{E_2P_2}) \leftrightarrow \mathcal{V}_{P_1P_2}$ , i.e., when  $\mathcal{V}_{E_1P_1}, \mathcal{V}_{E_2P_2}$  are coupled through  $\mathcal{V}_{P_1P_2}$ , we obtain  $\mathcal{V}_{E_1E_2}$ . It is desirable from a computational as well as theoretical point of view, that  $P_1$  and  $P_2$  have the minimum size. A related question is where we are given  $\mathcal{V}_{E_1E_2}$  and the pair  $\mathcal{V}_{E_1P_1}, \mathcal{V}_{E_2P_2}$  and asked whether the former can be obtained by ‘coupling’ the latter pair through some unspecified  $\mathcal{V}_{P_1P_2}$ . Both these questions can be answered by using the implicit inversion theorem (Theorem 3).

#### 3.1.1. Minimum number of ports between complementary subsets of edges

By the implicit inversion theorem, we can couple  $\mathcal{V}_{E_1P_1}, \mathcal{V}_{E_2P_2}$  to obtain  $\mathcal{V}_{E_1E_2}$ , iff  $(\mathcal{V}_{E_1P_1} \times E_1) \oplus (\mathcal{V}_{E_2P_2} \times E_2) = (\mathcal{V}_{E_1P_1} \oplus \mathcal{V}_{E_2P_2}) \times E_1E_2 \subseteq \mathcal{V}_{E_1E_2}$  and  $\mathcal{V}_{E_1E_2} \subseteq (\mathcal{V}_{E_1P_1} \oplus \mathcal{V}_{E_2P_2}) \circ E_1E_2 = (\mathcal{V}_{E_1P_1} \circ E_1) \oplus (\mathcal{V}_{E_2P_2} \circ E_2)$ . This is equivalent to  $\mathcal{V}_{E_iP_i} \times E_i \subseteq \mathcal{V}_{E_1E_2} \times E_i, i = 1, 2$ , together with  $\mathcal{V}_{E_iP_i} \circ E_i \supseteq \mathcal{V}_{E_1E_2} \circ E_i, i = 1, 2$ . Now by Theorem 1 we have,  $r(\mathcal{V}_{E_iP_i} \circ E_i) + r(\mathcal{V}_{E_iP_i} \times P_i) = r(\mathcal{V}_{E_iP_i} \times E_i) + r(\mathcal{V}_{E_iP_i} \circ P_i), i = 1, 2$ , so that  $|P_i| \geq r(\mathcal{V}_{E_iP_i} \circ P_i) - r(\mathcal{V}_{E_iP_i} \times P_i) = r(\mathcal{V}_{E_iP_i} \circ E_i) - r(\mathcal{V}_{E_iP_i} \times E_i) \geq r(\mathcal{V}_{E_1E_2} \circ E_1) - r(\mathcal{V}_{E_1E_2} \times E_1) = r(\mathcal{V}_{E_1E_2} \circ E_2) - r(\mathcal{V}_{E_1E_2} \times E_2)$ . In Subsection 3.1.3, we show how to achieve the above lower bound.

#### 3.1.2. Port decomposition of $\mathcal{V}_{E_1E_2}$ equivalent to port decomposition of $\mathcal{V}_{E_1E_2}^\perp$

Next, it is desirable that the decomposition of  $\mathcal{V}_{E_1E_2}$  also leads to a corresponding decomposition of  $\mathcal{V}_{E_1E_2}^\perp$ , since by Tellegen’s Theorem, the voltage and current spaces of a graph are complementary orthogonal. Implicit duality theorem (Theorem 5) assures us that this is automatic, i.e.,  $\mathcal{V}_{E_1E_2} = (\mathcal{V}_{E_1P_1} \oplus \mathcal{V}_{E_2P_2}) \leftrightarrow \mathcal{V}_{P_1P_2}$  is equivalent to  $\mathcal{V}_{E_1E_2}^\perp = (\mathcal{V}_{E_1P_1}^\perp \oplus \mathcal{V}_{E_2P_2}^\perp) \leftrightarrow \mathcal{V}_{P_1P_2}^\perp$ .

#### 3.1.3. Construction of minimum multiport decomposition of $\mathcal{V}_{E_1E_2}$

We show below how to construct a multiport decomposition of  $\mathcal{V}_{E_1E_2}$ , where  $|P_i| = r(\mathcal{V}_{E_1E_2} \circ E_1) - r(\mathcal{V}_{E_1E_2} \times E_1) = r(\mathcal{V}_{E_1E_2} \circ E_2) - r(\mathcal{V}_{E_1E_2} \times E_2)$ .

We first construct bases  $b_{E_1}, b_{E_2}$ , of  $\mathcal{V}_{E_1E_2} \times E_1, \mathcal{V}_{E_1E_2} \times E_2$ , which are the sets of rows of  $(R_{11}), (R_{22})$ , respectively. Note that rows of  $(R_{11} | \mathbf{0}_{1E_2}), (\mathbf{0}_{2E_1}, R_{22})$  are vectors in  $\mathcal{V}_{E_1E_2}$ , which are together linearly independent. Extend this set to a basis of  $\mathcal{V}_{E_1E_2}$ , by adding vectors which form the set of rows of  $(R_{31} | R_{32})$ .

This basis is the set of rows of the matrix  $\begin{pmatrix} R_{11} & \mathbf{0}_{1E_2} \\ \mathbf{0}_{2E_1} & R_{22} \\ R_{31} & R_{32} \end{pmatrix}$ . It is easy to show that rows of  $\begin{pmatrix} R_{11} \\ R_{31} \end{pmatrix}, \begin{pmatrix} R_{22} \\ R_{32} \end{pmatrix}$ , are bases for  $\mathcal{V}_{E_1E_2} \circ E_1, \mathcal{V}_{E_1E_2} \circ E_2$ , respectively.

Next, build the matrices  $\begin{pmatrix} R_{11} & \mathbf{0}_{1P_1} \\ R_{31} & \mathbf{I}_{3P_1} \end{pmatrix}, \begin{pmatrix} \mathbf{I}_{3P_2} & R_{32} \\ \mathbf{0}_{2P_2} & R_{22} \end{pmatrix}$ , where  $\mathbf{I}_{3P_1}, \mathbf{I}_{3P_2}$  both denote an identity matrix of size equal to the number of rows of  $(R_{31} | R_{32})$ . Let their rows form bases of  $\mathcal{V}_{E_1P_1}, \mathcal{V}_{P_2E_2}$ , respectively. Let the rows of  $(\mathbf{I}_{3P_1} \quad \mathbf{I}_{3P_2})$  form the bases for a vector space  $\mathcal{V}_{P_1P_2}$ .

We now show that  $(\mathcal{V}_{E_1P_1} \oplus \mathcal{V}_{P_2E_2}) \leftrightarrow \mathcal{V}_{P_1P_2} = \mathcal{V}_{E_1E_2}$ .

Suppose  $(f_{E_1}, f_{E_2}) \in \mathcal{V}_{E_1E_2}$ . Then we can write  $(f_{E_1}, f_{E_2}) = \lambda_1^T(R_{11}|\mathbf{0}_{1E_2}) + \lambda_2^T(\mathbf{0}_{2E_1}|R_{22}) + \lambda_3^T(R_{31}|R_{32})$ . Now  $\lambda_1^T(R_{11}|\mathbf{0}_{1P_1}) + \lambda_3^T(R_{31}|\mathbf{1}_{3P_1}) \in \mathcal{V}_{E_1P_1}$ ,  $\lambda_3^T(\mathbf{1}_{3P_2}|R_{32}) + \lambda_2^T(\mathbf{0}_{2P_2}|R_{22}) \in \mathcal{V}_{P_2E_2}$  and  $\lambda_3^T(\mathbf{1}_{3P_1}|\mathbf{1}_{3P_2}) \in \mathcal{V}_{P_1P_2}$ . Therefore  $(f_{E_1}, \lambda_3^T) \in \mathcal{V}_{E_1P_1}$ ,  $(\lambda_3^T, f_{E_2}) \in \mathcal{V}_{P_2E_2}$  and  $(\lambda_3^T, \lambda_3^T) \in \mathcal{V}_{P_1P_2}$ .

Thus  $(\mathcal{V}_{E_1P_1} \oplus \mathcal{V}_{P_2E_2}) \leftrightarrow \mathcal{V}_{P_1P_2} \supseteq \mathcal{V}_{E_1E_2}$ .

Next, let  $(f_{E_1}, \lambda_3^T) \in \mathcal{V}_{E_1P_1}$ ,  $(\lambda_3^T, f_{E_2}) \in \mathcal{V}_{P_2E_2}$  and  $(\lambda_3^T, \lambda_3^T) \in \mathcal{V}_{P_1P_2}$ . It is clear that  $(f_{E_1}, f_{E_2}) = \lambda_1^T(R_{11}|\mathbf{0}_{1E_2}) + \lambda_2^T(\mathbf{0}_{2E_1}|R_{22}) + \lambda_3^T(R_{31}|R_{32})$ .

Therefore  $(\mathcal{V}_{E_1P_1} \oplus \mathcal{V}_{P_2E_2}) \leftrightarrow \mathcal{V}_{P_1P_2} \subseteq \mathcal{V}_{E_1E_2}$ .

We have shown how to construct vector spaces  $\mathcal{V}_{E_1P_1}$ ,  $\mathcal{V}_{P_2E_2}$ ,  $\mathcal{V}_{P_1P_2}$  by linear algebraic methods. When  $\mathcal{V}_{E_1E_2} = \mathcal{V}^v(\mathcal{G}_{E_1E_2})$  we can build graphs  $\mathcal{G}_{E_1P_1}$ ,  $\mathcal{G}_{P_2E_2}$ ,  $\mathcal{G}_{P_1P_2}$  such that  $|P_i|$  is minimal under  $\mathcal{V}^v(\mathcal{G}_{E_1E_2}) = (\mathcal{V}^v(\mathcal{G}_{E_1P_1}) \oplus \mathcal{V}^v(\mathcal{G}_{P_2E_2})) \leftrightarrow \mathcal{V}^v(\mathcal{G}_{P_1P_2})$ ,  $\mathcal{V}^i(\mathcal{G}_{E_1E_2}) = (\mathcal{V}^i(\mathcal{G}_{E_1P_1}) \oplus \mathcal{V}^i(\mathcal{G}_{P_2E_2})) \leftrightarrow \mathcal{V}^i(\mathcal{G}_{P_1P_2})$ , by linear time algorithms ([13], [16]), [24]. We note that minimal multiport decomposition can be counter intuitive. For instance, in Figure 1, the subgraphs on  $E_1$  and  $E_2$  have four common nodes. At first sight, one might be tempted to put three common ports between the corresponding multiports. But the algorithm would come up with the single common port as shown in the figure.

#### 3.1.4. Use of multiport decomposition in circuit simulators

We now indicate how this method is used in practice for network analysis. Let the network have graph  $\mathcal{G}_{E_1E_2}$ . Suppose the network constraints have the form

$$(R_1 \quad R_2) \begin{pmatrix} i_{E_1} \\ i_{E_2} \end{pmatrix} = \mathbf{0} \quad (\text{KCE for } \mathcal{G}_{E_1E_2}) \quad (1)$$

$$\begin{pmatrix} v_{E_1} \\ v_{E_2} \end{pmatrix} - \begin{pmatrix} R_1^T \\ R_2^T \end{pmatrix} v_n = \mathbf{0} \quad (\text{KVE for } \mathcal{G}_{E_1E_2}) \quad (2)$$

$$M_1 v_{E_1} + N_1 i_{E_1} = s_1 \quad (\text{device characteristic for } E_1) \quad (3)$$

$$M_2 v_{E_2} + N_2 i_{E_2} = s_2 \quad (\text{device characteristic for } E_2) \quad (4)$$

Let us suppose that  $\mathcal{G}_{E_1E_2}$  has been decomposed into the two multiport graphs  $\mathcal{G}_{E_1P_1}$ ,  $\mathcal{G}_{E_2P_2}$  and the port connection graph  $\mathcal{G}_{P_1P_2}$ . For the two multiports and the port connection graph, the KCE equations will translate to

$$(\hat{R}_1 \quad \hat{R}_{P_1}) \begin{pmatrix} i_{E_1} \\ i_{P_1} \end{pmatrix} = \mathbf{0}, (\hat{R}_2 \quad \hat{R}_{P_2}) \begin{pmatrix} i_{E_2} \\ i_{P_2} \end{pmatrix} = \mathbf{0}, (R_{P_1} \quad R_{P_2}) \begin{pmatrix} i_{P_1} \\ i_{P_2} \end{pmatrix} = \mathbf{0}. \quad (5)$$

The KV equations will translate to

$$\begin{pmatrix} v_{E_1} \\ v_{P_1} \end{pmatrix} - \begin{pmatrix} \hat{R}_1^T \\ \hat{R}_{P_1}^T \end{pmatrix} v_n^1 = \mathbf{0}, \begin{pmatrix} v_{E_2} \\ v_{P_2} \end{pmatrix} - \begin{pmatrix} \hat{R}_2^T \\ \hat{R}_{P_2}^T \end{pmatrix} v_n^2 = \mathbf{0}, \begin{pmatrix} v_{P_1} \\ v_{P_2} \end{pmatrix} - \begin{pmatrix} R_{P_1}^T \\ R_{P_2}^T \end{pmatrix} v_n^P = \mathbf{0}. \quad (6)$$

Thus we have two multiport networks with constraints

$$(\hat{R}_1 \quad \hat{R}_{P_1}) \begin{pmatrix} i_{E_1} \\ i_{P_1} \end{pmatrix} = \mathbf{0}, \begin{pmatrix} v_{E_1} \\ v_{P_1} \end{pmatrix} - \begin{pmatrix} \hat{R}_1^T \\ \hat{R}_{P_1}^T \end{pmatrix} v_n^1 = \mathbf{0}, M_1 v_{E_1} + N_1 i_{E_1} = s_1, \quad (7)$$

$$(\hat{R}_2 \quad \hat{R}_{P_2}) \begin{pmatrix} i_{E_2} \\ i_{P_2} \end{pmatrix} = \mathbf{0}, \begin{pmatrix} v_{E_2} \\ v_{P_2} \end{pmatrix} - \begin{pmatrix} \hat{R}_2^T \\ \hat{R}_{P_2}^T \end{pmatrix} v_n^2 = \mathbf{0}, M_2 v_{E_2} + N_2 i_{E_2} = s_2. \quad (8)$$

and the port connection graph with constraints

$$(R_{P_1} \quad R_{P_2}) \begin{pmatrix} i_{P_1} \\ i_{P_2} \end{pmatrix} = \mathbf{0}, \begin{pmatrix} v_{P_1} \\ v_{P_2} \end{pmatrix} - \begin{pmatrix} R_{P_1}^T \\ R_{P_2}^T \end{pmatrix} v_n^P = \mathbf{0}. \quad (9)$$

If  $P_1, P_2$ , are chosen minimally, under mild genericity conditions on the entries of  $M_1, M_2, N_1, N_2$  we may assume that the two multiports have unique solutions for arbitrary values for  $i_{P_1}, i_{P_2}$ . If we solve the two multiports for  $i_{P_1} = 0, i_{P_2} = 0$  but with device- characteristic as  $M_1 v_{E_1} + N_1 i_{E_1} = s_1, M_2 v_{E_2} + N_2 i_{E_2} = s_2$ , and then with  $s_i = 0, i = 1, 2$ , but with  $i_{P_1}, i_{P_2}$  as columns of the identity matrix, we can obtain the relations  $v_{E_1} = K_1 i_{P_1} + \alpha_1, i_{E_1} = Q_1 i_{P_1} + \beta_1, v_{E_2} = K_2 i_{P_2} + \alpha_2, i_{E_2} = Q_2 i_{P_2} + \beta_2$ , and  $M_{P_1} v_{P_1} + N_{P_1} i_{P_1} = s_{P_1}, M_{P_2} v_{P_2} + N_{P_2} i_{P_2} = s_{P_2}$ . We treat the relation involving  $i_{P_j}, v_{P_j}, j = 1, 2$ , as the device- characteristic of a network with  $\mathcal{G}_{P_1 P_2}$  as its graph. Solving this network yields  $i_{P_1}, i_{P_2}$ , and from this using the above first two relations we can find  $i_{E_1}, v_{E_1}, i_{E_2}, v_{E_2}$ .

**Remark 3.** Usually, the two multiports are chosen so that when the ports are deleted, they would have several disconnected pieces. When  $i_{P_1}, i_{P_2}$  are treated as current sources, this would result in block diagonal coefficient matrix for Equations 7, 8.

Certain real life problems, which can be converted to electrical network solution problems, naturally lend themselves to solution by this method because the relevant multiports are immediately available (for instance when the problem involves the topological coupling of a number of partial differential equations ([27])).

**Remark 4.** A basic problem involving practical electrical networks is the computation of port behaviour of multiports with affine device- characteristics using widely available circuit simulators. These latter can only compute the solution of an affine electrical network provided it is unique. The port behaviour of the above mentioned multiport is computed by solving networks obtained by suitable terminations of the multiport. The general solution of this problem is by terminating the multiport by suitable affine translations of its ‘adjoint’ (see Section 5 of the present paper and subsection on ‘Electrical multiports and adjoints’ in [24]). This problem is discussed in [24] (the subsections on ‘rigid multiports’ and ‘Computation of port behaviour of rigid multiports:generalized Thevenin-Norton Theorem’).

### 3.2. Topological transformation

In the method of *topological transformation* [15], an electrical circuit is treated as though it has a more convenient topology with the same device- characteristic but with some additional constraints. The idea is to exploit the new topology while meeting the additional constraints. In the language of this paper, we have two vector spaces  $\mathcal{V}_E^1, \mathcal{V}_E^2$  and we need to find a space  $\mathcal{V}_{EP}$  and vector spaces  $\mathcal{V}_P^1, \mathcal{V}_P^2$ , such that

$$\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P^1 = \mathcal{V}_E^1, \quad \mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P^2 = \mathcal{V}_E^2, \quad \text{and } |P| \text{ is the minimum possible.}$$

If we find such a space, by implicit duality theorem (Theorem 5) it follows that

$$\mathcal{V}_{EP}^\perp \leftrightarrow (\mathcal{V}_P^1)^\perp = (\mathcal{V}_E^1)^\perp, \quad \mathcal{V}_{EP}^\perp \leftrightarrow (\mathcal{V}_P^2)^\perp = (\mathcal{V}_E^2)^\perp,$$

and  $|P|$  is the minimum possible. We will now show that this minimum size is  $d(\mathcal{V}_E^1, \mathcal{V}_E^2) := r(\mathcal{V}_E^1 + \mathcal{V}_E^2) - r(\mathcal{V}_E^1 \cap \mathcal{V}_E^2)$ .

By the implicit inversion theorem (Theorem 3), it follows that  $\mathcal{V}_{EP} \circ E \supseteq \mathcal{V}_E^i, \mathcal{V}_{EP} \times E \subseteq \mathcal{V}_E^i, i = 1, 2$ . Therefore  $r(\mathcal{V}_{EP} \circ E) \geq r(\mathcal{V}_E^1 + \mathcal{V}_E^2)$  and  $r(\mathcal{V}_{EP} \times E) \leq r(\mathcal{V}_E^1 \cap \mathcal{V}_E^2)$ . Now by Theorem 1, we have  $r(\mathcal{V}_{EP}) = r(\mathcal{V}_{EP} \circ E) + r(\mathcal{V}_{EP} \times P) = r(\mathcal{V}_{EP} \circ P) + r(\mathcal{V}_{EP} \times E)$ , so that  $r(\mathcal{V}_{EP} \circ E) - r(\mathcal{V}_{EP} \times E) = r(\mathcal{V}_{EP} \circ P) - r(\mathcal{V}_{EP} \times P) \leq |P|$ . Therefore  $r(\mathcal{V}_E^1 + \mathcal{V}_E^2) - r(\mathcal{V}_E^1 \cap \mathcal{V}_E^2) \leq |P|$ . In the next subsection, we show how to achieve this lower bound.

#### 3.2.1. Construction of minimal $\mathcal{V}_{EP}$ with $\mathcal{V}_E^1, \mathcal{V}_E^2$ , as generalized minors

We will now construct a  $\mathcal{V}_{EP}$  such that  $r(\mathcal{V}_E^1 + \mathcal{V}_E^2) - r(\mathcal{V}_E^1 \cap \mathcal{V}_E^2) = |P|$ . Suppose we build a basis for  $(\mathcal{V}_E^1 + \mathcal{V}_E^2)$  by starting with a basis  $b^\cap$  for  $\mathcal{V}_E^1 \cap \mathcal{V}_E^2$ , extending it to a basis  $b^1$  for  $\mathcal{V}_E^1$  and then extending  $b^\cap$  to a basis  $b^2$  for  $\mathcal{V}_E^2$ . It is easily seen that  $b^1 \cup b^2$  is a basis for  $\mathcal{V}_E^1 + \mathcal{V}_E^2$ . Now consider the matrix

$$R_{EP_1 P_2} = \begin{pmatrix} R_1 & I & \mathbf{0} \\ R_\cap & \mathbf{0} & \mathbf{0} \\ R_2 & \mathbf{0} & I \end{pmatrix} \quad (10)$$

where the first set of columns of  $R_{EP_1P_2}$  corresponds to  $E$ , and the sets of rows  $R_\cap, R_1, R_2$ , are equal respectively to  $b^\cap, (b^1 - b^\cap), (b^2 - b^\cap)$ . The second set of columns of  $R_{EP_1P_2}$  corresponds to a subset  $P_1$  and the third set to a disjoint subset  $P_2$ . Let  $\mathcal{V}_{EP}$  be the row space of  $R_{EP_1P_2}$ . By an examination of the matrix  $R_{EP_1P_2}$ , it is clear that  $\mathcal{V}_{EP} \leftrightarrow (\mathcal{F}_{P_1} + \mathbf{0}_{P_2}) = \mathcal{V}_E^1$ ,  $\mathcal{V}_{EP} \leftrightarrow (\mathbf{0}_{P_1} + \mathcal{F}_{P_2}) = \mathcal{V}_E^2$ . Therefore, if we take  $\mathcal{V}_P^1 \equiv \mathcal{F}_{P_1} + \mathbf{0}_{P_2}$ ,  $\mathcal{V}_P^2 \equiv \mathbf{0}_{P_1} + \mathcal{F}_{P_2}$ , it follows that  $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P^1 = \mathcal{V}_E^1$ ,  $\mathcal{V}_{EP} \leftrightarrow \mathcal{V}_P^2 = \mathcal{V}_E^2$ . Further,  $|P_1 \uplus P_2|$  is the number of rows of  $\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ , i.e., equal to  $r(\mathcal{V}_E^1 + \mathcal{V}_E^2) - r(\mathcal{V}_E^1 \cap \mathcal{V}_E^2)$ .

Details of this method for the analysis of electrical networks may be found in [15] and in Section 7.3.3 of [16]. 3.2.2 describes how these ideas are used in circuit simulators.

A much harder problem is where  $\mathcal{V}_E^1$  is given and  $\mathcal{V}_E^2$  belongs to a class of vector spaces on  $E$ , closed under sum and intersection, and we have to pick  $\mathcal{V}_E^2$  to minimize  $d(\mathcal{V}_E^1, \mathcal{V}_E^2) \equiv r(\mathcal{V}_E^1 + \mathcal{V}_E^2) - r(\mathcal{V}_E^1 \cap \mathcal{V}_E^2)$ . Special cases of this problem have been solved in Chapter 14 of [16].

### 3.2.2. Use of topological transformation in circuit simulators

We note that circuit simulators are expected to be very fast (in practice better than  $O(n^{1.5})$ ) for networks with  $n$  nodes and with average degree of nodes  $\leq 4$ ). Therefore purely linear algebraic algorithms are ruled out. Primarily the algorithms have to be graph theoretic. The problem is to solve the ‘original network’ by repeatedly solving the ‘desired network’ in terms of some extra variables  $q$  which are  $|P|$  in number and solving the  $q$  variables through certain derived equations. When the computed values of  $q$  are substituted back into the desired network equations we would get the currents and voltages of the original network.

Suppose the current equations of the original network are  $Ri_E = \mathbf{0}$  and that of the desired network are  $R_2i_E = \mathbf{0}$ . These could be written simply using the incidence matrix of the graphs of the networks. The current equations of the original network are rewritten as

$$\begin{pmatrix} R_1 & I & \mathbf{0} \\ R_2 & \mathbf{0} & R_{P_2} \end{pmatrix} \begin{pmatrix} i_E \\ i_{P_1} \\ i_{P_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad (11)$$

where  $i_{P_1} = \mathbf{0}$  and  $i_{P_2}$  is arbitrary. The voltage equations of the original network are

$$(I \quad -R^T) \begin{pmatrix} v_E \\ v_n \end{pmatrix} = \begin{pmatrix} \mathbf{0} \end{pmatrix}, \quad (12)$$

They can be rewritten as

$$\begin{pmatrix} I & -R_2^T & -R_1^T & \mathbf{0} \\ \mathbf{0} & -R_{P_2}^T & \mathbf{0} & I \end{pmatrix} \begin{pmatrix} v_E \\ v'_n \\ v_{P_1} \\ v_{P_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad (13)$$

where  $v_{P_2} = \mathbf{0}$  and  $v_{P_1}$  is arbitrary. (Note that the voltage equations of the desired network are  $(I \quad -R_2^T) \begin{pmatrix} v_E \\ v'_n \end{pmatrix} = \mathbf{0}$ .) If the desired network can be obtained from the original network, by fusing or cutting nodes, then  $R_1, R_{P_2}$  can be computed graph theoretically. (Graph theoretic construction of  $R_1, R_{P_2}$ , in these cases is described in [15], Problem 7.9 of [16], also in the section on ‘Topological transformation’ in [24].)

Suppose the original network constraints are

$$\begin{pmatrix} R & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & -R^T \\ M & N & \mathbf{0} \end{pmatrix} \begin{pmatrix} i_E \\ v_E \\ v'_n \\ i_{P_2} \\ v_{P_1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \alpha \end{pmatrix}, \quad (14)$$

where the last set of rows is the device- characteristic equations. This is rewritten as

$$\left( \begin{array}{ccc|cc} R_2 & \mathbf{0} & \mathbf{0} & R_{P_2} & \mathbf{0} \\ \mathbf{0} & I & -R_2^T & \mathbf{0} & -R_1^T \\ M & N & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \hline R_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -R_{P_2}^T & \mathbf{0} & \mathbf{0} \end{array} \right) \begin{pmatrix} i_E \\ v_E \\ v'_n \\ i_{P_2} \\ v_{P_1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \alpha \\ - \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad (15)$$

Note that the submatrix of the coefficient matrix in Equation 15 within dotted lines would be the coefficient matrix for the network equations of the desired network. By solving the desired network repeatedly ( $|P_1 \uplus P_2|$  times) for values of  $i_{P_2}, v_{P_1}$  corresponding to columns of  $R_{P_2}, R_1^T$ , respectively, we can write

$\begin{pmatrix} i_E \\ v'_n \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} i_{P_2} \\ v_{P_1} \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ . Using the last two sets of rows of the coefficient matrix of Equation 15, we get  $\begin{pmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & R_{P_2}^T \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} i_{P_2} \\ v_{P_1} \end{pmatrix} = - \begin{pmatrix} R_1 & \mathbf{0} \\ \mathbf{0} & R_{P_2}^T \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ . Solving this equation yields  $i_{P_2}, v_{P_1}$  and substituting in the first three rows of Equation 15, and solving the desired network once more yields  $i_E, v'_n$  and therefore also  $v_E$ . This procedure is valid since it is only a way of ordering the elimination of variables. It only depends upon Equation 14 having a unique solution, and therefore also Equation 15 having one.

Usual choices of desired network correspond to the topology  $\oplus \mathcal{G} \times E_i$  or  $\oplus \mathcal{G} \circ E_i$ , for some appropriate choices of partition  $\{E_1, \dots, E_k\}$  of  $E$ , chosen through heuristics.

#### 4. Transpose

Many large scale systems are defined by equations of the following form.

$$(R_Z \quad R_U \quad R_Y) \begin{pmatrix} z \\ u \\ y \end{pmatrix} = \mathbf{0}, \quad (16)$$

$$(K_Z) z = \mathbf{0}. \quad (17)$$

Equation 16 would be topological in origin. For instance, in the case of electrical networks they would be Kirchoff's current and voltage equations. In the case of block diagrams, they would be a set of vector connection point equations of the form  $z_1 = \dots = z_r = z_{out}$  and a set of vector summer equations of the form  $\Sigma z'_i = z'_{out}$ , with no common variables between the sets.

Equation 17 can be called 'device- characteristic' and would be usually composed of a large number of devices with no variables in common, i.e., of the form

$$K_{\hat{z}_1} \hat{z}_1 = \mathbf{0}, \quad (18)$$

⋮

$$K_{\hat{z}_k} \hat{z}_k = \mathbf{0}, \quad (19)$$

where  $\hat{z}_1, \dots, \hat{z}_k$  are vector variables with no entries in common.

Usually, there would be available a convenient simulator, which when fed the topological and device-characteristic constraints, and a specific value of the vector  $u$ , would compute the vector  $y$  provided it is unique. Let the solution space of Equation 16 be  $\mathcal{V}_{ZUY}$  and that of Equation 17, be  $\mathcal{V}_Z$ . Let the collection of vectors  $(u, y)$ , which satisfy both the equations be  $\mathcal{V}_{UY}$ , i.e.,  $\mathcal{V}_{UY} = \mathcal{V}_{ZUY} \leftrightarrow \mathcal{V}_Z$ . Under the assumption that the equations have a unique solution for every possible value of  $u$ ,  $\mathcal{V}_{UY}$  would be the solution space of an equation of the form  $y = Ku$ .

#### 4.1. How to use the simulator repeatedly to compute the matrix $K$ when rows small in number

Let  $k$  be the number of columns of  $K$ . If we set the vector  $u^T = (0, \dots, u_j, \dots, 0)$ , with  $u_j = 1, j = 1 \dots k$ , the corresponding  $y$  vector would be the  $j^{\text{th}}$  column of  $K$ . If the number of columns of  $K$  is much greater than its number of rows, it would be convenient if we could feed the simulator a system which has an input-output relation  $\hat{y} = \pm K^T \hat{u}$ .

It is easy to build  $\mathcal{V}_{ZUY}^\perp$  using topological methods. For instance, in the case of graphs we have to interchange the KVL and KCL equations and in the case of block diagrams we have to interchange summers and connection points and reverse all arrows. In the case of  $\mathcal{V}_Z$ , it is usually the direct sum of many smaller spaces of the form  $\oplus_i \mathcal{V}_{Z_i}$ . So it is easy to build  $\mathcal{V}_Z^\perp = \oplus_i \mathcal{V}_{Z_i}^\perp$ .

By implicit duality theorem (Theorem 5), we have  $\mathcal{V}_{UY}^\perp = \mathcal{V}_{ZUY}^\perp \leftrightarrow \mathcal{V}_Z^\perp$  and  $\mathcal{V}_{UY}^\perp$  would be the solution space of the equation  $u = -K^T y$ . So we redefine the topological space to be  $\hat{\mathcal{V}}_{Z\hat{U}\hat{Y}} := (\mathcal{V}_{ZUY}^\perp)_{Z\hat{Y}\hat{U}}$ , the device- characteristic space to be  $\hat{\mathcal{V}}_Z := \mathcal{V}_Z^\perp$ , and  $\hat{\mathcal{V}}_{\hat{U}\hat{Y}} := (\mathcal{V}_{UY}^\perp)_{\hat{Y}\hat{U}}$ . We would then have  $\hat{\mathcal{V}}_{\hat{U}\hat{Y}} = \hat{\mathcal{V}}_{Z\hat{U}\hat{Y}} \leftrightarrow \hat{\mathcal{V}}_Z$  and  $\hat{\mathcal{V}}_{\hat{U}\hat{Y}}$  would be governed by the equation  $\hat{y} = -K^T \hat{u}$ . We now feed the simulator  $\hat{\mathcal{V}}_{Z\hat{U}\hat{Y}}$  and  $\hat{\mathcal{V}}_Z$  and set  $\hat{u}^T = (0, \dots, \hat{u}_j, \dots, 0), \hat{u}_j = 1, j = 1 \dots r$ , where  $r$  is the number of rows of  $K$ . This would give the columns of  $-K^T$ .

##### 4.1.1. \*Illustration through the case of an electrical multiport $\mathcal{N}_{EP}$ .

\*(This subsection depends on Section 3).

Let  $\mathcal{N}_{EP}$  have topological constraints  $(v_E, v_P) \in \mathcal{V}^v(\mathcal{G}_{EP}), (i_{E'}, i_{P'}) \in \mathcal{V}^i(\mathcal{G}_{E'P'})$  (where  $E, E'$  and  $P, P'$  are copies of each other and  $E, E', P, P'$  are pairwise disjoint) and device- characteristic constraints  $v_E = R_{int} i_{E'}$ . Suppose the resulting port characteristic is  $v_P = R_{port} i_{P'}$ . If the device- characteristic is changed to  $v_E = R_{int}^T i_{E'}$ , we will show that the port characteristic would become  $v_P = R_{port}^T i_{P'}$ .

Let the solution space of the original device- characteristic constraints be  $\mathcal{V}_{EE'}$ , and that of the original port characteristic be  $\mathcal{V}_{PP'}$ . Let us denote  $(\mathcal{V}^v(\mathcal{G}_{EP}) \oplus \mathcal{V}^i(\mathcal{G}_{E'P'}))$  by  $\mathcal{V}_{EPE'P'}$ . We then have  $\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'} = \mathcal{V}_{PP'}$ . Therefore by Theorem 5,  $\mathcal{V}_{EPE'P'}^\perp \leftrightarrow \mathcal{V}_{EE'}^\perp = \mathcal{V}_{PP'}^\perp$  and  $(\mathcal{V}_{EPE'P'}^\perp)_{(-E')(-P')EP} \leftrightarrow (\mathcal{V}_{EE'}^\perp)_{(-E')E} = (\mathcal{V}_{PP'}^\perp)_{(-P')P}$ .

Since  $(\mathcal{V}^v(\mathcal{G}_{EP}))^\perp = \mathcal{V}^i(\mathcal{G}_{EP})$ , we have  $(\mathcal{V}_{EPE'P'}^\perp)_{(-E')(-P')EP} = (\mathcal{V}^v(\mathcal{G}_{EP}))_{(-E')(-P')}^\perp \oplus (\mathcal{V}^i(\mathcal{G}_{E'P'}))_{E'}^\perp = (\mathcal{V}^i(\mathcal{G}_{E'P'})) \oplus \mathcal{V}^v(\mathcal{G}_{EP}) = \mathcal{V}_{EPE'P'}$ . Now  $\mathcal{V}_{EE'}^\perp$  is the solution space of the constraint  $v_{E'} = -R_{int}^T i_E$ , so that  $(\mathcal{V}_{EE'}^\perp)_{(-E')E}$  is the solution space of the constraint  $v_E = R_{int}^T i_{E'}$ . Similarly,  $\mathcal{V}_{PP'}^\perp$  is the solution space of the constraint  $v_{P'} = -R_{port}^T i_P$ , so that  $(\mathcal{V}_{PP'}^\perp)_{(-P')P}$  is the solution space of the constraint  $v_P = R_{port}^T i_{P'}$ . The result follows.

#### 4.2. Block diagrams

The above ideas can also be used to prove the following well known fact about block diagrams (see Section 7.3.4 of [16], also [24]).

*Let a block diagram  $B$  made up of vector summers, vector connection points and vector block constraints  $z_y^j = K_j z_u^j$ , have the overall input-output relationship  $y = Ku$ . Then the block diagram  $B'$  obtained by reversing all arrows, replacing connection points by summers, summers by connection points, block constraints  $z_y^j = K_j z_u^j$  by  $\hat{z}_y^j = K_j^T \hat{z}_u^j$  has the overall input-output relationship  $\hat{y} = K^T \hat{u}$ .*

#### 4.3. Reciprocal and Dirac structures

The ideas in this subsection have been discussed in some detail in [19]. The essential idea is to show, using the implicit duality theorem (Theorem 5), that if a network and the device- characteristic both have certain special properties, namely reciprocity and dirac, these properties are also inherited by the port characteristics.

A vector space  $\mathcal{V}_{EE'}$ , where  $E, E'$  are disjoint copies of each other, is said to be *reciprocal* iff  $\mathcal{V}_{EE'}^\perp = (\mathcal{V}_{EE'})_{(-E')E}$  and *dirac* iff  $\mathcal{V}_{EE'}^\perp = (\mathcal{V}_{EE'})_{E'E}$ . Consider the special case where  $\mathcal{V}_{EE'}$  is the solution space of the equation  $v_E = R i_{E'}$ , i.e.,  $(v_E^T, i_{E'}^T) \in \mathcal{V}_{EE'}$  iff  $(v_E^T, i_{E'}^T) = \lambda^T (R^T | I)$ . Then  $\mathcal{V}_{EE'}^\perp$  is the

space of all vectors spanned by the rows of  $(I - R)$ . Now  $(\mathcal{V}_{EE'})_{(-E')E}$  is the space of all vectors  $(\hat{i}_E^T, \hat{v}_{E'}^T)$  spanned by the rows of  $(-I|R^T)$ , i.e., by the rows of  $(I - R^T)$ . Therefore  $\mathcal{V}_{EE'}$  is reciprocal iff  $R = R^T$  and dirac iff  $R = -R^T$ . In particular, if  $R$  is a diagonal matrix with positive or negative entries, then  $\mathcal{V}_{EE'}$  is reciprocal. If  $R$  is block diagonal with  $2 \times 2$  blocks  $\begin{pmatrix} 0 & -r_j \\ r_j & 0 \end{pmatrix}$ , with  $r_j$  positive or negative, then  $\mathcal{V}_{EE'}$  is dirac.

When  $\mathcal{V}_{EE'}$  is both reciprocal and dirac, it is an *ideal transformer*. For ideal transformers, we have  $\mathcal{V}_{EE'}^\perp = (\mathcal{V}_{EE'})_{(-E')E} = (\mathcal{V}_{EE'})_{E'E}$ . It follows that if  $(g_E, f_{E'}) \in \mathcal{V}_{EE'}$ , then  $(f_E, g_{E'}) \in (\mathcal{V}_{EE'})_{E'E}$  and  $(-f_E, g_{E'}) \in (\mathcal{V}_{EE'})_{E'E}$  and therefore  $(0_E, g_{E'}) \in (\mathcal{V}_{EE'})_{E'E}$ . Thus, if  $g_E \in \mathcal{V}_{EE'} \circ E$ , then  $g_{E'} \in (\mathcal{V}_{EE'})_{E'E} \times E'$ , i.e.,  $g_E \in \mathcal{V}_{EE'} \times E$ . Therefore  $\mathcal{V}_{EE'}$  can be written as  $\mathcal{V}_E^1 \oplus \mathcal{V}_{E'}^2$ . Since we also have  $\mathcal{V}_{EE'}^\perp = (\mathcal{V}_{EE'})_{E'E}$ , it is clear that  $\mathcal{V}_{E'}^2 = (\mathcal{V}_E^1)^\perp_{E'}$ .

#### 4.3.1. Inheritance of reciprocity and Dirac property

We will show that if vector spaces  $\mathcal{V}_{EPE'P'}$ ,  $\mathcal{V}_{EE'}$ ,  $(E, E', P, P'$ , pairwise disjoint with  $E, E'$  and  $P, P'$  being copies of each other) are both reciprocal (dirac) then  $\mathcal{V}_{PP'} := \mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'}$ , is also reciprocal (dirac).

Observe that  $(\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'})_{(-P')P} = (\mathcal{V}_{EPE'P'})_{(-E')(-P')EP} \leftrightarrow (\mathcal{V}_{EE'})_{(-E')E}$  and  $(\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'})_{P'P} = (\mathcal{V}_{EPE'P'})_{E'P'EP} \leftrightarrow (\mathcal{V}_{EE'})_{E'E}$ . Therefore if  $\mathcal{V}_{EPE'P'}$ ,  $\mathcal{V}_{EE'}$ , are reciprocal, we have  $(\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'})_{(-P')P} = (\mathcal{V}_{EPE'P'})_{(-E')(-P')EP} \leftrightarrow (\mathcal{V}_{EE'})_{(-E')E} = \mathcal{V}_{EPE'P'}^\perp \leftrightarrow \mathcal{V}_{EE'}^\perp = \mathcal{V}_{PP'}^\perp$ , (using Theorem 5) i.e.,  $\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'}$  is reciprocal.

Similarly, if  $\mathcal{V}_{EPE'P'}$ ,  $\mathcal{V}_{EE'}$ , are dirac, we have  $(\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'})_{P'P} = (\mathcal{V}_{EPE'P'})_{E'P'EP} \leftrightarrow (\mathcal{V}_{EE'})_{E'E} = \mathcal{V}_{EPE'P'}^\perp \leftrightarrow \mathcal{V}_{EE'}^\perp = \mathcal{V}_{PP'}^\perp$ , i.e.,  $\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'}$  is dirac.

It follows that if  $\mathcal{V}_{EPE'P'}$ ,  $\mathcal{V}_{EE'}$ , are ideal transformers, so is  $\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'}$ .

#### 4.3.2. \*Inheritance of properties in the case of electrical multiports

\*(This subsection depends on Section 3).

In the case of electrical networks, we may take  $\mathcal{V}_{EPE'P'}$  to be  $(\mathcal{V}^v(\mathcal{G}_{EP}) \oplus \mathcal{V}^i(\mathcal{G}_{E'P'}))$ . It is clear that here  $\mathcal{V}_{EPE'P'}^\perp = (\mathcal{V}_{EPE'P'})_{E'P'EP} = (\mathcal{V}_{EPE'P'})_{(-E')(-P')EP}$ , so that  $\mathcal{V}_{EPE'P'}$  is both reciprocal and dirac. Therefore in this case  $\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'}$  is reciprocal or dirac provided  $\mathcal{V}_{EE'}$  is reciprocal or dirac, respectively. This is true even if the ports are not minimal, i.e., contain loops or cutsets in  $\mathcal{G}_{EP}$ .

We note however, that if  $v_E = R_{int}i_{E'}$ , it does not follow that  $v_P = R_{port}i_{P'}$ , for some  $R_{port}$ . Thus, if  $R_{int}$  is symmetric we can only say that  $(\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'})_{(-P')P} = (\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'})^\perp$ . This guarantees, among other things, that the dimension of  $(\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'})_{(-P')P}$  is equal to  $|P| = |P'|$  and not that it is spanned by the rows of a matrix of the form  $(-R_{port}^T|I)$ , if  $\mathcal{V}_{EE'}$  is spanned by the rows of  $(-R_{int}^T|I)$ . Consider, for instance, the case of resistors  $R, -R$  connected in parallel across nodes  $a, b$  and the port branch also across the same pair of nodes. In this case the port characteristic  $\mathcal{V}_{PP'} := (\mathcal{V}_{EPE'P'} \leftrightarrow \mathcal{V}_{EE'})$  is spanned by the row  $(1|0)$ , i.e.,  $v_P$  is arbitrary but  $i_{P'}$  is zero, which does not have the form  $v_P = R_{port}i_{P'}$ . (When  $R_{int}$  is symmetric, positive definite and the ports are minimal, we can show using positive definiteness ideas, that  $R_{port}$  is symmetric and positive definite.)

## 5. Dynamical systems

This section is based on references [20] and [21]. We consider the *dynamical system*  $\mathcal{V}_{W\dot{W}UY}$  to be defined by an equation of the kind

$$\begin{bmatrix} S_{\dot{W}} & S_W & S_U & S_Y \end{bmatrix} \begin{pmatrix} \dot{w} \\ w \\ u \\ y \end{pmatrix} = 0, \quad (20)$$

where  $W, \dot{W}$  are disjoint copies of each other and  $u, y$  are the *manifest* variables, namely inputs and outputs visible external to the system. Sometimes the partition of the manifest variables into input and output may

not be natural, in which case we will refer to the system simply by  $\mathcal{V}_{W\dot{W}M}$ . If we impose conditions on the variables  $u, y$ , by insisting that  $(u, y) \in \mathcal{V}_{UY}$ , we get an *autonomous system*  $\mathcal{V}_{W\dot{W}} := \mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY}$ .

### 5.1. State feedback and output injection

State feedback and output injection are fundamental notions of control theory that have arisen from practical considerations. For a system defined by

$$\dot{w} = Aw + Bu \quad (21)$$

$$y = Cw + Du, \quad (22)$$

if we apply state feedback  $u = Fw$ , the system  $\dot{w} = (A + BF)w$ , would result and if we apply output injection  $\dot{w} = Gy$ , the system  $\dot{w} = (A - GC)w$ , would result.

In our notation state feedback would be called  $(W, U)$ – feedback and output injection,  $(\dot{W}, Y)$ – injection. They would be represented by spaces  $\mathcal{V}_{WU}, \mathcal{V}_{\dot{W}Y}$ , respectively (corresponding to the solution space of feedback and injection constraints). Applying  $(W, U)$ – feedback  $\mathcal{V}_{WU}$  to a dynamical system  $\mathcal{V}_{W\dot{W}UY}$  will result in the autonomous system  $\mathcal{V}_{W\dot{W}}(\text{feedback}) := (\mathcal{V}_{W\dot{W}UY} \cap \mathcal{V}_{WU}) \circ W \dot{W}$ .

Applying  $(\dot{W}, Y)$ – injection  $\mathcal{V}_{\dot{W}Y}$  to the dynamical system  $\mathcal{V}_{W\dot{W}UY}$  will result in the autonomous system  $\mathcal{V}_{W\dot{W}}(\text{injection}) := (\mathcal{V}_{W\dot{W}UY} + \mathcal{V}_{\dot{W}Y}) \times W \dot{W}$ . It can be verified that for the system defined by Equation 21, these definitions imply respectively, the autonomous systems defined by  $\dot{w} = (A + BF)w$ ,  $\dot{w} = (A - GC)w$  (see the subsection on ‘State feedback and output injection’ in [24]).

The question we address ourselves to is ‘given a desired autonomous dynamical system  $\mathcal{V}_{W\dot{W}}$ , how do we obtain it from a specified dynamical system  $\mathcal{V}_{W\dot{W}UY}$  by  $(W, U)$ – feedback ( $(\dot{W}, Y)$ – injection)?’ The answer is given in the following theorems whose proof uses Theorem 3. Proof is available in [21] and in the subsection on ‘Capability of state feedback and output injection’ in [24]. Theorem 7 can also be proved by using Theorem 5, starting from Theorem 6 (see the above mentioned subsection in [24]). The theorems contain the requirement on the  $(W, U)$ – feedback and  $(\dot{W}, Y)$ – injection and also algorithms ([24]) for building the feedback and injection when  $\mathcal{V}_{W\dot{W}}$  is defined by  $\dot{w} = Kw$  and  $\mathcal{V}_{W\dot{W}UY}$  is defined by Equation 21.

**Theorem 6.** *Given  $\mathcal{V}_{W\dot{W}UY}, \mathcal{V}_{W\dot{W}}$ , there exists  $\mathcal{V}_{WU}$  such that  $(\mathcal{V}_{W\dot{W}UY} \cap \mathcal{V}_{WU}) \circ W \dot{W} = \mathcal{V}_{W\dot{W}}$  iff*

$$\mathcal{V}_{W\dot{W}UY} \circ W \dot{W} \supseteq \mathcal{V}_{W\dot{W}}; \quad \mathcal{V}_{W\dot{W}UY} \circ W\dot{W}U \times \dot{W} \subseteq \mathcal{V}_{W\dot{W}} \times \dot{W}. \quad (23)$$

Let the conditions in Equation 23 hold and let  $\mathcal{V}_{WU} := (\mathcal{V}_{W\dot{W}UY} \cap \mathcal{V}_{W\dot{W}}) \circ WU$ .

Then  $(\mathcal{V}_{W\dot{W}UY} \cap \mathcal{V}_{WU}) \circ W \dot{W} = \mathcal{V}_{W\dot{W}}$ .

**Theorem 7.** *Given  $\mathcal{V}_{W\dot{W}UY}, \mathcal{V}_{W\dot{W}}$ , there exists  $\mathcal{V}_{\dot{W}Y}$  such that  $(\mathcal{V}_{W\dot{W}UY} + \mathcal{V}_{\dot{W}Y}) \times W \dot{W} = \mathcal{V}_{W\dot{W}}$  iff*

$$\mathcal{V}_{W\dot{W}UY} \times W \dot{W} \subseteq \mathcal{V}_{W\dot{W}}; \quad \mathcal{V}_{W\dot{W}UY} \times W\dot{W}Y \circ W \supseteq \mathcal{V}_{W\dot{W}} \circ W. \quad (24)$$

Let the conditions in Equation 24 hold and let  $\mathcal{V}_{\dot{W}Y} := (\mathcal{V}_{W\dot{W}UY} + \mathcal{V}_{W\dot{W}}) \times \dot{W}Y$ .

Then  $(\mathcal{V}_{W\dot{W}UY} + \mathcal{V}_{\dot{W}Y}) \times W \dot{W} = \mathcal{V}_{W\dot{W}}$ .

### 5.2. Conditioned and controlled invariant subspaces

The notions of conditioned and controlled invariance have proved very important in control theory ([36], [31], [3]). Essentially, if the system state  $w$  of  $\mathcal{V}_{W\dot{W}UY}$  is within a *conditioned invariant* subspace,

it cannot leave it whatever be the input and if  $w$  is within a *controlled invariant* subspace, it can be kept within it using a suitable input. In the notation of the present book, we say  $\mathcal{V}_W$  is conditioned invariant in  $\mathcal{V}_{W\dot{W}} := \mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_U$ , iff  $\mathcal{V}_{W\dot{W}} \leftrightarrow \mathcal{V}_W \subseteq (\mathcal{V}_W)_{\dot{W}}$  and  $\mathcal{V}_W$  is controlled invariant in  $\mathcal{V}_{W\dot{W}}$ , iff  $\mathcal{V}_{W\dot{W}} \leftrightarrow (\mathcal{V}_W)_{\dot{W}} \supseteq \mathcal{V}_W$ .

We will now motivate these definitions following [20]. Let us assume, for simplicity, that the system is such that  $w(t), u(t)$  together uniquely fix  $\dot{w}(t), y(t)$  at any time instant  $t$ . Note that this is true for the conventional state-input-output system defined by Equation 21.

Let us take  $\mathcal{V}_M = \mathbf{0}_U \oplus \mathbf{0}_Y$ . Consider the situation at time  $t$ . Let the vector  $(w(t), \dot{w}(t), m(t)) \in \mathcal{V}_{W\dot{W}M}$ . Now  $(w(t), \dot{w}(t)) \in \mathcal{V}_{W\dot{W}} := \mathcal{V}_{W\dot{W}M} \leftrightarrow \mathcal{V}_M$  provided  $m(t) \in \mathcal{V}_M$ .

Our definition of conditioned invariance states that if  $w(t) \in \mathcal{V}_W$  and  $\dot{w}(t)$  belongs to  $\mathcal{V}_{W\dot{W}} \leftrightarrow \mathcal{V}_W$  then  $\dot{w}(t)$  must belong to the copy  $\mathcal{V}_{\dot{W}}$  of  $\mathcal{V}_W$ . As the system evolves, if  $w(t)$  and  $\dot{w}(t)$  are in a fixed vector space for  $t \leq t_0$ , with  $m(t) \in \mathcal{V}_M = \mathbf{0}_U \oplus \mathbf{0}_Y$ , then  $w(t)$  will continue to remain in that space, that is, for  $t > t_0$ ,  $\lim_{t \rightarrow t_0} w(t)$  will be in that space for all  $u(t)$  with the output  $y(t) = 0$ .

Next, let us take  $\mathcal{V}_M = \mathcal{F}_U \oplus \mathcal{F}_Y$ . Our definition of controlled invariance states that if  $w(t) \in \mathcal{V}_W$  then there is a vector  $\dot{w}(t)$  in  $\mathcal{V}_{\dot{W}}$  such that  $w(t)$  belongs to  $\mathcal{V}_{W\dot{W}} \leftrightarrow \{\dot{w}(t)\}$ , i.e., there is a vector  $\dot{w}(t) \in \mathcal{V}_{\dot{W}}$  and  $m(t) \in \mathcal{F}_U \oplus \mathcal{F}_Y$  such that  $(w(t), \dot{w}(t), m(t)) \in \mathcal{V}_{W\dot{W}M}$ . This implies that we can choose the input  $u(t)$  such that  $\dot{w}(t)$  continues to be in the same space  $\mathcal{V}_{\dot{W}}$ . Thus if the system starts in  $\mathcal{V}_W$  it can be kept in the same space subsequently, through suitable choice of input.

We now give necessary and sufficient conditions for a conditioned (controlled) invariant space to be also controlled (conditioned) invariant.

**Theorem 8.** 1. Let  $\mathcal{V}_W$  be conditioned invariant in  $\mathcal{V}_{W\dot{W}}$ , i.e.,  $\mathcal{V}_{W\dot{W}} \leftrightarrow \mathcal{V}_W \subseteq (\mathcal{V}_W)_{\dot{W}}$ . Then it is controlled invariant in  $\mathcal{V}_{W\dot{W}}$ , i.e.,  $\mathcal{V}_{W\dot{W}} \leftrightarrow (\mathcal{V}_W)_{\dot{W}} \supseteq \mathcal{V}_W$  iff  $\mathcal{V}_{\dot{W}W} \circ W \supseteq \mathcal{V}_W$ .  
2. Let  $\mathcal{V}_W$  be controlled invariant in  $\mathcal{V}_{W\dot{W}}$ . Then it is conditioned invariant in  $\mathcal{V}_{W\dot{W}}$  iff  $\mathcal{V}_{W\dot{W}} \times \dot{W} \subseteq (\mathcal{V}_W)_{\dot{W}}$ .

The theorem is an immediate consequence of the following lemma which can be regarded as a variation of Theorem 3.

**Lemma 9.** Let  $\mathcal{K}_{SP}, \mathcal{K}_S, \mathcal{K}_P$  be zero-member collections of vectors on  $S \uplus P, S, P$  respectively with  $\mathcal{K}_{SP}$  closed under subtraction and  $\mathcal{K}_S, \mathcal{K}_P$  closed under addition. We have the following.

1. Let  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_S \subseteq \mathcal{K}_P$ . We then have  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \supseteq \mathcal{K}_S$ , iff  $\mathcal{K}_{SP} \circ S \supseteq \mathcal{K}_S$ .
2. Let  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \supseteq \mathcal{K}_S$ . We then have  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_S \subseteq \mathcal{K}_P$ , iff  $\mathcal{K}_{SP} \times P \subseteq \mathcal{K}_P$ .

*Proof.* 1. (only if). This is immediate since  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \supseteq \mathcal{K}_S$ .

(if) Let  $f_S \in \mathcal{K}_S$ . Since  $\mathcal{K}_{SP} \circ S \supseteq \mathcal{K}_S$ , there exists  $f_P$  such that  $(f_S, f_P) \in \mathcal{K}_{SP}$ . Further since  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_S \subseteq \mathcal{K}_P$ , we must have  $f_P \in \mathcal{K}_P$ . Therefore  $f_S \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ .

2. (only if). This is immediate since  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_S \subseteq \mathcal{K}_P$ .

(if) Let  $f_P \in \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_S$ . Then there exists  $f_S \in \mathcal{K}_S$  such that  $(f_S, f_P) \in \mathcal{K}_{SP}$ . Since  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \supseteq \mathcal{K}_S$ , there exists  $f'_P \in \mathcal{K}_P$  such that  $(f_S, f'_P) \in \mathcal{K}_{SP}$ . Since  $\mathcal{K}_{SP}$  is closed under subtraction,  $(0_S, (f_P - f'_P)) \in \mathcal{K}_{SP}$ , i.e.,  $(f_P - f'_P) \in \mathcal{K}_{SP} \times P \subseteq \mathcal{K}_P$ . Since  $\mathcal{K}_P$  is closed under addition, it follows that  $f_P \in \mathcal{K}_P$ .  $\square$

Using  $(W, U)$ - feedback we can make a controlled invariant space into a conditioned invariant space without losing its controlled invariance and using  $(\dot{W}, Y)$ - injection we can make a conditioned invariant space into a controlled invariant space without losing its conditioned invariance. We state the results without proof here. The proof is given in the subsection on ‘Controlled and conditioned invariant spaces strengthened through feedback and injection’ in [24].

**Theorem 10.** Let  $\mathcal{V}_{W\dot{W}UY}, \mathcal{V}_W$  be such that  $\mathcal{V}_{W\dot{W}UY} \circ W \dot{W} U \times \dot{W} \subseteq \mathcal{V}_{\dot{W}} := (\mathcal{V}_W)_{\dot{W}}$ . Then  $\mathcal{V}_W$  is controlled invariant in  $(\mathcal{V}_{W\dot{W}UY} \circ W \dot{W} U) \leftrightarrow \mathcal{V}_U$  iff  $\mathcal{V}_W$  is both controlled invariant and conditioned invariant in  $((\mathcal{V}_{W\dot{W}UY} \circ W \dot{W} U) \cap \mathcal{V}_{WU}) \leftrightarrow \mathcal{V}_U$ , where  $\mathcal{V}_{WU} := ((\mathcal{V}_{W\dot{W}UY} \circ W \dot{W} U) \cap \mathcal{V}_U) \leftrightarrow \mathcal{V}_{\dot{W}}$ .

A consequence of this result is that for the system  $\mathcal{V}_{W\dot{W}U}$  defined by  $\dot{w} = Aw + Bu$ ,  $\mathcal{V}_W$  is controlled invariant in  $\mathcal{V}_{W\dot{W}U} \leftrightarrow \mathcal{V}_U$ , where  $\mathcal{V}_U = \mathcal{F}_U$ , iff there exists a matrix  $F$ , such that  $(A + BF)\mathcal{V}_W \subseteq \mathcal{V}_W$  (see subsection on ‘Controlled and conditioned invariant spaces strengthened through feedback and injection’ in [24]).

**Theorem 11.** *Let  $\mathcal{V}_{W\dot{W}UY}$ ,  $\mathcal{V}_W$  be such that  $\mathcal{V}_{W\dot{W}UY} \times W \dot{W} Y \circ W \supseteq \mathcal{V}_W$ . Then  $\mathcal{V}_W$  is conditioned invariant in  $(\mathcal{V}_{W\dot{W}UY} \times W \dot{W} Y) \leftrightarrow \mathcal{V}_Y$  iff  $\mathcal{V}_W$  is both controlled invariant and conditioned invariant in  $((\mathcal{V}_{W\dot{W}UY} \times W \dot{W} Y) + \mathcal{V}_{\dot{W}Y}) \leftrightarrow \mathcal{V}_Y$ , where  $\mathcal{V}_{\dot{W}Y} := ((\mathcal{V}_{W\dot{W}UY} \times W \dot{W} Y) + \mathcal{V}_Y) \leftrightarrow \mathcal{V}_W$ .*

A consequence of this result is that for the system  $\mathcal{V}_{W\dot{W}Y}$  defined by  $\dot{w} = Aw, y = Cw$ ,  $\mathcal{V}_W$  is conditioned invariant in  $\mathcal{V}_{W\dot{W}Y} \leftrightarrow \mathcal{V}_Y$ , where  $\mathcal{V}_Y = \mathbf{0}_Y$ , iff there exists a matrix  $G$ , such that  $(A - GC)\mathcal{V}_W \subseteq \mathcal{V}_W$  (see subsection on ‘Controlled and conditioned invariant spaces strengthened through feedback and injection’ in [24]).

### 5.3. Emulators

Dynamical systems do not always come in a form that is suitable for theoretical and practical studies. We usually convert the original system into another that is more convenient for our purpose. The dynamical system as described by state variables is a very good example. We call such transformed systems, ‘emulators’.

More formally, a dynamical system  $\mathcal{V}_{P\dot{P}UY}$  ( $P, \dot{P}$  being disjoint copies of each other, disjoint also from  $W, \dot{W}$ ) is said to be an *emulator* of  $\mathcal{V}_{W\dot{W}UY}$ ,  $\mathcal{E}$ -linked through  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ , iff  $\mathcal{V}_{W\dot{W}UY} = \mathcal{V}_{P\dot{P}UY} \leftrightarrow (\mathcal{V}_{WP}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2)$ , and  $\mathcal{V}_{WP}^1 \supseteq (\mathcal{V}_{\dot{W}\dot{P}}^2)_{WP}$ . If  $\mathcal{V}_{W\dot{W}UY}$  is an emulator of  $\mathcal{V}_{P\dot{P}UY}$ ,  $\mathcal{E}$ -linked through  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ , it is clear that (using Theorem 2) that so is  $\mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY}$ , an emulator of  $\mathcal{V}_{P\dot{P}UY} \leftrightarrow \mathcal{V}_{UY}$ ,  $\mathcal{E}$ -linked through  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ . If both  $\mathcal{V}_{W\dot{W}UY}, \mathcal{V}_{P\dot{P}UY}$  are emulators of each other  $\mathcal{E}$ -linked through  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ , we refer to them as *mutual emulators*  $\mathcal{E}$ -linked through  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ . We refer to  $\mathcal{V}_{WP}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2$  as an  $\mathcal{E}$ -**linkage** for the mutual emulators  $\mathcal{V}_{W\dot{W}UY}, \mathcal{V}_{P\dot{P}UY}$  ( $\mathcal{V}_{W\dot{W}}, \mathcal{V}_{P\dot{P}}$ ).

To illustrate, the dynamical system  $\mathcal{V}_{W\dot{W}UY}$  could be defined by an equation of the kind

$$\begin{bmatrix} S_{\dot{W}} & S_W & S_U & S_Y \end{bmatrix} \begin{pmatrix} \dot{w} \\ w \\ u \\ y \end{pmatrix} = 0. \quad (25)$$

The second dynamical system  $\mathcal{V}_{P\dot{P}UY}$  could be defined by

$$\dot{p} = Ap + Bu \quad (26a)$$

$$y = Cp + Du, \quad (26b)$$

$\mathcal{V}_{WP}^1$  could be defined by  $(I \quad -K) \begin{pmatrix} w \\ p \end{pmatrix} = 0$  and  $\mathcal{V}_{\dot{W}\dot{P}}^2$  could be defined by  $\begin{pmatrix} I & -K \\ R_{\dot{w}} & R_{\dot{p}} \end{pmatrix} \begin{pmatrix} \dot{w} \\ \dot{p} \end{pmatrix} = 0$ .

Note that the constraints on  $w, p$ , carry over to  $\dot{w}, \dot{p}$ , by differentiation. In addition, the dynamical systems could perhaps impose some other indirect constraints on  $\dot{w}, \dot{p}$ .

We now have the following consequence of Theorem 3.

**Theorem 12.** *Let  $\mathcal{V}_{P\dot{P}UY}$  be an emulator of  $\mathcal{V}_{W\dot{W}UY}$ ,  $\mathcal{E}$ -linked through  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ . We then have the following.*

1.  $\mathcal{V}_{WP}^1 \circ W \supseteq \mathcal{V}_{W\dot{W}UY} \circ W, \mathcal{V}_{WP}^1 \times W \subseteq \mathcal{V}_{W\dot{W}UY} \times W,$   
 $\mathcal{V}_{\dot{W}\dot{P}}^2 \circ \dot{W} \supseteq \mathcal{V}_{W\dot{W}UY} \circ \dot{W}, \mathcal{V}_{\dot{W}\dot{P}}^2 \times \dot{W} \subseteq \mathcal{V}_{W\dot{W}UY} \times \dot{W}.$

2.  $\mathcal{V}_{W\dot{W}UY}$  is an emulator of  $\mathcal{V}_{P\dot{P}UY}$ ,  $\mathcal{E}$ -linked through  $\mathcal{V}_{\dot{W}P}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$  iff
- $$\mathcal{V}_{\dot{W}P}^1 \circ P \supseteq \mathcal{V}_{P\dot{P}UY} \circ P, \mathcal{V}_{\dot{W}P}^1 \times P \subseteq \mathcal{V}_{P\dot{P}UY} \times P,$$
- $$\mathcal{V}_{\dot{W}\dot{P}}^2 \circ \dot{P} \supseteq \mathcal{V}_{P\dot{P}UY} \circ \dot{P}, \mathcal{V}_{\dot{W}\dot{P}}^2 \times \dot{P} \subseteq \mathcal{V}_{P\dot{P}UY} \times \dot{P}.$$

*Proof.* 1. We have  $\mathcal{V}_{W\dot{W}UY} = \mathcal{V}_{P\dot{P}UY} \leftrightarrow (\mathcal{V}_{\dot{W}P}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2)$ . By Theorem 3, it follows that

$$(\mathcal{V}_{\dot{W}P}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2) \circ W\dot{W} \supseteq \mathcal{V}_{W\dot{W}UY} \circ W\dot{W}, (\mathcal{V}_{\dot{W}P}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2) \times W\dot{W} \subseteq \mathcal{V}_{W\dot{W}UY} \times W\dot{W},$$

This is equivalent to

$$\mathcal{V}_{\dot{W}P}^1 \circ W \supseteq \mathcal{V}_{W\dot{W}UY} \circ W, \mathcal{V}_{\dot{W}P}^1 \times W \subseteq \mathcal{V}_{W\dot{W}UY} \times W,$$

$$\mathcal{V}_{\dot{W}\dot{P}}^2 \circ \dot{W} \supseteq \mathcal{V}_{W\dot{W}UY} \circ \dot{W}, \mathcal{V}_{\dot{W}\dot{P}}^2 \times \dot{W} \subseteq \mathcal{V}_{W\dot{W}UY} \times \dot{W}.$$

2. This follows from part 2 of Theorem 3.  $\square$

When  $\mathcal{V}_{P\dot{P}UY}, \mathcal{V}_{W\dot{W}UY}$  are mutual emulators, operations such as  $(W, U)$ -feedback and  $(\dot{W}, Y)$ -injection can be performed on one of them and the result transferred to the other dynamical system through the  $\mathcal{E}$ -linkage (see [21] and the subsection on ‘Emulators’ in [24]).

**Theorem 13.** Let  $\mathcal{V}_{P\dot{P}UY}, \mathcal{V}_{W\dot{W}UY}$  be mutual emulators  $\mathcal{E}$ -linked through  $\mathcal{V}_{\dot{W}P}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ .

Let  $\mathcal{V}_{WU}, \mathcal{V}_{\dot{W}Y}, \mathcal{V}_{PU}, \mathcal{V}_{\dot{P}Y}$  be vector spaces on  $W \uplus U, \dot{W} \uplus Y, P \uplus U, \dot{P} \uplus Y$  respectively such that

$$\mathcal{V}_{\dot{W}P}^1 \leftrightarrow \mathcal{V}_{WU} = \mathcal{V}_{PU}, \mathcal{V}_{\dot{W}\dot{P}}^2 \leftrightarrow \mathcal{V}_{\dot{W}Y} = \mathcal{V}_{\dot{P}Y}.$$

We then have the following.

1.  $(\mathcal{V}_{\dot{W}P}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2) \leftrightarrow ((\mathcal{V}_{P\dot{P}UY} \cap \mathcal{V}_{PU}) \circ P \dot{P}) = (\mathcal{V}_{W\dot{W}UY} \cap \mathcal{V}_{WU}) \circ W \dot{W};$
2.  $(\mathcal{V}_{\dot{W}P}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2) \leftrightarrow ((\mathcal{V}_{P\dot{P}UY} + \mathcal{V}_{\dot{P}Y}) \times P \dot{P}) = (\mathcal{V}_{W\dot{W}UY} + \mathcal{V}_{\dot{W}Y}) \times W \dot{W}.$

Proof is given in [21] and in the subsection on ‘Emulators’ in [24].

Controlled and conditioned invariant spaces transfer from the emulator to the original dynamical system through the  $\mathcal{E}$ -linkage.

**Theorem 14.** Let  $\mathcal{V}_{W\dot{W}}, \mathcal{V}_{P\dot{P}}$  be mutual emulators  $\mathcal{E}$ -linked through  $\mathcal{V}_{\dot{W}P}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ .

1. Let  $\mathcal{V}_W$  be controlled invariant in  $\mathcal{V}_{W\dot{W}}$ . Then  $\mathcal{V}_{\dot{W}P}^1 \leftrightarrow \mathcal{V}_W, (\mathcal{V}_{\dot{W}\dot{P}}^2)_{WP} \leftrightarrow \mathcal{V}_W$  are controlled invariant in  $\mathcal{V}_{P\dot{P}}$ .
2. Let  $\mathcal{V}_W$  be conditioned invariant in  $\mathcal{V}_{W\dot{W}}$ . Then  $\mathcal{V}_{\dot{W}P}^1 \leftrightarrow \mathcal{V}_W, (\mathcal{V}_{\dot{W}\dot{P}}^2)_{WP} \leftrightarrow \mathcal{V}_W$  are conditioned invariant in  $\mathcal{V}_{P\dot{P}}$ .

Proof is given in [21] and in the subsection on ‘Emulators’ in [24].

**Remark 5.** The definition of emulators permits a convenient emulator to be built efficiently (near linear time) in practical situations such as in the case of network based dynamical systems. This captures all the relevant spectral properties of the system (see the section on ‘Network based dynamical systems’ in [24]). For large systems, the special case of emulators that corresponds to state variables is not easy to build.

#### 5.4. Adjoint systems

Adjoint dynamical systems have been natural constructs in mathematics and ever since Kalman’s seminal work [8], have been an important component of multivariable control theory. Kalman’s original work on adjoint systems brought out the duality between controllability and observability. Subsequently, controlled invariance and conditioned invariance were seen to be dual notions [2]. These were proved for dynamical systems defined in terms of state- input- output equations as in Equation 21. We will show in this subsection that such ideas follow routinely by use of Theorem 5, even when dynamical systems are defined more generally as in Equation 20. In what follows, for better readability we have omitted the brackets in a term like  $(\mathcal{V}_{AB})_{(-A)B}$  and written it more simply as  $(\mathcal{V}_{AB})_{-AB}$ .

**Definition 15.** Let  $\mathcal{V}_{W\dot{W}M}$  be a dynamical system. Let the set  $M$  be partitioned as  $U \uplus Y$ . We define the adjoint  $(\mathcal{V}_{W\dot{W}UY})^a$  of  $\mathcal{V}_{W\dot{W}UY}$  as

$$(\mathcal{V}_{W\dot{W}UY})^a := \mathcal{V}_{W\dot{W}U'Y'}^a := (\mathcal{V}_{W\dot{W}UY}^\perp)_{-\dot{W}W-Y'U'},$$

where  $U', Y'$  are disjoint sets (also disjoint from  $W, \dot{W}$ ), which are copies respectively of  $Y, U$ . The adjoint  $(\mathcal{V}_{W\dot{W}U'Y'}^a)^a$  of  $\mathcal{V}_{W\dot{W}U'Y'}^a$  is defined to be on  $W \uplus \dot{W} \uplus U \uplus Y$ , with  $W, \dot{W}, U', Y'$ , being copied onto  $-\dot{W}, W, -Y, U$ , respectively.

Let  $\mathcal{V}_{W\dot{W}}$  be an autonomous system. We define its adjoint as

$$(\mathcal{V}_{W\dot{W}})^a := \mathcal{V}_{W\dot{W}}^a := (\mathcal{V}_{W\dot{W}}^\perp)_{-\dot{W}W}.$$

The adjoint of the dynamical system defined by Equation 20 would be

$$\begin{bmatrix} S_{\dot{W}}^\perp & S_{\dot{W}}^\perp & S_{Y'}^\perp & S_{U'}^\perp \end{bmatrix} \begin{pmatrix} w \\ -\dot{w} \\ -y' \\ u' \end{pmatrix} = 0, \quad (27)$$

where the rows of the matrices,  $\begin{bmatrix} S_{\dot{W}} & S_W & S_U & S_Y \end{bmatrix}$ ,  $\begin{bmatrix} S_{\dot{W}}^\perp & S_{\dot{W}}^\perp & S_{Y'}^\perp & S_{U'}^\perp \end{bmatrix}$ , span complementary orthogonal spaces.

Suppose the original dynamical system is defined by

$$\dot{p} = Ap + Bu \quad (28a)$$

$$y = Cp + Du, \quad (28b)$$

This can be rewritten as

$$\begin{bmatrix} I & 0 & -A & -B \\ 0 & I & -C & -D \end{bmatrix} \begin{bmatrix} \dot{p} \\ y \\ p \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (29)$$

The adjoint of this system is

$$\begin{bmatrix} A^T & C^T & I & 0 \\ B^T & D^T & 0 & I \end{bmatrix} \begin{bmatrix} p \\ u' \\ -\dot{p} \\ -y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (30)$$

which can be seen to be the same as

$$\dot{p} = A^T p + C^T u' \quad (31a)$$

$$y' = B^T p + D^T u'. \quad (31b)$$

**Theorem 16.** Let  $\mathcal{V}_{W\dot{W}UY}$  be a dynamical system, let  $\mathcal{V}_{W\dot{W}}$  be an autonomous system,  $\mathcal{V}_{UY}$ , be a vector space. We have

1.  $\mathcal{V}_{W\dot{W}UY} = (\mathcal{V}_{W\dot{W}U'Y'}^a)^a$ .
2.  $\mathcal{V}_{W\dot{W}} := \mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY}$  iff  $\mathcal{V}_{W\dot{W}}^a = \mathcal{V}_{W\dot{W}U'Y'}^a \leftrightarrow (\mathcal{V}_{UY}^\perp)_{-Y'U'}$ .
3.  $\mathcal{V}_{W\dot{W}} = (\mathcal{V}_{W\dot{W}}^a)^a$ .

- Proof.* 1. We have  $(\mathcal{V}_{W\dot{W}U'Y'}^a)^a = ((\mathcal{V}_{W\dot{W}UY}^\perp)_{-\dot{W}W-Y'U'})_{-\dot{W}-\dot{W}-U-Y}^\perp = (\mathcal{V}_{W\dot{W}UY})_{-\dot{W}-\dot{W}-U-Y} = \mathcal{V}_{W\dot{W}UY}$ , (noting that  $(\mathcal{V}_Z)_{-Z} = \mathcal{V}_Z$ ).
2. We have  $\mathcal{V}_{W\dot{W}}^a := (\mathcal{V}_{W\dot{W}}^\perp)_{-\dot{W}W}$ . So  $\mathcal{V}_{W\dot{W}} := \mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY}$  iff  $\mathcal{V}_{W\dot{W}}^a = (\mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY})_{-\dot{W}W}^\perp = (\mathcal{V}_{W\dot{W}UY})_{-\dot{W}W-Y'U'}^\perp \leftrightarrow (\mathcal{V}_{UY})_{-Y'U'}^\perp$  (using Theorem 5) i.e., iff  $\mathcal{V}_{W\dot{W}}^a = \mathcal{V}_{W\dot{W}U'Y'}^a \leftrightarrow (\mathcal{V}_{UY}^\perp)_{-Y'U'}$ .
3. Follows from the first part if we omit the  $U, Y$  variables.

□

Interesting consequences of Theorem 5 (which in a sense justify the definitions of adjoints and emulators) are that

- if  $\mathcal{V}_W$  is controlled (conditioned) invariant in the original system,  $\mathcal{V}_W^\perp$  is conditioned (controlled) invariant in the adjoint system;
- the state equations of adjoint systems are adjoints of the original state equations.

We prove these below.

**Theorem 17.**  $\mathcal{V}_W$  is conditioned (controlled) invariant in  $\mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY}$  iff  $\mathcal{V}_W^\perp$  is controlled (conditioned) invariant in  $\mathcal{V}_{W\dot{W}U'Y'}^a \leftrightarrow (\mathcal{V}_{UY}^\perp)_{(-Y')U'}$ .

*Proof.* By Theorem 5 we have that  $\mathcal{V}_{W\dot{W}} \leftrightarrow \mathcal{V}_W \subseteq \mathcal{V}_{\dot{W}}$  iff  $\mathcal{V}_{W\dot{W}}^\perp \leftrightarrow \mathcal{V}_W^\perp \supseteq \mathcal{V}_{\dot{W}}^\perp$ . Therefore  $(\mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY}) \leftrightarrow \mathcal{V}_W \subseteq \mathcal{V}_{\dot{W}}$  iff  $(\mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY})^\perp \leftrightarrow \mathcal{V}_W^\perp \supseteq \mathcal{V}_{\dot{W}}^\perp$ , i.e., iff  $(\mathcal{V}_{W\dot{W}UY}^\perp \leftrightarrow \mathcal{V}_{UY}^\perp) \leftrightarrow \mathcal{V}_W^\perp \supseteq \mathcal{V}_{\dot{W}}^\perp$ , i.e., iff  $((\mathcal{V}_{W\dot{W}UY})_{(-\dot{W})W(-Y')U'}^\perp \leftrightarrow (\mathcal{V}_{UY}^\perp)_{(-Y')U'}) \leftrightarrow (\mathcal{V}_W^\perp)_{-\dot{W}} \supseteq (\mathcal{V}_{\dot{W}}^\perp)_W$  i.e., iff  $(\mathcal{V}_{W\dot{W}U'Y'}^a \leftrightarrow (\mathcal{V}_{UY}^\perp)_{(-Y')U'}) \leftrightarrow (\mathcal{V}_W^\perp)_{\dot{W}} \supseteq \mathcal{V}_{\dot{W}}^\perp$ .

Thus  $\mathcal{V}_W$  is conditioned invariant in  $\mathcal{V}_{W\dot{W}UY} \leftrightarrow \mathcal{V}_{UY}$  iff  $\mathcal{V}_W^\perp$  is controlled invariant in  $\mathcal{V}_{W\dot{W}U'Y'}^a \leftrightarrow (\mathcal{V}_{UY}^\perp)_{(-Y')U'}$ .

The ‘controlled in primal’ to ‘conditioned in dual’ part of the theorem follows from the facts that for any vector space  $\mathcal{V}$ ,  $(\mathcal{V})^{\perp\perp} = \mathcal{V}$ , and for any  $\mathcal{V}_{W\dot{W}UY}$ , we have  $\mathcal{V}_{W\dot{W}UY} = (\mathcal{V}_{W\dot{W}U'Y'}^a)^a$ . □

We next show, using Theorem 5, that the adjoint of an emulator of a dynamical system is an emulator of the adjoint of the original system.

We introduce the following definition primarily as a typographical simplification.

**Definition 18.** Let  $\mathcal{V}_{W\dot{W}UY}, \mathcal{V}_{P\dot{P}UY}$ , be mutual emulators  $\mathcal{E}$ -linked through  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$  and let  $\mathcal{V}_{W\dot{W}}, \mathcal{V}_{P\dot{P}}$ , also be mutual emulators  $\mathcal{E}$ -linked through  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$ .

Let

$$\begin{aligned} \tilde{\mathcal{V}}_{WP}^1 &:= (\mathcal{V}_{\dot{W}\dot{P}}^2)^a := (\mathcal{V}_{\dot{W}\dot{P}}^2)_{-WP}^\perp = (\mathcal{V}_{\dot{W}\dot{P}}^2)_{\dot{W}-P}^\perp, \\ \tilde{\mathcal{V}}_{\dot{W}\dot{P}}^2 &:= (\mathcal{V}_{WP}^1)^a := (\mathcal{V}_{WP}^1)_{-\dot{W}\dot{P}}^\perp = (\mathcal{V}_{WP}^1)_{\dot{W}-\dot{P}}^\perp, \\ \tilde{\mathcal{V}}_{W\dot{W}U'Y'} &:= (\mathcal{V}_{W\dot{W}UY})^a := (\mathcal{V}_{W\dot{W}UY})_{-\dot{W}W-Y'U'}^\perp, \\ \tilde{\mathcal{V}}_{P\dot{P}U'Y'} &:= (\mathcal{V}_{P\dot{P}UY})^a := (\mathcal{V}_{P\dot{P}UY})_{-\dot{P}P-Y'U'}^\perp, \\ \tilde{\mathcal{V}}_{W\dot{W}} &:= (\mathcal{V}_{W\dot{W}})^a := (\mathcal{V}_{W\dot{W}})_{-\dot{W}W}^\perp, \quad \tilde{\mathcal{V}}_{P\dot{P}} := (\mathcal{V}_{P\dot{P}})^a := (\mathcal{V}_{P\dot{P}})_{-\dot{P}P}^\perp. \end{aligned}$$

**Theorem 19.** 1. Let  $\mathcal{V}_{P\dot{P}UY} = (\mathcal{V}_{WP}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2) \leftrightarrow \mathcal{V}_{W\dot{W}UY}$ . Then  $(\tilde{\mathcal{V}}_{WP}^1)_{\dot{W}\dot{P}} \supseteq \tilde{\mathcal{V}}_{\dot{W}\dot{P}}^2$  and

$$\tilde{\mathcal{V}}_{P\dot{P}U'Y'} = (\tilde{\mathcal{V}}_{WP}^1 \oplus \tilde{\mathcal{V}}_{\dot{W}\dot{P}}^2) \leftrightarrow \tilde{\mathcal{V}}_{W\dot{W}U'Y'}.$$

2. Let  $\mathcal{V}_{P\dot{P}} = (\mathcal{V}_{WP}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2) \leftrightarrow \mathcal{V}_{W\dot{W}}$ . Then  $\tilde{\mathcal{V}}_{P\dot{P}} = (\tilde{\mathcal{V}}_{WP}^1 \oplus \tilde{\mathcal{V}}_{\dot{W}\dot{P}}^2) \leftrightarrow \tilde{\mathcal{V}}_{W\dot{W}}$ .

*Proof.* 1. We have  $(\mathcal{V}_{WP}^1)_{\dot{W}\dot{P}} \supseteq \mathcal{V}_{\dot{W}\dot{P}}^2$  so that  $(\tilde{\mathcal{V}}_{WP}^1)_{\dot{W}\dot{P}} = ((\mathcal{V}_{\dot{W}\dot{P}}^2)^\perp)_{\dot{W}\dot{P}} \supseteq (\mathcal{V}_{WP}^1)_{\dot{W}\dot{P}}^\perp = \tilde{\mathcal{V}}_{\dot{W}\dot{P}}^2$ .

Next, we have  $\tilde{\mathcal{V}}_{P\dot{P}U'Y'} := (\mathcal{V}_{P\dot{P}UY})^a = (\mathcal{V}_{P\dot{P}UY})_{-\dot{P}P-Y'U'}^\perp$ ,

$$= (((\mathcal{V}_{WP}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2) \leftrightarrow \mathcal{V}_{W\dot{W}UY})^\perp)_{-\dot{P}P-Y'U'}$$

$$= ((\mathcal{V}_{WP}^1 \oplus \mathcal{V}_{\dot{W}\dot{P}}^2)^\perp)_{-\dot{P}P-Y'U'} = (\mathcal{V}_{W\dot{W}UY})_{-\dot{P}P-Y'U'}^\perp \text{ (using Theorem 5),}$$

$$= ((\mathcal{V}_{WP}^1)^\perp \oplus (\mathcal{V}_{\dot{W}\dot{P}}^2)^\perp)_{-\dot{P}P-Y'U'}$$

$$= ((\mathcal{V}_{WP}^1)_{\dot{W}\dot{P}}^\perp \oplus (\mathcal{V}_{\dot{W}\dot{P}}^2)_{WP}^\perp)_{-\dot{P}P-Y'U'}$$

(now on left side of ' $\Leftarrow$ ' we change sign of  $W$ , on right side, sign of  $\dot{W}$  and replace ' $\Leftarrow$ ' by ' $\leftrightarrow$ ')  
 $= ((\mathcal{V}_{WP}^1)_{\dot{W}\dot{P}}^\perp \oplus (\mathcal{V}_{\dot{W}\dot{P}}^2)_{-WP}^\perp) \leftrightarrow (\mathcal{V}_{W\dot{W}UY})_{-\dot{W}W-Y'U'}^\perp$ , i.e.,

$$\tilde{\mathcal{V}}_{P\dot{P}U'Y'} = (\tilde{\mathcal{V}}_{\dot{W}\dot{P}}^2 \oplus \tilde{\mathcal{V}}_{WP}^1) \leftrightarrow \tilde{\mathcal{V}}_{W\dot{W}U'Y'} = (\tilde{\mathcal{V}}_{WP}^1 \oplus \tilde{\mathcal{V}}_{\dot{W}\dot{P}}^2) \leftrightarrow \tilde{\mathcal{V}}_{W\dot{W}U'Y'}.$$

2. The proof is identical to the previous part, but omitting the  $U, Y$ , variables. □

To illustrate Theorem 19, consider the case where the mutual emulators  $\mathcal{V}_{W\dot{W}UY}, \mathcal{V}_{P\dot{P}UY}$  are defined respectively through Equations 25 and 26. We can proceed as in [14] and define  $\mathcal{V}_{WP}^1, \mathcal{V}_{\dot{W}\dot{P}}^2$  by  $w = Kp$  and  $\dot{w} = K\dot{p}$ . In other words, the state equations of the system  $\mathcal{V}_{W\dot{W}UY}$  are given by Equation 26. The adjoint  $\tilde{\mathcal{V}}_{W\dot{W}U'Y'}$  of  $\mathcal{V}_{W\dot{W}UY}$  is defined by Equation 27 and has the emulator  $\tilde{\mathcal{V}}_{P\dot{P}U'Y'}$ , defined by Equation 31, i.e.,  $\tilde{\mathcal{V}}_{W\dot{W}U'Y'}$  has the state equations given by the latter equation. This is the adjoint of the system defined by state equations in Equation 26. The two systems are  $\mathcal{E}$ -linked through  $\tilde{\mathcal{V}}_{WP}^1 = (\mathcal{V}_{\dot{W}\dot{P}}^2)_{W(-P)}^\perp, \tilde{\mathcal{V}}_{\dot{W}\dot{P}}^2 = (\mathcal{V}_{WP}^1)_{\dot{W}(-\dot{P})}^\perp$ , i.e., through  $p = K^T w$  and  $\dot{p} = K^T \dot{w}$ .

Thus the state equations of adjoint dynamical systems are adjoints of each other and if  $w = Kp$  and  $\dot{w} = K\dot{p}$  in the original system, in the adjoint  $p = K^T w$  and  $\dot{p} = K^T \dot{w}$  [14].

Another consequence of Theorem 5 is that the operation of state feedback in the original dynamical system is equivalent to that of output injection in the adjoint (see [21], the subsection on 'Dynamical system duality' in [24]).

### 5.5. Spectral theory for generalized operators

The spectral theory of operators is fundamental to the study of classical dynamical systems. When  $\mathcal{V}_{W\dot{W}}$  satisfies the properties  $\mathcal{V}_{W\dot{W}} \circ W \supseteq (\mathcal{V}_{W\dot{W}} \circ \dot{W})_W, \mathcal{V}_{W\dot{W}} \times W \supseteq (\mathcal{V}_{W\dot{W}} \times \dot{W})_W$ , we call it a generalized operator (genop). The usual operators satisfy these properties trivially, since in their case  $\mathcal{V}_{W\dot{W}} \circ W = \mathcal{F}_W$  and  $\mathcal{V}_{W\dot{W}} \times \dot{W} = \mathbf{0}_{\dot{W}}$ . For genops, a spectral theory can be built paralleling that for the usual operators. This is done in the sections on 'Linkages and maps', 'Polynomials of generalized autonomous systems and operators' in [24]. The advantage of working with this formulation is that the usual eigenvalue related computations can be performed directly on the appropriate genop derived from the original dynamical system without putting it in the state variable form.

## 6. Behavioural systems

Let us define a dynamical system  $\mathcal{B}_{WCM}$ , more generally, to be a collection of vectors  $(w(\cdot), c(\cdot), m(\cdot))$ , where  $w(\cdot)$  are the internal variables,  $c(\cdot)$  are the variables available for control and  $m(\cdot)$  are the manifest

variables which will be exposed to the outside world after the controller is connected. We will assume that the variables are infinitely differentiable functions with respect to the time variable  $t$  and that  $\mathcal{B}_{WCM}$  is closed under subtraction. This agrees with the point of view of the behaviourists ([26]). Let us call  $\mathcal{B}_{WCM}$ , the ‘plant’. We have a ‘desired’ dynamical system  $\mathcal{B}_{WM}$ . We are allowed to ‘connect’ a ‘controller’ dynamical system  $\mathcal{B}_C$  to the plant. If  $(w(\cdot), c(\cdot), m(\cdot))$  belongs to  $\mathcal{B}_{WCM}$ , when  $\mathcal{B}_C$  is connected to it, we must have  $c(\cdot)$  belonging to  $\mathcal{B}_C$  and  $(w(\cdot), m(\cdot))$  belonging to  $\mathcal{B}_{WM}$ . This amounts to the equation  $\mathcal{B}_{WCM} \leftrightarrow \mathcal{B}_C = \mathcal{B}_{WM}$ , where  $\mathcal{B}_{WCM}, \mathcal{B}_{WM}$ , are specified and  $\mathcal{B}_C$  has to be found. (Here  $\mathcal{B}_{WCM} \leftrightarrow \mathcal{B}_C$  denotes the collection of all  $(w(\cdot), m(\cdot))$  where there exists a  $c(\cdot)$  such that  $(w(\cdot), c(\cdot), m(\cdot))$  belongs to  $\mathcal{B}_{WCM}$ , and  $c(\cdot) \in \mathcal{B}_C$ .) Observe that  $\mathcal{B}_{WCM}, \mathcal{B}_{WM}, \mathcal{B}_C$ , are over an abelian group. Let us assume that they are closed under subtraction. By Lemma 28 of the Appendix (generalization of the implicit inversion theorem), we know that the necessary and sufficient condition for existence of such a  $\mathcal{B}_C$  is that the conditions “ $\mathcal{B}_{WCM} \circ WM \supseteq \mathcal{B}_{WM}$  and  $\mathcal{B}_{WCM} \times WM \subseteq \mathcal{B}_{WM}$ ” hold and if they do, we have  $\mathcal{B}_{WCM} \leftrightarrow (\mathcal{B}_{WCM} \leftrightarrow \mathcal{B}_{WM}) = \mathcal{B}_{WM}$ , so that a possible solution for  $\mathcal{B}_C$  is  $\mathcal{B}_{WCM} \leftrightarrow \mathcal{B}_{WM}$ . This is in line with the thinking of behaviourists ([34], [35], [26]).

The usual dynamical systems that we encounter are defined through differential equations of the form  $(R_Z(\xi)) z(t) = 0$ , where the entries in  $(R_Z(\xi))$  are real polynomials in ‘ $\frac{d}{dt}$ ’. The solution space of the equation  $(R_Z(\xi)) z(t) = 0$ , is also the collection of vectors  $z(t)$  which satisfy  $\lambda(\xi)^T (R_Z(\xi)) z(t) = 0$ , for arbitrary vectors  $\lambda(\xi)^T$  with entries which are real polynomials in  $\xi$ . The collection of vectors of the form  $\lambda(\xi)^T (R_Z(\xi))$  is a ‘module’ closed under addition and multiplication by a real polynomial in  $\xi$  (and therefore also closed under subtraction). We will denote the module of vectors of the form  $\lambda(\xi)^T (R_Z(\xi))$  by  $\hat{\mathcal{K}}_Z$ . The operations of addition and intersection of modules as well as contraction and restriction of modules are defined as for vector spaces.

Let  $\hat{\mathcal{B}}_Z$  be the above solution space of the equation  $(R_Z(\xi)) z(t) = 0$ . We will henceforth confine ourselves to such dynamical systems. It is clear that  $\hat{\mathcal{B}}_Z$  is closed under subtraction. The operation of a row  $r_Z(\xi)$  of  $(R_Z(\xi))$  on a solution vector  $z(t)$ , yielding another vector  $y(t)$ , can be treated as a ‘generalized dot product’  $\langle r_Z(\xi), z(t) \rangle$ . This operation satisfies the properties

$$\begin{aligned} \langle -r_Z(\xi), z(t) \rangle &= \langle r_Z(\xi), -z(t) \rangle = -\langle r_Z(\xi), z(t) \rangle; \\ \langle r_Z^1(\xi) + r_Z^2(\xi), z(t) \rangle &= \langle r_Z^1(\xi), z(t) \rangle + \langle r_Z^2(\xi), z(t) \rangle; \\ \langle r_Z(\xi), z^1(t) + z^2(t) \rangle &= \langle r_Z(\xi), z^1(t) \rangle + \langle r_Z(\xi), z^2(t) \rangle. \end{aligned}$$

We will denote the module of vectors of the form  $\lambda(\xi)^T (R_Z(\xi))$  by  $\hat{\mathcal{K}}_Z$ . It can be shown that  $\hat{\mathcal{K}}_Z, \hat{\mathcal{B}}_Z$  uniquely fix each other [25]. We therefore have the bijection, denoting it by ‘\*’,

$$\hat{\mathcal{K}}_Z^* = \hat{\mathcal{B}}_Z, \hat{\mathcal{B}}_Z = \hat{\mathcal{K}}_Z^*, (\hat{\mathcal{K}}_Z^*)^* = \hat{\mathcal{K}}_Z, (\hat{\mathcal{B}}_Z^*)^* = \hat{\mathcal{B}}_Z.$$

Given dynamical systems  $\hat{\mathcal{B}}_{YZ}, \overline{\mathcal{B}}_{YZ}, \hat{\mathcal{B}}_{YQ}$  we note that

- $\hat{\mathcal{B}}_{YZ} + \overline{\mathcal{B}}_{YZ}$ , is the collection of all vectors  $(y''(\cdot), z''(\cdot))$  such that  $(y''(\cdot), z''(\cdot)) = (y(\cdot), z(\cdot)) + (\bar{y}(\cdot), \bar{z}(\cdot))$ , where  $(y(\cdot), z(\cdot)) \in \hat{\mathcal{B}}_{YZ}, (\bar{y}(\cdot), \bar{z}(\cdot)) \in \overline{\mathcal{B}}_{YZ}$ ;
- $\hat{\mathcal{B}}_{YZ} \circ Z$ , is the collection of all vectors  $z(\cdot)$  such that there exists a  $(y(\cdot), z(\cdot)) \in \hat{\mathcal{B}}_{YZ}$ ;
- $\hat{\mathcal{B}}_{YZ} \times Z$ , is the collection of all vectors  $z(\cdot)$  such that there exists a  $(0(\cdot), z(\cdot)) \in \hat{\mathcal{B}}_{YZ}$ .
- $\hat{\mathcal{B}}_{YZ} \leftrightarrow \hat{\mathcal{B}}_{YQ} \equiv (\hat{\mathcal{B}}_{YZ} \cap \hat{\mathcal{B}}_{YQ}) \circ ZQ = (\hat{\mathcal{B}}_{YZ} + (\hat{\mathcal{B}}_{YQ})_{(-Y)Q}) \circ ZQ$ .

For modules of the kind  $\hat{\mathcal{K}}_Z$ , we have polynomial time algorithms (based on the Euclidean algorithm for gcd) which given  $\hat{\mathcal{K}}_{YZ}, \overline{\mathcal{K}}_{YZ}, \hat{\mathcal{K}}_{YQ}$  can compute  $\hat{\mathcal{K}}_{YZ} \circ Z, \hat{\mathcal{K}}_{YZ} \times Z, \hat{\mathcal{K}}_{YZ} + \overline{\mathcal{K}}_{YZ}, \hat{\mathcal{K}}_{YZ} \cap \overline{\mathcal{K}}_{YZ}$ , and since  $\hat{\mathcal{K}}_{YZ} \leftrightarrow \hat{\mathcal{K}}_{YQ} = (\hat{\mathcal{K}}_{YZ} \cap \hat{\mathcal{K}}_{YQ}) \circ ZQ = (\hat{\mathcal{K}}_{YZ} + (\hat{\mathcal{K}}_{YQ})_{(-Y)Q}) \times ZQ$ , we also can compute  $\hat{\mathcal{K}}_{YZ} \leftrightarrow \hat{\mathcal{K}}_{YQ}$  in polynomial time.

We now have the following lemma.

**Lemma 20.** *Let  $\hat{\mathcal{B}}_{YZ}, \overline{\mathcal{B}}_{YZ}$ , be dynamical systems. We then have the following.*

1.  $(\hat{\mathcal{B}}_{YZ} + \overline{\mathcal{B}}_{YZ})^* = \hat{\mathcal{B}}_{YZ}^* \cap \overline{\mathcal{B}}_{YZ}^*$ ,  $(\hat{\mathcal{B}}_{YZ}^* + \overline{\mathcal{B}}_{YZ}^*)^* = \hat{\mathcal{B}}_{YZ} \cap \overline{\mathcal{B}}_{YZ}$ ;
2.  $(\hat{\mathcal{B}}_{YZ} \cap \overline{\mathcal{B}}_{YZ})^* = \hat{\mathcal{B}}_{YZ}^* + \overline{\mathcal{B}}_{YZ}^*$ ,  $(\hat{\mathcal{B}}_{YZ}^* \cap \overline{\mathcal{B}}_{YZ}^*)^* = \hat{\mathcal{B}}_{YZ} + \overline{\mathcal{B}}_{YZ}$ ;
3.  $(\hat{\mathcal{B}}_{YZ} \circ Z)^* = \hat{\mathcal{B}}_{YZ}^* \times Z$ ,  $(\hat{\mathcal{B}}_{YZ}^* \circ Z)^* = \hat{\mathcal{B}}_{YZ} \times Z$ ;
4.  $(\hat{\mathcal{B}}_{YZ} \times Z)^* = \hat{\mathcal{B}}_{YZ}^* \circ Z$ ,  $(\hat{\mathcal{B}}_{YZ}^* \times Z)^* = \hat{\mathcal{B}}_{YZ} \circ Z$ .

*Proof.* If  $\mathcal{B}_{YZ}^1, \mathcal{B}_{YZ}^2$ , are behaviours so are  $\mathcal{B}_{YZ}^1 + \mathcal{B}_{YZ}^2$ ,  $\mathcal{B}_{YZ}^1 \cap \mathcal{B}_{YZ}^2$ , and for any behaviour  $\mathcal{B}_{YZ}$ , we have a unique  $\mathcal{B}_{YZ}^*$  and  $(\mathcal{B}_{YZ}^*)^* = \mathcal{B}_{YZ}$  [25]. Similarly, if  $\mathcal{K}_{YZ}^1, \mathcal{K}_{YZ}^2$  are modules of constraints so are  $\mathcal{K}_{YZ}^1 + \mathcal{K}_{YZ}^2$ ,  $\mathcal{K}_{YZ}^1 \cap \mathcal{K}_{YZ}^2$ , and for any module of constraints  $\mathcal{K}_{YZ}$ , we have a unique behaviour  $\mathcal{K}_{YZ}^*$  and  $(\mathcal{K}_{YZ}^*)^* = \mathcal{K}_{YZ}$ .

We will prove only the first half of each statement.

1. Since  $\mathcal{B}_{YZ}^*$  is unique for any behaviour  $\mathcal{B}_{YZ}$ , it is clear that  $\mathcal{B}_{YZ}^1 \supseteq \mathcal{B}_{YZ}^2$  iff  $(\mathcal{B}_{YZ}^1)^* \subseteq (\mathcal{B}_{YZ}^2)^*$ . (If rows of constraints corresponding to  $(\mathcal{B}_{YZ}^*)^*$  are adjoined to those corresponding to  $(\mathcal{B}_{YZ}^2)^*$ , the behaviour  $\mathcal{B}_{YZ}^2$  would be unaffected.) Therefore, since  $\hat{\mathcal{B}}_{YZ} + \overline{\mathcal{B}}_{YZ} \supseteq \hat{\mathcal{B}}_{YZ}, \overline{\mathcal{B}}_{YZ}$ ,  $(\hat{\mathcal{B}}_{YZ} + \overline{\mathcal{B}}_{YZ})^* \subseteq \hat{\mathcal{B}}_{YZ}^* \cap \overline{\mathcal{B}}_{YZ}^*$ . On the other hand every member of  $\hat{\mathcal{B}}_{YZ}$  as well as of  $\overline{\mathcal{B}}_{YZ}$  would satisfy any constraint in  $\hat{\mathcal{B}}_{YZ}^* \cap \overline{\mathcal{B}}_{YZ}^*$ . Therefore the sum of a member of  $\hat{\mathcal{B}}_{YZ}$  and a member of  $\overline{\mathcal{B}}_{YZ}$ , i.e., every member of  $\hat{\mathcal{B}}_{YZ} + \overline{\mathcal{B}}_{YZ}$ , would satisfy any constraint in  $\hat{\mathcal{B}}_{YZ}^* \cap \overline{\mathcal{B}}_{YZ}^*$ . Thus  $(\hat{\mathcal{B}}_{YZ} + \overline{\mathcal{B}}_{YZ})^* \supseteq \hat{\mathcal{B}}_{YZ}^* \cap \overline{\mathcal{B}}_{YZ}^*$  and we conclude that  $(\hat{\mathcal{B}}_{YZ} + \overline{\mathcal{B}}_{YZ})^* = \hat{\mathcal{B}}_{YZ}^* \cap \overline{\mathcal{B}}_{YZ}^*$ .

2. This follows from part 1 and the fact that for any behaviour  $\mathcal{B}$  we have  $(\mathcal{B}^*)^* = \mathcal{B}$ .

3. Any member of  $\hat{\mathcal{B}}_{YZ} \circ Z$  would satisfy every constraint in  $\hat{\mathcal{B}}_{YZ}^* \times Z$ . Therefore  $(\hat{\mathcal{B}}_{YZ} \circ Z)^* \supseteq \hat{\mathcal{B}}_{YZ}^* \times Z$ . On the other hand every member of  $\hat{\mathcal{B}}_{YZ}$  would satisfy the constraints in  $(\hat{\mathcal{B}}_{YZ} \circ Z)^* \oplus \mathbf{0}_Y$ . Therefore  $(\hat{\mathcal{B}}_{YZ} \circ Z)^* \subseteq \hat{\mathcal{B}}_{YZ}^* \times Z$ .

4. This follows from part 3 and the fact that for any behaviour  $\mathcal{B}$  we have  $(\mathcal{B}^*)^* = \mathcal{B}$ . □

Using Lemma 20, we can prove the version of implicit duality theorem (Theorem 5), given below. The proof is a line by line translation of the proof of Theorem 5, replacing ‘ $\perp$ ’ wherever it occurs, by ‘ $*$ ’.

**Theorem 21.**  $(\hat{\mathcal{B}}_{WCM} \leftrightarrow \hat{\mathcal{B}}_C)^* = \hat{\mathcal{B}}_{WCM}^* \leftrightarrow \hat{\mathcal{B}}_C^*$ ,  $(\hat{\mathcal{B}}_{WCM}^* \leftrightarrow \hat{\mathcal{B}}_C^*)^* = \hat{\mathcal{B}}_{WCM} \leftrightarrow \hat{\mathcal{B}}_C$ .

Thus the problem of finding a  $\hat{\mathcal{B}}_C$ , such that  $\hat{\mathcal{B}}_{WCM} \leftrightarrow \hat{\mathcal{B}}_C = \hat{\mathcal{B}}_{WM}$ , given  $\hat{\mathcal{B}}_{WCM}, \hat{\mathcal{B}}_{WM}$  reduces to finding a  $\hat{\mathcal{B}}_C^*$ , such that  $\hat{\mathcal{B}}_{WCM}^* \leftrightarrow \hat{\mathcal{B}}_C^* = \hat{\mathcal{B}}_{WM}^*$ , given  $\hat{\mathcal{B}}_{WCM}^*, \hat{\mathcal{B}}_{WM}^*$ . We merely need to check if  $\hat{\mathcal{B}}_{WCM}^* \circ WM \supseteq \hat{\mathcal{B}}_{WM}^*$  and  $\hat{\mathcal{B}}_{WCM}^* \times WM \subseteq \hat{\mathcal{B}}_{WM}^*$ , which can be done in polynomial time and if these hold, compute in polynomial time  $\hat{\mathcal{B}}_C^* := \hat{\mathcal{B}}_{WCM}^* \leftrightarrow \hat{\mathcal{B}}_{WM}^*$ . This amounts to computing  $\hat{\mathcal{B}}_C$ , since  $(\hat{\mathcal{B}}_C^*)^* = \hat{\mathcal{B}}_C$ .

**Remark 6.** *The above discussion holds even if  $\xi$  denotes  $(\frac{d}{dt}, \dots, \frac{\partial}{\partial z_i}, \dots)$  and the entries of  $(R_Z(\xi))$  in  $(R_Z(\xi))(z(t)) = 0$ , are multivariate polynomials in  $\xi$ . However the algorithms are not polynomial time [18]*

## 7. General framework for implicit duality

A framework for a general version of implicit duality theorem based on the dot product is given below. There are other situations (eg. involving matroids) where a version of the theorem holds even when no dot product is defined (see Section 8).

Let  $G^1, G^2, G^3$  be abelian groups with a function  $\hat{f} : G^1 \times G^2 \rightarrow G^3$ , such that  $\hat{f}(a_1, b_2 +_2 c_2) = \hat{f}(a_1, b_2) +_3 \hat{f}(a_1, c_2)$  and  $\hat{f}(a_1 +_1 b_1, c_2) = \hat{f}(a_1, c_2) +_3 \hat{f}(b_1, c_2)$ ,  $a_1, b_1 \in G^1, b_2, c_2 \in G^2$  where  $+_i, i = 1, 2, 3$ , denote respectively the addition operations of  $G^1, G^2, G^3$ . (It follows that  $\hat{f}(a_1, 0) = \hat{f}(0, b_2) = 0, a_1 \in G^1, b_2 \in G^2$  and  $\hat{f}(-a, b) = \hat{f}(a, -b) = -\hat{f}(a, b)$ .)

Let  $H \subseteq G^3$  be closed under addition with  $0 \in H$ . A vector  $h_Z^i$  is a function  $h_Z^i : Z \rightarrow G^i, i = 1, 2$ . We say  $h_Z^i$  is a function on  $Z$  over  $G^i$ . We will treat  $h_Z^i$  as a row vector with columns indexed by  $Z$  and entries from  $G^i$ . A zero vector denoted  $0_Z^i$  has all its entries equal to  $0^i \in G^i$ . Addition, subtraction of vectors are defined in the usual way.

Let  $\mathcal{V}_Z^1, \mathcal{V}_Z^2$ , be collections of vectors on  $Z$  with entries from  $G^1, G^2$ , respectively, closed under subtraction. We will refer to  $\mathcal{V}_Z^1, \mathcal{V}_Z^2$ , respectively as a chain group on  $Z$  over  $G^1, G^2$ . A chain group on  $S \uplus P$  over  $G^i$  is denoted by  $\mathcal{V}_{SP}^i, i = 1, 2$ . The symbols  $\mathcal{J}_{SP}, \mathcal{K}_{SP}$ , respectively, would denote a collection of vectors on  $S \uplus P$ , over  $G^1, G^2$ , that contains the zero vector over  $G^1, G^2$ . The notions of restriction, contraction, sum, intersection, matched and skewed composition are defined as in Subsection 2.1. We remind the reader that  $\mathcal{L}_{SP} \leftrightarrow \mathcal{L}_{PQ} \equiv \{(h_S, g_Q) : (h_S, x_P) \in \mathcal{L}_{SP}, (x_P, g_Q) \in \mathcal{L}_{PQ}, \text{ for some } x_P\} = (\mathcal{L}_{SP} \cap \mathcal{L}_{PQ}) \circ SQ = (\mathcal{L}_{SP} + \mathcal{L}_{(-P)Q}) \times SQ$  and  $\mathcal{L}_{SP} \rightleftharpoons \mathcal{L}_{PQ} \equiv \{(h_S, g_Q) : (h_S, x_P) \in \mathcal{L}_{SP}, (-x_P, g_Q) \in \mathcal{L}_{PQ}, \text{ for some } x_P\} = (\mathcal{L}_{SP} \cap \mathcal{L}_{(-P)Q}) \circ SQ = (\mathcal{L}_{SP} + \mathcal{L}_{PQ}) \times SQ$ .

Let  $f : \mathcal{V}_Y^1 \times \mathcal{V}_Y^2 \rightarrow G^3, Y \subseteq Z$ , be defined by  $f(x, y) = \sum_{e \in Y} \hat{f}(x_e, y_e), x \in \mathcal{V}_Y^1, y \in \mathcal{V}_Y^2$ , where  $\Sigma$  denotes the addition operation of  $G^3$ . We will use  $\langle x, y \rangle$  in place of  $f(x, y)$  and call it a q-dot product of  $x, y$ . Let  $H \subseteq G^3$  be closed under addition with  $0 \in H$ . For  $\mathcal{J}_Z \subseteq \mathcal{V}_Z^1, \mathcal{K}_Z \subseteq \mathcal{V}_Z^2$ , we define  $\mathcal{J}_Z^* \equiv \{y, \langle x, y \rangle \in H, x \in \mathcal{J}_Z\}, \mathcal{K}_Z^* \equiv \{x, \langle x, y \rangle \in H, y \in \mathcal{K}_Z\}$ .

**Definition 22.** Let  $\mathcal{J}_Z \subseteq \mathcal{V}_Z^1, \mathcal{K}_Z \subseteq \mathcal{V}_Z^2$ , be closed under addition with the zero vector present in them. We say  $\mathcal{J}, \mathcal{K}$  are q-orthogonal (quasi orthogonal) iff  $\langle x, y \rangle \in H$ , whenever  $x \in \mathcal{J}_Z, y \in \mathcal{K}_Z$ . We say  $\mathcal{J}_Z, \mathcal{K}_Z$  are closed under q-orthogonality iff  $(\mathcal{J}_Z^*)^* = \mathcal{J}_Z$  and  $(\mathcal{K}_Z^*)^* = \mathcal{K}_Z$ .

**Definition 23.** Let  $\mathcal{V}_Z^1, \mathcal{V}_Z^2$ , be chain groups over  $G^1, G^2$ , respectively. We say a family  $\mathcal{S}^1 (\mathcal{S}^2)$  of collections of vectors  $\mathcal{J}_Y \subseteq \mathcal{V}_Z^1 \circ Y (\mathcal{K}_Y \subseteq \mathcal{V}_Z^2 \circ Y)$  on  $Y \subseteq Z$  over  $G^1 (G^2)$  is regular relative to  $\mathcal{V}_Z^1 (\mathcal{V}_Z^2)$ , respectively iff

- each  $\mathcal{J}_Y (\mathcal{K}_Y)$  is closed under addition and contains the zero vector;
- each  $\mathcal{J}_Y (\mathcal{K}_Y)$  is closed under q-orthogonality;
- whenever  $\mathcal{J}_Y (\mathcal{K}_Y)$  is in  $\mathcal{S}^1 (\mathcal{S}^2)$ , so are its restriction and contraction to  $T \subseteq Y$ ;
- whenever  $\mathcal{J}'_Y, \mathcal{J}''_Y (\mathcal{K}'_Y, \mathcal{K}''_Y)$  are in  $\mathcal{S}^1 (\mathcal{S}^2)$ , so are  $\mathcal{J}'_Y + \mathcal{J}''_Y, \mathcal{J}'_Y \cap \mathcal{J}''_Y (\mathcal{K}'_Y + \mathcal{K}''_Y, \mathcal{K}'_Y \cap \mathcal{K}''_Y)$ .

We now have a version of implicit duality theorem in this framework.

**Theorem 24.** Let  $(\mathcal{S}^1, \mathcal{S}^2)$  be regular relative to  $(\mathcal{V}_Z^1, \mathcal{V}_Z^2)$ . We then have the following.

1.  $\mathcal{J}_Y \rightarrow (\mathcal{J}_Y)^*$  is a bijection between  $\mathcal{S}^1, \mathcal{S}^2 (\mathcal{K}_Y \rightarrow (\mathcal{K}_Y)^*$  is the inverse bijection between  $\mathcal{S}^2, \mathcal{S}^1)$ .
2. Let  $\mathcal{J}_{SP}, \mathcal{J}_{PQ} (\mathcal{K}_{SP}, \mathcal{K}_{PQ})$  be members of  $\mathcal{S}^1 (\mathcal{S}^2)$ . We then have  $(\mathcal{J}_{SP} \leftrightarrow \mathcal{J}_{PQ})^* = (\mathcal{J}_{SP} \rightleftharpoons \mathcal{J}_{PQ})^* ((\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{PQ})^* = (\mathcal{K}_{SP} \rightleftharpoons \mathcal{K}_{PQ})^*)$ .

Proof is given in [24].

We list below instances, where the above general version of the implicit duality theorem is applicable.

1. Vector spaces  $\mathcal{V}_Y, Y \subseteq Z$ , with the usual definition of orthogonality taking  $H = \{0\}$ , where 0 is the zero of the scalar field and  $x_Y, y_Y$  orthogonal iff  $\langle x_Y, y_Y \rangle = 0$ . Here  $\mathcal{S}_1 = \mathcal{S}_2$ . Implicit duality theorem is one of the two basic results of implicit linear algebra in the present book (see also [21]).
2. Group codes with a definition of orthogonality taking  $H = \{0\}$ , where 0 is the zero of an abelian group  $G_3$  ([7]). An element of a group code is a vector whose entries are members of a topological abelian group. Here  $\mathcal{S}_1 = \mathcal{S}_2$ . Implicit duality is a basic result called conditioned code duality (Theorem 4.5 in the above reference). It follows by Theorem 24.
3. Modules over real polynomials in  $\xi \equiv \frac{d}{dt}$  with the usual definition of orthogonality. Here  $H = \{0\}$ , where 0 is the zero polynomial and  $x_Y, y_Y$  orthogonal iff  $\langle x_Y, y_Y \rangle = 0$ .

In this case  $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$  is the collection of all modules  $\mathcal{K}_Y, Y \subseteq Z$ , which satisfy  $(\mathcal{K}_Y^*)^* = \mathcal{K}_Y$ . Restrictions and contractions are dual notions under q-orthogonality for members of  $\mathcal{S}$ , so also are the operations of sums and intersections of members of  $\mathcal{S}$ . It is clear that sum, intersection, contraction, restriction of members of  $\mathcal{S}$ , remain in it.  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{PQ}$  is defined in the usual manner. Implicit

duality theorem follows by Theorem 24.

This result can be used to construct a version of duality for behavioural systems (see [17], [18],[24]).

The behaviour  $\mathcal{B}_Y$  is taken to be the collection of vectors  $w(t)$  which satisfy

$(R_Y(\xi))(w(t)) = 0$ , where  $(R_Y(\xi))$  is a matrix with entries which are polynomials in  $\xi := \frac{d}{dt}$  and columns  $Y$ . These ideas carry through even if we take  $\xi := (\frac{d}{dt}, \dots, \frac{\partial}{\partial z_i}, \dots)$ .

4. The family  $\mathcal{S}_1$  is taken to be that of behaviours  $\mathcal{B}_Y, Y \subseteq Z$ , which satisfy  $(R_Y(\xi))(w(t)) = 0$ , where  $(R_Y(\xi))$  is a matrix with entries which are real polynomials in  $\xi := \frac{d}{dt}$  and columns  $Y$ . The family  $\mathcal{S}_2$  is taken to be that of modules  $\mathcal{K}_Y, Y \subseteq Z$ , which are collections of all vectors of the form  $\lambda(\xi)^T (R_Y(\xi))$  (see [17]). A row vector  $k_Y(\xi)$  of  $\mathcal{K}_Y$  is q-orthogonal to  $w(t)$  iff  $k_Y(\frac{d}{dt})(w(t)) = 0$ . Here  $G_1$  is the abelian group of real polynomials,  $G_2 = G_3$  is the abelian group of real infinitely differentiable functions of time.

We say  $\mathcal{K}_Y^* = \mathcal{B}_Y$ , iff  $\mathcal{B}_Y$  is the collection of all vectors  $w(t)$  such that  $k_Y(\frac{d}{dt})(w(t)) = 0$ , where  $k_Y(\frac{d}{dt}) \in \mathcal{K}_Y$ . We say  $\mathcal{B}_Y^* = \mathcal{K}_Y$ , iff  $\mathcal{K}_Y$  is the collection of all vectors  $k_Y(\frac{d}{dt})$  such that  $k_Y(\frac{d}{dt})(w(t)) = 0, w(t) \in \mathcal{B}_Y$ . If  $\mathcal{B}_Y$  is the collection of vectors  $w(t)$  which satisfy  $(R_Y(\xi))(w(t)) = 0$ , it can be shown that the module  $\mathcal{B}_Y^*$  is the collection of vectors of the form  $\lambda(\xi)^T (R_Y(\xi))$  (see ([25])). Implicit duality theorem follows by Theorem 24.

In Section 6, these ideas were used to compute a controller  $\mathcal{B}_C$  that satisfies  $\mathcal{B}_{WCM} \leftrightarrow \mathcal{B}_C = \mathcal{B}_{WM}$ , where  $\mathcal{B}_{WCM}, \mathcal{B}_{WM}$  are respectively the plant and the desired behaviour. One solves the equation  $\mathcal{B}_{WCM}^* \leftrightarrow \mathcal{B}_C^* = \mathcal{B}_{WM}^*$ , which can be done in polynomial time and then computes  $\mathcal{B}_C = (\mathcal{B}_C^*)^*$ .

These ideas carry through even if we take  $\xi := (\frac{d}{dt}, \dots, \frac{\partial}{\partial z_i}, \dots)$  (see [18]). However computations are not necessarily polynomial time.

5. *Finitely generated real cones on subsets  $Y \subseteq Z$ .* Here  $\mathcal{S}_1 = \mathcal{S}_2$ . Two vectors  $x_Y, y_Y$  are q-orthogonal ('polar') iff  $\langle x_Y, y_Y \rangle \leq 0$ , i.e.,  $H$  is the set of nonpositive real numbers. The cone  $\mathcal{C}_Y^p$  is the set of all vectors polar to vectors in the cone  $\mathcal{C}_Y^p$ . Restrictions and contractions of finitely generated cones remain finitely generated, so also do sums and intersections of finitely generated cones. Restrictions and contractions are dual notions under polarity for finitely generated cones, so also are the operations of sums and intersections of finitely generated cones. Farkas Lemma implies that  $(\mathcal{C}_Y^p)^p = \mathcal{C}_Y$ . Matched composition  $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_{PQ}$  and skewed composition  $\mathcal{C}_{SP} \Rightarrow \mathcal{C}_{PQ}$  are defined in the usual manner. Implicit duality theorem follows by Theorem 24. The basic result in [1] can be proved by using these ideas (see [16]), [22]).
6.  $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$  is the family of collections of vectors  $x_Y = \lambda^T R_Y + \sigma^T J_Y, Y \subseteq Z$ , generated by a pair of rational matrices  $R_Y, J_Y$ , where  $\lambda^T$  is a rational vector and  $\sigma^T$  is an integral vector. It is clear that sum, intersection, contraction, restriction of members of  $\mathcal{S}$ , remain in it. We define  $x_Y, y_Y$  to be q-orthogonal iff  $\langle x_Y, y_Y \rangle$  is an integer, i.e.,  $H$  is the set of integers. It can be shown that if  $\mathcal{K}_Y \in \mathcal{S}$ , then  $\mathcal{K}_Y^* \in \mathcal{S}$  and  $(\mathcal{K}_Y^*)^* = \mathcal{K}_Y$ . Matched composition  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{PQ}$  and skewed composition  $\mathcal{K}_{SP} \Rightarrow \mathcal{K}_{PQ}$ , are defined in the usual manner. Restrictions and contractions are dual notions under q-orthogonality for members of  $\mathcal{S}$ , so also are the operations of sums and intersections of members of  $\mathcal{S}$ . Implicit duality theorem follows by Theorem 24 (see [16], [22]).

## 8. Implicit duality and implicit inversion for matroids

A matroid  $\mathcal{M}_S$  on a finite set  $S$  is an ordered pair  $(S, \mathcal{B})$ , where  $\mathcal{B}$  is a collection of subsets of  $S$ , called 'bases', satisfying the following axiom.

- If  $b_1, b_2 \in \mathcal{B}$  and if  $e_2 \in b_2 - b_1$ , then there exists  $e_1 \in b_1 - b_2$  such that  $(b_1 - \{e_1\}) \cup \{e_2\}$  is a member of  $\mathcal{B}$ .

A consequence of the axiom is that no base can contain another. Subsets of bases of  $\mathcal{M}_S$  are called **independent** sets of  $\mathcal{M}_S$ . For any subset  $b_S$  of  $S$ , its complement w.r.t.  $S$  is denoted by  $\bar{b}_S = S - b_S$ . Given a matroid  $\mathcal{M} = (S, \mathcal{B})$ , its **dual** matroid, denoted  $\mathcal{M}^*$ , is defined by  $(S, \mathcal{B}^*)$ , where  $\mathcal{B}^*$  is the collection of complements of the subsets present in  $\mathcal{B}$ . For a matroid  $\mathcal{M}$  on  $S$ , given a subset  $T \subseteq S$ , we

define **restriction** of  $\mathcal{M}$  to  $T$ , denoted  $\mathcal{M} \circ T$ , as a matroid, whose independent sets are all subsets of  $T$  which are independent in  $\mathcal{M}$  (equivalently, bases in  $\mathcal{M} \circ T$  are maximal intersections of bases of  $\mathcal{M}$  with  $T$ ). **Contraction** of  $\mathcal{M}$  to  $T$ , denoted  $\mathcal{M} \times T$ , is defined to be the matroid whose independent sets are precisely those  $X \subseteq T$  which satisfy the property that  $X \cup b_{S-T}$  is independent in  $\mathcal{M}$  whenever  $b_{S-T}$  is a base of  $\mathcal{M} \circ (S - T)$ .

Let  $\mathcal{M}_{SP}$  and  $\mathcal{M}_{PQ}$  be matroids on  $S \uplus P, P \uplus Q$ , respectively, where  $S, P, Q$ , are pairwise disjoint. The **union** of these matroids, denoted  $\mathcal{M}_{SP} \vee \mathcal{M}_{PQ}$ , is defined to be  $(S \uplus P \uplus Q, \mathcal{B}_\vee)$ , where  $\mathcal{B}_\vee$  is the collection of maximal sets of the form  $b_{SP} \cup b_{PQ}$ , where  $b_{SP}$  is a base of  $\mathcal{M}_{SP}$  and  $b_{PQ}$  is a base of  $\mathcal{M}_{PQ}$ . It can be shown that  $\mathcal{M}_{SP} \vee \mathcal{M}_{PQ}$  is again a matroid. The **linking** of  $\mathcal{M}_{SP}, \mathcal{M}_{PQ}$ , denoted  $\mathcal{M}_{SP} \leftrightarrow_{\mathcal{M}} \mathcal{M}_{PQ}$  is defined on the set  $S \uplus Q$  to be

$$(\mathcal{M}_{SP} \vee \mathcal{M}_{PQ}) \times (S \uplus Q).$$

The following ‘implicit duality theorem for matroids’ is from [29]. A proof is given in [24].

**Theorem 25.** (*Implicit Duality Theorem for matroids*)

$$(\mathcal{M}_{SP} \leftrightarrow_{\mathcal{M}} \mathcal{M}_{PQ})^* = \mathcal{M}_{SP}^* \leftrightarrow_{\mathcal{M}} \mathcal{M}_{PQ}^* \quad (32)$$

We now have a corollary on ‘self dual matroids’ which is analogous to the result on reciprocal and dirac structures in Subsection 4.3.

**Corollary 26.** *Let  $\mathcal{M}_{SP} \leftrightarrow_{\mathcal{M}} \mathcal{M}_{PQ}$  be matroids such that  $\mathcal{M}_{SP}^* = \mathcal{M}_{SP}$  and  $\mathcal{M}_{PQ}^* = \mathcal{M}_{PQ}$ . Then  $(\mathcal{M}_{SP} \leftrightarrow_{\mathcal{M}} \mathcal{M}_{PQ})^* = \mathcal{M}_{SP} \leftrightarrow_{\mathcal{M}} \mathcal{M}_{PQ}$ .*

We can take ‘self dual’ to mean that  $\mathcal{M}, \mathcal{M}^*$  are isomorphic, provided the isomorphism between  $\mathcal{M}_{SP}, \mathcal{M}_{SP}^*$  takes  $S$  to  $S$  and  $P$  to  $P$ . We note that there is a version of implicit duality theorem that holds for polymatroid rank functions ([29]).

There is a result analogous to the implicit inversion theorem for matroids but it works only in some special situations ([23]). We say a matroid  $\mathcal{M}_{SQ}$  is  $\{S, Q\}$ –complete iff the family of all independent sets of  $\mathcal{M}_{SQ} \circ S$  ( $\mathcal{M}_{SQ} \circ Q$ ), that contain bases of  $\mathcal{M}_{SQ} \times S$  ( $\mathcal{M}_{SQ} \times Q$ ), can be partitioned into an equivalence class  $\mathcal{E}_S(\mathcal{M}_{SQ})$  ( $\mathcal{E}_Q(\mathcal{M}_{SQ})$ ), such that if  $\mathbf{I}_j$  is one of the blocks of  $\mathcal{E}_S(\mathcal{M}_{SQ})$  ( $\mathcal{E}_Q(\mathcal{M}_{SQ})$ ) there is a unique block  $\hat{\mathbf{I}}_j$  of  $\mathcal{E}_Q(\mathcal{M}_{SQ})$  ( $\mathcal{E}_S(\mathcal{M}_{SQ})$ ) such that if  $I_1 \in \mathbf{I}_j$ ,  $I_1 \uplus I_2$  is a base of  $\mathcal{M}_{SQ}$  iff  $I_2 \in \hat{\mathbf{I}}_j$ .

$\{S, Q\}$ –complete matroids have occurred in the literature, for example the free product  $\mathcal{M}_S \square \mathcal{M}_Q$  of [6] or the principal sum of [5].

We have the following version of implicit inversion theorem for matroids.

**Theorem 27.** *Let matroids  $\mathcal{M}_{SP}, \mathcal{M}_{PQ}$  be  $\{S, P\}$ –complete,  $\{P, Q\}$ –complete respectively and let every block of  $\mathcal{E}_P(\mathcal{M}_{PQ})$  be a union of blocks of  $\mathcal{E}_P(\mathcal{M}_{SP}^*)$ . Then  $\mathcal{M}_{PQ} = \mathcal{M}_{SP}^* \leftrightarrow_{\mathcal{M}} (\mathcal{M}_{SP} \leftrightarrow_{\mathcal{M}} \mathcal{M}_{PQ})$ .*

Proof may be found in [23].

## Appendix A. Proof of the implicit inversion theorem (Theorem 3)

The following lemma is based on Theorem 11 of [15]. The proof is a slightly modified version of the one given in solution to Problem 7.5 in [16]. Theorem 3 follows immediately.

A zero-member collection  $\mathcal{K}_T$  of vectors has entries from an abelian group with  $0_T$  as a member.

**Lemma 28.** 1. *Let  $\mathcal{K}_{SP}, \mathcal{K}_{SQ}$  be zero-member collections of vectors on  $S \uplus P, S \uplus Q$  respectively.*

*Then there exists a zero-member collection of vectors  $\mathcal{K}_{PQ}$  on  $P \uplus Q$  s.t.  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{PQ} = \mathcal{K}_{SQ}$ , only if  $\mathcal{K}_{SP} \circ S \supseteq \mathcal{K}_{SQ} \circ S$  and  $\mathcal{K}_{SP} \times S \subseteq \mathcal{K}_{SQ} \times S$ .*

2. *Let  $\mathcal{K}_{SP}$  be a zero-member collection of vectors closed under subtraction on  $S \uplus P$  and let  $\mathcal{K}_{SQ}$  be a zero-member collection of vectors on  $S \uplus Q$ , closed under addition. Then the collection of vectors  $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{SQ}$ , is closed under addition, with  $0_{PQ}$  as a member.*

*Further let  $\mathcal{K}_{SP} \times S \subseteq \mathcal{K}_{SQ} \times S$  and  $\mathcal{K}_{SP} \circ S \supseteq \mathcal{K}_{SQ} \circ S$ . Then  $\mathcal{K}_{SP} \leftrightarrow (\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_{SQ}) = \mathcal{K}_{SQ}$ .*

3. Let  $\mathcal{H}_{SP} \circ S \supseteq \mathcal{H}_{SQ} \circ S$  and  $\mathcal{H}_{SP} \times S \subseteq \mathcal{H}_{SQ} \times S$ . Then  $\mathcal{H}_{SP} \leftrightarrow (\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{SQ}) = \mathcal{H}_{SQ}$  is the only solution to the equation  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{PQ} = \mathcal{H}_{SQ}$ , under the condition  $\mathcal{H}_{SP} \circ P \supseteq \mathcal{H}_{PQ} \circ P$  and  $\mathcal{H}_{SP} \times P \subseteq \mathcal{H}_{PQ} \times P$ .

*Proof.* 1. Suppose  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{PQ} = \mathcal{H}_{SQ}$  and  $0_{PQ} \in \mathcal{H}_{PQ}$ .

It is clear from the definition of the matched composition operation that  $\mathcal{H}_{SP} \circ S \supseteq \mathcal{H}_{SQ} \circ S$ . Since  $0_{PQ} \in \mathcal{H}_{PQ}$ , if  $(f_S, 0_P) \in \mathcal{H}_{SP}$ , we must have that  $(f_S, 0_Q) \in \mathcal{H}_{SQ}$ . Thus,  $\mathcal{H}_{SP} \times S \subseteq \mathcal{H}_{SQ} \times S$ .

2. Let  $\mathcal{H}_{PQ} := \mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{SQ}$ , i.e.,  $\mathcal{H}_{PQ}$  is the collection of all vectors  $(f_P, f_Q)$  s.t. for some vector  $f_S$ ,  $(f_S, f_Q) \in \mathcal{H}_{SQ}$ ,  $(f_S, f_P) \in \mathcal{H}_{SP}$ . Since  $\mathcal{H}_{PQ} = \mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{SQ}$ , and both  $\mathcal{H}_{SP}, \mathcal{H}_{SQ}$ , are closed under addition, so is  $\mathcal{H}_{PQ}$ . Since  $(0_S, 0_P) \in \mathcal{H}_{SP}$ ,  $(0_S, 0_Q) \in \mathcal{H}_{SQ}$ , we must have that  $0_{PQ} \in \mathcal{H}_{PQ}$ . Thus  $\mathcal{H}_{PQ}$  is a zero-member collection of vectors closed under addition.

Suppose  $\mathcal{H}_{SP} \times S \subseteq \mathcal{H}_{SQ} \times S$  and  $\mathcal{H}_{SP} \circ S \supseteq \mathcal{H}_{SQ} \circ S$ .

Let  $(f_S, f_Q) \in \mathcal{H}_{SQ}$ . Since  $\mathcal{H}_{SP} \circ S \supseteq \mathcal{H}_{SQ} \circ S$ , for some  $f_P$ , we must have that  $(f_S, f_P) \in \mathcal{H}_{SP}$ . By the definition of  $\mathcal{H}_{PQ}$ , we have that  $(f_P, f_Q) \in \mathcal{H}_{PQ}$ . Hence,  $(f_S, f_Q) \in \mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{PQ}$ . Thus,  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{PQ} \supseteq \mathcal{H}_{SQ}$ . Next, let  $(f_S, f_Q) \in \mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{PQ}$ , i.e., for some  $f_P$ ,  $(f_S, f_P) \in \mathcal{H}_{SP}$  and  $(f_P, f_Q) \in \mathcal{H}_{PQ}$ . We know, by the definition of  $\mathcal{H}_{PQ}$ , that there exists  $(f'_S, f_Q) \in \mathcal{H}_{SQ}$  s.t.  $(f'_S, f_P) \in \mathcal{H}_{SP}$ . Since  $\mathcal{H}_{SP}$  is closed under subtraction, we must have,  $(f_S - f'_S, 0_P) \in \mathcal{H}_{SP}$ . Hence,  $(f_S - f'_S) \in \mathcal{H}_{SP} \times S \subseteq \mathcal{H}_{SQ} \times S$  and therefore  $((f_S - f'_S), 0_Q) \in \mathcal{H}_{SQ}$ . Since  $\mathcal{H}_{SQ}$  is closed under addition and  $(f'_S, f_Q) \in \mathcal{H}_{SQ}$ , it follows that  $((f_S - f'_S), 0_Q) + (f'_S, f_Q) = (f_S, f_Q)$  also belongs to  $\mathcal{H}_{SQ}$ . Thus,  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{PQ} \subseteq \mathcal{H}_{SQ}$ .

3. From parts (1) and (2) above, the equation  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{PQ} = \mathcal{H}_{SQ}$  can be satisfied by some  $\mathcal{H}_{PQ}$  if and only if  $\mathcal{H}_{SP} \circ S \supseteq \mathcal{H}_{SQ} \circ S$  and  $\mathcal{H}_{SP} \times S \subseteq \mathcal{H}_{SQ} \times S$ .

Next, let  $\mathcal{H}'_{PQ}$  satisfy the equation  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}'_{PQ} = \mathcal{H}_{SQ}$  and be closed under addition.

From part (2), we know that if  $\mathcal{H}'_{PQ}$  satisfies  $\mathcal{H}_{SP} \circ P \supseteq \mathcal{H}'_{PQ} \circ P$  and  $\mathcal{H}_{SP} \times P \subseteq \mathcal{H}'_{PQ} \times P$ ,

then  $\mathcal{H}_{SP} \leftrightarrow (\mathcal{H}_{SP} \leftrightarrow \mathcal{H}'_{PQ}) = \mathcal{H}'_{PQ}$ . But  $\mathcal{H}'_{PQ}$  satisfies  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}'_{PQ} = \mathcal{H}_{SQ}$ .

It follows that for any such  $\mathcal{H}'_{PQ}$ , we have  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{SQ} = \mathcal{H}'_{PQ}$ .

This proves that  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{SQ}$  is the only solution to the equation  $\mathcal{H}_{SP} \leftrightarrow \mathcal{H}_{PQ} = \mathcal{H}_{SQ}$ , under the condition  $\mathcal{H}_{SP} \circ P \supseteq \mathcal{H}_{PQ} \circ P$  and  $\mathcal{H}_{SP} \times P \subseteq \mathcal{H}_{PQ} \times P$ . □

## Appendix B. Proof of the implicit duality theorem (Theorem 5)

From the definition of matched composition, when  $(X, Y, Z)$  are disjoint,

$$\mathcal{V}_{XY} \leftrightarrow \mathcal{V}_{YZ} = (\mathcal{V}_{XY} + \mathcal{V}_{(-Y)Z}) \times (X \cup Z)$$

and also equal to

$$(\mathcal{V}_{XY} \cap \mathcal{V}_{YZ}) \circ (X \cup Z).$$

Similarly, from the definition of skewed composition,

$$\mathcal{V}_{XY} \rightrightarrows \mathcal{V}_{YZ} = (\mathcal{V}_{XY} + \mathcal{V}_{YZ}) \times (X \cup Z)$$

and also equal to

$$(\mathcal{V}_{XY} \cap \mathcal{V}_{(-Y)Z}) \circ (X \cup Z).$$

Hence we have

$$\begin{aligned} (\mathcal{V}_{XY} \leftrightarrow \mathcal{V}_{YZ})^\perp &= [(\mathcal{V}_{XY} + \mathcal{V}_{(-Y)Z}) \times (X \cup Z)]^\perp \\ &= (\mathcal{V}_{XY} + \mathcal{V}_{(-Y)Z})^\perp \circ (X \cup Z) \\ &= (\mathcal{V}_{XY}^\perp \cap \mathcal{V}_{(-Y)Z}^\perp) \circ (X \cup Z) \\ &= \mathcal{V}_{XY}^\perp \rightrightarrows \mathcal{V}_{YZ}^\perp \end{aligned}$$

In particular,

$$(\mathcal{V}_{XY} \leftrightarrow \mathcal{V}_Y)^\perp = \mathcal{V}_{XY}^\perp \leftrightarrow \mathcal{V}_Y^\perp$$

since  $\mathcal{V}_Y = \mathcal{V}_{(-Y)}$ .

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