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Image = Cartoon+Texture

How Yves Meyer's "Oscillating patterns in image processing and in some nonlinear evolution equations" ended up in a computer vision model

Antonin Chambolle*and Jean-Michel Morel†

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Summary *While analyzing variational restoration methods in imaging with the eye of an harmonic analyst, Yves Meyer proposed a seminal reformulation of image decomposition methods, which aim at separating the "cartoon" and "texture" parts of an image. In this note, we describe Meyer's analysis of the image denoising problem. This analysis occupied an important part of his book "Oscillating patterns in image processing and in some nonlinear evolution equations". Starting from the famous Rudin, Osher and Fatemi (ROF) Total Variation-based model for natural image denoising, Meyer tries to bracket the space BV , of functions with bounded variation, and its dual between spaces that can be characterized by the decay of their wavelet coefficients. In that way, he first attempts to build a bridge between the Donoho and Johnstone wavelet shrinkage model for image denoising and the ROF model. This discussion leads him to explore the consistency of the ROF model on examples. He proves that the ROF model sometimes finds the right decomposition, namely $BV +$ an oscillatory component, and sometimes not. This leads him to pay attention to the dual of BV and eventually to the discovery that small norms in this space characterize oscillatory signals very well. This exploration yields a reinterpretation and extension of the ROF model, with the proposal of decomposing a distribution $f = u + v$ into two components, the first one u in BV and another one in a dual space where oscillating components have a small norm, in the spirit of interpolation theory. We illustrate the resulting cartoon+texture model and its developments with striking experiments which show its impact on medicine and society.*

Keywords Digital images, total variation, Rudin-Osher-Fatemi model, wavelet shrinkage, Besov spaces, oscillating patterns, image decomposition, cartoon, TV+L1 model.

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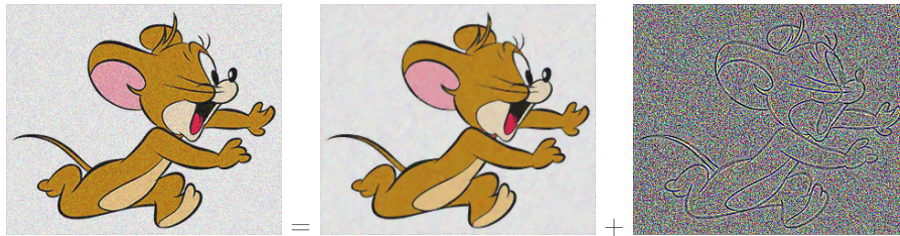


Figure 1: An “image =cartoon + texture” algorithm aims at decomposing any image into two parts, its geometric part or texture, and its noise or textured part. This experiment illustrates the decomposition in an ideal case. The left image is a scan of a real cartoon drawing. The middle image is its “BV” part, and the image on the right its texture. Many surprising applications of this model will be illustrated in section 4.

AMS classification numbers 26A45, 30H25, 49Q20, 68U10, 65D18, 65T60.

1 Historical background

The starting motivation of Yves Meyer, in his celebrated book [36], was to study a well known, at that time relatively recent model for image denoising and restoration, based on the Total Variation (TV). In the continuous setting, one assumes that the image is represented by a scalar valued grey-level $u(x) \in L^\infty(\Omega)$ (usually, Ω is a square or a rectangle, a bounded Lipschitz domain of \mathbb{R}^2 , or a periodicity cell, yet in most of [36] it is the whole plane \mathbb{R}^2), and the total variation is simply defined as

$$\int_{\Omega} |Du| = \sup \left\{ \int_{\Omega} u(x) \operatorname{div} g(x) dx : g \in C_c^\infty(\Omega; \mathbb{R}^2), \|g(x)\| \leq 1 \ \forall x \in \Omega \right\}, \quad (1)$$

and is the total mass (or variation) of the distributional derivative Du , which has to be a measure in order for this quantity to be finite.

Throughout this paper we handle the case where the values of $u(x)$ are scalar and represent the grey level at x . Yet, a straightforward extension of our discussion and algorithms applies to color images. Color images are vector images $u(x) = (R(x), G(x), B(x))$ with three scalar values standing for the amount of red, green, blue at each pixel. A possible definition of the total variation of a color image is to take the sum of the total variations of its color components. Nevertheless, in all numerical experiments, the sum is replaced by the Frobenius norm of Du .

The space of functions in $L^1(\Omega)$ with bounded variation is usually denoted $BV(\Omega)$. However in most of [36], Ω is \mathbb{R}^2 and BV functions are not required to be summable. Sobolev’s embedding only implies L^2 summability.

The TV-based image restoration or denoising model, proposed in 1992 by Rudin, Osher and Fatemi in [40], consists in recovering u for a noisy image g by minimizing the energy

$$\min_u \int_{\Omega} |Du| + \frac{\lambda}{2} \int_{\Omega} (u(x) - g(x))^2 dx. \quad (2)$$

More general inverse reconstruction problems were also considered. The groundbreaking model (2), known as “ROF” or “ $TV - L^2$ ”, is one of the first nonlinear edge-preserving image restoration models (and arguably the simplest one).

Meyer’s point of view was to reformulate (2) as a “decomposition” problem inspired by the classical definitions of interpolation spaces, as follows (*cf* [36, Def. 7]):

$$\inf_{(u,v) \in BV \times L^2 | f=u+v} \left(\int |Du| + \lambda |v|^2 \right). \quad (3)$$

In this beautiful interpretation, the model is seen to decompose any given image f into the sum of two components: the underlying BV image u , and an oscillating component v which would be the noise and texture. It is nonlinear, but remains a convex minimization problem with a unique solution. Yet finding an algorithm computing efficiently and precisely this unique minimum has proved challenging. Standard proximal splitting methods, relying on a dual [13, 7] or saddle-point [16, 9] formulation, are the basis of the most efficient methods by now, even if purely discrete formulations based on max-flow/min-cut duality yield (fast and) exact solutions [31, 14] for graph-based variants of the problem.

The ROF model is the first image analysis model where the “prior” on an image is described in terms of an ℓ^1 norm (here in practice of the discrete gradients), enforcing sparsity of the discontinuities: this general philosophy has become in the 2000’s the core of modern approaches to image and signal analysis (and more general data science), where the notion of sparsity is now overwhelming. Due to its simplicity, the total variation has also become one of the most convenient “regularizers” for large scale inverse problems in imaging, such as satellite imaging or 2D and 3D medical image reconstruction.

In the book [36], Yves Meyer developed a systematic analysis of the solutions of (3) and their properties, introducing appropriate functional spaces of oscillating functions in particular to represent the “ v ” component. In an attempt to unify this approach and the statisticians’ point of view, at that time based on wavelets and *wavelet shrinkage* [23, 24, 15], he tried to compare precisely such oscillating functions with functions in appropriate Besov spaces, defined by the decay of their wavelet coefficients. Of course, there is much more than this in [36], such as a survey on wavelet bases, on Littlewood-Paley and wavelet analysis, or improved Gagliardo-Nirenberg inequalities involving Besov seminorms, as well as applications of harmonic analysis to non-linear PDEs with in particular a few striking results on the Navier-Stokes equation [10, 12, 33]. Also, the last part of the book develops a theory of “chirps”, or highly oscillating signals, with tools very similar to the one developed for the modeling of oscillations in images.

In this note, we focus on Meyer's analysis of problem (3), and the connection to wavelet shrinkage. We now describe, first, his analysis.

2 Yves Meyer's analysis of the "ROF" model

2.1 The core of the problem: incompatibility of wavelet shrinkage with BV

"Wavelet shrinkage" Meyer writes, "is more than a new denoising algorithm. It provides a new way of understanding wavelet analysis. Indeed wavelet shrinkage is deeply related to "nonlinear approximation". In both cases wavelet coefficients are sorted out. In the first situation (denoising algorithms), the first N terms of the wavelet expansion provide us with the denoised signal or image and, in the second case, we obtain the optimal nonlinear approximation."

The wavelet shrinkage was introduced and studied by several groups in the early 90's, both from the point of view of estimation in statistics [25, 26, 23, 24] and interpolation theory [22]. Yves Meyer summarizes the main statistical properties of the approach into the following recipe, attributed to Donoho and collaborators:

Theorem 1 (*Theorem 7 of [36]*) *If we are given an $N \times N$ image $u(x)$ which is being corrupted by a white noise with level σ , then a wavelet shrinkage with threshold*

$$\tau = c\sigma \frac{\sqrt{\log N}}{N} \tag{16.8}$$

yields an estimator \hat{u}_N^ enjoying the following properties*

- (a) *With high probability, \hat{u}_N^* is as smooth as u is, with smoothness measured in a wide class of function spaces*
- (b) *This estimator \hat{u}_N^* is nearly minimax.*

The above theorem is not exactly a theorem: it requires clarification for "in a wide class of function spaces" and a precision for "nearly minimax". The only clear point is given in Lemma 18 of [36], which says that the wavelet coefficients of the additive noise do not exceed $c'\sigma \frac{\sqrt{\log N}}{N}$. (Again for c' large enough, and "with high probability"). But in this paper Meyer meets and addresses a core difficulty: the only functional space that has received general approval for modeling images is BV . Indeed, BV contains "shapes", namely characteristic functions of sets with finite perimeter. Such functions are obviously reasonable models for the silhouettes of objects photographed in a scene. Hence, even if by Lemma 18 a high enough wavelet threshold "kills the noise", the question is: what damage is it causing to a BV function? Meyer does not give a unique answer to this dilemma. We might summarize his exploration as an attempt to marry the ROF model with wavelet thresholding. But the match is not perfect. On the one hand, wavelet thresholding is uncompromising: it can just be "hard"

or "soft", that's its only concession. Thus the focus will rather be on the ROF side: Hence Meyer considers adapting the ROF model by

- changing the space BV into another space of the "wide class" of wavelet compatible spaces;
- changing the space L^2 into a space with finer characterizations for the wavelet thresholding.

Yet, since nothing of that will prove quite convincing, Meyer returns to the ROF model, examines the pros and cons of this model when applied to a sum of a BV function perturbed with an oscillatory signal. And this analysis leads to the emergence of the space G , a pseudo dual of BV and the one space prone to texture modeling.

2.2 Functional alternatives: the dual spaces $\dot{B}_1^{1,1}$ and $\dot{B}_\infty^{-1,\infty}$

As we said, a first attempt to marry wavelet shrinkage and the ROF model is to develop "good" alternatives for BV and L^2 . These alternative functional spaces should be characterized by the decay of wavelet coefficients and therefore prone to wavelet thresholding. Meyer first notices that the Banach space BV is contained in $L^{n/n-1}(\mathbb{R}^n)$ and that this embedding is sharp. Then he examines a first candidate to replace the BV bridgegroom for a better match with wavelet thresholding. In the next definition, $c_{j,k}$, $(j,k) \in \mathbb{Z} \times \mathbb{Z}^n$, are wavelet (frequency and space) coefficients in a basis consisting of smooth and compactly supported wavelets, see [36] for details.

Definition 1 (Definition 4 of [36]) *The Banach space $\dot{B}_1^{1,1}$ is the subspace of $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ defined by*

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |c_{j,k}| 2^{j(1-n/2)} < \infty \quad (12.3)$$

The dual space of $\dot{B}_1^{1,1}$ is the Banach space $\dot{B}_\infty^{-1,\infty}$. If $n = 2$, the latter space is characterized by the fact that the wavelet coefficients of a generalized function in $\dot{B}_\infty^{-1,\infty}$ belong to $l^\infty(\mathbb{Z} \times \mathbb{Z}^2)$.

The attempt goes back to DeVore and Lucier [22]. These authors proposed to replace the Banach space BV by the homogeneous Besov space $B = \dot{B}_1^{1,1}$ of Definition 1 in the ROF model. The above definition gives the norm of a function f in $B = \dot{B}_1^{1,1}$ as the l^1 norm of its wavelet coefficients. In this proposal, the second term of the ROF model is still the L^2 -norm of v . Hence the ROF functional becomes $\tilde{J}(u) = \|u\|_B + \lambda \|v\|_2^2$. Its minimization can be easily solved by a wavelet transform and a shrinkage of the coefficients, with the threshold $\frac{1}{2\lambda}$. A consequence is that if (u_λ, v_λ) mimimizes the DeVore-Lucier functional, then the norm of v_λ in the Besov space $\dot{B}_\infty^{-1,\infty}$ does not exceed $\frac{1}{2\lambda}$. Meyer notices that an equivalent statement holds true for the Osher-Rudin model (Theorem 2 below), where the same threshold is met.

Meyer nevertheless resists this first temptation: "The DeVore-Lucier model suffers from an obvious drawback. Indeed indicator functions of smooth domains do not belong to $\dot{B}_1^{1,1}$ and edges have disappeared from images." Hence, Meyer considers another variant: keeping u in BV but using the Besov norm $\dot{B}_\infty^{-1,\infty}$ for analyzing the v component in the $f = u + v$ decomposition leads to the new splitting algorithm by variational method,

$$\tilde{\omega}_\lambda(f) = \inf\{J_0(u) = \|u\|_{BV} + \lambda\|v\|_*; \quad f = u + v\} \quad (14.38)$$

where $\|\cdot\|_*$ now denotes the norm in the Besov space $\dot{B}_\infty^{-1,\infty}$.

The good point is that the Besov space $\dot{B}_\infty^{-1,\infty}$ admits a trivial characterization by the wavelet coefficient decay :

Lemma 1 (*Lemma 11 of [36]*) *Let $2^j\psi(2^jx - k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^2$, $\psi \in F$, be an orthonormal wavelet basis of $L^2(\mathbb{R}^2)$ where F is a finite set consisting of three analyzing wavelets belonging to \mathcal{C}^2 and compactly supported. Then a generalized function f belongs to the homogeneous Besov space $\dot{B}_\infty^{-1,\infty}$ if and only if its wavelet coefficients belong to $l^\infty(\mathbb{Z}^3 \times F)$.*

Meyer writes: "this lemma is extremely attractive since it nicely relates the functional norm in $\dot{B}_\infty^{-1,\infty}$ to an algorithm named wavelet shrinkage and which consists in putting to zero all wavelet coefficients which are less than a given threshold."

2.3 After all, can't we apply wavelet thresholding to BV ?

Left in doubt about the above attempts to smuggle in wavelet compatible spaces into image analysis, Meyer cyclically returns to the desire of "applying Theorem 1" to the space BV . The problem is that

Corollary 1 (*Corollary 2 of [36]*) *The space BV cannot be characterized by size properties on wavelet coefficients.*

Nevermind! For Meyer, "The fact that BV can be accepted comes from the decision to neglect logarithmic factors like $\log N$ in the minimax risk. If this decision is accepted, we can cheat and claim that the characterization of BV will be the same as the one of two simpler Besov spaces. Indeed we have

$$\dot{B}_1^{1,1} \subset BV \subset \dot{B}_\infty^{1,1}. \quad (16.9)$$

Then Donoho observed that the estimates of the minimax risks he could prove for these two Besov spaces differ only by logarithmic factors. That is why BV can be treated by the wavelet shrinkage algorithm."

Is that correct? In fact, the only results so far comparing the solution of (2) to wavelet thresholding algorithms are results of Cohen, Petrushev, Xu [18]. These authors, in particular, establish that in the Haar basis, given a L^2 function f (in the unit cube), the function u obtained by a *hard* thresholding of the smaller wavelet coefficients has an energy (2) which compares to the minimal

value (up to some explicit, yet large multiplicative constant), see [18, Thm. 9.3]. In any other case, wavelet shrinkage or thresholding actually mimicks solving (2) without being really equivalent, and it is even not known whether the solution will be of bounded variation (see [18, §3] for such results in the 2D Haar basis). Since BV is not a good space for wavelet analysis, Meyer will then turn to examine its dual space, or rather an *Ersatz* G of its dual space, in the hope that it will be more amenable.

2.4 The emergence of the space G , "dual of BV "

Among the many functional thorns of BV , Meyer notices that BV is not separable (i.e., does not contain a numerable dense subspace). Hence, the dual space of BV is not a function space. "However, there exists a closed subspace of BV which has a simple dual." Meyer considers the closed subspace¹ \mathcal{BV} of BV consisting of all $f(x)$ such that $\partial_j f \in L^1(\mathbb{R}^n)$, $j = 1, \dots, n$. Then the dual space of \mathcal{BV} is the Banach space G of Definition 2.

Definition 2 (*Definition 10 of [36]*) Let G denote the Banach space consisting of all generalized functions $f(x)$ which can be written as

$$f(x) = \partial_1 g_1(x) + \partial_2 g_2(x), \quad g_1, g_2 \in L^\infty(\mathbb{R}^2) \quad (4)$$

The norm $\|f\|_*$ of f in G is defined as the lower bound of all L^∞ norms of the functions $|g|$ where $g = (g_1, g_2)$, $|g(x)| = (|g_1|^2 + |g_2|^2)^{1/2}(x)$ and where the infimum is computed over all decompositions (4) of f .

Since $L^\infty(\mathbb{R}^2)$ is a dual space, it is easily seen that there exists an optimal decomposition (4) which provides the norm. A second definition of this norm $\|f\|_*$ is given by the following obvious lemma:

Lemma 2 (*Lemma 2 of [36]*) Let \mathcal{BV} be defined as being the closure in BV of the Schwartz class. Then the Banach space G is the dual space \mathcal{BV}^* of \mathcal{BV} .

For Meyer, "the Banach space G will play a key role in modeling the structured component of an image". It will in fact measure how oscillatory the non BV component is. Here Meyer goes away from the initial ROF model, which was designed as a means to retrieve the noise free BV component of an image. The focus will be to model *texture* as the necessary complement of the BV component.

This leads Meyer to return to the idea of dethroning BV by bracketing it by more amenable spaces: "Finally we consider the already mentioned Besov space $E = \dot{B}_\infty^{-1, \infty}$. We then have $G \subset E$ and these embeddings are easily obtained if one observes that these [...] spaces G and E are the dual spaces of \mathcal{BV} , and $\dot{B}_1^{1,1}$. This latter sequence is decreasing and the dual one is increasing." [36,

¹In most of [36], the domain of the functions is \mathbb{R}^2 , as is standard in the wavelet community, and Meyer calls BV the space of L^2 ($L^{n/(n-1)}$ in higher dimension n) functions with finite total variation, rather than the usual definition which would require f to be in $L^1(\mathbb{R}^2)$. Similarly, in his notation, \mathcal{BV} differs from $W^{1,1}(\mathbb{R}^2)$ as functions in that space are merely L^2 .

p. 54] "We are happy to learn" Meyer says, "that BV^* is nearly characterized by size estimates on wavelet coefficients:"

Proposition 1 (*Proposition 3 of [36]*) *If one is using an orthonormal wavelet basis (Theorem 10 of [36] of $L^2(\mathbb{R}^2)$), then the wavelet coefficients $c_{j,k}$ of any f in BV^* are uniformly bounded. Conversely if $c_{j,k}$, $(j,k) \in \mathbb{Z} \times \mathbb{Z}^2$, are the wavelet coefficients of f and if the non-increasing rearrangement c_n^* of $|c_{j,k}|$ satisfy $\sum_{n \geq 1} c_n^*/n < \infty$, then $f \in BV^*$.*

At the end of the day, Meyer doesn't like cheating, and he will return to examine again his thorny contender, BV rather than replacing it by another space. In the next section we summarize his formalization of the solutions of the ROF problem. This formalization leads him to discover that the pseudo dual of BV comes up naturally as a new functional model for what he will define as "texture".

2.5 Analysis of the consistency of the ROF model

Meyer returns to his formulation of the ROF model (3) with some more precise notation.

Definition 3 (*Definition 7 of [36]*) *For a positive parameter λ and a given function $f(x)$ in $L^2(\mathbb{R}^2)$, the Osher-Rudin model selects the decomposition $f = u_\lambda + v_\lambda$ which is the solution of the following variational problem*

$$\inf\{J(u); \quad f = u + v\} \quad (13.1)$$

where

$$J(u) = \|u\|_{BV} + \lambda \|v\|_2^2 \quad (13.2)$$

This infimum which is denoted by $\omega_\lambda(f)$ is computed over all possible decompositions of f into a sum $f = u + v$ between a function u in $BV(\mathbb{R}^2)$ and a function v in $L^2(\mathbb{R}^2)$.

A first meaningful observation is that the norm of the "dual of BV " appears naturally when attempting to characterize the noise part of the ROF decomposition. Indeed,

Lemma 3 (*Lemma 3 of [36]*) *If $g \in L^2(\mathbb{R}^2)$, then*

$$\left| \int f(x)g(x) dx \right| \leq \|f\|_{BV} \|g\|_* \quad (14.3)$$

Meyer then investigates the mathematical properties of the optimal pair (u, v) in the Osher-Rudin model.

Theorem 2 (*Theorem 3 and Lemma 4 of [36]*) *Let f, u and v be three functions in $L^2(\mathbb{R}^2)$ and let $\|\cdot\|_*$ be defined by Definition 2.*

If $\|f\|_ > 1/(2\lambda)$, then the Osher-Rudin decomposition $f = u + v$ is characterized by the following two conditions*

$$\|v\|_* = \frac{1}{2\lambda} \quad \text{and} \quad \int u(x)v(x) dx = \frac{1}{2\lambda} \|u\|_{BV} \quad (14.6)$$

If $\|f\|_* \leq 1/(2\lambda)$, then $u = 0$, $v = f$.

"Roughly speaking" says Meyer, "this result says that an oscillating image $f(x)$ will be treated as a texture by the Osher-Rudin algorithm and that the correct norm for detecting textures is the G -norm".

Relapsing into his pet subject, Meyer writes that "Theorem 2 paves the road for Donoho's 'wavelet shrinkage'. This algorithm consists in (a) writing the wavelet series $\sum c_\lambda \psi_\lambda(x)$ of f , (b) putting to 0 all the wavelet coefficients $|c_\lambda|$ which do not exceed $1/(2\lambda)$, (c) retaining the other terms in the wavelet expansion of f but shrinking c_λ to $c_\lambda - \eta/(2\lambda)$ where $\eta = \pm$ is the sign of c_λ . Roughly speaking, this algorithm yields the u component in the Osher-Rudin model. Indeed Proposition 1 tells us that $\|v\|_*$ is almost the l^∞ norm of the wavelet coefficients of v and Theorem 2 says that $\|v\|_* = 1/(2\lambda)$."

But, finally leaving behind the "almosts" and the "roughly speakings", Meyer returns to the solution of the ROF model to observe that we always have $|\int g(x)h(x) dx| \leq \|g\|_{BV} \|h\|_*$ by Lemma 3 while (14.6) implies $\int u(x)v(x) dx = \|u\|_{BV} \|v\|_*$. This means that u is an element of BV where the linear form $g \mapsto \int g(x)v(x) dx$ attains its upper bound. This coupling between u and v leads to the notion of *extremal pair*.

Definition 4 (Definition 11 of [36]) *Let us assume that u and v are real valued functions in $L^2(\mathbb{R}^2)$ and $u \in BV$. We say that (u, v) is an extremal pair if*

$$\int u(x)v(x) dx = \|u\|_{BV} \|v\|_* \quad (14.7)$$

If u and v are two functions in $L^2(\mathbb{R}^2)$ and if (u, v) is an extremal pair, then the Osher-Rudin algorithm applied to $f = u+v$ with λ defined by $\|v\|_* = 1/(2\lambda)$ yields back this pair, as Theorem 2 shows.

Then Meyer proves a remarkable list of consistency and inconsistency results for the ROF model, given that this model should ideally give back the BV part of a function. For example

Lemma 4 (Lemma 8 of [36]) *If Ω is any smooth domain, then there exists an extremal pair (u, v) such that $u = \chi_\Omega$.*

Proposition 2 (Proposition 8 of [36]) *Let us apply the Osher-Rudin algorithm to a textured image modeled by $f_N(x) = m(Nx)f_0(x)$ where $f_0(x)$ is a simple geometrical form and $m(x)$ is a periodic pattern. Let us assume that N tends to infinity. Then, up to an error term whose L^2 -norm is $O(N^{-1/2})$, the Osher-Rudin algorithm yields the same geometrical component u_0 as if the texture were averaged (f_N is replaced by μf_0 where μ is the mean value of $m(x)$).*

These positive results contrast with the two negative statements proved next:

Lemma 5 (Lemma 9 of [36]) *The product $\omega\chi_\Omega$ between a smooth function $\omega \in C^\infty(\mathbb{R}^2)$ and the indicator function of a smooth domain Ω is not, in general, the u component of an extremal pair.*

Proposition 3 (Proposition 6 of [36]) *If Ω is the unit square, then the indicator function χ_Ω of Ω cannot be the u component arising in the Osher-Rudin algorithm.*

Last but not least, Meyer’s analysis leads him to return to the virtues of soft thresholding:

Lemma 6 (Lemma 10 of [36]) *If f belongs to $L^2(\mathbb{R}^2)$ and if (u, v) is the minimizer of the Osher-Rudin functional (13.1), then all wavelet coefficients of v are less than γ/λ where γ is an absolute constant. Conversely if a soft thresholding is applied to f with threshold C/λ , then the resulting function \hat{f} is a good substitute for being the optimal u . Indeed $f = \hat{f} + R$ where $\lambda\|R\|_2^2 \leq C\omega_\lambda(f)$. Here also C is some absolute constant.*

This “good substitute” is nevertheless discussable. If in the preceding lemma the constant C were equal to 1 and if $\omega_\lambda(f)$ were also bounding the BV norm of \hat{f} , then we would get an optimal pair and the bridge would be complete, but such a strong result is obviously not true. A conjecture was that \hat{f} obtained from f by soft thresholding belongs to BV . Yet, even this is unlikely to be true in general. It was and still is only proven for the hard thresholding of Haar wavelets coefficients, as mentioned earlier. Precisely, [18, Theorem 9.3] by Petrushev, Cohen, Xu, DeVore actually guarantees that the BV norm of \hat{f} is bounded after (hard) thresholding.

3 The legacy

3.1 The Cartoon + Texture model

In image analysis, the most influential part of Yves Meyer’s book, beyond the general idea of measuring oscillatory signals by appropriate (dual) norms, is a variant of (3). The new idea of Meyer is found in [36, eq. (14:38)] (see p. 72), where he proposes to replace (3) with a “new splitting algorithm [...] provided by a variational problems which reads

$$\tilde{\omega}_\lambda(f) = \inf \{J_0(u) = \|u\|_{BV} + \lambda\|v\|_* : f = u + v\}$$

where $\|\cdot\|_*$ now denotes the norm in the Besov space $\dot{B}_\infty^{-1,\infty}$.” What is very interesting, here, is the idea to directly look for a decomposition of the image into a regular/cartoon part and an oscillatory part, without relying on convex duality results such as Theorem 2.

While the actual use of the latter space would yield a solution u obtained by a complex algorithm involving wavelet shrinkage, and in particular not invariant by translation (see [5, Sec. 5] for a sub-optimal approach), the first authors who developed this idea for applications are L. Vese and S. Osher [43] who replaced (as suggested by Meyer), the $\dot{B}_\infty^{-1,\infty}$ norm with the so-called G -norm, leading

to the variational formulation

$$\inf_{\substack{(u,v) \in BV \times G, \\ f=u+v}} \left(\int |Du| + \lambda |v|_G \right). \quad (5)$$

This approach took advantage of Meyer’s next result, which he summarized in the saying, “oscillating patterns have small norms in G ”:

Lemma 7 (Lemma 14 of [36]) *Let $f_n, n \geq 1$ be a sequence of functions in $L^2(D)$ with the following three properties*

- (a) *there exists a compact set K such that the supports of $f_n, n \geq 1$ are contained in K*
- (b) *there exists an exponent $q > 2$ and a constant C such that $\|f_n\|_q \leq C$*
- (c) *the sequence f_n tends to 0 in the distributional sense.*

Then $\|f_n\|_G$ tends to 0 as n tends to infinity.

In the Vese and Osher [43] work, the G -norm was approximated by dual norms of $W^{1,p}$ spaces with various choices of p (ideally, one would let $p \rightarrow 1$), and a third intermediate L^2 penalization was introduced for computational purposes. More modern techniques would simply rely on standard Douglas-Rachford [35] or primal-dual splitting [16], for instance, the results in Fig. 2 have been obtained by such approaches which are relatively efficient. In another paper with A. Solé [38], the same authors developed an alternate model to extract textures, based on the squared H^{-1} seminorm, which is computationally easier to implement. Further numerical experiments based on this model were also presented in [44].

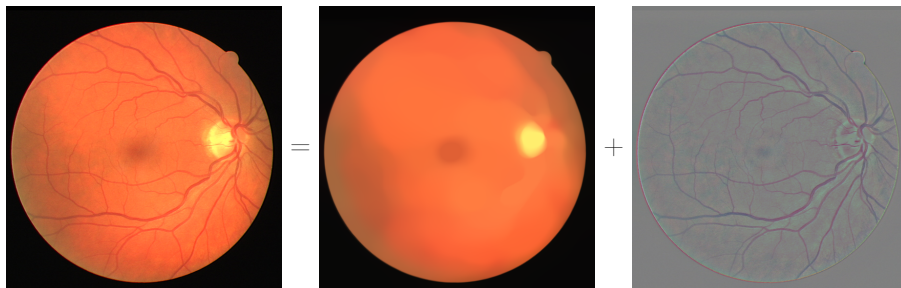


Figure 2: The left image is decomposed into a BV (middle) and “ G ” (right) components. The decomposition extracts perfectly, in this example, the pattern from the background.

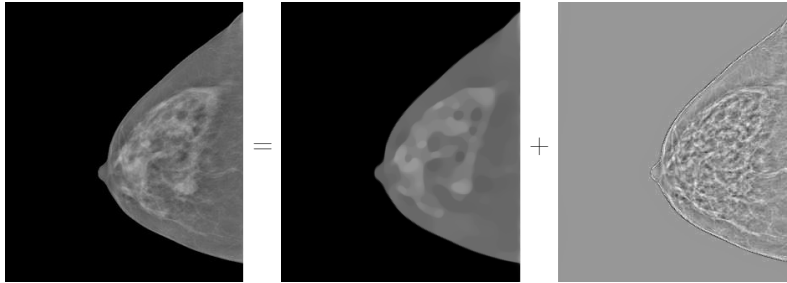


Figure 3: Another example: extraction of oscillating patterns in a mammography.

This general approach and its interpretation by Osher, Solé and Vese inspired many authors in the subsequent years. In [4], the authors proposed a variant of the problem studied in [43, 44], based on a decomposition problem of the form

$$\min_{u,v} \left\{ \int |Du| + \frac{1}{2\lambda} \|f - u - v\|_{L^2}^2 : |v|_G \leq \mu \right\}$$

replacing the penalization in the G -norm with a constraint. The main reason to do so was that it becomes very easy to develop an alternating minimization algorithm with respect to u and v to tackle this problem: each minimization amounts to solving one problem of the form (3), as is obvious with respect to the u variable and follows from easy convex duality arguments for the v variable. The paper [2] developed further on this idea. In [5] was presented a systematic study of some dual Sobolev norms for measuring oscillatory components. Also, the authors proposed a 3-components decomposition approach based on the G -seminorm and a dual Besov norm for scale separation, all models coming with algorithms and numerical experiments. Interestingly, the latter model was then successfully exploited in [20] to denoise ancient documents and separate the text and texture from the noise.

Further ahead, Theorem 2 found another application, as it was understood that by varying the parameter λ , one could easily define a nonlinear “scale-space” analysis extracting oscillatory components at various scales (see Fig. 4 for an example). This idea of a “spectral” Total Variation analysis was proposed in [28, 11, 41]. Thanks to this approach, one can modify in an image only the details at a particular scale without affecting the other components, see Fig. 5 for an example. This idea has had a lot of influence recently in the imaging community and is developed in for instance, for various applications, in [8, 3, 32].

3.2 The $TV+L^1$ model

The solution of (5) requires relatively costly numerical computations, and in the early 2000s it was not well understood how it should be tackled. A solution

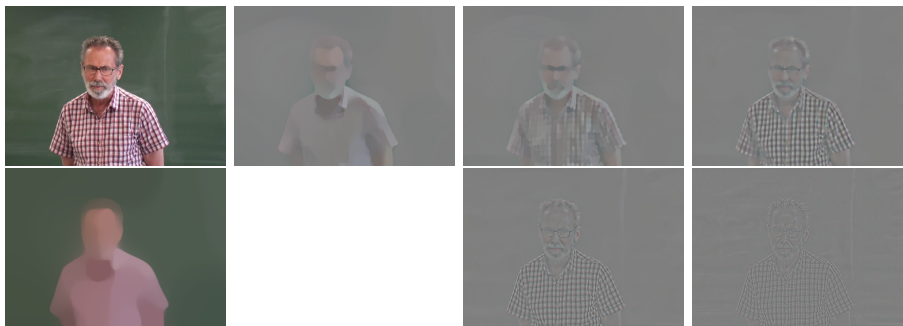


Figure 4: Nonlinear $TV - G$ decomposition: top left, the original image, bottom left, the lower frequency component. The other pictures are the G components at different scales of increasing frequencies.

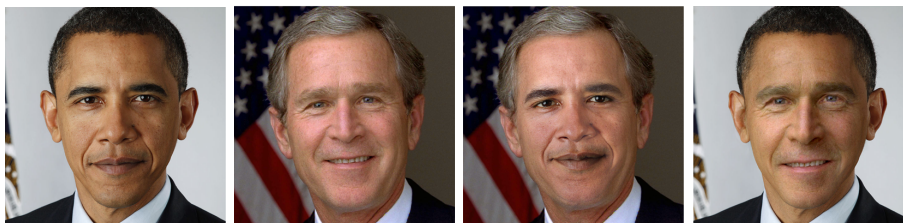


Figure 5: Swap of two presidential “high frequency” components. (*Images courtesy of M. Benning, see [8] for further examples.*)

to this dilemma between mathematical consistency and an efficient numerical implementation stems from the works of Mila Nikolova. Just one year after Meyer, Nikolova introduced a discrete version of what would be later called the “ $TV - L^1$ ” model [37, 17]. Jean-François Aujol analysed the continuous $TV - L^1$ model and proved formal properties that implied a correct cartoon + texture separation, like for the $TV - G$ model [6]. The model reads

$$\inf_{(u,v) \in BV \times L^1 | f=u+v} \left(\int |Du| + \lambda|v| \right), \quad (6)$$

where the only difference with the ROF $TV - L^2$ model is the replacement of the exponent 2 by the exponent 1.

This minimization process was not obvious either at that time. Fortunately, the progress in nonsmooth optimization (as well as the dissemination of old powerful ideas, such as operator splitting, proximal methods [19, 39] or Augmented Lagrangian type algorithms [9, 30]), led to a few powerful and simple methods to solve efficiently large classes of singular convex optimization problems such as the $TV - L^2$ and $TV - L^1$ models, see [7, 16]. Graph-cuts can

also be useful [21, 29]. Specific techniques were also introduced for the $TV - L^1$ model, based on its “morphological” character, see [27].

4 Impact of decompositions model

We have an ideal observatory to realize the impact on society of the cartoon+texture model. While it is generally difficult to demonstrate such an impact, this has become quite simple in image processing thanks to the online publication of algorithms. In 2014 Vincent Le Guen published his implementation based on [16] of the $TV - L^1$ model [34], in the Image Processing on Line (IPOL) journal www.ipol.im. This online publication has the main interest that users from all sectors of science and society can upload their own images to the executable paper’s demo website and get an immediate numerical simulation of the result. Furthermore, each IPOL paper has its archive where all online experiments are recorded if the users allow it. This enables us to illustrate four real applications of the $TV - L^1$ model that are simply selected from the article’s online archive where they can still be found.

Our brain’s ability to separate a drawing from its underlying filigrane is illustrated in the next figure. This application is crucial for the analysis and control of fiducial images such as banknotes and identity documents. The ability of this model is well illustrated in Fig. 6. The second application, for which

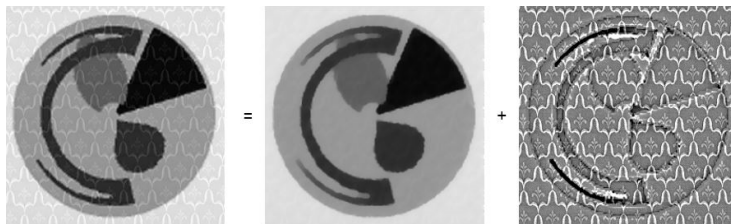


Figure 6: A basic (but far from simple) cartoon + texture separation

many instances are present in the archive, is medical image detail enhancement. Here the user focuses on the textural part of the image, not its background. Indeed, in many image analysis we are in a situation where the cartoon is a mere background to be removed. The next experiment found in the IPOL archive shows a decomposition of a mammography, where the enhancement of the interesting structure is obvious in the textural part. See Fig. 7. Also the image in Figure 2, left, was taken from this archive and can be successfully decomposed with this model, even if on such examples, the results of (5) seem slightly superior.

Perhaps the most recurrent application of the cartoon+texture model is its systematic use in police inquests to recognize challenging fingerprint samples. Here, the separation of the texture from the background delivers a much clearer

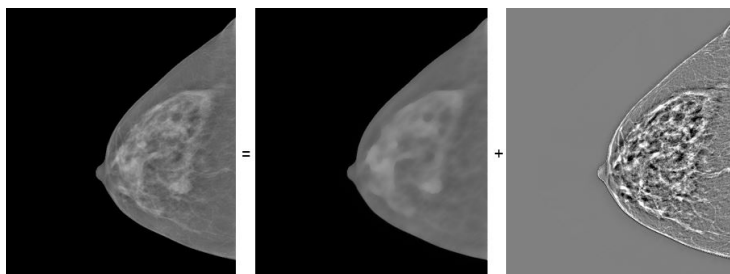


Figure 7: Extraction of the meaningful patterns in a mammography. To be compared with Fig. 3.

picture of the fingerprint's minutiae. It increases the chances that the police's automatic fingerprint identification system (AFIS) succeeds. The fact that so many such fingerprint experiments are being made on line by important police departments indicates that this invention has not yet be inserted in the most current AFISs. The first experiment below illustrates a simple case study of extraction of a fingerprint on a textured wooden background (which goes into the cartoon part), see Fig. 8, top. The second experiment (Fig. 8, bottom)

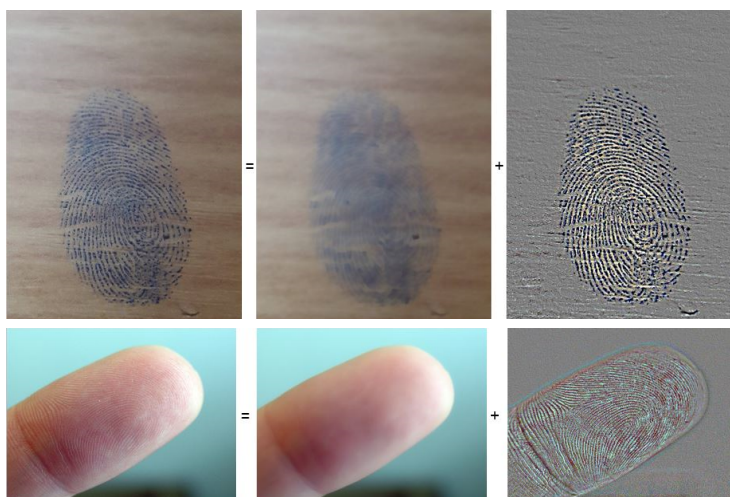


Figure 8: Analysis of fingerprints with the $TV+L^1$ model

illustrates the use of the decomposition in new fingerprint verification sytems, that directly take a picture of the finger. Here again the separation is amazing. The next experiment, Fig. 9 shows a spectacular extraction. It seems that the police officers did not care too much about leaving a trace of their activity in an image processing journal; many FBI experiments are also present in the archive.



Figure 9: Extraction of a fingerprint from a document with the $TV+L^1$ model

But the usage of the cartoon+texture model by police is not limited to fingerprint extraction, and our last experiment extracted from the IPOL archive shows, in Fig. 10 how the very same algorithm helps reading licence plates in night images.

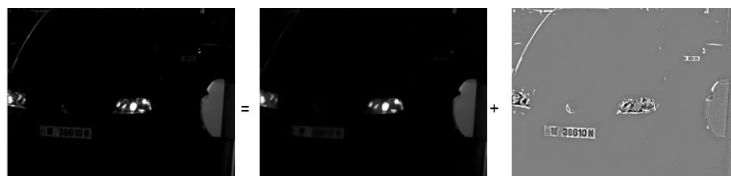


Figure 10: Extraction of a licence plate

5 Conclusion

The discussion on the right function space to model natural images and to perform denoising by wavelet shrinkage may now seem futile. Indeed, technology has led to learn implicitly and intensively fine statistics of natural images by endowing it in the weights of large convolutional networks [46, 45, 42]. These denoising algorithms perform beautifully and beat by several decibels the best handcrafted algorithms. Yet, as we have seen, the cartoon+texture model proposed by Meyer has become a classic in image processing. It is extremely practical, as it separates essential shapes in an image from the accidents of their textures. In 1954 the visionary psychologist Attneave [1] sketched the program of what would decades later become computer vision. He noticed that our visual perception of images should rely on the extraction of piecewise smooth silhouettes (that would be encoded by their extrema of curvature) and by their “texture”. Texture was defined by Attneave as an additional characteristic of shapes, like the color. So our visual perception amounted, according to Attneave, to perform an $f = u + v$ decomposition, thus separating two immiscible

phases. In some sense Meyer’s vision and work, first directed at making compatible two functional models for image denoising, ended up in a realization of Attneave’s program.

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