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Duality of the existing geometric variable strain models for the dynamic modelling of continuum robots

A. Ouyoucef¹, Q. Peyron¹, V. Lebastard², F. Renda³, G. Zheng¹ and F. Boyer²

Abstract—The Cosserat rod theory has become a gold standard for modeling the statics and dynamics of serial and parallel continuum robots. Recently, a weak form of these Cosserat rod models called the geometric variable strain model has been derived where the robot deformations are projected on finite-dimensional basis functions. This model has very interesting features for continuum robotics, such as a Lagrangian form close to classical rigid robots and the ability to tune its performances in terms of computation time and accuracy. Two approaches have been proposed to obtain and compute it. The first is based on the Newton-Euler recursive algorithm and the second, on the projection of the strong form equations using Jacobian matrices. Although these approaches yield identical model forms, their disparate implementations and numerical schemes render each uniquely suited to specific applications. Notably, underlying these disparities lies a profound duality between these models, prompting our quest for a comprehensive overview of this duality along with an analysis of their algorithmic differences. Finally, we discuss perspectives for these two approaches, in particular their hybridization, based on the current knowledge of rigid robotics.

Index Terms—Continuum robot, Cosserat rod theory, dynamics, geometric variable strain model, duality.

I. INTRODUCTION

CONTINUUM robots (CRs) have gained increasing popularity in recent years, finding diverse applications in many fields such as healthcare [1]. In this context, several modeling techniques exist and have already been the subject of reviews such as [2]. To perform faster movements, the robot must be designed, analyzed, and controlled taking into account its dynamic behavior [3]. In this context, it is desirable to have accurate dynamic models to compute the deformed shape of the robot under specific actuation inputs and external forces. The finite element method (FEM) is one approach to model such slender soft robots [4]. In this model, the robot is divided into several sub-domains or "elements", and its deformations are described by the displacement of the elements' nodes. Another approach is to use beam/rod theories. Linear theories assume geometric linearity and account for small rotations, such as the Euler–Bernoulli (E-B) beam theory

which has been a cornerstone for modeling CR systems over the years [5]. It characterizes a beam's behavior under axial forces and bending. Incorporating shear effects and rotation, the Timoshenko beam model improves upon the E-B beam model to model CRs [6].

For applications involving large deflections, rod theories capable of handling geometric non-linearities, such as Kirchhoff rod theory, come into play. The Kirchhoff model extends E-B theory and can be used to describe bending and torsion deformations of general designs of CRs [7]. The Cosserat rod theory extends this model further by encompassing bending, torsion, shear, and extension [8]. As a consequence, it has become a gold standard to accurately model serial [2] and parallel [9] CRs, in quasi-static and dynamic regimes. There are two main classes of implementations for the Cosserat rod theory, as detailed in [10]: the Newtonian and the Lagrangian approaches. The Newtonian approach uses only Newton's three laws, including the action-reaction principle, to establish local relationships between robot deformation and actuation along the rod. The resulting model can then be put into the form of a boundary value problem (BVP), which is usually solved using shooting methods [9]. On the other hand, the Lagrangian approach derives from the virtual work (VW) principle (VWP) and reduces the strong form of the partial differential equations (PDEs) of the Cosserat model to a set of ordinary differential equations (ODEs). To do this, the strain field, initially defined in an infinite dimensional space, is projected onto a finite-dimensional functional basis using the Ritz method. Several function bases can be used and applied either between pre-defined nodes [11, 12] as in the FEM, or along the whole robots backbone, leading to a model referred to as the Geometric Variable Strain (GVS) model.

The GVS model proved valuable in the general context of CRs featuring an assembly of multiple rods, rigid bodies, and joints as they enable the direct derivation of a concise set of equations [10]. However, the relative nature of the parameterization of the GVS model, considerably, increases the non-linearities in the inertia and external force matrices. These matrices take the form of multiple space integrals that have to be computed, at each time step. Computing these matrices with a low computation time while ensuring high accuracy, represents a real challenge. To date, two approaches have been used to solve this problem. The first approach is based on the projection of the Cosserat PDEs with kinematic Jacobian matrices [13], in accordance with the general Kane method. The second approach [14], is based on Newton-Euler (NE) recursive algorithms, as developed in [15] for rigid multi-body systems. Both approaches are based on the same modeling assumptions and produce for a same state, the same

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mapping between motion accelerations and actuation forces. However, the implementation of this mapping is radically different, and the relationships between these two realizations of the inverse dynamics have never been studied before, nor has their equivalence been mathematically proven. What's more, these GVS models have mutual advantages and disadvantages that can be exploited and combined as required.

In the continuity of our previous work [10] where we explained didactically and compared the Newtonian and Lagrangian static Cosserat rod model for CRs, we propose in this paper to focus on the two approaches leading to the GVS dynamic model. The contributions of this paper are three-fold:

- A didactical elaboration of the two existing GVS approaches with the same notations and in the framework of the VWP.
- A proof of their mathematical duality
- A comparative overview of the most recent implementation of the two approaches, and new perspectives.

II. BASIC CONCEPTS

Here we define the spaces of a CR, the duality that structures the models, and the VWP as a machine for changing spaces.

A. Kinematic spaces: from strains to poses

The concept of kinematic space is one of the most important ones of robotics as evidenced by the emblematic joint space and workspace of rigid manipulators. In the case of CR, more than two spaces are needed to describe these systems whatever the considered actuation strategy [1]. In Fig. 1, we have represented an extended version of the different spaces subtended by a classical tendon actuated continuum robot (TACR), for a better and easy understanding, as it is a well known design in the community. From left to right, we find

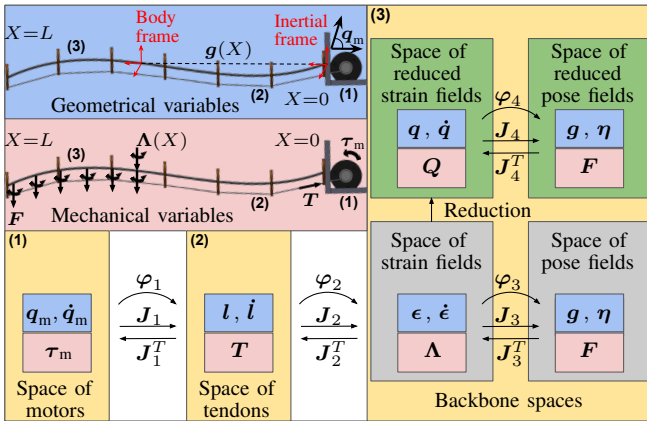


Fig. 1. Flow charts of kinematic spaces for a TACR. Geometric transformations and their tangents are defined symbolically by applications φ , and their Jacobian matrices J . In yellow, we have the location of the robot's spaces. The red and the blue indicate the dual variables. The grey and the green represent the backbone spaces before and after reduction, respectively.

the space of motor joints q_m , the space of tendon lengths l , the space of rod strains ϵ , with a possible reduction of it (q), as in the case of the GVS model for example, and the space of pose g defined by the Lie group $SE(3)$. Once these spaces are defined, conventional robotics models can be used to shift

from one to another according to the user's need. For instance, $g = \varphi(q_m)$ defines the forward model that maps the motor joint space onto the workspace. i.e. a nonlinear map between two manifolds. More generally, let be q^* and q the robot coordinates in two different spaces, the map between them and its tangent are generally noted φ , so that $q^* = \varphi(q)$ and $J = (\partial\varphi/\partial q)$ respectively. This latter matrix is a Jacobian operator that transforms velocities between the two spaces.

In this work, we exploit mainly the mapping relating the poses and their associated speed and accelerations $(g, \eta, \dot{\eta})$, to the strain field and its time derivatives $(\epsilon, \dot{\epsilon}, \ddot{\epsilon})$. This mapping can be deduced from the continuous geometric model along the robot:

$$g' = g\xi^\wedge, \quad g(0) = g_0, \quad (1)$$

where $\xi = (g^{-1}g')^\vee$ is the X -rate twist of cross-sections along the robot, related to the strain field ϵ by $\epsilon = \xi - \xi_0$, with ξ_0 the value of ξ in a reference (stress-less) configuration of the rod. Time-differentiating twice (1), provides the continuous model of velocities $\eta = (g^{-1}\dot{g})^\vee$ and accelerations $\dot{\eta}$:

$$\eta' = \dot{\xi} - \text{ad}_\xi \eta, \quad \eta(0) = \mathbf{0}, \quad (2)$$

$$\dot{\eta}' = \ddot{\xi} - \text{ad}_\xi \eta - \text{ad}_\xi \dot{\eta}, \quad \dot{\eta}(0) = \mathbf{0}, \quad (3)$$

where ad is the adjoint operator of an element of $se(3)$. Replacing $\partial./\partial t$ by variation δ . in (2) defines compatible displacement fields $\delta\zeta = (g^{-1}\delta g)^\vee$, as:

$$(\delta\zeta)' = \delta\xi - \text{ad}_\xi \delta\zeta, \quad \delta\zeta(0) = \mathbf{0}. \quad (4)$$

Equations (1-4) define the differential kinematics of the CR. Alternatively, one can integrate them with respect to the X -variable, to obtain equivalent kinematics in integral form. The X -ODE (1) can be integrated in the following form which preserves the group structure of $SE(3)$:

$$g = g_0 \exp(\Omega(X)), \quad (5)$$

where $\exp(\cdot)$ denotes the exponential map, and $\Omega(X)$ is an infinite series of multiple integrals of nested Lie Brackets of ξ , named Magnus expansion [16]. For the remaining X -ODEs, we use the relationships between ad and Ad , where Ad is the adjoint operator of an element of $SE(3)$ [17]:

$$\text{ad}_\xi = -\text{Ad}_g^{-1} \text{Ad}_g, \quad (6)$$

$(\xi, \partial./\partial X)$ can be indifferently replaced by $(\eta, \partial./\partial t)$, and $(\delta\zeta, \delta)$. Using these relations in (2)-(3) and (4) gives:

$$\eta = \text{Ad}_g^{-1} \int_0^X \text{Ad}_g \dot{\epsilon} dY, \quad \delta\zeta = \text{Ad}_g^{-1} \int_0^X \text{Ad}_g \delta\epsilon dY, \quad (7)$$

$$\dot{\eta} = \text{Ad}_g^{-1} \int_0^X \text{Ad}_g \ddot{\epsilon} dY + \text{Ad}_g^{-1} \int_0^X \text{Ad}_g \text{ad}_\eta \dot{\epsilon} dY. \quad (8)$$

Now that all the kinematic fields are parameterized by strains, and because these latter are in a vector space, one can approximate them by a truncated Ritz decomposition in:

$$\epsilon(X, t) = \Phi(X)q(t), \quad (9)$$

where Φ is a matrix of "strain modes" [13, 14], while q defines a finite (discrete) vector of generalized (strain) coordinates. Using this strain parametrization in (7) and (8), one can shift

δq , \dot{q} and \ddot{q} out of the integrals and obtain the change of space relations in usual finite dimensional form:

$$\eta = J\dot{q}, \quad \delta\zeta = J\delta q, \quad \dot{\eta} = \dot{J}\dot{q} + J\ddot{q} \quad (10)$$

where we introduced the Jacobian matrix and its time derivative:

$$J = \text{Ad}_g^{-1} \int_0^X \text{Ad}_g \Phi dY \quad (11)$$

$$\dot{J} = \text{Ad}_g^{-1} \int_0^X \text{Ad}_g \text{ad}_{J\dot{q}} \Phi dY. \quad (12)$$

B. Force-displacement duality and VW

In Lagrange mechanics, a (generalised) force Q is what acts along a small (generalised) displacement dq (or velocity \dot{q}), to produce a work $dW = dq^T Q$ (a power $P = \dot{q}^T Q$). Since works are scalars, and displacements (or velocities) are vectors, thus forces are vectors in the dual space of displacements (velocities), i.e. co-vectors. Referring to Fig. 1, in the case of a TACR, the dual of motor joint velocities, is the vector of motor torques τ_m , those of tendon length velocities are the vectors of the tensions T . The dual fields to the strain velocity field $\dot{\epsilon}$ are the stress fields Λ . Those of inertial twists η are the fields of force-wrenches F , and those of generalized strain velocities \dot{q} are generalized stress forces Q .

This duality strongly structures Lagrangian mechanics and becomes apparent when we change forces from one space to another. To apply such a change of space, we use the fundamental property of VWs of being invariant by any change of coordinates or kinematic space. For instance, if $q^* = \varphi(q)$ is invertible, we have for any variation of q and q^* , noted δq and δq^* , and such that $\delta q^* = J\delta q$:

$$\delta W = \delta q^T Q = \delta q^{*T} Q^* = (J\delta q)^T Q^* = \delta q^T J^T Q^*, \quad (13)$$

which holding for any δq , provides the change rule for generalized forces:

$$Q = J^T Q^*, \quad Q^* = J^{-T} Q. \quad (14)$$

Note that, while velocities transform forwards with Jacobian matrices, forces transform backwards with transpositions (or duals) of the Jacobians, as illustrated in Fig. 1.

C. Changing the space of the dynamics equations

Once the kinematics so defined, we can move on to the dynamics. The equations of motion of a CR can be derived by applying one of the principles of dynamics (Newton, D'Alembert, Hamilton). In this article, we will use only D'Alembert's VWP, the statement of which is reminded below. At any moment, and for any variation or "virtual displacements" of its configuration, parametrized by a set of generalized coordinates q , we have the VW balance:

$$\forall \delta q, \quad 0 = \delta W_{\text{acc}} - \delta W_{\text{int}} - \delta W_{\text{ext}}, \quad (15)$$

where from left to right we find the VWs of accelerations quantities, that of internal forces and that of external forces respectively. In practice, the dynamics in the configuration space of qs , is obtained by expressing all the VWs of (15) as $\delta W = \delta q^T Q$. Then, using the fact that (15) is true for any

δq , one can remove δq in (15), to get $Q_{\text{acc}} = Q_{\text{int}} + Q_{\text{ext}}$, which defines the dynamic balance of the system in the space of qs .

Throughout the article, the VWP is used as a machine for changing the dynamic model of CRs from one kinematic space to another. To illustrate such a change, consider a rigid manipulator whose dynamics in a generalized coordinate system q , is:

$$\tau = M(q)\ddot{q} + Q(q, \dot{q}), \quad (16)$$

with τ and Q the vectors of control and Coriolis and centrifugal torques, and M the mass matrix. We want to reexpress (16) in another kinematic space defined by q^* . To this end, we have thanks to the kinematic invariance of works:

$$\begin{aligned} \forall \delta q: \quad 0 &= \delta W_{\text{acc}} - \delta W_{\text{int}} - \delta W_{\text{ext}} \\ &= \delta q^T (\tau - M\ddot{q} - Q) \\ &= \delta q^{*T} (\tau^* - M^*\ddot{q}^* - Q^*). \end{aligned} \quad (17)$$

Then, introducing the differential consequences of the kinematic space change:

$$\delta q^* = J\delta q, \quad \dot{q}^* = J\dot{q}, \quad \ddot{q}^* = J\ddot{q} + \dot{J}\dot{q} \quad (18)$$

into (17), we obtain by simple identifications:

$$\tau^* = J^{-T} \tau, \quad M^* = J^{-T} M J^{-1}, \quad Q^* = J^{-T} Q, \quad (19)$$

where J, Q, M in (19) depend (q, \dot{q}) that need to be calculated from (q^*, \dot{q}^*) with $q = \varphi^{-1}(q^*)$ and $\dot{q} = J^{-1}\dot{q}^*$.

In the case of finite dimensional systems, q is an ordinary vector of discrete generalized coordinates, as are the joint coordinates of a rigid manipulator. However, the VWP can be applied to the infinite dimensional configuration space of a deformable body \mathcal{B} , where the discrete sums of $\delta q^T Q$ are replaced by integrals $\int_{\mathcal{B}} \delta q^T Q d\mathcal{B}$ and $q, \delta q$ and Q are now vector fields¹ over the continuous material domain of \mathcal{B} . In this later case, the dynamics turns to be a set of partial differential equations, or "strong form", that governs the field q , while the balance of VWs defines the "weak form" of the same dynamics.

By way of illustration, for a TACR, the VWP can be applied in any kinematic space in the flowchart in Fig. 1. In the remainder of the article, we will do so in three of these spaces. Thus, the model of a CR will first be derived in the space of poses g , then moved to that of strains ϵ and finally to that of modal coordinates q of (9). Note that while the g coordinates are absolute, i.e. they refer to the inertial frame, the ϵ and q coordinates, are relative, i.e. they refer to the cross-sectional frames. As we move forward, we will look at the two main approaches that can be used to achieve these changes of space.

III. STRONG FORM OF CR DYNAMICS

Serial and parallel continuum manipulators are typically modelled by considering one or several Cosserat rods cantilevered at one end and free, or assembled at a platform,

¹In the remainder of the article, the notations q and Q will be reserved for finite vectors, while others will be introduced for such fields.

at the other. As a sub-theory of continuous media mechanics, the dynamic model of such rods is naturally defined in a space of absolute coordinates, here realized by the field of inertial poses \mathbf{g} . It consists of the following ingredients:

- A definition of (small) strain measurements:

$$\epsilon = \xi - \xi_0, \quad (20)$$

- The dynamic balance in the medium:

$$\Lambda' = \mathcal{M}\dot{\eta} - \text{ad}_{\eta}^T \mathcal{M}\eta + \text{ad}_{\xi}^T \Lambda - \bar{\mathbf{F}}_{\text{ext}}, \quad (21)$$

- Boundary conditions (BCs):

$$\mathbf{g}(0) = \mathbf{g}_0, \quad \Lambda(l) = \mathbf{F}_+ \quad (22)$$

- A constitutive law, e.g. for an elastic material:

$$\Lambda = \mathcal{H}\epsilon. \quad (23)$$

where Λ is the field of stress wrenches, $\bar{\mathbf{F}}_{\text{ext}}$ and \mathbf{F}_+ are the density of external wrench per unit of X , and the concentrated external wrench applied at the tip $X = l$ of the rod, respectively, while \mathcal{H} and \mathcal{M} are its cross-sectional Hooke tensor and its density of inertia tensors respectively, all expressed in the cross-section frames along the rod.

The model of a continuum manipulator needs also to incorporate a model of actuation, that can be entered through:

- The model of the stresses, which can be the sum of the internal stresses given by the constitutive law and distributed internal actuation wrenches expressed in the cross-sectional frame, like in the case of TACR [13, 14]:

$$\Lambda = \Lambda_{\text{act}} + \mathcal{H}\epsilon. \quad (24)$$

with Λ_{act} the field of stress wrench produced by the actuation.

- The model of distributed external wrenches, if we have an external actuation possibly depending on the robot pose, such as magnetic CRs with ferromagnetic elastic bodies [18]:

$$\bar{\mathbf{F}}_{\text{ext}} = \bar{\mathbf{F}}_{\text{act}} + \bar{\mathbf{F}}_{\text{env}} \quad (25)$$

with $\bar{\mathbf{F}}_{\text{act}}$ and $\bar{\mathbf{F}}_{\text{env}}$, the (\mathbf{g}, η) -dependent models of the density field of force wrench produced by the actuation and the external environment respectively.

- The model of BCs, if actuation is applied through boundaries. For instance, if the manipulator is fully actuated at its basis by an external wrench $\mathbf{F}_{\text{act},-}$, which is the case for concentric tube robots (CTRs) [19] and most designs of continuum parallel robots (CPRs) [20], one can replace the geometric boundary conditions at $\mathbf{g}(0) = \mathbf{g}_0$, by the condition on wrenches:

$$\Lambda(0) = -\mathbf{F}_{\text{act},-} \quad (26)$$

As a note, $\mathbf{F}_{\text{act},-}$ can be either directly imposed in case of torque control of the robot actuators or induced by geometrical constraints through Lagrange multipliers in case of displacement control.

The dynamic balance (21) can be deduced by applying the

VWP in the space of cross-sectional poses \mathbf{g} , i.e. by imposing that the balance (15) holds for any field of virtual displacements $\delta\zeta = (\mathbf{g}^{-1}\delta\mathbf{g})^\vee$, compatible with the geometric BC of (22), i.e. s.t. $\delta\zeta(0) = \mathbf{0}$. To see how this can be achieved, let us consider the case of a CR internally actuated with a controlled stress-field Λ_{act} , and decompose the VW of external forces in (15), as:

$$\delta W_{\text{ext}} = \delta W_{\text{env}} + \delta W_{\text{act}} \quad (27)$$

where the VW of environmental forces is:

$$\delta W_{\text{env}} = \int_0^l \delta\zeta^T \bar{\mathbf{F}}_{\text{env}} dX + \delta\zeta(l)^T \mathbf{F}_+ \quad (28)$$

while, for the sake of genericity, the actuation forces are considered a part of the external forces for all actuation strategies:

$$\delta W_{\text{act}} = - \int_0^l \delta\epsilon^T \Lambda_{\text{act}} dX. \quad (29)$$

As regards the two other contributions δW_{int} and δW_{acc} , we have:

$$\delta W_{\text{int}} = - \int_0^l \delta\epsilon^T (\mathcal{H}\epsilon) dX, \quad (30)$$

$$\delta W_{\text{acc}} = \int_0^l \delta\zeta^T \bar{\mathbf{F}}_{\text{in}} dX \quad (31)$$

with $\bar{\mathbf{F}}_{\text{in}}$ is the density of inertial forces :

$$\bar{\mathbf{F}}_{\text{in}} = \mathcal{M}\dot{\eta} - \text{ad}_{\eta}^T \mathcal{M}\eta \quad (32)$$

Now let us remark that (28) and (31), being naturally expressed in the space of poses, they need no further algebraic manipulation. However, this is not the case of the two others (29) and (30), which being naturally defined in the space of strains, need to be transferred to the space of poses, to obtain all the works of (15) in the same space. To achieve this transfer, we use (4) to express $\delta\epsilon = \delta\xi$ in terms of $\delta\zeta$ in (30) and (29), that can then be by-part integrated with respect to X . Once gathered with other contributions coming from (28) and (31), the integral terms of these by-part integrations directly provide² the PDEs (21), while the residual boundary terms of the by-part integrations can be gathered with the VW of \mathbf{F}_+ , to form the second BC of (22).

IV. REDUCED MODEL IN STRAIN PARAMETRIZATION

A. From the strong form to the reduced strain-based model

In this section, we detail how the reduced dynamic model of CRs in the space of strain coordinates \mathbf{q} , can be obtained from the above strong form in the space of \mathbf{g} . In this finite dimensional discrete space, this model takes the usual form of a set of ODEs in Lagrangian form:

$$\mathbf{Q}_{\text{act}}(\mathbf{q}) = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}\mathbf{q} \quad (33)$$

where \mathbf{M} is the matrix of generalized inertia, \mathbf{Q}_{act} is the vector of generalized actuation forces, \mathbf{Q} is the vector of external (e.g. gravity, contacts...), Coriolis and centrifugal forces, $\mathbf{K}\mathbf{q}$ is the vector of generalized restoring forces, with

²Thanks to the fact that (15) holds for any $\delta\zeta$ compatible with the geometric BC.

\mathbf{K} the generalized stiffness matrix.

As presented in Section II.C, the reduced model (33) can be deduced from the above strong form by using the VWP (15), to move from the space of (absolute) poses \mathbf{g} to that of the (relative) strain coordinates \mathbf{q} . Technically, there are two ways to achieve this change of space:

- By converting absolute virtual twists $\delta\zeta$ into relative ones $\delta\epsilon$, using kinematic Jacobian matrices, according to the so-called Kane's method, here referred to as the Jacobian based approach [11].
- By converting the force wrenches \mathbf{F} into stress wrenches $\mathbf{\Lambda}$ through a generalization of the NE inverse dynamics algorithm from rigid to continuous robots [14].

In this section, we provide the computational principle of each of these two ways.

B. Jacobian based approach

The Jacobian-based approach is the reciprocal of the computational process evoked at the end of section III, i.e. it aims to move (28) and (31) from the space of poses to the space of strains. Practically, this is achieved by introducing (10) in the weak form (15), and thanks to the kinematic invariance of VWs, to identify the result with the expected expression of the weak form in the space of \mathbf{q} s:

- $\forall \delta\mathbf{q} \in \mathbb{R}^n$, we have:

$$\begin{aligned} 0 &= \delta W_{\text{int}} + \delta W_{\text{ext}} - \delta W_{\text{acc}} & (34) \\ &= \delta\zeta(l)^T \mathbf{F}_+ + \int_0^l \delta\zeta^T (\bar{\mathbf{F}}_{\text{env}} - \bar{\mathbf{F}}_{\text{in}}) dX \\ &\quad - \int_0^l \delta\epsilon^T (\mathbf{\Lambda}_{\text{act}} + \mathcal{H}\epsilon) dX \\ &= \delta\mathbf{q}^T (\mathbf{Q}_{\text{act}}(\mathbf{q}) - \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{K}\mathbf{q}). \end{aligned}$$

To apply the above identity, we need to consider separately the stress and force wrenches as distinguished by the notations $\mathbf{\Lambda}$ and \mathbf{F} in all the above, and defined first in Section II.B.

1) Generalized forces generated by stress-wrenches $\mathbf{\Lambda}$:

The VW developed by restoring and actuation stresses being directly defined in the space of strains, we simply have with $\epsilon = \Phi\mathbf{q}$ and $\delta\epsilon = \Phi\delta\mathbf{q}$:

$$\begin{aligned} \delta W_{\text{int}} + \delta W_{\text{act}} &= - \int_0^l \delta\epsilon^T (\mathbf{\Lambda}_{\text{act}} + \mathcal{H}\epsilon) dX \\ &= -\delta\mathbf{q}^T \int_0^l \Phi^T (\mathbf{\Lambda}_{\text{act}} + \mathcal{H}\Phi\mathbf{q}) dX \\ &= \delta\mathbf{q}^T (\mathbf{Q}_{\text{act}} - \mathbf{K}\mathbf{q}). \end{aligned} \quad (35)$$

which gives by identification:

$$\mathbf{K} = \int_0^l \Phi^T \mathcal{H} \Phi dX, \quad \mathbf{Q}_{\text{act}} = - \int_0^l \Phi^T \mathbf{\Lambda}_{\text{act}} dX. \quad (36)$$

2) Generalized forces generated by force-wrenches \mathbf{F} :

The VW developed by environmental and inertial forces needs to be shifted from the space of poses to that of strains. Using the integrated kinematics of (10), we find:

$$\begin{aligned} \delta W_{\text{acc}} - \delta W_{\text{env}} &= -\delta\mathbf{q}^T \mathbf{J}^T(l) \mathbf{F}_+ - \\ \delta\mathbf{q}^T \int_0^l \mathbf{J}^T (\bar{\mathbf{F}}_{\text{env}} - \bar{\mathbf{F}}_{\text{in}}) dX &= \delta\mathbf{q}^T (\mathbf{M}\ddot{\mathbf{q}} + \mathbf{Q}). \end{aligned} \quad (37)$$

Now replacing $\bar{\mathbf{F}}_{\text{in}}$ by its detailed expression (32) and using (10), (37) gives by identification:

$$\begin{aligned} \mathbf{M}(\mathbf{q}) &= \int_0^l \mathbf{J}^T \mathcal{M} \mathbf{J} dX, \\ \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}) &= \int_0^l \mathbf{J}^T (\mathcal{M} \dot{\mathbf{J}} - \text{ad}_{\mathbf{J}\dot{\mathbf{q}}}^T \mathcal{M} \mathbf{J}) dX \dot{\mathbf{q}} \\ &\quad - \int_0^l \mathbf{J}^T \bar{\mathbf{F}}_{\text{env}} dX - \mathbf{J}^T(l) \mathbf{F}_+. \end{aligned} \quad (38)$$

Let us remark that all these expressions depend on \mathbf{g} through the model of $(\bar{\mathbf{F}}_{\text{env}}, \mathbf{F}_+)$ (e.g. due to gravity), as well as through $\text{Ad}_{\mathbf{g}}$ in the expressions (11), (12) of \mathbf{J} and $\dot{\mathbf{J}}$.

C. NE based approach

In the previous approach, we started from the VW balance (15),(27)-(29) and converted the absolute virtual displacement fields along with the force wrenches (inertial and environmental) work, into strain variation fields. Reciprocally, one can first compute the stress-wrench field $\mathbf{\Lambda}$ generated by the force-wrenches exerted onto the CR, and use it in the following alternative identity, which is equivalent to (34):

- $\forall \delta\mathbf{q} \in \mathbb{R}^n$, we have:

$$\begin{aligned} 0 &= \delta W_{\text{int}} + \delta W_{\text{ext}} - \delta W_{\text{acc}} & (39) \\ &= \int_0^l \delta\epsilon^T (\mathbf{\Lambda} - \mathbf{\Lambda}_{\text{act}} - \mathcal{H}\epsilon) dX \\ &= \delta\mathbf{q}^T (\mathbf{Q}_{\text{act}}(\mathbf{q}) - \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{K}\mathbf{q}). \end{aligned}$$

In this alternative view, which corresponds to the NE-based approach of rigid robotics, the generalized forces generated by inertial and environmental force-wrenches, take the form:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}) = - \int_0^l \Phi^T \mathbf{\Lambda} dX, \quad (40)$$

where at any $X \in [0, l]$, $\mathbf{\Lambda}(X)$ is the stress-wrench that balances all the inertial and environmental forces exerted onto the piece of rod $[X, l]$. To calculate this stress-wrench field, one can introduce the dual relationship to (6):

$$\text{ad}_{\xi}^T = -\text{Ad}_{\mathbf{g}}^T \text{Ad}_{\mathbf{g}}^{-T} \quad (41)$$

into the original dynamic balance (21) that can then be integrated backward from l to X as:

$$\begin{aligned} \mathbf{\Lambda}(X) &= \text{Ad}_{\mathbf{g}}^T \text{Ad}_{\mathbf{g}(l)}^{-T} \mathbf{F}_+ \\ &\quad + \text{Ad}_{\mathbf{g}}^T \int_X^l \text{Ad}_{\mathbf{g}(Y)}^{-T} (\bar{\mathbf{F}}_{\text{env}} - \bar{\mathbf{F}}_{\text{in}})(Y) dY. \end{aligned} \quad (42)$$

Following the NE approach of rigid manipulators, one can first compute $(\mathbf{g}, \boldsymbol{\eta}, \dot{\boldsymbol{\eta}})$ from $(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$ with the integrated (5) and (10), or differential kinematics (1)-(3), and use a $(\mathbf{g}, \boldsymbol{\eta})$ -dependent model of environmental forces with that of inertial ones (32), to compute all these force-wrenches along the CR. These values of $(\bar{\mathbf{F}}_{\text{in}}, \bar{\mathbf{F}}_{\text{env}}, \mathbf{F}_+)$ are then introduced into (42) which is projected onto the Ritz basis Φ through (40). In the language of NE, this computational process defines a two-pass (one called "forward kinematics", the other called "backward dynamics") algorithm, named *IDM*, for "inverse dynamic model". Referring to (40), one has:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}) = \text{IDM}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}). \quad (43)$$

We will see later how this algorithm can be implemented numerically or symbolically.

V. SIMILARITIES AND DIFFERENCES

The two approaches are based on the same physical model (Cosserat), with a same parametrization (strain coordinates), and both lead to the inverse dynamic model of a CR, i.e. to a model of the actuation forces \mathbf{Q}_{act} as some functions of $(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$. From this highest point of view, they only differ by the computational algorithm used to obtain these functions, as shown on the flow chart diagram in Fig. 2. As introduced before, they are extensions of the Kane's method and Luh's algorithm that have both been extensively studied in the context of rigid robotics. Therefore, after demonstrating the duality of the Jacobian and NE approaches and their algorithmic differences, we will provide perspectives on hybridizing both based on current rigid robotics knowledge.

A. Mathematical duality

In this section, we prove the equivalence of the two approaches presented above. Since they differ from each other only through the treatment of force-wrenches, we will concentrate on them, and show that starting from their model in the NE-approach, one can deduce their model in the Jacobian based one. To go further into details, remind that in the Jacobian based approach, the vector of generalized forces generated by any distribution of force-wrenches $(\bar{\mathbf{F}}, \mathbf{F}_+)$, is given by:

$$\mathbf{Q} = -\mathbf{J}^T(l)\mathbf{F}_+ + \int_0^l \mathbf{J}^T \bar{\mathbf{F}} dX. \quad (44)$$

While in the NE-one, the same vector is given by:

$$\mathbf{Q} = - \int_0^l \Phi^T \Lambda dX \quad (45)$$

where Λ is the field of stress-wrench that balances $\bar{\mathbf{F}}$ and \mathbf{F}_+ .

As a result, to prove the equivalence of the two approaches, it suffices to prove that (44) and (45) are equal for any given pair $(\bar{\mathbf{F}}, \mathbf{F}_+)$. The proof below goes from (45) to (44) and is segmented in two steps. In the first step, we will explicitly state the expression (42) of Λ as a function of $(\bar{\mathbf{F}}, \mathbf{F}_+)$. In the second, we will explicitly demonstrate the identity of (44) and (45).

1) *Proof of (42):* Introducing (41) into (21), with $\bar{\mathbf{F}} = \bar{\mathbf{F}}_{\text{in}} - \bar{\mathbf{F}}_{\text{ext}}$, gives:

$$\Lambda' + \text{Ad}_g^T \text{Ad}_g^{-T'} \Lambda = \bar{\mathbf{F}}. \quad (46)$$

Now multiplying (46) by Ad_g^{-T} , gives:

$$\text{Ad}_g^{-T} \Lambda' + \text{Ad}_g^{-T'} \Lambda = \text{Ad}_g^{-T} \bar{\mathbf{F}}. \quad (47)$$

whose left-hand side can be written as:

$$(\text{Ad}_g^{-T} \Lambda)' = \text{Ad}_g^{-T} \bar{\mathbf{F}}. \quad (48)$$

Integrating (48) from X to l and using the BC (22):

$$\Lambda(X) = \text{Ad}_{g(X)}^T \text{Ad}_{g(l)}^{-T} \mathbf{F}_+ - \text{Ad}_{g(X)}^T \int_X^l \text{Ad}_g^{-T} \bar{\mathbf{F}} dY. \quad (49)$$

2) *Proof of (44) = (45):* We start by introducing (49) into (45), which gives:

$$- \int_0^l \Phi^T \Lambda dX = - \left(\int_0^l \Phi^T \text{Ad}_{g(X)}^T dX \right) \text{Ad}_{g(l)}^{-T} \mathbf{F}_+ + \int_0^l \Phi^T \text{Ad}_{g(X)}^T \left(\int_X^l \text{Ad}_{g(Y)}^{-T} \bar{\mathbf{F}} dY \right) dX. \quad (50)$$

In the first term of the right-hand side of this expression, we recognize the expression (11) of \mathbf{J} , and we have:

$$- \left(\int_0^l \Phi^T \text{Ad}_g^T dX \text{Ad}_g^{-T}(l) \right) \mathbf{F}_+ = -\mathbf{J}^T(l)\mathbf{F}_+. \quad (51)$$

Applying a by-part integration to the second term of the right-hand side of (50), noted \mathbf{I} , gives:

$$\mathbf{I} = - \left[\int_0^X \Phi^T \text{Ad}_g^T dY \int_X^l \text{Ad}_g^{-T} \bar{\mathbf{F}} dY \right]_0^l + \int_0^l \left(\int_0^X \Phi^T \text{Ad}_g^T dY \text{Ad}_g^{-T} \right) \bar{\mathbf{F}} dX. \quad (52)$$

The first term equals to 0 as:

$$- \int_0^l \Phi^T \text{Ad}_g^T dY \int_l^l \text{Ad}_g^{-T} \bar{\mathbf{F}} dY + \int_0^0 \Phi^T \text{Ad}_g^T dY \int_0^l \text{Ad}_g^{-T} \bar{\mathbf{F}} dY = 0. \quad (53)$$

Using the expression (11) of the Jacobian matrix in the second term of (52), gives:

$$\mathbf{I} = \int_0^l \mathbf{J}^T \bar{\mathbf{F}} dX. \quad (54)$$

Finally, using (51) and (54) in (50), we do have:

$$- \int_0^l \Phi^T \Lambda dX = -\mathbf{J}^T(l)\mathbf{F}_+ + \int_0^l \mathbf{J}^T \bar{\mathbf{F}} dX. \quad (55)$$

As announced, the identity (55) proves the equivalence of the two approaches. Moreover, the computational process leading to (55), being entirely based on the duality of twists and wrenches (matrix transposition, by part integration), the two algorithms can be said to be dual of each other.

B. Formulation and implementation

The Jacobian-based approach leads naturally to a model with an explicit formulation, meaning it provides access to each vector and matrix in Eq. (33). Indeed, by forward integrating (from $X = 0$ to l) the equations of the Jacobian and its time derivative (11-12), obtained from the analytical integration of the kinematic models (2-3), one can integrate the equations of the reduced model matrices (38). Moreover, in its latest version [13], this approach uses spatial integration schemes that have analytical formulations, to symbolically compute each model matrix, and export it into a single function. As such, one can obtain not only every matrices but also their analytical derivatives with respect to design or control parameters. This makes this model particularly interesting for the performance analysis and design of CRs, where for example the matrices must be combined and differentiated to compute singularity detection criteria [20]. For control, this enables also reshaping Eq. (33) in the usual state-space form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, where $\mathbf{x} = (\mathbf{q}^T, \dot{\mathbf{q}}^T)^T$ is the state vector and \mathbf{u} is the control vector,

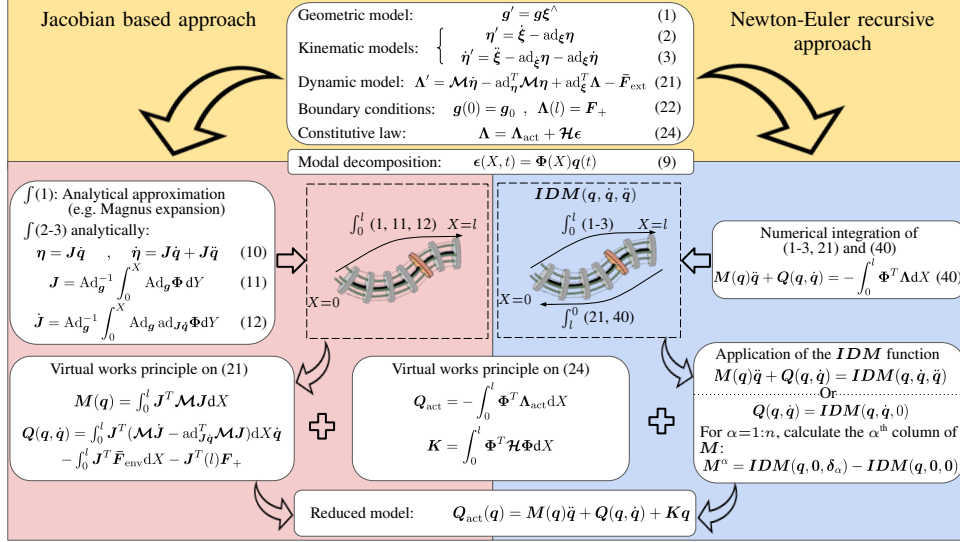


Fig. 2. Synthesis of the steps in both approaches side by side. The blue and red areas are relative to the NE recursive approach and Jacobian based approach, respectively. The yellow part corresponds to the full Cosserat rod model.

for which most work on the control of smooth dynamical systems has been built. On the contrary, the NE-based approach uses an implicit formulation, where a part of the matrices is not directly available, in particular the mass matrix. The set of non-linear equations (43) needs to be solved with a numerical solver to get \ddot{q} from (q, \dot{q}) and the actuation and external forces, as illustrated in Fig. 2. If one wants to obtain the mass matrix and forces of (33) with such a generalized *IDM*, it suffices to feed it with some specific inputs. For $\alpha = 1 : n$, calculate the α^{th} column of M :

$$M^\alpha = IDM(q, 0, \delta_\alpha) - IDM(q, 0, 0) \quad (56)$$

with δ_α a unit vector with zero entries, except the α^{th} . This algorithm was implemented in a pure numerical setting in [10, 14], leveraging numerical spatial integration schemes, to validate the GVS against the geometrically exact FEM with explicit time-integrators. Although simple to implement, this further NE algorithm remains too slow to be used beyond this objective.

C. Space and time integration

Concerning time integration, the Jacobian-based approach naturally leads to using explicit schemes, for which the mass matrix of the Lagrangian model (33) needs to be explicitly computed and inverted. However, the dynamics of flexible systems lead to numerically stiff ODE systems whose explicit integration is problematic. The practical manifestation of this limitation is that, to preserve stability, the user has to add artificially high damping and/or use very small time steps, which slows down the simulation considerably, if it doesn't simply explode. As a consequence, using explicit schemes may not efficiently capture certain dynamic behaviors specific to some robot architectures, such as high-frequency vibrations in CRs with a low mass-to-stiffness ratio, the buckling phenomenon in CPRs [20], and the snapping phenomenon in CTRs [19]. Therefore, to preserve stability and maintain large time steps (and save computations), it is strongly recommended

to use implicit time integration to simulate CR dynamics. In this context, the simulator turns to be a predictive-corrective algorithm that requires both the residual vector of the dynamics (33), and its Jacobian to be computed at each step of a Newton loop. Recently, the NE-based GVS has been extended to this further context in [21]. In this case, NE's implicit formulation has proved to be a valuable aid in coping with the considerable increase in nonlinearities introduced by the linearization of dynamics (the Jacobian of its residual vector). More precisely, the *IDM* of (43) has been exactly differentiated to produce a tangent *IDM* or *TIDM*, that allows the Jacobian of the residual vector to be computed efficiently. As shown in [21], although the computation of the tangent is more demanding in terms of complexity, the approach is sufficiently stable to allow large time steps and reduce simulation times.

Concerning space integration, while (10)-(12) are usual integrals in vector spaces that could be integrated using standard numerical quadrature (e.g. Gauss, Simpson...), (1) needs to be integrated on the Lie group $SE(3)$. In [13], this integration has been achieved with the Jacobian-based approach by applying a Fourth-order Zannah quadrature approximation of the Magnus expansion (5), as proposed in [16]. To avoid the double space integrations due to the introduction of (10)-(12) into (38), the approximation of (5) is time-differentiated twice to evaluate $\delta\zeta$, η , and $\dot{\eta}$ at the Gauss points of the (unique) remaining integral of (38). In contrast, the NE approach has been developed so far with purely numerical spatial integration schemes. To cope with the Lie group structure of (1), quaternions are used, which enables using standard integration schemes in \mathbb{R}^4 . In [14], this has been first performed with usual explicit finite difference schemes (e.g. Runge Kutta) as this is done in the Newtonian approach of [9]. More recently, the differential properties of the forward and backward ODEs have been exploited to integrate them with a spectral method [10]. However, despite its high performance in terms of convergence and accuracy, the spectral integration requires more nodes than the Zannah quadrature. Due to its numerical character it is

much less optimized in terms of computations (redundancies of operations, multiplication by zeros...). While this is not a problem for analysis and design applications, it can be for online control or planning applications.

VI. HYBRIDIZATION OF THE TWO APPROACHES: CONCLUSION AND PERSPECTIVES FOR THE GVS

A. Simulation

Based on the above discussion, we can conclude that the Jacobian approach is more advanced than the NE approach from the point of view of spatial integration, while the NE approach is more advanced than the Jacobian approach from the point of view of time integration. Therefore, for the purpose of efficient simulation, it is natural to say that the first approach has to be extended to implicit integration while the NE based one needs to be implemented in symbolic form. In the first case, the extension requires to cope with the complex nonlinear calculation of the tangent dynamics. To this end, one can take inspiration from the recent progress accomplished in rigid robots dynamics [22], or use the symbolic differentiation of the recursions proposed in [21] to derive the *TIDM* from the *IDM* algorithm. Reciprocally, the spatial integration based on Magnus expansion and quadrature developed in [13] for the Jacobian-based approach can naturally replace the spectral integration of [10]. The result would be a purely symbolic implementation of the *IDM* and *TIDM* algorithms. This would be in line with rigid robot dynamics, for which it has been concluded that the most efficient implementations of inverse and forward rigid robot dynamics are based on NE algorithms implemented and optimized with symbolic computation engines such as Mathematica or Maple [23].

B. Control

In rigid robotics, it is now well established that the Lagrangian and NE models need not be opposed, but are in fact complementary. In detail, the explicit form (33) is a valuable aid to designing nonlinear controllers in the first instance, which can then be implemented in implicit form in fast, compact NE algorithms. The archetypal example of this idea is the computed-torque controller of a rigid manipulator, whose synthesis and analysis are based on the Lagrangian model in explicit form, while its most efficient implementation uses its implicit NE algorithmic realization. Exploiting this complementarity requires to change control laws into equivalent NE algorithms. While this equivalence is straightforward in the case of a rigid manipulator, it can be much more difficult in the case of CRs. However, in [14] this idea was first applied to a 3-segment in-plane TACR, with the aim of testing a non-linear two-time-scale controller, like those developed in the 90s to control flexible arms with motor-actuated localized joints [24]. This simple example shows that this complementarity between Lagrangian and NE points of view could also be applied to CRs, in particular by exploiting the structure they share with the flexible segment and concentrated joint robot models of the 90s. From this point of view, the recent article [25] did a further step towards this idea, since it provides an algorithm for testing whether the dynamics of a CR can be put into the collocated control form, of these other robots.

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