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► **To cite this version:**

| Ismaël Bailleul, Yvain Bruned. Locality for singular stochastic PDEs. 2025. hal-04893027

HAL Id: hal-04893027

<https://hal.science/hal-04893027v1>

Preprint submitted on 17 Jan 2025

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Locality for singular stochastic PDEs

I. Bailleul¹, Y. Bruned²

Abstract. This work deals with singular stochastic PDEs driven by non-translation invariant differential operators. We describe the renormalized equation for a very large class of spacetime dependent renormalization schemes. Our approach bypasses in particular the use of decorated trees with extended decorations.

1. INTRODUCTION

Denote by $x = (x_0, x_1) \in \mathbb{R} \times \mathbf{T}$ a typical spacetime point over the one dimensional periodic torus \mathbf{T} . Let

$$\mathcal{L}^p v := a^p(\cdot) \partial_{x_1}^2 v \quad (1 \leq p \leq p_0),$$

stand for a finite family of second order differential operators with variable smooth coefficients. We consider systems of parabolic equations of the form

$$(\partial_{x_0} - \mathcal{L}^p) u_p = f_p(u) \xi + g_p(x, u, \partial_{x_1} u) \quad (1 \leq p \leq p_0), \quad (1)$$

with $u := (u_1, \dots, u_{p_0})$ and each u_p taking values in a finite dimensional space \mathbb{R}^{d_p} , and with $\xi = (\xi_1, \dots, \xi_{l_0})$ an l_0 -dimensional spacetime ‘noise’. The quantity $f_p(u) \xi$ is thus to be understood as $\sum_{l=1}^{l_0} f_p^l(u) \xi_l$. An initial condition $u(0)$ of positive Hölder regularity is given and fixed throughout that work. We consider the question of the renormalization of this system of non-translation invariant singular stochastic PDEs.

Non-translation invariance refers to the fact that the operators \mathcal{L}^p in the system (1) may have non-constant coefficients. Working in a non-translation invariant setting does not influence the algebraic side of the theory of regularity structures encoding the mechanics of the local description of modelled distributions/functions. We can use in the non-translation invariant setting the same algebraic structure, with no extended decorations, as in the translation invariant setting. The translation invariance of the operators involved in the equations studied so far using regularity structures manifests itself in two different ways. On a technical level one uses of a number of translation invariant kernels $q(x - y)$ rather than just kernels $q(x, y)$. On a more fundamental level, it manifests itself in the renormalization process in the fact that renormalization *constants* are involved rather than renormalization *functions*, as expected in a non-translation invariant setting, as found for instance in the paracontrolled approach to singular stochastic PDEs developed in [2–4]. In a non-translation invariant setting the very formulation of the renormalization scheme of Bruned, Hairer and Zambotti [18] does not make obvious sense anymore. Denote by $\mathbf{M} = (\Pi, \mathbf{g})$ the canonical smooth admissible model associated with a given system of singular stochastic PDEs driven by a continuous noise, acting on the regularity structure $((T, \Delta), (T^+, \Delta^+))$ associated with that system after Bruned, Hairer and Zambotti’s work [18]. The renormalized admissible continuous models built in [18] are associated with some characters ℓ of an algebra to which one associates some linear maps $M_\ell : T \rightarrow T$ and some renormalized interpretation operators

$$\Pi_\ell := \Pi \circ M_\ell.$$

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The map Π_ℓ defines a unique admissible model. The transform $\Pi \mapsto \Pi \circ M$ is called a renormalization scheme. There is no canonical way of associating a model to a non-continuous noise ξ . One then regularizes the noise into a continuous noise ξ^ε so it has an associated canonical model Π^ε . This model typically diverges as $\varepsilon > 0$ goes to 0. One then look for an ε -dependent character ℓ^ε such that the model associated with the renormalized interpretation map $\Pi^\varepsilon \circ M_{\ell^\varepsilon}$ is converging.

Such a renormalization scheme cannot work in a non-translation invariant setting, even by replacing ℓ by a character-valued function $\ell(x)$ of the state space variable x . Indeed such a map cannot account for some possible internal renormalization indexed by functions of the internal variables of the multiple integral expressions defining the renormalized interpretation map, not by the base/external variable x . Let us illustrate with an example why we need to have a local renormalisation. We consider the following integral

$$\int K(y - x_1)G(x_1, x_2)F(x_2)dx_1dx_2.$$

We suppose that G is singular along the diagonal $x_1 = x_2$ but it is not translation invariant. Therefore one has

$$\int G(x_1, x_2)dx_2 \neq \int G(0, x_2)dx_2$$

In order to make sense of the previous integral one has to renormalize it; in this case one has to subtract a Taylor jet to F at the point x_1 . This gives the renormalized integral

$$\int K(y - x_1)G(x_1, x_2)(F(x_2) - F(x_1))dx_1dx_2$$

where we have supposed that subtracting only $F(x_1)$ is enough for counter-balancing the singularity of $G(x_1, x_2)$. The counter-term is

$$\int K(y - x_1)F(x_1)G(x_1, x_2)dx_1dx_2.$$

In that case the renormalisation function is given by

$$\int G(x_1, x_2)dx_2$$

which is clearly not equal to $\int G(0, x_2)dx_2$. This situation will happen when one wants to have bounds on the moments of some stochastic iterated integrals. These moments will be given by some iterated integrals of the form above with kernels singular on some diagonals.

So a different approach of renormalization is needed in a non-translation invariant setting. It is one of the aims of the present work to provide such an alternative. This is first done in Section 3 and Section 4 in a translation invariant setting after we set the algebraic background in Section 2. *Our approach involves in particular no extended decorations and does not rest on the pre-Lie mechanics* that was the workhorse of Bruned, Chandra, Chevyrev & Hairer in their seminal work [14]. Instead we rely on some important algebraic properties of a family of maps called preparation maps and a new right morphism property that they satisfy. This simplifies the general setting of regularity structures. We turn in Section 5 to the general setting of systems of singular stochastic PDEs involving non-translation invariant differential operators. Section 5.1 describes the analytic ingredient needed to run the machinery of regularity structures in a non-translation invariant setting. The natural extension of our approach to the renormalized equation works perfectly in a non-translation invariant setting. This involves state space dependent preparation maps

and their associated models, both of which are discussed in Section 5.2. Our main result reads as follows. The notions involved here are explained in the body of the text.

Theorem 1.1. *Let ξ be a continuous noise. Let $R : (\mathbb{R} \times \mathbf{T}) \times T \rightarrow T$ be a strong preparation map such that $R\tau = \tau$ for all polynomials and planted trees τ in $\bigoplus_{a \in \mathfrak{S}^+ \times \{0\}} \mathcal{I}_a(T)$. Denote by \mathbb{M}^R the admissible model associated with R and write $\mathbb{R}^{\mathbb{M}^R}$ for the corresponding reconstruction map. Let \mathbf{u}^R stand for the modelled distribution solution of the regularity structure lift of system (1) associated with \mathbb{M}^R , with initial condition $u(0)$. Then*

$$u := \mathbb{R}^{\mathbb{M}^R}(\mathbf{u}^R)$$

is a solution of the renormalized system

$$(\partial_{x_0} - \mathcal{L}^p)u_p = f_p(u)\xi + g_p(u, \partial_{x_1}u) + \sum_{l=0}^{l_0} \Upsilon_p\left((R(\cdot)^* - \text{Id})\zeta_l\right)(u, \partial_{x_1}u)\xi_l \quad (1 \leq p \leq p_0), \quad (2)$$

for some explicit functions $\Upsilon_p(\tau)(u, \partial_{x_1}u)$ indexed by $\tau \in T$.

The reader will see from the proof below that this statement holds in a multidimensional space setting as well. Branched rough paths provide a particular example of simple regularity structures whose elements have no polynomial decorations. The class of renormalization maps of branched rough paths is indexed by a set of multi-pre-Lie morphisms. We prove in Section 6 that this class is exactly parametrized by the set of preparation maps.

We finish this introduction by a point of methodology. So far, in a singular SPDE setting, renormalization was mainly viewed as the task of finding some linear maps M of the model space T that turn any admissible model Π into an admissible model $\Pi \circ M$. Our introduction of preparation maps has a different aim. It provides a mechanics for *building* inductively some admissible models.

Notations – *The letter ζ will be used exclusively for the noise symbol in a regularity structure. The letters σ, τ, μ, ν will denote some decorated trees.*

Define the parabolic distance d on $\mathbb{R} \times \mathbb{R}$ by

$$d(x, x') := \sqrt{|x_0 - x'_0|} + |x_1 - x'_1|,$$

for some arbitrary points $x = (x_0, x_1), x' = (x'_0, x'_1)$ in \mathbb{R}^2 . For $k = (k_0, k_1) \in \mathbb{N} \times \mathbb{N}$ set

$$|k|_{\mathfrak{s}} := 2k_0 + k_1.$$

2. DECORATED TREES

We recall from Bruned, Hairer & Zambotti's work [18] and Bruned & Manchon's work [19] what we need from the combinatorial setting used to build regularity structures associated with some system of singular stochastic PDEs. A fundamental morphism property of the \star product introduced below is proved in Proposition 2.2. This statement will play later the role played by a fundamental multi-pre-Lie morphism introduced in [14]. Our proof of the renormalized equation will in particular not involve any pre-Lie morphisms. This will be crucial for the results of Section 5 that hold in the general setting of systems of singular stochastic PDEs with non-translation invariant differential operators, for which the renormalization strategy of [18] has no obvious counterpart.

Recall system (1) with its noises ξ_1, \dots, ξ_{l_0} and its operators $\mathcal{L}^1, \dots, \mathcal{L}^{p_0}$. Let

$$\mathfrak{T}_- = (\mathfrak{t}_-^1, \dots, \mathfrak{t}_-^{l_0}), \text{ and } \mathfrak{T}_+ = (\mathfrak{t}_+^1, \dots, \mathfrak{t}_+^{p_0})$$

be some finite sets representing the noise types and the operator types, respectively. Denote by

$$\mathfrak{T} := \mathfrak{T}_- \cup \mathfrak{T}_+$$

the set of all types. We consider some decorated trees $(\tau, \mathfrak{n}, \mathfrak{e})$ where τ is a non-planar rooted tree with node set N_τ and edge set E_τ . The maps $\mathfrak{n} : N_\tau \rightarrow \mathbb{N}^{d+1}$ and $\mathfrak{e} = (\mathfrak{t}(\cdot), \mathfrak{q}(\cdot)) : E_\tau \rightarrow \mathfrak{T} \times \mathbb{N}^{d+1}$ are node decorations and edge decorations, respectively. The \mathbb{N}^{d+1} -part $\mathfrak{q}(e)$ in the edge decoration of an edge e encodes some possible derivatives acting on the operator associated with the given edge type $\mathfrak{t}(e)$. Let us stress that we describe here a setting where the noises are represented by edges. (There is a slightly different setting where noises are represented by a vertex type decoration.) We will frequently abuse notations and simply denote by τ a decorated tree, using a symbolic notation.

- An edge decorated by $(\mathfrak{t}, q) \in \mathfrak{T} \times \mathbb{N}^{d+1}$ is denoted by $\mathcal{I}_{(\mathfrak{t}, q)}$. The symbol $\mathcal{I}_{(\mathfrak{t}, q)}$ is also viewed as the operation that grafts a tree onto a new root via a new edge with edge decoration (\mathfrak{t}, q) .
- A factor X^k encodes a single node decorated by $k \in \mathbb{N}^{d+1}$. Denote by

$$(\varepsilon_1, \dots, \varepsilon_{d+1})$$

the canonical basis of \mathbb{N}^{d+1} . For $1 \leq i \leq d+1$, write X_i for X^{ε_i} . The element X^0 is identified with $\mathbf{1}$.

We have $d = 1$ and second order differential operators \mathcal{L}^p in the particular setting of system (1). We describe below a setting adapted to deal with the general case where $d \geq 1$ and the operators \mathcal{L}^p may be of order different from 2. We require that every decorated tree τ contains at most one edge decorated by (\mathfrak{t}, q) with $\mathfrak{t} \in \mathfrak{T}_-$ and any $q \in \mathbb{N}^{d+1}$, at each node. This encodes the fact that no product of two noises are involved in the analysis of the system (1). We suppose that these edges of noise type lead directly to leaves of the tree; we denote them by ζ_l , for $1 \leq l \leq l_0$. By convention we set

$$\zeta_0 := \mathbf{1}.$$

Any decorated tree τ has a unique decomposition

$$\tau = X^k \zeta_l \prod_{i=1}^n \mathcal{I}_{a_i}(\tau_i),$$

where $0 \leq l \leq l_0$ and \prod_i is the tree product, the τ_i are decorated trees and the a_i belong to $\mathfrak{T}_+ \times \mathbb{N}^{d+1}$, so no factor in the product is a noise symbol ζ_l . The root of the particular tree $X^k \zeta_l$, with $1 \leq l \leq l_0$, has for instance an \mathfrak{n} -decoration equal to k . Here is an example of a decorated tree

$$X^\beta \zeta_2 \mathcal{I}_b(X^\gamma \zeta_3) \mathcal{I}_a(X^\alpha \zeta_1) =$$

where we have represented the decoration on the edges and the nodes. For a noise edge, we have chosen to put ζ_i as a decoration on the decorated tree.

The algebraic symmetry factor $S(\tau)$ of a decorated tree $\tau = X^k \zeta_l \prod_{j=1}^m \mathcal{I}_{a_j}(\tau_j)^{\beta_j}$ is defined grouping terms uniquely in such a way that $(a_i, \tau_i) \neq (a_j, \tau_j)$ for $i \neq j$, and setting inductively

$$S(\tau) = k! \left(\prod_{j=1}^m S(\tau_j)^{\beta_j} \beta_j! \right),$$

starting with $S(\zeta_l) = 1$ for $0 \leq l \leq l_0$. We define an inner product on the set of all decorated trees setting for all σ, τ

$$\langle \sigma, \tau \rangle := S(\tau) \mathbf{1}_{\sigma=\tau}.$$

We also set

$$\langle \sigma_1 \otimes \sigma_2, \tau_1 \otimes \tau_2 \rangle := \langle \sigma_1, \tau_1 \rangle \langle \sigma_2, \tau_2 \rangle.$$

The linear span of the set of decorated trees will be denoted by T . A *planted tree* is a tree of the form $\mathcal{I}_a(\sigma)$, for a decorated tree σ and $a \in \mathfrak{T}_+ \times \mathbb{N}^{d+1}$; we denote by $\mathcal{I}(T)$ the linear span of planted trees.

We now associate some numbers to every decorated tree. Fix a scaling vector $\mathfrak{s} \in \mathbb{N}^{d+1}$ and a map

$$|\cdot|_{\mathfrak{s}} : \mathfrak{T} \rightarrow \mathbb{R}$$

which is negative on the noise types \mathfrak{T}_- and positive on the operator types \mathfrak{T}_+ . This map accounts for the regularity of the noises and the gain of regularity of some operators encoded in the Schauder-type estimates that they satisfy. We extend the map $|\cdot|_{\mathfrak{s}}$ to $\mathfrak{T} \times \mathbb{N}^{d+1}$ setting

$$|q|_{\mathfrak{s}} := \sum_{i=1}^{d+1} \mathfrak{s}_i q_i, \quad \text{and} \quad |(\mathfrak{t}, q)|_{\mathfrak{s}} := |\mathfrak{t}|_{\mathfrak{s}} - |q|_{\mathfrak{s}}.$$

The degree of a decorated rooted tree $(\tau, \mathfrak{n}, \mathfrak{e})$ is defined by

$$\deg(\tau, \mathfrak{n}, \mathfrak{e}) := \sum_{v \in N_T} |\mathfrak{n}(v)|_{\mathfrak{s}} + \sum_{e \in E_T} |(\mathfrak{t}(e), \mathfrak{q}(e))|_{\mathfrak{s}}. \quad (3)$$

For cross reference, note that the ‘Degree’ is called ‘homogeneity’ in Hairer’s work [33]. We use the degree to introduce the space of ‘positive’ decorated trees T^+ . It is the linear span of trees of the form $X^k \prod_{i=1}^n \mathcal{I}_{a_i}(\tau_i)$, where $\deg(\mathcal{I}_{a_i}(\tau_i)) > 0$ and $k \in \mathbb{N}^{d+1}$. We also consider the linear space T^- spanned by the decorated trees with negative degree, and denote by $\mathbb{R}[T^-]$ the linear space spanned by forests of elements of T^- .

Given $k \in \mathbb{N}^{d+1}$ denote by \uparrow_v^k the derivation on decorated trees that adds k to the decoration at the node v , so $\uparrow_v^k(\sigma\tau) = \uparrow_v^k(\sigma)\tau + \sigma \uparrow_v^k(\tau)$ for all $\sigma, \tau \in T$. For $a = (\mathfrak{t}, p)$, one writes $a - m$ for $(\mathfrak{t}, p - m)$. We introduce a family of pre-Lie products of grafting type setting for all decorated trees $\sigma, \tau \in T$, with $\tau = (\tau, \mathfrak{n}, \mathfrak{e})$ and $a \in \mathfrak{T}_+ \times \mathbb{N}^{d+1}$,

$$\sigma \curvearrowright_a \tau := \sum_{v \in N_\tau} \sum_{m \in \mathbb{N}^{d+1}} \binom{\mathfrak{n}_v}{m} \sigma \curvearrowright_{a-m}^v (\uparrow_v^{-m} \tau),$$

where $\mathfrak{n}_v = ((\mathfrak{n}_v)_1, \dots, (\mathfrak{n}_v)_{d+1}) \in \mathbb{N}^{d+1}$ is the decoration at the node v , the binomial coefficient $\binom{\mathfrak{n}_v}{m}$ equals $\prod_{1 \leq i \leq d+1} \binom{(\mathfrak{n}_v)_i}{m_i}$, and \curvearrowright_{a-m}^v grafts σ onto τ at the node v with an edge decorated by $a - m$. The above sum is finite due to the binomial coefficient $\binom{\mathfrak{n}_v}{m}$

which is equal to zero if m is greater than \mathfrak{n}_v , by convention. It is forbidden to graft on the top of noise edges which are terminal edges. Here is an example of this grafting product

The pre-Lie products \curvearrowright_a are non-commutative; they were first introduced in Bruned, Chandra, Chevyrev and Hairer's work [14]. We recall one universal result that we will use in the sequel; it was first established in Corollary 4.23 of [14]. It can be viewed as an extension of the universal result of Chapoton-Livernet [25] on pre-Lie algebras. (Such a result becomes immediate when one constructs \curvearrowright_a as a deformation, as in Section 2.1 of [19]. See also Foissy's work [27] for the case with no deformation.)

Proposition 2.1. *The space T is freely generated by the elements*

$$\left\{ X^k \zeta_l; 1 \leq l \leq l_0, k \in \mathbb{N}^{d+1} \right\}$$

and the operations $\{\curvearrowright_a; a \in \mathfrak{T}_+ \times \mathbb{N}^{d+1}\}$.

Let us mention that we will not use in the sequel the pre-Lie character of the maps \curvearrowright_a ; this property will only play an implicit role via the use of Proposition 2.1 in the proof of Proposition 2.2. We define a product

$$\curvearrowright: \mathcal{I}(T) \times T \rightarrow T$$

setting for all $a \in \mathfrak{T}_+ \times \mathbb{N}^{d+1}, \sigma, \tau \in T$

$$\mathcal{I}_a(\sigma) \curvearrowright \tau := \sigma \curvearrowright_a \tau. \quad (4)$$

We also set $\mathbf{1} \curvearrowright \tau := \tau$. We invite the reader to check that if we graft n trees σ_i along some grafting operators \curvearrowright_{a_i} on τ , *not on one another*, then the resulting element of T does not depend on the order in which we grafted our trees. We can then extend the \curvearrowright to products of planted trees $\prod_{i=1}^n \mathcal{I}_{a_i}(\sigma_i)$, so

$$\prod_{i=1}^n \mathcal{I}_{a_i}(\sigma_i) \curvearrowright \tau$$

means that we perform the simultaneous grafting of the decorated trees σ_i onto σ along some edges decorated by the a_i . Some nodes of τ can received several grafting and we sum over all the possibilities. The order of the grafting does not matter. Below, we provide an example of computation

where $a_k = a - k_1$ and $b_k = b - k_2$. In this example only one node is allowed for the grafting which is the root decorated by k . One can observe the symmetry between k_1 and k_2 that illustrate that the order of the grafting does not matter.

Following Bruned and Manchon's construction in [19], for $B \subset N_\tau$, consider the derivation map \uparrow_B^k defined as

$$\uparrow_B^k \tau = \sum_{\sum_{v \in B} k_v = k} \prod_{v \in B} \uparrow_v^{k_v} \tau. \quad (5)$$

In the sequel, we will use the following short hand notation: $\uparrow_{N_\tau}^k \tau = \uparrow^k \tau$. Below, we provide an example of computation

$$\uparrow^k \begin{array}{c} \bullet \\ \zeta_3 \\ \bullet \\ \zeta_2 \quad b \\ \bullet \\ \beta \end{array} \gamma = \sum_{k=k_1+k_2} \binom{k}{k_1, k_2} \begin{array}{c} \bullet \\ \zeta_3 \\ \bullet \\ \zeta_2 \quad b \\ \bullet \\ \beta + k_1 \end{array} \gamma + k_2.$$

We define the product

$$\star : T \times T \rightarrow T$$

for all $\sigma = X^k \prod_i \mathcal{I}_{a_i}(\sigma_i) \in T$ and $\tau \in T$ by the formula

$$\sigma \star \tau := \uparrow_{N_\tau}^k \left(\left(\prod_i \mathcal{I}_{a_i}(\sigma_i) \right) \curvearrowright \tau \right). \quad (6)$$

Let us emphasize that the operator $\uparrow_{N_\tau}^k$ only acts on the nodes of τ . One has for instance

$$X^k \zeta_l \prod_{i=1}^n \mathcal{I}_{a_i}(\tau_i) = \left(X^k \prod_{i=1}^n \mathcal{I}_{a_i}(\tau_i) \right) \star \zeta_l, \quad (7)$$

and

$$\left\{ X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \right\} \star (X^m \zeta_l) = \zeta_l \sum_{r_1, \dots, r_n \in \mathbb{N}^{d+1}} X^{m - \sum_j r_j + k} \prod_{j=1}^n \binom{m}{r_j} \mathcal{I}_{a_j - r_j}(\tau_j).$$

Note that our product \star corresponds to the \star_2 product in [19]. One has also

$$\begin{array}{c} \bullet \\ \zeta_1 \\ \bullet \\ a \\ \bullet \\ k \end{array} \alpha \star \begin{array}{c} \bullet \\ \zeta_3 \\ \bullet \\ \zeta_2 \quad b \\ \bullet \\ \beta \end{array} \gamma = \sum_{k=k_1+k_2} \binom{k}{k_1, k_2} \left(\sum_{m \in \mathbb{N}^{d+1}} \binom{\beta}{m} \begin{array}{c} \bullet \\ \zeta_3 \\ \bullet \\ \zeta_2 \quad b \\ \bullet \\ \beta + k_1 - m \end{array} \gamma + k_2 \right) \\ + \sum_{m \in \mathbb{N}^{d+1}} \binom{\gamma}{m} \begin{array}{c} \bullet \\ \zeta_1 \\ \bullet \\ \zeta_3 \quad a - m \\ \bullet \\ \zeta_2 \quad b \\ \bullet \\ \beta + k_1 \end{array} \gamma + k_2 - m.$$

It has been proved in Section 3.3 of [19] that this product is associative; this can be obtained by applying the Guin-Oudom procedure [31, 32] to a well-chosen pre-Lie product. When $\sigma \in T^+$ and $\tau, \mu \in T$ one has from Theorem 4.2 in [19]

$$\langle \sigma \star \tau, \mu \rangle = \langle \tau \otimes \sigma, \Delta \mu \rangle \quad (8)$$

where

$$\Delta : T \rightarrow T \otimes T^+$$

is a co-action first introduced in Hairer's seminal work [33] – see also [18] and [8] where it plays a prominent role. So the restriction to $T^+ \times T$ of the product \star is the $\langle \cdot, \cdot \rangle$ -dual of the splitting map Δ . The restriction to $T^- \times T$ of the product \star is also the dual of the splitting map δ_r introduced by Bruned in Section 4 of [11], in formula (23) and Definition 4.2 therein, in the sense that one has for all $(\tau, \sigma) \in T^- \times T$ and $\mu \in T$ the identity

$$\langle \sigma \star \tau, \mu \rangle = \langle \tau \otimes \sigma, \delta_r \mu \rangle.$$

One can take this relation as a definition of the map δ_r . The index r stands here for the word 'root'. Note here as in (8) the different orders of σ and τ in the left and right hand sides of the equalities.

Recall $(\varepsilon_i)_{1 \leq i \leq d+1}$ stands for the canonical basis of \mathbb{N}^{d+1} . With a view on the system (1) assume we are given a family

$$(\Upsilon_p^l)_{1 \leq p \leq p_0, 0 \leq l \leq l_0}$$

of functions of a finite number of abstract variables

$$(Z_a)_{a \in (\mathfrak{T}_+ \cup \{*\}) \times \mathbb{N}^{d+1}}. \quad (9)$$

We impose that the $Z_{(*, \varepsilon_i)}$ are all equal to 1 and $Z_{(*, k)} = 0$ for $k \in \mathbb{N}^{d+1}$ not in the canonical basis. In the particular setting of the system (1) the functions $\Upsilon_p^l((Z_a)_a)$ are place holders for the functions f_p^l , when $1 \leq l \leq l_0$, and the function g_p when $l = 0$. The variables Z_a play the roles of x and u and its derivatives, with Z_* in the role of x and $Z_{\mathfrak{t}_+^p}$ in the role of u_p , and $Z_{(\mathfrak{t}_+^p, q)}$ in the role of $\partial^q u_p$, for instance. This interpretation explains our choice for the variables $Z_{(*, k)}$. We define the partial derivative operators

$$D_a = D_{Z_a}$$

with respect to the variable Z_a in the usual way and set

$$\partial^{\varepsilon_i} := \sum_a Z_{a+\varepsilon_i} D_a.$$

where $a + \varepsilon_i$ is defined by $(\mathfrak{t}, p + \varepsilon_i)$ for $a = (\mathfrak{t}, p)$. These operators commute since

$$\begin{aligned} \partial^{\varepsilon_i} \partial^{\varepsilon_j} &= \sum_{a_1, a_2} Z_{a_1+\varepsilon_1} D_{a_1} (Z_{a_2+\varepsilon_2} D_{a_2}) \\ &= \sum_{a_1, a_2} \left(Z_{a_1+\varepsilon_1} \delta_{a_1}(a_2 + \varepsilon_2) D_{a_2} + Z_{a_1+\varepsilon_1} Z_{a_2+\varepsilon_2} D_{a_1} D_{a_2} \right) \\ &= \sum_a Z_{a+\varepsilon_1+\varepsilon_2} D_a + \sum_{a_1, a_2} Z_{a_1+\varepsilon_1} Z_{a_2+\varepsilon_2} D_{a_1} D_{a_2} \end{aligned}$$

is symmetric in $\varepsilon_i, \varepsilon_j$. So one can define for $k = (k_1, \dots, k_{d+1}) \in \mathbb{N}^{d+1}$

$$\partial^k := \prod_{i=1}^{d+1} (\partial^{\varepsilon_i})^{k_i}.$$

These operators will act on functions that depend on finitely many variables Z_a so their action will not involve infinite sums, unlike what the definition of ∂^{ε_i} may suggest.

We define inductively a family $\Upsilon = (\Upsilon_p)_{1 \leq p \leq p_0}$ of functions of the variables $(Z_a)_a$, indexed by T , setting for $\tau = X^k \zeta_l \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j)$, with $a_j = (t_j, k_j)$ and $1 \leq l_j \leq p_0$

$$\begin{aligned} \Upsilon_p(\zeta_l) &:= \Upsilon_p^l \\ \Upsilon_p(\tau) &:= \left\{ \partial^k D_{a_1} \dots D_{a_n} \Upsilon_p(\zeta_l) \right\} \prod_{j=1}^n \Upsilon_{l_j}(\tau_j) \end{aligned} \quad (10)$$

for all $1 \leq p \leq p_0$. The maps Υ_p are extended to T by linearity. The following fact plays a key role in the proof of Proposition 2.2.

Lemma 2.1. *One has for all $m \in \mathbb{N}^{d+1}$ and $a = (t, q) \in (\mathfrak{T}_+ \cup \{*\}) \times \mathbb{N}^{d+1}$ the identity*

$$\sum_{r_1, \dots, r_n \in \mathbb{N}^{d+1}} \partial^{m - \sum_{j=1}^n r_j} \prod_{j=1}^n \binom{m}{r_j} D_{a_j - r_j} = \left\{ \prod_{j=1}^n D_{a_j} \right\} \partial^m. \quad (11)$$

Proof. One can check that for an element ε_i of the canonical basis of \mathbb{N}^{d+1} , one has

$$\sum_{r_1, \dots, r_n \in \mathbb{N}^{d+1}} \partial^{\varepsilon_i - \sum_{j=1}^n r_j} \prod_{j=1}^n \binom{\varepsilon_i}{r_j} D_{a_j - r_j} = \left\{ \prod_{j=1}^n D_{a_j} \right\} \partial^{\varepsilon_i}.$$

This is a consequence of the fact that

$$D_{a_j} \partial^{\varepsilon_i} = \partial_{\varepsilon_i} D_{a_j} + D_{a_j - \varepsilon_i}$$

coming from the definitions of ∂^{ε_i} and D_{a_j} . We proceed by recurrence and we suppose true for $m \in \mathbb{N}^{d+1}$. One has

$$\begin{aligned} \left\{ \prod_{j=1}^n D_{a_j} \right\} \partial^{m + \varepsilon_i} &= \sum_{r_1, \dots, r_n \in \mathbb{N}^{d+1}} \partial^{m - \sum_{j=1}^n r_j} \prod_{j=1}^n \binom{m}{r_j} D_{a_j - r_j} \partial^{\varepsilon_i} \\ &= \sum_{r_1, \bar{r}_1, \dots, r_n, \bar{r}_n \in \mathbb{N}^{d+1}} \partial^{m - \sum_{j=1}^n r_j} \prod_{j=1}^n \binom{m}{r_j} \binom{\varepsilon_i}{\bar{r}_j} D_{a_j - r_j - \bar{r}_j} \\ &= \sum_{r_1, \dots, r_n \in \mathbb{N}^{d+1}} \partial^{m - \sum_{j=1}^n r_j} \prod_{j=1}^n \binom{m + \varepsilon_i}{r_j} D_{a_j - r_j} \end{aligned}$$

where we have used the Chu-Vandermonde identity that tells us that

$$\sum_{r_j + \bar{r}_j = k_j} \binom{m}{r_j} \binom{\varepsilon_i}{\bar{r}_j} = \binom{m + \varepsilon_i}{k_j}.$$

□

The next statement is a morphism property of the map $(\tau \in T) \mapsto \Upsilon(\tau)$ for the \star product.

Proposition 2.2. *For every $1 \leq p \leq p_0$, for every $\sigma \in T$, and $\tau = X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \in T$ with $a_j = (t_j, q_j)$ and $1 \leq l_k \leq p_0$, one has*

$$\Upsilon_p \left(\left\{ X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \right\} \star \sigma \right) = \left\{ \partial^k D_{a_1} \dots D_{a_n} \Upsilon_p(\sigma) \right\} \prod_{j=1}^n \Upsilon_{l_j}(\tau_j). \quad (12)$$

We see from identity (7) that the defining inductive property (10) is a particular case of (12). In the simple case where $\tau = \mathcal{I}_{b_1}(\sigma_1)$ with $b_1 = (t_b, q_b)$ formula (12) reads

$$\Upsilon_p(\mathcal{I}_{b_1}(\sigma_1) \star \sigma) = D_{b_1} \Upsilon_p(\sigma) \Upsilon_b(\sigma_1).$$

Proof. We proceed by induction on $|\sigma|_\zeta$. It follows from Proposition 2.1 that it suffices to prove the statement for $\sigma = X^m \zeta_l$ and $\sigma = \mathcal{I}_b(\sigma_1) \curvearrowright \sigma_2$ with σ_1 and σ_2 satisfying (12) for all τ as above.

– The case $\sigma = \zeta_l$ is part of the definition (10). For $\sigma = X^m \zeta_l$, we have

$$\left\{ X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \right\} \star (X^m \zeta_l) = \zeta_l \sum_{r_1, \dots, r_n \in \mathbb{N}^{d+1}} X^{k+m-\sum_{j=1}^n r_j} \prod_{j=1}^n \binom{m}{r_j} \mathcal{I}_{a_j-r_j}(\tau_j).$$

Using Lemma 2.1 and the linearity of Υ_p one has

$$\begin{aligned} \Upsilon_p \left(\left\{ X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \right\} \star (X^m \zeta_l) \right) &= \left\{ \prod_{j=1}^n \Upsilon_{l_j}(\tau_j) \right\} \sum_{r_1, \dots, r_n \in \mathbb{N}^{d+1}} \partial^{k+m-\sum_{j=1}^n r_j} \prod_{j=1}^n \binom{m}{r_j} D_{a_j-r_j} \Upsilon_p(\zeta_l) \\ &= \left\{ \prod_{j=1}^n \Upsilon_{l_j}(\tau_j) \right\} \left\{ \partial^k \prod_{j=1}^n D_{a_j} \partial^m \Upsilon_p(\zeta_l) \right\} \\ &= \left\{ \prod_{j=1}^n \Upsilon_{l_j}(\tau_j) \right\} \left\{ \partial^k \prod_{j=1}^n D_{a_j} \Upsilon_p(X^m \zeta_l) \right\}, \end{aligned}$$

which is indeed (12) in the particular case where $\sigma = X^m \zeta_l$.

– We now assume that $\sigma = \mathcal{I}_{b_1}(\sigma_1) \curvearrowright \sigma_2 = \mathcal{I}_{b_1}(\sigma_1) \star \sigma_2$ with $b_1 = (t_b, q_b)$ and σ_1 and σ_2 satisfying (12) for all τ . Pick $\tau = X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \in T$ with $a_j = (t_{l_j}, q_j)$. One has from the associativity of \star the equality

$$\begin{aligned} \tau \star \sigma &= (\tau \star \mathcal{I}_{b_1}(\sigma_1)) \star \sigma_2 \\ &= \left\{ \sum_{I \subset \{1, \dots, n\}} \sum_{k_1+k_2=k} \binom{k}{k_1, k_2} X^{k_1} \prod_{i \in I} \mathcal{I}_{a_i}(\tau_i) \mathcal{I}_{b_1} \left(\left(X^{k_2} \prod_{j \in I^c} \mathcal{I}_{a_j}(\tau_j) \right) \star \sigma_1 \right) \right\} \star \sigma_2. \end{aligned}$$

The identity for $\tau \star \mathcal{I}_{b_1}(\sigma_1)$ comes from the fact that

$$\tau \star \mathcal{I}_{b_1}(\sigma_1) = \uparrow_{N_\sigma}^k \left(\left(\prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \right) \curvearrowright \mathcal{I}_{b_1}(\sigma_1) \right)$$

Then, one grafts some of the decorated trees τ_j at the root of σ via the term $\prod_{i \in I} \mathcal{I}_{a_i}(\tau_i)$. The rest of the decorated trees are grafted on the other nodes given. This term is given by $\left(\prod_{j \in I^c} \mathcal{I}_{a_j}(\tau_j) \right) \curvearrowright \sigma_1$. One is summing over all the possibilities of such splitting. We do the same reasoning for justifying the splitting of X^k into X^{k_1} and X^{k_2} . One has to be careful to not forget the combinatorial factor $\binom{k}{k_1, k_2}$.

We apply the induction hypothesis to get for $\Upsilon_p(\tau \star \sigma)$ the formula

$$\begin{aligned}
& \sum_{I \subset \{1, \dots, n\}} \sum_{k_1 + k_2 = k} \binom{k}{k_1, k_2} \left\{ \left(\partial^{k_1} \prod_{i \in I} D_{a_i} D_{b_1} \right) \Upsilon_p(\sigma_2) \right\} \prod_{i \in I} \Upsilon_{l_i}(\tau_i) \Upsilon_b \left((X^{k_2} \prod_{j \in I^c} \mathcal{I}_{a_j}(\tau_j)) \star \sigma_1 \right) \\
&= \sum_{\substack{I \subset \{1, \dots, n\} \\ k_1 + k_2 = k}} \binom{k}{k_1, k_2} \left\{ \left(\partial^{k_1} \prod_{i \in I} D_{a_i} D_{b_1} \right) \Upsilon_p(\sigma_2) \right\} \prod_{i \in I} \Upsilon_{l_i}(\tau_i) \left\{ \left(\partial^{k_2} \prod_{j \in I^c} D_{a_j} \right) \Upsilon_b(\sigma_1) \right\} \prod_{j \in I^c} \Upsilon_{l_j}(\tau_j) \\
&= \prod_{j=1}^n \Upsilon_{l_j}(\tau_j) \sum_{I \subset \{1, \dots, n\}} \sum_{k_1 + k_2 = k} \binom{k}{k_1, k_2} \left\{ \left(\partial^{k_1} \prod_{i \in I} D_{a_i} D_{b_1} \right) \Upsilon_p(\sigma_2) \right\} \left\{ \left(\partial^{k_2} \prod_{j \in I^c} D_{a_j} \right) \Upsilon_b(\sigma_1) \right\}.
\end{aligned}$$

Using the fact that ∂^k and D_{a_i} satisfy the Leibniz rule one then gets

$$\begin{aligned}
\Upsilon_p(\tau \star \sigma) &= \prod_{j=1}^n \Upsilon_{l_j}(\tau_j) \left\{ \partial^k \prod_{i=1}^n D_{a_i} \right\} \left(\Upsilon_b(\sigma_1) D_{b_1} \Upsilon_p(\sigma_2) \right) \\
&= \prod_{j=1}^n \Upsilon_{l_j}(\tau_j) \left\{ \partial^k \prod_{i=1}^n D_{a_i} \right\} \Upsilon_p(\mathcal{I}_{b_1}(\sigma_1) \curvearrowright \sigma_2) \\
&= \prod_{j=1}^n \Upsilon_{l_j}(\tau_j) \left\{ \partial^k \prod_{i=1}^n D_{a_i} \right\} \Upsilon_p(\sigma),
\end{aligned}$$

as we have from the induction assumption that σ_2 satisfies the conclusion of the proposition, so $\Upsilon_p(\mathcal{I}_{b_1}(\sigma_1) \curvearrowright \sigma_2) = \Upsilon_b(\sigma_1) D_{b_1} \Upsilon_p(\sigma_2)$. This concludes the proof. \square

Lemma 2.1 was first noticed in the proof of Proposition 33 in Bailleul and Hoshino's work [8]. It lead the authors to a simple proof of the fact that for $a = (\mathfrak{t}_j, p_a)$ and all p

$$\Upsilon_p(\mathcal{I}_a(\sigma) \curvearrowright \tau) = \Upsilon_j(\sigma) D_a \Upsilon_p(\tau);$$

a special case of (12). This was a huge simplification in comparison to the original proof given in Bruned, Chandra, Chevyrev & Hairer's work [14], where the authors had to go through an extended space of rooted trees in Section 4 therein. Identity (12) was observed in the simpler context of rough differential equations in Lemma 3.4 of Bonnefoi, Chandra, Moinat & Weber's work [10]. The \star product happens to be the adjoint of the Butcher-Connes-Kreimer coproduct in that setting.

3. PREPARATION MAPS AND THEIR ASSOCIATED MODELS

Preparation maps were introduced by Bruned in [11] as the building block of a family of renormalization procedures encompassing the BHZ renormalization procedure from [18]. Their dual action on the equation was not investigated so far. We introduce in the present section a notion of strong preparation map characterized by a morphism property for the \star product.

3.1. Strong preparation maps. For $\tau \in T$ denote by $|\tau|_\zeta$ the number of noise symbols that appear in τ . Recall from [11] the following definition.

Definition 3.1. A **preparation map** is a linear map $R : T \rightarrow T$ such that $R(X^k \tau) = X^k R(\tau)$, for all $k \in \mathbb{N}^{d+1}$, $\tau \in T$, and $R\mathcal{I}_a = \mathcal{I}_a$ for all a , and

$$(R \otimes \text{Id}) \Delta = \Delta R. \tag{13}$$

and for each $\tau \in T$ there exist finitely many $\tau_i \in T$ and constants λ_i such that one has

$$R(\tau) = \tau + \sum_i \lambda_i \tau_i, \quad \text{with } \deg(\tau_i) \geq \deg(\tau) \quad \text{and} \quad |\tau_i|_\zeta < |\tau|_\zeta. \quad (14)$$

Preparation maps act non-trivially on trees with multiple edges at the root. This accounts for the fact that such trees represent products of analytical quantities, some of which need to be renormalized to be given sense. The ‘deformed product’ provided by $R(\tau)$ for such trees τ makes precisely that. Preparation maps were named for that reason ‘local product renormalization maps’ in Chandra, Moinat & Weber’s work [23] on the $\phi_{4-\delta}^4$ equation, as well as in Bruned’s work [12] on the renormalization of branched rough paths. A preparation map is in particular a perturbation of the identity by elements that are not more singular ($\deg(\tau_i) \geq \deg(\tau)$) and defined with strictly less noises ($|\tau_i|_\zeta < |\tau|_\zeta$). Note that the linear map $R - \text{Id}$ is nilpotent as a consequence of the condition (14). Recall that the recentering operators Γ_{yx} in a regularity structure are given by some maps of the form $\Gamma_{yx} = (\text{Id} \otimes \mathbf{g}_{yx})\Delta$, for some character \mathbf{g}_{yx} on T^+ and arbitrary points x, y in the state space. Identity (13) encodes the fact that the recentering operators and the preparation map commute

$$\Gamma_{zy}R = R\Gamma_{zy}$$

for all z, y since

$$\Gamma_{zy}(R\tau) = (\text{Id} \otimes \mathbf{g}_{zy})\Delta R(\tau) = (\text{Id} \otimes \mathbf{g}_{zy})(R \otimes \text{Id})\Delta\tau = (R \otimes \mathbf{g}_{zy})\Delta\tau = R\Gamma_{zy}(\tau).$$

The next statement is a direct consequence of the duality relation (8) between the product \star and the splitting map Δ .

Proposition 3.1. *Identity (13) is equivalent to having*

$$R^*(\sigma \star \tau) = \sigma \star (R^*(\tau)) \quad (15)$$

for all $\sigma \in T^+$ and $\tau \in T$.

Proof. We use the duality relation (8) between Δ and the restriction to $T \otimes T^+$ of the \star product to write for $\mu, \nu \in T$ and $\sigma \in T^+$

$$\langle \mu \otimes \sigma, \Delta R\nu \rangle = \langle \sigma \star \mu, R\nu \rangle = \langle R^*(\sigma \star \mu), \nu \rangle.$$

The result is thus a consequence of the identity

$$\langle \mu \otimes \sigma, (R \otimes \text{Id}) \Delta\nu \rangle = \langle (R^*(\mu)) \otimes \sigma, \Delta\nu \rangle = \langle \sigma \star (R^*(\mu)), \nu \rangle.$$

□

Definition 3.2. A **strong preparation map** is a preparation map satisfying the identity (15) for all $\sigma \in T$ and $\tau \in T$ – and not only for $\sigma \in T^+$.

Taking some specific σ in (15) for strong preparation maps yields some special identities. For $\sigma = \mathcal{I}_a(\sigma_1)$ the identity (15) reads

$$R^*(\mathcal{I}_a(\sigma_1) \curvearrowright \tau) = \mathcal{I}_a(\sigma_1) \curvearrowright (R^*(\tau)),$$

that is

$$R^*(\sigma_1 \curvearrowright_a \tau) = \sigma_1 \curvearrowright_a (R^*(\tau)) \quad (16)$$

Another interesting case is when σ is equal to X^k . In that case one has for all $k \in \mathbb{N}^{d+1}$ and $\tau \in T$

$$R^*(X^k \star \tau) = (X^k \star R^*\tau) = \uparrow^k R^*\tau \quad (17)$$

Note that the universality property of T stated in Proposition 2.1 implies that the identities (16) and (17) characterize the map R^* once its values on the generators $X^k \zeta_l$ are given.

Denote by \mathcal{B}^- the canonical basis of T^- and recall from [18] or [8] that $\mathbb{R}[T^-]$ is equipped with a Hopf algebra structure. We follow [11] and define for any character ℓ of $\mathbb{R}[T^-]$ and all $\sigma \in T$

$$R_\ell^*(\sigma) := \sum_{\tau \in \mathcal{B}^-} \frac{\ell(\tau)}{S(\tau)} (\sigma \star \tau). \quad (18)$$

This definition corresponds to the dual of its usual definition

$$R_\ell(\mu) = \sum_{\tau \in \mathcal{B}^-, \sigma \in \mathcal{B}} \frac{\ell(\tau)}{S(\tau)} (\tau \otimes \sigma, \delta_r \mu) \sigma,$$

where the map δ_r can be defined from the duality relation $\langle \mu, R_\ell^*(\sigma) \rangle = \langle R_\ell(\mu), \sigma \rangle$.

Proposition 3.2. *Assume that ℓ is null on planted trees and trees of T^- of the form $X^k \sigma$. The map R_ℓ is a strong preparation map.*

Proof. From definition (18) one has for any $\mu, \tau \in T$

$$R_\ell^*(\mu \star \tau) = \sum_{\sigma \in \mathcal{B}^-} \frac{\ell(\sigma)}{S(\sigma)} (\mu \star \tau) \star \sigma.$$

By using the associativity of \star one gets

$$R_\ell^*(\mu \star \tau) = \sum_{\sigma \in \mathcal{B}^-} \frac{\ell(\sigma)}{S(\sigma)} \mu \star (\tau \star \sigma) = \mu \star R_\ell^*(\tau),$$

so Proposition 3.1 ensures that the commutation condition (13) is in particular satisfied. The condition $\deg(\tau_i) \geq \deg(\tau)$ comes from the fact that we are summing over decorated trees with negative degree in the definition of R_ℓ^* . The condition on $|\cdot|_\zeta$ is a consequence of the fact that the δ_r map breaks any decorated tree into a sum of tensor products of trees with a strictly smaller number of noises. \square

The BPHZ renormalization map from [18, 22] is a map built from R_ℓ for a particular choice of character ℓ on T^- . Although elementary the next statement will play a crucial role in the proof of our main result, Theorem 1.1, in the next section.

Proposition 3.3. *Let R be a strong preparation map. For every $1 \leq p \leq p_0$, for every $\tau = X^k \zeta_l \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j)$ with $a_j = (\mathfrak{t}_+^j, p_j) \in \mathfrak{T}^+ \times \mathbb{N}^{d+1}$, one has*

$$\Upsilon_p(R^*(\tau)) = \left\{ \partial^k D_{a_1} \dots D_{a_n} \Upsilon_p(R^*(\zeta_l)) \right\} \prod_{j=1}^n \Upsilon_{l_j}(\tau_j)$$

Proof. Writing $\tau = \left(X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \right) \star \zeta_l$ and using the property (15) one gets

$$R^* \left(\left(X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \right) \star \zeta_l \right) = \left(X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \right) \star (R^*(\zeta_l)).$$

Identity (12) in Proposition 2.2 then yields

$$\Upsilon_p \left(X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j) \star (R^*(\zeta_l)) \right) = \left\{ \partial^k D_{a_1} \dots D_{a_n} (\Upsilon_p(R^*(\zeta_l))) \right\} \prod_{j=1}^n \Upsilon_{l_j}(\tau_j).$$

\square

3.2. Models associated with a preparation map. We recall now from [11] the construction of a continuous admissible model associated with a preparation map. This allows to emphasize in (20) and (21) below a fundamental feature of its associated reconstruction map.

Assume we are given some kernels $(K^p)_{1 \leq p \leq p_0}$ with a polynomial singularity at 0 satisfying Assumption (5.1) in Hairer's seminal work [33], and some continuous noises $(\xi_l)_{1 \leq l \leq l_0}$ on the state space. Following [11] one can associate to a preparation map R an admissible model \mathbf{M}^R on T . It is defined from a side family $((\Pi_x^{R \times} \tau)(\cdot))_{x, \tau}$ of continuous functions on the state space satisfying

$$(\Pi_x^{R \times} \mathbf{1})(y) = 1, \quad (\Pi_x^{R \times} \zeta_l)(y) = \xi_l(y), \quad (\Pi_x^{R \times} X_i)(y) = y_i - x_i,$$

the multiplicativity condition

$$(\Pi_x^{R \times} (\sigma \tau))(y) = (\Pi_x^{R \times} \sigma)(y) (\Pi_x^{R \times} \tau)(y),$$

and the condition

$$\begin{aligned} (\Pi_x^{R \times} \mathcal{I}_a(\tau))(y) &= (D^q K^p * \Pi_x^{R \times}(R\tau))(y) \\ &\quad - \sum_{|r|_s \leq \deg(\mathcal{I}_a \tau)} \frac{(y-x)^r}{r!} (D^{q+r} K^p * \Pi_x^{R \times}(R\tau))(x), \end{aligned} \quad (19)$$

for $a = (\mathfrak{t}_+^p, q)$. Define for all x and τ a continuous function on the state space

$$(\Pi_x^R \tau)(\cdot) := (\Pi_x^{R \times}(R\tau))(\cdot).$$

Bruned gave in Proposition 3.16 of [11] an explicit construction of an R -dependent admissible model $\mathbf{M}^R = (\Pi, \mathfrak{g}) = (\Pi^R, \mathfrak{g}^R)$ on T , with values in the space of continuous functions, such that the operators Π_x^R are indeed associated with the model \mathbf{M}^R in the sense that one has for all $\tau \in T$ and x

$$\Pi_x^R \tau = (\Pi^R \otimes (\mathfrak{g}_x^R)^{-1}) \Delta \tau.$$

(You will find some details in Section 5.2 in a more general setting.) Since the model \mathbf{M}^R takes values in the space of continuous functions its associated reconstruction operator $\mathbf{R}^{\mathbf{M}^R}$ is given by the explicit formula

$$(\mathbf{R}^{\mathbf{M}^R} \mathfrak{v})(x) = (\Pi_x^{\mathfrak{g}} \mathfrak{v}(x))(x)$$

for any modelled distribution \mathfrak{v} with positive regularity, so

$$(\mathbf{R}^{\mathbf{M}^R} \mathfrak{v})(x) = (\Pi_x^{R \times}(R\mathfrak{v}(x)))(x). \quad (20)$$

Since $R\mathcal{I}_a = \mathcal{I}_a$ one has in particular

$$\Pi_x^R(\mathcal{I}_a \tau) = \Pi_x^{R \times}(\mathcal{I}_a \tau).$$

The point here is that $\Pi_x^{R \times}$ is multiplicative while Π_x^R is not multiplicative. Denote by $T_X \subset T$ the linear space spanned by the polynomials in T . Modelled distribution \mathfrak{u} with values in the subspace $\mathcal{I}(T) \oplus T_X$ of T satisfy in that case the identity

$$(\mathbf{R}^{\mathbf{M}^R} \mathfrak{u})(x) = (\Pi_x^{R \times} \mathfrak{u}(x))(x). \quad (21)$$

It follows further from the relation (19) that the model \mathbf{M} is admissible.

4. A SHORT PROOF FOR THE RENORMALIZED EQUATION FOR TRANSLATION INVARIANT OPERATORS

For $\gamma > 0$ denote by \mathcal{Q}_γ the natural projection from T to the linear subspace $T_{<\gamma}$ of elements of T of degree less than γ . Recall that for a smooth function G , with derivatives $G^{(k)}$, and a modelled function-like distribution \mathbf{v} of regularity γ the formula

$$\widehat{G}(\mathbf{v}) = \mathcal{Q}_\gamma \sum_{k \in \mathbb{N}^{d+1}} \frac{G^{(k)}(\langle \mathbf{v}, \mathbf{1} \rangle)}{k!} (\mathbf{v} - \langle \mathbf{v}, \mathbf{1} \rangle \mathbf{1})^k$$

defines a modelled distribution $\widehat{G}(\mathbf{v})$ of regularity γ . This section contains the statement and proof of a particular case of our main result, Theorem 1.1 when the operators \mathcal{L}^p are translation invariant. Theorem 4.1 below describes the autonomous dynamics satisfied by $\mathbf{R}^{\mathbf{M}^R}(\mathbf{u}^R)$ when \mathbf{M}^R is the model constructed from a preparation map R and \mathbf{u}^R is the solution to the lift of system (1) to its associated regularity structure

$$\mathbf{u}_p^R = \mathcal{K}_p^{\mathbf{M}^R} \left(\mathcal{Q}_{\gamma-2}(\widehat{F}_p(\mathbf{u}^R, D\mathbf{u}^R)\zeta) \right) + \mathcal{P}_{p,\gamma}(u_i(0)), \quad (22)$$

where $1 \leq p \leq p_0$. The real number γ is here strictly bigger than $3/2$. The notation $\widehat{F}_p(\mathbf{u}^R, D\mathbf{u}^R)\zeta$ is a shorthand notation for $\widehat{f}_p(\mathbf{u}^R)\zeta + \widehat{g}_p(\mathbf{u}^R)(D\mathbf{u}^R)^2$. The operator $\mathcal{P}_{p,\gamma}$ stands here for the projection on the subspace of elements of degree less than γ of the natural lift in the polynomial regularity structure T_X of the map $x = (x_0, x_1) \mapsto e^{-x_0 \mathcal{L}^p}(u(0))(x_1)$. The operator $\mathcal{Q}_{\gamma-2}$ is the natural projection from T to the linear subspace $T_{<\gamma-2}$ of elements of T of degree less than $\gamma - 2$. The \mathbf{M}^R -dependent maps $\mathcal{K}_p^{\mathbf{M}^R}$ are the regularity structure lifts of the operators $(\partial_{x_0} - \mathcal{L}^p)^{-1}$. From the definition of the operators $\mathcal{K}_p^{\mathbf{M}^R}$, for a solution $\mathbf{u}^R = (\mathbf{u}_1^R, \dots, \mathbf{u}_{p_0}^R)$ of the system (22) each map \mathbf{u}_p^R takes values in $\mathcal{I}_{(\mathfrak{t}_+^p, 0)}(T) \oplus T_X$. Write

$$\mathbf{u}_p^R =: \sum u_{p,\sigma} \sigma$$

for a sum over trees σ in the canonical basis of $\mathcal{I}_{(\mathfrak{t}_+^p, 0)}(T) \oplus T_X$ – monomials are seen as trees with just one vertex.

One can get a general expansion for the derivatives of \mathbf{u}_p^R as

$$\mathbf{u}_a^R = \sum u_{p,\sigma} D_a \sigma = \sum u_{a,\sigma} \sigma$$

where $a = (\mathfrak{t}_+^p, q)$. Then, the variable Z_a introduced in (9) corresponds to

$$u_{a,X^k} = \langle \mathbf{u}_a^R, X^k \rangle.$$

It follows from [14, Lem. 3.21] that

$$\mathcal{Q}_{\gamma-2}(\widehat{F}_p(\mathbf{u}^R, D\mathbf{u}^R)\zeta) = \sum_{\deg(\tau) < \gamma-2} \frac{\Upsilon_p(\tau)(u, \partial_{x_1} u)}{S(\tau)} \tau;$$

a fact called *coherence*. In fact, one has for $\sigma \in \mathcal{I}_{(\mathfrak{t}_+^p, 0)}(T)$

$$u_{a,\sigma} = u_{p,\sigma} = \frac{\Upsilon_p(\tau)(u, \partial_{x_1} u)}{S(\tau)}$$

and

$$u_{a,X^k} = u_{p,X^{k+q}}.$$

Set

$$u := \mathbf{R}^{\mathbf{M}^R}(\mathbf{u}^R) = (u_1, \dots, u_{p_0}).$$

Theorem 4.1. *Let $\xi = (\xi_1, \dots, \xi_{l_0})$ stand for a continuous function. Let R be a strong preparation map with associated admissible model \mathbb{R}^R . Then u is a solution of the renormalized p_0 -dimensional system*

$$(\partial_t - \mathcal{L}^p)u_p = f_p(u)\xi + g_p(u)(\partial_{x_1}u)^2 + \sum_{l=0}^{l_0} \Upsilon_p\left((R^* - \text{Id})\zeta_l\right)(u, \nabla u)\xi_l. \quad (23)$$

Proof. As we are working with an admissible model we have

$$\begin{aligned} (\partial_t - \mathcal{L}^p)u_p &= \mathbb{R}^{\mathbb{R}^R}\left(\mathcal{Q}_{\gamma-2}(\widehat{F}_p(u^R, Du^R)\zeta)\right) \\ &= \mathbb{R}^{\mathbb{R}^R}\left(\sum_{\deg(\tau) < \gamma-2} \frac{\Upsilon_p(\tau)(u, \partial_{x_1}u)}{S(\tau)} \tau\right) =: \mathbb{R}^{\mathbb{R}^R}(v_p) \end{aligned}$$

for a sum over the canonical basis of T , from the coherence condition (4). One has by construction

$$\mathbb{R}^{\mathbb{R}^R}(v_p)(x) = \left(\Pi_x^{\mathbb{R}^R \times} (Rv_p(x))\right)(x)$$

and

$$Rv_p = \sum_{\deg(\tau) < \gamma-2} \langle Rv_p, \tau \rangle \tau = \sum_{\deg(\tau) < \gamma-2} \langle v_p, R^*\tau \rangle \tau = \sum_{\deg(\tau) < \gamma-2} \frac{\Upsilon_p(R^*\tau)(u, \partial_{x_1}u)}{S(\tau)} \tau.$$

Writing each τ in its canonical form $\tau = X^k \zeta_l \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j)$, with $a_j = (\mathfrak{t}_+^{l_j}, k_j)$, and using the morphism property

$$\Upsilon_p(R^*\tau)(u, \partial_{x_1}u) = \left\{ \partial^k D_{a_1} \cdots D_{a_n} \Upsilon_p(R^*\zeta_l)(u, \partial_{x_1}u) \right\} \prod_{j=1}^n \Upsilon_{l_j}(\tau_j)(u, \partial_{x_1}u)$$

of the Υ_p maps obtained in Proposition 3.3 one gets

$$\begin{aligned} Rv_p &= \sum_{\sigma = X^k \zeta_l \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j)} \frac{k! \prod_{j=1}^n S(\tau_j)}{S(X^k \zeta_l \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j))} \frac{X^k}{k!} \prod_{j=1}^n \frac{\Upsilon_{l_j}(\tau_j)(u, \partial_{x_1}u)}{S(\tau_j)} \mathcal{I}_{a_j}(\tau_j) \\ &\quad \left\{ \partial^k \prod_{j=1}^n D_{a_j} \Upsilon_p(R^*\zeta_l)(u, \partial_{x_1}u) \zeta_l \right\}, \end{aligned}$$

that is Rv_p is equal to

$$\sum_{\sigma = X^k \zeta_l \prod_{j=1}^m \mathcal{I}_{a_j}(\tau_j)^{\beta_j}} \frac{X^k}{k!} \prod_{j=1}^m \frac{1}{\beta_j!} \left(\frac{\Upsilon_{l_j}(\tau_j)(u, \partial_{x_1}u)}{S(\tau_j)} \mathcal{I}_{a_j}(\tau_j) \right)^{\beta_j} \left\{ \partial^k \prod_{j=1}^m (D_{a_j})^{\beta_j} \Upsilon_p(R^*\zeta_l)(u, \partial_{x_1}u) \zeta_l \right\}$$

for some integer $m \leq n$, grouping together the terms that coincide and using the recursive definition of the symmetry factor $S(\cdot)$. Recall now that if G stands for a function of some variables Z_{a_1}, \dots, Z_{a_r} that are a qfinite subset of the Z_a given in (9). The Faà di Bruno formula from Lemma A.1 in [14] states that

$$\frac{\partial^k G}{k!} = \sum_{a_1, \dots, a_r} \sum_{k = \sum_{j=1}^r \beta_j k_j} \prod_{j=1}^r \frac{1}{\beta_j!} \left(\frac{Z_{a_j + k_j}}{k_j!} \right)^{\beta_j} \prod_{j=1}^r (D_{b_j})^{\beta_j} G.$$

where the k_j in the previous sum are non-zero. We can apply this identity to the functions $G = \Upsilon_p(R^* \zeta_l)$ of the variables $Z_{b_i} = \langle \mathbf{u}_{b_i}^R, \mathbf{1} \rangle$. One obtains

$$\begin{aligned} R\mathbf{v}_p &= \sum_{\sigma = X^k \zeta_l} \sum_{\prod_{j=1}^m \mathcal{I}_{a_j}(\tau_j)^{\beta_j} \mathbf{a}_{m+1}, \dots, \mathbf{a}_{m+r}} \sum_{k = \sum_{j=m+1}^{m+r} \beta_j k_j} \frac{X^k}{k!} \prod_{j=1}^m \frac{1}{\beta_j!} \left(\frac{\Upsilon_{l_j}(\tau_j)(u, \partial_{x_1} u)}{S(\tau_j)} \mathcal{I}_{a_j}(\tau_j) \right)^{\beta_j} \\ &= \prod_{j=m+1}^r \frac{1}{\beta_j!} \left(\frac{\langle \mathbf{u}_{a_j}^R, X^{k_j} \rangle}{k_j!} X^{k_j} \right)^{\beta_j} \left\{ \prod_{j=1}^{m+m'} (D_{a_j})^{\beta_j} \Upsilon_p(R^* \zeta_l)(u, \partial_{x_1} u) \zeta_l \right\}. \end{aligned}$$

We notice that for $a = (\mathbf{t}_+, k)$ one has

$$\mathbf{u}_a^R - \langle \mathbf{u}_a^R, \mathbf{1} \rangle = \sum_{k \neq 0} \frac{\langle \mathbf{u}_a^R, X^k \rangle}{k!} X^k + \sum_{\tau} \frac{\Upsilon_l(\tau)(u, \partial_{x_1} u)}{S(\tau)} \mathcal{I}_{a_j}(\tau) \tau$$

Using the binomial identity, one gets

$$\begin{aligned} R\mathbf{v}_p &= \sum_{l, n} \sum_{\substack{a_1, \dots, a_n \\ \beta_1, \dots, \beta_n}} \prod_{j=1}^n \frac{1}{\beta_j!} \left(\mathbf{u}_{a_j}^R - \langle \mathbf{u}_{a_j}^R, \mathbf{1} \rangle \right)^{\beta_j} \prod_{j=1}^n (D_{a_j})^{\beta_j} \Upsilon_p(R^* \zeta_l)(u, \partial_{x_1} u) \zeta_l \\ &= \sum_{l=0}^{l_0} \Upsilon_p(\widehat{R^* \zeta_l})(\mathbf{u}^R, D\mathbf{u}^R) \zeta_l. \end{aligned}$$

where $\Upsilon_p(\widehat{R^* \zeta_l})$ has to be understood in the sense of (4). Using the multiplicativity of $\Pi_x^{M^R}$ and identity (21) identifying $\mathbf{R}^{M^R}(\mathbf{u}^R)(x)$ we obtain

$$\begin{aligned} ((\partial_{x_0} - \mathcal{L}^p)u_p)(x) &= \mathbf{R}^{M^R}(\mathbf{v}_p)(x) = \Pi_x^{R \times} (R\mathbf{v}_p)(x)(x) \\ &= \sum_{l=0}^{l_0} \Pi_x^{R \times} \left(\Upsilon_p(\widehat{R^* \zeta_l})(\mathbf{u}^R(x), D\mathbf{u}^R(x)) \zeta_l \right)(x) \\ &= \sum_{l=0}^{l_0} \Upsilon_p(R^* \zeta_l) \left((\Pi_x^{R \times} \mathbf{u}^R(x))(x), \partial_{x_1} (\Pi_x^{R \times} \mathbf{u}^R(x))(x) \right) \Pi_x^{R \times} \zeta_l \\ &= \sum_{l=0}^{l_0} \Upsilon_p(R^* \zeta_l)(u(x), \partial_{x_1} u(x)) \xi_l. \end{aligned}$$

This identity is equivalent to (23). \square

We invite the reader to note right now that the above proof works verbatim if R were x -dependent and we knew that there is indeed a model \mathbf{M}^R with reconstruction operator \mathbf{R}^{M^R} and some multiplicative maps $\Pi_x^{R \times}$ such that

$$\mathbf{R}^{M^R}(\mathbf{v})(x) = \Pi_x^{R \times} (R(x)\mathbf{v}(x))(x) \quad (24)$$

for all modelled distribution \mathbf{v} . We will see in Section 5 that this is indeed the case.

In the particular case where the preparation map R is of BPHZ form (18) a direct computation shows that the renormalized system (23) takes the form

$$(\partial_t - \mathcal{L}^p)u_p = f_p(u)\xi + g_p(u)(\partial_{x_1} u)^2 + \sum_{\tau \in \mathcal{B}^- \setminus \{\mathbf{1}\}} \ell(\tau) \frac{\Upsilon_p(\tau)(u, \partial_{x_1} u)}{S(\tau)}$$

if we further assume that $R_\ell^*(\zeta_l) = \zeta_l$ for all $l \neq 0$ – recall $\zeta_0 = \mathbf{1}$. This assumption accounts for the fact that we never need to subtract a multiple of one of the noises $(\zeta_l)_{1 \leq l \leq l_0}$ to any tree-indexed quantity in our renormalization algorithm. We note that Chandra, Moinat & Weber used in [23] a similar strategy to get back the renormalized equation in the particular case of the $\Phi_{4-\delta}^4$ equation.

Remarks – 1. *The pattern of proof of the renormalized equation devised in [14] has its roots in Bruned, Chevyrev, Friz & Preiss’ work [15] and rests heavily on the formalism and properties of multi-pre-Lie algebras. The renormalized equation is obtained from the fact that one can associate to a BHZ preparation map R_ℓ a renormalization map M_ℓ whose dual has a pre-Lie morphism property. Writing Π_x for the canonical interpretation map associated with a continuous noise, and Π_x^ℓ for the interpretation map of the renormalized model, the core of the mechanics of [14] boils down to the identity*

$$\begin{aligned} \Pi_x^\ell(v_p) &= \sum_{\deg(\tau) < \gamma - 2} \frac{\Upsilon_p(\tau)(u, \partial_{x_1} u)}{S(\tau)} (\Pi_x M_\ell \tau) \\ &= \sum_{\deg(\tau) < \gamma - 2} \frac{\Upsilon_p^{M_\ell}(\tau)(u, \partial_{x_1} u)}{S(\tau)} (\Pi_x \tau) \end{aligned}$$

in which the functions $\Upsilon_p^{M_\ell}$ are constructed from the pre-Lie property. This pattern of proof cannot work in a non-translation invariant setting as one cannot build some renormalized models for which $\Pi_x^R = \Pi_x \circ M_R(x)$.

2. *The proof of Theorem 4.1 has been used in many places as it provides a shorter proof than the original proof given in [14, Thm 2.2, Eq (2.21)]. This proof is very robust when one considers a different framework. For example, this result has been used for quasilinear equations in [9, Prop. 24] and [17, Thm 3.4], where a variant of the proof is given with non-commutative derivatives. In a geometric context Hairer & Singh provide a proof of the renormalised equation in [37, Prop. 14.9] which follows the main steps of the present proof. Ideas of this proof were also used for composition and substitution for Regularity Structures B-series [13] and Multi-indices B-series [16].*

5. RENORMALIZATION SCHEMES AND RENORMALIZED EQUATION IN A NON-TRANSLATION INVARIANT SETTING

5.1. The analytic setting for non-translation invariant operators. Recall the operators

$$\mathcal{L}^p v := a_p(\cdot) \partial_{x_1}^2 v$$

have some smooth coefficients. The analysis of the regularity structure lift of the resolvent operator $(\partial_t - \Delta)^{-1}$ done by Hairer in Section 5 of [33] rests entirely on a notion of β -regularizing singular kernel whose definition takes profit of the translation-invariance of the heat operator – see e.g. Assumption 5.1 in Section 5 of [33]. A slightly different approach is needed in a translation invariant setting to build the resolvent of the operator $\partial_t - \mathcal{L}^p$ and one can for instance follow the alternative heat kernel approach of Bailleul & Hoshino’s ‘*Tourist Guide*’ [8] which is fully developed in Hoshino’s work [38].

Define the non-positive symmetric fourth order elliptic differential operator on $\mathbb{R} \times \mathbb{R}$

$$\mathcal{G}^p := \partial_{x_0}^2 - (\mathcal{L}^p)^2,$$

and denote by $K^p(t, x, x')$ the fundamental solution of the operator $\partial_t - \mathcal{G}^p$. We learn from Theorem 5.12 in [38] that all we need to build a setting where to prove a version

of the multilevel Schauder estimates is that K^p satisfies some (anisotrope) Gaussian type estimates. These estimates ensure in particular that one has the uniform estimate

$$\int_{\mathbb{R}^2} |\partial_x^n K^p(t, x, x')| d(x, x')^c dx \lesssim t^{\frac{c-|n|_6}{4}} \quad (\forall x' \in \mathbb{R}^2, n \in \mathbb{N} \times \mathbb{N}, c \in \mathbb{R}_+). \quad (25)$$

This estimate gives in some technical computations (e.g. in Section 5 of [38]) the estimates provided by Hairer's decompositions of the β -regularizing singular kernels as a sum of kernels with support in some annuli. The fact that the heat kernel of \mathcal{G}^p satisfies the above mentioned Gaussian type estimate is folklore in the PDE litterature. One can find all the details fully spelled out in Appendix A of Bailleul, Hoshino & Kusuoka's work [9].

5.2. State space dependent preparation maps and their associated models. Preparation maps can be state space dependent. We define this notion in this section and show that one can associate to such preparation maps some admissible models when the noise ξ is continuous. They satisfy the relation (24) so it follows from the proof of Theorem 4.1 and the remark following it that the renormalized equation associated with system (1) is indeed given by Theorem 1.1.

Definition 5.1. A (state space dependent) **preparation map** is a map

$$R : (\mathbb{R} \times \mathbf{T}) \times T \rightarrow T$$

such that all the maps $R(x, \cdot)$ fix the polynomials and $R(x, \cdot)\mathcal{I}_a = \mathcal{I}_a$ for all a , and such that

- for each $\tau \in T$ there exist finitely many $\tau_i \in T$ and some continuous real-valued bounded functions λ_i on $\mathbb{R} \times \mathbf{T}$ such that for every $x \in \mathbb{R} \times \mathbf{T}$

$$R(x, \tau) = \tau + \sum_i \lambda_i(x) \tau_i, \quad \text{with } \deg(\tau_i) \geq \deg(\tau) \quad \text{and} \quad |\tau_i|_\zeta < |\tau|_\zeta,$$

- one has for all $x \in \mathbb{R} \times \mathbf{T}$

$$(R(x, \cdot) \otimes \text{Id})\Delta = \Delta R(x, \cdot).$$

A **strong preparation map** is a preparation map satisfying

$$R(x, \cdot)^* (\sigma \star \tau) = \sigma \star (R(x, \cdot)^* \tau)$$

for all $\sigma \in T$ and $\tau \in T$ and all $x \in \mathbb{R} \times \mathbf{T}$.

As an example one sees from Proposition 3.2 that for any x -dependent character $\ell(x, \cdot)$ of the algebra T^- that is null on planted trees and trees of the form $X^k \sigma$ the map

$$R_\ell(x) := (\ell(x, \cdot) \otimes \text{Id})\delta_r$$

is a strong preparation map. Recall that the co-action $\Delta : T \rightarrow T \otimes T^+$ satisfies in our setting the induction relation

$$\Delta(\bullet) := \bullet \otimes \mathbf{1}, \quad \text{for } \bullet \in \{\mathbf{1}, X_i, \zeta\},$$

$$\Delta(\mathcal{I}_a \tau) := (\mathcal{I}_a \otimes \text{Id})\Delta + \sum_{|k+m|_s < \deg(\mathcal{I}_a \tau)} \frac{X^k}{k!} \otimes \frac{X^m}{m!} \mathcal{I}_{a+k+m}^+(\tau).$$

Denote by S^+ the antipode of the coproduct $\Delta^+ : T^+ \rightarrow T^+ \otimes T^+$ and let ξ stand for a continuous noise. Set

$$\Pi^R(\zeta) = \Pi^{R \times}(\zeta) = \xi,$$

and define inductively the maps Π^R and $\Pi^{R \times}$ by the relations

$$\Pi^R = \Pi^{R \times} R, \quad \Pi^{R \times}(\tau \bar{\tau}) = (\Pi^{R \times} \tau)(\Pi^{R \times} \bar{\tau}), \quad \Pi^{R \times}(\mathcal{I}_a \tau) = D^a K * (\Pi^R \tau).$$

It follows from this definition and the fact that $R\mathcal{I}_a = \mathcal{I}_a$ that the map Π^R satisfies the admissibility condition $\Pi^R(\mathcal{I}_a\tau) = D^a K * (\Pi^R\tau)$. Set

$$\Pi_x^R \tau = (\Pi^R \otimes (\mathfrak{g}_x^R)^{-1}) \Delta$$

with

$$(\mathfrak{g}_x^R)^{-1}(\mathcal{I}_a^+ \tau) := -(D^a K * \Pi_x^R \tau)(x)$$

and

$$\widehat{\mathfrak{g}}_{yx}^R := (\text{Id} \otimes \mathfrak{g}_{yx}^R) \Delta$$

and

$$\mathfrak{g}_{yx}^R := \left((\mathfrak{g}_y^R)^{-1} \circ S^+ \otimes (\mathfrak{g}_x^R)^{-1} \right) \Delta^+.$$

An induction on $\deg(\tau)$ shows that Π_x^R is the solution of the induction relation

$$\begin{aligned} (\Pi_x^R \tau)(y) &= \left(\Pi_x^{R \times} (R(y)\tau) \right)(y), \\ \Pi_x^{R \times} (\tau \bar{\tau}) &= (\Pi_x^{R \times} \tau) (\Pi_x^{R \times} \bar{\tau}), \\ (\Pi_x^{R \times} (\mathcal{I}_a \tau))(y) &= (D^q K^p * \Pi_x^R \tau)(y) - \sum_{|k|_s \leq \deg(\mathcal{I}_a \tau)} \frac{(y-x)^k}{k!} (D^{q+k} K^p * \Pi_x^R \tau)(x), \end{aligned}$$

for $a = (t_+^p, q)$. This was done in Section 3.2 of [11] in the case where the preparation map R does not depend on its \mathbb{R}^2 argument. The proof carries over verbatim to the present setting. It turns out to be useful to have the following recursive identity.

Lemma 5.2. *One has the identity*

$$\widehat{\mathfrak{g}}_{yx}^R(\mathcal{I}_a(\sigma)) = \mathcal{I}_a(\widehat{\mathfrak{g}}_{yx}^R(\sigma)) - \sum_{|k|_s < \deg(\mathcal{I}_a(\sigma))} \frac{(X+x-y)^k}{k!} \Pi_x^R(\mathcal{I}_{a+k}(\widehat{\mathfrak{g}}_{yx}^R(\sigma)))(y).$$

Its proof follows the same lines as the proof of Lemma 12 in the authors' previous work [5].

Proposition 5.1. *The pair $M^R = (\mathfrak{g}^R, \Pi^R)$ defines a continuous admissible model.*

Proof. We proceed in two steps.

1. We prove that

$$|(\Pi_x^R \tau)(y)| \lesssim |y-x|_s^{\deg(\tau)}$$

by induction on $\deg(\tau) - |\tau|_\zeta$. Since we subtract in the definition $\Pi_x^{R \times}(\mathcal{I}_a \sigma)$ the correct Taylor expansion we have

$$|\Pi_x^{R \times}(\mathcal{I}_a \sigma)(y)| \lesssim |y-x|_s^{\deg(\mathcal{I}_a \sigma)}.$$

For $\tau = X^k \zeta_l \prod_{i=1}^n \mathcal{I}_{a_i}(\sigma_i)$, with $R(y)(\tau) = \tau + \sum \lambda_j(y) \tau_j$, the continuity of ζ_l and the multiplicativity of $\Pi_x^{R \times}$, together with the inductive assumption, give

$$\begin{aligned} |(\Pi_x^R \tau)(y)| &\lesssim |(\Pi_x^{R \times} \tau)(y)| + \sum_j |(\Pi_x^{R \times} \tau_j)(y)| \\ &\lesssim |y-x|_s^{|k| + \sum_j \deg(\mathcal{I}_{a_i}(\sigma_i))} + \sum_j |y-x|_s^{\deg(\tau_j)} \lesssim |y-x|_s^{\deg(\tau)}. \end{aligned}$$

2. We use the identity of Lemma 5.2 to get the bounds on $\mathfrak{g}_{yx}(\tau)$ inductively on $\deg(\tau)$, using the bounds on Π_x obtained in the first point. This type of argument was first implemented in the proof of Proposition 3.16 in [11]; we recall the details here.

For β one of the finitely many values of $\deg(\tau)$ less than a fixed constant β_1 denote by ν_β the canonical projection of an element $\nu \in T$ on the elements of degree β . The bounds on \mathfrak{g}_{yx} are equivalent to the inequality

$$|\{\widehat{\mathfrak{g}}_{yx}^R(\tau)\}_\beta| \lesssim |y-x|_s^{\deg(\tau)-\beta}$$

for all $\beta < \deg(\tau) \leq \beta_1$. By multiplicativity of $\widehat{\mathfrak{g}}_{yx}^R$ it suffices to prove this estimate for τ of the form $\mathcal{I}_a(\sigma)$. If β is not an integer

$$\{\widehat{\mathfrak{g}}_{yx}^R(\mathcal{I}_a(\sigma))\}_\beta = \{\mathcal{I}_a(\widehat{\mathfrak{g}}_{yx}^R(\sigma))\}_\beta$$

and we get from the induction assumption $\widehat{\mathfrak{g}}_{yx}^R(\sigma) = \sigma + \sum_j c_{yx}^j \sigma_j$, with $|c_{yx}^j| \lesssim |y-x|_s^{\deg(\sigma)-\deg(\sigma_j)}$, so the conclusion follows in that case. If $\beta = n < \deg(\tau)$ is an integer we use Lemma 5.2, step 1 and the induction assumption to see that

$$\begin{aligned} |\{\widehat{\mathfrak{g}}_{yx}^R(\tau)\}_n| &\lesssim \sum_{n \leq |r| < \deg(\tau)} |y-x|_s^{|r|-n} |\Pi_x^R(\mathcal{I}_{a+r}(\widehat{\mathfrak{g}}_{yx}^R(\sigma)))(y)| \\ &\lesssim \sum_{n \leq |r| < \deg(\tau)} |y-x|_s^{|r|-n} \sum_{\beta} |\{\widehat{\mathfrak{g}}_{yx}^R(\sigma)\}_\beta| |y-x|_s^{2-|a|-|r|+\beta} \lesssim |y-x|_s^{|\tau|-n}. \end{aligned}$$

□

Since the model \mathbb{M}^R takes values in the space of continuous functions its associated reconstruction operator $\mathbb{R}^{\mathbb{M}^R}$ is given by

$$(\mathbb{R}^{\mathbb{M}^R} \mathbf{v})(x) = (\Pi_x^R \mathbf{v}(x))(x)$$

for any modelled distribution \mathbf{v} with positive regularity, so

$$(\mathbb{R}^{\mathbb{M}^R} \mathbf{v})(x) = \left(\Pi_x^{R \times} (R(x) \mathbf{v}(x)) \right)(x).$$

As we saw in the proof of Theorem 4.1 the multiplicative character of the map $\Pi_x^{R \times}(\cdot)(x)$ is the crucial feature of this factorization of the reconstruction map $\mathbb{R}^{\mathbb{M}^R}$ that leads to the proof of Theorem 1.1.

5.3. BHZ renormalization. In a translation-invariant setting one can recast the renormalization scheme introduced by Bruned, Hairer and Zambotti in [18] in the setting of preparation maps by associating to any character ℓ of the algebra T^- that is null on planted trees and trees of the form $X^k \sigma$ a strong preparation map R_ℓ whose dual R_ℓ^* is given by

$$R_\ell^*(\tau) = \sum_{\sigma \in \mathcal{B}^-} \frac{\ell(\sigma)}{S(\sigma)} (\tau \star \sigma),$$

where \mathcal{B}^- stands for the canonical basis of T^- . If the random noise ξ in (1) takes values in the space of continuous functions and has a translation-invariant distribution the BHZ renormalization corresponds to taking

$$\ell(\tau) = \mathbb{E}[\Pi(\widetilde{\mathcal{A}}_- \tau)(0)],$$

where $\widetilde{\mathcal{A}}_-$ is a ‘twisted antipode’ associated with a co-action $\delta : T \rightarrow T^- \otimes T$. This character is the BPHZ character introduced in Section 6.3 of [18] – see also Section 7 of

[8]. The renormalized system (23) then takes the form

$$(\partial_{x_0} - \mathcal{L}^p)u_p = f_p(u)\xi + g(x, u, \partial_{x_1}u) + \sum_{\tau \in \mathcal{B}^- \setminus \{\mathbf{1}\}} \ell(\tau) \frac{\Upsilon_p(\tau)(u, \partial_{x_1}u)}{S(\tau)} \quad (1 \leq p \leq p_0).$$

In the general non-translation invariant setting, and for any choice of strong preparation map R such that $R(x)^*\zeta_l = \zeta_l$ for all $l \neq 0$ and all $x \in \mathbb{R} \times \mathbf{T}$ the renormalized system takes the form

$$(\partial_{x_0} - \mathcal{L}^p)u_p = f_p(u)\xi + g_p(x, u, \partial_{x_1}u) + \Upsilon_p(R^*(\mathbf{1}) - \mathbf{1})(u, \partial_{x_1}u) \quad (1 \leq p \leq p_0).$$

In particular for any choice of state space dependent character ℓ on T^- that is null on planted trees and trees of the form $X^k\sigma$, one has indeed $R_\ell(x)^*\zeta_l = \zeta_l$ for all $l \neq 0$ and all $x \in \mathbb{R} \times \mathbf{T}$, and the renormalized system takes the form

$$(\partial_{x_0} - \mathcal{L}^p)u_p = f_p(u)\xi + g_p(x, u, \partial_{x_1}u) + \sum_{\tau \in \mathcal{B}^- \setminus \{\mathbf{1}\}} \ell(\cdot, \tau) \frac{\Upsilon_p(\tau)(u, \partial_{x_1}u)}{S(\tau)} \quad (1 \leq p \leq p_0).$$

As a final remark note that one has the following result in the translation invariant setting.

Proposition 5.2. *For every $\tau \in \mathcal{B}_- \setminus \{\mathbf{1}\}$*

$$\ell(\tau) = -\mathbb{E}[(\tilde{\Pi}^{R_\ell}\tau)(0)] \quad (26)$$

where

$$(\tilde{\Pi}^{R_\ell}\tau)(0) = (\Pi^{R_\ell}\tau)(0) - \ell(\tau)$$

with Π^{R_ℓ} the renormalized smooth model associated with R_ℓ .

Proof. We first recall that one has from [11, Prop. 3.12] and [11, Cor. 4.5]

$$(\Pi^{R_\ell}\tau)(0) = (\Pi M_\ell\tau)(0).$$

Then

$$\mathbb{E}[(\Pi M_\ell\tau)(0)] = 0$$

due to the choice of the character ℓ (see [18, Thm. 6.18, Eq. (6.26)]) which gives the result. \square

This suggest to choose in a non-translation invariant setting a character defined inductively for all x and all τ by the identity

$$\ell(x, \tau) := -\mathbb{E}[(\tilde{\Pi}^{R_\ell}\tau)(x)]. \quad (27)$$

Such a definition makes sense as one can rewrite $(\tilde{\Pi}^{R_\ell}\tau)(x)$ as

$$\begin{aligned} (\tilde{\Pi}^{R_\ell}\tau)(x) &= (\Pi^{R_\ell}\tau)(x) - \ell(x, \tau) \\ &= \left(\Pi_x^{R_\ell \times} (R_{\ell(x, \cdot)}\tau) \right)(x) - \ell(x, \tau) \\ &= \left(\Pi_x^{R_\ell \times} (\tilde{R}_{\ell(x, \cdot)}\tau) \right)(x) \end{aligned}$$

where one has

$$\tilde{R}_{\ell(x, \cdot)}\tau = (\ell(x, \cdot) \otimes \text{id})(\delta_r\tau - \tau \otimes \mathbf{1}).$$

It is an immediate consequence from the definition (27) that one has

$$\mathbb{E}[(\Pi^{R_\ell}\tau)(x)] = 0.$$

Indeed, one has

$$\mathbb{E}[(\Pi^{R_\ell}\tau)(x)] = \mathbb{E}[(\tilde{\Pi}^{R_\ell}\tau)(x)] + \ell(x, \tau) = -\ell(x, \tau) + \ell(x, \tau) = 0.$$

Remarks – 1. *How far are we from having a framework for dealing with systems of real-valued functions on a 2 or 3-dimensional closed manifold satisfying singular stochastic PDEs of the form (1) with the differential of u in the role of $\partial_{x_1}u$? The proof of the estimate (25) is robust enough to be adapted to a closed manifold setting. The algebraic results of Section 5 are insensitive to the fact that we could be on a manifold, and we know from Dahlqvist, Diehl & Driver’s work [26] what notion of model to take. The only point where analysis enters the scene is when using the reconstruction theorem. The proof of the latter given in the ‘Tourist guide’ [8] works verbatim on a manifold setting once one has the estimates (25). Alternatively one can use the manifold version of Caravenna & Zambotti’s approach to the reconstruction theorem [20] given by Rinaldi and Sclavi in [39]. However the present setting is not sufficient to deal with more general systems of singular stochastic PDEs like the scalar Φ_3^4 equation on a closed manifold. In that setting some more geometrical background needs to be introduced (jets in particular) to deal with local expansions of order higher than 1. See the work [37] of Hairer & Singh on the subject.*

2. *The group G^- of characters of the algebra T^- comes equipped with a convolution product derived from the extraction/contraction coproduct $\Delta^- : T^- \rightarrow T^- \otimes T^-$:*

$$\ell \circ \bar{\ell} := (\ell \otimes \bar{\ell}) \Delta^-, \quad \ell^{-1}(\cdot) = \ell(S\cdot),$$

where S^- is the antipode associated to Δ^- . In the translation invariant setting of all previous works, this group has an explicit representation in the space of endomorphisms on T given by

$$M_\ell := (\ell \otimes \text{Id}) \delta,$$

for which $\delta : T \rightarrow T^- \otimes T$ is a co-action and

$$M_\ell \circ M_{\bar{\ell}} = M_{\ell \circ \bar{\ell}}.$$

Although we cannot build such a representation of the group G^- in a non-translation invariant setting we can associate

$$R_\ell(x) = (\ell(x, \cdot) \otimes \text{Id}) \delta_r$$

to any state space dependent character ℓ and any state space point x . For any two characters $\ell, \bar{\ell}$ on T^- set

$$\ell \circ \bar{\ell} := (\ell \otimes \bar{\ell}) \delta_r,$$

pointwise in x . One then has

$$R_\ell \circ R_{\bar{\ell}} = R_{\ell \circ \bar{\ell}}.$$

6. RENORMALIZATION MAPS FOR BRANCHED ROUGH PATHS

The setting of subcritical singular stochastic PDEs devised in [18] defines *renormalization maps* over a given regularity structure as pairs of maps (M, M^+) that turn any admissible models (Π, \mathfrak{g}) into an admissible model $(\Pi \circ M, \mathfrak{g} \circ M^+)$. Branched rough paths provide an example of regularity structure with no polynomial decorations and for which $T = T^+$ and $M = M^+$ – we refer the reader to Gubinelli’s original article [29], Hairer and Kelly’s work [36] or [1, 21] for some background on this class of rough paths. This definition of a renormalization map takes its roots in the work [15] of Bruned, Chevyrev, Friz & Preiss where the role of pre-Lie structures for rough differential equations was first noticed.

We give in this section an alternative picture of renormalization maps on branched rough paths and prove in Theorem 6.2 that they are all obtained from a preparation map. It is not clear yet whether this statement holds for an arbitrary regularity structure. However

as part of the arguments below hold in full generality part of this section is written for a general regularity structure. Things are only specialized to branched rough paths in Corollary 6.1 where the generators are ζ_l (there is no monomial decoration and \uparrow^k is not used). Moreover, in this case, we assume that \mathfrak{T}^+ is a singleton. One has only one grafting product. Indeed, convolution with several kernels is replaced by integration in time.

Definition 6.1. A **good multi-pre-Lie morphism on T** is a map $A : T \rightarrow T$ such that one has for all $\sigma, \tau \in T$ and $k \in \mathbb{N}^{d+1}$, and all $a \in \mathfrak{T}^+ \times \mathbb{N}^{d+1}$,

$$\begin{aligned} A(\sigma \curvearrowright_a \tau) &= (A\sigma) \curvearrowright_a (A\tau), \\ A\uparrow^k &= \uparrow^k A. \end{aligned} \tag{28}$$

In the particular case of branched rough paths there are no polynomial decoration and good multi-pre-Lie morphisms are just multi-pre-Lie morphisms, that is they satisfy only the first identity in (28). We associate to a strong preparation map R a linear map

$$M_R^\times : T \rightarrow T$$

defined by the requirement that $M_R^\times(\mathbf{1}) = \mathbf{1}$, that M_R^\times is multiplicative, by the data of the $M_R^\times(\zeta_l)$, for $1 \leq l \leq l_0$, and by the induction relation

$$M_R^\times(\mathcal{I}_{(t,k)}\tau) = \mathcal{I}_{(t,k)}(M_R^\times(R\tau)) \tag{29}$$

for all $\tau \in T$ and $(t, k) \in \mathcal{L}^+ \times \mathbb{N}^{d+1}$. Following Section 3.1 of [11] define

$$M_R := M_R^\times R. \tag{30}$$

Theorem 6.2.

- Let R be a strong preparation map. The map M_R^* is a good multi-pre-Lie morphism satisfying $M_R^*(\zeta_l) = R^*(\zeta_l)$, for all $1 \leq l \leq l_0$.
- Let M be a good multi-pre-Lie morphism then it is obtained from a preparation map R using the construction (29)-(30). The preparation map R is defined for $\tau = \zeta_l \bar{\tau}$ with $\bar{\tau} = \prod_{i=1}^n \mathcal{I}_{a_i}(\tau_i)$ by

$$R^*(\zeta_l) := M^*(\zeta_l), \quad R^*(\tau) = R^*(\bar{\tau} \star \zeta_l) := \bar{\tau} \star R^* \zeta_l. \tag{31}$$

Proof. We start with R a strong preparation map and we proceed in two steps.

– We show that the map $(M_R^\times)^*$ is also multiplicative. One has

$$\begin{aligned} \langle (M_R^\times)^* \mathcal{I}_a(\sigma), \mathcal{I}_a(\tau) \rangle &= \langle \mathcal{I}_a(\sigma), M_R^\times \mathcal{I}_a(\tau) \rangle = \langle \mathcal{I}_a(\sigma), \mathcal{I}_a(M_R \tau) \rangle \\ &= \langle \sigma, M_R \tau \rangle = \langle M_R^* \sigma, \tau \rangle = \langle \mathcal{I}_a(M_R^* \sigma), \mathcal{I}_a(\tau) \rangle, \end{aligned}$$

which implies $(M_R^\times)^* \mathcal{I}_a(\sigma) = \mathcal{I}_a(M_R^* \sigma)$. Denote by Δ_d the deconcatenation coproduct

$$\Delta_d \mathcal{I}_a(\tau) = \mathcal{I}_a(\tau) \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{I}_a(\tau), \quad \Delta_d X^k = X^k \otimes \mathbf{1} + \mathbf{1} \otimes X^k,$$

extended multiplicatively not to the tree product but to the product between a decorated tree and a decorated tree with no polynomial decorations at the root. It follows then from the multiplicativity of M_R^\times and the identity $M_R^\times \mathcal{I}_a(\sigma) = \mathcal{I}_a(M_R \sigma)$, that

$$\langle (M_R^\times)^* \mathcal{I}_a(\sigma) \mu, \tau \rangle = \langle \mathcal{I}_a(\sigma) \mu, M_R^\times \tau \rangle = \langle \mathcal{I}_a(\sigma) \otimes \mu, \Delta_d M_R^\times \tau \rangle,$$

where we have used the fact that Δ_d is the adjoint of the joint root product \mathcal{M} . We have $\mathcal{I}_a(\sigma) \mu = \mathcal{M}(\mathcal{I}_a(\sigma) \otimes \mu)$. One has also

$$\Delta_d M_R^\times = (M_R^\times \otimes M_R^\times) \Delta_d.$$

Then, we get

$$\begin{aligned} \langle \mathcal{I}_a(\sigma) \otimes \mu, \Delta_d M_R^\times \tau \rangle &= \langle \mathcal{I}_a(\sigma) \otimes \mu, (M_R^\times \otimes M_R^\times) \Delta_d \tau \rangle \\ &= \langle (M_R^\times)^* \mathcal{I}_a(\sigma) \otimes (M_R^\times)^* \mu, \Delta_d \tau \rangle = \langle (M_R^\times)^* \mathcal{I}_a(\sigma) (M_R^\times)^* \mu, \tau \rangle \end{aligned}$$

which concludes the proof that $(M_R^\times)^*$ is multiplicative.

– It will be useful for our purpose to decompose the grafting map \curvearrowright_a into the sum of a grafting map at the root and a grafting map outside the root

$$\curvearrowright_a = \curvearrowright_a^{\text{root}} + \curvearrowright_a^{\text{non-root}},$$

with, for $\tau = X^k \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j)$ and $a_i \in \mathfrak{T} \times \mathbb{N}^{d+1}$,

$$\sigma \curvearrowright_a^{\text{root}} \tau := \sum_{m \in \mathbb{N}^{d+1}} \binom{k}{m} X^{k-m} \mathcal{I}_{a-m}(\sigma) \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j)$$

and

$$\sigma \curvearrowright_a^{\text{non-root}} \tau := X^k \sum_{i=1}^n \mathcal{I}_{a_i}(\sigma \curvearrowright_a \tau_i) \prod_{j \neq i} \mathcal{I}_{a_j}(\tau_j).$$

We proceed by induction on the size of the trees appearing in the product. In the induction hypothesis, we include the two following identities for $\sigma, \tau \in \mathcal{T}$

$$M_R^*(\sigma \curvearrowright_a \tau) = (M_R^* \sigma) \curvearrowright_a (M_R^* \tau) \quad (32)$$

and

$$(M_R^\times)^*(\sigma \curvearrowright_a \tau) = (M_R^* \sigma) \curvearrowright_a ((M_R^\times)^* \tau). \quad (33)$$

The base cases are given when one applies $(M_R^\times)^*$ and $(M_R^*)^*$ to the generators $X^k \zeta_l$. One has

$$(M_R^\times)^* X^k \zeta_l = X^k \zeta_l, \quad (M_R^*)^* X^k \zeta_l = R^*(M_R^\times)^* X^k \zeta_l = R^* X^k \zeta_l = \uparrow^k R^* \zeta_l$$

where for the last identity we have used that

$$R^* X^k \zeta_l = R^*(X^k \star \zeta_l) = X^k \star R \zeta_l = \uparrow^k R \zeta_l$$

that is a consequence of the right morphism property of the strong preparation map with the product \star (see (15)). Let $\sigma, \tau \in \mathcal{T}$, one has

$$\begin{aligned} M_R^*(\sigma \curvearrowright_a \tau) &= R^*(M_R^\times)^*(\sigma \curvearrowright_a \tau) \\ &= R^*(M_R^* \sigma \curvearrowright_a (M_R^\times)^* \tau) \\ &= (M_R^* \sigma \curvearrowright_a R^*(M_R^\times)^* \tau) = (M_R^* \sigma \curvearrowright_a M_R^* \tau) \end{aligned}$$

where we have applied the induction hypothesis given by (33) on $(M_R^\times)^*$ and the right-morphism property of R^* , given by the assumption that R is a *strong* preparation map. We consider $\tau = X^k \bar{\tau}$ where $\bar{\tau} = \prod_{i=1}^n \mathcal{I}_{a_i}(\tau_i)$. The multiplicativity property of $(M_R^\times)^*$ and the fact that $(M_R^\times)^* \mathcal{I}_a(\sigma) = \mathcal{I}_a(M_R^* \sigma)$ yield

$$\begin{aligned} (M_R^\times)^*(\sigma \curvearrowright_a^{\text{root}} \tau) &= (M_R^\times)^* \sum_{\ell \in \mathbb{N}^{d+1}} \binom{k}{\ell} X^{k-\ell} \mathcal{I}_{a-\ell}(\sigma) \bar{\tau} \\ &= \sum_{\ell \in \mathbb{N}^{d+1}} \binom{k}{\ell} X^{k-\ell} \mathcal{I}_{a-\ell}(M_R^* \sigma) (M_R^\times)^* \bar{\tau} \\ &= M_R^* \sigma \curvearrowright_a^{\text{root}} (M_R^\times)^* \tau \end{aligned}$$

For the grafting outside the root, we use the induction hypothesis. One has

$$\begin{aligned}
(M_R^\times)^*(\sigma \curvearrowright_a^{\text{non-root}} \tau) &= (M_R^\times)^* X^k \mathcal{I}_{a_i}(\sigma \curvearrowright_a \tau_i) \prod_{j \neq i} \mathcal{I}_{a_j}(\tau_j) \\
&= (M_R^\times)^*(X^k \mathcal{I}_{a_i}(M_R^*(\sigma \curvearrowright_a \tau_i))) \prod_{j \neq i} \mathcal{I}_{a_j}(M_R^* \tau_j) \\
&= X^k \mathcal{I}_{a_i}(M_R^* \sigma \curvearrowright_a M_R^* \tau_i) \prod_{j \neq i} \mathcal{I}_{a_j}(M_R^* \tau_j) \\
&= M_R^* \sigma \curvearrowright_a^{\text{non-root}} (M_R^\times)^* \tau.
\end{aligned}$$

where we have used the induction hypothesis (32) on σ and τ_i . The proof that $M_R^* \uparrow^k = \uparrow^k M_R^*$, works the same by decomposing insertion of polynomial decorations at the root and outside the root. For the second part of the theorem, let M be a good multi-pre-Lie morphism. The map R defined uniquely by (31) is actually a strong preparation map. Then, from the first part M_R built out of R is a multi-pre-Lie morphism which coincide with M a multi-pre-Lie morphism on the generator. Therefore, one has $M_R = M$ which concludes the proof. \square

We now specialize the previous result to the setting of branched rough paths. Recall $T = T^+$ is in that case the Butcher-Connes-Kreimer Hopf algebra and all preparation maps are strong in that setting. We can then apply Theorem 6.2 to obtain the following fact.

Corollary 6.1. *In the Butcher-Connes-Kreimer setting the family of (automatically good) multi-pre-Lie morphisms are in bijection with the set of preparation maps.*

In this particular setting Theorem 1.1 takes the following form. Let X stand for a branched rough path over \mathbb{R}^e , where $e \geq 2$. In a regularity structure terminology X provides a model over the regularity structure (T^+, T^+) with $\mathbf{g}_{ts} = \Pi_s(\cdot)(t) = \mathsf{X}_{ts}$. Denote by $\zeta_0, \zeta_1, \dots, \zeta_e$ the generators of T^+ . If we are given some regular enough vector field V_0, \dots, V_e on \mathbb{R}^d the map

$$V : \zeta_i \mapsto V_i$$

has a unique extension as a pre-Lie morphism from (T^+, \curvearrowright) to the the set of vector fields on \mathbb{R}^d with the pre-Lie bracket $(W_1, W_2) \mapsto (DW_2)(W_1)$. Given a preparation map R on T^+ the path (z_t) is a solution of the differential equation

$$dz_t = V(z_t) d\mathsf{X}_t^R$$

driven by the renormalized branched rough path $\mathsf{X}^R = \mathsf{X} \circ M_R$ if and only if

$$dz_t = V(z_t) d\mathsf{X}_t + V((M_R^* - \text{Id})\mathbf{1})(z_t) dt.$$

Funding: I.B. acknowledges support from the CNRS and PIMS and the ANR-16-CE40-0020-01 grant. Y.B. gratefully acknowledges funding support from the European Research Council (ERC) through the ERC Starting Grant Low Regularity Dynamics via Decorated Trees (LoRDeT), grant agreement No. 101075208.

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