



HAL
open science

Nonlinear Functional Analysis and Applications

Jesus Garcia-Falset, Khalid Latrach

► **To cite this version:**

Jesus Garcia-Falset, Khalid Latrach. Nonlinear Functional Analysis and Applications. Nonlinear Functional Analysis and Applications, 41, De Gruyter, pp.VII-X, 2023, Series In Nonlinear Analysis and Applications, 978-3-11-103096-8. 10.1515/9783111031811-201 . hal-04890687

HAL Id: hal-04890687

<https://hal.science/hal-04890687v1>

Submitted on 16 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Jesus Garcia-Falset & Khalid Latrach

Nonlinear Functional Analysis and Applications

De Gruyter Studium

Übergreifende Fragestellung zu Methode und Darstellungsweise

Studien und Voruntersuchungen zur kommentierten Überlieferung
und Wirkung der Abweichungen vom Text der Ausgabe und
Fassung

⁵ Herausgegeben von
Max Mustermann und Moritz Maier

Band 5/31

Jesus Garcia-Falset & Khalid Latrach

Nonlinear Functional Analysis and Applications

Unterscheidung als Hilfs- und Orientierungsbegriff zur
Dekodierung

Herausgegeben von
Max Mustermann als Editor

5

In Zusammenarbeit mit
Mustermann Stiftung

2. Auflage



Mathematics Subject Classification 2010

35A01, 35F31, 46B20, 46E35, 47G10, 47H06, 47H08, 47H10, 47G10, 47H20, 47J35

Auch beim Konzept der ästhetischen Bildung von Wissen und dessen möglichst rasche und erfolgsorientierte Anwendung verspielen Einsichten und Gewinne ohne den Bezug auf die um 1900 entwickelten Argumentationen.

- 5 Auch umfasst die Untersuchung in der Hauptsache den Zeitraum zwischen dem Inkrafttreten und der Darstellung in seiner heute geltenden Fassung. Ihre Funktion als Teil der literarischen Darstellung und narrativen Technik enthält auch einen vollständigen textkritischen Apparat und ein Verzeichnis der Stellen.

ISBN 978-3-11-021808-4

e-ISBN (PDF) 978-3-11-021809-1

10 ISSN 0179-0986

Library of Congress Cataloging-in-Publication Data

A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliografische Information der Deutschen Nationalbibliothek

Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte

- 15 bibliografische Daten sind im Internet über

<http://dnb.dnb.de> abrufbar.

© 2025 Copyright-Text, Walter de Gruyter GmbH, Berlin/Boston

Umschlaggestaltung: Cover-Firma

Satz: le-tex publishing services GmbH, Leipzig

- 20 Druck und Bindung: Druckerei XYZ

☻ Gedruckt auf säurefreiem Papier

Printed in Germany

www.degruyter.com

Dedications

To our beloved families with affection and gratitude

Preface

Nonlinear functional analysis is a new area that was born and has matured from abundant research developed in studying nonlinear problems. In the past thirty years, nonlinear analysis has undergone rapid growth; it has become part of the mainstream research fields in contemporary mathematical analysis. It is now an important field of study in both pure and applied mathematics. Many nonlinear analysis problems have their roots in geometry, astronomy, fluid and elastic mechanics, physics, chemistry, biology, control theory, image processing, optimization, game theory and economics. Their development provided nonlinear functional analysis with new concepts, tools and theories that enriched the subject considerably. Nowadays nonlinear functional analysis is a well established mathematical discipline, which is characterized by a remarkable mixture of analysis, topology and applications. It is exactly the fact that the subject combines in a beautiful way these items that makes it attractive to mathematicians. The notions and techniques of nonlinear functional analysis provide the appropriate tools to develop more realistic and accurate models describing various phenomena. Today the more theoretically inclined nonmathematician (engineer, economist, biologist or chemist) needs a working knowledge of at least a part of nonlinear analysis in order to be able to conduct a complete qualitative analysis of their models. Of course actually the subject is vast and it is not possible to include in a book all its theoretical and applied parts. Thus, in this manuscript, we have selected some techniques and applications where we have been working recently. We have focused essentially on two topics which are pertinent to evolution equations and boundary value problems: Accretive operators and Fixed point theory.

The theory of generation of semigroups of linear contractions, which is the basis of the theory of evolution equations governed by linear operators, was developed by E. Hille and K. Yosida in 1948 and by W. Feller, I. Miyadera and R. S. Phillips for the general case. Theorems of generation of semigroups of Hille-Yosida and Lumer-Phillips and the Hille-Yosida exponential formula are presented. The theory of evolution equations governed by accretive operators started in the late 1960s by the works of Y. Komura who established a generation theorem of nonlinear semigroups in a Hilbert space in 1967. Afterwards, in 1971, the theory was extended to general Banach spaces with the seminal work by M. G. Crandall and T. Liggett. We present an account of various aspects of the (m -)accretive operators theory (old and recent results). The exponential formula of M. G. Crandall and T. Liggett and the generation theorem for accretive operators in Banach spaces are established. The case of quasi-accretive operators is also considered. Crandall-Liggett's generation result is applied to discuss the well-posedness of Cauchy problems (existence uniqueness of solutions and continuous dependence of the initial data of the solutions) governed by accretive operators. The concepts of mild, weak and strong solutions are discussed. These results are applied to concrete evolution problems arising in dynamic of population and neutron transport theory.

For stationary problems, the fixed point theory plays a crucial role because, in general, the solutions of such problems can be expressed as fixed points of operators derived from the problems. We present the remarkable Banach's fixed point theorem known as the Banach contraction principle. It is fundamental in fixed point theory and it is a simple tool in establishing existence and uniqueness

results for functional and operator equations. In the literature they are numerous generalizations of Banach's fixed point theorem for various kind of contractive mappings (we refer, for example, to the works by E. Rakotch, V. M. Sehgal, D.W. Boyd and J. S. W. Wong, M. Edelstein, L. Janos, A. Meir and E. Keeler, R. Kannan, S. Reich, D. Wardowski, S. P. Singh, L. B. Ćirić, Y. Liu and Z. Li, T. Suzuki, etc.). We discuss the fixed point property for nonexpansive mappings in some classes of Banach spaces such as Hilbert spaces, reflexive Banach spaces with the Opial property, uniformly convex Banach spaces, reflexive Banach spaces with normal structure. A detailed discussion of fixed points for ϕ -expansive mappings and the zeros of m -accretive mappings is presented. 5

A detailed exposition of topological fixed point theory for the strong as well as the weak topology is given. For the strong topology, the classical results such as Brouwer's fixed point theorem, Schauder's fixed point theorem, Darbo's fixed point theorem and Sadovskii's fixed point theorem for set contractive mappings (with respect to a measure of noncompactness) are presented. Afterwards, some recent results of Dardo-Sadovskii's type are discussed. For the weak topology, we start with the Tychonoff fixed points theorem in a Hausdorff locally convex topological vector space and its formulation in Banach spaces equipped with the weak topology together with its extension for weakly sequentially mappings in metrizable locally convex topological vector spaces. We introduce the classes of ws -compact and ww -compact operators and derive some results related to them. Various fixed point theorems of Dardo-Sadovskii's type for set contractive mappings (with respect to a measure of weak noncompactness) are shown. Afterwards, we present a fixed point theorems of Schaefer's type and Leray-Schauder's type. Next, fixed points results for sum of mappings in bounded sets, unbounded sets and in Banach algebras are discussed. These results are applied to concrete boundary value problems arising in dynamic of population and neutron transport theory. 15 20

The book consists in eight chapters. Chapter 1 collects preliminary ideas and gathers most of definitions and concepts which will be needed throughout the book, i.e. elements of topological linear vector spaces, linear operators theory, weak topology, geometry of Banach spaces, Bochner integral, duality mapping, etc. In order to avoid permanent referrals to other works, we give definitions and precise statements of many results of functional analysis. 25

In Chapter 2 we expose an introduction to the theory of C_0 -semigroups for linear operators. We introduce the concepts such as, the infinitesimal generator, semigroup of contractions, Yosida approximation, exponential formula, required in Chapters 3 and 4. Theorems of generation (Hille-Yosida's theorem and Lumer-Phillips's theorem) are established. 30

Chapter 3 deals with the concept of accretive operators in Banach spaces. We give the main properties of accretive operators and m -accretive operators. We present a proof of Crandall-Liggett's exponential formula. We close this chapter by introducing the class of quasi-accretive operators. 35

In Chapter 4, we deal with abstract Cauchy problems governed by accretive operators. We discuss existence and uniqueness of (strong, integral, weak, mild) solutions to both homogeneous and inhomogeneous Cauchy problems as well as the relationship between them. We end this chapter by discussing abstract Cauchy problems governed by quasi-accretive and m -quasi-accretive operators and derive some of their elementary properties. 40

In Chapter 5, we present several applications of general theory to nonlinear Cauchy problems governed by m -accretive or m -quasi-accretive operators illustrating the ideas and general existence theory developed in the previous chapter.

Chapter 6 is dedicated to the metric fixed point theory. The first part is concerned with the Banach contraction principle and some of its generalizations. In Sections 6.5 we have gathered some results about the fixed point property for nonexpansive mappings in some classes of Banach spaces. Section 6.6 is devoted to fixed point properties of ϕ -expansive mappings while Sections 6.7 and 6.8 treat with zeros of m -accretive and ϕ -expansive operators. The results of the last three sections of this chapter are recent.

In Chapter 7, we consider the topological fixed point theory in normed spaces. Section 7.2 is concerned with the Brouwer and the Schauder fixed point theorems while Section 7.3 is dedicated to study set contractive mappings with respect to a measure of noncompactness. In addition to the classical Darbo's and Sadovskii's theorems we present some recent results of Darbo's and Sadovskii's type. In Section 7.4 we give two recent results concerning the existence of approximate fixed point sequence for mappings f such that $I - f$ is ϕ -expansive. In Section 7.5 we present Thychonoff's fixed point theorem and its formulation for weakly sequentially maps. We introduce in Section 7.6 the classes of ws -compact and ww -compact mappings. In Sections 7.7, 7.8 and 7.10 the concept of measure of weak noncompactness is considered and we give some fixed point results for set contractive mappings with respect to a measure of weak noncompactness. In Section 7.11 our goal is to present some fixed point theorems of Leray-Schauder's type for the weak topology while Sections 7.12 deals with fixed point theorems for sum of mappings involving the weak topology in bounded as well as unbounded sets. The results presented in Sections 7.11 and 7.12 are recent.

In Chapter 8, we present some examples of nonlinear boundary value problems to illustrate the field of applications of the results presented in Chapters 6 and 7.

Valencia, June 2022

Jesus Garcia-Falset

Clermont-Ferrand, June 2022

Khalid Latrach

Inhalt

Dedications — I

Preface — II

1 Fundamentals of Functional Analysis — 1

1.1 Topological spaces — 1	5
1.1.1 Topology and base for a topology — 1	
1.1.2 Continuity of mappings — 3	
1.1.3 Compact spaces — 4	
1.2 Metric spaces — 4	
1.2.1 Generalities — 4	10
1.2.2 Completeness — 6	
1.2.3 Compactness and completeness — 6	
1.2.4 The Baire theorem — 7	
1.3 Topological vector spaces — 9	
1.3.1 Vector spaces — 9	15
1.3.2 Topology on a vector space — 10	
1.3.3 Separation properties — 11	
1.3.4 Metrizable topological vector spaces — 12	
1.3.5 Seminorms and local convexity — 13	
1.4 Normed vector spaces — 14	20
1.4.1 Generalities — 14	
1.4.2 Finite dimensional normed spaces — 15	
1.5 Linear operators — 16	
1.5.1 Generalities — 16	
1.5.2 Continuity of linear operators — 17	25
1.5.3 Uniform boundedness principle — 18	
1.5.4 The open mapping theorem — 20	
1.5.5 The closed graph theorem — 21	
1.5.6 Linear functionals — 22	
1.5.7 The adjoint of an operator — 23	30
1.6 The Hahn-Banach theorem — 24	
1.6.1 The analytic form of Hahn-Banach's theorem — 24	
1.6.2 Geometric forms of the Hahn-Banach theorem — 25	
1.7 Weak topology — 28	
1.8 Weak* topology — 32	35
1.9 Some classes of Banach spaces — 35	
1.9.1 Reflexive spaces — 35	
1.9.2 Uniformly convex Banach spaces — 36	

	1.9.3	Strictly convex Banach spaces	— 38
	1.9.4	Smooth Banach spaces	— 38
	1.9.5	Banach spaces with the Dunford-Pettis property	— 39
	1.10	Duality mapping	— 42
5	1.11	Convex function and its subdifferentials	— 45
	1.11.1	Convex functions	— 45
	1.11.2	Lower semicontinuous functions	— 46
	1.11.3	Subdifferentials	— 46
	1.12	Retracts of normed spaces	— 48
10	1.13	Banach algebras	— 49
	1.14	Bochner integral	— 51
	1.14.1	Measurable functions	— 51
	1.14.2	Bochner integrable functions	— 53
	1.14.3	Komura theorem	— 55
15	1.15	Superposition operators	— 59
	1.15.1	Scalar-valued functions	— 59
	1.15.2	Vector-valued functions	— 60
	1.15.3	The case of weighed Lebesgue spaces	— 60
	1.16	Sobolev functions	— 61
20	1.17	Sobolev spaces	— 62
	1.17.1	Sobolev space in dimension one	— 62
	1.17.2	Sobolev spaces in higher dimension	— 64
	1.18	Bibliographical remarks	— 66
	2A	short introduction to semigroups of linear operators	— 67
25	2.1	Strongly continuous semigroups	— 67
	2.2	Uniformly continuous semigroups	— 70
	2.3	The infinitesimal generator of a semigroup	— 72
	2.4	Generation results	— 75
	2.4.1	The Hille-Yosida theorem	— 75
30	2.4.2	The Lumer-Phillips theorem	— 80
	2.5	The Hille-Yosida exponential formula	— 81
	2.6	Bibliographical remarks	— 83
	3	Accretive operators in Banach spaces	— 84
35	3.1	Introduction	— 84
	3.1.1	Linear case	— 84
	3.1.2	Nonlinear case	— 88
	3.2	Accretive operators	— 89
	3.3	Examples of accretive operators	— 91
	3.4	Properties of accretive operators	— 93

3.5	<i>m</i> -accretive operators	— 96	
3.6	Examples of <i>m</i> -accretive operators	— 99	
3.7	Properties of <i>m</i> -accretive operators	— 104	
3.8	Domains of <i>m</i> -accretive operators	— 109	
3.9	Perturbation of <i>m</i> -accretive operators	— 114	5
3.10	The Crandall-Liggett exponential formula	— 119	
3.11	Quasi-accretive operators	— 126	
3.12	Bibliographical remarks	— 127	
4	Abstract Cauchy Problem	— 128	
4.1	Introduction	— 128	10
4.2	Strong solutions for the Cauchy problem	— 130	
4.3	The homogeneous problem for accretive operators	— 132	
4.4	Integral solutions	— 136	
4.5	The inhomogeneous problem for accretive operators	— 150	
4.5.1	Linear case	— 157	15
4.5.2	Regularization of solutions	— 162	
4.6	The Abstract Cauchy problem for quasi-accretive operators	— 165	
4.6.1	Regularization	— 172	
4.6.2	Lipschitzian perturbations of <i>m</i> -quasi-accretive operators	— 176	
4.6.3	Local Lipschitzian perturbations	— 177	20
4.7	Bibliographical remarks	— 179	
5	Solvability of nonlinear evolution equations	— 180	
5.1	Lebowitz-Rubinow's model	— 180	
5.1.1	Lebowitz-Rubinow's model for finite maximum cells cycle length	— 181	
5.1.2	Lebowitz-Rubinow's model for infinite maximum cells cycle length	— 197	25
5.2	Rotenberg's model	— 215	
5.2.1	The functional setting of the problem and preliminaries	— 216	
5.2.2	Local boundary conditions	— 219	
5.2.3	Nonlocal boundary conditions	— 226	
5.3	A transport equation with delayed neutrons	— 236	30
5.3.1	The functional setting of the problem and preliminaries	— 237	
5.3.2	Existence and uniqueness results	— 240	
5.3.3	Some lemmas	— 243	
5.3.4	Proofs	— 248	
5.4	Bibliographical remarks	— 251	35
6	Metric Fixed Point Theorems	— 254	
6.1	Introduction	— 254	
6.2	The Banach contraction principle	— 254	

	6.3	Some generalizations of the BCP —	259
	6.3.1	Nonlinear contraction mappings —	259
	6.3.2	Separate contraction mappings —	260
	6.3.3	Expansive mappings —	261
5	6.4	Nonlinear expansive mappings —	262
	6.5	Nonexpansive mappings —	264
	6.5.1	Generalities —	264
	6.5.2	Hilbert spaces —	266
	6.5.3	Uniformly convex Banach spaces —	267
10	6.5.4	Normal structure —	269
	6.5.5	Fixed points in unbounded sets —	272
	6.6	Fixed points for ϕ -expansive mappings —	274
	6.7	Zeros of m -accretive operators —	279
	6.8	Zeros of m -accretive and ϕ -expansive operators —	286
15	6.9	Bibliographical remarks —	290
	7	Topological Fixed Point Theorems —	291
	7.1	Introduction —	291
	7.2	Brouwer and Schauder fixed point theorems —	292
	7.2.1	Brouwer's fixed point theorem —	292
20	7.2.2	Schauder's fixed point theorem —	295
	7.3	Set contractive mappings —	299
	7.3.1	Measure of noncompactness —	299
	7.3.2	Darbo's type fixed point theorems —	304
	7.3.3	Sadovskii's type fixed point theorems —	307
25	7.4	On maps f such that $I - f$ is ϕ -expansive —	314
	7.5	Tychonoff's type fixed point theorems —	318
	7.5.1	Tychonoff's fixed point theorem —	318
	7.5.2	Weakly sequentially continuous maps —	321
	7.6	ws-compact and ww-compact mappings —	322
30	7.7	Measure of weak noncompactness —	327
	7.7.1	Axiomatic approach —	327
	7.7.2	Measure of weak noncompactness of De Blasi —	328
	7.7.3	Measure of weak noncompactness of Banaś-Knap —	334
	7.8	Darbo-Sadovskii's type fixed point theorems —	335
35	7.9	A Schaefer's type fixed point theorem —	344
	7.10	Convex-power condensing mappings —	345
	7.11	Leray-Schauder's type fixed point theorems —	348
	7.12	Fixed point theorems for sum of mappings —	353
	7.12.1	Krasnosel'skii's type fixed point theorems —	353
40	7.12.2	Krasnosel'skii-Leray-Schauder's type fixed point theorems —	377

- 7.12.3 Krasnoselskii-Schafer's type fixed point theorems — 380
 7.13 Bibliographical notes — 384

8 Solvability of nonlinear boundary value problems — 387

- 8.1A Dirichlet problem — 387
 8.2 Existence results for a nonlinear functional integral equation — 390 5
 8.3A nonlinear transport equation with delayed neutrons — 395
 8.3.1 Introduction — 395
 8.3.2 Notations and preliminaries — 396
 8.3.3 Regular collision operators — 399
 8.3.4 Compactness — 404 10
 8.3.5A measure of weak noncompactness — 408
 8.3.6 Existence results — 411
 8.4A nonlinear singular neutron transport equation — 415
 8.4.1 Introduction — 415
 8.4.2 Notations and preliminaries — 415 15
 8.4.3 Preparatory results — 418
 8.4.4 Existence results — 425
 8.5 Bibliographical remarks — 429

Literatur — 431

Stichwortverzeichnis — 440 20