TRANSPORT OF LOW REGULARITY GAUSSIAN MEASURES FOR THE 1D QUINTIC NONLINEAR SCHR ÖDINGER EQUATION

Alexis Knezevitch

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Abstract. We consider the 1d nonlinear Schrödinger equation (NLS) on the torus with initial data distributed according to the Gaussian measure with covariance operator $(1-\Delta)^{-s}$, where Δ is the Laplace operator. We prove that the Gaussian measures are quasi-invariant along the flow of (NLS) for the full range $s > \frac{3}{2}$. This improves a previous result obtained by Planchon, Tzvetkov and Visciglia in [25], where the quasi-invariance is proven for $s = 2k$ for all integers $k \geq 1$. In our approach, to prove the quasi-invariance, we directly establish an explicit formula for the Radon-Nikodym derivative $G_s(t,.)$ of the transported measures, which is obtained as the limit of truncated Radon-Nikodym derivatives $G_{s,N}(t,.)$ for transported measures associated with a truncated system. We also prove that the Radon-Nikodym derivatives belong to L^p, $p > 1$, with respect to $H^1(\mathbb{T})$-cutoff Gaussian measures, relying on the introduction of weighted Gaussian measures produced by a normal form reduction, following Sun-Tzvetkov [29]. Additionally, we prove that the truncated densities $G_{s,N}(t,.)$ converges to $G_s(t,.)$ in L^p (with respect to the $H^1(\mathbb{T})$-cutoff Gaussian measures).

Contents

1. Introduction

In this paper, we contribute to the program initiated by Tzvetkov in [33] on the transport of Gaussian measures under the flow of Hamiltonian partial differential equations (PDEs). We consider the defocusing quintic nonlinear Schrödinger equation on the torus :

$$\begin{cases} i\partial_t u + \partial_x^2 u = |u|^4 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T} \\ u|_{t=0} = u_0 \end{cases} \tag{1.1}$$

This is a Hamiltonian PDE with the associated Hamiltonian :

$$H(u) := \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 dx + \frac{1}{6}\int_{\mathbb{T}} |u|^6 dx \tag{1.2}$$

1.1. Description of the problem.

In the present work, we consider the situation where (1.1) is globally well posed in a certain Banach space X. With such a Banach space X, we can invoke, for every time $t \in \mathbb{R}$, the flow of (1.1) :

$$\Phi(t) : X \longrightarrow X$$

which is the continuous map that for any initial data $u_0 \in X$ associates the solution of (1.1) evaluated at time t.
Given a Gaussian measure μ on X (defined on $\mathcal{B}(X)$, the σ-algebra of Borel sets of X), we can consider the push-forward measure of μ under $\Phi(t)$, denoted by $\Phi(t)_{\#}\mu$, and defined for all $A \in \mathcal{B}(X)$ as

$$\Phi(t)_{\#}\mu(A) := \mu(\Phi(t)^{-1}A)$$

We say that $\Phi(t)_{\#}\mu$ is the *transported measure* of μ under the flow $\Phi(t)$. This object is of interest because properties on the measure $\Phi(t)_{\#}\mu$ provide a macroscopic description of the flow. Following the problem raised by Tzvetkov for Hamitlonian PDEs in [33], one wonders if, for every time t, the measure $\Phi(t)_{\#}\mu$ is absolutely continuous with respect to μ. If that is indeed the case, we use the notation $\Phi(t)_{\#}\mu \ll \mu$. In other words, one wonders if, for any Borel set A in X, the following assertion

$$\mu(A) = 0 \implies \Phi(t)_{\#}\mu(A) = 0 \tag{1.3}$$

is true. If the answer is positive, we say that the measure μ is *quasi-invariant* under the flow $\Phi(t)$. In that case, we can invoke the Radon-Nikodym derivative $F_t \in \mathbb{L}^1(d\mu)$ which satisfies:

$$\Phi(t)_{\#}\mu = F_t(u)d\mu$$

and often denoted as $\frac{d\Phi(t)_{\#}\mu}{d\mu}$. The absolute continuity (1.3) is only a qualitative result because we only obtain the existence of the Radon-Nikodym derivative F_t. A more quantitative result would be providing additional information on F_t, such as an explicit formula which should be suitably interpreted.

The initial data spaces X under consideration will be Sobolev spaces on the torus. We can define Gaussian measures on Sobolev spaces as follows. For any given $s \in \mathbb{R}$, we define the Gaussian measure μ_s as the law of the random varibale:

$$S : \ \omega \longmapsto \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx} \tag{1.4}$$

where $\langle n\rangle := (1+n^2)^{\frac{1}{2}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ are independent standard complex-valued Gaussian measures[1] on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. More precisely, for $\sigma \in \mathbb{R}$, we have that

$$\mathbb{E}\left[\sum_{n\in\mathbb{Z}} \langle n\rangle^{2\sigma} \left|\frac{g_n}{\langle n\rangle^s}\right|^2\right] < +\infty \iff \sigma < s - \frac{1}{2}$$

so the random series in (1.4) converges in $\mathbb{L}^2(\Omega, H^\sigma(\mathbb{T}))$ if and only if $\sigma < s - \frac{1}{2}$. Thus, $\mu_s = S_{\#}\mathbb{P}$ is a probability measure on $\mathcal{B}(H^\sigma(\mathbb{T}))$ for all $\sigma < s - \frac{1}{2}$. For more details on Gaussian measures, we refer to [17] (see also [2]). Furthermore, it is well-known that

$$S \in H^{(s-\frac{1}{2})-}(\mathbb{T}) := \bigcap_{\sigma<s-\frac{1}{2}} H^\sigma(\mathbb{T}) \quad \text{almost surely,}$$

so the transported measure:

$$\Phi(t)_{\#}\mu_s = (\Phi(t)\circ S)_{\#}\mathbb{P}$$

makes sense if the flow $\Phi(t)$ is well defined on $H^{(s-\frac{1}{2})-}(\mathbb{T})$ almost surely. In the situation where $s > \frac{3}{2}$, we have $H^{(s-\frac{1}{2})-}(\mathbb{T}) \subset H^1(\mathbb{T})$. And, at the regularity $H^1(\mathbb{T})$, equation (1.1) is globally well posed. It follows from the combination of an elementary local wellposedness (thanks to the algebra property of $H^1(\mathbb{T})$) and the use of the conservation of the Hamiltonian (1.2) and of the mass:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 dx + \frac{1}{6}\int_{\mathbb{T}} |u|^6 dx, \qquad M(u) := \int_{\mathbb{T}} |u|^2 dx$$

In conclusion, the transported measure $\Phi(t)_{\#}\mu_s$ is well defined whenever $s > \frac{3}{2}$, and we can legitimately wonder if it is absolutely continuous with respect to μ_s. For some $s \leq \frac{3}{2}$, it is still possible to construct the transported measure $\Phi(t)_{\#}\mu_s$, but our method in this paper only works for $s > \frac{3}{2}$.

[1] in the sense that $g_n = h_n + il_n$, where h_n and l_n are two independent real Gaussian measures on $\mathbb{R}$ with law $\mathcal{N}(0, \frac{1}{2})$

1.2. **Formal computation and main results.** Formally, we can see the Gaussian measure μ_s as the measure

$$\frac{1}{Z_s}e^{-\frac{1}{2}\|u\|_{H^s}^2}du$$

where du is formally the Lebesgue measure (which does not exist on infinite dimensional vector spaces). Let us compute formally $\Phi(t)_\#\mu_s$ in order to predict what the Radon-Nikodym derivative of $\Phi(t)_\#\mu_s$ with respect to μ_s could be:

$$\Phi(t)_\#\mu_s = \Phi(t)_\#\big(\frac{1}{Z_s}e^{-\frac{1}{2}\|u\|_{H^s}^2}du\big) = \frac{1}{Z_s}e^{-\frac{1}{2}\left\|\Phi(t)^{-1}u\right\|_{H^s}^2}\Phi(t)_\#du$$

Since (1.1) is a Hamiltonian PDE, we may formally write that the (non existent) Lebesgue measure is preserved by the flow $\Phi(t)$, that is $\Phi(t)_\#du = du$. Moreover, from the additivity of the flow, we have $\Phi(t)^{-1} = \Phi(-t)$. Hence,

$$\Phi(t)_\#\mu_s = \frac{1}{Z_s}e^{-\frac{1}{2}\|\Phi(-t)u\|_{H^s}^2}du = e^{-\frac{1}{2}(\|\Phi(-t)u\|_{H^s}^2-\|u\|_{H^s}^2)}d\mu_s$$

Then, we expect that the actual density of $\Phi(t)_\#\mu_s$ with respect to μ_s is:

$$G_s(t,u) := e^{-\frac{1}{2}(\|\Phi(-t)u\|_{H^s}^2-\|u\|_{H^s}^2)}$$

However, since it is known that $\mu_s(H^{s-\frac{1}{2}}(\mathbb{T})) = 0$, we have:

$$\|\Phi(-t)u\|_{H^s}^2 = +\infty \text{ and } \|u\|_{H^s}^2 = +\infty, \ \mu_s - \text{almost surely}$$

so this is not even clear that the density $G_s(t,u)$ is well defined on the support of μ_s. But the hope is to observe some cancellation in the difference between $\|\Phi(-t)u\|_{H^s}^2$ and $\|u\|_{H^s}^2$. In order to analyze this difference, we first consider instead an approximated system for $N \in \mathbb{N}$:

$$\begin{cases} i\partial_t u + \partial_x^2 u = \Pi_N\left(|\Pi_N u|^4\Pi_N u\right), \ \ (t,x) \in \mathbb{R}\times\mathbb{T} \\ u|_{t=0} = u_0 \end{cases}$$

where Π_N is the Dirichlet projector. Denoting by $\Phi_N(t)$ its flow (called the *truncated flow*), we will be able, thanks to a finite-dimensional-type computation in Section 5, to prove rigorously that the transported measure $\Phi_N(t)_\#\mu_s$ is indeed:

$$\Phi_N(t)_\#\mu_s = e^{-\frac{1}{2}(\|\Pi_N\Phi_N(-t)u\|_{H^s}^2-\|\Pi_N u\|_{H^s}^2)}d\mu_s = G_{s,N}(t,u)d\mu_s$$

and the challenge will be to take the limit $N\to\infty$ into this formula. In order to do so, an integration by parts will give rise to a rewriting of the difference $\|\Pi_N\Phi_N(-t)u\|_{H^s}^2 - \|\Pi_N u\|_{H^s}^2$ as:

$$\begin{aligned} -\frac{1}{2}\big(\|\Pi_N\Phi_N(-t)u\|_{H^s}^2 - \|\Pi_N u\|_{H^s}^2\big) &= -\int_0^{-t}\frac{d}{d\tau}\frac{1}{2}\|\Pi_N\Phi_N(\tau)u\|_{H^s}^2\,d\tau \\ &= R_{s,N}(\Phi_N(-t)u) - R_{s,N}(u) - \int_0^{-t}Q_{s,N}(\Phi_N(\tau)u)d\tau \end{aligned}$$

where $R_{s,N}$ and $Q_{s,N}$ will emerge in Section 2 from a normal form reduction. Fortunately, we will see, respectively in Section 3 and 4, that $R_{s,N}$ and $Q_{s,N}$ are continuous functions on $H^\sigma(\mathbb{T})$ (for $\sigma < s-\frac{1}{2}$ close enough to $s-\frac{1}{2}$) that converge pointwisely[2] to continuous functions respectively

[2]Actually, we will see that the convergence holds uniformly on compact sets of $H^\sigma(\mathbb{T})$, which is stronger. See Propositions 3.3 and 4.4

denoted R_s and Q_s(the proof of those facts will be postponed to Section 10). Hence, the a priori ill-defined quantity $-\frac{1}{2}(\|\Phi(-t)u\|_{H^s}^2 - \|u\|_{H^s}^2)$ will be seen as:

$$-\frac{1}{2}\big(\|\Phi(-t)u\|_{H^s}^2 - \|u\|_{H^s}^2\big) := R_s(\Phi(-t)u) - R_s(u) - \int_0^{-t} Q_s(\Phi(\tau)u)d\tau$$
$$= \lim_{N\to\infty}\big(R_{s,N}(\Phi_N(-t)u) - R_{s,N}(u) - \int_0^{-t} Q_{s,N}(\Phi_N(\tau)u)d\tau\big)$$

And from this, we will be able in Section 6 to prove that indeed:

$$\Phi(t)_{\#}\mu_s = G_s(t,u)d\mu_s$$

More precisely, assuming that $R_{s,N}$, R_s and $Q_{s,N}$, Q_s have been constructed (see Section 3 and 4), we will prove the following result:

Theorem 1.1. *Let $s > \frac{3}{2}$ and $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$. Let $t \in \mathbb{R}$. Then, for every $N \in \mathbb{N}$, the transported measure $\Phi_N(t)_{\#}\mu_s$ has a density $G_{s,N}(t,.)$ with respect to μ_s given by:*

$$G_{s,N}(t,u) = \exp\big(-\frac{1}{2}(\|\Pi_N\Phi_N(-t)u\|_{H^s}^2 - \|\Pi_N u\|_{H^s}^2)\big)$$
$$= \exp\left(R_{s,N}(\Phi_N(-t)u) - R_{s,N}(u) - \int_0^{-t} Q_{s,N}(\Phi_N(\tau)u)d\tau\right)$$

Moreover, the transported measure $\Phi(t)_{\#}\mu_s$ has a density $G_s(t,.)$ with respect to μ_s given by:

$$G_s(t,u) = \exp\left(R_s(\Phi(-t)u) - R_s(u) - \int_0^{-t} Q_s(\Phi(\tau)u)d\tau\right)$$

which is continuous on $H^\sigma(\mathbb{T})$. In addition, the densities $G_{s,N}(t,.)$ converge to $G_s(t,.)$ uniformly on compact sets of $H^\sigma(\mathbb{T})$.

As a consequence,

Corollary 1.2. *Let $s > \frac{3}{2}$. Then, the Gaussian measure μ_s is quasi-invariant along the flow of (1.1).*

Results of this type were proven recently for many models, see [6, 9, 10, 11, 12, 14, 13, 15, 16, 20, 21, 22, 23, 24, 26, 25, 28, 33].
The quasi-invariance of μ_s along the flow of (1.1) has already been proven in [25] when $s = 2k$, for all integers $k \geq 1$, where the authors relied on modified energy estimates (see Theorem 1.4 in [25]). Here, our approach is different because our aim is to obtain directly the Radon-Nikodym derivative of the transported measures. Such an approach was adopted by Debussche and Tsutsumi in [9] and later by Genovese-Lucà-Tzvetkov [15] and by Forlano and Seong in [10]. More importantly, we are inspired by the method employed by Sun-Tzvetkov in [29] in the context of the 3d energy critical nonlinear Schrödinger equation. However, in order to reach the full range $s > \frac{3}{2}$ for the quasi-invariance, we will need to employ sharper estimates, notably by incorporating dispersive effects through Strichartz estimates. In addition, the 1d case will allow us to benefit from deterministic properties – through convergence on compact sets of *truncated densities* – in order to obtain the explicit formula for the Radon-Nikodym derivative of the transported measure $\Phi(t)_{\#}\mu_s$. It is however worth noticing that we will not need the *remarkable cancellation* presented in [29].
It would be interesting to prove the quasi-invariance below the threshold $s > \frac{3}{2}$ of this paper. Indeed, the question of quasi-invariance for (1.1) still arises for smaller s because, thanks to Bourgain in [5], we know that (1.1) is still globally well-posed in $H^\sigma(\mathbb{T})$ for $\sigma > \sigma^*$ with $\sigma^* < \frac{1}{2}$.

More precisely, from more recent works we know that (1.1) is globally well-posed for $\sigma > \frac{2}{5}$, see [18] and [1]. Besides, we know that the quasi-invariance is true for $s = 1$. It is a consequence of an other result obtained by Bourgain in [4] which states that the Gibbs measure:

$$Gb := \frac{1}{Z} e^{-\frac{1}{6}\|u\|^6_{\mathbb{L}^6(\mathbb{T})}} d\mu_1$$

is invariant under the flow of (1.1) for every time $t \in \mathbb{R}$ (meaning that $\Phi(t)_{\#} Gb = Gb$).
It would be also interesting to see if the recent work of Coe-Tolomeo in [8] may be used to identify a sharp threshold s_0 above which the quasi-invariance holds, and under which the transported measure and the initial Gaussian measure are mutually singular for every time.

Let us observe that the formula $\Phi(t)_{\#}\mu_s = G_s(t,.)d\mu_s$ from Theorem 1.1 implies that $G_s(t,.)$ belongs to $\mathbb{L}^1(d\mu_s)$. Thus, it is legitimate to ask if $G_s(t,.)$ belongs to $\mathbb{L}^p(d\mu_s)$, with $p > 1$. In the second result of this paper, we provide a partial answer to this question. We prove in Section 8 that if we add a $H^1(\mathbb{T})$-cutoff to the Gaussian measure μ_s, defining the *restricted Gaussian measure*:

$$\mu_{s,R} := \mathbb{1}_{\{\mathcal{C}(u) \leq R\}} \mu_s \tag{1.5}$$

where $\mathcal{C}(u)$ is the (conserved by the flow) quantity:

$$\mathcal{C}(u) := \frac{1}{2}\|u\|^2_{\mathbb{L}^2(\mathbb{T})} + H(u), \text{ with } H \text{ the Hamiltonian (1.2)} \tag{1.6}$$

then, (with the same $G_{s,N}$ and G_s as before) we have the following result:

Theorem 1.3. *Let $s > \frac{3}{2}$ and $R > 0$. Let $t \in \mathbb{R}$. Then, for every $N \in \mathbb{N}$:*

$$\Phi_N(t)_{\#}\mu_{s,R} = G_{s,N}(t,u)d\mu_{s,R} \qquad \text{and,} \qquad \Phi(t)_{\#}\mu_{s,R} = G_s(t,u)d\mu_{s,R}$$

Moreover, the densities $G_{s,N}(t,.)$, $G_s(t,.)$ belongs to $\mathbb{L}^p(d\mu_{s,R})$; and $G_{s,N}$ converge to $G_s(t,.)$ in $\mathbb{L}^p(d\mu_{s,R})$.

We stress the fact that $s > \frac{3}{2}$ is the energy threshold where it is still possible to use the cutoff $\mathcal{C}$. For $s \leq \frac{3}{2}$, $H^1(\mathbb{T})$ is strictly contained in the full measure space $H^{(s-\frac{1}{2})-}(\mathbb{T})$, and we would need to consider a renormalized cutoff as in [34].

Our approach to prove Theorem 1.3 is to work (instead of directly with the Gaussian measures $\mu_{s,R}$) with *weighted Gaussian measures*, that we define in Section 7. Formally, the idea is to replace the restricted Gaussian measure:

$$\mu_{s,R} = \text{"}\frac{1}{Z_s}\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-\frac{1}{2}\|u\|^2_{H^s}} du\text{" by } \rho_{s,R} := \text{"}\frac{1}{Z_s}\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-E_s(u)} du\text{"}$$

where $E_s(u)$ is a *modified energy* of the form:

$$E_s(u) = \frac{1}{2}\|u\|^2_{H^s} + R_s(u)$$

and where $R_s(u)$ is a correction term due to the non-linearity in (1.1), which will be produced by the normal form reduction from Section 2. For the weighted Gaussian measures, we will be able to prove a quantitative inequality (see Proposition 8.5) that could be transferred afterwards to $\mu_{s,R}$ (see Proposition 8.7).
Besides, contrary to the proof of Theorem 1.1 (which requires only deterministic considerations), the proof of Theorem 1.3 will require in addition probabilistic tools, relying on the fact that the initial data are distributed according to Gaussian measures. Mainly, in order to prove $\mathbb{L}^p$-estimates on R_s and Q_s (respectively in Section 11 and 12), we will use the independence between high and

low frequency Gaussians, along with a conditional Wiener chaos estimate (see Lemma 9.9). This method was adopted before in [29]. However, we point out that we will use the Wiener chaos estimate with respect to three high-frequency Gaussians (that is with $m = 3$ in Lemma 9.9) whereas in [29] the authors performed the Wiener chaos with respect to two high-frequency Gaussians (that is with $m = 2$ in Lemma 9.9). This remark is in fact significant because it will imply in our analysis that a "pairing between generations" (see Section 5 of [29]) cannot occur. On the contrary, such a pairing could occur in [29], and the authors dealt with it by emphasizing a "remarkable cancellation" (see Section 7 of [29]).

Organization of the paper. We organize this paper as follows:
In Section 2, we perform a normal form reduction where energy-type quantities will emerge. In particular, the normal form reduction will produce two crucial quantities: one called the *energy correction*, denoted $R_{s,N}$, and the other called the *derivative of the modified energy at 0*, denoted $Q_{s,N}$. Section 3 and Section 4 are respectively dedicated to deterministic properties of $R_{s,N}$ and $Q_{s,N}$. In Section 5, we prove that (for every $s > \frac{3}{2}$) μ_s is quasi-invariant along the truncated flow $\Phi_N(t)$ (for $N \in \mathbb{N}$). More precisely, we prove the formula $\Phi_N(t)_{\#}\mu_s = G_{s,N}(t,.)d\mu_s$, providing an explicit formula for $G_{s,N}(t,u)$. In Section 6, we extend this formula to the flow $\Phi(t)$; we prove that $\Phi(t)_{\#}\mu_s = G_s(t,.)d\mu_s$, where the density $G_s(t,.)$ will be the pointwise limit of the truncated densities $G_{s,N}(t,.)$. In section 8, we prove that the densities $G_{s,N}(t,.)$ and $G_s(t,.)$ belong to L^p with respect to the restrictions of μ_s on bounded sets of $H^1(\mathbb{T})$; we also prove that, with respect to these measures, $G_{s,N}(t,.)$ converges to $G_s(t,.)$ in L^p. To do so, we will rely on the introduction, in Section 7, of weighted Gaussian measures. Then, the remaining part of the paper will be dedicated to the proof of the energy estimates we used in the previous sections. In Section 9, we gather the deterministic and probabilistic tools of this paper. Section 10 is dedicated to the proof of the deterministic properties stated in Sections 3 and 4. Section 11 and Section 12 are respectively dedicated to the proof of L^p estimates on $R_{s,N}$ and $Q_{s,N}$. Finally, in Appendix A, we provide a local and global Cauchy theory for (1.1) and (2.1) on $H^\sigma(\mathbb{T})$, for $\sigma \geq 1$, and we then prove an approximation property of $\Phi(t)$ by $\Phi_N(t)$. Besides, we will prove a decomposition for the truncated flow $\Phi_N(t)$.

1.3. **Further remarks.**

Remark 1.4. In our approach, we view the difference $\|\Phi(t)u\|_{H^s}^2 - \|u\|_{H^s}^2$ as the limit of continuous functions on $H^\sigma(\mathbb{T})$, with $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$. An alternative to give a meaning to this quantity (for $u \in H^\sigma(\mathbb{T})$) could have been to use a *nonlinear smoothing* for the solutions of (1.1). In the context of this paper, with $s > \frac{3}{2}$, the result in Theorem 1 from the interesting work [19] implies that for $\sigma < s - \frac{1}{2}$ (close enough to $s - \frac{1}{2}$), we have the following nonlinear smoothing (for $u \in H^\sigma(\mathbb{T})$):

$$v(t) := \Phi(t)u - e^{it\partial_x^2} e^{-i\frac{3}{2\pi}\int_0^t \|\Phi(\tau)u\|_{\mathbb{L}^4}^4 d\tau} u \in \mathcal{C}([-T,T]; H^{\sigma+1-\varepsilon}(\mathbb{T}))$$

for $0 < \varepsilon < 1$, and at least for small time $T > 0$. Hence, we could have tried to define the difference $\|\Phi(t)u\|_{H^s}^2 - \|u\|_{H^s}^2$ as :

$$\|\Phi(t)u\|_{H^s}^2 - \|u\|_{H^s}^2 := \|v(t)\|_{H^s}^2 + 2\mathrm{Re}\Big\langle \langle\nabla\rangle^{2s-\sigma} v(t), \langle\nabla\rangle^{\sigma}\big(e^{it\partial_x^2} e^{-i\frac{3}{2\pi}\int_0^t \|\Phi(\tau)u\|_{\mathbb{L}^4}^4 d\tau} u\big)\Big\rangle_{\mathbb{L}^2(\mathbb{T})}$$

since this formula is true when u is smooth (with $\langle .,. \rangle_{\mathbb{L}^2(\mathbb{T})}$ the $\mathbb{L}^2(\mathbb{T})$-scalar product). With ε close to 0, the $1-\varepsilon$ gain of regularity in the smoothing above implies that for $u \in H^\sigma(\mathbb{T})$, $\|v(t)\|_{H^s(\mathbb{T})}^2 < +\infty$ because $\sigma + 1 - \varepsilon$ is close to $s + \frac{1}{2} > s$. However, this smoothing is not enough

to define the term $\langle\nabla\rangle^{2s-\sigma}v(t)$ because $\sigma+1-\varepsilon<2s-\sigma$, no matter how close ε and σ are close to 0 and $s-\frac{1}{2}$ respectively. For $\langle\nabla\rangle^{2s-\sigma}v(t)$ to be well-defined, we need a gain of $1+\varepsilon'$ regularity, where $\varepsilon'>0$.

Remark 1.5. In this paper, we deal with the defocusing NLS, but it would have also been possible to consider the focusing NLS. However, in the focusing case, we would have needed an additional cut-off on small $\mathbb{L}^2(\mathbb{T})$-initial data, ensuring that the flow of (1.1) is global and a control on the $H^1(\mathbb{T})$-norm of the solutions.

Acknowledgments. This work is partially supported by the ANR project Smooth ANR-22-CE40-0017. The author is grateful to his advisors Chenmin Sun and Nikolay Tzvetkov for suggesting this problem and for their valuable advice. The author would also like to thank Tristan Robert for pointing out the reference [19].

2. Poincaré-Dulac normal form reduction and modified energy

2.1. The truncated system.

Let $N\in\mathbb{N}$. We work with the following equation :

$$\begin{cases} i\partial_t u+\partial_x^2 u=\Pi_N\left(|\Pi_N u|^4\Pi_N u\right), \quad (t,x)\in\mathbb{R}\times\mathbb{T} \\ u|_{t=0}=u_0 \end{cases} \tag{2.1}$$

called the *truncated equation*, where Π_N is the projector on frequencies $\leq N$. More precisely,

$$\Pi_N\left(\sum_{k\in\mathbb{Z}}u_k e^{ikx}\right):=\sum_{|k|\leq N}u_k e^{ikx}$$

We also define $\Pi_N^\perp:=Id-\Pi_N$ as :

$$\Pi_N^\perp\left(\sum_{k\in\mathbb{Z}}u_k e^{ikx}\right):=\sum_{|k|>N}u_k e^{ikx}$$

Equation (2.1) is a smoothly approximated system of (1.1). We denote by Φ_N the flow of (2.1), called the *truncated flow*. Sometimes, we might use the notation Φ_∞ instead of Φ to refer to the flow of (1.1). If we set

$$E_N:=\Pi_N\mathbb{L}^2(\mathbb{T}), \qquad E_N^\perp:=\Pi_N^\perp\mathbb{L}^2(\mathbb{T})=(Id-\Pi_N)\mathbb{L}^2(\mathbb{T}),$$

the truncated flow $\Phi_N(t)$ can be factorized as $(\widetilde{\Phi}_N(t),e^{it\partial_x^2})$ on $E_N\times E_N^\perp$ in the sense that

$$\Phi_N(t)u_0=\underbrace{\widetilde{\Phi}_N(t)\Pi_N u_0}_{\in E_N}+\underbrace{e^{it\partial_x^2}\Pi_N^\perp u_0}_{\in E_N^\perp} \tag{2.2}$$

where $\widetilde{\Phi}_N$ is the flow of a (finite dimensional) ordinary differential equation (ODE). In Appendix A, we show the local and global well-posedness of the truncated flow as well as its structure.

In the same vein, we can decompose the Gaussian measure μ_s as

$$\mu_s=\mu_{s,N}\otimes\mu_{s,N}^\perp \tag{2.3}$$

where $\mu_{s,N}$ and $\mu_{s,N}^\perp$ are respectively the law of

$$\omega\longmapsto\sum_{|n|\leq N}\frac{g_n(\omega)}{\langle n\rangle^s}e^{inx} \qquad \text{and} \qquad \omega\longmapsto\sum_{|n|>N}\frac{g_n(\omega)}{\langle n\rangle^s}e^{inx}$$

Equivalently, $\mu_{s,N}$ is the probability measure on E_N given by:

$$\mu_{s,N} = \frac{1}{Z_N} \prod_{|n|\leq N} e^{-\frac{1}{2}\langle n\rangle^2 |\widehat{u}(n)|^2} d\widehat{u}(n) = \frac{1}{Z_N} e^{-\frac{1}{2}\|\Pi_N u\|^2_{H^s(\mathbb{T})}} \left(\prod_{|n|\leq N} d\widehat{u}(n) \right) \tag{2.4}$$

where $Z_N > 0$ is a normalizing constant, and $d\widehat{u}(n)$ is the Lebesgue measure on $\mathrm{Span}(e^{inx}) = \mathbb{C}\cdot e^{inx}$.

2.2. **Normal form reduction and modified energy.** We consider a smooth solution u_N of (2.1). By factoring through the linear flow, we introduce a new unknown

$$v_N := e^{-it\partial_x^2} u_N$$

Note that from (2.2), we have $v_N = \Pi_N v_N + \Pi_N^{\perp} u_0$, so $\partial_t v_N = \partial_t \Pi_N v_N$. In other words, if we invoke

$$w_N := \Pi_N v_N,$$

we have $\partial_t v_N = \partial_t w_N$. We denote respectively by $u_k(t), v_k(t)$ and $w_k(t)$ the k-th Fourier coefficient of $u_N(t), v_N(t)$ and $w_N(t)$, and for better readability, we will simply write u_k, v_k and w_k, omitting the variable t.

Now, we observe that v_N satisfies the equation

$$i\partial_t v = e^{-it\partial_x^2} \Pi_N \left(|e^{it\partial_x^2} \Pi_N v|^4 e^{it\partial_x^2} \Pi_N v \right) \tag{2.5}$$

Since the Fourier transform converts the product into convolution, we deduce from (2.5) that

$$\begin{aligned} i\partial_t v_k = i\partial_t w_k &= \mathbb{1}_{|k|\leq N} \sum_{k_1 - k_2 + k_3 - k_4 + k_5 = k} e^{-it\Omega(\vec{k})} \left(\prod_{j=1}^{5} \mathbb{1}_{|k_j|\leq N} \right) v_{k_1} \overline{v_{k_2}} ... v_{k_5} \\ &= \mathbb{1}_{|k|\leq N} \sum_{k_1 - k_2 + k_3 - k_4 + k_5 = k} e^{-it\Omega(\vec{k})} w_{k_1} \overline{w_{k_2}} ... w_{k_5} \end{aligned} \tag{2.6}$$

where $\vec{k} = (k_1, ..., k_5, k)$ and,

$$\Omega(\vec{k}) := \sum_{j=1}^{5} (-1)^{j-1} k_j^2 - k^2$$

is the so-called *resonant function*. In the sequel, we use the equivalent Sobolev norm for $s \geq 0$

$$|||f|||^2_{H^s(\mathbb{T})} := \sum_{k\in\mathbb{Z}} (1 + |k|^{2s}) |\widehat{f}(k)|^2$$

which is more convenient for our purpose. Let us now compute $\frac{1}{2}\frac{d}{dt}|||u_N|||^2_{H^s(\mathbb{T})}$:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|||u_N|||^2_{H^s(\mathbb{T})} = \frac{1}{2}\frac{d}{dt}|||v_N|||^2_{H^s(\mathbb{T})} = \frac{1}{2}\frac{d}{dt}|||w_N|||^2_{H^s(\mathbb{T})} &= \mathrm{Im}\,(i\partial_t w_N | w_N)_{H^s\times H^s} \\ &= \mathrm{Im} \sum_{\substack{k_6\in\mathbb{Z} \\ |k_6|\leq N}} (1 + |k_6|^{2s}) (i\partial_t w_{k_6}) \overline{w_{k_6}} \end{aligned}$$

Plugging (2.6) into this, we get

$$\frac{1}{2}\frac{d}{dt}|||w_N|||^2_{H^s(\mathbb{T})} = \mathrm{Im} \sum_{k_1 - k_2 + ... - k_6 = 0} (1 + |k_6|^{2s}) e^{-it\Omega(\vec{k})} w_{k_1} \overline{w_{k_2}} ... w_{k_5} \overline{w_{k_6}}$$

(where now $\vec{k} = (k_1, ..., k_5, k_6)$ and still $\Omega(\vec{k}) = \sum_{j=1}^{6}(-1)^{j-1}k_j^2$).

The above formula can be symmetrized using the symmetries of the resonant function and of the indices. Firstly, the change of variables $k_4 \leftrightarrow k_6$ and $k_2 \leftrightarrow k_6$ respectively yield,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||w_N|||^2_{H^s(\mathbb{T})} &= \mathrm{Im}\sum_{k_1-k_2+...-k_6=0}(1+|k_4|^{2s})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}}\\ &= \mathrm{Im}\sum_{k_1-k_2+...-k_6=0}(1+|k_2|^{2s})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}}\end{aligned}$$

so that,

$$\frac{1}{2}\frac{d}{dt}|||w_N|||^2_{H^s(\mathbb{T})} = \mathrm{Im}\sum_{k_1-k_2+...-k_6=0}\frac{3+|k_2|^{2s}+|k_4|^{2s}+|k_6|^{2s}}{3}e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}} \tag{2.7}$$

Secondly, the change of variables $(k_1, k_3, k_5) \leftrightarrow (k_2, k_4, k_6)$ and $(k_1, k_5, k_3) \leftrightarrow (k_2, k_4, k_6)$ and $(k_5, k_3, k_1) \leftrightarrow (k_2, k_4, k_6)$ respectively yield,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||w_N|||^2_{H^s(\mathbb{T})} &= -\mathrm{Im}\sum_{k_1-k_2+...-k_6=0}(1+|k_5|^{2s})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}}\\ &= -\mathrm{Im}\sum_{k_1-k_2+...-k_6=0}(1+|k_3|^{2s})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}}\\ &= -\mathrm{Im}\sum_{k_1-k_2+...-k_6=0}(1+|k_1|^{2s})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}}\end{aligned}$$

so that,

$$\frac{1}{2}\frac{d}{dt}|||w_N|||^2_{H^s(\mathbb{T})} = -\mathrm{Im}\sum_{k_1-k_2+...-k_6=0}\frac{3+|k_1|^{2s}+|k_3|^{2s}+|k_5|^{2s}}{3}e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}} \tag{2.8}$$

Finally, from combining (2.7) and (2.8), we obtain the symmetrized formula

$$\frac{1}{2}\frac{d}{dt}|||w_N|||^2_{H^s(\mathbb{T})} = -\frac{1}{6}\mathrm{Im}\sum_{k_1-k_2+...-k_6=0}\psi_{2s}(\vec{k})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}} \tag{2.9}$$

where,

$$\psi_{2s}(\vec{k}) := \sum_{j=1}^{6}(-1)^{j-1}|k_j|^{2s}$$

In order to perform a differentiation by parts, we decompose the set of indices according to whether $\Omega(\vec{k}) = 0$ or not :

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|||w_N|||^2_{H^s(\mathbb{T})} &= -\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}} \\
&\quad - \frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \psi_{2s}(\vec{k})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}} \\
&= -\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}} \\
&\quad - \frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{-i\Omega(\vec{k})}\partial_t\left(e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\right) \\
&\quad + \frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{-i\Omega(\vec{k})}e^{-it\Omega(\vec{k})}\partial_t\left(w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}|||w_N|||^2_{H^s(\mathbb{T})} + \frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{-i\Omega(\vec{k})}e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\right) \\
&= -\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}} + \frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{-i\Omega(\vec{k})}e^{-it\Omega(\vec{k})}\partial_t\left(w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\right)
\end{aligned} \tag{2.10}$$

Now, motivated by the above formula, we define the following quantities:

Definitions 2.1 (Energy correction and modified energy)**.** Let $N \in \mathbb{N}$.

(1) For $u \in \mathcal{S}'(\mathbb{T})$, we define:

$$R_{s,N}(u) := \frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{-i\Omega(\vec{k})}\left(\prod_{j=1}^{6}\mathbb{1}_{|k_j|\leq N}\right)u_{k_1}\overline{u_{k_2}}\ldots\overline{u_{k_6}} \tag{2.11}$$

and,

$$E_{s,N}(u) := \frac{1}{2}|||\Pi_N u|||^2_{H^s(\mathbb{T})} + R_{s,N}(u) \tag{2.12}$$

(2) For $u \in \mathcal{C}^\infty(\mathbb{T})$, we define:

$$R_{s,\infty}(u) := \frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{-i\Omega(\vec{k})}u_{k_1}\overline{u_{k_2}}\ldots\overline{u_{k_6}} \tag{2.13}$$

and,

$$E_{s,\infty} := \frac{1}{2}|||u|||^2_{H^s(\mathbb{T})} + R_{s,\infty}(u) \tag{2.14}$$

For $N \in \mathbb{N} \cup \{\infty\}$, $R_{s,N}$ and $E_{s,N}$ are respectively called the *energy correction* and the *modified energy*. Moreover, we will use interchangeably the notation R_s and E_s to refer respectively to $R_{s,\infty}$ and $E_{s,\infty}$.

Remark 2.2. In Section 3 (in Proposition 3.1), we will see that for $s > \frac{3}{2}$ and $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{3}{2}$, we can extend R_s to $H^\sigma(\mathbb{T})$, because we will be able to prove that the sum in (2.13) is absolutely convergent for every $u \in H^\sigma(\mathbb{T})$.

Using the modified energy (2.12), we can rewrite (2.10) as

$$\begin{aligned}\frac{d}{dt}E_{s,N}(\Pi_N u_N(t)) = &-\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\\ &+\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{-i\Omega(\vec{k})}e^{-it\Omega(\vec{k})}\partial_t\left(w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\right)\end{aligned}$$

Furthermore, expanding the time derivation $\partial_t\left(w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\right)$ and performing the change of variables $k_1 \leftrightarrow k_3$, $k_1 \leftrightarrow k_5$, $k_2 \leftrightarrow k_4$ and $k_2 \leftrightarrow k_6$, we get that

$$\begin{aligned}\frac{d}{dt}E_{s,N}(\Pi_N u_N(t)) = &-\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\\ &+\frac{1}{2}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}e^{-it\Omega(\vec{k})}\left(i\partial_t w_{k_1}\right)\overline{w_{k_2}}\ldots\overline{w_{k_6}}\\ &+\frac{1}{2}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}e^{-it\Omega(\vec{k})}w_{k_1}\left(\overline{-i\partial_t w_{k_2}}\right)\ldots\overline{w_{k_6}}\end{aligned}$$

At this point, we can make use of the formula (2.6) so that the above formula can be rewritten as

$$\begin{aligned}\frac{d}{dt}E_{s,N}(\Pi_N u_N(t)) = &-\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})e^{-it\Omega(\vec{k})}w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\\ &+\frac{1}{2}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ p_1-p_2+\ldots+p_5=k_1\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}e^{-it(\Omega(\vec{k})+\Omega(\vec{p}))}w_{p_1}\overline{w_{p_2}}\ldots w_{p_5}\overline{w_{k_2}}\ldots\overline{w_{k_6}}\\ &-\frac{1}{2}\mathrm{Im}\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ q_1-q_2+\ldots+q_5=k_2\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}e^{-it(\Omega(\vec{k})-\Omega(\vec{q}))}w_{k_1}\overline{w_{q_1}}w_{q_2}\ldots\overline{w_{q_5}}w_{k_3}\ldots\overline{w_{k_6}}\end{aligned} \tag{2.15}$$

where $\vec{p} = (p_1, ..., p_5, k_1)$ and $\vec{q} = (q_1, ..., q_5, k_2)$. If we come back to the variable u_N on the right hand side of (2.15), we obtain

$$\begin{aligned}\frac{d}{dt}E_{s,N}(\Pi_N u_N(t)) = &-\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})u_{k_1}\overline{u_{k_2}}...\overline{u_{k_6}}\left(\prod_{j=1}^{6}\mathbb{1}_{|k_j|\leq N}\right)\\ &+\frac{1}{2}\mathrm{Im}\sum_{\substack{k_1-k_2+...-k_6=0\\ p_1-p_2+...+p_5=k_1\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}u_{p_1}\overline{u_{p_2}}...u_{p_5}\overline{u_{k_2}}...\overline{u_{k_6}}\left(\prod_{j=1}^{6}\mathbb{1}_{|k_j|\leq N}\right)\left(\prod_{j=1}^{5}\mathbb{1}_{|p_j|\leq N}\right)\\ &-\frac{1}{2}\mathrm{Im}\sum_{\substack{k_1-k_2+...-k_6=0\\ q_1-q_2+...+q_5=k_2\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}u_{k_1}\overline{u_{q_1}}u_{q_2}...\overline{u_{q_5}}u_{k_3}...\overline{u_{k_6}}\left(\prod_{j=1}^{6}\mathbb{1}_{|k_j|\leq N}\right)\left(\prod_{j=1}^{5}\mathbb{1}_{|p_j|\leq N}\right)\end{aligned} \tag{2.16}$$

This quantity evaluated at time 0 will play an important role because it will appear in the explicit formula for the Radon-Nikodym derivatives of the transported measures. Motivated by the above formula, we define the following quantities:

Definitions 2.3 (Modified energy derivative at 0)**.** Let $N \in \mathbb{N}$. Let $\Phi_N(t)$ the flow of (2.1).

(1) For $u \in \mathcal{S}'(\mathbb{T})$, we define:

$$Q_{s,N}(u) := \frac{d}{dt}E_{s,N}(\Pi_N\Phi_N(t)u)|_{t=0} = (\mathrm{RHS})|_{t=0} \text{ of (2.16)}$$

(2) For $u \in \mathcal{C}^\infty(\mathbb{T})$, we define:

$$\begin{aligned}Q_{s,\infty}(u) &:= (\mathrm{RHS})|_{t=0} \text{ of (2.16) with } N=\infty\\ &= -\frac{1}{6}\mathrm{Im}\sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})u_{k_1}\overline{u_{k_2}}...\overline{u_{k_6}} + \frac{1}{2}\mathrm{Im}\sum_{\substack{k_1-k_2+...-k_6=0\\ p_1-p_2+...+p_5=k_1\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}u_{p_1}\overline{u_{p_2}}...u_{p_5}\overline{u_{k_2}}...\overline{u_{k_6}}\\ &-\frac{1}{2}\mathrm{Im}\sum_{\substack{k_1-k_2+...-k_6=0\\ q_1-q_2+...+q_5=k_2\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}u_{k_1}\overline{u_{q_1}}u_{q_2}...\overline{u_{q_5}}u_{k_3}...\overline{u_{k_6}}\end{aligned} \tag{2.17}$$

For $N \in \mathbb{N}\cup\{\infty\}$, $Q_{s,N}$ is called the *derivative of the modified energy at 0.* Moreover, we will use interchangeably the notation Q_s to refer to $Q_{s,\infty}$.

Remark 2.4. For $N \in \mathbb{N}$, note that from the additivity of the flow we have:

$$\begin{aligned}Q_{s,N}(\Phi_N(\tau)u) = \frac{d}{dt}E_{s,N}\left(\Pi_N\Phi_N(t)\Phi_N(\tau)u\right)|_{t=0} &= \frac{d}{dt}E_{s,N}\left(\Pi_N\Phi_N(t+\tau)u\right)|_{t=0}\\ &= \frac{d}{dt}E_{s,N}\left(\Pi_N\Phi_N(t)u\right)|_{t=\tau}\end{aligned} \tag{2.18}$$

Remark 2.5. In Section 4 (in Proposition 4.1), we will see that for $s > \frac{3}{2}$ and $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$, we can extend Q_s to $H^\sigma(\mathbb{T})$, because we will be able to prove that the sum in (2.17) is absolutely convergent for every $u \in H^\sigma(\mathbb{T})$.

3. Definition and deterministic properties of the energy correction

In Section 2, an energy correction $R_{s,N}$ (for $N \in \mathbb{N} \cup \{\infty\}$) has emerged from the normal form reduction (see Definition 2.1). We dedicate this section to the study of this quantity on $H^\sigma(\mathbb{T})$ for $\sigma < s - \frac{1}{2}$ (close enough to $s - \frac{1}{2}$), provided that $s > \frac{3}{2}$. In particular, we show that we are able to extend $R_{s,\infty}$ (which is defined on $\mathcal{C}^\infty(\mathbb{T})$) to $H^\sigma(\mathbb{T})$, because we will prove that the right hand side in (2.13) is actually an absolutely convergent sum for every $u \in H^\sigma(\mathbb{T})$. This section is composed of two results; firstly, we state that for $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$, the $R_{s,N}$ are continuous functions on $H^\sigma(\mathbb{T})$ given by the diagonal of a continuous multi-linear form; secondly, we state that $R_{s,N}$ converges to $R_{s,\infty}$ uniformly on compact sets of $H^\sigma(\mathbb{T})$.

Proposition 3.1. *Let $s > \frac{3}{2}$. For $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$, there exists a constant $C > 0$ such that for every $u^{(1)}, ..., u^{(6)} \in H^\sigma(\mathbb{T})$:*

$$\sum_{\substack{k_1 - k_2 + ... - k_6 = 0 \\ \Omega(\vec{k}) \neq 0}} \left| \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} \right| \left| u^{(1)}_{k_1} \overline{u^{(2)}_{k_2}} ... \overline{u^{(6)}_{k_6}} \right| \leq C \prod_{j=1}^{6} \left\| u^{(j)} \right\|_{H^\sigma} \tag{3.1}$$

Hence, the map:

$$\begin{array}{rccc} \mathcal{R} : & H^\sigma(\mathbb{T})^6 & \longrightarrow & \mathbb{C} \\ & (u^{(1)}, ..., u^{(6)}) & \longmapsto & \displaystyle\sum_{\substack{k_1 - k_2 + ... - k_6 = 0 \\ \Omega(\vec{k}) \neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} u^{(1)}_{k_1} \overline{u^{(2)}_{k_2}} ... \overline{u^{(6)}_{k_6}} \end{array}$$

is a continuous multi-linear form. Then, for $N \in \mathbb{N} \cup \{\infty\}$[3], setting [4]:

$$\begin{array}{rccc} R_{s,N} : & H^\sigma(\mathbb{T}) & \longrightarrow & \mathbb{C} \\ & u & \longmapsto & \frac{1}{6} \mathrm{Re}\, \mathcal{R}(\Pi_N u, ..., \Pi_N u) \end{array} , \tag{3.2}$$

we deduce that $R_{s,N}$ is a continuous map on $H^\sigma(\mathbb{T})$ that satisfies for all $u, v \in H^\sigma(\mathbb{T})$:

$$|R_{s,N}(u) - R_{s,N}(v)| \leq C \|u - v\|_{H^\sigma} \left(\|u\|^5_{H^\sigma} + \|v\|^5_{H^\sigma} \right)$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$.

Notation 3.2. We also use the notation R_s to refer to $R_{s,\infty}$.

We postpone the proof of this proposition for Section 10, where a detailed analysis is provided. Instead, assuming this proposition, we are able to prove now the following approximation result:

Proposition 3.3. *Let $s > \frac{3}{2}$. Let $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$ so that the conclusion of Proposition 3.1 holds. Then, for every compact set $K \subset H^\sigma(\mathbb{T})$,*

$$\sup_{u \in K} |R_s(u) - R_{s,N}(u)| \underset{N \to \infty}{\longrightarrow} 0$$

In other words, $R_{s,N}$ converges to R_s uniformly on compact sets of $H^\sigma(\mathbb{T})$.

[3] Using the notation $\Pi_\infty = id$

[4] By abuse of notation, we still denote by $R_{s,N}$ this new function, even though it has already been defined in Definition 2.1. The $R_{s,N}$ of Definition 2.1 and the $R_{s,N}$ of this proposition coincide on $\mathcal{C}^\infty(\mathbb{T})$.

Proof of Proposition 3.3 assuming Proposition 3.1. Here we assume the statements in Proposition 3.1. Thus, we invoke $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$ along with the constant $C > 0$ from this proposition. Next, we observe that for all $N \in \mathbb{N}$:

$$R_s(u) - R_{s,N}(u) = R_s(\Pi_N^{\perp} u)$$

Then, it follows from Proposition 3.1 that:

$$|R_s(u) - R_{s,N}(u)| = |R_s(\Pi_N^{\perp} u)| \leq C \left\|\Pi_N^{\perp} u\right\|_{H^\sigma}^6$$

On the other hand, we have that for every $u \in H^\sigma$, $\left\|\Pi_N^{\perp} u\right\|_{H^\sigma} \underset{N\to\infty}{\longrightarrow} 0$ and $\left\|\Pi_N^{\perp}\right\|_{H^\sigma \to H^\sigma} \leq 1$. So, using the following general abstract lemma in the inequality above finishes the proof:

Lemma 3.4. *Let E, F two Banach spaces. Let $\{T_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear maps with the uniform (in $N \in \mathbb{N}$) bound $\|T_N\|_{E\to F} \leq M$, for some constant $M > 0$. If for every $u \in E$:*

$$\|T(u) - T_N(u)\|_F \underset{N\to\infty}{\longrightarrow} 0$$

for some linear map $T : E \to F$, then for every compact set $K \subset E$, we have:

$$\sup_{u\in K} \|T(u) - T_N(u)\|_F \underset{N\to\infty}{\longrightarrow} 0$$

□

4. Definition and deterministic properties of the modified energy derivative at 0

In Section 2, we defined a quantity $Q_{s,N}$ (for $N \in \mathbb{N} \cup \{\infty\}$) in Definition 2.3, called the modified energy derivative at 0. We dedicate this section to the study of this quantity on $H^\sigma(\mathbb{T})$ for $\sigma < s - \frac{1}{2}$ (close enough to $s - \frac{1}{2}$), provided that $s > \frac{3}{2}$. In particular, we show that we are able to extend $Q_{s,\infty}$ (which is defined on $\mathcal{C}^\infty(\mathbb{T})$) to $H^\sigma(\mathbb{T})$, because we will prove that the right hand side in (2.17) is actually the sum of three absolutely convergent sum for every $u \in H^\sigma(\mathbb{T})$. This section is composed of two results; firstly, we state that for $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$, the $Q_{s,N}$ are continuous functions on $H^\sigma(\mathbb{T})$ given by the diagonal of a sum of three continuous multi-linear forms; secondly, we state that $Q_{s,N}$ converges to $Q_{s,\infty}$ uniformly on compact sets of $H^\sigma(\mathbb{T})$. We point out that the analysis here is similar to the one from Section 3.

Proposition 4.1. *Let $s > \frac{3}{2}$. For $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$, there exists a constant $C > 0$ such that for every $u^{(1)}, ..., u^{(6)}, v^{(1)}, ..., v^{(5)} \in H^\sigma(\mathbb{T})$:*

$$\sum_{\substack{k_1-k_2+...-k_6=0 \\ \Omega(\vec{k})=0}} \left|\psi_{2s}(\vec{k})\right| \left|u_{k_1}^{(1)} \overline{u_{k_2}^{(2)}} ... \overline{u_{k_6}^{(6)}}\right| \leq C \prod_{j=1}^{6} \left\|u^{(j)}\right\|_{H^\sigma} \tag{4.1}$$

and,

$$\sum_{\substack{k_1-k_2+...-k_6=0 \\ p_1-p_2+...+p_5=k_1 \\ \Omega(\vec{k})\neq 0}} \left|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\right| \left|v_{p_1}^{(1)} \overline{v_{p_2}^{(2)}} ... v_{p_5}^{(5)} \overline{u_{k_2}^{(2)}} ... \overline{u_{k_6}^{(6)}}\right| \leq C \prod_{j\in\{2,...,6\}} \left\|u^{(j)}\right\|_{H^\sigma} \prod_{l=1}^{5} \left\|v^{(l)}\right\|_{H^\sigma} \tag{4.2}$$

and,

$$\sum_{\substack{k_1-k_2+...-k_6=0\\ q_1-q_2+...+q_5=k_2\\ \Omega(\vec{k})\neq 0}} \Big|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\Big| \big|u_{k_1}^{(1)}\overline{v_{q_1}^{(1)}}v_{q_2}^{(2)}...\overline{v_{q_5}^{(5)}}u_{k_3}^{(3)}...\overline{u_{k_6}^{(6)}}\big| \leq C \prod_{j\in\{1,3,...,6\}} \big\|u^{(j)}\big\|_{H^\sigma} \prod_{l=1}^{5} \big\|v^{(l)}\big\|_{H^\sigma} \tag{4.3}$$

Hence, the maps:

$$\begin{aligned} \mathcal{T}_0 : \quad H^\sigma(\mathbb{T})^6 &\longrightarrow \mathbb{C} \\ u^{(1)},...,u^{(6)} &\longmapsto \sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k})u_{k_1}^{(1)}\overline{u_{k_2}^{(2)}}...\overline{u_{k_6}^{(6)}}\ , \end{aligned}$$

$$\begin{aligned} \mathcal{T}_1 : \quad H^\sigma(\mathbb{T})^{10} &\longrightarrow \mathbb{C} \\ u^{(2)},...,u^{(6)},v^{(1)},...,v^{(5)} &\longmapsto \sum_{\substack{k_1-k_2+...-k_6=0\\ p_1-p_2+...+p_5=k_1\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}v_{p_1}^{(1)}\overline{v_{p_2}^{(2)}}...v_{p_5}^{(5)}\overline{u_{k_2}^{(2)}}...\overline{u_{k_6}^{(6)}} \end{aligned}$$

$$\begin{aligned} \mathcal{T}_2 : \quad H^\sigma(\mathbb{T})^{10} &\longrightarrow \mathbb{C} \\ u^{(1)},u^{(3)},...,u^{(6)},v^{(1)},...,v^{(5)} &\longmapsto \sum_{\substack{k_1-k_2+...-k_6=0\\ q_1-q_2+...+q_5=k_2\\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}u_{k_1}^{(1)}\overline{v_{q_1}^{(1)}}v_{q_2}^{(2)}...\overline{v_{q_5}^{(5)}}u_{k_3}^{(3)}...\overline{u_{k_6}^{(6)}} \end{aligned}$$

are continuous multi-linear forms. Then, for $j=0,1,2$, setting for $u\in H^\sigma(\mathbb{T})$:

$$\mathcal{Q}_j(u) := \mathcal{T}_j(u,...,u)$$

we deduce that each Q_j is continuous on $H^\sigma(\mathbb{T})$, and that for $N\in\mathbb{N}\cup\{\infty\}$, the map[5]*:*

$$Q_{s,N} := \mathrm{Im}(-\frac{1}{6}\mathcal{Q}_0\circ\Pi_N + \frac{1}{2}\mathcal{Q}_1\circ\Pi_N - \frac{1}{2}\mathcal{Q}_2\circ\Pi_N) \tag{4.4}$$

is continuous on $H^\sigma(\mathbb{T})$ and satisfies for all $u,v\in H^\sigma(\mathbb{T})$:

$$\begin{aligned} |Q_{s,N}(u)-Q_{s,N}(v)| &\leq C\,\|u-v\|_{H^\sigma}\left(\|u\|_{H^\sigma}^6+\|v\|_{H^\sigma}^6+\|u\|_{H^\sigma}^9+\|v\|_{H^\sigma}^9\right) \\ &\leq C\,\|u-v\|_{H^\sigma}\left(1+\|u\|_{H^\sigma}+\|v\|_{H^\sigma}\right)^9 \end{aligned} \tag{4.5}$$

uniformly in $N\in\mathbb{N}\cup\{\infty\}$.

Notation 4.2. We also use the notation Q_s to refer to $Q_{s,\infty}$.

Remark 4.3. Note that we cannot a priori define $Q_s(u)$ on the support of μ_s as $\frac{d}{dt}E_s(\Phi(t)u)|_{t=0}$, because the expression

$$E_s(\Phi(t)u) = \frac{1}{2}|||\Phi(t)u|||_{H^s(\mathbb{T})}^2 + R_s(\Phi(t)u)$$

is a priori ill defined for initial data u in the support of μ_s since:

$$|||\Phi(t)u|||_{H^s(\mathbb{T})}^2 = +\infty, \quad \mu_s\text{-almost surely.}$$

[5]By abuse of notation, we still denote by $Q_{s,N}$ this new function, even though it has already been defined in Definition 2.3. The $Q_{s,N}$ of Definition 2.3 and the $Q_{s,N}$ of this proposition coincide on $\mathcal{C}^\infty(\mathbb{T})$.

Indeed, for $\sigma < s - \frac{1}{2}$, it follows from the fact that the flow $\Phi(t)$ is a bijection from $H^\sigma(\mathbb{T})$ to itself that:

$$\mu_s(\{u \in H^\sigma(\mathbb{T}) : \ |||\Phi(t)u|||^2_{H^s(\mathbb{T})} < +\infty\}) = \mu_s(\{v \in H^\sigma(\mathbb{T}) : \ |||v|||^2_{H^s(\mathbb{T})} < +\infty\}) = \mu_s(H^s(\mathbb{T})) = 0$$

We postpone the proof of Proposition 4.1 for Section 10 where a detailed analysis is provided. Instead, we prove now the following approximation result:

Proposition 4.4. *Let $s > \frac{3}{2}$. Let $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$ so that the conclusion of Proposition 4.1 holds. Then, for every compact set $K \subset H^\sigma(\mathbb{T})$,*

$$\sup_{u \in K} |Q_s(u) - Q_{s,N}(u)| \underset{N \to \infty}{\longrightarrow} 0$$

In other words, $Q_{s,N}$ converges to Q_s uniformly on compact sets of $H^\sigma(\mathbb{T})$.

Proof of Proposition 4.4. Assuming the statements in Proposition 4.1, the proof goes exactly the same as the proof of Proposition 3.3. □

5. Transport of Gaussian measures under the truncated flow

In this section, we prove that the transported measure $\Phi_N(t)_\# \mu_s$ is absolutely continuous with respect to μ_s. To do so, we directly establish an explicit formula for the Radon-Nikodym derivative of $\Phi_N(t)_\# \mu_s$ with respect to μ_s. Our method relies on a change-of-variable formula. Analogous change-of-variables have already been used: see for example [23] Proposition 6.6, or [33] Section 4.

5.1. A change-of-variable formula.

We dedicate this paragraph to this change-of-variable. It is equivalent to the statement that the truncated flow preserves a certain measure.
On the Euclidean space E_N, equipped with the orthonormal basis $\{e^{ikx}\}_{|k| \leq N}$, we consider the Lebesgue measure $\prod_{|n| \leq N} d\widehat{u}_k$, where $d\widehat{u}_k$ is the Lebesgue measure on $\mathbb{C} \cdot e^{ikx}$.

Proposition 5.1. *The measure*

$$\prod_{|k| \leq N} d\widehat{u}_k \otimes \mu^\perp_{s,N}$$

is invariant under the truncated flow. In other words, for every $N \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$(\Phi_N(t))_\# \left(\prod_{|k| \leq N} d\widehat{u}_k \otimes \mu^\perp_{s,N} \right) = \prod_{|k| \leq N} d\widehat{u}_k \otimes \mu^\perp_{s,N}$$

A reformulation of this proposition is:

Corollary 5.2 (change-of-variable formula)**.** *Let A a Borel measurable set. Let $f : H^\sigma(\mathbb{T}) \longrightarrow \mathbb{R}_+$ be a positive measurable function. Then,*

$$\int_{\Phi_N(t)A} f(v) \prod_{|k| \leq N} d\widehat{v}_k \otimes \mu^\perp_{s,N}(v) = \int_A f(\Phi_N(t)u) \prod_{|k| \leq N} d\widehat{u}_k \otimes \mu^\perp_{s,N}(u)$$

Proof of Proposition 5.1. Let $\widetilde{\Phi}_N(t)$ be the restriction of the truncated flow $\Phi_N(t)$ on the finite-dimensional space E_N, which maps E_N to itself (see Proposition A.12). We factorize $\Phi_N(t)$ as $(\widetilde{\Phi}_N(t), e^{it\partial_x^2})$ on $E_N \times E_N^\perp$, in the sense that $\Phi_N(t)u_0 = \widetilde{\Phi}_N(t)\Pi_N u_0 + e^{it\partial_x^2}\Pi_N^\perp u_0$ (see again Proposition A.12). Now, we provide a proof in two steps.

Step 1 : Firstly, we show the formula

$$(\Phi_N(t))_{\#}\left(\prod_{|k|\leq N} d\widehat{u}_k \otimes \mu^{\perp}_{s,N}\right) = \left(\prod_{|k|\leq N} d\widehat{u}_k\right) \otimes (e^{it\partial_x^2})_{\#}\mu^{\perp}_{s,N} \tag{5.1}$$

To do so, we rely on the fact $\mathcal{B}(H^\sigma(\mathbb{T})) = \mathcal{B}(E_N) \otimes \mathcal{B}(E_N^\perp)$[6]. Thanks to that point, if we show that for every $A_{low} \in \mathcal{B}(E_N)$ and $A_{high} \in \mathcal{B}(E_N^\perp)$, we have:

$$\Big(\prod_{|k|\leq N} d\widehat{u}_k \otimes \mu^{\perp}_{s,N}\Big)\big(\Phi_N(t)^{-1}(A_{low} \times A_{high})\big) = \Big(\prod_{|k|\leq N} d\widehat{u}_k\Big)(A_{low})\mu^{\perp}_{s,N}(e^{-it\partial_x^2}A_{high}), \tag{5.2}$$

then, the property of uniqueness of product measures[7] will ensure that the two measures in (5.1) coincide on $\mathcal{B}(E_N) \otimes \mathcal{B}(E_N^\perp)$, that is on $\mathcal{B}(H^\sigma(\mathbb{T}))$. Hence, let us prove (5.2). Let $A_{low} \in \mathcal{B}(E_N)$ and $A_{high} \in \mathcal{B}(E_N^\perp)$. On the one hand, from the factorization of the truncated flow, we have:

$$\Phi_N(t)^{-1}(A_{low} \times A_{high}) = \widetilde{\Phi}_N(t)^{-1}(A_{low}) \times e^{-it\partial_x^2}A_{high} \subset E_N \times E_N^\perp$$

so, using the definition of product measures we obtain:

$$\Big(\prod_{|k|\leq N} d\widehat{u}_k \otimes \mu^{\perp}_{s,N}\Big)\big(\Phi_N(t)^{-1}(A_{low} \times A_{high})\big) = \Big(\prod_{|k|\leq N} d\widehat{u}_k\Big)(\widetilde{\Phi}_N(t)^{-1}(A_{low}))\mu^{\perp}_{s,N}(e^{-it\partial_x^2}A_{high})$$

Thus, to prove (5.2), it remains to show that:

$$\Big(\prod_{|k|\leq N} d\widehat{u}_k\Big)\big(\widetilde{\Phi}_N(t)^{-1}(A_{low})\big) = \Big(\prod_{|k|\leq N} d\widehat{u}_k\Big)\big(A_{low}\big) \tag{5.3}$$

To do so, we recall that on the other hand, $\widetilde{\Phi}_N(t)$ is the flow of the Hamiltonian equation:

$$\begin{cases} i\partial_t u = \frac{\partial H_N}{\partial \overline{u}}(u) \\ u|_{t=0} = u_0 \in E_N \end{cases} \tag{FNLS}$$

on the finite-dimensional space E_N (see Proposition A.12). So the equality (5.3) follows from the application of Liouville's theorem, which states that the flow of finite-dimensional Hamiltonian equation preserves the Lebesgue measure. To sum up, we obtained (5.1). Let us now turn to the second step of the proof:

Step 2 : Secondly, we show the following invariance property:

$$(e^{it\partial_x^2})_{\#}\mu^{\perp}_{s,N} = \mu^{\perp}_{s,N}$$

We recall that the probability measure $\mu^{\perp}_{s,N}$ is the law of the random variable

$$\begin{aligned} X : \Omega &\longrightarrow H^\sigma(\mathbb{T}) \\ \omega &\longmapsto \sum_{|n|>N} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx} \end{aligned}$$

[6]Indeed, whenever X and Y are two topological separable spaces, we have $\mathcal{B}(X \times Y) = \mathcal{B}(X) \otimes \mathcal{B}(Y)$, where $\otimes$ is the symbol for tensor-product of sigma-algebra

[7]Both $\prod_{|k|\leq N} d\widehat{u}_k$ and $(e^{it\partial_x^2})_{\#}\mu^{\perp}_{s,N}$ being σ-finite measures.

where the sum converges in the space $\mathbb{L}^2(\Omega, H^\sigma(\mathbb{T}))$. Since $e^{it\partial_x^2}$ is a linear isometry on $\mathbb{L}^2(\mathbb{T})$, we deduce that

$$e^{it\partial_x^2} \circ X(\omega) = \sum_{|n|>N} \frac{e^{-itn^2} g_n(\omega)}{\langle n \rangle^s} e^{inx}$$

where the sum still converges in $\mathbb{L}^2(\Omega, H^\sigma(\mathbb{T}))$. Moreover, the Gaussian measures are invariant under rotations, so the family $\{e^{-itn^2} g_n\}_{n\in\mathbb{Z}}$ is still a family of independent standard complex Gaussian measures. Consequently, we obtain that X and $e^{it\partial_x^2} \circ X$ have the same law, and this means that

$$(e^{it\partial_x^2})_\# \mu_{s,N}^\perp = \mu_{s,N}^\perp$$

which is the invariance property of Step 2.

Conclusion : We get the desired result by combining Step 1 and Step 2. □

5.2. **The Radon-Nikodym derivative for the truncated transported Gaussian measures.** In this paragraph, we fix $N \in \mathbb{N}$. We will use Proposition 5.1 in order to obtain the Radon-Nikodym derivative of $\Phi_N(t)_\# \mu_s$ with respect to μ_s.

Proposition 5.3. *Let $s > \frac{3}{2}$, $R > 0$ and $N \in \mathbb{N}$. For every $t \in \mathbb{R}$, we have*

$$\Phi_N(t)_\# \mu_s = \exp\Big(-\frac{1}{2}(\|\Pi_N \Phi_N(-t)u\|_{H^s}^2 - \|\Pi_N u\|_{H^s}^2)\Big) d\mu_s$$

Moreover, we can rewrite this formula as

$$\Phi_N(t)_\# \mu_s = \exp\left(R_{s,N}(\Phi_N(-t)u) - R_{s,N}(u) - \int_0^{-t} Q_{s,N}(\Phi_N(\tau)u) d\tau \right) d\mu_s$$

where $R_{s,N}$ is defined in Definition 2.1 (see also (3.2)) and $Q_{s,N}$ is defined in (4.4) (see also Definition 2.3).

Remark 5.4. Such a formula for the density has also been obtained for the different models in [9], [10], [15] and [11].

Proof of Proposition 5.3. Firstly, we decompose μ_s as:

$$d\mu_s = \left(\frac{1}{Z_N} e^{-\frac{1}{2}\|\Pi_N u\|_{H^s}^2} \prod_{|k|\leq N} d\widehat{u}_k \right) \otimes d\mu_{s,N}^\perp$$

(see (2.3) and (2.4)). Then, thanks to a general feature for the transport of density measure, we have :

$$\begin{aligned}
\Phi_N(t)_\# d\mu_s &= \Phi_N(t)_\# \left(\frac{1}{Z_N} e^{-\frac{1}{2}\|\Pi_N u\|_{H^s}^2} \prod_{|k|\leq N} d\widehat{u}_k \right) \otimes d\mu_{s,N}^\perp \\
&= \frac{1}{Z_N} e^{-\frac{1}{2}\left\|\Pi_N(\Phi_N(t)^{-1}u)\right\|_{H^s}^2} \cdot \Phi_N(t)_\# \left(\prod_{|k|\leq N} d\widehat{u}_k \right) \otimes d\mu_{s,N}^\perp \\
&= \frac{1}{Z_N} e^{-\frac{1}{2}\|\Pi_N(\Phi_N(-t)(u)\|_{H^s}^2} \cdot \left(\prod_{|k|\leq N} d\widehat{u}_k \right) \otimes d\mu_{s,N}^\perp \\
&= e^{-\frac{1}{2}\left(\|\Pi_N(\Phi_N(-t)(u)\|_{H^s}^2 - \|\Pi_N u\|_{H^s}^2\right)} d\mu_s
\end{aligned}$$

where we used in the third line the invariance property from Proposition 5.1. Hence, the first statement of Proposition 5.3 is proven. To achieve the proof, we rewrite the term inside the exponential thanks to the definition of the modified energy (2.12) and the identity (2.18):

$$\begin{aligned}-\frac{1}{2}\big(\|\Pi_N(\Phi_N(-t)(u)\|_{H^s}^2-\|\Pi_N u\|_{H^s}^2\big)&=\int_0^{-t}-\frac{1}{2}\frac{d}{d\tau}\|\Pi_N(\Phi_N(\tau)(u)\|_{H^s}^2\,d\tau\\&=\int_0^{-t}\Big(\frac{d}{d\tau}\left(R_{s,N}(\Phi_N(\tau)u)\right)-\frac{d}{d\tau}(E_{s,N}(\Pi_N\Phi_N(\tau)u))\Big)d\tau\\&=\int_0^{-t}\Big(\frac{d}{d\tau}\left(R_{s,N}(\Phi_N(\tau)u)\right)-Q_{s,N}(\Phi_N(\tau)u)\Big)d\tau\\&=R_{s,N}(\Phi_N(-t)u)-R_{s,N}(u)-\int_0^{-t}Q_{s,N}(\Phi_N(\tau)u)d\tau\end{aligned}$$

which is the desired rewriting. □

6. Transport of Gaussian measures under the flow

In Section 5, we have seen that for every $t\in\mathbb{R}$, and every $N\in\mathbb{N}$:

$$\Phi_N(t)_\#\mu_s=G_{s,N}(t,.)\mu_s \tag{6.1}$$

where,

$$G_{s,N}(t,u)=\exp\left(R_{s,N}(\Phi_N(-t)u)-R_{s,N}(u)-\int_0^{-t}Q_{s,N}(\Phi_N(\tau)u)d\tau\right) \tag{6.2}$$

Our goal in this section is to "take the limit" $N\to\infty$ in order to extend this formula to $N=\infty$. Thus, we invoke:

$$G_s(t,u):=\exp\left(R_s(\Phi(-t)u)-R_s(u)-\int_0^{-t}Q_s(\Phi(\tau)u)d\tau\right) \tag{6.3}$$

and we aim to show the following proposition:

Proposition 6.1. *Let $s>\frac{3}{2}$ and $R>0$. Let $t\in\mathbb{R}$. Then,*

$$\Phi(t)_\#\mu_s=G_s(t,u)d\mu_s$$

In particular, μ_s is quasi-invariant under the flow of (1.1).

Remark 6.2. For $\sigma<s-\frac{1}{2}$ close enough to $s-\frac{1}{2}$, we deduce from Proposition 3.1, Proposition 4.1 (whose proofs are provided in Section 10), and from the continuity properties of the flow, that the map :

$$(t,u)\in\mathbb{R}\times H^\sigma(\mathbb{T})\mapsto\exp\left(R_s(\Phi(-t)u)-R_s(u)-\int_0^{-t}Q_{s,N}(\Phi_N(\tau)u)d\tau\right)$$

is continuous, for any $N\in\mathbb{N}\cup\{\infty\}$.
Moreover, on $H^\sigma(\mathbb{T})$, the a priori ill-defined object $-\frac{1}{2}(\|\Phi(-t)u\|_{H^s}^2-\|u\|_{H^s}^2)$ can be seen as the well-defined object:

$$-\frac{1}{2}(\|\Phi(-t)u\|_{H^s}^2-\|u\|_{H^s}^2):=R_s(\Phi(-t)u)-R_s(u)-\int_0^{-t}Q_s(\Phi(\tau)u)d\tau$$

In this section, we assume the statements from Proposition 3.1 and Proposition 4.1. Hence, we work with a $\sigma<s-\frac{1}{2}$ close enough to $s-\frac{1}{2}$ so that the pointwise properties from Proposition 3.1 and 4.1 are satisfied.

6.1. **Approximation properties.** Our main ingredients to "take the limit" $N \to \infty$ in (6.1) is two approximation properties. The first one is the inner regularity satisfied by probability measures on $H^\sigma(\mathbb{T})$:

Proposition 6.3 (inner regularity)**.** *Let μ be a finite measure on $(H^\sigma(\mathbb{T}), \mathcal{B}(H^\sigma(\mathbb{T})))$. Then, for any Borel set $A \subset H^\sigma(\mathbb{T})$, we have*

$$\mu(A) = \sup\{\mu(K) : \ K \subset A, \ K \text{ compact set in } H^\sigma(\mathbb{T})\}$$

Proof. This follows from the general fact that finite measures on Polish spaces are regular. □

As a consequence, we have:

Corollary 6.4. *Let μ and ν be two finite measures on $(H^\sigma(\mathbb{T}), \mathcal{B}(H^\sigma(\mathbb{T})))$. Assume that for every compact set $K \subset H^\sigma(\mathbb{T})$, we have:*

$$\mu(K) = \nu(K)$$

Then, $\mu = \nu$.

Proof. Let $A \in \mathcal{B}(H^\sigma(\mathbb{T}))$. Let us prove that $\mu(A) = \nu(A)$. Let $\varepsilon > 0$. From the inner regularity of μ (see Proposition 6.3), we invoke a compact set K of $H^\sigma(\mathbb{T})$ such that $K \subset A$ and $\mu(A) - \varepsilon \leq \mu(K)$. Thus,

$$\mu(A) - \varepsilon \leq \mu(K) = \nu(K) \leq \nu(A)$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\mu(A) \leq \nu(A)$. By interchanging the roles of μ and ν, we obtain the converse inequality. □

The second approximation property that we will use is the approximation of the expected density $G_s(t,.)$ by the *truncated densities* $G_{s,N}(t,.)$:

Proposition 6.5. *Let $s > \frac{3}{2}$. Let $K \subset H^\sigma(\mathbb{T})$ a compact set. Then, for every $t \in \mathbb{R}$:*

$$\sup_{u \in K} |G_s(t,u) - G_{s,N}(t,u)| \underset{N \to \infty}{\longrightarrow} 0$$

In other words, $G_{s,N}(t,.)$ converges to $G_s(t,.)$ uniformly on compact sets.

Proof. Let $t \in \mathbb{R}$. Recall that $G_{s,N}(t,.)$ and $G_s(t,.)$ are respectively defined in (6.2) and (6.3). We prove separately that:

(a) : $R_{s,N}(\Phi_N(t)u) - R_{s,N}(u)$ converges to $R_s(\Phi(t)u) - R_s(u)$ uniformly on compact sets as $N \to \infty$

and,

(b) : $\int_0^t Q_{s,N}(\Phi_N(\tau)u)d\tau$ converges to $\int_0^t Q_s(\Phi(\tau)u)d\tau$ uniformly on compact sets as $N \to \infty$.

Indeed, if we do so, we will obtain that:

$R_{s,N}(\Phi_N(t)u) - R_{s,N}(u) - \int_0^t Q_{s,N}(\Phi(_N\tau)u)d\tau$ converges to $R_s(\Phi(t)u) - R_s(u) - \int_0^t Q_s(\Phi(\tau)u)d\tau$ uniformly on compact sets as $N \to \infty$.

Then, since the exponential is continuous, this will lead to the result. In order not to repeat the same argument, and since (a) is similar to (b) and a little bit easier, we will only prove (b). Then, let $K \subset H^\sigma(\mathbb{T})$ be a compact set. Then,

$$\begin{aligned} &\Big| \int_0^t Q_s(\Phi(\tau)u)d\tau - \int_0^t Q_{s,N}(\Phi_N(\tau)u)d\tau \Big| \\ &\leq \int_0^t |Q_s(\Phi(\tau)u) - Q_{s,N}(\Phi(\tau)u)|d\tau + \int_0^t |Q_{s,N}(\Phi(\tau)u) - Q_{s,N}(\Phi_N(\tau)u)|d\tau \\ &\leq |t| \sup_{\tau\in[0,t]} |Q_s(\Phi(\tau)u) - Q_{s,N}(\Phi(\tau)u)| + |t| \sup_{\tau\in[0,t]} |Q_{s,N}(\Phi(\tau)u) - Q_{s,N}(\Phi_N(\tau)u)| \end{aligned} \tag{6.4}$$

On the one hand, from the continuity of $\Phi : \mathbb{R} \times H^\sigma \to H^\sigma$, the set $\{\Phi(\tau)u;\ \tau \in [0,t], u \in K\}$ is compact. Combining this with the fact that $Q_{s,N} \underset{N}{\longrightarrow} Q_s$ uniformly on compact sets (see Proposition 4.4) yields:

$$\sup_{u\in K} \sup_{\tau\in[0,t]} |Q_s(\Phi(\tau)u) - Q_{s,N}(\Phi(\tau)u)| \underset{N\to\infty}{\longrightarrow} 0$$

On the other hand, using (4.5) from Proposition 4.1 we can invoke a constant $C_s > 0$ independent of N such that:

$$\begin{aligned} &\sup_{u\in K} \sup_{\tau\in[0,t]} |Q_{s,N}(\Phi(\tau)u) - Q_{s,N}(\Phi_N(\tau)u)| \\ &\leq C_s \sup_{(\tau,u)\in[0,t]\times K} \|\Phi(\tau)u - \Phi_N(\tau)u\|_{H^\sigma} \left(1 + \|\Phi(\tau)u\|_{H^\sigma} + \|\Phi_N(\tau)u\|_{H^\sigma}\right)^9 \end{aligned} \tag{6.5}$$

Besides, from the Cauchy theory, there exists a constant $C > 0$ independent of N such that:

$$\sup_{(\tau,u)\in[0,t]\times K} \|\Phi(\tau)u\|_{H^\sigma} + \|\Phi_N(\tau)u\|_{H^\sigma} \leq C$$

(see Proposition A.5). And, from Proposition A.10, we also have:

$$\sup_{(\tau,u)\in[0,t]\times K} \|\Phi(\tau)u - \Phi_N(\tau)u\|_{H^\sigma} \underset{N\to\infty}{\longrightarrow} 0$$

Using these two facts in (6.5) yields:

$$\sup_{(\tau,u)\in[0,t]\times K} |Q_{s,N}(\Phi(\tau)u) - Q_{s,N}(\Phi_N(\tau)u)| \underset{N\to\infty}{\longrightarrow} 0$$

Finally, coming back to (6.4), we conclude that:

$$\sup_{u\in K} \Big| \int_0^t Q_s(\Phi(\tau)u)d\tau - \int_0^t Q_{s,N}(\Phi_N(\tau)u)d\tau \Big| \underset{N\to\infty}{\longrightarrow} 0$$

This completes the proof. □

6.2. **The Radon-Nikodym derivative for the transported Gaussian measure.** In this paragraph, we prove Proposition 6.1 based on the combination of (6.1) with the approximation properties above.

Remark 6.6. We will be able to use Corollary 6.4 with the measures $\Phi(t)_\# \mu_s$ and $G_s(t,.)d\mu_s$. Indeed, both are finite measures on $(H^\sigma(\mathbb{T}), \mathcal{B}(H^\sigma(\mathbb{T})))$. On the one hand, $\Phi(t)_\# \mu_s$ is a probability measure; on the other hand, Fatou's lemma provide the following a priori bound for

$G_s(t,.)d\mu_s(H^\sigma(\mathbb{T}))$:

$$G_s(t,.)d\mu_s(H^\sigma(\mathbb{T})) = \int_{H^\sigma} G_s(t,u)d\mu_s = \int_{H^\sigma} \lim_N G_{s,N}(t,u)d\mu_s \leq \liminf_N \int_{H^\sigma} G_{s,N}(t,u)d\mu_s = 1$$

Remark 6.7. The proof we provide below for Proposition 6.1 is similar to the one for Theorem 1.4 in [26]. However, it is worth noting that our proof do not require any $\mathbb{L}^p$-integrability for the truncated densities $G_{s,N}(t,.)$.

Proof of Proposition 6.1. Let $t \in \mathbb{R}$. Relying on Corollary 6.4, it suffices to prove that for every compact set K of $H^\sigma(\mathbb{T})$, we have:

$$G_s(t,u)d\mu_s(K) = \Phi(t)_\# \mu_s(K), \text{ that is, } \int_K G_s(t,u)d\mu_s = \mu_s(\Phi(-t)K) \tag{6.6}$$

Fix K a compact of $H^\sigma(\mathbb{T})$.
We invoke two real numbers σ_1 and σ_2 such that $\sigma < \sigma_1 < \sigma_2 < s - \frac{1}{2}$. Moreover, we invoke for $k \in \mathbb{N}$:

$$B_k^{H^{\sigma_2}} := \{u \in H^{\sigma_2}(\mathbb{T}) : \|u\|_{H^{\sigma_2}(\mathbb{T})} \leq k\}$$

the closed centered ball in $H^{\sigma_2}(\mathbb{T})$ of radius k. Note that $B_k^{H^{\sigma_2}}$ is compact in $H^{\sigma_1}(\mathbb{T})$ because $\sigma_1 < \sigma_2$. To establish (6.6), it suffices to prove that for all $k \in \mathbb{N}$:

$$\int_{K \cap B_k^{H^{\sigma_2}}} G_s(t,u)d\mu_s = \mu_s(\Phi(-t)(K \cap B_k^{H^{\sigma_2}})) \tag{6.7}$$

Indeed, if we do so, we will obtain that:

$$\begin{aligned}
&\int_{K \cap H^{\sigma_2}(\mathbb{T})} G_s(t,u)d\mu_s = \int_{\bigcup_{k\in\mathbb{N}}(K\cap B_k^{H^{\sigma_2}})} G_s(t,u)d\mu_s = \lim_{k\to\infty} \nearrow \int_{K\cap B_k^{H^{\sigma_2}}} G_s(t,u)d\mu_s \\
&= \lim_{k\to\infty} \nearrow \mu_s(\Phi(-t)(K \cap B_k^{H^{\sigma_2}})) = \mu_s\Big(\bigcup_{k\in\mathbb{N}} \Phi(-t)(K \cap B_k^{H^{\sigma_2}})\Big) = \mu_s\Big(\Phi(-t)\big(\bigcup_{k\in\mathbb{N}}(K \cap B_k^{H^{\sigma_2}})\big)\Big) \\
&= \mu_s\big(\Phi(-t)(K \cap H^{\sigma_2}(\mathbb{T}))\big)
\end{aligned}$$

Besides, since $\mu_s(H^\sigma(\mathbb{T}) \setminus H^{\sigma_2}(\mathbb{T})) = 0$, we have:

$$\int_K G_s(t,u)d\mu_s = \int_{K\cap H^{\sigma_2}(\mathbb{T})} G_s(t,u)d\mu_s$$

and,

$$\mu_s(\Phi(-t)K) = \mu_s\big((\Phi(-t)K) \cap H^{\sigma_2}(\mathbb{T})\big) = \mu_s\big(\Phi(-t)(K \cap H^{\sigma_2}(\mathbb{T}))\big)$$

Hence, if we prove (6.7), then we will obtain (6.6), and the proof will be complete. Thus, we now move on to the proof of (6.7). Let $k \in \mathbb{N}$ and denote $K_2 := K \cap B_k^{H^{\sigma_2}}$.

– Firstly, we prove that:

$$\int_{K_2} G_s(t,u)d\mu_s \leq \mu_s(\Phi(-t)(K_2)) \tag{6.8}$$

Since K_2 is compact in $H^\sigma(\mathbb{T})$, we obtain from Proposition 6.5 that:

$$\int_{K_2} G_s(t,u)d\mu_s = \lim_{N\to\infty} \int_{K_2} G_{s,N}(t,u)d\mu_s = \lim_{n\to\infty} \Phi_N(t)_\# \mu_s(K_2) = \lim_{N\to\infty} \mu_s(\Phi_N(-t)K_2) \tag{6.9}$$

Let $\varepsilon > 0$. Thanks to Corollary A.11, we invoke $N_0 \in \mathbb{N}$ such that:

$$N \geq N_0 \implies \Phi_N(-t)(K_2) \subset \Phi(-t)(K_2) + B_\varepsilon^{H^\sigma}$$

where $B_\varepsilon^{H^\sigma}$ is the closed centered ball in $H^\sigma(\mathbb{T})$ of radius ε. As a consequence, we have:

$$\limsup_N \mu_s(\Phi_N(-t)K_2) \leq \mu_s(\Phi(-t)(K_2) + B_\varepsilon^{H^\sigma})$$

Plugging this into (6.9), we obtain:

$$\int_{K_2} G_s(t,u)d\mu_s \leq \mu_s(\Phi(-t)(K_2) + B_\varepsilon^{H^\sigma})$$

Since $\varepsilon > 0$ is arbitrary, we deduce that[8]:

$$\int_{K_2} G_s(t,u)d\mu_s \leq \lim_{\varepsilon\to 0} \searrow \mu_s(\Phi(-t)(K_2) + B_\varepsilon^{H^\sigma}) = \mu_s\big(\bigcap_{\varepsilon>0}(\Phi(-t)(K_2) + B_\varepsilon^{H^\sigma})\big) = \mu_s(\Phi(-t)K_2)$$

So we have proven (6.8).

– Secondly, we prove that:

$$\int_{K_2} G_s(t,u)d\mu_s \geq \mu_s(\Phi(-t)(K_2)) \tag{6.10}$$

Let us first observe that K_2 is compact in $H^{\sigma_1}(\mathbb{T})$: if $\{u_n\}_{n\in\mathbb{N}} \in K_2^{\mathbb{N}}$ is a sequence in $K_2 = K \cap B_k^{H^{\sigma_2}}$, then, from the compactness of $B_k^{H^{\sigma_2}}$ in $H^{\sigma_1}(\mathbb{T})$ (because $\sigma_1 < \sigma_2$), there exists a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ and an element $u \in B_k^{H^{\sigma_2}}$ such that:

$$\big\|u_{n_j} - u\big\|_{H^{\sigma_1}} \underset{j\to\infty}{\longrightarrow} 0$$

In particular, we have $\big\|u_{n_j} - u\big\|_{H^\sigma} \underset{j\to\infty}{\longrightarrow} 0$ because $\sigma < \sigma_1$. Since K is closed in $H^\sigma(\mathbb{T})$, it implies that $u \in K$. Then $u \in K \cap B_k^{H^{\sigma_2}}$, and K_2 is compact in $H^{\sigma_1}(\mathbb{T})$.

Now, let $\varepsilon > 0$. Thanks to Corollary A.11, and the fact that K_2 is compact in $H^{\sigma_1}(\mathbb{T})$, we are able to invoke $N_1 \in \mathbb{N}$ such that:

$$N \geq N_1 \implies \Phi(-t)(K_2) \subset \Phi_N(-t)(K_2 + B_\varepsilon^{H^{\sigma_1}})$$

where $B_\varepsilon^{H^{\sigma_1}}$ is the closed centered ball in $H^{\sigma_1}(\mathbb{T})$ of radius ε. It is now important to notice that $K_2 + B_\varepsilon^{H^{\sigma_1}}$ is compact in $H^\sigma(\mathbb{T})$. It follows from the fact that both K_2 and $B_\varepsilon^{H^{\sigma_1}}$ are compact in $H^\sigma(\mathbb{T})$. As a consequence, we obtain from Proposition 6.5 that for $N \geq N_1$:

$$\mu_s(\Phi(-t)(K_2)) \leq \mu_s(\Phi_N(-t)(K_2 + B_\varepsilon^{H^{\sigma_1}})) = \int_{K_2+B_\varepsilon^{H^{\sigma_1}}} G_{s,N}(t,u)d\mu_s \underset{N\to\infty}{\longrightarrow} \int_{K_2+B_\varepsilon^{H^{\sigma_1}}} G_s(t,u)d\mu_s$$

Since $\varepsilon > 0$ is arbitrary, we obtain that:

$$\mu_s(\Phi(-t)(K_2)) \leq \lim_{\varepsilon\to 0} \searrow \int_{K_2+B_\varepsilon^{H^{\sigma_1}}} G_s(t,u)d\mu_s = \int_{K_2} G_s(t,u)d\mu_s$$

So we have proven (6.10). This completes the proof of Proposition 6.1. □

[8]Since $\Phi(-t)K_2$ is closed in $H^\sigma(\mathbb{T})$, we have $\bigcap_{\varepsilon>0}(\Phi(-t)(K_2) + B_\varepsilon^{H^\sigma}) = \Phi(-t)K_2$

7. Weighted Gaussian measures, L^p-estimates on the weight, and transport along the flows

In Section 6, we proved that for every $s > \frac{3}{2}$ and every $t \in \mathbb{R}$, we have:

$$\Phi(t)_{\#}\mu_s = G_s(t,u)\mu_s$$

where $G_s(t,u)$ is the continuous function on $H^\sigma(\mathbb{T})$ (for a given $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$) given by:

$$G_s(t,u) = \exp\left(R_s(\Phi(-t)u) - R_s(u) - \int_0^{-t} Q_s(\Phi(\tau)u)d\tau \right)$$

In particular, it implies that $G_s(t,.)$ belongs to $\mathbb{L}^1(d\mu_s)$ (with a $\mathbb{L}^1(d\mu_s)$-norm equal to 1). Thus, it is interesting to wonder if $G_s(t,.)$ belongs to $\mathbb{L}^p(d\mu_s)$ for $p \in (1,\infty)$. We will see that the answer is positive for all $p \in (1,\infty)$ if we restrict μ_s on bounded sets of $H^1(\mathbb{T})$, that is at the level where we can make use of the conservation of the Hamiltonian and of the $\mathbb{L}^2(\mathbb{T})$-norm. To do so, we need to introduce *weighted Gaussian measures*.

In Section 2, we identified a modified energy in Definition 2.1. Based on this modified energy, we define the weighted Gaussian measures. Formally, the idea is to replace the Gaussian measure $\frac{1}{Z_s}e^{-\frac{1}{2}\|u\|^2_{H^s(\mathbb{T})}}du$ by $\frac{1}{Z'_s}e^{-E_s(u)}du$. However, we need to add a cut-off at the energy level, where the Hamiltonian and the $\mathbb{L}^2(\mathbb{T})$-norm are conserved by the flow. In this section, we introduce the weighted Gaussian measures as density measures with respect to the Gaussian measure μ_s. We also provide $\mathbb{L}^p$-estimates on these densities, ensuring in particular that the weighted Gaussian measure are well-defined probability measures on $H^\sigma(\mathbb{T})$, $\sigma < s - \frac{1}{2}$.

7.1. Definitions.

We start by invoking the following quantity :

$$\mathcal{C}(u) := \frac{1}{2}\|u\|^2_{\mathbb{L}^2} + H(u)$$

which is conserved by the flow of (1.1). Next, for $R > 0$, and for every $N \in \mathbb{N}$, we define the weighted Gaussian measures as

$$d\rho_{s,R,N} := \frac{1}{Z_{s,R,N}}\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_{s,N}(u)}d\mu_s, \qquad d\rho_{s,R} := \frac{1}{Z_{s,R}}\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_s(u)}d\mu_s \tag{7.1}$$

where,

$$Z_{s,R,N} := \int_{H^{s-\frac{1}{2}-}} \mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_{s,N}(u)}d\mu_s, \qquad Z_{s,R} := \int_{H^{s-\frac{1}{2}-}} \mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_s(u)}d\mu_s$$

are normalizing constants ensuring that $\rho_{s,R,N}$ and $\rho_{s,R}$ are probability measures (if they are positive and finite, see Remark 7.5). We recall that $R_{s,N}(u)$ and $R_s(u)$ have been defined in Definition 2.1 (see also (3.2)).

Notation 7.1. We also use the notations $\rho_{s,R,\infty}$ and $Z_{s,R,\infty}$ to respectively refer to $\rho_{s,R}$ and $Z_{s,R}$.

Remark 7.2 ($H^1(\mathbb{T})$ cut-off)**.** We can rewrite $\mathcal{C}(u)$ as :

$$\mathcal{C}(u) = \frac{1}{2}\|u\|^2_{H^1} + \frac{1}{6}\|u\|^6_{\mathbb{L}^6}$$

From the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow \mathbb{L}^6(\mathbb{T})$, we have

$$\frac{1}{2}\|u\|^2_{H^1} \leq \mathcal{C}(u) \leq C(1 + \|u\|_{H^1})^6$$

It means that the cut-off $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}$ is a $H^1(\mathbb{T})$ cut-off. In particular, (for $R \geq 2$)

$$\mathcal{C}(u) \leq R \implies \|u\|_{H^1} \leq \sqrt{2R} \leq R$$

so,

$$\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \leq \mathbb{1}_{B_R^{H^1}}(u)$$

where $B_R^{H^1}$ is the closed center ball of radius R in $H^1(\mathbb{T})$.
The additional nice property is that the quantity $\mathcal{C}(u)$ is conserved by the flow of (1.1).

Remark 7.3. (Passing from $\rho_{s,R,N}$ to μ_s) Let $R > 0$. The measure $\mu_s|_{\{\mathcal{C}\leq R\}}$ coincide with the measure $Z_{s,N,R}e^{R_{s,N}(u)}\rho_{s,R,N}|_{\{\mathcal{C}\leq R\}}$. In other words, for every Borel set $A \subset H^1(\mathbb{T})$ such that $A \subset \{\mathcal{C} \leq R\}$, we have

$$Z_{s,N,R}e^{R_{s,N}(u)}\rho_{s,R,N}(A) = \mu_s(A)$$

Indeed, it results from

$$\begin{aligned} Z_{s,N,R}e^{R_{s,N}(u)}\rho_{s,R,N}(A) &= \int_A Z_{s,N,R}e^{R_{s,N}(u)}d\rho_{s,R,N}(u) \\ &= \int_A Z_{s,N,R}e^{R_{s,N}(u)}\underbrace{\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}}_{=1}e^{-R_{s,N}(u)}\frac{1}{Z_{s,N,R}}d\mu_s(u) \\ &= \int_A d\mu_s(u) = \mu_s(A) \end{aligned}$$

Now, we state the following crucial proposition, whose proof is postponed to the dedicated Section 11.

Proposition 7.4. *Let $s > \frac{3}{2}$ and $R > 0$. Then for any $p \in [1, +\infty)$, there exists a constant $C(s, p, R) > 0$ such that for every $N \in \mathbb{N} \cup \{\infty\}$, we have:*

$$\left\|\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{|R_{s,N}(u)|}\right\|_{\mathbb{L}^p(d\mu_s)} \leq C(s, p, R) \tag{7.2}$$

Moreover,

$$\left\|\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_s(u)} - \mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_{s,N}(u)}\right\|_{\mathbb{L}^p(d\mu_s)} \underset{N\to\infty}{\longrightarrow} 0$$

In particular, $Z_{s,N,R} \underset{N}{\longrightarrow} Z_{s,R}$ so that we also have:

$$\left\|\frac{1}{Z_{s,R}}\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_s(u)} - \frac{1}{Z_{s,R,N}}\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_{s,N}(u)}\right\|_{\mathbb{L}^p(d\mu_s)} \underset{N\to\infty}{\longrightarrow} 0 \tag{7.3}$$

Remark 7.5. The inequality (7.2) in Proposition 7.4 ensures that for all $N \in \mathbb{N} \cup \{\infty\}$:

$$Z_{s,R,N} = \int_{H^{s-\frac{1}{2}}} \mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_{s,N}(u)}d\mu_s < +\infty$$

Besides, we also have $Z_{s,R,N} > 0$ because one can show that $\mu_s(\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_{s,N}(u)} > 0) > 0$. To see this, we write on the one hand:

$$\mu_s\left(\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}e^{-R_{s,N}(u)} > 0\right) = \mu_s\left(\{\mathcal{C}(u) \leq R\} \cap R_{s,N} < +\infty\right) = \mu_s\left(\{\mathcal{C}(u) \leq R\}\right)$$

and on the other hand, we have that $\mu_s(\{\mathcal{C}(u) \leq R\}) > 0$, for any $R > 0$, since μ_s charges all open sets of $H^1(\mathbb{T})$. For this latter point, we refer to [7], Proposition 1.2. Note also that since $Z_{s,R,N} \to Z_{s,R}$, there exists a constant $C_{s,R} > 0$ such that:

$$\frac{1}{C_{s,R}} \leq Z_{s,R,N} \leq C_{s,R} \tag{7.4}$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$.

7.2. **Transport of weighted Gaussian measures along the flows.** For $N \in \mathbb{N} \cup \{\infty\}$, now that we know explicitly the density of $\Phi_N(t)_\# \mu_s$ with respect to μ_s, we are able to obtain the density of $\Phi_N(t)_\# \rho_{s,R,N}$ with respect to $\rho_{s,R,N}$ without re-performing the analysis of Section 5 and 6.

We stress the fact that the following proposition holds for $N \in \mathbb{N}$ **and** $N = \infty$.

Proposition 7.6. *Let $s > \frac{3}{2}$, $R > 0$ and $N \in \mathbb{N} \cup \{\infty\}$. For every $t \in \mathbb{R}$, we have:*

$$\Phi_N(t)_\# \rho_{s,R,N} = F_{s,N}(t,u) d\rho_{s,R,N}$$

for a function $F_{s,N}(t,.)$ given by the explicit formula:

$$F_{s,N}(t,u) = \exp\left(- \int_0^{-t} Q_{s,N}(\Phi_N(\tau)u) d\tau \right)$$

where $Q_{s,N}$ is defined in (4.4) (see also Definition 2.3). In particular, $\rho_{s,R}$ is quasi-invariant along the flow of (1.1); and when $N \in \mathbb{N}$, $\rho_{s,R,N}$ is quasi-invariant along the flow of (2.1).

Remark 7.7. When $N \in \mathbb{N}$, we have:

$$\int_0^t Q_{s,N}(\Phi_N(\tau)u) d\tau = E_{s,N}(\Pi_N \Phi_N(t)u) - E_{s,N}(\Pi_N u)$$

Hence, the a priori ill-defined object $E_s(\Phi(t)u) - E_s(u)$ (on the support of μ_s) can be seen as:

$$E_s(\Phi(t)u) - E_s(u) := \int_0^t Q_s(\Phi(\tau)u) d\tau$$

which is a continuous function on $H^\sigma(\mathbb{T})$ for $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$.

Proof of Proposition 7.6. Let $N \in \mathbb{N} \cup \{\infty\}$. We start by applying a general feature for the transport of density measures:

$$\begin{aligned} \Phi_N(t)_\# \rho_{s,R,N} &= \Phi_N(t)_\# \left(\frac{1}{Z_{s,R,N}} \mathbb{1}_{\{\mathcal{C}(u) \leq R\}} e^{-R_{s,N}(u)} d\mu_s \right) \\ &= \frac{1}{Z_{s,R,N}} \mathbb{1}_{\{\mathcal{C}(\Phi_N(t)^{-1}u) \leq R\}} e^{-R_{s,N}(\Phi_N(t)^{-1}u)} \Phi_N(t)_\# d\mu_s \end{aligned}$$

Next, thanks to Proposition 5.3 and 6.1, along with the facts that $\Phi_N(t)^{-1} = \Phi_N(-t)$ and that $\mathcal{C}$ is conserved by the flows, we obtain:

$$\begin{aligned} \Phi_N(t)_\# \rho_{s,R,N} &= \frac{1}{Z_{s,R,N}} \mathbb{1}_{\{\mathcal{C}(u) \leq R\}} \exp\left(-R_{s,N}(u) - \int_0^{-t} Q_{s,N}(\Phi_N(\tau)u) d\tau \right) d\mu_s \\ &= \exp\left(- \int_0^{-t} Q_{s,N}(\Phi_N(\tau)u) d\tau \right) d\rho_{s,R,N} \end{aligned}$$

which completes the proof. □

Remark 7.8. Let $s > \frac{3}{2}$ and $\sigma < s - \frac{1}{2}$. Then, for every $t \in \mathbb{R}$, $F_{s,N}(t,.)$ converges to $F_s(t,.)$ uniformly on compact sets of $H^\sigma(\mathbb{T})$ (see the proof of Proposition 6.5).

8. Densities in L^p and convergence in L^p of the truncated densities

Let $s > \frac{3}{2}$. In Section 5 and 6, we have seen that for every $N \in \mathbb{N}$ and every $t \in \mathbb{R}$,

$$\Phi_N(t)_\# \mu_s = G_{s,N}(t,.)\mu_s, \qquad \text{and,} \qquad \Phi(t)_\# \mu_s = G_s(t,.)\mu_s$$

where $G_{s,N}$ and G_s are known explicitly (see Proposition 5.3 and 6.1). In this section, our goal is to prove that for every fixed $R > 0$, and every $t \in \mathbb{R}$:

$$G_s(t,.),\ G_{s,N}(t,.) \in \mathbb{L}^p(d\mu_{s,R}) \text{ and: } \|G_s(t,.) - G_{s,N}(t,.)\|_{\mathbb{L}^p(d\mu_{s,R})} \underset{N\to\infty}{\longrightarrow} 0$$

where $\mu_{s,R}$ is the *restricted Gaussian measure* define by:

$$\mu_{s,R} := \mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mu_s \tag{8.1}$$

This is equivalent to the fact that:

$$\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}G_s(t,.),\ \mathbb{1}_{\{\mathcal{C}(u)\leq R\}}G_{s,N}(t,.) \in \mathbb{L}^p(d\mu_s) \text{ and: } \left\|\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}(G_s(t,.) - G_{s,N}(t,.))\right\|_{\mathbb{L}^p(d\mu_s)} \underset{N\to\infty}{\longrightarrow} 0$$

Remark 8.1. Since $\mathcal{C}$ is conserved by the flow, we still have:

$$\Phi_N(t)_\# \mu_{s,R} = G_{s,N}(t,.)\mu_{s,R}, \qquad \text{and,} \qquad \Phi(t)_\# \mu_{s,R} = G_s(t,.)\mu_{s,R} \tag{8.2}$$

In our approach, we do not consider directly $\mu_{s,R}$ but we consider instead the weighted Gaussian measures $\rho_{s,R,N}$. We will then be able to prove a *quantitative inequality* in Proposition 8.5 thanks to suitable $\mathbb{L}^p$-estimates on $Q_{s,N}$. In a second step, we will be able to go back to $\mu_{s,R}$ by proving the same quantitative inequaltiy for $\mu_{s,R}$.
The quantitative inequaltiy (8.6) is significant in itself; indeed, it is often used to obtain the quasi-invariance without knowing the Radon-Nikodym derivative, see for example [13], [23], [33],[15]. Here, we know explicitly the Radon-Nikodym derivative of the transported measure, and from this point it will be easier to establish (8.6).

8.1. Quantitative quasi-invariance.

Recall that for convenience we use the notations $\rho_{s,R,\infty} = \rho_{s,R}, Q_s = Q_{s,\infty}$, etc.
We start this paragraph by providing L^p estimates for $Q_{s,N}$. These estimates will help us to establish the *quantitative quasi-invariance property* in Proposition 8.5. We postpone the proof of the following proposition to Section 12 where a detailed analysis is provided.

Proposition 8.2. *Let $s > \frac{3}{2}$. There exists $\beta \in (0,1)$ such that for every $R > 0$, there exists a constant $C(s,R) > 0$, such that for any $p \in [2, +\infty)$,*

$$\left\|\mathbb{1}_{\{\mathcal{C}u\leq R\}}Q_{s,N}(u)\right\|_{\mathbb{L}^p(d\mu_s)} \leq C(s,R)p^\beta \tag{8.3}$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$.

Combining estimate (8.3) with estimate (7.2) from Proposition 7.4, we also have:

Proposition 8.3. *Let $s > \frac{3}{2}$. There exists $\beta \in (0,1)$ such that for every $R > 0$, there exists a constant $C(s,R) > 0$, such that for any $p \in [1, +\infty)$,*

$$\|Q_{s,N}(u)\|_{\mathbb{L}^p(d\rho_{s,R,N})} \leq C(s,R)p^\beta \tag{8.4}$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$.

Proof of Proposition 8.3 assuming (8.3) and (7.2). It results from Cauchy-Schwarz that:

$$\begin{aligned}\|Q_{s,N}(u)\|_{\mathbb{L}^p(d\rho_{s,R,N})} &= \left\| \frac{1}{Z_{s,R,N}} \mathbb{1}_{\{\mathcal{C}u \leq R\}} |Q_{s,N}(u)|^p e^{-R_{s,N}(u)} \right\|^{1/p}_{\mathbb{L}^1(d\mu_s)} \\ &\leq \frac{1}{Z_{s,R,N}} \left\| \mathbb{1}_{\{\mathcal{C}u \leq R\}} Q_{s,N}(u) \right\|_{\mathbb{L}^{2p}(d\mu_s)} \left\| \mathbb{1}_{\{\mathcal{C}u \leq R\}} e^{-R_{s,N}(u)} \right\|^{1/p}_{\mathbb{L}^2(d\mu_s)} \\ &\leq C(s,R) 2^\beta p^\beta \cdot C(s,R,2)^{1/p} \leq C(s,R) p^\beta\end{aligned}$$

where the constant $C(s,R)$ has changed but still depends only on s and R. Note also that we used (7.4). □

The following identity will be our starting point in order to obtain the forthcoming inequality (8.6).

Proposition 8.4. *Let $s > \frac{3}{2}$, $R > 0$, $N \in \mathbb{N} \cup \{\infty\}$ and $t \in \mathbb{R}$. For every Borel set $A \subset H^\sigma(\mathbb{T})$, we have :*

$$\frac{d}{dt} \rho_{s,R,N} \left(\Phi_N(t)(A) \right) = - \int_{\Phi_N(t)A} Q_{s,N}(u) d\rho_{s,R,N} \tag{8.5}$$

Proof of Proposition 8.4. We use the explicit formula for the density from Proposition 7.6, so that we obtain:

$$\begin{aligned}\frac{d}{dt} \rho_{s,R,N} \left(\Phi_N(t)(A) \right) &= \frac{d}{dt} \left(\Phi_N(-t)_\# \rho_{s,R,N} \right)(A) = \frac{d}{dt} \int_A e^{-\int_0^t Q_{s,N}(\Phi_N(\tau)u) d\tau} d\rho_{s,R,N} \\ &= - \int_A Q_{s,N}(\Phi_N(t)u) d \left(\Phi_N(-t)_\# \rho_{s,R,N} \right)(u) = - \int_{\Phi_N(t)A} Q_{s,N}(u) d\rho_{s,R,N}\end{aligned}$$

where the last equality follows from the definition of a push-forward measure. □

Proposition 8.5. *Let $s > \frac{3}{2}$, $R > 0$ and $t \in \mathbb{R}$. Then, there exists $\beta \in (0,1)$ such that for every $\alpha \in (0,1)$, there exists a constant $C = C_{s,R,\alpha,\beta} > 0$ such that for all Borel set $A \subset H^\sigma(\mathbb{T})$:*

$$\rho_{s,R,N}(\Phi_N(t)A) \leq \rho_{s,R,N}(A)^{1-\alpha} \exp\left(C(1+|t|)^{\frac{1}{1-\beta}} \right) \tag{8.6}$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$.

Proof. Let $s > \frac{3}{2}$, $R > 0$ and $N \in \mathbb{N} \cup \{\infty\}$. Let $A \subset H^\sigma(\mathbb{T})$ be a Borel set. Using (8.5), Hölder inequality and the energy estimate from Proposition 8.3, we obtain :

$$\left| \frac{d}{dt} \rho_{s,R,N}(\Phi_N(t)A) \right| \leq \|Q_{s,N}\|_{\mathbb{L}^p(d\rho_{s,R,N})} \rho_{s,R,N}(\Phi_N(t)A)^{1-\frac{1}{p}} \leq C_{s,R}\, p^\beta \rho_{s,R,N}(\Phi_N(t)A)^{1-\frac{1}{p}}$$

for all $p \in [1, +\infty)$. It means that the function $F(t) := \rho_{s,R,N}(\Phi_N(t)A)$ satisfies the differential inequality :

$$|F'(t)| \leq C_{s,R}\, p^\beta F(t)^{1-\frac{1}{p}}$$

Integrating this yields :

$$\begin{aligned}F(t) \leq \left(F(0)^{\frac{1}{p}} + C_{s,R}|t| p^{\beta-1} \right)^p &= F(0) \left(1 + C_{s,R}|t| F(0)^{-\frac{1}{p}} p^{\beta-1} \right)^p \\ &= F(0) \exp\left(p \log(1 + C_{s,R}|t| F(0)^{-\frac{1}{p}} p^{\beta-1}) \right)\end{aligned}$$

Then, using the inequality $\log(1+x) \leq x$ implies :

$$F(t) \leq F(0) \exp\left(C_{s,R}|t| F(0)^{-\frac{1}{p}} p^\beta) \right)$$

Now, we choose $p := 1 + \log(\frac{1}{F(0)})$ so that $F(0)^{-\frac{1}{p}} = \exp(\frac{\log(F(0))}{\log(F(0))-1}) \leq \exp(1)$. Hence,

$$F(t) \leq F(0)\exp\left(C_{s,R}|t|(1-\log F(0))^{\beta}\right) \tag{8.7}$$

Let us mention the following elementary lemma, whose proof is provided afterwards.

Lemma 8.6. *For every $\alpha \in (0,1)$, there exists a constant $C_{s,R,\alpha,\beta} > 0$ such that for all $x \geq 0$:*

$$C_{s,R}|t|\,(1+x)^{\beta} \leq \alpha x + C_{s,R,\alpha,\beta}\,(1+|t|)^{\frac{1}{1-\beta}}$$

Using Lemma 8.6 in (8.7) with $x = -\log(F(0)) \geq 0$ yields :

$$F(t) \leq F(0)^{1-\alpha}\exp\left(C_{s,R,\alpha,\beta}(1+|t|)^{\frac{1}{1-\beta}}\right)$$

which is the desired inequality, so the proof is completed. □

Here, we provide a proof of Lemma 8.6:

Proof of Lemma 8.6. Let $\alpha \in (0,1)$. We invoke the function $f(x) := C_{s,R}|t|(1+x)^{\beta} - \alpha x$. We aim to show that $f(x) \leq C_{s,R,\alpha,\beta}\,(1+|t|)^{\frac{1}{1-\beta}}$. Recall that $\beta \in (0,1)$. We have :

$$f'(x) = \beta C_{s,R}|t|(1+x)^{\beta-1} - \alpha \geq 0 \iff (1+x)^{\beta-1} \geq \frac{\alpha}{\beta C_{s,R}|t|} \iff x \leq \left(\frac{\beta C_{s,R}|t|}{\alpha}\right)^{\frac{1}{1-\beta}} - 1$$

This implies that f has a maximum at the point $x = \left(\frac{\beta C_{s,R}|t|}{\alpha}\right)^{\frac{1}{1-\beta}} - 1$. Thus, for all $x \geq 0$:

$$\begin{aligned} f(x) &\leq C_{s,R}|t|\left(\frac{\beta C_{s,R}|t|}{\alpha}\right)^{\frac{\beta}{1-\beta}} - \alpha\left(\frac{\beta C_{s,R}|t|}{\alpha}\right)^{\frac{1}{1-\beta}} + \alpha \\ &= \alpha + (C_{s,R}|t|)^{\frac{1}{1-\beta}}\left(\left(\frac{\beta}{\alpha}\right)^{\frac{\beta}{1-\beta}} - \alpha\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\beta}}\right) \\ &\leq C_{s,R,\alpha,\beta}(1+|t|)^{\frac{1}{1-\beta}} \end{aligned}$$

This completes the proof of Lemma 8.6. □

Now, we can go back to the measure $\mu_{s,R}$ (see (8.1) for the definition). Indeed, we are able to deduce from Proposition 8.5 the analogous proposition for $\mu_{s,R}$.

Proposition 8.7. *Let $s > \frac{3}{2}$, $R > 0$ and $t \in \mathbb{R}$. Then, there exists $\beta \in (0,1)$ such that for every $\alpha \in (0,1)$, there exists a constant $C = C_{s,R,\alpha,\beta} > 0$ such that for all Borel set $A \subset H^1(\mathbb{T})$:*

$$\mu_{s,R}(\Phi_N(t)A) \leq \mu_{s,R}(A)^{1-\alpha}\exp\left(C(1+|t|)^{\frac{1}{1-\beta}}\right) \tag{8.8}$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$.

Proof. Let $N \in \mathbb{N} \cup \{\infty\}$. Recall that $\mu_{s,R}$ is defined in (8.1), and observe that we have:

$$\mu_{s,R} = Z_{s,N,R}e^{R_{s,N}(u)}d\rho_{s,R,N}$$

(see also Remark 7.3). Let $\alpha \in (0,1)$. Let us invoke the $\beta \in (0,1)$ and the constant $C > 0$ from Proposition 8.5. Then, for every $q_1 \in (1,\infty)$:

$$\begin{aligned} \mu_{s,R}(\Phi_N(t)A) = Z_{s,R,N}\int_{\Phi(t)A} e^{R_{s,N}(u)}d\rho_{s,R,N} &\leq Z_{s,R,N}\left\|e^{R_{s,N}}\right\|_{\mathbb{L}^{q_1}(d\rho_{s,R,N})}\rho_{s,R,N}(\Phi(t)A)^{1-\frac{1}{q_1}} \\ &\leq C_{s,R,q_1}\rho_{s,R,N}(\Phi(t)A)^{1-\frac{1}{q_1}} \end{aligned}$$

where in the last inequality we used (7.4) and Proposition 7.4. Using now Proposition 8.5, we obtain:

$$\mu_{s,R}(\Phi_N(t)A) \leq \rho_{s,R,N}(A)^{(1-\alpha)(1-\frac{1}{q_1})}\exp\left(C(1+|t|)^{\frac{1}{1-\beta}}\right) \tag{8.9}$$

where the constant $C>0$ depends on s, R, α and q_1 (and also β). On the other hand, for every $q_2 \in (1,\infty)$:

$$\begin{aligned}\rho_{s,R,N}(A) = \frac{1}{Z_{s,R,N}}\int_A e^{-R_{s,N}}d\mu_{s,R} &\leq \frac{1}{Z_{s,R,N}}\left\|e^{-R_{s,N}}\right\|_{\mathbb{L}^{q_2}(d\mu_{s,R})}\mu_{s,R}(A)^{1-\frac{1}{q_2}}\\ &\leq C_{s,R,q_2}\mu_{s,R}(A)^{1-\frac{1}{q_2}}\end{aligned}$$

where again in the last inequality we used (7.4) and Proposition 7.4. Then, plugging this into (8.9) yields:

$$\mu_{s,R}(\Phi_N(t)A) \leq \rho_{s,R,N}(A)^{(1-\alpha)(1-\frac{1}{q_1})(1-\frac{1}{q_2})}\exp\left(C(1+|t|)^{\frac{1}{1-\beta}}\right)$$

where now, the constant $C>0$ depends on s, R, α, q_1 and q_2. Finally, since $\alpha \in (0,1)$ and $q_1, q_2 \in (1,\infty)$ are arbitrary, we have that $(1-\alpha)(1-\frac{1}{q_1})(1-\frac{1}{q_2})$ takes all the values in $(0,1)$. This completes the proof. □

8.2. **Consequences: uniform L^p integrability and convergence.** In this paragraph, we perform an analysis similar to the one from Section 7 of [15]. As a consequence of Proposition 8.7, we are now able to prove the following proposition. Here, for clarity, we denote $G_{s,N,t} = G_{s,N}(t,.)$ for $N \in \mathbb{N}\cup\{\infty\}$ (still $G_{s,\infty,t} = G_{s,t}$).

Proposition 8.8. *Let $s > \frac{3}{2}$, $R>0$ and $t \in \mathbb{R}$. For every $p \in [1,+\infty)$, there exists a constant $C(s,R,p,t) \in (0,+\infty)$ such that :*

$$\|G_{s,N,t}\|_{\mathbb{L}^p(d\mu_{s,R})} \leq C(s,R,p,t)$$

uniformly in $N \in \mathbb{N}\cup\{\infty\}$.

Proof. Let $p \in [1,+\infty)$ and let us prove that $G_{s,N,t}$ belongs to $\mathbb{L}^p(d\mu_{s,R})$. To do so, we use Cavalieri's principle :

$$\|G_{s,N,t}\|^p_{L^p(d\mu_{s,R})} = p\int_0^\infty \lambda^{p-1}\mu_{s,R}(G_{s,N,t}>\lambda)d\lambda \tag{8.10}$$

We can estimate $\mu_{s,R}(G_{s,N,t}>\lambda)$ as follows :

$$\begin{aligned}\mu_{s,R}(G_{s,N,t}>\lambda) &= \int \mathbb{1}_{\{G_{s,N,t}>\lambda\}}d\mu_{s,R} = \int \frac{1}{G_{s,N,t}}\mathbb{1}_{\{G_{s,N,t}>\lambda\}}G_{s,N,t}d\mu_{s,R}\\ &= \int \frac{1}{G_{s,N,t}}\mathbb{1}_{\{G_{s,N,t}>\lambda\}}d\left(\Phi_N(t)_\#\mu_{s,R}\right) \leq \frac{1}{\lambda}\int \mathbb{1}_{\{G_{s,N,t}>\lambda\}}d\left(\Phi_N(t)_\#\mu_{s,R}\right) = \frac{1}{\lambda}\mu_{s,R}(\Phi_N(-t)(G_{s,N,t}>\lambda))\end{aligned}$$

Here, we can make use of estimate (8.8) and deduce that for any $\alpha \in (0,1)$, there exists $C = C_{s,R,\alpha} > 0$ such that

$$\mu_{s,R}(G_{s,N,t}>\lambda) \leq \frac{1}{\lambda}\mu_{s,R}(G_{s,N,t}>\lambda)^{1-\alpha}\exp\left(C(1+|t|)^{\frac{1}{1-\beta}}\right)$$

So,

$$\mu_{s,R}(G_{s,N,t}>\lambda) \leq \lambda^{-1/\alpha}\exp\left(C'(1+|t|)^{\frac{1}{1-\beta}}\right) =: \lambda^{-1/\alpha}C_{s,R,\alpha,t}$$

(Where $C' = \frac{C}{\alpha}$ still depends only on s, R and α). Plugging this into (8.10) yields

$$\|G_{s,N,t}\|^p_{L^p(d\mu_{s,R})} \leq p + \int_1^\infty \lambda^{p-1}\mu_{s,R}(G_{s,N,t} > \lambda)d\lambda \leq p + C_{s,R,\alpha,t}\int_1^\infty \lambda^{p-1-1/\alpha}d\lambda$$

Choosing $\alpha \in (0,1)$ such that $p-1-\frac{1}{\alpha} < -1$, that is $\alpha < \frac{1}{p}$, we have that the integral $\int_1^\infty \lambda^{p-1-1/\alpha}d\lambda$ is finite. Hence, denoting by

$$C(s,R,p,t) := p + C_{s,R,\alpha,t}\int_1^\infty \lambda^{p-1-1/\alpha}d\lambda$$

leads the result. □

Proposition 8.9. *Let $s > \frac{3}{2}$ and $R > 0$. Then, for every $t \in \mathbb{R}$ and every $p \in [1, +\infty)$, $G_{s,N}(t,.)$ converges to $G_s(t,.)$ in $\mathbb{L}^p(d\mu_{s,R})$.*

Proof. Let $q \in (1, +\infty)$. From Proposition 8.8 and 6.5, we know that the two following facts hold:

$$\begin{cases} \sup_{N\in\mathbb{N}} \|G_{s,N}(t,.)\|_{L^q(d\mu_{s,R})} < +\infty \\ G_{s,N}(t,.) \text{ converges in measure to } G_s(t,.) \text{ (with respect to } \mu_{s,R}) \end{cases}$$

Then – see for example [31] Remark 3.8 – it implies that for every $p \in [1,q)$, $G_{s,N}(t,.)$ converges in $\mathbb{L}^p(d\mu_{s,R})$ to $G_s(t,.)$. Since $q \in (1,+\infty)$ is arbitrary, we deduce that this convergence holds for every $p \in [1,+\infty)$. □

We conclude this section by a short remark:

Remark 8.10. Starting from Proposition 8.5, it would have also been possible to prove (with the notations of Proposition 7.6) that the densities $F_{s,N}(t,.)$, $F_s(t,.)$ of the transported weighted Gaussian measures belongs to $\mathbb{L}^p(d\rho_{s,R})$ and that $F_{s,N}(t,.)$ converges to $F_s(t,.)$ in $\mathbb{L}^p(d\rho_{s,R})$.

9. Tools for the energy estimates

In this section, we gather the main tools that we will use in the forthcoming Section 10, 11 and 12.

9.1. Deterministic tools.

Notations 9.1 (Number ordering)**.** We will intensively use the following notations:

- Given a set of frequencies $k_1, ..., k_m \in \mathbb{Z}$, we denote by $k_{(1)}, ..., k_{(m)}$ a rearrangement of the $k_j's$ such that
$$|k_{(1)}| \geq |k_{(2)}| \geq ... \geq |k_{(m)}|$$
- Similarly, given a set of dyadic integers $N_1, ..., N_m \in 2^{\mathbb{N}}$, we denote by $N_{(1)}, ..., N_{(m)}$ a non-decreasing rearrangement of the N_j.

Examples 9.2. For example, we have :

- If $k_1 = -1$, $k_2 = 0$, $k_3 = 2$ then $k_{(1)} = k_3$, $k_{(2)} = k_1$, and $k_{(3)} = k_2$
- If $N_1 = 8$, $N_2 = 2$, $N_3 = 4$ then $N_{(1)} = N_1$, $N_{(2)} = N_3$, and $N_{(3)} = N_2$

Lemma 9.3 (Counting bound)**.** *There exists a constant $C > 0$ such that for every dyadic integers $N_1, ..., N_m$, every $\varepsilon_1, ..., \varepsilon_m \in \{-1, +1\}$, and every $\kappa \in \mathbb{Z}$,*

$$\sum_{k_1,...,k_m\in\mathbb{Z}} \mathbb{1}_{\varepsilon_1 k_1+\varepsilon_2 k_2+...+\varepsilon_m k_m=\kappa} \cdot \left(\prod_{j=1}^m \mathbb{1}_{|k_j|\sim N_j}\right) \leq C N_{(2)}N_{(3)}...N_{(m)} \tag{9.1}$$

Proof. Without loss of generality, we assume that $N_j = N_{(j)}$. The idea of the proof is to let free the variables $k_2, ..., k_m$ and to freeze the variable k_1 thanks to the constraint $\varepsilon_1 k_1 = \kappa - \varepsilon_2 k_2 - ... - \varepsilon_m k_m$:

$$\sum_{\substack{k_1,...,k_m \\ |k_j|\sim N_j}} \mathbb{1}_{\varepsilon_1 k_1 + \varepsilon_2 k_2 + ... + \varepsilon_m k_m = \kappa} = \sum_{\substack{k_1,...,k_m \\ |k_j|\sim N_j, j\geq 2}} \underbrace{\sum_{|k_1|\sim N_1} \mathbb{1}_{\varepsilon_1 k_1 = \kappa - \varepsilon_2 k_2 - ... - \varepsilon_m k_m}}_{\leq 1} \leq \sum_{\substack{k_1,...,k_m \\ |k_j|\sim N_j, j\geq 2}} 1 \lesssim N_2 N_3 ... N_m$$

□

In the proof of energy estimates, we will use the following estimate on ψ_{2s} as a starting point.

Lemma 9.4. *Set*

$$\psi_{2s}(\vec{k}) := \sum_{j=1}^{6} (-1)^{j-1} |k_j|^{2s}, \qquad \Omega(\vec{k}) = \sum_{j=1}^{6} (-1)^{j-1} k_j^2$$

There exists a constant $C(s) > 0$ such that for every $k_1 - k_2 + k_3 - k_4 + k_5 - k_6 = 0$,

$$|\psi_{2s}(\vec{k})| \leq C(s) |k_{(1)}|^{2s-2} \left(|\Omega(\vec{k})| + |k_{(3)}|^2 \right)$$

Proof. Essentially, we have to consider two cases : when $k_{(1)} = k_1$, $k_{(2)} = k_2$ and $k_{(1)} = k_1$, $k_{(2)} = k_3$. In any case, we can assume that $|k_{(3)}| \leq \frac{1}{2}|k_{(2)}|$. Otherwise, using the fact that the constraint $k_1 - k_2 + k_3 - k_4 + k_5 - k_6 = 0$ implies that $|k_{(2)}| \sim |k_{(1)}|$, we would deduce that $|k_{(3)}| \sim |k_{(1)}|$. Therefore, the a priori bound $|\psi_{2s}(\vec{k})| \lesssim |k_{(1)}|^{2s}$ would guarantee the desired inequality.

Case 1 : Firstly, we consider the case $k_{(1)} = k_1$ and $k_{(2)} = k_2$. We use the mean value theorem :

$$\begin{aligned}
|\psi_{2s}(\vec{k})| &\leq k_1^{2s} - k_2^{2s} + |k_3^{2s} - k_4^{2s} + k_5^{2s} - k_6^{2s}| \\
&\leq \sup_{t \in [k_2^2, k_1^2]} \frac{d}{dt}(t^s)(k_1^2 - k_2^2) + 4k_{(3)}^{2s} \\
&\leq s|k_{(1)}|^{2(s-1)} (\Omega(\vec{k}) - k_3^2 + k_4^2 - k_5^2 + k_6^2) + 4k_{(1)}^{2(s-1)} k_{(3)}^2 \\
&\leq C(s)|k_{(1)}|^{2(s-1)} (|\Omega(\vec{k})| + |k_{(3)}|^2)
\end{aligned}$$

which is the desired inequality.

Case 2 : Secondly, we consider the case $k_{(1)} = k_1$ and $k_{(2)} = k_3$. Since we assume that $k_{(3)} \leq \frac{1}{2}|k_{(2)}| = \frac{1}{2}|k_3|$, we have

$$\Omega(\vec{k}) \geq k_1^2 + k_3^2 - |k_2^2 + k_4^2 - k_5^2 + k_6^2| \geq k_1^2 + k_3^2 - 4k_{(3)}^2 \geq k_1^2$$

Therefore, the a priori bound $|\psi_{2s}(\vec{k})| \lesssim |k_{(1)}|^{2s}$ guarantees the desired inequality. □

In order to make our analysis work for the full range $s > \frac{3}{2}$, we will use the followinw Strichartz estimate for the linear propagator of the Schrödinger equation–see [3].

Theorem 9.5 (Strichartz estimate)**.** *For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for any function g on $\mathbb{T}$:*

$$\left\| e^{it\partial_x^2} g \right\|_{\mathbb{L}^6(\mathbb{T}_t \times \mathbb{T}_x)} \leq C_\varepsilon \|g\|_{H^\varepsilon(\mathbb{T})}$$

As a consequence, we can prove the following useful estimate :

Lemma 9.6. *Let $\varepsilon > 0$. There exists a constant $C_\varepsilon > 0$ such that for any dyadic integers $N_1, ..., N_6$ and any sequences $\left(f_{k_1}^{(1)}\right)_{k_1\in\mathbb{Z}}, ..., \left(f_{k_6}^{(6)}\right)_{k_6\in\mathbb{Z}}$ of complex numbers that satisfy $f_{k_j}^{(j)} = \mathbb{1}_{|k_j|\sim N_j} f_{k_j}^{(j)}$, we have for any $\kappa \in \mathbb{Z}$:*

$$\sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=\kappa}} \prod_{j=1}^{6} |f_{k_j}^{(j)}| \leq C_\varepsilon N_{(1)}^{\varepsilon} \prod_{j=1}^{6} \left\|f^{(j)}\right\|_{l^2}$$

Proof. We set

$$F_0^{(1)}(x) := \sum_{k_1\in\mathbb{Z}} |f_{k_1}^{(1)}| e^{ik_1x}, \qquad F^{(1)}(t,x) := e^{it\partial_x^2} e^{it\kappa} F_0^{(1)}(x)$$

and for $j \geq 2$,

$$F_0^{(j)}(x) := \sum_{k_j\in\mathbb{Z}} |f_{k_j}^{(j)}| e^{ik_jx}, \qquad F^{(j)}(t,x) := e^{it\partial_x^2} F_0^{(j)}(x)$$

Let us now prove the identity :

$$\sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=\kappa}} \prod_{j=1}^{6} |f_{k_j}^{(j)}| = \int_{\mathbb{T}_t}\int_{\mathbb{T}_x} \left(F^{(1)}\overline{F^{(2)}}F^{(3)}\overline{F^{(4)}}F^{(5)}\overline{F^{(6)}}\right)(t,x)\frac{dxdt}{(2\pi)^2} \tag{9.2}$$

Firstly, we expand

$$\left(F^{(1)}\overline{F^{(2)}}F^{(3)}\overline{F^{(4)}}F^{(5)}\overline{F^{(6)}}\right)(t,x) = \sum_{k_1,...,k_6\in\mathbb{Z}} |f_{k_1}^{(1)}...f_{k_6}^{(6)}| e^{ix(k_1-k_2+...-k_6)} e^{-it(\Omega(\vec{k})-\kappa)}$$

Secondly, we integrate with respect to x and t

$$\begin{aligned}\int_{\mathbb{T}_t}\int_{\mathbb{T}_x} F^{(1)}\overline{F^{(2)}}F^{(3)}\overline{F^{(4)}}F^{(5)}\overline{F^{(6)}}(t,x)\frac{dxdt}{(2\pi)^2}&\\ = \sum_{k_1,...,k_6\in\mathbb{Z}} |f_{k_1}^{(1)}...f_{k_6}^{(6)}| \left(\int_{\mathbb{T}_t}\int_{\mathbb{T}_x} e^{ix(k_1-k_2+...-k_6)} e^{-it(\Omega(\vec{k})-\kappa)}\frac{dxdt}{(2\pi)^2}\right)&\end{aligned}$$

Then, the formula (9.2) follows from the fact that

$$\int_{\mathbb{T}_t}\int_{\mathbb{T}_x} e^{ix(k_1-k_2+...-k_6)} e^{-it(\Omega(\vec{k})-\kappa)}\frac{dxdt}{(2\pi)^2} = \mathbb{1}_{k_1-k_2+...-k_6=0}\mathbb{1}_{\Omega(\vec{k})=\kappa}$$

Starting now from (9.2) and using the Hölder's inequality and Theorem 9.5 we get that

$$\sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=\kappa}} \prod_{j=1}^{6} |f_{k_j}^{(j)}| \leq \prod_{j=1}^{6} \left\|F^{(j)}\right\|_{\mathbb{L}^6(\mathbb{T}_t\times\mathbb{T}_x)} \leq C_\varepsilon \prod_{j=1}^{6} \left\|F_0^{(j)}\right\|_{H^{\frac{\varepsilon}{6}}(\mathbb{T}_x)}$$

We also used the fact that $\left\|F^{(1)}\right\|_{\mathbb{L}^6(\mathbb{T}_t\times\mathbb{T}_x)} = \left\|e^{it\kappa}e^{it\partial_x^2}F_0^{(1)}\right\|_{\mathbb{L}^6(\mathbb{T}_t\times\mathbb{T}_x)} = \left\|e^{it\partial_x^2}F_0^{(1)}\right\|_{\mathbb{L}^6(\mathbb{T}_t\times\mathbb{T}_x)}$.

On the other hand, with the localizations of the sequences $\left(f_{k_j}^{(j)}\right)_{k_j\in\mathbb{Z}}$, we can write

$$\left\|F_0^{(j)}\right\|_{H^{\frac{\varepsilon}{6}}(\mathbb{T}_x)} \lesssim N_j^{\frac{\varepsilon}{6}} \left\|f^{(j)}\right\|_{l^2} \lesssim N_{(1)}^{\frac{\varepsilon}{6}} \left\|f^{(j)}\right\|_{l^2}$$

Coming back to the previous estimate we obtain

$$\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})=\kappa}} \prod_{j=1}^{6} |f_{k_j}^{(j)}| \leq C'_{\varepsilon} N_{(1)}^{\varepsilon} \prod_{j=1}^{6} \left\| f^{(j)} \right\|_{l^2}$$

which is the desired estimate. □

Finally, we conclude this section with the following suitable lemma. Although its significance will become apparent later (in the proof of energy estimates in Section 10, 11 and 12), we prove it now for a better clarity.

Lemma 9.7. *Let $s \in \mathbb{R}$ and $\varepsilon > 0$. There exists a constant $C_{\varepsilon,s} > 0$ such that for any dyadic integers $N_1, \ldots, N_6$ and any sequences $\left(f_{k_1}^{(1)}\right)_{k_1\in\mathbb{Z}}, \ldots, \left(f_{k_6}^{(6)}\right)_{k_6\in\mathbb{Z}}$ of complex numbers that satisfy $f_{k_j}^{(j)} = \mathbb{1}_{|k_j|\sim N_j} f_{k_j}^{(j)}$, we have*

$$\sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \left| \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} \right| \prod_{j=1}^{6} |f_{k_j}^{(j)}| \leq C_{\varepsilon,s} N_{(1)}^{2s-2+\varepsilon} N_{(3)}^{2} \prod_{j=1}^{6} \left\| f^{(j)} \right\|_{l^2}$$

Proof. We start by applying Lemma 9.4 :

$$\begin{aligned} \sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \left| \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} \right| \prod_{j=1}^{6} |f_{k_j}^{(j)}| &\lesssim \sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} |k_{(1)}|^{2s-2} \left(1 + \frac{|k_{(3)}|^2}{|\Omega(\vec{k})|} \right) \prod_{j=1}^{6} |f_{k_j}^{(j)}| \\ &\lesssim \quad \mathbf{I} + \mathbf{II} \end{aligned}$$

where we denote

$$\mathbf{I} := N_{(1)}^{2s-2} \sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \prod_{j=1}^{6} |f_{k_j}^{(j)}|$$

and,

$$\mathbf{II} := N_{(1)}^{2s-2} N_{(3)}^{2} \sum_{\substack{k_1-k_2+\ldots-k_6=0\\ \Omega(\vec{k})\neq 0}} \frac{1}{|\Omega(\vec{k})|} \prod_{j=1}^{6} |f_{k_j}^{(j)}|$$

We estimate separately **I** and **II**.

•**Estimate of I :** Removing the constraint $\Omega(\vec{k}) \neq 0$ and using the Cauchy-Schwarz inequality in the $k_{(1)}, k_{(2)}$ summations, we get that

$$\begin{aligned} \mathbf{I} &\leq N_{(1)}^{2s-2} \left\| f^{((1))} \right\|_{l^2} \left\| f^{((2))} \right\|_{l^2} \prod_{j=3}^{6} \left\| f^{((j))} \right\|_{l^1} \\ &\leq N_{(1)}^{2s-2} \left(N_{(3)} N_{(4)} N_{(5)} N_{(6)} \right)^{\frac{1}{2}} \prod_{j=1}^{6} \left\| f^{(j)} \right\|_{l^2} \\ &\leq N_{(1)}^{2s-2} N_{(3)}^{2} \prod_{j=1}^{6} \left\| f^{(j)} \right\|_{l^2} \end{aligned}$$

which is even better than the desired bound.

•**Estimate of II :** Firstly, we observe that $|\Omega(\vec{k})| \lesssim N_{(1)}^2$ so we can write

$$\mathbf{II} \leq N_{(1)}^{2s-2} N_{(3)}^2 \sum_{N_{(1)}^2 \gtrsim |\kappa| \geq 1} \frac{1}{|\kappa|} \sum_{\substack{k_1 - k_2 + \ldots - k_6 = 0 \\ \Omega(\vec{k}) = \kappa}} \prod_{j=1}^{6} |f_{k_j}^{(j)}|$$

Secondly, we use Lemma 9.6 that says

$$\sum_{\substack{k_1 - k_2 + \ldots - k_6 = 0 \\ \Omega(\vec{k}) = \kappa}} \prod_{j=1}^{6} |f_{k_j}^{(j)}| \leq C_\varepsilon N_{(1)}^{\frac{\varepsilon}{2}} \prod_{j=1}^{6} \left\| f^{(j)} \right\|_{l^2}$$

Finally, we invoke the well-known estimate

$$\sum_{N_{(1)}^2 \gtrsim |\kappa| \geq 1} \frac{1}{|\kappa|} \lesssim \log\left(N_{(1)}\right) \leq C'_\varepsilon N_{(1)}^{\frac{\varepsilon}{2}}$$

and the combination of those inequalities yields

$$\mathbf{II} \leq C''_\varepsilon N_{(1)}^{2s-2+\varepsilon} N_{(3)}^2 \prod_{j=1}^{6} \left\| f^{(j)} \right\|_{l^2}$$

which is the desired bound. This completes the proof of Lemma 9.7. □

9.2. **Some properties of Gaussian measures.** Our analysis for the energy estimates in Section 11 and 12 will also require the following probabilistic tools.

Lemma 9.8 (Moments of Gaussian measures)**.** *Let $s \in \mathbb{R}$ and $\sigma < s - \frac{1}{2}$. Then, there exists $C = C(s, \sigma) > 0$ such that for all $m \geq 1$:*

$$\left(\int_{H^\sigma} \|u\|_{H^\sigma}^m \, d\mu_s \right)^{\frac{1}{m}} \leq C m^{\frac{1}{2}} \tag{9.3}$$

Proof. From Fernique's theorem (see for example [17], [2]), there exists $\alpha > 0$ such that:

$$\int_{H^\sigma} e^{\alpha \|u\|_{H^\sigma}^2} d\mu_s < +\infty$$

Then, by Markov's inequality, we obtain the following large deviation estimate:

$$\mu_s(\|u\|_{H^\sigma} > \lambda) = \mu_s(e^{\alpha\|u\|_{H^\sigma}^2} > e^{\alpha\lambda^2}) \leq e^{-\alpha\lambda^2} \int_{H^\sigma} e^{\alpha\|u\|_{H^\sigma}^2} d\mu_s \leq C e^{-\alpha\lambda^2}$$

Combining this estimate with Cavalieri's principle, we have:

$$\int_{H^\sigma} \|u\|_{H^\sigma}^m \, d\mu_s = m \int_0^{+\infty} \lambda^{m-1} \mu_s(\|u\|_{H^\sigma} > \lambda) d\lambda \leq C m \int_0^{+\infty} \lambda^{m-1} e^{-\alpha\lambda^2} d\lambda$$

To conclude, we perform $\lfloor \frac{m}{2} \rfloor - 1$ integration by parts on the integral on the right hand side:

$$\begin{aligned}\int_0^{+\infty} \lambda^{m-1} e^{-\alpha\lambda^2} d\lambda &= \frac{m-2}{2\alpha} \int_0^{+\infty} \lambda^{m-3} e^{-\alpha\lambda^2} d\lambda \\ &= ... = \frac{(m-2)(m-4)...(m-2(\lfloor \frac{m}{2} \rfloor - 1))}{(2\alpha)^{\lfloor \frac{m}{2} \rfloor - 1}} \int_0^{+\infty} \lambda^{m-2(\lfloor \frac{m}{2} \rfloor - 1)-1} e^{-\alpha\lambda^2} d\lambda \\ &\leq \Big(\frac{m}{2\alpha}\Big)^{\lfloor \frac{m}{2} \rfloor - 1} \sup_{\widetilde{m} \in [0,2]} \int_0^{+\infty} \lambda^{\widetilde{m}} e^{-\alpha\lambda^2} d\lambda\end{aligned}$$

□

Next, we state a conditional Wiener chaos estimate which will play a crucial role in the energy estimates. In the sequel, for any complex number z, we adopt the notation $z^+ = z$ and $z^- = \overline{z}$, called respectively positive and negative *signature* of z.

Lemma 9.9 (Conditional Wiener chaos estimate)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{B}$ be a σ-algebra on Ω such that $\mathcal{B} \subset \mathcal{A}$. Let $m \in \mathbb{N}$ and $\iota_1, ..., \iota_m \in \{-, +\}$. We consider the following expression :*

$$F(\omega) := \sum_{k_1,...,k_m} c_{k_1,...,k_m}(\omega) \cdot \prod_{j=1}^{m} g_{k_j}^{\iota_j}(\omega), \qquad \omega \in \Omega$$

where, the $g_{k_j}(\omega)$ are complex standard i.i.d Gaussians, independent of the σ-algebra $\mathcal{B}$, and the $c_{k_1,...,k_m}(\omega)$ are $\mathcal{B}$-mesurable complex random variables. Then, there exists $C > 0$ such that for every $p \geq 2$, we have :

$$\|F\|_{\mathbb{L}^p(\Omega|\mathcal{B})} \leq C p^{\frac{m}{2}} \|F\|_{\mathbb{L}^2(\Omega|\mathcal{B})}$$

where $\mathbb{L}^p(\Omega|\mathcal{B})$ is the $\mathbb{L}^p$-norm conditioned to the σ- algebra $\mathcal{B}$.

In the energy estimates, we will apply this lemma with $m = 3$, $\mathcal{B}$ the σ-algebra generated by low-frequency Gaussians, and the random variables $c_{k_1,..,k_m}$ will be some multi-linear expression of high-frequency Gaussians (independent of the low-frequency Gaussians). For a proof of this lemma, we refer to [27] (see also [32] and [30]).

Lemma 9.10. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $m \in \mathbb{N}$, $\iota_1, ..., \iota_m \in \{-, +\}$, and $g_{k_1}, ..., g_{k_m}$ be complex standard i.i.d Gaussians. We consider the following multi-linear expression of Gaussians:*

$$G(\omega) := \sum_{\substack{k_1,...,k_m \\ \forall \iota_i \neq \iota_j,\ k_i \neq k_j}} c_{k_1,...,k_m} \cdot \prod_{j=1}^{m} g_{k_j}^{\iota_j}(\omega), \qquad \omega \in \Omega \tag{9.4}$$

where $c_{k_1,...,k_m}$ is a sequence of $l^2(\mathbb{Z}^m; \mathbb{C})$. Then, there exists $C > 0$ such that:

$$\|G\|_{\mathbb{L}^2(\Omega)} \leq C \Bigg(\sum_{\substack{k_1,...,k_m \\ \forall \iota_i \neq \iota_j,\ k_i \neq k_j}} |c_{k_1,...,k_m}|^2 \Bigg)^{\frac{1}{2}} \tag{9.5}$$

Proof. For more readability, we perform the proof only for $m = 3$ (and that is the value of m we will use with this lemma). In addition, we assume that $\iota_1, \iota_3 = 1$ and $\iota_2 = -1$ (the other cases are similar). Thus, we want to prove that:

$$\Big\| \sum_{\substack{k_1,k_2,k_3 \\ k_2 \neq k_1,k_3}} c_{k_1,k_2,k_3} g_{k_1} \overline{g_{k_2}} g_{k_3} \Big\|_{\mathbb{L}^2(\Omega)}^2 \leq C \sum_{\substack{k_1,k_2,k_3 \\ k_2 \neq k_1,k_3}} |c_{k_1,k_2,k_3}|^2$$

We start by expanding the left hand side:

$$\Big\| \sum_{\substack{k_1,k_2,k_3\\ k_2\neq k_1,k_3}} c_{k_1,k_2,k_3} g_{k_1}\overline{g_{k_2}}g_{k_3}\Big\|^2_{\mathbb{L}^2(\Omega)} = \sum_{\substack{k_1,k_2,k_3,l_1,l_2,l_3\\ k_2\neq k_1,k_3 \ \& \ l_2\neq l_1,l_3}} c_{k_1,k_2,k_3}\overline{c_{l_1,l_2,l_3}}\,\mathbb{E}[g_{k_1}\overline{g_{k_2}}g_{k_3}\overline{g_{l_1}}g_{l_2}\overline{g_{l_3}}] \tag{9.6}$$

From the independence of the Gaussians g_n, and the fact that for a complex standard Gaussian variable g we have $\mathbb{E}[g] = \mathbb{E}[g^2] = \mathbb{E}[g^3] = 0$, the only non-zero contributions in the sum above is when for each $g^{\pm}_{k_j}$ there exists an $g^{\mp}_{l_i}$, with the opposite signature, such that $k_j = l_i$. Hence, the only non-zero contributions are of the form:

$$c_{k_1,k_2,k_3}\overline{c_{l_1,l_2,l_3}}\,\mathbb{E}[|g_{k_1}|^2|g_{k_2}|^2|g_{k_3}|^2] \text{ with } k_2 = l_2,\ \{k_1,k_3\} = \{l_1,l_3\} \text{ and } k_2 \neq k_1,k_3$$

Then, we invoke the following set of indices:

$$\begin{aligned}
D_1 &:= \{(k_1,k_2,k_3,l_1,l_2,l_3)\in\mathbb{Z}^6 \ :\ k_2=l_2,\ k_1=l_1,\ k_3=l_3,\ k_2\neq k_1,k_3,\ k_1\neq k_3\}\\
D_2 &:= \{(k_1,k_2,k_3,l_1,l_2,l_3)\in\mathbb{Z}^6 \ :\ k_2=l_2,\ k_1=l_3,\ k_3=l_1,\ k_2\neq k_1,k_3,\ k_1\neq k_3\}\\
D_3 &:= \{(k_1,k_2,k_3,l_1,l_2,l_3)\in\mathbb{Z}^6 \ :\ k_2=l_2,\ k_1=k_3=l_1=l_3,\ k_2\neq k_1,k_3\}
\end{aligned}$$

so that, coming back to (9.6), we have (for any complex standard Gaussian g):

$$\begin{aligned}
\Big\| \sum_{\substack{k_1,k_2,k_3\\ k_2\neq k_1,k_3}} c_{k_1,k_2,k_3} g_{k_1}\overline{g_{k_2}}g_{k_3}\Big\|^2_{\mathbb{L}^2(\Omega)} &= \mathbb{E}[|g|^2]^3\sum_{D_1}|c_{k_1,k_2,k_3}|^2 + \mathbb{E}[|g|^2]^3\sum_{D_2} c_{k_1,k_2,k_3}\overline{c_{k_3,k_2,k_1}}\\
&\quad + \mathbb{E}[|g|^2]\mathbb{E}[|g|^4]\sum_{D_3}|c_{k_1,k_2,k_1}|^2
\end{aligned}$$

This concludes the proof since:

$$\sum_{D_1}|c_{k_1,k_2,k_3}|^2 + \sum_{D_3}|c_{k_1,k_2,k_1}|^2 = \sum_{\substack{k_1,k_2,k_3\\ k_2\neq k_1,k_3 \ \& \ k_1\neq k_3}}|c_{k_1,k_2,k_3}|^2 + \sum_{\substack{k_1,k_2\\ k_2\neq k_1}}|c_{k_1,k_2,k_1}|^2 = \sum_{\substack{k_1,k_2,k_3\\ k_2\neq k_1,k_3}}|c_{k_1,k_2,k_3}|^2$$

and,

$$\sum_{D_2} c_{k_1,k_2,k_3}\overline{c_{k_3,k_2,k_1}} \leq \frac{1}{2}\sum_{\substack{k_1,k_2,k_3\\ k_2\neq k_1,k_3 \ \& \ k_1\neq k_3}}|c_{k_1,k_2,k_3}|^2 + \frac{1}{2}\sum_{\substack{k_1,k_2,k_3\\ k_2\neq k_1,k_3 \ \& \ k_1\neq k_3}}|c_{k_3,k_2,k_1}|^2 \leq \sum_{\substack{k_1,k_2,k_3\\ k_2\neq k_1,k_3}}|c_{k_1,k_2,k_3}|^2$$

□

Remark 9.11. When $\iota_i \neq \iota_j$ **and** $k_i = k_j$, we say that k_i and k_j are paired. If one allows such parings in the sum in (9.4), then the inequality (9.5) does not hold anymore in general. In our analysis in the upcoming sections, we will not encounter such pairings. However, in [29], this situation occurred, and the authors needed to study these pairing contributions separately.

10. Proofs of the deterministic properties

This section is dedicated to the proof of the deterministic properties of the energy correction $R_{s,N}$ and of the derivative of the modified energy $Q_{s,N}$. More precisely, we prove here Proposition 3.1 and Proposition 4.1. To do so, we are going to use the deterministic tools from Section 9.

Deterministic estimate for the energy correction. In this paragraph, we prove Proposition 3.1. Recall that we want to estimate the sum of positive terms in (3.1).

Proof of Proposition 3.1. Let $\sigma < s - \frac{1}{2}$ and let $u^{(1)}, ..., u^{(6)} \in H^\sigma(\mathbb{T})$. First, we decompose the sum in (3.1) dyadically:

$$\sum_{\substack{k_1-k_2+...-k_6=0 \\ \Omega(\vec{k})\neq 0}} |\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}||u^{(1)}_{k_1}u^{(2)}_{k_2}...u^{(6)}_{k_6}| = \sum_{N_1,...,N_6} \mathfrak{R}(N_1, ..., N_6) \tag{10.1}$$

where the summations are performed on the dyadic values of $N_1, ..., N_6$ and,

$$\mathfrak{R}(N_1, ..., N_6) := \sum_{\substack{k_1-k_2+...-k_6=0 \\ \Omega(\vec{k})\neq 0}} |\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}| \prod_{j=1}^{6} \mathbb{1}_{|k_j|\sim N_j}|u^{(j)}_{k_j}|$$

Now, using Lemma 9.7 yields:

$$\mathfrak{R}(N_1, ..., N_6) \lesssim_\varepsilon N_{(1)}^{2s-2+\varepsilon} N_{(3)}^2 \prod_{j=1}^{6} \left\|P_{N_j} u^{(j)}\right\|_{\mathbb{L}^2(\mathbb{T})} \lesssim_\varepsilon N_{(1)}^{2s-2+\varepsilon} N_{(3)}^2 (N_{(1)}...N_{(6)})^{-\sigma} \prod_{j=1}^{6} \left\|u^{(j)}\right\|_{H^\sigma} \tag{10.2}$$

where P_N is the projector onto frequencies $|k| \sim N$. Note that we have $N_{(1)} \sim N_{(2)}$ because the constraint $k_1 - k_2 + ... - k_6 = 0$ implies that $|k_{(2)}| \sim |k_{(1)}|$. Besides, we crudely estimate $(N_{(4)}N_{(5)}N_{(3)})^{-\sigma} \lesssim 1$. Then, we rewrite the inequality above as:

$$\mathfrak{R}(N_1, ..., N_6) \lesssim_\varepsilon N_{(1)}^{2(s-1-\sigma)+\varepsilon} N_{(3)}^{2-\sigma} \prod_{j=1}^{6} \left\|u^{(j)}\right\|_{H^\sigma} \tag{10.3}$$

Next, let us observe that for $\sigma < s - \frac{1}{2}$ and $\varepsilon > 0$:

$$\begin{cases} 2(s-1-\sigma)+\varepsilon > -1 \\ 2(s-1-\sigma)+\varepsilon \longrightarrow -1 \text{ as } \begin{cases} \sigma \to s - \frac{1}{2} \\ \varepsilon \to 0 \end{cases} \end{cases} \quad \text{and,} \quad \begin{cases} 2-\sigma > \frac{5}{2} - s \\ 2-\sigma \longrightarrow \frac{5}{2} - s \text{ as } \begin{cases} \sigma \to s - \frac{1}{2} \\ \varepsilon \to 0 \end{cases} \end{cases}$$

In particular, in (10.3), $N_{(1)}$ is accompanied by a negative exponent[9].
Finally, if:

$$-1 + (\frac{5}{2} - s) < 0, \qquad \text{that is if:} \qquad s > \frac{3}{2},$$

then, for $\sigma < s - \frac{1}{2}$ and $\varepsilon > 0$ respectively close enough to $s - \frac{1}{2}$ and 0, we deduce from (10.3) that[10]:

$$\mathfrak{R}(N_1, ..., N_6) \lesssim N_{(1)}^{-} \cdot \prod_{j=1}^{6} \left\|u^{(j)}\right\|_{H^\sigma}$$

which is summable in the N_j. Coming back to (10.1) finishes the proof. □

[9]In dimension $d \geq 2$, we have $\sigma < s - \frac{d}{2}$, and this scenario would be worse because we would have $2(s-1-\sigma) > 0$, so $N_{(1)}$ would be accompanied by a positive exponent. This scenario is encountered in [29], with $d = 3$.

[10]The notation N^- means that the power of N is $-\gamma$ for a certain $\gamma > 0$.

Deterministic estimate for the derivative of the modified energy. In this paragraph, we prove Proposition 4.1. Recall that we want to estimate the sums of positive terms in (4.1), (4.2) and (4.3).

Proof of Proposition 4.1. Let $\sigma < s - \frac{1}{2}$.

•**Estimate for $\mathcal{T}_0$:** The analysis is similar to the one for $\mathcal{R}$. Indeed, let us fix $u^{(1)}, ..., u^{(6)} \in H^\sigma(\mathbb{T})$. Then, decomposing the sum in (4.1) dyadically and using Lemma 9.4 (with $\Omega(\vec{k}) = 0$), leads to:

$$\sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=0}} |\psi_{2s}(\vec{k})||u^{(1)}_{k_1}u^{(2)}_{k_2}...u^{(6)}_{k_6}| \lesssim \sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=0}} |k_{(1)}|^{2s-2}|k_{(3)}|^2|u^{(1)}_{k_1}u^{(2)}_{k_2}...u^{(6)}_{k_6}|$$
$$\lesssim \sum_{N_1,...,N_6} \mathfrak{T}_0(N_1, ..., N_6)$$

where,

$$\mathfrak{T}_0(N_1, ..., N_6) := \sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})=0}} N_{(1)}^{2s-2}N_{(3)}^2 \prod_{j=1}^{6} \mathbb{1}_{|k_j|\sim N_j}|u^{(j)}_{k_j}|$$

Next, from Lemma 9.6 (with $f^j_{k_j} = \mathbb{1}_{|k_j|\sim N_j}|u^{(j)}_{k_j}|$), we have:

$$\mathfrak{T}_0(N_1, ..., N_6) \lesssim_\varepsilon N_{(1)}^{2s-2+\varepsilon}N_{(3)}^2 \prod_{j=1}^{6} \left\|P_{N_j}u^{(j)}\right\|_{\mathbb{L}^2(\mathbb{T})}$$

From that point the proof goes the same as the one for $\mathcal{R}$ (see the estimate (10.2)).

•**Estimate of $\mathcal{T}_j$ for $j = 1, 2$:** We only prove the estimate (4.2) since the analysis for the estimate (4.3) is similar. Here, the computations are a little more delicate. For the sake of readability, we restrict ourselves to prove the estimate:

$$\sum_{\substack{k_1-k_2+...-k_6=0\\ p_1-p_2+...+p_5=k_1\\ \Omega(\vec{k})\neq 0}} \left|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\right||u_{p_1}\overline{u_{p_2}}...u_{p_5}\overline{u_{k_2}}...\overline{u_{k_6}}| \leq C_s \|u\|^{10}_{H^\sigma}$$

In other words, we just prove the estimate (4.2) where all the $u^{(j)}$ and $v^{(l)}$ are equal to a single u. To do so, we will use the following lemma:

Lemma 10.1. *There exists a constant $C > 0$ such that for any dyadic integers $P_1, ..., P_5$ and any sequences $\left(g^{P_1}_{p_1}\right)_{p_1\in\mathbb{Z}}, ..., \left(g^{P_5}_{p_5}\right)_{p_5\in\mathbb{Z}}$ of complex numbers that satisfy for $j = 1, ..., 5$, $g^{P_j}_{p_j} = \mathbb{1}_{|p_j|\sim P_j}g_{p_j}$, if we set for $k_1 \in \mathbb{Z}$:*

$$f_{k_1} := \sum_{p_1-p_2+...+p_5=k_1} g^{P_1}_{p_1}g^{P_2}_{p_2}g^{P_3}_{p_3}g^{P_4}_{p_4}g^{P_5}_{p_5}$$

Then,

$$\|f\|_{l^2(\mathbb{Z})} \leq C\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{\frac{1}{2}} \prod_{j=1}^{5} \left\|g^{P_j}\right\|_{l^2(\mathbb{Z})}$$

For clarity, we postpone the proof of this lemma for the end of this section. Now we are ready to prove the estimate. Once again, we start by decomposing the sum as:

$$\sum_{\substack{k_1-k_2+...-k_6=0\\ p_1-p_2+...+p_5=k_1\\ \Omega(\vec{k})\neq 0}} \Big|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\Big||u_{p_1}\overline{u_{p_2}}...u_{p_5}\overline{u_{k_2}}...\overline{u_{k_6}}| = \sum_{\substack{M_1,...,M_6\\ P_1,...,P_5}} \mathfrak{T}_1(M_1,...,M_6,P_1,...,P_5)$$

where $M_1,...,M_6,P_1,...,P_5$ are dyadic-valued and

$$\mathfrak{T}_1(M_1,...,M_6,P_1,...,P_5) := \sum_{\substack{k_1-k_2+...-k_6=0\\ p_1-p_2+...+p_5=k_1\\ \Omega(\vec{k})\neq 0}} \Big|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\Big||u_{p_1}^{P_1}...u_{p_5}^{P_5}u_{k_2}^{M_2}...u_{k_6}^{M_6}| \cdot \mathbb{1}_{|k_1|\sim M_1} \tag{10.4}$$

with the practical notation $u_l^N := \mathbb{1}_{|l|\sim N}u_l$. Next, we rewrite $\mathfrak{T}_1$ as:

$$\mathfrak{T}_1(M_1,...,P_5) = \sum_{\substack{k_1-k_2+...-k_6=0\\ \Omega(\vec{k})\neq 0}} \Big|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\Big| \prod_{j=1}^{6} |f_{k_j}^{(j)}| \tag{10.5}$$

where, for $j=2,...,6$ we denote $f_{k_j}^{M_j} := u_{k_j}^{M_j}$, and:

$$f_{k_1}^{(1)} := \mathbb{1}_{|k_1|\sim M_1} \cdot \sum_{p_1-p_2+...+p_5=k_1} |u_{p_1}^{P_1}...u_{p_5}^{P_5}|$$

At this stage, let us recall that $M_{(1)} \geq ... \geq M_{(6)}$ and $P_{(1)} \geq ... \geq P_{(5)}$ are respectively a non-increasing rearrangement of $M_1,...,M_6$ and $P_1,...,P_5$. We also introduce:

$$N_{(1)} \geq N_{(2)} \geq ... \geq N_{(10)} \text{ a non-increasing rearrangement of } M_2,...,M_6,P_1,...,P_5$$

Note that the constraints in the sum in $\mathfrak{T}_1$ imply that

$$M_{(1)} \sim M_{(2)}, \text{ and: } P_{(1)} \sim P_{(2)} \text{ or } P_{(1)} \sim M_1, \text{ and: } N_{(1)} \sim N_{(2)}.$$

Now, we use Lemma 9.7 to estimate $\mathfrak{T}_1$ in (10.5); we obtain that for any $\varepsilon > 0$:

$$\mathfrak{T}_1(M_1,...,P_5) \lesssim_\varepsilon M_{(1)}^{2s-2+\varepsilon}M_{(3)}^2 \prod_{j=1}^{6} \left\|f^{(j)}\right\|_{l^2} \lesssim_\varepsilon M_{(1)}^{2s-2+\varepsilon}M_{(3)}^2 \left\|f^{(1)}\right\|_{l^2} \prod_{j=2}^{6} \left\|u^{M_j}\right\|_{\mathbb{L}^2}$$

Applying the well-suited estimate of $\left\|f^{(1)}\right\|_{l^2}$ from Lemma 10.1 leads to

$$\mathfrak{T}_1(M_1,...,P_5) \lesssim_\varepsilon M_{(1)}^{2s-2+\varepsilon}M_{(3)}^2 (P_{(2)}P_{(3)}P_{(4)}P_{(5)})^{\frac{1}{2}} \prod_{j=1}^{5} \left\|u^{P_j}\right\|_{\mathbb{L}^2} \prod_{j=2}^{6} \left\|u^{M_j}\right\|_{\mathbb{L}^2} \tag{10.6}$$

and it follows that

$$\mathfrak{T}_1(M_1,...,P_5) \lesssim_\varepsilon M_{(1)}^{2s-2+\varepsilon}M_{(3)}^2 \left(M_2M_3M_4M_5M_6\right)^{-\sigma} P_{(1)}^{-\sigma} \left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{\frac{1}{2}-\sigma} \|u\|_{H^\sigma}^{10}$$

In the remaining part of the proof, we show that it implies that:

$$\mathfrak{T}_1(M_1,...,P_5) \lesssim_\varepsilon N_{(1)}^{-} \|u\|_{H^\sigma}^{10} \tag{10.7}$$

for $\sigma < s - \frac{1}{2}$ close enough to $s - \frac{1}{2}$, which will complete the estimate for $\mathcal{T}_1$. We start from the inequality above and we use the fact that $M_1 \lesssim P_{(1)}$ due to the constraint $p_1 - p_2 + ... + p_5 = k_1$, along with the rough estimates $(M_{(4)}M_{(5)}M_{(6)})^{-\sigma} \leq 1$ and $(P_{(3)}P_{(4)}P_{(5)})^{\frac{1}{2}-\sigma} \leq 1$, in order to obtain:

$$\begin{aligned}\mathfrak{T}_1(M_1, ..., P_5) &\lesssim_\varepsilon M_{(1)}^{2s-2+\varepsilon} M_{(3)}^2 (M_1 M_2 M_3 M_4 M_5 M_6)^{-\sigma} (M_1^\sigma P_{(1)}^{-\sigma})(P_{(2)}P_{(3)}P_{(4)}P_{(5)})^{\frac{1}{2}-\sigma} \|u\|_{H^\sigma}^{10} \\ &\lesssim_\varepsilon M_{(1)}^{2(s-1-\sigma)+\varepsilon} M_{(3)}^{2-\sigma} P_{(2)}^{\frac{1}{2}-\sigma} \|u\|_{H^\sigma}^{10}\end{aligned} \tag{10.8}$$

Besides, when $\sigma < s - \frac{1}{2}$ and $\varepsilon > 0$ are respectively arbitrarily close to $s - \frac{1}{2}$ and 0, we have that $2(s-1-\sigma)+\varepsilon$, $2-\sigma$ and $\frac{1}{2}-\sigma$ are respectively arbitrarily close to

$$2(s-1-(s-\frac{1}{2})) = -1, \qquad 2-(s-\frac{1}{2}) = \frac{5}{2} - s, \qquad \frac{1}{2} - (s - \frac{1}{2}) = 1 - s$$

Hence, $M_{(1)}$ is accompanied by a negative exponent, $P_{(2)}$ is accompanied by a negative exponent (since $s > \frac{3}{2} > 1$), and $M_{(3)}$ is accompanied by a negative exponent when $s \geq \frac{5}{2}$ and by a non-negative one when $s < \frac{5}{2}$.

–In the situation where $M_{(1)} \sim N_{(1)}$, we use the rough estimate $P_{(2)}^{\frac{1}{2}-\sigma} \leq 1$ in (10.8), and we obtain:

$$\mathfrak{T}_1(M_1, ..., P_5) \lesssim_\varepsilon N_{(1)}^{2(s-1-\sigma)+\varepsilon} M_{(3)}^{2-\sigma} \|u\|_{H^\sigma}^{10}$$

which is conclusive as far as

$$2(s-1-(s-\frac{1}{2})) + 2 - (s - \frac{1}{2}) < 0$$

that is, as far as $s > \frac{3}{2}$.
–In the other situation, where $M_{(1)} \ll N_{(1)}$, we necessarily have $P_{(1)} \sim P_{(2)} \sim N_{(1)}$, so we deduce from (10.8) that:

$$\mathfrak{T}_1(M_1, ..., P_5) \lesssim_\varepsilon N_{(1)}^{\frac{1}{2}-\sigma} M_{(3)}^{2(s-1-\sigma)+2-\sigma+\varepsilon} \|u\|_{H^\sigma}^{10}$$

Then, for $\sigma < s - \frac{1}{2}$ and $\varepsilon > 0$ respectively arbitrarily close to $s - \frac{1}{2}$ and 0, we have:

$$\mathfrak{T}_1(M_1, ..., P_5) \lesssim N_{(1)}^{(1-s)+} M_{(3)}^{(\frac{3}{2}-s)+} \|u\|_{H^\sigma}^{10} \lesssim N_{(1)}^{-} \|u\|_{H^\sigma}^{10}$$

since $s > \frac{3}{2}$.
In conclusion, (10.7) is proven and the proof of Proposition 4.1 is completed. □

We finish this section by a proof of Lemma 10.1.

Proof of Lemma 10.1. We have to estimate :

$$\|f\|_{l^2(\mathbb{Z})}^2 = \sum_{k_1 \in \mathbb{Z}} \Big| \sum_{p_1 - p_2 + ... + p_5 = k_1} g_{p_1}^{P_1} g_{p_2}^{P_2} g_{p_3}^{P_3} g_{p_4}^{P_4} g_{p_5}^{P_5} \Big|^2$$

Using the Cauchy-Schwarz inequality we obtain:

$$\|f\|_{l^2(\mathbb{Z})}^2 \leq \sum_{k_1 \in \mathbb{Z}} \Big(\sum_{p_1,...,p_5} \mathbb{1}_{p_1 - p_2 + ... + p_5 = k_1} \prod_{j=1}^5 \mathbb{1}_{|p_j| \sim P_j} \Big) \Big(\sum_{p_1,...,p_5} \mathbb{1}_{p_1 - p_2 + ... + p_5 = k_1} |g_{p_1}^{P_1} g_{p_2}^{P_2} g_{p_3}^{P_3} g_{p_4}^{P_4} g_{p_5}^{P_5}|^2 \Big)$$

Plugging the counting bound (9.1) into this formula yields

$$\|f\|^2_{l^2(\mathbb{Z})} \lesssim P_{(2)}P_{(3)}P_{(4)}P_{(5)} \sum_{p_1,\dots,p_5} |g^{P_1}_{p_1} g^{P_2}_{p_2} g^{P_3}_{p_3} g^{P_4}_{p_4} g^{P_5}_{p_5}|^2 \underbrace{\Big(\sum_{k_1\in\mathbb{Z}} \mathbb{1}_{p_1-p_2+\dots+p_5=k_1} \Big)}_{\leq 1}$$

Thus,

$$\|f\|^2_{l^2(\mathbb{Z})} \lesssim P_{(2)}P_{(3)}P_{(4)}P_{(5)} \prod_{j=1}^{5} \left\|g^{P_j}\right\|^2_{l^2(\mathbb{Z})}$$

which is the desired estimate. □

11. Estimates for the weight of the weighted Gaussian measures

This section is dedicated to estimates on the weight $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_{s,N}(u)}$ of the weighted Gaussian measure $\rho_{s,R,N}$ (defined in (7.1)). In particular, we show that :

$$\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_{s,N}(u)} \in \mathbb{L}^1(d\mu_s)$$

which ensures that $\rho_{s,R,N}$ is a finite measure on $H^\sigma(\mathbb{T})$, $\sigma < s - \frac{1}{2}$. More precisely, we prove Proposition 7.4. Yet, before doing so, we will need the two following lemmas :

Lemma 11.1. *Let $s > \frac{3}{2}$. Then, there exists $\beta \in (0,1)$ such that for every $R > 0$, there exists $C(s,R) > 0$, such that for any $p \in [2, +\infty)$,*

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} R_{s,N} \right\|_{\mathbb{L}^p(d\mu_s)} \leq C(s,R) p^\beta$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$.

Recall that we denote $R_{s,\infty} = R_s$ for convenience.

Lemma 11.2. *Let $(X, \mathcal{A}, d\nu)$ be a probability space and let $F : X \longrightarrow \mathbb{C}$ be a mesurable function. Assume that there exist constants $C_0 > 0$ and $\beta \in (0,1)$ such that for every $p \in [2, +\infty)$,*

$$\|F\|_{\mathbb{L}^p(d\nu)} \leq C_0 p^\beta$$

Then, there exist $\delta > 0$ and $C_1 > 0$ only depending on β and C_0 such that :

$$\int_X e^{\delta |F(v)|^{\frac{1}{\beta}}} d\nu(v) \leq C_1$$

We will only prove Lemma 11.1, Lemma 11.2 being just a slightly different version of Lemma 4.5 from [32] (where a proof is given).

Before doing the proof of Lemma 11.1, let us briefly show how it implies Proposition 7.4 when combining with Lemma 11.2.

Proof of Proposition 7.4. Here, we assume the statements in Lemma 11.1 and Lemma 11.2.
–We start by proving the first statement in Proposition 7.4. Lemma 11.1 shows that $F := \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} R_{s,N}(u) : H^\sigma(\mathbb{T}) \longrightarrow \mathbb{C}$ satisfies the assumptions of Lemma 11.2 (with $\nu = \mu_s$). Then, applying Lemma 11.2, we obtain that there exist $\delta = \delta(s,R) > 0$ and $C_1(s,R) > 0$ such that :

$$\int_{H^\sigma} e^{\delta \left| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} R_{s,N}(u) \right|^{\frac{1}{\beta}}} d\mu_s \leq C_1(s,R)$$

A fortiori we have,

$$\int_{H^\sigma} \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{\delta \left|R_{s,N}(u)\right|^{\frac{1}{\beta}}} d\mu_s \leq C_1(s,R)$$

Thanks to the fact that $\frac{1}{\beta} > 1$, we deduce from the above inequality that for every $p \in [1,+\infty)$:

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{|R_{s,N}(u)|} \right\|^p_{\mathbb{L}^p(d\mu_s)} = \int_{H^\sigma} \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{p|R_{s,N}(u)|} d\mu_s \leq C(s,R,p) < +\infty$$

for a certain constant $C(s,R,p) > 0$.
–As a consequence, we are now able to prove the second statement of Proposition 7.4, that is:

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_s(u)} - \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_{s,N}(u)} \right\|_{\mathbb{L}^p(d\mu_s)} \underset{N\to\infty}{\longrightarrow} 0$$

Firstly, we know from Proposition 3.3 that we have the pointwise convergence:

$$\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} R_s(u) = \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \cdot \frac{1}{6} \mathrm{Re} \sum_{\substack{k_1-k_2+\ldots-k_6=0 \\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} u_{k_1} \overline{u_{k_2}} \ldots \overline{u_{k_6}} = \lim_{N\to\infty} \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} R_{s,N}(u)$$

Consequently, from the continuity of the exponential, we also have the pointwise convergence:

$$\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_s(u)} = \lim_{N\to\infty} \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_{s,N}(u)}$$

In particular, $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_{s,N}(u)}$ converges to $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_s(u)}$ in measure. In addition, for a fixed $q \in (1,+\infty)$, we just proved that the functions $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_{s,N}(u)}$ are uniformly bounded in $\mathbb{L}^q(d\mu_s)$ (with respect to $N \in \mathbb{N}$). Then, (using the same argument as the one from the proof of Proposition 8.9) we can conclude that $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_{s,N}(u)}$ converges to $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} e^{-R_s(u)}$ in $\mathbb{L}^p(d\mu_s)$ for any $p \in [1,q)$; and since $q \in (1,+\infty)$ is arbitrary, the convergence holds for any $p \in [1,+\infty)$. This completes the proof of Proposition 7.4. □

However, it remains to prove Lemma 11.1. Our analysis will rely on a decomposition into two parts of $R_{s,N}$. We will be able to treat the first part deterministically thanks to suitable "exchanges of derivatives". For the second part, those "exchanges of derivatives" will fail because we will be in a *high-high-high-low-low-low* regime of frequency (where the three highest frequencies in the sum (3.2) defining $R_{s,N}$ are in fact much more higher than the three lowest frequencies). Instead, we will handle the second part using the independence between the high frequency Gaussians and the low frequency Gaussians, using Wiener chaos estimate.

11.1. **Decomposition.** Recall that we have (see Section 3, Proposition 3.1):

$$R_{s,N}(u) = \frac{1}{6} \mathrm{Re}\, \mathcal{R}(w)$$

where $w := \Pi_N u$ (with the convention $\Pi_\infty = id$), and

$$\mathcal{R}(w) := \sum_{\substack{k_1-k_2+\ldots-k_6=0 \\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{k_1} \overline{w_{k_2}} \ldots \overline{w_{k_6}}$$

It suffices to show the estimate of Lemma 11.1 for $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathcal{R}(w)$ because $|R_{s,N}(u)| \leq \frac{1}{6} |\mathcal{R}(w)|$.

Next, we split the set of indices over which we sum. We invoke the following sets of indices :

$$\Lambda_D := \{(k_1,...,k_6) \in \mathbb{Z}^6 : \sum_{j=1}^{6}(-1)^{j-1}k_j = 0,\ \sum_{j=1}^{6}(-1)^{j-1}k_j^2 \neq 0, \\ |k_{(3)}| < |k_{(1)}|^{1-\delta_0} \textbf{ or } |k_{(4)}| \geq |k_{(3)}|^{\delta_0}\} \tag{11.1}$$

and,

$$\Lambda_W := \{(k_1,...,k_6) \in \mathbb{Z}^6 : \sum_{j=1}^{6}(-1)^{j-1}k_j = 0,\ \sum_{j=1}^{6}(-1)^{j-1}k_j^2 \neq 0, \\ |k_{(3)}| \geq |k_{(1)}|^{1-\delta_0} \textbf{ and } |k_{(4)}| < |k_{(3)}|^{\delta_0}\} \tag{11.2}$$

where $\delta_0 \in (0,1)$[11]. Then, we split $\mathcal{R}$ as :

$$\mathcal{R}(w) = \mathcal{R}^{(D)}(w) + \mathcal{R}^{(W)}(w)$$

where,

$$\mathcal{R}^{(D)}(w) := \sum_{\Lambda_D} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}}, \qquad \mathcal{R}^{(W)}(w) := \sum_{\Lambda_W} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}}$$

To estimate $\mathcal{R}^{(D)}$ we will use the deterministic tools from Section 9. To estimate $\mathcal{R}^{(W)}$, we note that in the sum we are in a *high-high-high-low-low-low* regime because $|k_{(1)}|, |k_{(2)}|, |k_{(3)}| \gg |k_{(4)}|, |k_{(5)}|, |k_{(6)}|$. It will then be possible to make use of the independence between the Gaussians $g_{k_{(1)}}, g_{k_{(2)}}, g_{k_{(3)}}$ and $g_{k_{(4)}}, g_{k_{(5)}}, g_{k_{(6)}}$ via Wiener chaos.

11.1.1. *Absence of pairing:*

Definition 11.3 (Pairing)**.** Consider a constraint under the form :

$$\varepsilon_1 k_1 + \varepsilon_2 k_2 + ... + \varepsilon_m k_m = 0$$

where $k_j \in \mathbb{Z}$ and $\varepsilon_j \in \{-,+\}$. We say that k_j and k_l are paired if

$$\varepsilon_j k_j + \varepsilon_l k_l = 0$$

Remark 11.4. In Λ_W (see (11.2) above), there is (for $k_{(1)}$ large enough) no pairing within the three highest frequencies (relatively to the constraint $k_1 - k_2 + ... - k_6 = 0$). Indeed, suppose there is a pairing between two of the three highest frequencies. Then, the constraint would take the form:

$$\text{the remaining high frequency} = \text{sum of three low frequencies}$$

which is impossible because in Λ_W we have:

$$|\text{high frequencies}| \gg |\text{low frequencies}|$$

[11]We will see in the forthcoming proof that there is no constraint on δ_0, so we can chose it as any number in $(0,1)$

11.2. **Proof of the estimate.** We are now ready to prove Lemma 11.1.

Proof of Lemma 11.1. We organize the proof in two steps. In the first step, we provide a (deterministic) estimate for $\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathcal{R}$. In the second step, we show that this estimate is only conclusive for the contribution $\mathcal{R}^{(D)}$, and that for the remaining contribution $\mathcal{R}^{(W)}$, further analysis using Wiener chaos is required.

Step 1, Deterministic estimate : We rely on the estimate (10.2) we obtained in Section 10 (which we do not reprove here). Thus, we start our analysis from the following estimates:

$$|\mathcal{R}(w)| \lesssim \sum_{N_1,...,N_6} \mathfrak{R}(N_1,...,N_6) \quad \text{and:} \quad \mathfrak{R}(N_1,...,N_6) \lesssim_\varepsilon N_{(1)}^{2s-2+\varepsilon} N_{(3)}^2 \prod_{j=1}^6 \left\|P_{N_j} w\right\|_{\mathbb{L}^2(\mathbb{T})} \tag{11.3}$$

where $N_1,...,N_6$ are dyadic valued, P_N is the projector onto frequencies $|k| \sim N$, and:

$$\mathfrak{R}(N_1,...,N_6) := \sum_{\substack{k_1-k_2+...-k_6=0 \\ \Omega(\vec{k})\neq 0}} \left|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\right| \prod_{j=1}^6 \mathbb{1}_{|k_j|\sim N_j} |w_{k_j}|$$

We decompose $\mathfrak{R}$ as $\mathfrak{R} = \mathfrak{R}^{(D)} + \mathfrak{R}^{(W)}$, according to the decomposition of $\mathcal{R}$.

Now, let $1 < \sigma' < \sigma < s - \frac{1}{2}$. We have:

$$\prod_{j=1}^6 \left\|P_{N_j} w\right\|_{\mathbb{L}^2(\mathbb{T})} \lesssim \prod_{j=1}^6 \left\|P_{N_j} u\right\|_{\mathbb{L}^2(\mathbb{T})} \lesssim (N_{(1)}^{-\sigma} \|u\|_{H^\sigma})(N_{(2)}^{-\sigma'} \|u\|_{H^{\sigma'}})(N_{(3)}N_{(4)}N_{(5)}N_{(6)})^{-1} \|u\|_{H^1}^4 \tag{11.4}$$

Recall that the constraints in the sum in $\mathfrak{R}$ above implies that $N_{(1)} \sim N_{(2)}$. Moreover, if we write:

$$\sigma' = \alpha 1 + (1-\alpha)\sigma$$

for $\alpha \in (0,1)$, then, by interpolating $H^{\sigma'}$ between H^1 and H^σ, we obtain from (11.4) that:

$$\begin{aligned}\prod_{j=1}^6 \left\|P_{N_j} u\right\|_{\mathbb{L}^2(\mathbb{T})} &\lesssim N_{(1)}^{-2\sigma+\alpha(\sigma-1)} (N_{(3)}N_{(4)}N_{(5)}N_{(6)})^{-1} \|u\|_{H^1}^{4+\alpha} \|u\|_{H^\sigma}^{2-\alpha} \\ &\lesssim N_{(1)}^{-2\sigma+\alpha(s-\frac{3}{2})} (N_{(3)}N_{(4)}N_{(5)}N_{(6)})^{-1} \|u\|_{H^1}^{4+\alpha} \|u\|_{H^\sigma}^{2-\alpha}, \quad \text{(because } \alpha(\sigma-1) < \alpha(s-3/2)\text{)}\end{aligned}$$

Plugging this into (11.3), we obtain

$$\mathfrak{R}(N_1,...,N_6) \lesssim_\varepsilon N_{(1)}^{2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2})} N_{(3)} (N_{(4)}N_{(5)}N_{(6)})^{-1} \|u\|_{H^1}^{4+\alpha} \|u\|_{H^\sigma}^{2-\alpha}$$

In particular, thanks to Remark 7.2,

$$\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathfrak{R}(N_1,...,N_6) \lesssim_\varepsilon N_{(1)}^{2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2})} N_{(3)} (N_{(4)}N_{(5)}N_{(6)})^{-1} R^{4+\alpha} \|u\|_{H^\sigma}^{2-\alpha}$$

And since we will not need the smallness provided by $(N_{(5)}N_{(6)})^{-1}$, we simply write :

$$\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathfrak{R}(N_1,...,N_6) \lesssim_{\varepsilon,R} N_{(1)}^{2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2})} N_{(3)} N_{(4)}^{-1} \|u\|_{H^\sigma}^{2-\alpha}$$

Using now (9.3), we conclude that

$$\left\|\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathfrak{R}(N_1,...,N_6)\right\|_{\mathbb{L}^p(d\mu_s)} \leq C N_{(1)}^{2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2})} N_{(3)} N_{(4)}^{-1} p^{\frac{2-\alpha}{2}} \tag{11.5}$$

Where the constant $C>0$ depends on R, s, σ and ε. This is the estimate we will work with later in the proof. We stress the fact that (11.5) is true for $\mathfrak{R}$ and also for $\mathfrak{R}^{(D)}$ and $\mathfrak{R}^{(W)}$ (with the exact same proof).

Step 2, Estimates for $\mathcal{R}^{(D)}$ and $\mathcal{R}^{(W)}$:

•We start with $\mathcal{R}^{(D)}$. Recalling that $\sigma < s-\frac{1}{2}$, we see that the exponent of $N_{(1)}$ in (11.5) satisfies:

$$\begin{cases} 2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2}) > -1 \\ 2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2}) \longrightarrow -1 \ \text{ as } \begin{cases} \varepsilon \to 0 \\ \alpha \to 0 \\ \sigma \to s-\frac{1}{2} \end{cases} \end{cases} \tag{11.6}$$

Moreover, for a fixed $\delta_0 \in (0,1)$, the conditions on the indices lying in Λ_D imply that $N_{(3)} \lesssim N_{(1)}^{1-\delta_0}$ or $N_{(4)} \gtrsim N_{(3)}^{\delta_0}$. In both cases, we deduce from (11.5) that for ε and α close enough to 0 and σ close enough to $s-\frac{1}{2}$, we have[12]

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathfrak{R}^{(D)}(N_1,...,N_6) \right\|_{\mathbb{L}^p(d\mu_s)} \leq C N_{(1)}^{-} p^{\frac{2-\alpha}{2}}$$

Consequently,

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathcal{R}^{(D)} \right\|_{\mathbb{L}^p(d\mu_s)} \leq C p^{\frac{2-\alpha}{2}}$$

•We continue our analysis with the term $\mathcal{R}^{(W)}$. Once again, we start by decomposing $\mathcal{R}^{(W)}$ dyadically as $\mathcal{R}^{(W)}(w) = \sum_{N_1,...,N_6} \mathcal{R}^{(W)}_{N_1,...,N_6}(w)$, where[13]:

$$\mathcal{R}^{(W)}_{N_1,...,N_6}(w) := \sum_{\substack{k_1-k_2+...-k_6=0 \\ \Omega(\vec{k})\neq 0 \\ |k_{(4)}|\leq |k_{(3)}|^{\delta_0}}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{k_1}\overline{w_{k_2}}...\overline{w_{k_6}} \prod_{j=1}^{6} \mathbb{1}_{|k_j|\sim N_j}$$

Henceforth, we denote $w_{k_j}^{N_j} := \mathbb{1}_{|k_j|\sim N_j} w_{k_j}$ for better readability.

Without loss of generality, we assume that $\{N_1,N_2,N_3\} = \{N_{(1)},N_{(2)},N_{(3)}\}$, meaning that the three highest frequencies are k_1,k_2,k_3. The other cases are similar or simpler.
We denote $\mathcal{B}_{\ll N_{(3)}}$ the σ-algebra generated by the Gaussians $(g_k)_{|k|\leq N_{(3)}/100}$. We only need to consider the contribution when $N_{(1)}$ is large because when $N_{(1)}$ is small, all the N_j's are small, and we can use (11.5) without fearing issues of summability in the N_j's. In particular, we assume that $N_{(1)}$ is sufficiently large so that $N_{(3)}$ (which is $\gtrsim N_{(1)}^{1-\delta_0}$) is large enough to ensure that $N_{(4)}$ (which is $\lesssim N_{(3)}^{\delta_0}$) is $\leq N_{(3)}/100$. As a consequence, we have that the random variables :

$$\begin{aligned} &w_{k_4}^{N_4}, w_{k_4}^{N_4}, w_{k_5}^{N_6} \ \text{ are } \mathcal{B}_{\ll N_{(3)}} \text{mesurable} \\ &w_{k_1}^{N_1}, w_{k_2}^{N_2}, w_{k_3}^{N_3} \ \text{ are independent of } \mathcal{B}_{\ll N_{(3)}} \end{aligned}$$

[12]The notation N^- means that the power of N is $-\gamma$ for a certain $\gamma>0$.

[13]For the analysis below, we need to keep the complex conjugation bars, so here we don't use $\mathfrak{R}^{(W)}$

With this set up, we can now begin the analysis. Thanks to Remark 7.2, we have

$$\begin{aligned}\left\|\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s)} &\leq \left\|\mathbb{1}_{B^{H^1}_R}(u)\cdot\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s)}\\ &\leq \left\|\mathbb{1}_{B^{H^1}_R}(P_{N_{(3)/100}}u)\cdot\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s)}\end{aligned}$$

And, $\mathbb{1}_{B^{H^1}_R}(P_{N_{(3)/100}}u)$ is $\mathcal{B}_{\ll N_{(3)}}$ mesurable. So, using the $\mathbb{L}^p$-norm conditioned to the σ-algebra $\mathcal{B}_{\ll N_{(3)}}$, denoted $\mathbb{L}^p(d\mu_s|\mathcal{B}_{\ll N_{(3)}})$, we obtain :

$$\begin{aligned}\left\|\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s)} &\leq \left\|\left\|\mathbb{1}_{B^{H^1}_R}(P_{N_{(3)/100}}u)\cdot\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s|\mathcal{B}_{\ll N_{(3)}})}\right\|_{\mathbb{L}^\infty(d\mu_s)}\\ &\leq \left\|\left\|\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s|\mathcal{B}_{\ll N_{(3)}})}\cdot\mathbb{1}_{B^{H^1}_R}(P_{N_{(3)/100}}u)\right\|_{\mathbb{L}^\infty(d\mu_s)}\end{aligned} \tag{11.7}$$

Now, the conditional Wiener-chaos estimate from Lemma 9.9 (with $m=3$), followed by Lemma 9.10 combined with the absence of paring (see Remark 11.4), allows us to obtain :

$$\begin{aligned}\left\|\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s|\mathcal{B}_{\ll N_{(3)}})} &\lesssim p^{\frac{3}{2}}\left\|\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^2(d\mu_s|\mathcal{B}_{\ll N_{(3)}})}\\ &\lesssim p^{\frac{3}{2}}(N_{(1)}N_{(2)}N_{(3)})^{-s}\Big(\sum_{k_1,k_2,k_3}\Big|\sum_{k_4,k_5,k_6}C(\vec{k})\cdot\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\overline{w^{N_4}_{k_4}}w^{N_5}_{k_5}\overline{w^{N_6}_{k_6}}\Big|^2\Big)^{\frac{1}{2}}\end{aligned} \tag{11.8}$$

where we gathered all the constraints into the term :

$$C(\vec{k}) := \mathbb{1}_{k_1-k_2+\dots-k_6=0}\cdot\mathbb{1}_{\Omega(\vec{k})\neq 0}\cdot\prod_{j=1}^{6}\mathbb{1}_{|k_j|\sim N_j}$$

Next, by Cauchy-Schwarz,

$$\sum_{k_1,k_2,k_3}\Big|\sum_{k_4,k_5,k_6}C(\vec{k})\cdot\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\overline{w^{N_4}_{k_4}}w^{N_5}_{k_5}\overline{w^{N_6}_{k_6}}\Big|^2 \lesssim \sum_{k_1,k_2,k_3}\Big(\sum_{k_4,k_5,k_6}C(\vec{k})\cdot\Big|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\Big|^2|w^{N_4}_{k_4}w^{N_5}_{k_5}|^2\Big)\Big(\sum_{k_4,k_5,k_6}C(\vec{k})|w^{N_6}_{k_6}|^2\Big)$$

From Lemma 9.4 we get $\left|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\right| \lesssim |k_{(1)}|^{2s-2}(1+\frac{|k_{(3)}|^2}{|\Omega(\vec{k})|}) \lesssim |k_{(1)}|^{2s-2}|k_{(3)}|^2 \lesssim N^{2s-2}_{(1)}N^2_{(3)}$. Then, on the one hand :

$$\begin{aligned}\sum_{k_4,k_5,k_6}C(\vec{k})\cdot\Big|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\Big|^2|w^{N_4}_{k_4}w^{N_5}_{k_5}|^2 &\lesssim (N^{2s-2}_{(1)}N^2_{(3)})^2\sum_{k_4,k_5,k_6}C(\vec{k})\cdot|w^{N_4}_{k_4}w^{N_5}_{k_5}|^2\\ &\lesssim (N^{2s-2}_{(1)}N^2_{(3)})^2\sum_{k_4,k_5}|w^{N_4}_{k_4}w^{N_5}_{k_5}|^2\cdot\underbrace{\sum_{k_6}\mathbb{1}_{k_6=k_1-k_2\dots+k_5}}_{\leq 1}\\ &\lesssim (N^{2s-2}_{(1)}N^2_{(3)}\left\|w^{N_4}\right\|_{\mathbb{L}^2}\left\|w^{N_5}\right\|_{\mathbb{L}^2})^2\end{aligned}$$

and on the other hand,

$$\sum_{k_1,k_2,k_3}\Big(\sum_{k_4,k_5,k_6}C(\vec{k})|w^{N_6}_{k_6}|^2\Big) = \sum_{k_6}|w^{N_6}_{k_6}|^2\sum_{k_1,k_2,k_3,k_4,k_5}C(\vec{k}) \lesssim N_{(2)}N_{(3)}N_{(4)}N_{(5)}\cdot\left\|w^{N_6}\right\|^2_{\mathbb{L}^2}$$

where we used the counting bound from Lemma 9.3. We deduce from the two inequalities above that

$$\begin{aligned}\Big(\sum_{k_1,k_2,k_3}\Big|\sum_{k_4,k_5,k_6} C(\vec{k})\cdot\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\overline{w_{k_4}^{N_4}}w_{k_5}^{N_5}\overline{w_{k_6}^{N_6}}\Big|^2\Big)^{\frac12} &\lesssim N_{(1)}^{2s-2}N_{(3)}^2N_{(1)}^{\frac12}N_{(3)}^{\frac12}(N_{(4)}N_{(5)})^{\frac12}\left\|w^{N_4}\right\|_{\mathbb{L}^2}\left\|w^{N_5}\right\|_{\mathbb{L}^2}\left\|w^{N_6}\right\|_{\mathbb{L}^2}\\ &\lesssim N_{(1)}^{2s-2+\frac12}N_{(3)}^{\frac52}(N_{(4)}N_{(5)})^{-\frac12}N_{(6)}^{-1}\left\|w^{N_4}\right\|_{H^1}\left\|w^{N_5}\right\|_{H^1}\left\|w^{N_6}\right\|_{H^1}\\ &\lesssim N_{(1)}^{2s-\frac32}N_{(3)}^{\frac52}\left\|w^{N_4}\right\|_{H^1}\left\|w^{N_5}\right\|_{H^1}\left\|w^{N_6}\right\|_{H^1}\end{aligned}$$

Coming back to (11.8), we deduce that

$$\left\|\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s|\mathcal{B}_{\ll N_{(3)}})}\lesssim p^{\frac32}N_{(1)}^{-\frac32}N_{(3)}^{\frac52-s}\left\|w^{N_4}\right\|_{H^1}\left\|w^{N_5}\right\|_{H^1}\left\|w^{N_6}\right\|_{H^1}$$

And plugging this into (11.7), we obtain:

$$\left\|\mathbb{1}_{\{\mathcal{C}(u)\le R\}}\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s)}\lesssim_R p^{\frac32}N_{(1)}^{-\frac32}N_{(3)}^{\frac52-s}$$

Interpolating the above inequality with (11.5) (more precisely (11.5) with $\mathfrak{R}^{(W)}$ instead of $\mathfrak{R}$), we have that for any $\theta\in(0,1)$:

$$\left\|\mathbb{1}_{\{\mathcal{C}(u)\le R\}}\mathcal{R}^{(W)}_{N_1,\dots,N_6}\right\|_{\mathbb{L}^p(d\mu_s)}\lesssim p^{\frac32\theta+\frac{2-\alpha}{2}(1-\theta)}N_{(1)}^{-\frac32\theta+\left(2(s-1-\sigma)+\varepsilon+\alpha(s-\frac32)\right)(1-\theta)}N_{(3)}^{(\frac52-s)\theta+(1-\theta)}\tag{11.9}$$

Finally, we use the following lemma to finish the proof of Lemma 11.1:

Lemma 11.5. *There exist $\sigma<s-\frac12$ close enough to $s-\frac12$, $\varepsilon>0$ close enough to 0, $\theta\in(0,1)$ and $\alpha\in(0,1)$, such that*

$$\frac32\theta+\frac{2-\alpha}{2}(1-\theta)<1,\qquad N_{(1)}^{-\frac32\theta+\left(2(s-1-\sigma)+\varepsilon+\alpha(s-\frac32)\right)(1-\theta)}N_{(3)}^{(\frac52-s)\theta+(1-\theta)}\lesssim N_{(1)}^-$$

Let us provide a proof of this lemma:

Proof of Lemma 11.5. Firstly, we have:

$$\begin{aligned}\frac32\theta+\frac{2-\alpha}{2}(1-\theta)<1 &\iff \theta\left(\frac{1+\alpha}{2}\right)<1+\frac{\alpha-2}{2}=\frac{\alpha}{2}\\ &\iff \theta<\frac{\alpha}{1+\alpha}\end{aligned}\tag{11.10}$$

Secondly, regarding the exponent of $N_{(3)}$, we have:

$$(\frac52-s)\theta+(1-\theta)\ge0\iff\theta\le\frac{1}{s-\frac32}\tag{11.11}$$

Let us take $\theta\le\frac{1}{s-\frac32}$ (always true when $s\le\frac52$). Then,

$$N_{(1)}^{-\frac32\theta+\left(2(s-1-\sigma)+\varepsilon+\alpha(s-\frac32)\right)(1-\theta)}N_{(3)}^{(\frac52-s)\theta+(1-\theta)}\lesssim N_{(1)}^{-\frac32\theta+\left(2(s-1-\sigma)+\varepsilon+\alpha(s-\frac32)\right)(1-\theta)}N_{(1)}^{(\frac52-s)\theta+(1-\theta)}$$

We want to have appropriate parameters such that the conditions on θ given in (11.10) and (11.11) are satisfied along with the following one:

$$-\frac32\theta+\left(2(s-1-\sigma)+\varepsilon+\alpha(s-\frac32)\right)(1-\theta)+(\frac52-s)\theta+(1-\theta)<0$$

Since $2(s-1-\sigma)+\varepsilon \longrightarrow -1$ as $\sigma \to s-\frac{1}{2}$ and $\varepsilon \to 0$, the previous condition will be true for σ and ε respectively close enough to $s-\frac{3}{2}$ and 0 if:

$$-\frac{3}{2}\theta + \left(-1+\alpha(s-\frac{3}{2})\right)(1-\theta) + (\frac{5}{2}-s)\theta + (1-\theta) < 0$$

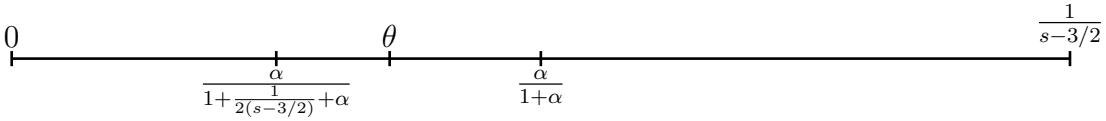
And this condition is equivalent to the following one:

$$\begin{aligned} &\theta\left(1-s-\alpha(s-\frac{3}{2})\right) < -\alpha(s-\frac{3}{2}) \\ \iff &\theta > \frac{\alpha(s-\frac{3}{2})}{s-1+\alpha(s-\frac{3}{2})} = \frac{\alpha}{\frac{s-1}{s-\frac{3}{2}}+\alpha} = \frac{\alpha}{1+\frac{1}{2(s-\frac{3}{2})}+\alpha} \end{aligned} \tag{11.12}$$

To conclude, if we first take α small enough such that $\frac{\alpha}{1+\alpha} < \frac{1}{s-\frac{3}{2}}$, and then θ such that

$$\frac{\alpha}{1+\frac{1}{2(s-3/2)}+\alpha} < \theta < \frac{\alpha}{1+\alpha},$$

the conditions given in (11.10), (11.11) and (11.12) are satisfied. We illustrate this with the following drawing :



With those parameters, Lemma 11.5 is proven. □

Hence, coming back to (11.9), the proof of Lemma 11.1 is completed. □

12. Estimates for the modified energy derivative at 0

This section is dedicated to $L^p(d\mu_s)$ estimates on $Q_{s,N}$ (defined in (4.4), see also Definition 2.3). More precisely, we prove Proposition 8.2. The strategy of our proof is the same as the one of Lemma 11.1. We first obtain a deterministic estimate that will be conclusive except for a frequency regime *high-high-high-low-...-low* (where the three highest frequencies are in fact much more higher than the others). This will lead us to decompose $Q_{s,N}$ into two parts, one that captures the *high-high-high-low-...-low* regime, and one that captures the other regime of frequencies. We will handle the *high-high-high-low-...-low* regime using the independence of Gaussians via Wiener-chaos estimate. It will be slightly more complicated than the proof of Lemma 11.1 because of the presence of more indices. However, in our situation, we will not encounter the possibility of a "pairing between generations"[14] which could have required the "remarkable cancellation" that has been presented in [29] (sections 5 and 7). The reason why we do not encounter such a pairing stems from the fact that we perform the Wiener chaos estimate with respect to three high-frequency Gaussians, that is with $m=3$ in Lemma 9.9. In doing so, we prevent a pairing between generations from occurring (see Paragraph 12.1.1). In [29], the Wiener chaos estimates are performed with respect to two high-frequency Gaussians, that is with $m=2$ in Lemma 9.9, and in this situation, a pairing between generation may occur.

[14]For example, considering the constraint $k_1-k_2+...-k_6=0$ & $p_1-p_2+...+p_5=k_1$, then according to Definition 11.3, a "pairing between generations" corresponds to the situation when one of the k_j is paired with one of the p_l.

Preliminaries. Fix $N \in \mathbb{N} \cup \{\infty\}$. Recall that from (4.4) we have:

$$Q_{s,N}(u) = Q_{s,N}(w) = \mathrm{Im}(-\frac{1}{6}\mathcal{Q}_0(w) + \frac{1}{2}\mathcal{Q}_1(w) - \frac{1}{2}\mathcal{Q}_2(w))$$

where $w := \Pi_N u$ (with the convention $\Pi_\infty = id$), and:

$$\mathcal{Q}_0(w) := \sum_{\substack{k_1-k_2+\ldots-k_6=0 \\ \Omega(\vec{k})=0}} \psi_{2s}(\vec{k}) w_{k_1}\overline{w_{k_2}}\ldots\overline{w_{k_6}}$$

and,

$$\mathcal{Q}_1(w) := \sum_{\substack{k_1-k_2+\ldots-k_6=0 \\ p_1-p_2+\ldots+p_5=k_1 \\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{p_1}\overline{w_{p_2}}\ldots w_{p_5}\overline{w_{k_2}}\ldots\overline{w_{k_6}}$$

and,

$$\mathcal{Q}_2(w) := \sum_{\substack{k_1-k_2+\ldots-k_6=0 \\ q_1-q_2+\ldots+q_5=k_2 \\ \Omega(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{k_1}\overline{w_{q_1}} w_{q_2}\ldots\overline{w_{q_5}} w_{k_3}\ldots\overline{w_{k_6}}$$

It suffices to show the estimates for $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ because $|Q_{s,N}(u)| \leq |\mathcal{Q}_0(w)|+|\mathcal{Q}_1(w)|+|\mathcal{Q}_2(w)|$.

Estimate for $\mathcal{Q}_0$: Actually, the estimate :

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathcal{Q}_0 \right\|_{\mathbb{L}^p(d\mu_s)} \leq C(s,R)p^\beta \tag{12.1}$$

has somehow already been proven in the proof of Lemma 11.1. Indeed, the proof is very similar, and we only sketch the beginning of it.

Sketch of the proof of (12.1). We decompose $\mathcal{Q}_0$ dyadically and we use Lemma 9.4 (with $\Omega(\vec{k}) = 0$), and we obtain :

$$|\mathcal{Q}_0(w)| \lesssim \sum_{N_1,\ldots,N_6} \mathfrak{Q}_0(N_1,\ldots,N_6)$$

where,

$$\mathfrak{Q}_0(N_1,\ldots,N_6) := \sum_{\substack{k_1-k_2+\ldots-k_6=0 \\ \Omega(\vec{k})=0}} N_{(1)}^{2s-2} N_{(3)}^2 \prod_{j=1}^6 \mathbb{1}_{|k_j|\sim N_j} |w_{k_j}|$$

To estimate $\mathfrak{Q}_0(N_1,\ldots,N_6)$, we use Lemma 9.6 (with $f_{k_j}^j = \mathbb{1}_{|k_j|\sim N_j}|w_{k_j}|$), which yields:

$$\mathfrak{Q}_0(N_1,\ldots,N_6) \lesssim_\varepsilon N_{(1)}^{2s-2+\varepsilon} N_{(3)}^2 \prod_{j=1}^6 \left\| P_{N_j} w \right\|_{\mathbb{L}^2(\mathbb{T})}$$

with P_N the projector onto frequencies $|k| \sim N$. This estimate is the analogue of (11.3). And from this point, the proof of (12.1) goes exactly the same as the proof of Lemma 11.1. □

To sum up, in order to prove Proposition 8.2, it remains to establish the estimates in the following lemma :

Lemma 12.1. *Let $s > \frac{3}{2}$. Then, there exists $\beta \in (0,1)$ such that for every $R > 0$, there exists a constant $C(s,R) > 0$, such that for any $p \in [2, +\infty)$,*

$$\left\| \mathbb{1}_{\{\mathcal{C}(u) \leq R\}} \mathcal{Q}_j \right\|_{\mathbb{L}^p(d\mu_s)} \leq C(s,R) p^{\beta} \tag{12.2}$$

for $j = 1, 2$.

Since the analysis for $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is the same, we only prove the estimate for $\mathcal{Q}_1$.

12.0.1. *Notations and remarks on set of indices:* Before decomposing $\mathcal{Q}$, let us first introduce some notations :

Notations 12.2. Given a set of frequency $k_1, k_2, ..., k_6, p_1, p_2, ..., p_5 \in \mathbb{Z}$, we denote :

- $k_{(1)}, ..., k_{(6)}$ a rearrangement of the k_j such that
$$|k_{(1)}| \geq |k_{(2)}| \geq ... \geq |k_{(6)}|$$
- $p_{(1)}, ..., p_{(6)}$ a rearrangement of the p_j such that
$$|p_{(1)}| \geq |p_{(2)}| \geq ... \geq |p_{(5)}|$$
- $n_{(1)}, n_{(2)}, ..., n_{(10)}$ a rearrangement of $k_2, ..., k_6, p_1, ..., p_5$ such that
$$|n_{(1)}| \geq |n_{(2)}| \geq ... \geq |n_{(10)}|$$

Also, in the sequel we will use :

- the letter M_j for the localization of the frequency k_j,
- the letter P_j for the localization of the frequency p_j,

Finally, $N_{(1)} \geq N_{(2)} \geq ... \geq N_{(10)}$ will be a non-increasing rearrangement of $P_1, ..., P_5, M_2, ..., M_6$.

Remark 12.3. – when $k_1 - k_2 + ... - k_6 = 0$, we have $|k_{(1)}| \sim |k_{(2)}|$.
– when $k_1 = p_1 - p_2 + ... + p_5$, we have $|p_{(1)}| \sim |p_{(2)}|$ or $|p_{(1)}| \sim |k_1|$.
– when $k_1 - k_2 + ... - k_6 = 0$ and $k_1 = p_1 - p_2 + ... + p_5$, we have $|n_{(1)}| \sim |n_{(2)}|$.

12.1. **Decomposition.** For convenience, we will denote $\mathcal{Q}_1$ simply as $\mathcal{Q}$.

Next, in the same spirit as the decomposition in Section 11, we decompose the set of indices over which we sum in $\mathcal{Q}$. We invoke the following set of indices :

$$\mathcal{I}_D := \{(k_1, ..., k_6, p_1, ..., p_5) \in \mathbb{Z}^{11} : \sum_{j=1}^{6} (-1)^{j-1} k_j = 0, \ k_1 = \sum_{j=1}^{5} (-1)^{j-1} p_j, \ \sum_{j=1}^{6} (-1)^{j-1} k_j^2 \neq 0, \\ (n_{(1)}, n_{(2)}) \in \{p_1, ..., p_5\}^2 \text{ or } |n_{(3)}| < |n_{(1)}|^{1-\delta_0} \text{ or } |n_{(4)}| \geq |n_{(3)}|^{\delta_0}\} \tag{12.3}$$

and,

$$\mathcal{I}_W := \{(k_1, ..., k_6, p_1, ..., p_5) \in \mathbb{Z}^{11} : \sum_{j=1}^{6} (-1)^{j-1} k_j = 0, \ k_1 = \sum_{j=1}^{5} (-1)^{j-1} p_j, \ \sum_{j=1}^{6} (-1)^{j-1} k_j^2 \neq 0, \\ (n_{(1)}, n_{(2)}) \notin \{p_1, ..., p_5\}^2 \text{ and } |n_{(3)}| \geq |n_{(1)}|^{1-\delta_0} \text{ and } |n_{(4)}| < |n_{(3)}|^{\delta_0}\} \tag{12.4}$$

for a fixed $\delta_0 \in (0,1)$[15]. Note that what differs from the decomposition in Section 11 is the additional constraint $(n_{(1)}, n_{(2)}) \notin \{p_1, ..., p_5\}^2$ in $\mathcal{I}_W$. This is because if $(n_{(1)}, n_{(2)}) \in \{p_1, ..., p_5\}^2$,

[15]Once again, δ_0 can be any number in $(0,1)$

we will not need to use Wiener-chaos.
In the same vein, we split $\mathcal{Q}$ as :

$$\mathcal{Q}(w) = \mathcal{Q}^{(D)}(w) + \mathcal{Q}^{(W)}(w)$$

where,

$$\mathcal{Q}^{(D)}(w) := \sum_{\mathcal{I}_D} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{p_1}\overline{w_{p_2}}...w_{p_5}\overline{w_{k_2}}...\overline{w_{k_6}}, \quad \mathcal{Q}^{(W)}(w) := \sum_{\mathcal{I}_W} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{p_1}\overline{w_{p_2}}...w_{p_5}\overline{w_{k_2}}...\overline{w_{k_6}} \quad (12.5)$$

12.1.1. *Absence of pairing.* Recall that we have defined what a "pairing" is in Definition 11.3. In $\mathcal{I}_W$ (see (12.4) above), there is (for $n_{(1)}$ large enough) no pairing within the three highest frequencies, relatively to the constraint:

$$p_1 - p_2... + p_5 - k_2 + k_3... - k_6 = 0$$

Indeed, suppose there is a pairing between two of the three highest frequencies. Then, the constraint would take the form:

$$\text{the remaining high frequency} = \text{sum of seven low frequencies}$$

which is impossible because in $\mathcal{I}_W$ we have:

$$|\text{high frequencies}| \gg |\text{low frequencies}|$$

12.2. **Proof of the estimate.** We are now prepared to prove Lemma 12.1 (for $j = 1$). The forthcoming proof is organized as follows. Firstly, we establish an estimate that will be conclusive only for the contribution $\mathcal{Q}^{(D)}$. For the contribution $\mathcal{Q}^{(W)}$, we will also exploit the independence between high frequency Gaussians and low frequency Gaussians using Wiener chaos estimate.

Proof of Lemma 12.1. Step 1, Deterministic estimate : Let $w \in H^{s-\frac{1}{2}-}(\mathbb{T})$. For convenience, for any dyadic number N we denote $w_k^N := \mathbb{1}_{|k|\sim N}|w_k|$. We start by taking the absolute value and summing over dyadic blocks

$$|\mathcal{Q}(w)| \lesssim \sum_{M_1,...,M_6,P_1,...,P_5} \mathfrak{Q}(M_1,...,M_6,P_1,...,P_5)$$

where $M_1,...,M_6,P_1,...,P_5$ are dyadic-valued and

$$\mathfrak{Q}(M_1,...,M_6,P_1,...,P_5) := \sum_{\substack{k_1-k_2+...-k_6=0\\ p_1-p_2+...+p_5=k_1\\ \Omega(\vec{k})\neq 0}} \Big|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\Big| |w_{p_1}^{P_1}...w_{p_5}^{P_5}w_{k_2}^{M_2}...w_{k_6}^{M_6}| \cdot \mathbb{1}_{|k_1|\sim M_1}$$

We decompose $\mathfrak{Q}$ as $\mathfrak{Q} = \mathfrak{Q}^{(D)} + \mathfrak{Q}^{(W)}$, according to the decomposition of $\mathcal{Q}$.

In regard to (10.4) and (10.6) from the proof of Proposition 4.1, we have:

$$\mathfrak{Q}(M_1,...,P_5) \lesssim M_{(1)}^{2s-2+\varepsilon} M_{(3)}^2 \left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{\frac{1}{2}} \prod_{j=1}^{5} \left\|w^{P_j}\right\|_{\mathbb{L}^2} \prod_{j=2}^{6} \left\|w^{M_j}\right\|_{\mathbb{L}^2}$$

Using Notations 12.2, it means that

$$\mathfrak{Q}(M_1,...,P_5) \lesssim M_{(1)}^{2s-2+\varepsilon} M_{(3)}^2 \left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{\frac{1}{2}} \prod_{j=1}^{10} \left\|w^{N_j}\right\|_{\mathbb{L}^2} \quad (12.6)$$

Moreover, for $1 < \sigma' < \sigma < s - \frac{1}{2}$, we have :

$$\prod_{j=1}^{10} \left\| w^{N_j} \right\|_{\mathbb{L}^2} \lesssim (N_{(1)}^{-\sigma} \|u\|_{H^\sigma})(N_{(2)}^{-\sigma'} \|u\|_{H^{\sigma'}})(N_{(3)}N_{(4)}...N_{(10)})^{-1} \|u\|_{H^1}^8 \tag{12.7}$$

Then, since $N_{(1)} \sim N_{(2)}$, and writing σ' as:

$$\sigma' = \alpha 1 + (1-\alpha)\sigma$$

for $\alpha \in (0,1)$, we deduce from (12.7), by interpolating $H^{\sigma'}$ between H^1 and H^σ, that:

$$\begin{aligned}\prod_{j=1}^{10} \left\| w^{N_j} \right\|_{\mathbb{L}^2} &\lesssim N_{(1)}^{-2\sigma+\alpha(\sigma-1)}(N_{(3)}N_{(4)}...N_{(10)})^{-1} \|u\|_{H^1}^{8+\alpha} \|u\|_{H^\sigma}^{2-\alpha} \\ &\lesssim N_{(1)}^{-2\sigma+\alpha(s-\frac{3}{2})}(N_{(3)}N_{(4)}...N_{(10)})^{-1} \|u\|_{H^1}^{8+\alpha} \|u\|_{H^\sigma}^{2-\alpha}, \quad \text{(because } \alpha(\sigma-1) < \alpha(s-3/2) \text{)}\end{aligned}$$

Plugging this into (12.6), we obtain

$$\mathfrak{Q}(M_1,...,P_5) \lesssim N_{(1)}^{-2\sigma+\alpha(s-\frac{3}{2})} M_{(1)}^{2s-2+\varepsilon} M_{(3)}^2 \left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{\frac{1}{2}} (N_{(3)}N_{(4)}...N_{(10)})^{-1} \|u\|_{H^1}^{8+\alpha} \|u\|_{H^\sigma}^{2-\alpha}$$

Using the facts $M_{(1)} \lesssim N_{(1)}$ and $N_{(1)} \sim N_{(2)}$, and also Remark 7.2, we deduce that

$$\begin{aligned}&\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathfrak{Q}(M_1,...,P_5) \\ &\lesssim_{\varepsilon,R} N_{(1)}^{2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2})} M_{(3)}^2 \left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{\frac{1}{2}} N_{(1)}^2 (\overbrace{N_{(1)}N_{(2)}N_{(3)}N_{(4)}...N_{(10)}}^{=M_2...M_6P_1...P_5})^{-1} \|u\|_{H^\sigma}^{2-\alpha} \\ &\lesssim N_{(1)}^{2(s-\sigma)+\varepsilon+\alpha(s-\frac{3}{2})} M_{(3)}^2 (M_2...M_6)^{-1} P_{(1)}^{-1} \left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}} \|u\|_{H^\sigma}^{2-\alpha} \\ &\lesssim \begin{cases} N_{(1)}^{2(s-\sigma)-\frac{3}{2}+\varepsilon+\alpha(s-\frac{3}{2})} \|u\|_{H^\sigma}^{2-\alpha}, & \text{if } (N_{(1)},N_{(2)}) \in \{P_1,...,P_5\}^2 \\ N_{(1)}^{2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2})} N_{(3)}N_{(4)}^{-\frac{1}{2}} \|u\|_{H^\sigma}^{2-\alpha}, & \text{if } (N_{(1)},N_{(2)}) \notin \{P_1,...,P_5\}^2 \end{cases}\end{aligned} \tag{12.8}$$

Here, we used estimates on $M_{(3)}^2(M_2...M_6)^{-1}P_{(1)}^{-1}\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}}$ in terms of the N_j, which are gathered in the following lemma.

Lemma 12.4. – *If* $(N_{(1)},N_{(2)}) \in \{P_1,...,P_5\}^2$, *then*

$$M_{(3)}^2(M_2...M_6)^{-1}P_{(1)}^{-1}\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}} \lesssim N_{(1)}^{-\frac{3}{2}}$$

– *If* $(N_{(1)},N_{(2)}) \notin \{P_1,...,P_5\}^2$, *then*

$$M_{(3)}^2(M_2...M_6)^{-1}P_{(1)}^{-1}\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}} \lesssim N_{(1)}^{-2}N_{(3)}N_{(4)}^{-\frac{1}{2}}$$

For clarity, we postpone the proof of Lemma 12.4 for the end of this section.
To conclude the first step of the proof, using (9.3) in (12.8) yields:

$$\left\|\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathfrak{Q}(M_1,...,P_5)\right\|_{\mathbb{L}^p(d\mu_s)} \lesssim \begin{cases} N_{(1)}^{2(s-\sigma)-\frac{3}{2}+\varepsilon+\alpha(s-\frac{3}{2})} p^{\frac{2-\alpha}{2}}, & \text{if } (N_{(1)},N_{(2)}) \in \{P_1,...,P_5\}^2 \\ N_{(1)}^{2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2})} N_{(3)}N_{(4)}^{-\frac{1}{2}} p^{\frac{2-\alpha}{2}}, & \text{if } (N_{(1)},N_{(2)}) \notin \{P_1,...,P_5\}^2 \end{cases} \tag{12.9}$$

where the constant depends on s, R, σ and ε. This is the deterministic estimate we will work with later in the proof. We stress the fact that (12.9) is true for $\mathfrak{Q}$ and also for $\mathfrak{Q}^{(D)}$ and $\mathfrak{Q}^{(W)}$ (with

the exact same proof).

Step 2, Estimates for $\mathcal{Q}^{(D)}$ and $\mathcal{Q}^{(W)}$:

•Let us begin with the estimate for $\mathcal{Q}^{(D)}$. Recall that in $\mathcal{Q}^{(D)}$ we sum over the set $\mathcal{I}_D$ defined in (12.3). And, the constraints in $\mathcal{I}_D$ imply that the dyadic integers (from the dyadic decomposition) must satisfy one of the following conditions :

$$(N_{(1)}, N_{(2)}) \in \{P_1, ..., P_5\}^2 \qquad \textbf{or} \qquad N_{(3)} \lesssim N_{(1)}^{1-\delta_0} \qquad \textbf{or} \qquad N_{(4)} \gtrsim N_{(3)}^{\delta_0}$$

In the first situation, when $(N_{(1)}, N_{(2)}) \in \{P_1, ..., P_5\}^2$, (12.9) (with $\mathfrak{Q}^{(D)}$ instead of $\mathfrak{Q}$) yields:

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathfrak{Q}^{(D)}(M_1, ..., P_5) \right\|_{\mathbb{L}^p(d\mu_s)} \lesssim N_{(1)}^{2(s-\sigma)-\frac{3}{2}+\varepsilon+\alpha(s-\frac{3}{2})} p^{\frac{2-\alpha}{2}}$$

And, the exponent of $N_{(1)}$ satisfy :

$$\begin{cases} 2(s-\sigma) - \frac{3}{2} + \varepsilon + \alpha(s-\frac{3}{2}) > -\frac{1}{2} \\ \\ 2(s-\sigma) - \frac{3}{2} + \varepsilon + \alpha(s-\frac{3}{2}) \longrightarrow -\frac{1}{2} \ \text{ as } \begin{cases} \varepsilon \to 0 \\ \alpha \to 0 \\ \sigma \to s - \frac{1}{2} \end{cases} \end{cases}$$

Thus, for ε and α close enough to 0 and σ close enough to $s - \frac{1}{2}$, we have

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathfrak{Q}^{(D)}(M_1, ..., P_5) \right\|_{\mathbb{L}^p(d\mu_s)} \lesssim N_{(1)}^{-} p^{\frac{2-\alpha}{2}}$$

Now, in the second situation when $(N_{(1}, N_{(2)}) \notin \{P_1, ..., P_5\}^2$ **and :** $N_{(3)} \lesssim N_{(1)}^{1-\delta_0}$ **or** $N_{(4)} \gtrsim N_{(3)}^{\delta_0}$, (12.9) yields :

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathfrak{Q}^{(D)}(M_1, ..., P_5) \right\|_{\mathbb{L}^p(d\mu_s)} \lesssim N_{(1)}^{2(s-\sigma-1)+\varepsilon+\alpha(s-\frac{3}{2})} N_{(3)} N_{(4)}^{-\frac{1}{2}} p^{\frac{2-\alpha}{2}}$$

And since the exponent of $N_{(1)}$ satisfy :

$$\begin{cases} 2(s-\sigma-1) + \varepsilon + \alpha(s-\frac{3}{2}) > -1 \\ \\ 2(s-\sigma-1) + \varepsilon + \alpha(s-\frac{3}{2}) \longrightarrow -1 \ \text{ as } \begin{cases} \varepsilon \to 0 \\ \alpha \to 0 \\ \sigma \to s - \frac{1}{2} \end{cases} \end{cases}$$

we deduce that for a fixed $\delta_0 \in (0, 1)$ and for ε and α close enough to 0 and σ close enough to $s - \frac{1}{2}$, we have again

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathfrak{Q}^{(D)}(M_1, ..., P_5) \right\|_{\mathbb{L}^p(d\mu_s)} \lesssim N_{(1)}^{-} p^{\frac{2-\alpha}{2}}$$

Finally, summing over the dyadic integers $M_1, ..., M_6, P_1, ..., P_5$, this estimate for both situations leads to

$$\left\| \mathbb{1}_{\{\mathcal{C}(u)\leq R\}} \mathcal{Q}^{(D)} \right\|_{\mathbb{L}^p(d\mu_s)} \lesssim p^{\frac{2-\alpha}{2}}$$

which is the desired estimate.

•Next, in order to finish the proof of Lemma 12.1, we need the same estimate for the contribution $\mathcal{Q}^{(W)}$. For this term, while the estimate (12.9) is not conclusive, we are in a situation where we can make use of Wiener chaos. Recall that in $\mathcal{Q}^{(W)}$, we sum over the set $\mathcal{I}_W$ defined in (12.4). Even though the following method is very similar to the one used for estimating $\mathcal{R}^{(W)}$ in Section 11 (see

the proof of Lemma 11.1), we re-perform the analysis in detail.

As usual, we start by decomposing $\mathcal{Q}^{(W)}$ as[16] $\mathcal{Q}^{(W)} = \sum_{M_1,...,M_6,P1,...,P_5} \mathcal{Q}^{(W)}_{M_1,...,M_6,P1,...,P_5}$, where :

$$\mathcal{Q}^{(W)}_{M_1,...,M_6,P1,...,P_5} := \sum_{\mathcal{I}_W} \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{p_1} \overline{w_{p_2}} ... w_{p_5} \overline{w_{k_2}} ... \overline{w_{k_6}} \left(\prod_{j=1}^{6} \mathbb{1}_{|k_j| \sim M_j} \right) \left(\prod_{j=1}^{5} \mathbb{1}_{p_j \sim P_j} \right)$$

Henceforth, we denote $w_{k_j}^{M_j} := \mathbb{1}_{|k_j| \sim N_j} w_{k_j}$ and $w_{p_j}^{P_j} := \mathbb{1}_{|p_j| \sim P_j} w_{p_j}$ for better readability.

Without loss of generality, we only prove the estimate for the contribution in $\mathcal{Q}^{(W)}$ where the three highest frequencies are k_2, k_3 and k_4, because the other cases are identical.
Again we denote $\mathcal{B}_{\ll N_{(3)}}$ the σ-algebra generated by Gaussians $(g_k)_{|k| \leq N_{(3)}/100}$. Since we only need to consider the contribution when $N_{(1)}$ is large, we can assume that $N_{(1)}$ is sufficiently large to ensure $N_{(4)} \leq N_{(3)}/100$. This follows from the constraints in $\mathcal{I}_W$ which imply $N_{(3)} \gtrsim N_{(1)}^{1-\delta_0}$ and $N_{(4)} \lesssim N_{(3)}^{\delta_0}$. Thus, for $N_{(1)}$ large enough, we have $N_{(3)}^{\delta_0} \ll N_{(3)}$, ensuring that $N_{(4)} \leq N_{(3)}/100$.
As a consequence, we have that the random variables :

$$w_{k_2}^{M_2}, w_{k_3}^{M_3}, w_{k_4}^{M_4} \text{ are independent of } \mathcal{B}_{\ll N_{(3)}},$$
$$w_{k_5}^{M_5}, w_{k_6}^{M_6}, w_{p_1}^{P_1}, w_{p_2}^{P_2}, w_{p_3}^{P_3}, w_{p_4}^{P_4}, w_{p_5}^{P_5} \text{ are } \mathcal{B}_{\ll N_{(3)}} \text{ mesurable.}$$

Now, identically to (11.7), we have

$$\left\| \mathbb{1}_{\{\mathcal{C}(u) \leq R\}} \mathcal{Q}^{(W)}_{M_1,...,M_6,P_1,...,P_5} \right\|_{\mathbb{L}^p(d\mu_s)} \leq \left\| \left\| \mathcal{Q}^{(W)}_{M_1,...,M_6,P_1,...,P_5} \right\|_{\mathbb{L}^p(d\mu_s|\mathcal{B}_{\ll N_{(3)}})} \cdot \mathbb{1}_{B_R^{H^1}}(P_{N_{(3)/100}} u) \right\|_{\mathbb{L}^\infty(d\mu_s)} \tag{12.10}$$

And, the conditional Wiener-chaos estimate from Lemma 9.9 (with $m = 3$), followed by Lemma 9.10 combined with the absence of paring (see Paragraph 12.1.1), allows us to obtain :

$$\begin{aligned} &\left\| \mathcal{Q}^{(W)}_{M_1,...,M_6,P_1,...,P_5} \right\|_{\mathbb{L}^p(d\mu_s|\mathcal{B}_{\ll N_{(3)}})} \lesssim p^{\frac{3}{2}} \left\| \mathcal{Q}^{(W)}_{M_1,...,M_6,P_1,...,P_5} \right\|_{\mathbb{L}^2(d\mu_s|\mathcal{B}_{\ll N_{(3)}})} \\ &\lesssim p^{\frac{3}{2}} (N_{(1)} N_{(2)} N_{(3)})^{-s} \left(\sum_{k_2,k_3,k_4} \left| \sum_{\substack{k_1,k_5,k_6 \\ p_1,p_2,p_3,p_4,p_5}} C(\vec{k},\vec{p}) \cdot \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{p_1}^{P_1} \overline{w_{p_2}^{P_2}} w_{p_3}^{P_3} \overline{w_{p_4}^{P_4}} w_{p_5}^{P_5} w_{k_5}^{M_5} \overline{w_{k_6}^{M_6}} \right|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{12.11}$$

where we gathered all the constraints into the term :

$$C(\vec{k},\vec{p}) := \mathbb{1}_{k_1 - k_2 + ... - k_6 = 0} \cdot \mathbb{1}_{p_1 - p_2 + ... + p_5 = k_1} \cdot \mathbb{1}_{\Omega(\vec{k}) \neq 0} \cdot \prod_{j=1}^{6} \mathbb{1}_{|k_j| \sim M_j} \prod_{j=1}^{5} \mathbb{1}_{|p_j| \sim P_j}$$

Next, by Cauchy-Schwarz,

$$\begin{aligned} &\sum_{k_2,k_3,k_4} \left| \sum_{\substack{k_1,k_5,k_6 \\ p_1,p_2,p_3,p_4,p_5}} C(\vec{k},\vec{p}) \cdot \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} w_{p_1}^{P_1} ... w_{p_5}^{P_5} w_{k_5}^{M_5} \overline{w_{k_6}^{M_6}} \right|^2 \\ &\lesssim \sum_{k_2,k_3,k_4} \left(\sum_{\substack{k_1,k_5,k_6 \\ p_1,p_2,p_2,p_4,p_5}} C(\vec{k},\vec{p}) \cdot \left| \frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})} \right|^2 |w_{p_1}^{P_1} ... w_{p_5}^{P_5} w_{k_5}^{M_5}|^2 \right) \left(\sum_{\substack{k_1,k_5,k_6 \\ p_1,p_2,p_2,p_4,p_5}} C(\vec{k},\vec{p}) |w_{k_6}^{M_6}|^2 \right) \end{aligned}$$

[16] For the analysis below, we need to keep the complex conjugation bars, so here we don't use $\mathfrak{Q}^{(W)}$

From Lemma 9.4 we get $\left|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\right| \lesssim |k_{(1)}|^{2s-2}(1+\frac{|k_{(3)}|^2}{|\Omega(\vec{k})|}) \lesssim |k_{(1)}|^{2s-2}|k_{(3)}|^2 \lesssim N_{(1)}^{2s-2}N_{(3)}^2$. Then, on the one hand:

$$\begin{aligned}\sum_{\substack{k_1,k_5,k_6\\p_1,p_2,p_2,p_4,p_5}} C(\vec{k},\vec{p})\cdot\left|\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}\right|^2|w_{p_1}^{P_1}...w_{p_5}^{P_5}w_{k_5}^{M_5}|^2 &\lesssim (N_{(1)}^{2s-2}N_{(3)}^2)^2 \sum_{\substack{k_1,k_5,k_6\\p_1,p_2,p_2,p_4,p_5}} C(\vec{k},\vec{p})\cdot|w_{p_1}^{P_1}...w_{p_5}^{P_5}w_{k_5}^{M_5}|^2\\ &\lesssim (N_{(1)}^{2s-2}N_{(3)}^2)^2 \sum_{\substack{k_5\\p_1,p_2,p_2,p_4,p_5}} |w_{p_1}^{P_1}...w_{p_5}^{P_5}w_{k_5}^{M_5}|^2\cdot \underbrace{\sum_{k_1,k_6}\mathbb{1}_{k_6=k_1-k_2...+k_5}\cdot\mathbb{1}_{p_1-p_2+...+p_5=k_1}}_{\leq 1}\\ &\lesssim (N_{(1)}^{2s-2}N_{(3)}^2\left\|w^{P_1}\right\|_{\mathbb{L}^2}...\left\|w^{P_5}\right\|_{\mathbb{L}^2}\left\|w^{M_5}\right\|_{\mathbb{L}^2})^2\\ &\lesssim \left(N_{(1)}^{2s-2}N_{(3)}^2(P_1...P_5M_5)^{-1}\left\|w^{P_1}\right\|_{H^1}...\left\|w^{P_5}\right\|_{H^1}\left\|w^{M_5}\right\|_{H^1}\right)^2\end{aligned}$$

and on the other hand,

$$\sum_{\substack{k_1,k_2,k_3,k_4,k_5,k_6\\p_1,p_2,p_2,p_4,p_5}} C(\vec{k},\vec{p})|w_{k_6}^{N_6}|^2 = \sum_{k_6}|w_{k_6}^{N_6}|^2\cdot\sum_{\substack{k_1,k_2,k_3,k_4,k_5\\p_1,p_2,p_2,p_4,p_5}} C(\vec{k},\vec{p}) \lesssim N_{(2)}N_{(3)}P_1...P_5M_5\cdot(M_6^{-1}\left\|w^{M_6}\right\|_{H^1})^2$$

where we used the counting bound from Lemma 9.3. We deduce from the two inequalities above that

$$\begin{aligned}&\left(\sum_{k_2,k_3,k_4}\Big|\sum_{\substack{k_1,k_5,k_6\\p_1,p_2,p_3,p_4,p_5}} C(\vec{k},\vec{p})\cdot\frac{\psi_{2s}(\vec{k})}{\Omega(\vec{k})}w_{p_1}^{P_1}\overline{w_{p_2}^{P_2}}w_{p_3}^{P_3}\overline{w_{p_4}^{P_4}}w_{p_5}^{P_5}w_{k_5}^{M_5}\overline{w_{k_6}^{M_6}}\Big|^2\right)^{\frac{1}{2}}\\ &\qquad\lesssim N_{(1)}^{2s-2}N_{(3)}^2N_{(1)}^{\frac{1}{2}}N_{(3)}^{\frac{1}{2}}\left\|w^{P_1}\right\|_{H^1}...\left\|w^{P_5}\right\|_{H^1}\left\|w^{M_5}\right\|_{H^1}\left\|w^{M_6}\right\|_{H^1}\\ &\qquad\lesssim N_{(1)}^{2s-\frac{3}{2}}N_{(3)}^{\frac{5}{2}}\left\|w^{P_1}\right\|_{H^1}...\left\|w^{P_5}\right\|_{H^1}\left\|w^{M_5}\right\|_{H^1}\left\|w^{M_6}\right\|_{H^1}\end{aligned}$$

Coming back to (12.11), we deduce that

$$\left\|\mathcal{Q}^{(W)}_{M_1,...,M_6,P_1,...,P_5}\right\|_{\mathbb{L}^p(d\mu_s|\mathcal{B}_{\ll N_{(3)}})} \lesssim p^{\frac{3}{2}}N_{(1)}^{-\frac{3}{2}}N_{(3)}^{\frac{5}{2}-s}\left\|w^{P_1}\right\|_{H^1}...\left\|w^{P_5}\right\|_{H^1}\left\|w^{M_5}\right\|_{H^1}\left\|w^{M_6}\right\|_{H^1}$$

And plugging this into (12.10), we obtain :

$$\left\|\mathbb{1}_{\{\mathcal{C}(u)\leq R\}}\mathcal{Q}^{(W)}_{M_1,...,M_6,P_1,...,P_5}\right\|_{\mathbb{L}^p(d\mu_s)} \lesssim_R p^{\frac{3}{2}}N_{(1)}^{-\frac{3}{2}}N_{(3)}^{\frac{5}{2}-s}$$

Interpolating the above inequality with (12.9) (more precisely (12.9) with $\mathfrak{Q}^{(W)}$ instead of $\mathfrak{Q}$), we can conclude in the same way as what we did with $\mathcal{R}^{(W)}$ in the proof of Lemma 11.1 (in particular using again Lemma 11.5). □

In order to complete the proof of Lemma 12.1, we provide in the next paragraph a proof of the technical Lemma 12.4.

Proof of Lemma 12.4.

Proof of Lemma 12.4. Recall that we use Notations 12.2.

•Assume first that $(N_{(1)},N_{(2)})\in\{P_1,...,P_5\}^2$. Then, $P_{(1)}^{-1}P_{(2)}^{-\frac{1}{2}}\lesssim N_{(1)}^{-\frac{3}{2}}$. Moreover, we have

$$M_{(3)}^2(M_2...M_6)^{-1}\leq M_{(3)}^2(M_{(2)}M_{(3)}...M_{(6)})^{-1}\leq(M_{(4)}M_{(5)}M_{(6)})^{-1}\lesssim 1$$

Combining these two inequalities, we obtain

$$M_{(3)}^2(M_2...M_6)^{-1}P_{(1)}^{-1}\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}} \lesssim N_{(1)}^{-\frac{3}{2}}(P_{(3)}P_{(4)}P_{(5)})^{-\frac{1}{2}} \lesssim N_{(1)}^{-\frac{3}{2}}$$

which is the desired estimate.

•Now, assume that $(N_{(1)}, N_{(2)}) \notin \{P_1, ..., P_5\}^2$. It means that :

$$(N_{(1)}, N_{(2)}) \in \{M_2, ..., M_6\}^2 \text{ or } (\ N_{(1)} \in \{M_2, ..., M_6\} \text{ and } N_{(2)} \in \{P_1, ..., P_5\}\ )$$
$$\text{or } (\ N_{(2)} \in \{M_2, ..., M_6\} \text{ and } N_{(1)} \in \{P_1, ..., P_5\}\ )$$

– Fristly, we suppose that $(N_{(1)}, N_{(2)}) \in \{M_2, ..., M_6\}^2$. Then, we note that $N_{(3)} \in \{M_2, ..., M_6\}$ **or** $P_{(1)} = N_{(3)}$. And also,

$$N_{(4)} \in \{M_2, ..., M_6\} \quad \text{or} \quad N_{(4)} \in \{P_1, ..., P_5\}$$

Considering these facts (and still the fact that $N_{(1)} \sim N_{(2)}$), we deduce that

$$M_{(3)}^2(M_2...M_6)^{-1}P_{(1)}^{-1}\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}} \lesssim M_{(3)}^2N_{(1)}^{-2}N_{(3)}^{-1}N_{(4)}^{-\frac{1}{2}}$$

Using then the fact that $M_{(3)} \lesssim N_{(3)}$, we have

$$M_{(3)}^2(M_2...M_6)^{-1}P_{(1)}^{-1}\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}} \lesssim N_{(1)}^{-2}N_{(3)}N_{(4)}^{-\frac{1}{2}}$$

which is the desired inequality.
–Secondly, we suppose that $N_{(1)} \in \{M_2, ..., M_6\}$ **and** $N_{(2)} \in \{P_1, ..., P_5\}$. We consider two cases.
a) The first one is when $M_1 \in \{M_{(1)}, M_{(2)}, M_{(3)}\}$. In that case, $M_{(4)}, M_{(5)}, M_{(6)}, P_{(2)}, ..., P_{(5)}$ are seven indices out of the ten indices N_j (that would not have been the case if $M_{(1)} \in \{M_{(4)}, M_{(5)}, M_{(6)}\}$, because according to Notations 12.2, M_1 is not one of the N_j). As a consequence,

$$\max\{M_{(4)}, M_{(5)}, M_{(6)}, P_{(2)}, ..., P_{(5)}\} \geq N_{(4)} \quad \text{so,} \quad (M_{(4)}M_{(5)}M_{(6)})^{-1}(P_{(2)}...P_{(5)})^{-\frac{1}{2}} \lesssim N_{(4)}^{-\frac{1}{2}}$$

On the other hand, using the fact that $M_{(1)} \sim M_{(2)}$, we have

$$M_{(3)}^2(M_2M_3...M_6)^{-1} \lesssim M_{(3)}^2(M_{(2)}M_{(3)}...M_{(6)})^{-1} \lesssim M_{(1)}^{-1}M_{(3)}(M_{(4)}M_{(5)}M_{(6)})^{-1}$$

Combining these two inequalities and recalling that $M_{(1)}, P_{(1)} \sim N_{(1)}$, we obtain

$$M_{(3)}^2(M_2...M_6)^{-1}P_{(1)}^{-1}\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}} \lesssim N_{(1)}^{-2}M_{(3)}N_{(4)}^{-\frac{1}{2}} \lesssim N_{(1)}^{-2}N_{(3)}N_{(4)}^{-\frac{1}{2}}$$

b) The second case we consider is when $M_1 \in \{M_{(4)}, M_{(5)}, M_{(6)}\}$. In that case, $\{M_{(1)}, M_{(2)}, M_{(3)}\} \subset \{M_2, M_3, ..., M_6\}$. Thus,

$$(M_2...M_6)^{-1} \lesssim (M_{(1)}M_{(2)}M_{(3)}M_{(5)}M_{(6)})^{-1}$$

so,

$$M_{(3)}^2(M_2...M_6)^{-1} \lesssim N_{(1)}^{-2}M_{(3)}$$

and,

$$M_{(3)}^2(M_2...M_6)^{-1}P_{(1)}^{-1}\left(P_{(2)}P_{(3)}P_{(4)}P_{(5)}\right)^{-\frac{1}{2}} \lesssim N_{(1)}^{-3}M_{(3)} \lesssim N_{(1)}^{-3}N_{(3)}$$

which is an even better estimate than the one we desired.
–Finally, the case $N_{(2)} \in \{M_2, ..., M_6\}$ **and** $N_{(1)} \in \{P_1, ..., P_5\}$ is identical to the previous one.
Hence, the proof of Lemma 12.4 is finished.

□

Appendix A. Construction and properties of the flow and the truncated flow

In this appendix, we study the local and global wellposedness in $H^\sigma(\mathbb{T})$, $\sigma \geq 1$, for both equation (1.1) and truncated equation (2.1). We also study approximation properties of the truncated flow along with its structure.

A.1. **Local and global wellposedness.** We begin this paragraph by recalling some facts about the space :

$$W(\mathbb{T}) := \{u \in \mathcal{D}'(\mathbb{T}) : \sum_{k\in\mathbb{Z}} |\widehat{u}(k)| < +\infty\}$$

of absolutely convergent Fourier series, called the *Wiener algebra*, and equipped with the norm

$$\|u\|_{W(\mathbb{T})} := \sum_{k\in\mathbb{Z}} |\widehat{u}(k)|$$

(1) $W(\mathbb{T})$ is a Banach algebra with the estimate

$$\|uv\|_{W(\mathbb{T})} \leq \|u\|_{W(\mathbb{T})} \|v\|_{W(\mathbb{T})}$$

(2) For every $\sigma \geq 0$, $W(\mathbb{T}) \cap H^\sigma(\mathbb{T})$ is an algebra. Furthermore, there exists $C_\sigma > 0$ such that

$$\|uv\|_{H^\sigma} \leq C_\sigma(\|u\|_{H^\sigma} \|v\|_{W(\mathbb{T})} + \|v\|_{H^\sigma} \|u\|_{W(\mathbb{T})})$$

In particular, if $\sigma > \frac{1}{2}$, $H^\sigma(\mathbb{T})$ is a Banach algebra and there exists $C'_\sigma > 0$ such that

$$\|uv\|_{H^\sigma} \leq C'_\sigma \|u\|_{H^\sigma} \|v\|_{H^\sigma}$$

We are now ready to prove the following local existence theorem :

Theorem A.1. *Let $\sigma \geq 1$. Let $t_0 \in \mathbb{R}$. Both equation (1.1) and truncated equation (2.1) are locally well-posed in $H^\sigma(\mathbb{T})$, in the sense that for any $R_0 > 0$, there exists $T_0 > 0$ such that for every $\|u\|_{H^\sigma} \leq R_0$ there exists a unique $u \in \mathcal{C}([t_0 - T_0, t_0 + T_0], H^\sigma(\mathbb{T}))$ that satisfies the Duhamel formula :*

$$u(t) = e^{i(t-t_0)\partial_x^2} u_0 - i \int_{t_0}^t e^{i(t-\tau)\partial_x^2}(|u(\tau)|^4 u(\tau))d\tau \tag{A.1}$$

for all $|t - t_0| \leq T_0$. And, for every $N \in \mathbb{N}$, there exists a unique $v_N \in \mathcal{C}([t_0 - T_0, t_0 + T_0], H^\sigma(\mathbb{T}))$ that satisfies the Duhamel formula :

$$v_N(t) = e^{i(t-t_0)\partial_x^2} u_0 - i \int_{t_0}^t e^{i(t-\tau)\partial_x^2} \Pi_N(|\Pi_N v_N(\tau)|^4 \Pi_N v_N(\tau))d\tau \tag{A.2}$$

for all $|t - t_0| \leq T_0$.

Remark A.2. It is important to notice that the existence time $T_0 > 0$ is the same for both NLS and truncated equation and does not depend on the integer $N \in \mathbb{N}$.

Proof. To show Theorem A.1, we apply a fixed point argument. Fix $R_0 > 0$ and $||u_0||_{H^\sigma} \leq R_0$. For $T > 0$, we denote

$$X_T := \mathcal{C}([t_0 - T, t_0, +T], H^\sigma(\mathbb{T}))$$

which is a Banach space when endowed with the sup norm

$$\|u\|_{X_T} := \sup_{\tau\in[t_0-T,t_0,+T]} \|u(\tau)\|_{H^\sigma}$$

For every $N \in \mathbb{N} \cup \{\infty\}$, let us consider the well-defined map

$$\begin{aligned}\Gamma_N : X_T &\longrightarrow X_T \\ u &\longmapsto e^{i(t-t_0)\partial_x^2}u_0 - i\int_{t_0}^{t} e^{i(t-\tau)\partial_x^2}\Pi_N(|\Pi_N u(\tau)|^4\Pi_N u(\tau))d\tau\end{aligned}$$

where by convention $\Pi_\infty = id$. We recall the crucial point that $||\Pi_N u||_{H^\sigma} \leq \|u\|_{H^\sigma}$. Thanks to that point, every estimate below are uniform in N so that every constant below does not depend on N.

Using now the facts that $H^\sigma(\mathbb{T})$ is an algebra (since $\sigma \geq 1 > \frac{1}{2}$) and $e^{i(t-t_0)\partial_x^2}$ is a linear isometry on $H^\sigma(\mathbb{T})$, we have for some constant $C_0 > 0$,

$$\|\Gamma_N u\|_{X_T} \leq \|u_0\|_{H^\sigma} + C_0 T \|u\|_{X_T}^5 \tag{A.3}$$

In addition, using the multilinearity of $(u_1, u_2, u_3, u_4, u_5) \longmapsto u_1\overline{u_2}u_3\overline{u_4}u_5$ we have for some constant $C_1 > 0$,

$$\begin{aligned}\|\Gamma_N u - \Gamma_N v\|_{X_T} &= \left\| \int_{t_0}^{t} e^{i(t-\tau)\partial_x^2}\Pi_N(|\Pi_N u(\tau)|^4\Pi_N u(\tau) - |\Pi_N v(\tau)|^4\Pi_N v(\tau))d\tau \right\|_{X_T} \\ &\leq C_1 T(\|u\|_{X_T}^4 + \|v\|_{X_T}^4)\|u - v\|_{X_T}\end{aligned} \tag{A.4}$$

Let $R > 2R_0$ (for example $R = 1 + 2R_0$) and $T_0 := \frac{1}{3\max(C_0, C_1)R^4}$. Fix $0 < T \leq T_0$ and denote by $\overline{B}_R(T)$ the closed centered ball in X_T of radius R. We get from (A.3) and (A.4) and our choice of the parameters R and T_0 that for any $u, v \in \overline{B}_R(T)$,

$$||\Gamma_N u||_{\overline{B}_R(T)} \leq \frac{R}{2} + \frac{R}{2} \leq R \tag{A.5}$$

and,

$$||\Gamma_N u - \Gamma_N v||_{\overline{B}_R(T)} \leq 2C_1 T R^4 \|u - v\|_{\overline{B}_R(T)} \leq \frac{2}{3}\|u - v\|_{\overline{B}_R(T)} \tag{A.6}$$

These two estimates imply that the map Γ_N is a contraction from (the complete space) $\overline{B}_R(T)$ to itself. Hence, applying the Banach's fixed point theorem to the map

$$\Gamma_N : \overline{B}_R(T) \longrightarrow \overline{B}_R(T)$$

leads to the existence part of Theorem A.1. However, through classical considerations, we can prove that the uniqueness holds in the entire space $\mathcal{C}([t_0 - T, t_0 + T], H^\sigma(\mathbb{T}))$. So the proof is completed. □

Putting together the local solutions from Theorem A.1, we obtain the two following corollaries.

Corollary A.3. *Let $\sigma \geq 1$. For every $u_0 \in H^\sigma(\mathbb{T})$, there exists a unique maximal solution $u \in \mathcal{C}(I_{max}(u_0), H^\sigma(\mathbb{T}))$ of (1.1) where $I_{max}(u_0)$ is an open interval containing 0. Moreover, if $I_{max}(u_0)$ is strictly included in $\mathbb{R}$, then*

$$\|u(t)\|_{H^\sigma} \longrightarrow +\infty$$

as $t \longrightarrow \partial I_{max}(u_0)$.

Corollary A.4. *Let $\sigma \geq 1$. For every $u_0 \in H^\sigma(\mathbb{T})$ and every $N \in \mathbb{N}$, there exists a unique maximal solution $u_N \in \mathcal{C}(I_{max,N}(u_0), H^\sigma(\mathbb{T}))$ of (2.1) where $I_{max,N}(u_0)$ is an open interval containing 0. Moreover, if $I_{max,N}(u_0)$ is strictly included in $\mathbb{R}$, then*

$$||u_N(t)||_{H^\sigma} \longrightarrow +\infty$$

as $t \longrightarrow \partial I_{max,N}(u_0)$.

Thus, to obtain that the solutions of equations (1.1) and (2.1) are global, it suffices to show that the $H^\sigma(\mathbb{T})$-norm of the solutions do not blow up in finite time. Fortunately, we have :

Proposition A.5. *Let $u_0 \in H^\sigma(\mathbb{T})$. There exists a constant $C_0 > 0$ only depending on $\|u\|_{H^1(\mathbb{T})}$ and $\sigma \geq 1$, such that*

$$\|u(t)\|_{H^\sigma} \leq \|u\|_{H^\sigma} e^{C_0|t|}, \text{ for all } t \in I_{max}(u_0) \tag{A.7}$$

and,

$$||u_N(t)||_{H^\sigma} \leq \|u\|_{H^\sigma} e^{C_0|t|}, \text{ for all } t \in I_{max,N}(u_0) \tag{A.8}$$

where u is the maximal solution of (1.1) and u_N is the maximal solution of (2.1). Moreover, we can explicitly choose $C_0 \leq C_\sigma(1 + \|u\|_{H^1(\mathbb{T})})^{12}$ where $C_\sigma > 0$ depends only on σ.

Proof. We use the convention $\Pi_\infty = id$, $u_\infty = u$ and $I_{max,\infty}(u_0) = I_{max}(u_0)$. Let $N \in \mathbb{N} \cup \{\infty\}$ and $t \in I_{max,N}(u_0)$. Passing to the H^σ-norm in the Duhamel formula

$$u_N(t) = e^{i(t)\partial_x^2} u_0 - i \int_0^t e^{i(t-\tau)\partial_x^2} \Pi_N(|\Pi_N u_N(\tau)|^4 \Pi_N u_N(\tau)) d\tau$$

we get that, for some constant $C > 0$ (only depending on σ),

$$\begin{aligned} ||u_N(t)||_{H^\sigma} &\leq \|u\|_{H^\sigma} + | \int_0^t \left\| |\Pi_N u_N(\tau)|^4 \Pi_N u_N(\tau) \right\|_{H^\sigma} d\tau | \\ &\leq \|u\|_{H^\sigma} + C | \int_0^t \|u_N(\tau)\|_{H^\sigma} \|u_N(\tau)\|^4_{W(\mathbb{T})} d\tau | \end{aligned} \tag{A.9}$$

The crucial point of the proof is that $\|u_N(\tau)\|_{W(\mathbb{T})}$ is bounded by the $H^1(\mathbb{T})$-norm of the initial data. Indeed, using the mass and Hamiltonian conservation

$$\begin{aligned} \|u_N(\tau)\|_{W(\mathbb{T})} &\leq \|u_N(\tau)\|_{H^1(\mathbb{T})} \\ &\leq (\|u_N(\tau)\|^2_{\mathbb{L}^2} + \|\partial_x u_N(\tau)\|^2_{\mathbb{L}^2})^{\frac{1}{2}} \\ &\lesssim \left(\|u_N(\tau)\|^2_{\mathbb{L}^2} + \left(\frac{1}{2} \|\partial_x u_N(\tau)\|^2_{\mathbb{L}^2} + \frac{1}{6} \|u_N(\tau)\|^6_{\mathbb{L}^6} \right) \right)^{\frac{1}{2}} \\ &\lesssim (\|u\|^2_{\mathbb{L}^2} + \frac{1}{2} \|\partial_x u_0\|^2_{\mathbb{L}^2} + \frac{1}{6} \|u\|^6_{\mathbb{L}^6})^{\frac{1}{2}} \\ &\leq C_0' \end{aligned}$$

where we can choose $C_0' \sim (1 + \|u\|_{H^1(\mathbb{T})})^3$, because $H^1(\mathbb{T})$ embeds continuously in $\mathbb{L}^6(\mathbb{T})$. Now, plugging this bound into (A.9), we get

$$||u_N(t)||_{H^\sigma} \leq \|u\|_{H^\sigma} + C_0 | \int_0^t \|u_N(\tau)\|_{H^\sigma} d\tau |$$

where we can choose $C_0 \sim (1 + \|u\|_{H^1(\mathbb{T})})^{12}$. Then, applying Gronwall's inequality, we obtain

$$||u_N(t)||_{H^\sigma} \leq \|u\|_{H^\sigma} e^{C_0|t|}$$

which is the desired bound. □

Theorem A.6. *For every $u_0 \in H^\sigma(\mathbb{T})$, each maximal solution of (1.1) and (2.1) is global. In other words,*

$$I_{max}(u_0), I_{max,N}(u_0) = +\infty$$

for all $N \in \mathbb{N}$.

Proof. The estimates from Proposition A.5 imply that the H^σ-norm of both solutions does not blow up in finite time. Therefore, Corollaries A.3 and A.4 lead to the result. □

A.2. **Regularity of the flows and approximation properties.** We recall that we denote by $\Phi(t)$ the flow of (1.1) and by $\Phi_N(t)$ the flow of the truncated equation (2.1). We also use the notation $\Phi_\infty(t) = \Phi(t)$.

Proposition A.7. *Let $\sigma \geq 1$. Let $R > 0$ and $T > 0$. There exists a constant $\Lambda(R,T) > 0$ only depending on T, R and σ, such that for any $u_0 \in B_R^{H^\sigma}$,*

$$\sup_{|t|\leq T} \|\Phi(t)u_0\|_{H^\sigma} + \sup_{|t|\leq T} \|\Phi_N(t)u_0\|_{H^\sigma} \leq \Lambda(R,T), \quad \forall N \in \mathbb{N}$$

Proof. This is a consequence of Proposition A.5. For example we can take

$$\Lambda(R,T) = Re^{C_\sigma(1+R)^{12}T}$$

for a certain constant $C_\sigma > 0$ only depending on σ. □

We continue this paragraph with two significant regularity results for the flow of (1.1) and (2.1).

Proposition A.8. *Let $t \in \mathbb{R}$. For every $N \in \mathbb{N} \cup \{\infty\}$, the map*

$$\begin{aligned} \Phi_N(t) : H^\sigma(\mathbb{T}) &\longrightarrow H^\sigma(\mathbb{T}) \\ u &\longmapsto \Phi_N(t)u \end{aligned}$$

is continuous.

Proof. For better readability, we only provide the proof for $\Phi(t)$. Let $u, v \in H^\sigma(\mathbb{T})$. Without loss of generality, we can assume that $\|u\|_{H^\sigma} \leq R$ and $\|v\|_{H^\sigma} \leq R$ for some $R > 0$. We invoke $\Lambda(R,t) > 0$ from the Proposition A.7. Firstly, using the Duhamel formula and passing to the $H^\sigma(\mathbb{T})$-norm we get that for all $s \leq t$

$$\|\Phi(s)v - \Phi(s)u\|_{H^\sigma} \leq \|v-u\|_{H^\sigma} + C\Lambda(R,t)^4 \mid \int_0^s \|\Phi(\tau)v - \Phi(\tau)u\|_{H^\sigma}\, d\tau \mid$$

Secondly, the Gronwall's inequality yields

$$\|\Phi(t)v - \Phi(t)u\|_{H^\sigma} \leq \|v-u\|_{H^\sigma}\, e^{C\Lambda(R,t)^4|t|}$$

In particular,

$$\|\Phi(t)v - \Phi(t)u\|_{H^\sigma} \underset{v\to u}{\longrightarrow} 0$$

□

We can go even further with the following proposition:

Proposition A.9. *For every $N \in \mathbb{N} \cup \{\infty\}$, the map*

$$\begin{aligned} \Phi_N : \mathbb{R} \times H^\sigma(\mathbb{T}) &\longrightarrow H^\sigma(\mathbb{T}) \\ (t,u) &\longmapsto \Phi_N(t)u \end{aligned}$$

is continuous.

Proof. For better readability, we only provide the proof for Φ. We fix $t \in \mathbb{R}$ and $u \in H^\sigma(\mathbb{T})$ and we show that

$$\|\Phi(t+h)(u+\delta u) - \Phi(t)u\|_{H^\sigma} \longrightarrow 0, \quad \text{as } h \to 0 \text{ and } \delta u \to 0$$

We denote $F(\tau, v) := |\Phi(\tau)v|^4\Phi(\tau)v$. Let us compute the difference $\Phi(t+h)(u+\delta u) - \Phi(t)u$ using the Duhamel formula.

$$\begin{aligned}
&\Phi(t+h)(u+\delta u) - \Phi(t)u = \Big(\Phi(t+h)(u+\delta u) - \Phi(t+h)u\Big) + \Big(\Phi(t+h)u - \Phi(t)u\Big)\\
&= e^{i(t+h)\partial_x^2}\delta u - i\int_0^{t+h} e^{i(t+h-\tau)\partial_x^2}(F(\tau, u+\delta u) - F(\tau, u))d\tau\\
&+ e^{it\partial_x^2}(e^{ih\partial_x^2} - 1)u - i\int_0^{t+h} e^{i(t+h-\tau)\partial_x^2}F(\tau, u)d\tau + i\int_0^{t} e^{i(t-\tau)\partial_x^2}F(\tau, u)d\tau\\
&= e^{i(t+h)\partial_x^2}\delta u + e^{it\partial_x^2}(e^{ih\partial_x^2} - 1)u - i\int_0^{t+h} e^{i(t+h-\tau)\partial_x^2}(F(\tau, u+\delta u) - F(\tau, u))d\tau\\
&- i\int_t^{t+h} e^{i(t+h-\tau)\partial_x^2}F(\tau, u)d\tau - i\int_0^{t} e^{i(t-\tau)\partial_x^2}(e^{ih\partial_x^2} - 1)F(\tau, u)d\tau
\end{aligned}$$

Furthermore, without loss of generality we can assume that $t + h \leq t + 1 =: T$ and $\|u + \delta u\|_{H^\sigma} \leq \|u\|_{H^\sigma} + 1 =: R$. Now, we invoke $\Lambda(R, T) > 0$ from Proposition A.7. Passing to the $H^\sigma(\mathbb{T})$-norm in the above formula we have

$$\begin{aligned}
\|\Phi(t+h)(u+\delta u) - \Phi(t)u\|_{H^\sigma} \leq \|\delta u\|_{H^\sigma} &+ \Big\|(e^{ih\partial_x^2} - 1)u\Big\|_{H^\sigma}\\
&+ C\Lambda(R, T)^4 \,|\int_0^T \|\Phi(\tau)(u+\delta u) - \Phi(\tau)u\|_{H^\sigma}\, d\tau\,|\\
&+ |h|\Lambda(R, T)^5 + |\int_0^t \Big\|(e^{ih\partial_x^2} - 1)F(\tau, u)\Big\|_{H^\sigma} d\tau\,|
\end{aligned}$$

Using the continuity of $v \longmapsto \Phi(\tau)v$ (Proposition A.8) and the dominated convergence theorem, we deduce that the right hand side tends to 0 as $h \to 0$ and $\delta u \to 0$. So the proof of Proposition A.9 is complete. □

Proposition A.10 (Approximation property)**.** *Let $\sigma \geq 1$. Let K be a compact subset of $H^\sigma(\mathbb{T})$ and $T > 0$. Then, uniformly in $|t| \leq T$ and $u_0 \in K$,*

$$\lim_{N \to +\infty} \|\Phi(t)u_0 - \Phi_N(t)u_0\|_{H^\sigma} = 0,$$

In other words,

$$\sup_{|t| \leq T} \sup_{u_0 \in K} \|\Phi(t)u_0 - \Phi_N(t)u_0\|_{H^\sigma} \underset{N \to +\infty}{\longrightarrow} 0$$

Proof. First, from the compactness of K, we invoke $R > 0$ such that $K \subset B_R^{H^\sigma}$, where $B_R^{H^\sigma}$ is the closed centered ball of radius R in $H^\sigma(\mathbb{T})$. We then invoke $\Lambda(R, T) > 0$ from Proposition A.7. Now, we set $R_1 := 1 + 2\Lambda(R, T)$ and invoke $\delta = \delta(R_1) > 0$ the local existence time associated to R_1 from the local theory (Theorem A.1). These parameters ensure that for every $N \in \mathbb{N} \cup \{\infty\}$ and $u_0 \in B_R^{H^\sigma}$, the Duhamel map,

$$\begin{aligned}
\Gamma_N : \overline{B}_{R_1}(\delta) &\longrightarrow \overline{B}_{R_1}(\delta)\\
u &\longmapsto e^{it\partial_x^2}u_0 - i\int_0^t e^{i(t-\tau)\partial_x^2}\Pi_N(|\Pi_N u(\tau)|^4\Pi_N u(\tau))d\tau
\end{aligned}$$

is a contraction, with a universal contraction coefficient $0 < \gamma < 1$ (for example $\gamma = \frac{2}{3}$ as in the proof of Theorem A.1), where $\overline{B}_{R_1}(\delta)$ is the closed centered ball in $\mathcal{C}([-\delta, \delta], H^\sigma(\mathbb{T}))$ of radius R_1.

Step 1: Local-time convergence

Firstly, we prove the local property

$$\sup_{|t|\leq\delta} \sup_{u_0\in K} \|\Phi(t)u_0 - \Phi_N(t)u_0\|_{H^\sigma} \underset{N\to+\infty}{\longrightarrow} 0 \tag{A.10}$$

Let $u_0 \in B_R^{H^\sigma}$. We denote $u(t) = \Phi(t)u_0$ and $u_N(t) = \Phi_N(t)u_0$. Since u and u_N are respectively the fixed point of Γ_∞ and Γ_N, we have

$$\begin{aligned}\|u - u_N\|_{\overline{B}_{R_1}(\delta)} &= \|\Gamma_\infty u - \Gamma_N u_N\|_{\overline{B}_{R_1}(\delta)} \\ &\leq \|\Gamma_\infty u - \Gamma_N u\|_{\overline{B}_{R_1}(\delta)} + \|\Gamma_N u - \Gamma_N u_N\|_{\overline{B}_{R_1}(\delta)} \\ &\leq \|\Gamma_\infty u - \Gamma_N u\|_{\overline{B}_{R_1}(\delta)} + \gamma \|u - u_N\|_{\overline{B}_{R_1}(\delta)}\end{aligned}$$

Therefore,

$$\|u - u_N\|_{\overline{B}_{R_1}(\delta)} \leq \frac{1}{1-\gamma} \|\Gamma_\infty u - \Gamma_N u\|_{\overline{B}_{R_1}(\delta)}$$

Hence, to prove (A.10), it suffices to show that

$$\sup_{u_0\in K} \|\Gamma_\infty u - \Gamma_N u\|_{\overline{B}_{R_1}(\delta)} \underset{N\to+\infty}{\longrightarrow} 0$$

Next, for every $|t| \leq \delta$ we have,

$$\begin{aligned}&\Gamma_\infty u(t) - \Gamma_N u(t) = -i \int_0^t e^{i(t-\tau)\partial_x^2}(|u(\tau)|^4 u(\tau) - \Pi_N(|\Pi_N u(\tau)|^4 \Pi_N u(\tau)))d\tau \\ &= -i \int_0^t e^{i(t-\tau)\partial_x^2} \Pi_N^\perp(|u(\tau)|^4 u(\tau))d\tau - i \int_0^t e^{i(t-\tau)\partial_x^2} \Pi_N(|u(\tau)|^4 u(\tau) - |\Pi_N u(\tau)|^4 \Pi_N u(\tau))d\tau\end{aligned}$$

So passing to the sup norm we get,

$$\begin{aligned}\|\Gamma_\infty u - \Gamma_N u\|_{\overline{B}_{R_1}(\delta)} &\leq \delta \sup_{|\tau|\leq\delta} \left\|\Pi_N^\perp(|u(\tau)|^4 u(\tau))\right\|_{H^\sigma} \\ &\quad + C\delta \sup_{|\tau|\leq\delta} \|u(\tau) - \Pi_N u(\tau)\|_{H^\sigma} (\|u(\tau)\|_{H^\sigma}^4 + \|\Pi_N u(\tau)\|_{H^\sigma}^4) \\ &\leq C\delta(\sup_{|\tau|\leq\delta} \left\|\Pi_N^\perp(|u(\tau)|^4 u(\tau))\right\|_{H^\sigma} + \sup_{|\tau|\leq\delta} \left\|\Pi_N^\perp u(\tau)\right\|_{H^\sigma} \Lambda(R,T)^4)\end{aligned}$$

Taking the supremum over $u_0 \in K$ we then obtain,

$$\begin{aligned}&\sup_{u_0\in K} \|\Gamma_\infty u - \Gamma_N u\|_{\overline{B}_{R_1}(\delta)} \\ &\quad \leq C\delta(\sup_{u_0\in K} \sup_{|\tau|\leq\delta} \left\|\Pi_N^\perp(|u(\tau)|^4 u(\tau))\right\|_{H^\sigma} + \Lambda(R,T)^4 \sup_{u_0\in K} \sup_{|\tau|\leq\delta} \left\|\Pi_N^\perp u(\tau)\right\|_{H^\sigma})\end{aligned} \tag{A.11}$$

Besides, based on a classical result in functional analysis (see Lemma 3.4), $\Pi_N^\perp$ satisfies the key property of converging uniformly to 0 on compact sets as $N \to +\infty$. At the same time, the two following sets

$$K_1 := \{\Phi(\tau)u_0 : \ u_0 \in K, \ |\tau| \leq \delta\}, \qquad K_2 := \{|\Phi(\tau)u_0|^4 \Phi(\tau)u_0 : \ u_0 \in K, \ |\tau| \leq \delta\}$$

are compacts in $H^\sigma(\mathbb{T})$. Indeed, it results from the facts that the map

$$\begin{aligned}\mathbb{R}\times H^\sigma(\mathbb{T}) &\longrightarrow H^\sigma(\mathbb{T})\\ (t,u_0)&\longmapsto \Phi(t)u_0\end{aligned}$$

and the map

$$\begin{aligned}H^\sigma(\mathbb{T})^5 &\longrightarrow H^\sigma(\mathbb{T})\\ (u_1,u_2,u_3,u_4,u_5)&\longmapsto u_1\overline{u_2}u_3\overline{u_4}u_5\end{aligned}$$

are continuous and the fact that the image of a compact set under a continuous map is compact. Now, rewriting (A.11) we get

$$\begin{aligned}\sup_{u_0\in K}\|\Gamma_\infty u-\Gamma_N u\|_{\overline{B}_{R_1}(\delta)} &\leq C\delta\Big(\sup_{w_2\in K_2}\Pi_N^\perp w_2+\Lambda(R,T)^4\sup_{w_1\in K_1}\Pi_N^\perp w_1\Big)\\ &\underset{N\to\infty}{\longrightarrow} 0\end{aligned}$$

And this implies the desired local property (A.10).

Step 2: Long-time convergence

Secondly, we complete the proof of Proposition A.10 by iterating this local argument. Let $m:=\lfloor\frac{T}{\delta}\rfloor+1$, and for any integer $|k|\leq m$, let $I_k:=[k\delta-\delta,k\delta+\delta]$. We show that for any $|k|\leq$ m,

$$\sup_{t\in I_k}\sup_{u_0\in K}\|\Phi(t)u_0-\Phi_N(t)u_0\|_{H^\sigma}\underset{N\to+\infty}{\longrightarrow}0 \tag{A.12}$$

If we do so, the proof of Proposition A.10 will be completed. At this stage, we know that (A.12) is true when $k=0$. Now, we assume that (A.12) is true for some integer $|k|\leq m-1$ and we show that (A.12) is still true for every integer $|k'|=|k|+1$. Also, without loss of generality, we assume that $k\geq 0$ and $k'=k+1$. The crucial point here is that since we chose $R_1=1+2\Lambda(R,T)$, we have that for every $N\in\mathbb{N}\cup\{\infty\}$ and $u_0\in B_R^{H^\sigma}$, the Duhamel map

$$\begin{aligned}\Gamma_N:\overline{B}^{(k)}_{R_1}(\delta)&\longrightarrow\overline{B}^{k}_{R_1}(\delta)\\ u&\longmapsto e^{i(t-t_k)\partial_x^2}u_N(t_k)-i\int_{t_k}^t e^{i(t-\tau)\partial_x^2}\Pi_N(|\Pi_N u(\tau)|^4\Pi_N u(\tau))d\tau\end{aligned}$$

is a contraction with a universal contraction coefficient $0<\gamma<1$, where $t_k:=k\delta$, $u_\infty(t_k):=u(t_k)$, and $\overline{B}^{(k)}_{R_1}(\delta)$ is the closed centered ball in $\mathcal{C}([t_k-\delta,t_k+\delta],H^\sigma(\mathbb{T}))$ of radius R_1.

Now, doing the same calculations as in the first step of the proof we obtain,

$$\|u-u_N\|_{\overline{B}^{(k)}_{R_1}(\delta)}\leq\frac{1}{1-\gamma}\|\Gamma_\infty u-\Gamma_N u\|_{\overline{B}^{(k)}_{R_1}(\delta)}$$

and,

$$\begin{aligned}\sup_{u_0\in K}\|\Gamma_\infty u-\Gamma_N u\|_{\overline{B}^{(k)}_{R_1}(\delta)}&\leq\sup_{u_0\in K}\sup_{t\in I_k}\Big\|e^{i(t-t_k)\partial_x^2}(u(t_k)-u_N(t_k))\Big\|_{H^\sigma}\\ &+C\delta\Big(\sup_{u_0\in K}\sup_{\tau\in I_k}\big\|\Pi_N^\perp(|u(\tau)|^4u(\tau))\big\|_{H^\sigma}+\Lambda(R,T)^4\sup_{u_0\in K}\sup_{\tau\in I_k}\big\|\Pi_N^\perp u(\tau)\big\|_{H^\sigma}\Big)\end{aligned}$$

On the right hand side, the first term :

$$\sup_{u_0\in K}\sup_{t\in I_k}\Big\|e^{i(t-t_k)\partial_x^2}(u(t_k)-u_N(t_k))\Big\|_{H^\sigma}=\sup_{u_0\in K}\|u(t_k)-u_N(t_k)\|_{H^\sigma}$$

tends to 0 as $N \longrightarrow +\infty$ by our assumption. And, we handle the remaining term in the same way as in the first part of the proof. Finally (A.12) is true for all $|k| \leq m$, so the proof of Proposition A.10 is completed. □

Corollary A.11. *Let $\sigma \geq 1$, $T > 0$ and let K be a compact subset of $H^\sigma(\mathbb{T})$. Then,*

(1) for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\Phi_N(t)(K) \subset \Phi(t)(K) + B_\varepsilon^{H^\sigma}$$

for all $|t| \leq T$.

(2) for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$

$$\Phi(t)(K) \subset \Phi_N(t)(K + B_\varepsilon^{H^\sigma})$$

for all $|t| \leq T$.

Proof. Let $\varepsilon > 0$.

- For the first point, we take $u_0 \in K$ and we write

$$\Phi_N(t)u_0 = \Phi(t)u_0 + (\Phi_N(t)u_0 - \Phi(t)u_0) \tag{A.13}$$

From Proposition A.10, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\sup_{|t|\leq T} \sup_{v_0 \in K} \|\Phi_N(t)v_0 - \Phi(t)v_0\|_{H^\sigma} \leq \varepsilon$$

Thus, for all $N \geq N_0$, $\Phi_N(t)u_0 - \Phi(t)u_0 \in B_\varepsilon^{H^\sigma}$ for all $|t| \leq T$ and all $u_0 \in K$. Coming back to (A.13), it implies that for all $N \geq N_0$

$$\Phi_N(t)u_0 \in \Phi(t)(K) + B_\varepsilon^{H^\sigma}$$

for all $u_0 \in K$ and all $|t| \leq T$.

- The second point is a consequence of the first one. Let $|t| \leq T$. From the continuity of the map $\Phi(t)$ (see Proposition A.8) we have that $\Phi(t)(K)$ is a compact subset of $H^\sigma(\mathbb{T})$. So from the first point there exists $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$

$$\Phi_N(-\tau)\left(\Phi(t)(K)\right) \subset \Phi(-\tau)\Phi(t)(K) + B_\varepsilon^{H^\sigma}$$

for all $|\tau| \leq T$. In particular, for $\tau = t$ we have :

$$\Phi_N(-t)\left(\Phi(t)(K)\right) \subset \Phi(-t)\Phi(t)(K) + B_\varepsilon^{H^\sigma} = K + B_\varepsilon^{H^\sigma}$$

Applying $\Phi_N(t)$ to this we obtain

$$\Phi(t)(K) \subset \Phi_N(t)\left(K + B_\varepsilon^{H^\sigma}\right)$$

This concludes the proof of Corollary A.11. □

A.3. **Structure of the truncated flow.** We set

$$E_N := \Pi_N \mathbb{L}^2(\mathbb{T})$$
$$E_N^\perp := \Pi_N^\perp \mathbb{L}^2(\mathbb{T}) = (Id - \Pi_N)\mathbb{L}^2(\mathbb{T})$$

Proposition A.12. *We have the following properties ;*

(1) The truncated flow $\Phi_N(t)$ commute with the frequency projector Π_N, that is

$$\Phi_N(t) \circ \Pi_N = \Pi_N \circ \Phi_N(t)$$

As a consequence, the truncated flow $\Phi_N(t)$ maps the finite-dimensional space E_N to itself. Moreover, the induced map

$$\begin{aligned} \widetilde{\Phi}_N(t) : E_N &\longrightarrow E_N \\ u_0 &\longmapsto \Phi_N(t)(u_0) \end{aligned}$$

is the flow of the ODE

$$\begin{cases} i\partial_t u + \partial_x^2 u = \Pi_N \left(|u|^4 u \right) \\ u|_{t=0} = u_0 \in E_N \end{cases} \tag{FNLS}$$

which can be seen as the finite-dimensional Hamiltonian equation on E_N :

$$\begin{cases} i\partial_t u = \frac{\partial H_N}{\partial \overline{u}}(u) \\ u|_{t=0} = u_0 \in E_N \end{cases}$$

with Hamiltonian $H_N(u) := \frac{1}{2} \left\| \partial_x^2 u \right\|_{\mathbb{L}^2(\mathbb{T})}^2 + \frac{1}{6} \left\| u \right\|_{\mathbb{L}^6(\mathbb{T})}^6$, for $u \in E_N$.

(2) The truncated flow $\Phi_N(t)$ commute with $\Pi_N^\perp$, that is

$$\Phi_N(t) \circ \Pi_N^\perp = \Pi_N^\perp \circ \Phi_N(t)$$

As a consequence, the truncated flow $\Phi_N(t)$ maps the space $E_N^\perp$ to itself. Moreover, the induced map

$$\begin{aligned} \Phi_N^\perp(t) : E_N^\perp &\longrightarrow E_N^\perp \\ u_0 &\longmapsto \Phi_N(t)(u_0) \end{aligned}$$

is the solution of the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0 \\ u|_{t=0} = u_0 \in E_N^\perp \end{cases}$$

Hence, $\Phi_N^\perp(t)$ coincide with the linear operator $e^{it\partial_x^2}$ on $E_N^\perp$.

(3) The truncated flow $\Phi_N(t)$ can be factorized as $(\widetilde{\Phi}_N(t), e^{it\partial_x^2})$ on $E_N \times E_N^\perp$. In other words, for every $u_0 \in H^\sigma(\mathbb{T})$,

$$\Phi_N(t)(u_0) = \widetilde{\Phi}_N(t)\Pi_N u_0 + e^{it\partial_x^2}\Pi_N^\perp u_0$$

Let us now prove these three points.

Proof. Let $u_0 \in L^2(\mathbb{T})$.

(1) • $\Phi_N(t)$ is the flow of (2.1). And, when we apply Π_N to equation (2.1), we see that $\Pi_N \Phi_N(t) u_0$ is the solution of the equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = \Pi_N \left(|\Pi_N u|^4 \Pi_N u \right) = \Pi_N \left(|u|^4 u \right) \\ u|_{t=0} = \Pi_N u_0 \end{cases}$$

On the other hand, from the definition of the flow of (2.1), the solution of the equation above is none other than $\Phi_N(t)\Pi_N u_0$ itself. This means that

$$\Pi_N \Phi_N(t) u_0 = \Phi_N(t) \Pi_N u_0$$

• Next, we show that (FNLS) can be seen as a finite-dimensional Hamiltonian equation on E_N. Every element $u \in E_N$ can be decompose as

$$u = v + iw$$

where

$$v = \sum_{|k|\leq N} \mathrm{Re}(u_k)e^{ikx} =: \sum_{|k|\leq N} v_k e^{ikx} \in E_N$$
$$w = \sum_{|k|\leq N} \mathrm{Im}(u_k)e^{ikx} =: \sum_{|k|\leq N} w_k e^{ikx} \in E_N$$

Furthermore, we invoke the operators

$$\frac{\partial}{\partial \overline{u}_k} := \frac{\partial}{\partial v_k} + i\frac{\partial}{\partial w_k}$$
$$\frac{\partial}{\partial \overline{u}} := \frac{\partial}{\partial v} + i\frac{\partial}{\partial w} =: \sum_{|k|\leq N} e^{ikx}\frac{\partial}{\partial v_k} + i\sum_{|k|\leq N} e^{ikx}\frac{\partial}{\partial w_k}$$

and also the function

$$H_N : E_N \longrightarrow \mathbb{R}$$
$$u \longmapsto \frac{1}{2}\left\|\partial_x^2 u\right\|^2_{\mathbb{L}^2(\mathbb{T})} + \frac{1}{6}\left\|u\right\|^6_{\mathbb{L}^6(\mathbb{T})}$$

By performing elementary computations, we see that we can rewrite (FNLS) as

$$\begin{cases} i\partial_t u = \frac{\partial H_N}{\partial \overline{u}}(u) \\ u|_{t=0} = u_0 \end{cases} \tag{FNLS}$$

This means that (FNLS) is an Hamiltonian equation on E_N with associated Hamiltonian H_N.

(2) Once again, applying $\Pi_N^\perp$ to equation (2.1), we see that $\Pi_N^\perp \Phi_N(t)u_0$ is the solution of the equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0 \\ u|_{t=0} = \Pi_N^\perp u_0 \end{cases}$$

This means that

$$\Pi_N^\perp \Phi_N(t)u_0 = e^{it\partial_x^2}\Pi_N^\perp u_0 \tag{A.14}$$

Furthermore, $\Pi_N \Phi_N(t)\Pi_N^\perp u_0$ is the solution of the equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = \Pi_N\left(|\Pi_N u|^4 \Pi_N u\right) \\ u|_{t=0} = 0 \end{cases}$$

Thus, $\Pi_N \Phi_N(t)\Pi_N^\perp u_0$ is none other than 0, and we obtain from (A.14) that

$$\Phi_N(t)\Pi_N^\perp u_0 = \Pi_N^\perp \Phi_N(t)\Pi_N^\perp u_0 = e^{it\partial_x^2}\Pi_N^\perp u_0 = \Pi_N^\perp \Phi_N(t)u_0$$

(3) As a consequence of the two previous points, we can write

$$\begin{aligned} \Phi_N(t)u_0 &= \Pi_N \Phi_N(t)u_0 + \Pi_N^\perp \Phi_N(t)u_0 \\ &= \Phi_N(t)\Pi_N u_0 + \Phi_N(t)\Pi_N^\perp u_0 \\ &= \underbrace{\widetilde{\Phi}_N(t)\Pi_N u_0}_{\in E_N} + \underbrace{e^{it\partial_x^2}\Pi_N^\perp u_0}_{\in E_N^\perp} \end{aligned}$$

□

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