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Gabriel Dospinescu, Juan Esteban Rodriguez Camargo

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# A JACQUET-LANGLANDS FUNCTOR FOR $p$ -ADIC LOCALLY ANALYTIC REPRESENTATIONS

GABRIEL DOSPINESCU AND JUAN ESTEBAN RODRÍGUEZ CAMARGO

ABSTRACT. We study the locally analytic theory of infinite level local Shimura varieties. As a main result, we prove that in the case of a duality of local Shimura varieties, the locally analytic vectors of different period sheaves at infinite level are independent of the actions of the  $p$ -adic Lie groups  $G$  and  $G_b$  of the two towers; this generalizes a result of Pan for the Lubin-Tate and Drinfeld spaces for  $\mathbf{GL}_2$ . We apply this theory to show that the  $p$ -adic Jacquet-Langlands functor of Scholze commutes with the passage to locally analytic vectors, and is compatible with central characters of Lie algebras. We also prove that the compactly supported de Rham cohomology of the two towers are isomorphic as smooth representations of  $G \times G_b$ .

## CONTENTS

1. Introduction .....	1
Main results .....	3
Outline of the paper .....	8
Conventions .....	8
Acknowledgements .....	9
2. Preliminaries .....	9
2.1. The Fargues-Fontaine curve and sheaves of periods .....	9
2.2. Solid almost quasi-coherent sheaves .....	12
2.3. The décalage functor .....	14
2.4. Solid locally analytic representations .....	15
2.5. Equivariant sheaves over flag varieties .....	16
3. Local Shimura varieties .....	17
3.1. $\mathbf{G}$ -torsors over Fargues-Fontaine curves .....	18
3.2. Moduli space of shtukas of one leg .....	19
3.3. Local Shimura varieties .....	20
4. Geometric Sen operators of local Shimura varieties .....	22
4.1. The Kodaira-Spencer map .....	23
4.2. Pullbacks of equivariant vector bundles along $\pi_{\text{HT}}$ .....	24
4.3. Computation of the geometric Sen operators .....	24
5. Locally analytic vectors at infinite level .....	25
5.1. Locally analytic vectors of towers of rigid spaces .....	26
5.2. De Rham cohomology of the two towers .....	33
5.3. The Jacquet-Langlands functor for admissible locally analytic representations .....	36
References .....	40

## 1. INTRODUCTION

Let  $p$  be a prime number. The main objective of this work is to give some new insights in the locally analytic incarnation of the  $p$ -adic local Langlands correspondence, cf. [Bre10] [CDP14]. The objects of study in this paper are the infinite level Lubin-Tate and Drinfeld spaces (or more generally local Shimura varieties) and the locally analytic vectors for the action of the associated  $p$ -adic Lie groups on different

period sheaves. To motivate our main results let us recall the perfectoid geometry of the Lubin-Tate and Drinfeld spaces.

Let  $F$  be a finite extension of  $\mathbb{Q}_p$  with ring of integers  $\mathcal{O}$ , pseudo-uniformizer  $\varpi$  and residue field  $\mathbb{F}$ . Fix  $\overline{\mathbb{F}}$  an algebraic closure of  $\mathbb{F}$  and let  $\check{\mathcal{O}}$  be the completion of the maximal unramified extension of  $\mathcal{O}$  with residue field  $\overline{\mathbb{F}}$ . Write  $\check{F} = \check{\mathcal{O}}[\frac{1}{\varpi}]$ . Consider the group  $\mathbf{GL}_{n,F}$ , let  $\mu$  be the cocharacter  $(1, 0, \dots, 0)$  with  $n-1$  occurrences of 0 and let  $\mathbb{X}_b$  be a formal  $\mathcal{O}$ -module over  $\overline{\mathbb{F}}$  of dimension 1 and  $\mathcal{O}$ -height  $n$ . Denote by  $D$  the central division algebra over  $F$  with invariant  $1/n$ . Let  $\text{Def}_{\mathbb{X}_b}$  be the formal scheme over  $\check{\mathcal{O}}$  parametrizing deformations of the formal  $\mathcal{O}$ -module  $\mathbb{X}_b$ . The Lubin-Tate space of  $\mathbf{GL}_{n,F}$  at level  $\mathbf{GL}_n(\mathcal{O})$  is the rigid generic fiber  $\mathcal{LT}_{\check{F}}$  of  $\text{Def}_{\mathbb{X}_b}$ . The space  $\mathcal{LT}_{\check{F}}$  has a *dual* given by the Drinfeld space  $\Omega_{\check{F}} \subset \mathbb{P}_{\check{F}}^{n-1}$  defined as the complement of the  $F$ -rational hyperplanes of  $\mathbb{P}_{\check{F}}^{n-1}$ . These are particular examples of Rapoport-Zink spaces [RZ96] which are themselves special cases of local Shimura varieties [RV14] [SW20].

The spaces  $\mathcal{LT}_{\check{F}}$  and  $\Omega_{\check{F}}$  are intimately related via perfectoid geometry in a very clean way: let  $\mathcal{LT}_{\infty, \check{F}}$  be the Lubin-Tate space at infinite level obtained by trivializing the Tate module of the universal deformation of  $\mathbb{X}_b$ . It was shown in [SW13] that  $\mathcal{LT}_{\infty, \check{F}}$  has a natural structure of a perfectoid space which, by construction, is a proétale  $\mathbf{GL}_n(\mathcal{O})$ -torsor over  $\mathcal{LT}_{\check{F}}$ . Furthermore, a Hodge-Tate period map is constructed in *loc. cit.*

$$\pi_{\text{HT}} : \mathcal{LT}_{\infty, \check{F}} \rightarrow \mathbb{P}_{\check{F}}^{n-1}.$$

The image of the Hodge-Tate period map is the Drinfeld space  $\Omega_{\check{F}}$ , and the map  $\pi_{\text{HT}} : \mathcal{LT}_{\infty, \check{F}} \rightarrow \Omega_{\check{F}}$  is a proétale  $D^\times$ -torsor. When composing  $\mathcal{LT}_{\infty, \check{F}} \rightarrow \mathcal{LT}_{\check{F}}$  with the Grothendieck-Messing period map  $\mathcal{LT}_{\check{F}} \rightarrow \mathbb{P}_{\check{F}}^{n-1}$ , the morphism

$$\pi_{\text{GM}} : \mathcal{LT}_{\infty, \check{F}} \rightarrow \mathbb{P}_{\check{F}}^{n-1}$$

becomes a proétale  $\mathbf{GL}_n(F)$ -torsor.

Summarizing, we have a perfectoid space  $\mathcal{LT}_{\infty, \check{F}}$  endowed with an action of  $\mathbf{GL}_n(F) \times D^\times$  fitting in an equivariant diagram

$$\begin{array}{ccc} & \mathcal{LT}_{\infty, \check{F}} & \\ \pi_{\text{GM}} \swarrow & & \searrow \pi_{\text{HT}} \\ \mathbb{P}_{\check{F}}^{n-1} & & \Omega_{\check{F}} \end{array} \quad (1.1)$$

such that:

- $\pi_{\text{HT}}$  is a  $\mathbf{GL}_n(F)$ -equivariant  $D^\times$ -torsor for the natural action of  $\mathbf{GL}_n(F)$  on  $\Omega_{\check{F}}$ .
- $\pi_{\text{GM}}$  is a  $D^\times$ -equivariant  $\mathbf{GL}_n(F)$ -torsor where  $D^\times$  acts on  $\mathbb{P}_{\check{F}}^{n-1}$  via its embedding into  $\mathbf{GL}_n(\check{F})$ .
- The diagram carries a suitable Weil descent over  $F$ . Thus, its base change to  $\mathbb{C}_p$  carries an action of the Weil group  $W_F$ .

The diagram (1.1) actually encodes the isomorphism of the Lubin-Tate and Drinfeld towers which was previously established by Fargues in [Far08] and envisioned by Faltings in [Fal02].

In particular, there is an action of  $\mathbf{GL}_n(F) \times D^\times \times W_F$  on the infinite level Lubin-Tate space  $\mathcal{LT}_{\infty, \mathbb{C}_p}$  which makes natural the expectation that both the Jacquet-Langlands correspondence, relating  $\mathbf{GL}_n(F)$  and  $D^\times$ -representations, and the Langlands correspondence, relating  $\mathbf{GL}_n(F)$  and  $W_F$ -representations, can be realized in different cohomologies attached to  $\mathcal{LT}_{\infty, \mathbb{C}_p}$ , see for example [Far08], [DLB17], [CDN20], [CDN21], [CDN23].

In [Sch18], Scholze used the diagram (1.1) to construct a *p-adic Jacquet-Langlands functor* sending smooth admissible representations of  $\mathbf{GL}_n(F)$  to smooth admissible representations of  $D^\times$ . Let us be more precise; let  $\pi$  be an admissible representation of  $\mathbf{GL}_n(F)$  over an artinian ring  $A$  which is  $p$ -power torsion. Since  $\pi_{\text{GM}}$  is a proétale  $\mathbf{GL}_n(F)$ -torsor the representation  $\pi$  gives rise an étale sheaf  $\mathcal{F}_\pi$  on the rigid space  $\mathbb{P}_{\mathbb{C}_p}^{n-1}$ . Furthermore, this étale sheaf descends to the  $v$ -stack  $[\mathbb{P}_{\mathbb{C}_p}^{n-1}/D^\times]$  and so its cohomology carries a natural action of  $D^\times$ . The Jacquet-Langlands functor  $\mathcal{JL}$  is the functor sending a smooth admissible representation  $\pi$  of  $\mathbf{GL}_n(F)$  to the complex of smooth  $D^\times$ -representations over  $A$

$$\mathcal{JL}(\pi) = R\Gamma_{\text{ét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\pi).$$

By [Sch18, Theorem 1.1] the cohomology groups  $\mathcal{JL}^i(\pi)$  of  $\mathcal{JL}(\pi)$  are smooth admissible representations of  $D^\times$  over  $A$  and  $\mathcal{JL}^i(\pi) = 0$  for  $i > 2(n-1)$ . In addition, [Sch18, Theorem 1.3] says that this construction satisfies a local-global compatibility for  $\mathbf{GL}_2$  (though the main ideas should hold for general  $\mathbf{GL}_n$ ), justifying the compatibility with a more classical Jacquet-Langlands correspondence.

One can naturally extend the Jacquet-Langlands functor to unitary Banach representations and it is not hard to see that it also preserves admissible Banach representations, see Corollary 5.3.5. On the other hand, Schneider-Teitelbaum introduced a class of admissible locally analytic representations for  $p$ -adic Lie groups in [ST03]. A natural question arises:

**Question 1.1.** Is there a Jacquet-Langlands functor  $\pi \mapsto \mathcal{JL}(\pi)$  for admissible locally analytic representations of  $\mathbf{GL}_n(F)$ ? If so, is it compatible with the Jacquet-Langlands functor of Banach representations?

In this paper we give a partial answer to this question, namely, that the Jacquet-Langlands functor for admissible Banach representations is compatible with the passage to locally analytic vectors, see Theorem 1.8 for a more precise statement.

In a different direction, the works of Lue Pan [Pan22a, Pan22b] studying the locally analytic vectors of perfectoid modular curves use some special sheaves of locally analytic functions at infinite level. These sheaves encode, via the localization theory of Beilinson-Bernstein on the flag variety [BB81] and the Hodge-Tate period map, many aspects of the  $p$ -adic Hodge theory of Shimura varieties. In [RC23, RC24b] some of these features have been generalized to arbitrary global Shimura varieties under the name of *geometric Sen theory*; part of the goals of this paper is to extend the results in geometric Sen theory from the global to the local set up. It is then natural to ask what additional properties local Shimura varieties acquire after taking locally analytic vectors, in particular one can ask the following question:

**Question 1.2.** Let  $\mathcal{LT}_{\infty, \mathbb{C}_p}$  be the infinite level Lubin-Tate space and let  $\widehat{\mathcal{O}}_{\mathcal{LT}}$  be its structural sheaf as a perfectoid space. Let  $U_\infty \subset \mathcal{LT}_{\infty, \mathbb{C}_p}$  be an affinoid perfectoid and let  $K_U \subset \mathbf{GL}_n(F)$  and  $K_{U,D} \subset D^\times$  be the (open) stabilizers of  $U_\infty$ . Do we have an equality of locally analytic vectors

$$\widehat{\mathcal{O}}_{\mathcal{LT}}(U_\infty)^{K_U^{-la}} = \widehat{\mathcal{O}}_{\mathcal{LT}}(U_\infty)^{K_{U,D}^{-la}}$$

as subspaces of  $\widehat{\mathcal{O}}_{\mathcal{LT}}(U_\infty)$ ? Equivalently, are the locally analytic vectors of the structural sheaf at infinite level independent of the tower?

For the case of  $\mathbf{GL}_2$  this is proven by Pan in [Pan22b, Corollary 5.3.9] via explicit power series expansions. In this paper we prove a much more general result that holds for an arbitrary duality of local Shimura varieties and arbitrary period sheaves appearing in the affinoid charts of relative Fargues-Fontaine curves, see Corollary 1.2 for a precise statement. Then, the partial result towards Question 1.1 mentioned above will be a rather formal consequence of this independence of locally analytic vectors at infinite level, after applying enough technology coming from the theory of solid locally analytic representations [RJRC22, RJRC23]. We also apply this independence of locally analytic vectors to construct an equivariant isomorphism for the compactly supported de Rham cohomology between the two towers of a duality of local Shimura varieties, see Theorem 1.7.

In order to present the main results of this paper we have separated the introduction in different paragraphs, going from the general results on towers of rigid spaces, passing to the applications to local Shimura varieties, and finishing with the most specific applications to the Lubin-Tate and Drinfeld towers.

## Main results.

*Cohomology of towers of rigid spaces.* In this paragraph we explain the results of Section 5.1 about locally analytic vectors of period sheaves in towers of rigid spaces. Let  $\text{Perfd}$  be the category of perfectoid spaces and  $\text{Perf} \subset \text{Perfd}$  the full subcategory of perfectoid spaces in characteristic  $p$ . Following [Sch22] we see  $\text{Perfd}$  and  $\text{Perf}$  as sites endowed with the  $v$ -topology. Let  $\text{Perf}_\varpi$  be the category of perfectoid spaces in characteristic  $p$  endowed with a fixed pseudo-uniformizer  $\varpi$ , and with maps preserving the pseudo-uniformizer. We can define different period sheaves as follows:

- i. We have the structural sheaves  $\widehat{\mathcal{O}}$  and  $\widehat{\mathcal{O}}^+$  mapping an affinoid perfectoid  $\text{Spa}(R, R^+) \in \text{Perfd}$  to  $\widehat{\mathcal{O}}(R, R^+) = R$  and  $\widehat{\mathcal{O}}^+(R, R^+) = R^+$  respectively.

- ii. We have the tilted sheaves  $\mathcal{O}^b$  and  $\mathcal{O}^{b,+}$  mapping an affinoid perfectoid  $\mathrm{Spa}(R, R^+) \in \mathrm{Perfd}$  to  $\mathcal{O}^b(R, R^+) = R^b$  and  $\mathcal{O}^{b,+}(R, R^+) = R^{b,+}$  respectively.
- iii. We have the period sheaf  $\mathbb{A}_{\mathrm{inf}}$  mapping an affinoid perfectoid  $\mathrm{Spa}(R, R^+) \in \mathrm{Perfd}$  to  $\mathbb{A}_{\mathrm{inf}}(R, R^+) = W(R^{b,+})$  where  $W(-)$  is the functor of  $p$ -typical Witt vectors.
- iv. For  $I = [s, r] \subset (0, \infty)$  a compact interval with rational ends we define the period ring  $\mathbb{B}_I$  mapping an affinoid perfectoid  $\mathrm{Spa}(R, R^+) \in \mathrm{Perfd}_{\varpi}$  in characteristic  $p$  with fixed pseudo-uniformizer to the rational localization

$$\mathbb{B}_I(R, R^+) = \mathbb{A}_{\mathrm{inf}}(R, R^+) \left( \frac{p}{[\varpi]^{1/r}}, \frac{[\varpi]^{1/s}}{p} \right) \left[ \frac{1}{[\varpi]} \right].$$

The period sheaves in (i)-(iii) above are standard in  $p$ -adic Hodge theory. The sheaves in (iv) give rise to affinoid charts of families of Fargues–Fontaine curves as in [SW20] and [FS24].

Let  $K$  be a perfectoid field in characteristic 0 containing the  $p$ -th powers roots of unit and let  $X$  be a qcqs smooth rigid space over  $K$ . Let  $H$  be a compact  $p$ -adic Lie group acting on  $X$ . Let  $G$  be another compact  $p$ -adic Lie group and suppose we are given with an  $H$ -equivariant pro-finite-étale  $G$ -torsor  $\tilde{X} \rightarrow X^\diamond$  of the diamond attached to  $X$ . In particular,  $\tilde{X}$  is endowed with an action of the  $p$ -adic Lie group  $G \times H$ . The following theorem relates the locally analytic vectors of the  $v$ -cohomologies of period sheaves at infinite level.

**Theorem 1.3** (Theorem 5.1.1). *Let  $I \subset (0, \infty)$  be a compact interval with rational ends. Then the  $G$ -locally analytic vectors of the solid  $\mathbb{Q}_p$ -linear representation  $R\Gamma_v(\tilde{X}, \mathbb{B}_I)$  are  $H$ -locally analytic. More precisely, the natural map of solid  $G \times H$ -representations*

$$R\Gamma_v(\tilde{X}, \mathbb{B}_I)^{RG \times H\text{-la}} \xrightarrow{\sim} R\Gamma_v(\tilde{X}, \mathbb{B}_I)^{RG\text{-la}}$$

is an equivalence.

*Remark 1.1.* Theorem 1.3 holds for a larger class of  $\mathbb{B}_I$ -modules, including  $\mathbb{B}_I$  and  $\widehat{\mathcal{O}}$ -vector bundles, see Remark 5.1.5.

As a corollary, in the case when  $G = 1$ , we prove that pro-étale cohomologies of qcqs rigid varieties  $X$  endowed with actions of  $p$ -adic Lie groups  $H$  tend to be locally analytic:

**Corollary 1.1** (Corollary 5.1.6). *Let  $X$  be a qcqs smooth rigid space endowed with the action of a  $p$ -adic Lie group  $H$ . Then for  $I \subset (0, \infty)$  a compact interval with rational ends the solid  $H$ -representation  $R\Gamma_v(X, \mathbb{B}_I)$  is  $H$ -locally analytic.*

*Remark 1.2.* Corollary 1.1 implies that the cohomology groups of period sheaves on  $X$  admit an action of the Lie algebra of  $H$  obtained by derivations, we found this fact surprising since there is no finiteness or Hausdorff assumptions in the cohomology groups. This also suggests that there is a deeper structure in the period sheaves of rigid spaces that witness the locally analytic properties of their cohomologies. In a work in progress of Johannes Anschütz, Arthur–César le Bras, Peter Scholze and the second author we expect to give a conceptual explanation of these facts via the analytic prismaticization.

*Geometric Sen theory over local Shimura varieties.* In the next paragraph we state the main results of Section 4 extending those of [Pan22a] and [RJRC22] about the Sen operators of local Shimura varieties. In order to be more precise we need to introduce some notation, we shall follow [SW20]. Let  $(\mathbf{G}, b, \mu)$  be a local Shimura datum as in Lecture XXIV of *loc. cit.*, let  $E$  be the field of definition of  $\mu$  and  $\check{E}$  the completion of the maximal unramified extension of  $E$ . Let  $\mathrm{FL}_{\mathbf{G}, \mu, E}$  and  $\mathrm{FL}_{\mathbf{G}, \mu^{-1}, E}$  be the algebraic flag varieties parametrizing decreasing and increasing  $\mu$ -filtrations of the trivial  $\mathbf{G}$ -torsor respectively. We let  $\mathcal{F}\ell_{\mathbf{G}, \mu, E}$  and  $\mathcal{F}\ell_{\mathbf{G}, \mu^{-1}, E}$  be the analytification of the flag varieties to adic spaces [Hub94]. For  $K \subset \mathbf{G}(\mathbb{Q}_p)$  open compact subgroup we let  $\mathcal{M}_{\mathbf{G}, b, \mu, K, \check{E}}$  be the local Shimura variety over  $\check{E}$  at level  $K$ .

Let  $E'/E$  be a finite extension where the group  $\mathbf{G}$  is split and let us fix once and for all a cocharacter  $\mu : \mathbb{G}_{m, E'} \rightarrow \mathbf{G}_{E'}$  representing the conjugacy class of  $\mu$ . Let  $\mathbf{P}_\mu$  and  $\mathbf{P}_{\mu^{-1}}$  be the parabolic subgroups parametrizing decreasing and increasing filtrations of  $\mu$ , let  $\mathbf{N}_\mu \subset \mathbf{P}_\mu$  and  $\mathbf{N}_{\mu^{-1}} \subset \mathbf{P}_{\mu^{-1}}$  be their unipotent radicals respectively, and let  $\mathbf{M} = \mathbf{M}_\mu = \mathbf{M}_{\mu^{-1}}$  be the centralizer of  $\mu$  (eq. of  $\mu^{-1}$ ) in  $\mathbf{G}_{E'}$ . We have presentations for the flag varieties  $\mathrm{FL}_{\mathbf{G}, \mu, E'} = \mathbf{G}_{E'}/\mathbf{P}_\mu$  and  $\mathrm{FL}_{\mathbf{G}, \mu^{-1}, E'} = \mathbf{G}_{E'}/\mathbf{P}_{\mu^{-1}}$ , these presentations

give rise to an equivalence of  $\mathbf{G}$ -equivariant quasi-coherent sheaves on  $\mathrm{FL}_{\mathbf{G},\mu,E'}$  and  $\mathrm{FL}_{\mathbf{G},\mu^{-1},E'}$  and algebraic representations of  $\mathbf{P}_\mu$  and  $\mathbf{P}_{\mu^{-1}}$  respectively.

Let  $\mathfrak{n}_\mu \subset \mathfrak{p}_\mu \subset \mathfrak{g}$  be the Lie algebras of  $\mathbf{N}_\mu \subset \mathbf{P}_\mu \subset \mathbf{G}_{E'}$  and  $\mathfrak{m}_\mu = \mathfrak{p}_\mu/\mathfrak{n}_\mu$  the Lie algebra of the Levi quotient. We see these Lie algebras endowed with the adjoint action of  $\mathbf{P}_\mu$  and consider their corresponding  $\mathbf{G}$ -equivariant Lie algebroids  $\mathfrak{n}_\mu^0 \subset \mathfrak{p}_\mu^0 \subset \mathfrak{g}^0 = \mathcal{O}_{\mathrm{FL}_{\mathbf{G},\mu,E'}} \otimes_{\mathbb{Q}_p} \mathfrak{g}$  and  $\mathfrak{m}_\mu^0 = \mathfrak{p}_\mu^0/\mathfrak{n}_\mu^0$  appearing in the localization theory of Beilinson-Bernstein [BB81]. We denote in the same way their pullbacks to vector bundles over the analytic flag varieties, and use similar notation for the Lie algebras of the opposite parabolic and their associated Lie algebroids in  $\mathrm{FL}_{\mathbf{G},\mu^{-1},E'}$ .

Let  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}}^\diamond = \varprojlim_K \mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}}^\diamond$  be the infinite level local Shimura variety seen as a diamond over  $\check{E}$  and consider the Grothendieck-Messing and Hodge-Tate period maps

$$\begin{array}{ccc} & \mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}}^\diamond & \\ \pi_{\mathrm{GM}} \swarrow & & \searrow \pi_{\mathrm{HT}} \\ \mathcal{F}\ell_{\mathbf{G},\mu,\check{E}}^\diamond & & \mathcal{F}\ell_{\mathbf{G},\mu^{-1},\check{E}}^\diamond \end{array}$$

Let  $\tilde{G}_b$  be the group of automorphisms of the constant  $\mathbf{G}$ -torsor  $\mathcal{E}_b$  over the curve, see [FS24, III.5.1]. Then  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}}^\diamond$  is endowed with an action of  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$  for which both maps  $\pi_{\mathrm{GM}}$  and  $\pi_{\mathrm{HT}}$  are equivariant in a suitable sense (see Section 3 for more details).

By [SW20, Corollary 23.3.2], when  $b$  is basic there is a dual local Shimura datum  $(\check{\mathbf{G}}, \check{b}, \check{\mu})$ , an isomorphism  $G_b = \tilde{G}_b = \check{\mathbf{G}}(\mathbb{Q}_p)$  and a  $\mathbf{G}(\mathbb{Q}_p) \times \check{\mathbf{G}}(\mathbb{Q}_p)$ -equivariant isomorphism of infinite level local Shimura varieties

$$\mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}} \cong \mathcal{M}_{\check{\mathbf{G}},\check{b},\check{\mu},\infty,\check{E}}$$

that exchanges the Grothendieck-Messing and Hodge-Tate period maps (see also Proposition 3.2.3).

On the other hand, the map  $\pi_K : \mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}}^\diamond \rightarrow \mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}}^\diamond$  is a proétale  $K$ -torsor. Thus, for any ind-system  $V = \varinjlim_i V_i$  of  $p$ -adically complete continuous representations of  $K$  we can construct a  $v$ -sheaf  $\mathcal{F}_V$  on  $\mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}}^\diamond$  by first constructing the  $p$ -complete  $v$ -sheaves  $\mathcal{F}_{V_i}$  via descent  $V_i$  along  $\pi_K$  and then by extending by colimits  $\mathcal{F}_V = \varinjlim_i \mathcal{F}_{V_i}$  (see Definition 3.3.3). In particular, for  $V$  an algebraic representation of  $\mathbf{G}$  we have automorphic local systems  $\mathcal{F}_V$ , and for  $\pi$  a smooth admissible representation of  $\mathbf{G}(\mathbb{Q}_p)$  over a  $p$ -power torsion ring  $A$  the sheaf  $\mathcal{F}_\pi$  is the étale local system considered in [Sch18] for the  $p$ -adic Jacquet-Langlands functor.

Let  $\mathcal{F}_{\mathfrak{g}^\vee}$  be the local system over the local Shimura tower  $(\mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}'})_{K \subset \mathbf{G}(\mathbb{Q}_p)}$  attached to the dual of the adjoint representation of  $\mathbf{G}$ . Let us write by  $\mathcal{O}_{\mathcal{M}}$  for the structural sheaf of a finite level local Shimura variety and let  $\Omega_{\mathcal{M}}^1$  be its cotangent bundle. We now state the first theorem concerning the computation of the geometric Sen operators of local Shimura varieties extending [RC24b, Theorem 5.2.5].

**Theorem 1.4** (Theorem 4.3.1). *The geometric Sen operator  $\theta_{\mathcal{M}} : \mathcal{F}_{\mathfrak{g}^\vee} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}} \rightarrow \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \widehat{\mathcal{O}}(-1)$  of the tower  $(\mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}'})_{K \subset \mathbf{G}(\mathbb{Q}_p)}$  in the sense of [RC23, Theorem 3.3.4] is given by the pullback along  $\pi_{\mathrm{HT}}$  of the  $\mathbf{G}$ -equivariant map of vector bundles on  $\mathcal{F}\ell_{\mathbf{G},\mu^{-1},E'}$*

$$\mathfrak{g}^{0,\vee} \rightarrow \mathfrak{n}_{\mu^{-1}}^{0,\vee}$$

where the identification  $\pi_{\mathrm{HT}}^*(\mathfrak{n}_{\mu^{-1}}^{0,\vee}) \cong \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \widehat{\mathcal{O}}(-1)$  is through the opposite of the Kodaira-Spencer isomorphism, see Section 4.1. Here  $\mathcal{F}(n)$  is the  $n$ -th Hodge-Tate twist of  $\mathcal{F}$  by the  $n$ -th power of the cyclotomic character.

*Remark 1.3.* The previous theorem was stated for the base change of flag varieties and local Shimura varieties to  $E'$ . This base change can be avoided if one works without fixing a Hodge cocharacter  $\mu$ , indeed, the flag varieties and the Hodge-Tate period maps are already defined over  $E$ . Moreover, the Lie algebroids  $\mathfrak{n}_{\mu^{-1}}^0$ ,  $\mathfrak{p}_{\mu^{-1}}^0$  and  $\mathfrak{m}_{\mu^{-1}}^0$  are also defined over  $E$ , see Remark 2.5.1.

A first consequence of the computation of the geometric Sen operator is the vanishing of the higher locally analytic vectors of the structural sheaf  $\widehat{\mathcal{O}}$  at infinite level, as well as the computation of the arithmetic Sen

operator in terms of representation theory. Let  $C/E'$  be a completed algebraically closed extension and consider the  $C$ -base change  $\mathcal{M}_{\mathbf{G},b,\mu,K,C}$  of the local Shimura varieties. Let  $\mathcal{V}_K = C^{la}(K, \mathbb{Q}_p)_{\star_1}$  be the space of locally analytic functions of  $K$  endowed with the left regular action and let  $\mathcal{F}_{\mathcal{V}_K}$  be the  $v$ -sheaf over  $\mathcal{M}_{\mathbf{G},b,\mu,K,C}$  obtained by descent from infinite level. We have the following theorem, analogue to Proposition 6.2.8, Corollary 6.2.12 and Theorem 6.3.5 of [RC24b].

**Theorem 1.5** (Theorem 4.3.3). *Let  $U \subset \mathcal{M}_{\mathbf{G},b,\mu,K,C}$  be an open affinoid subspace admitting an étale map to a product of tori  $\mathbb{T}_C^d$  that factors as a finite composition of rational localizations and finite étale maps. Let  $U_\infty \subset \mathcal{M}_{\mathbf{G},b,\mu,\infty,C}^\diamond$  be the pullback of  $U$ , then the  $v$ -cohomology*

$$R\Gamma_v(U, \widehat{\mathcal{O}} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}_{\mathcal{V}_K}) \quad (1.2)$$

*sits in degree 0 and is equal to the locally analytic vectors  $\widehat{\mathcal{O}}(U_\infty)^{G-la}$  of  $\widehat{\mathcal{O}}(U_\infty)$ . Here the completed tensor product is a filtered colimit of  $p$ -completed tensor products obtained by writing  $\mathcal{F}_{\mathcal{V}_K}$  as a colimit of Banach sheaves (equivalently a solid tensor product as in [AM24]).*

*Furthermore, the action of  $\mathfrak{g}_{\mu^{-1}}^0 = \mathcal{O}_{\mathcal{F}^l_{\mathbf{G},\mu^{-1},C}} \otimes_{\mathbb{Q}_p} \mathfrak{g}$  on  $\widehat{\mathcal{O}}(U_\infty)^{G-la}$  by derivations kills  $\mathfrak{n}_{\mu^{-1}}^0$ . Thus, we have an horizontal action of  $\mathfrak{m}_{\mu^{-1}}^0$  on  $\widehat{\mathcal{O}}(U_\infty)^{G-la}$ . Moreover, the space  $\widehat{\mathcal{O}}(U_\infty)^{G-la}$  has an arithmetic Sen operator as in [RC24b, Theorem 6.3.5] given by the opposite of the derivative of the Hodge cocharacter  $-\theta_\mu = \theta_{\mu^{-1}} \in \mathfrak{m}_{\mu^{-1}}^0$ .*

*Locally analytic vectors of local Shimura varieties.* In this paragraph we apply Theorem 1.3 to prove the independence of locally analytic vectors for a duality of local Shimura varieties, generalizing a theorem of Pan for the Lubin-Tate tower of  $\mathbf{GL}_2$  [Pan22b, Corollary 5.3.9]. Let  $C/E$  be a complete algebraically closed field, we write  $G = \mathbf{G}(\mathbb{Q}_p)$  and let  $G_b$  be the profinite quotient of  $\widetilde{G}_b$ .

Let  $\widehat{\mathcal{O}}_{\mathcal{M}}$  be the restriction of the structural sheaf  $\widehat{\mathcal{O}}$  in the  $v$ -site of  $\mathcal{M}_{\mathbf{G},\mu,b,\infty,C}^\diamond$  to the underlying topological space  $|\mathcal{M}_{\mathbf{G},\mu,b,\infty,C}^\diamond|$ . Let  $\mathcal{O}_{\mathcal{M}}^{G-la} \subset \widehat{\mathcal{O}}_{\mathcal{M}}$  be the subsheaf whose values in a qcqs open subspace  $U_\infty \subset \mathcal{M}_{\mathbf{G},\mu,b,\infty,C}^\diamond$  are given by the  $G = \mathbf{G}(\mathbb{Q}_p)$ -locally analytic sections of  $\widehat{\mathcal{O}}_{\mathcal{M}}$ , namely, given by

$$\mathcal{O}_{\mathcal{M}}^{G-la}(U_\infty) = \widehat{\mathcal{O}}(U_\infty)^{K_{U_\infty} - la}$$

where  $K_{U_\infty} \subset G$  is the stabilizer of  $U_\infty$ . We have the following corollary:

**Corollary 1.2** (Corollary 5.1.9). *For any  $p$ -adic Lie group  $H \subset \widetilde{G}_b$  and any qcqs open subspace  $U_\infty \subset \mathcal{M}_{\mathbf{G},\mu,b,\infty,C}^\diamond$  the natural map*

$$\mathcal{O}_{\mathcal{M}}^{G-la}(U_\infty)^{RH-la} \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}}^{G-la}(U_\infty)$$

*from the derived  $H$ -locally analytic vectors is an equivalence. In particular, if  $b$  is basic we have an equality of subsheaves of  $\widehat{\mathcal{O}}_{\mathcal{M}}$*

$$\mathcal{O}_{\mathcal{M}}^{G-la} = \mathcal{O}_{\mathcal{M}}^{G_b-la}.$$

*More generally, for  $b$  basic and  $I \subset (0, \infty)$  a compact interval with rational ends, we have an equivalence of derived solid locally analytic representations of  $G \times G_b$*

$$R\Gamma_v(U_\infty, \mathbb{B}_I)^{RG_b-la} \xrightarrow{\sim} R\Gamma_v(U_\infty, \mathbb{B}_I)^{RG \times G_b-la} \xleftarrow{\sim} R\Gamma_v(U_\infty, \mathbb{B}_I)^{RG-la}.$$

From now on we shall focus in the case when  $b$  is basic. We will identify the  $G$  and  $G_b$ -locally analytic vectors of the structural sheaf  $\widehat{\mathcal{O}}_{\mathcal{W}}$  at infinite level and simply write  $\mathcal{O}_{\mathcal{W}}^{la}$ . We can then identify the horizontal actions of the Levi Lie algebras of Theorem 1.5. For this, we need some additional notation.

By Corollary 3.3.7 we have a natural  $G \times G_b$ -equivariant isomorphism of  $\widehat{\mathcal{O}}_{\mathcal{M}}$ -vector bundles on  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,C}^\diamond$

$$\mathfrak{m}_{\mu^{-1}}^0 \otimes_{\mathcal{O}_{\mathcal{F}^l_{\mathbf{G},\mu^{-1},C}}} \widehat{\mathcal{O}}_{\mathcal{M}} \cong \widehat{\mathcal{O}}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{F}^l_{\mathbf{G},\mu,C}}} \mathfrak{m}_\mu^0.$$

By taking locally analytic vectors we obtain an  $\mathcal{O}_{\mathcal{M}}^{la}$ -vector bundle which we shall denote as  $\mathfrak{m}^{0,la}$ , endowed with  $G \times G_b$ -equivariant isomorphisms

$$\mathfrak{m}_{\mu^{-1}}^0 \otimes_{\mathcal{O}_{\mathcal{F}^l_{\mathbf{G},\mu^{-1},C}}} \mathcal{O}_{\mathcal{M}}^{la} \cong \mathfrak{m}^{0,la} \cong \mathcal{O}_{\mathcal{M}}^{la} \otimes_{\mathcal{O}_{\mathcal{F}^l_{\mathbf{G},\mu,C}}} \mathfrak{m}_\mu^0. \quad (1.3)$$

We have the following theorem:

**Theorem 1.6** (Theorem 5.1.10). *The actions of  $\mathfrak{n}_\mu^0$  and  $\mathfrak{n}_{\mu^{-1}}^0$  on  $\mathcal{O}_{\mathcal{M}}^{la}$  vanish. Furthermore, the actions of  $\mathfrak{m}_\mu^0$  and  $\mathfrak{m}_{\mu^{-1}}^0$  on  $\mathcal{O}_{\mathcal{M}}^{la}$  by derivations are identified via (1.3). In particular, the central character of the actions of  $\mathfrak{m}_\mu^0$  and  $\mathfrak{m}_{\mu^{-1}}^0$  on  $\mathcal{O}_{\mathcal{M}}^{la}$  agree under the natural isomorphism of the center of the enveloping algebras  $\mathcal{Z}(\mathfrak{m}_{\mu,C}) \cong \mathcal{Z}(\mathfrak{m}_{\mu^{-1},C})$ , where  $\mathfrak{m}_{\mu,C}$  and  $\mathfrak{m}_{\mu^{-1},C}$  are the Levi subalgebras of  $\mathrm{Lie} G_b \otimes_{\mathbb{Q}_p} C$  and  $\mathrm{Lie} G \otimes_{\mathbb{Q}_p} C$  respectively.*

*De Rham cohomology of towers of local Shimura varieties.* Our next result is the comparison between compactly supported de Rham cohomologies of the two towers in a duality of local Shimura varieties. This theorem has been also independently obtained by Guido Bosco, Wiesława Nizioł and the first author. Let  $(\mathbf{G}, b, \mu)$  be a local Shimura datum with  $b$  basic and let  $(\check{\mathbf{G}}, \check{b}, \check{\mu})$  be the dual local Shimura datum. Consider the towers of rigid spaces  $(\mathcal{M}_{\mathbf{G}, b, \mu, K, \check{E}})_{K \subset \mathbf{G}(\mathbb{Q}_p)}$  and  $(\mathcal{M}_{\check{\mathbf{G}}, \check{b}, \check{\mu}, \check{K}, \check{E}})_{\check{K} \subset \check{\mathbf{G}}(\mathbb{Q}_p)}$ . We have the following theorem:

**Theorem 1.7** (Theorem 5.2.2). *There is a natural  $\mathbf{G}(\mathbb{Q}_p) \times \check{\mathbf{G}}(\mathbb{Q}_p)$ -equivariant isomorphism of compactly supported de Rham cohomology groups*

$$\varinjlim_{K \subset \mathbf{G}(\mathbb{Q}_p)} H_{dR,c}^i(\mathcal{M}_{\mathbf{G}, b, \mu, K, \check{E}}) \cong \varinjlim_{\check{K} \subset \check{\mathbf{G}}(\mathbb{Q}_p)} H_{dR,c}^i(\mathcal{M}_{\check{\mathbf{G}}, \check{b}, \check{\mu}, \check{K}, \check{E}}).$$

*Remark 1.4.* Theorem 1.7 should be seen as an evidence of the fact that there is a well defined analytic de Rham stack (in the sense of [RC24a]) for the infinite level Shimura variety, together with  $\mathbf{G}(\mathbb{Q}_p) \times \check{\mathbf{G}}(\mathbb{Q}_p)$ -equivariant equivalences

$$\varprojlim_K \mathcal{M}_{\mathbf{G}, b, \mu, K, \check{E}}^{\mathrm{dR}} = \mathcal{M}_{\mathbf{G}, b, \mu, \infty, \check{E}}^{\diamond, \mathrm{dR}} \cong \mathcal{M}_{\check{\mathbf{G}}, \check{b}, \check{\mu}, \infty, \check{E}}^{\diamond, \mathrm{dR}} = \varprojlim_{\check{K}} \mathcal{M}_{\check{\mathbf{G}}, \check{b}, \check{\mu}, \check{K}, \check{E}}^{\mathrm{dR}}.$$

Indeed, as it was explained by Scholze to the second author, one can prove sufficient descent for the formation of the analytic de Rham stack to be well defined for (suitable nice) diamonds, where the previous equivalence holds as analytic stacks. It is likely that purely motivic techniques as those appearing in [Vez19] are enough to show the equivalence of the de Rham cohomologies for the two towers, see Proposition 4.5 in *loc. cit.*; we thanks Arthur-César le Bras for this observation.

*Locally analytic Jacquet-Langlands functor in the Lubin Tate case.* We finish the presentation of the main results with the principal motivation that initiated this project, that is, the  $p$ -adic Jacquet-Langlands functor of the Lubin-Tate tower treated in [Sch18]. We shall keep the notation of the beginning of the introduction regarding the Lubin-Tate and Drinfeld towers. We have the following compatibility with the passage to locally analytic vectors:

**Theorem 1.8** (Theorem 5.3.6). *Let  $\pi$  be an admissible Banach representation of  $\mathbf{GL}_n(F)$  and let  $\Pi = (\pi[\frac{1}{p}])^{\mathbf{GL}_n(F)-la}$  be the space of locally analytic vectors seen as a colimit of Banach spaces. Let  $\mathcal{F}_\Pi$  be the proétale sheaf over  $\mathbb{P}_{\mathbb{C}_p}^{n-1}$  constructed via descent along  $\pi_{\mathrm{GM}}$  of the continuous representation  $\Pi$ . There is a natural equivalence of solid locally analytic  $H$ -representations*

$$(\mathcal{JL}(\pi)[\frac{1}{p}])^{RH-la} \cong R\Gamma_{\mathrm{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\Pi).$$

Furthermore, this equivalence induces an isomorphism of cohomology groups:

$$(\mathcal{JL}^i(\pi)[\frac{1}{p}])^{H-la} \cong H_{\mathrm{proét}}^i(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\Pi).$$

As a corollary of Theorems 1.6 and 1.8 we can show that the Jacquet-Langlands functor preserves central characters for the locally analytic vectors of admissible Banach representations.

**Corollary 1.3** (Corollary 5.3.7). *Let  $\pi$  be an admissible Banach representation of  $\mathbf{GL}_n(L)$  over a finite extension of  $\mathbb{Q}_p$  and suppose that  $\Pi = \pi^{\mathbf{GL}_n(L)-la}$  has central character  $\chi$ . Then, for all  $i \in \mathbb{Z}$ , the locally analytic  $H$ -representation  $\mathcal{JL}^i(\pi)^{H-la}$  has central character  $\chi$  under the natural identification  $\mathcal{Z}(\mathrm{Lie} H) \cong \mathcal{Z}(\mathrm{Lie} G)$ .*



**Outline of the paper.** Section 2 is a preliminary section where we introduce the main objects and tools used in the paper. In Section 2.1 we recall the definition of period sheaves and the construction of the different incarnations of families of Fargues-Fontaine curves following [SW20, §11.2] and [FS24, §II.1]. In 2.2 we briefly recall the construction of the categories of solid almost quasi-coherent sheaves on diamonds of [Man22b], and its relation with smooth representation theory of profinite groups. In 2.3 we recall the construction of the décalage operator of [BMS18] which will be relevant to perform a technical dévissage in the proof of Theorem 1.3. Then, in Section 2.4 we briefly recall the basics of the theory of solid locally analytic representations of [RJRC22, RJRC23], in particular we state the locally analytic criterion of Lemma 2.4.1 which is key in the proof of Theorem 1.3. Finally, in Section 2.5 we briefly summarize the relationship between representations of reductive groups and equivariant sheaves over flag varieties, making special emphasis in the Lie algebroids appearing in the localization theory of Beilinson-Bernstein [BB81].

We continue with Section 3 which concerns the definition of the local Shimura varieties and some basic Hodge theoretic features of them, we follow [SW20, Lecture XXIII] and [FS24, §III.4 and 5]. In Section 3.1 we discuss some facts about torsors on families of Fargues-Fontaine curves. In Section 3.2 we recall the definition of the moduli space of shtukas of one leg as well as the construction of the Grothendieck-Messing and Hodge-Tate period maps. Then, in Section 3.3 we specialize the previous construction to the situation of local Shimura varieties where we deduce from the general theory of [SW20, Lecture XXIII] a  $p$ -adic Riemann-Hilbert correspondence for automorphic local systems in Proposition 3.3.4; this formulation of the theory of Scholze-Weinstein will be useful in the computation of the geometric Sen operator of the next section.

Next, in Section 4 we compute the geometric and arithmetic Sen operators for local Shimura varieties. In Section 4.1 we explain the purely representation theoretic construction of the Kodaira-Spencer isomorphism for Shimura varieties which is essentially a reinterpretation of the anchor map of the reductive group acting on the flag variety. This point of view of the Kodaira-Spencer map will allow us to compute the pullback of equivariant sheaves of flag varieties via the Hodge-Tate period maps in terms of automorphic vector bundles and the Faltings extension in Section 4.2, see Theorem 4.2.1. Finally, we use this description of the pullbacks of automorphic vector bundles to compute the geometric and arithmetic Sen operators of Theorems 1.4 and 1.5. The analogue of these theorems for global Shimura varieties were achieved in [RC23, RC24b], and the proofs in the local situation follow exactly the same line of arguments.

We conclude with Section 5 where most of the main theorems stated in the introduction are proven. In Section 5.1 we prove Theorem 1.3; the strategy of the proof is to use the locally analytic criterion of Lemma 2.4.1. For this one has to implement a long dévissage up to the point where one is reduced to showing that the proétale (eq.  $v$ -) cohomology of  $\mathcal{O}^+/p$  on a qcqs smooth rigid space is small after applying a décalage operator  $L\eta_{p^\varepsilon}$  for some  $\varepsilon > 0$ , see Dévissage 4. As an immediate consequence we obtain Corollary 1.1. We conclude this section with the application of Theorem 1.3 to local Shimura varieties; we first obtain the independence of locally analytic vectors at infinite level for  $b$  basic of Corollary 1.2, then, with a more careful study of the horizontal actions arising from the two towers, we prove Theorem 1.6. In Section 5.2 we compare the de Rham cohomology of the two towers proving Theorem 1.7; here the strategy is to relate the de Rham complexes of each tower with a suitable de Rham complex of the sheaf  $\mathcal{O}_{\mathcal{M}}^{la}$  arising from the derivations of both groups  $G$  and  $G_b$ . Finally, in Section 5.3 we prove the compatibility of Scholze's Jacquet-Langlands functor with the passage to locally analytic vectors for admissible Banach representations proving Theorem 1.8; here the key strategy is to rewrite the proétale cohomology of the sheaf  $\mathcal{F}_\pi$  in terms of period sheaves  $\mathbb{B}_I$  and then to exploit the independence of locally analytic vectors at infinite level of Corollary 1.2 in order to jump between towers. Finally, using the proof of Theorem 1.8 and the compatibility of the horizontal characters for the sheaf  $\mathcal{O}_{\mathcal{M}}^{la}$  of Theorem 1.6, we obtain the compatibility of central characters of the Jacquet-Langlands functor of Corollary 1.3.

**Conventions.** In this paper we use the  $v$ -site of perfectoid spaces as introduced in [Sch22]. We use the theory of solid almost quasi-coherent sheaves of [Man22b], [AM24] and [AMLB]; the use of these cohomology theories is important in order to properly keep track to the condensed or topological structure of cohomology complexes. In particular, this work heavily depends on the theory of condensed mathematics of Clausen and Scholze [CS19, CS20], and in higher category theory for which we refer to [Lur09, Lur17]. A different reason to use condensed mathematics is to have access to the theory of solid locally analytic representations of [RJRC22, RJRC23]. This is important since, even though most of the main theorems involve classical

topological representations, the proofs will make appear very general solid representations which are not classical.

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## 2. PRELIMINARIES

In this section we introduce the main objects and techniques used in the paper. In Section 2.1 we recall the definition of families of Fargues-Fontaine curves following [SW20, §11.2] and [FS24, §II.1]. Some period sheaves associated to affinoid charts of the curves will be explicitly introduced for reference in later sections. Then in Section 2.2, we recall the definition of the derived categories of solid almost quasi-coherent sheaves of [Man22b]; we shall not need all the power of the six functor formalism, only the existence of these categories and their relation with smooth representations after [Man22b, §3.4]. We continue in Section 2.3 with some basic properties of the décalage operator  $L\eta_{\mathcal{I}}$  of [BMS18, §6]; their importance for us will be to kill some small enough torsion in order to apply a locally analytic criterion discussed in the next section. In Section 2.4 we briefly introduce the theory of solid locally analytic representations of [RJRC22, RJRC23], in particular we recall the criterion of [RJRC23, Proposition 3.3.3] for a solid representation of a  $p$ -adic Lie group to be locally analytic. Finally, in Section 2.5 we state the classical dictionary between representation theory and equivariant quasi-coherent sheaves on flag varieties; in particular we make emphasis in the Lie algebroids over the flag variety appearing in the localization theory of [BB81].

Apart (but not disjoint!) from condensed mathematics, we also use different aspects of  $p$ -adic Hodge theory. The main objects we study are period sheaves on diamonds [RC23] and the computation of the geometric Sen operators of Shimura varieties [RC24b]. Finally, in order to realize our cohomology groups as honest solid abelian groups we use the categories of solid quasi-coherent sheaves of diamonds of Mann [Man22b] though the full six functor formalism will not be necessary.

**2.1. The Fargues-Fontaine curve and sheaves of periods.** Let  $\text{Perfd}$  be the category of perfectoid spaces over  $\mathbb{Z}_p$  and let  $\text{Perf} \subset \text{Perfd}$  be the full subcategory of perfectoid spaces over  $\mathbb{F}_p$ . Following [Sch22], we consider the  $v$ -site  $\text{Perfd}_v$  of perfectoid spaces. Let  $\mathbb{F}_p((\varpi^{1/p^\infty}))$  be the perfectoid field parametrizing pseudo-uniformizers in  $\text{Perf}$  and let  $\text{Perf}_\varpi$  be the slice category  $\text{Perf}/_{\text{Spa}\mathbb{F}_p((\varpi^{1/p^\infty}))}$ , equivalently,  $\text{Perf}_\varpi$  is the category of perfectoid spaces in characteristic  $p$  with fixed pseudo-uniformizer  $\varpi$ , and with maps preserving the pseudo-uniformizer. We let  $\widehat{\mathcal{O}}$  and  $\widehat{\mathcal{O}}^+$  be the  $v$ -sheaves on  $\text{Perfd}$  mapping an affinoid perfectoid  $\text{Spa}(R, R^+)$  to  $R$  and  $R^+$  respectively. Similarly, we let  $\mathcal{O}^b$  and  $\mathcal{O}^{b,+}$  be the  $v$ -sheaves mapping  $\text{Spa}(R, R^+)$  to  $R^b$  and  $R^{b,+}$  respectively. Given  $S = \text{Spa}(R, R^+) \in \text{Perfd}$  we let  $\mathbb{A}_{\text{inf}}(S) := \mathbb{A}_{\text{inf}}(R^+) := W(R^{b,+})$  be the period ring of Fontaine, and denote by  $[-] : R^{b,+} \rightarrow \mathbb{A}_{\text{inf}}(S)$  the Teichmüller lift.

For  $S = \text{Spa}(R, R^+) \in \text{Perf}_\varpi$  consider the sous-perfectoid analytic adic space [SW20, Proposition 11.2.1]

$$\mathcal{Y}_S^{\text{FF}} := \{[[\varpi]] \neq 0\} \subset \text{Spa}(\mathbb{A}_{\text{inf}}(S)),$$

consisting on the locus where  $[\varpi]$  is a pseudo-uniformizer, we call this adic space the  $\mathcal{Y}^{\text{FF}}$ -curve over  $S$ .

The space  $\mathcal{Y}_S^{\text{FF}}$  has pseudo-uniformizer  $[\varpi]$ . Furthermore, the following properties hold (see [SW20, §11] and [FS24, §II.1])

- We have a natural equivalence of diamonds

$$\mathcal{Y}_S^{\text{FF}, \diamond} = S \times \text{Spd}\mathbb{Z}_p$$

where  $\text{Spd}\mathbb{Z}_p$  is the diamond parametrizing untilts of perfectoid spaces. In particular the formation of  $\mathcal{Y}_S^{\text{FF}}$  is independent of the pseudo-uniformizer of  $S$ . Moreover,  $\mathcal{Y}_S^{\text{FF}}$  has a natural Frobenius automorphism  $\varphi_S$  lifting the Frobenius of  $S$ .

- Let  $|\mathcal{Y}_S^{\text{FF}}|$  be the underlying topological space of the adic space. There is a (unique) continuous radius map

$$\text{rad} : |\mathcal{Y}_S^{\text{FF}}| \rightarrow [0, \infty)$$

sending a rank 1 point  $x$  to

$$\text{rad}(x) = \log_p |[\varpi]|_x / \log_p |p|_x = \log_{|p|_x} |[\varpi]|_x.$$

The radius map and the Frobenius endomorphism are related by the formula

$$\text{rad} \circ \varphi_S = p \text{rad}.$$

- For  $I = [s, r] \subset [0, \infty)$  a compact interval with rational ends one defines affinoid subspaces

$$\mathcal{Y}_{S,I}^{\text{FF}} = \mathcal{Y}_S^{\text{FF}} \left( \frac{p}{[\varpi]^{1/r}}, \frac{[\varpi]^{1/s}}{p} \right).$$

One has  $\text{rad}(|\mathcal{Y}_{S,I}^{\text{FF}}|) \subset I$  but the inclusion  $\mathcal{Y}_{S,I}^{\text{FF}} \subsetneq \text{rad}^{-1}(I)$  is strict.

- For  $I \subset (0, \infty)$  an open interval we let  $\mathcal{Y}_{S,I}^{\text{FF}} \subset \mathcal{Y}_S^{\text{FF}}$  be the inverse image of  $I$  by  $\text{rad}$ . This space can be also described as

$$\mathcal{Y}_{S,I} = \bigcup_{J \subset I} \mathcal{Y}_{S,J}^{\text{FF}}$$

where  $J$  runs over all the compact subintervals of  $I$  with rational ends.

- The relative *Fargues-Fontaine curve* over  $S$  is the sous-perfectoid space

$$\mathcal{X}_S^{\text{FF}} = \mathcal{Y}_{S,(0,\infty)}^{\text{FF}} / \varphi_S^{\mathbb{Z}}.$$

We shall call this space the  $\mathcal{X}^{\text{FF}}$ -curve over  $S$ .

**Convention 2.1.1.** From now on all closed rational intervals  $I = [s, r] \subset [0, \infty)$  will be assumed to have rational ends.

**Lemma 2.1.2.** *Let  $S, S' \in \text{Perf}_{\varpi}$  be affinoid perfectoids of characteristic  $p$  with fixed pseudo-uniformizer  $\varpi$ , let  $f : S' \rightarrow S$  be a map in  $\text{Perf}_{\varpi}$  and let  $f^{\text{FF}} : \mathcal{Y}_{S'}^{\text{FF}} \rightarrow \mathcal{Y}_S^{\text{FF}}$  be the corresponding map of  $\mathcal{Y}^{\text{FF}}$ -curves.*

- (1) *We have equivalences of sites  $\mathcal{Y}_{S,\text{ét}}^{\text{FF}} \cong \mathcal{Y}_{S,\text{ét}}^{\text{FF},\diamond}$  and  $\mathcal{Y}_{S,\text{fét}}^{\text{FF}} \cong \mathcal{Y}_{S,\text{fét}}^{\text{FF},\diamond}$ .*
- (2)  *$f^{\text{FF}}$  is an open immersion if and only if  $f$  is so. If  $f$  is a rational localization then so is  $f^{\text{FF}}$ .*
- (3) *For  $I \subset [0, \infty)$  a closed or open interval, the map  $f$  induces a cartesian square*

$$\begin{array}{ccc} \mathcal{Y}_{S',I}^{\text{FF}} & \longrightarrow & \mathcal{Y}_{S,I}^{\text{FF}} \\ \downarrow & & \downarrow \\ \mathcal{Y}_{S'}^{\text{FF}} & \longrightarrow & \mathcal{Y}_S^{\text{FF}}. \end{array}$$

*Proof.* Part (1) is a particular case of [Sch22, Lemma 15.6]. The first assertion of (2) follows from (1) and the fact that open immersions of analytic adic spaces are the same as étale maps  $f : Y \rightarrow X$  such that the diagonal  $Y \rightarrow Y \times_X Y$  is an equivalence. The second assertion of (2) follows from the fact that if  $S' = S \left( \frac{f_1, \dots, f_n}{g} \right)$  then

$$\mathcal{Y}_{S'}^{\text{FF}} = \mathcal{Y}_S^{\text{FF}} \left( \frac{[f_1], \dots, [f_n]}{[g]} \right).$$

For part (3) it suffices to deal with the case where  $I$  is closed, then it follows from the definition of  $\mathcal{Y}_{S,I}^{\text{FF}}$  as it is a rational localization involving only  $p$  and  $[\varpi]$ .  $\square$

Some of the main players in this paper are the sheaves of periods defined by the affinoid subspaces  $\mathcal{Y}_{S,I}^{\text{FF}}$  for  $I$  closed.

**Definition 2.1.3.** Let  $I \subset [0, \infty)$  be a compact interval. We define the  $v$ -sheaves  $\mathbb{B}_I$  and  $\mathbb{B}_I^+$  on  $\text{Perf}_\varpi$  to be the  $v$ -sheafification of the presheaves mapping an affinoid perfectoid  $S = \text{Spa}(R, R^+) \in \text{Perf}_\varpi$  to the rings

$$\begin{aligned}\mathbb{B}_I^+(S) &= \mathbb{B}_I^+(R, R^+) := \mathcal{O}^+(\mathcal{Y}_{S,I}^{\text{FF}}), \\ \mathbb{B}_I(S) &= \mathbb{B}_I(R, R^+) := \mathcal{O}(\mathcal{Y}_{S,I}^{\text{FF}}).\end{aligned}$$

We also define the  $v$ -sheaf  $\mathbb{A}_{\text{inf}}$  as the sheaf mapping  $S$  to  $\mathbb{A}_{\text{inf}}(S)$ .

In the following lemma we consider almost mathematics with respect to the  $(p, [\varpi])$ -completion of the  $\mathbb{A}_{\text{inf}}$ -ideal generated by  $([\varpi]^{1/p^n})_{n \in \mathbb{N}}$ .

**Lemma 2.1.4.** *Let  $S = \text{Spa}(R, R^+) \in \text{Perf}_\varpi$  be affinoid perfectoid.*

(1) *We have an almost equivalence of derived  $(p, [\varpi])$ -adically complete complexes*

$$\mathbb{A}_{\text{inf}}(R^+) =^a R\Gamma_v(S, \mathbb{A}_{\text{inf}}).$$

(2) *Let  $I = [0, r]$ , then we have an isomorphism of  $v$ -sheaves*

$$\mathbb{B}_I^+([\varpi^{1/r}]) = \widehat{\mathcal{O}}^+ / (\varpi^{1/r})[T]$$

*where the RHS term is a polynomial algebra with  $T$  the residue class of  $p/[\varpi]^{1/r}$ .*

(3) *Suppose that  $I = [0, r]$ , we have a natural almost equivalence of derived  $[\varpi]$ -adically complete complexes*

$$\mathbb{B}_I^+(S) =^a R\Gamma_v(S, \mathbb{B}_I^+).$$

(4) *For  $I \subset [0, \infty)$  a compact interval we have a natural equivalence*

$$\mathbb{B}_I(S) = R\Gamma_v(S, \mathbb{B}_I)$$

*Proof.* By [Sch22, Proposition 8.8] we have a natural almost equivalence  $R^+ =^a R\Gamma_v(S, \mathcal{O}^+)$  and so an almost equivalence modulo any pseudo-uniformizer. Since  $\mathbb{A}_{\text{inf}}/(p, [\varpi]) =^a \widehat{\mathcal{O}}^+ / \varpi$  as almost  $v$ -sheaves, by derived Nakayama's lemma we have an almost equivalence of derived  $(p, [\varpi])$ -complexes

$$\mathbb{A}_{\text{inf}}(R^+) =^a R\Gamma_v(S, \mathbb{A}_{\text{inf}})$$

proving (1).

Suppose that  $I = [0, r]$ , we have a short exact sequence of  $(p, [\varpi])$ -adically complete  $v$ -sheaves

$$0 \rightarrow \mathbb{A}_{\text{inf}}\langle T \rangle \xrightarrow{[\varpi]^{1/r}T - p} \mathbb{A}_{\text{inf}}\langle T \rangle \rightarrow \mathbb{B}_I^+ \rightarrow 0 \quad (2.1)$$

where  $\mathbb{A}_{\text{inf}}\langle T \rangle$  is the  $(p, [\varpi])$ -adic completion of the polynomial algebra over  $\mathbb{A}_{\text{inf}}$ . Indeed, this follows from the fact that one has the presentation for the ring  $\mathbb{B}_I^+(S)$  for any affinoid perfectoid  $S = \text{Spa}(R, R^+)$ :

$$\mathbb{B}_I^+(S) = \left\{ \sum_{n \in \mathbb{N}} [a_n] \frac{p^n}{[\varpi]^{n/r}} : a_n \in R^+ \text{ and } |a_n| \rightarrow 0 \text{ as } n \rightarrow \infty \right\}, \quad (2.2)$$

endowed with the  $[\varpi]$ -adic topology, see [SW20, proof of Proposition 11.2.1]. The equation (2.2) shows that  $[\varpi]^{1/r}$  is a regular element of  $\mathbb{B}_I^+$ , and taking quotients in (2.1) by  $[\varpi]^{1/r}$  yields an isomorphism of sheaves

$$\mathbb{B}_{[0,r]}^+ / ([\varpi]^{1/r}) = \widehat{\mathcal{O}}^+ / \varpi^{1/r}[T]$$

proving (2).

Then, part (2), the almost acyclicity of  $\widehat{\mathcal{O}}^+$  and derived Nakayama's lemma imply that

$$\mathbb{B}_I^+(S) =^a R\Gamma_v(S, \mathbb{B}_I^+)$$

proving (3).

Finally, part (4) for  $I = [0, r]$  follows from (3) by inverting pseudo-uniformizers. Moreover, as  $\mathbb{B}_{[0,r]}^+\langle T \rangle$  is the  $\varpi$ -adic completion of  $\mathbb{B}_{[0,r]}^+[T]$ , part (3) also implies that

$$\mathbb{B}_{[0,r]}^+(S)\langle T \rangle =^a R\Gamma_v(S, \mathbb{B}_{[0,r]}^+\langle T \rangle)$$

and  $\mathbb{B}_{[0,r]}(S)\langle T \rangle = R\Gamma_v(S, \mathbb{B}_{[0,r]}\langle T \rangle)$ . Let us now consider  $I = [s, r]$  with  $s > 0$ . By [KHH<sup>+</sup>19, Lemma 1.8.2], for all affinoid perfectoid  $\mathrm{Spa}(A, A^+) \in \mathrm{Perf}_\varpi$  we have a short exact sequence

$$0 \rightarrow \mathbb{B}_{[0,r]}(A, A^+)\langle T \rangle \xrightarrow{pT - [\varpi]^{1/s}} \mathbb{B}_{[0,r]}(A, A^+)\langle T \rangle \rightarrow \mathbb{B}_{[r,s]}(A, A^+) \rightarrow 0.$$

This gives rise to a short exact sequence of  $v$ -sheaves

$$0 \rightarrow \mathbb{B}_{[0,r]}\langle T \rangle \xrightarrow{pT - [\varpi]^{1/s}} \mathbb{B}_{[0,r]}\langle T \rangle \rightarrow \mathbb{B}_{[s,r]} \rightarrow 0. \quad (2.3)$$

Part (4) follows from part (3) after taking  $v$ -cohomology of (2.3).  $\square$

**Lemma 2.1.5.** *Let  $X$  be a locally spatial diamond over an affinoid perfectoid space  $S$  in characteristic  $p$ . Let  $I \subset [0, \infty)$  be a compact interval and  $\varpi$  a fixed pseudo-uniformizer of  $S$ . Then  $\mathbb{B}_I^+ / [\varpi]$  arises from an étale sheaf of  $X$  via the fully faithful embedding  $\mathrm{Sh}(X_{\text{ét}}, \Lambda) \rightarrow \mathrm{Sh}(X_v, \Lambda)$  of [Sch22, Proposition 14.10] with  $\Lambda = \mathbb{B}_I^+(S) / [\varpi]$ .*

*Proof.* This follows from [Sch22, Theorem 14.12] since  $\Lambda = \mathbb{B}_I^+ / [\varpi]$  is clearly an étale sheaf on perfectoid spaces. Indeed, it suffices to show that  $\mathbb{B}_I^+ / [\varpi]^b$  is an étale sheaf for a suitable  $b$ . If  $I = [0, r]$  this follows from Lemma 2.1.4 (2). For  $I = [s, r]$  with  $s > 0$  consider the short exact sequence of  $v$ -sheaves

$$0 \rightarrow \mathbb{B}_{[0,r]}\langle T \rangle \xrightarrow{pT - [\varpi]^{1/s}} \mathbb{B}_{[0,r]}\langle T \rangle \rightarrow \mathbb{B}_{[s,r]} \rightarrow 0.$$

Then,  $\mathbb{B}_{[s,r]}^+ / [\varpi]$  is a subquotient of  $\mathbb{B}_{[0,r]}\langle T \rangle / [\varpi] \mathbb{B}_{[0,r]}^+ \langle T \rangle$  which is étale by the previous case, proving that  $\mathbb{B}_{[s,r]}^+ / [\varpi]$  is étale itself.  $\square$

**2.2. Solid almost quasi-coherent sheaves.** In this paper we shall work with cohomologies of Banach sheaves on locally spatial diamonds such as  $\mathbb{B}_I$ . However, the sheaves we shall consider are not arbitrary; they are actually solid quasi-coherent sheaves over the  $\mathcal{Y}^{FF}$ -curve in the sense of [AMLB]. This promotion to solid sheaves helps to naturally endow their  $v$ -cohomologies with the structure of solid abelian groups as in [AM24, §4]. Since the sheaves we shall consider will be generic fibers of completed sheaves, it will be enough to use the formalism of solid almost quasi-coherent sheaves with torsion coefficients of [Man22b, §3] that we briefly recall in this section.

Let  $\mathrm{Perf}_\varpi$  be the category of perfectoids in characteristic  $p$  with fixed pseudo-uniformizer  $\varpi$ . Let  $I = [0, r] \subset [0, \infty)$  be a compact interval and  $b > 0$  a positive rational number. Consider the sheaf of coefficients  $\mathbb{B}_{I,b}^+ := \mathbb{B}_I^+ / ([\varpi]^b)$  on  $\mathrm{Perf}_\varpi$  with almost structure generated by  $([\varpi]^{1/p^k})_{k \in \mathbb{N}}$ .

**Definition 2.2.1.** Let  $X$  be a small  $v$ -stack with fixed pseudo-uniformizer  $\varpi$ . The  $\infty$ -category of solid almost quasi-coherent  $\mathbb{B}_{I,b}^+$ -modules  $\mathcal{D}_\square^a(X, \mathbb{B}_{I,b}^+)$  is the hypercompletion of the functor mapping an affinoid perfectoid  $\mathrm{Spa}(R, R^+) \in X_v$  to the almost category of solid  $\mathbb{B}_{I,b}^+(R, R^+)$ -modules  $\mathcal{D}^a(\mathbb{B}_{I,b}^+(R, R^+)_{\square})$ .

The category of almost solid modules satisfies strong descent properties:

**Proposition 2.2.2.** *Let  $X = \mathrm{Spa}(R, R^+)$  be a totally disconnected perfectoid space. Then the natural map*

$$\mathcal{D}^a(\mathbb{B}_{I,b}^+(R, R^+)_{\square}) \rightarrow \mathcal{D}_\square^a(X, \mathbb{B}_{I,b}^+)$$

*is an equivalence of  $\infty$ -categories.*

*Proof.* This follows essentially from [Man22b, Theorem 3.1.27]. Indeed, let  $Y_\bullet \rightarrow X$  be an hypercover of  $X$  by totally disconnected perfectoid spaces, we want to show that the natural map

$$\mathcal{D}^a(\mathbb{B}_{I,b}^+(R, R^+)_{\square}) \rightarrow \varprojlim_{[n] \in \Delta} \mathcal{D}^a(\mathbb{B}_{I,b}^+(Y_n)_{\square})$$

is an equivalence. Concretely, this amounts to show the following:

i. For  $M \in \mathcal{D}^a(\mathbb{B}_{I,b}^+(R, R^+)_{\square})$  the natural map

$$M \rightarrow \varprojlim_{[n] \in \Delta} (M \otimes_{\mathbb{B}_{I,b}^+(R, R^+)_{\square}}^L \mathbb{B}_{I,b}^+(Y_n)_{\square})$$

is an equivalence.

ii. For  $(M_n)_{[n] \in \Delta}$  a cocartesian section of  $\varprojlim_{[n] \in \Delta} \mathcal{D}^a(\mathbb{B}_{I,b}^+(Y_n)_\square)$  with totalization  $M$  the natural map

$$M \otimes_{\mathbb{B}_{I,b}^+(R,R^+)_\square}^L \mathbb{B}_{I,b}^+(Y_n)_\square \rightarrow M_n$$

is an equivalence for all  $n \in \mathbb{N}$ .

Suppose first that  $b = 1/r$ , then by Lemma 2.1.4 (3) we have that  $\mathbb{B}_{I,r}^+ = \mathcal{O}^+/\varpi^{1/r}[T]$  is a polynomial algebra over  $\mathcal{O}^+/\varpi^{1/r}$  where  $T$  is the class of  $p/[\varpi]^{1/r}$ . By [Man22b, Theorem 3.1.27] we have an equivalence of categories

$$\mathcal{D}^a(R_\square^+/\varpi^{1/r}) \xrightarrow{\sim} \varprojlim_{[n] \in \Delta} \mathcal{D}^a(\mathcal{O}^+(Y_n)_\square/\varpi^{1/r}). \quad (2.4)$$

Consider the maps of augmented cosimplicial diagrams of analytic rings  $(\mathcal{O}^+(Y_n)_\square/\varpi^{1/r})_{[n] \in \Delta_+} \rightarrow (\mathbb{B}_{I,b}^+(Y_n)_\square)_{[n] \in \Delta_+}$  with  $Y_{-1} = X$ . For any map  $\alpha : [n] \rightarrow [m]$  consider the commutative square provide by base change

$$\begin{array}{ccc} \mathcal{D}^a((\mathcal{O}^+(Y_n)_\square/\varpi^{1/r})) & \xrightarrow{f_\alpha^*} & \mathcal{D}^a((\mathcal{O}^+(Y_m)_\square/\varpi^{1/r})) \\ \downarrow h_n^* & & \downarrow h_m^* \\ \mathcal{D}^a(\mathbb{B}_{I,b}^+(Y_n)_\square) & \xrightarrow{g_\alpha^*} & \mathcal{D}^a(\mathbb{B}_{I,b}^+(Y_m)_\square). \end{array}$$

Let  $h_{n,*}$  be the right adjoint of  $h_n^*$  given by the forgetful functor. Since any map of discrete Huber pairs is steady [Man22b, Proposition 2.9.7 (ii)], the natural transformations  $f_\alpha^* h_{n,*} \xrightarrow{\sim} h_{m,*} g_\alpha^*$  of functors  $\mathcal{D}^a(\mathbb{B}_{I,b}^+(Y_n)_\square) \rightarrow \mathcal{D}^a((\mathcal{O}^+(Y_m)_\square/\varpi^{1/r}))$  is an equivalence. Therefore, the forgetful functors  $h_{n,*}$  preserve cocartesian sections and induce a functor

$$(h_{n,*})_{[n]} : \varprojlim_{[n] \in \Delta} \mathcal{D}^a(\mathbb{B}_{I,b}^+(Y_n)_\square) \rightarrow \varprojlim_{[n] \in \Delta} \mathcal{D}^a(\mathcal{O}^+(Y_n)_\square/\varpi^{1/r}) \cong \mathcal{D}^a(R^+/\varpi^{1/r})$$

which is the right adjoint of the natural base change along  $(\mathcal{O}^+(Y_n)_\square/\varpi^{1/r})_{[n] \in \Delta} \rightarrow (\mathbb{B}_{I,b}^+(Y_n)_\square)_{[n] \in \Delta}$ , and that fits in a commutative square

$$\begin{array}{ccc} \mathcal{D}^a(\mathbb{B}_{I,b}^+(R,R^+)_\square) & \longrightarrow & \varprojlim_{[n] \in \Delta} \mathcal{D}^a(\mathbb{B}_{I,b}^+(Y_n)_\square) \\ \downarrow h_* & & \downarrow (h_{n,*})_{[n]} \\ \mathcal{D}^a(R^+/\varpi^{1/r}) & \xrightarrow{\sim} & \varprojlim_{[n] \in \Delta} \mathcal{D}^a(\mathcal{O}^+(Y_n)_\square/\varpi^{1/r}). \end{array}$$

Therefore, since the functors  $h_{n,*}$  are conservative, in order to show (i) or (ii) we can apply  $(h_{n,*})_{[n]}$  where the claim follows from (2.4).

The case of general  $b$  follows from derived Nakayama's lemma: for either (i) or (ii) above we have to show that a map of solid  $\mathbb{B}_{I,b}^+(R,R^+)$ -modules  $N \rightarrow N'$  is an equivalence. For this, it suffices to check that it is an equivalence after taking derived quotients by  $[\varpi]^{1/r}$  where it was already proven.  $\square$

Finally, we recall how smooth representation theory appears in terms of solid almost quasi-coherent sheaves.

**Proposition 2.2.3.** *Let  $X = \mathrm{Spa}(R, R^+)$  be a totally disconnected perfectoid space with pseudo-uniformizer  $\varpi$ . Let  $\Pi$  be a locally profinite group acting on  $X$  and consider the  $v$ -stack  $X/\Pi$ . Then the pullback along the map  $f : X \rightarrow X/\Pi$  realizes  $\mathcal{D}_\square^a(X/\Pi, \mathbb{B}_{I,r}^+)$  as the derived  $\infty$ -category of semilinear smooth almost representations  $\mathrm{Rep}_{\mathbb{B}_{I,b}^+(R,R^+)_\square}^{sm,a}(\Pi)$  of  $\Pi$  (denoted as  $\mathcal{D}^{sm,a}(\mathbb{B}_{I,b}^+(R,R^+)_\square, \Pi)$  in [Man22b, Definition 3.4.11]).*

*Proof.* This follows from the same argument of [Man22b, Lemma 3.4.26].  $\square$

*Remark 2.2.4.* By construction the  $\infty$ -category  $\mathrm{Rep}_{\mathbb{B}_{I,b}^+(R,R^+)_\square}^{sm,a}(\Pi)$  is the derived category of its heart  $\mathrm{Rep}_{\mathbb{B}_{I,b}^+(R,R^+)_\square}^{sm,a,\heartsuit}(\Pi)$ . There is an obvious forgetful functor

$$\mathrm{Rep}_{\mathbb{B}_{I,b}^+(R,R^+)_\square}^{sm,a,\heartsuit}(\Pi) \rightarrow \mathrm{Mod}^a(\mathbb{B}_{I,b}^+(R,R^+)_\square[\Pi])$$

to the category of almost solid modules over the semilinear solid group algebra  $\mathbb{B}_{I,b}^+(R, R^+)_{\square}[\Pi]$ . This gives rise a map of derived  $\infty$ -categories

$$\mathrm{Rep}_{\mathbb{B}_{I,b}^+(R, R^+)_{\square}}^{sm,a}(\Pi) \rightarrow \mathcal{D}^a(\mathbb{B}_{I,b}^+(R, R^+)_{\square}[\Pi]).$$

This map is not in general fully faithful, see Remark [Man22b, 3.4.18] for a counter example when  $\Pi = \prod_{\mathbb{N}} \mathbb{F}_p$ . However, the fully faithfulness is expected when  $\Pi$  has a basis of compact open subgroups with uniformly bounded finite  $p$ -cohomological dimension, eg. when  $\Pi$  is a  $p$ -adic Lie group.

**2.3. The décalage functor.** In the next section we recall some facts about the décalage functor  $L\eta_{\mathcal{I}}$  of [BMS18]. For us it will suffice to consider this functor at the level of the homotopy category of the  $\infty$ -derived category of modules of an algebra in a topos as in *loc. cit.*

Let  $(T, \mathcal{O}_T)$  be a ringed topos. Let  $K(\mathcal{O}_T)$  be the category of complexes of  $\mathcal{O}_T$ -modules up to homotopy, and  $D(\mathcal{O}_T)$  the derived category of  $\mathcal{O}_T$ -modules obtained by inverting quasi-isomorphisms in  $K(\mathcal{O}_T)$ .

Let  $\mathcal{I} \subset \mathcal{O}_T$  be an invertible ideal, and let  $K^{\mathcal{I}\text{-free}}(\mathcal{O}_T)$  denote the full subcategory of  $K(\mathcal{O}_T)$  whose objects are  $\mathcal{I}$ -torsion free complexes. By [BMS18, Lemma 6.1] the (non  $\infty$ !) derived category  $D(\mathcal{O}_T)$  is the localization of  $K^{\mathcal{I}\text{-free}}(\mathcal{O}_T)$  along quasi-isomorphisms.

**Definition 2.3.1** ([BMS18, Definition 6.2]). Let  $C^{\bullet} \in K^{\mathcal{I}\text{-free}}(\mathcal{O}_T)$ . Define a new object  $\eta_{\mathcal{I}}C^{\bullet} = (\eta_{\mathcal{I}}C)^{\bullet} \in K^{\mathcal{I}\text{-free}}(\mathcal{O}_T)$  with terms

$$(\eta_{\mathcal{I}}C)^i = \{x \in C^i \mid dx \in \mathcal{I}C^{i+1}\} \otimes_{\mathcal{O}_T} \mathcal{I}^{\otimes i},$$

and differentials

$$d_{\eta_{\mathcal{I}}C, i} : (\eta_{\mathcal{I}}C)^i \rightarrow (\eta_{\mathcal{I}}C)^{i+1}$$

making the following diagram commute

$$\begin{array}{ccc} (\eta_{\mathcal{I}}C)^i & \xrightarrow{d_{C^i \otimes \mathcal{I}^{\otimes i}}} & \mathcal{I}C^{i+1} \otimes \mathcal{I}^{\otimes i} \\ d_{(\eta_{\mathcal{I}}C)^i} \downarrow & & \downarrow \simeq \\ (\eta_{\mathcal{I}}C)^{i+1} & \longrightarrow & C^{i+1} \otimes \mathcal{I}^{\otimes i+1}. \end{array}$$

By [BMS18, Corollary 6.5] the operator  $L\eta_{\mathcal{I}}$  preserves quasi-isomorphisms and extends to a filtered colimit preserving functor

$$L\eta_{\mathcal{I}} : D(\mathcal{O}_T) \rightarrow D(\mathcal{O}_T).$$

Moreover, the following properties hold:

- For  $C \in D(\mathcal{O}_T)$  there are natural isomorphisms [BMS18, Lemma 6.4]

$$H^i(L\eta_{\mathcal{I}}C) \cong H^i(C)/H^i(C)[\mathcal{I}] \otimes_{\mathcal{O}_T} \mathcal{I}^i.$$

- $L\eta_{\mathcal{I}}$  is lax symmetric monoidal, i.e. for  $C, D \in D(\mathcal{O}_T)$  there is a natural map

$$L\eta_{\mathcal{I}}C \otimes_{\mathcal{O}_T}^L L\eta_{\mathcal{I}}D \rightarrow L\eta_{\mathcal{I}}(C \otimes_{\mathcal{O}_T}^L D)$$

functorial in  $C$  and  $D$ , and symmetric in  $C$  and  $D$  [BMS18, Lemma 6.7].

- Suppose that the topos is replete. Let  $C \in D(\mathcal{O}_T)$ , then the natural maps

$$(L\eta_{\mathcal{I}}C)^{\wedge \mathcal{I}} \rightarrow L\eta_{\mathcal{I}}(C^{\wedge \mathcal{I}}) \rightarrow R\varprojlim_n (L\eta_{\mathcal{I}}(C \otimes_{\mathcal{O}_T}^L \mathcal{O}_T/\mathcal{I}^n))$$

are equivalences [BMS18, Lemma 6.20]. Here for an object  $M \in D(\mathcal{O}_T)$  we let  $M^{\wedge \mathcal{I}} = R\varprojlim_n (M \otimes_{\mathcal{O}_T}^L \mathcal{O}_T/\mathcal{I}^n)$  be the derived  $\mathcal{I}$ -adic completion.

*Remark 2.3.2.* The functor  $L\eta_{\mathcal{I}}$  preserves filtered colimits but is not exact, i.e. it does not preserve cones. For example, we have the short exact sequence

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_T/\mathcal{I}^2 \rightarrow \mathcal{O}_T/\mathcal{I} \rightarrow 0$$

but we also have that

$$L\eta_{\mathcal{I}}(\mathcal{I}/\mathcal{I}^2) = L\eta_{\mathcal{I}}(\mathcal{O}_T/\mathcal{I}) = 0$$

and

$$L\eta_{\mathcal{I}}(\mathcal{O}_T/\mathcal{I}^2) = \mathcal{O}_T/\mathcal{I}.$$

We shall need the following behavior of the décalage operator with respect to the passage to the special fiber.

**Lemma 2.3.3.** *Let  $C \in D(\mathcal{O}_T)$  and  $k \geq 1$ , then the natural map*

$$L\eta_{\mathcal{I}}C \otimes_{\mathcal{O}_T}^L \mathcal{O}_T/\mathcal{I}^k = L\eta_{\mathcal{I}}C \otimes_{\mathcal{O}_T}^L L\eta_{\mathcal{I}}(\mathcal{O}_T/\mathcal{I}^{k+1}) \xrightarrow{\sim} L\eta_{\mathcal{I}}(C \otimes_{\mathcal{O}_T}^L \mathcal{O}_T/\mathcal{I}^{k+1}) \quad (2.5)$$

is an equivalence.

*Proof.* Let  $C^\bullet$  be an  $\mathcal{I}$ -torsion free complex representing  $C$ . Then,  $C \otimes_{\mathcal{O}_T}^L \mathcal{O}_T/\mathcal{I}^{k+1}$  is represented by the complex  $D^\bullet$  with terms

$$D^i = C^i \oplus C^{i+1} \otimes_{\mathcal{O}_T} \mathcal{I}^{k+1}$$

and differentials

$$d_{D^i} = \begin{pmatrix} d_{C^i} & \text{id}_{C^{i+1}} \otimes \iota \\ 0 & (-1)^i d_{C^{i+1}} \otimes \mathcal{I}^{k+1} \end{pmatrix}$$

where  $\iota : \mathcal{I}^{k+1} \rightarrow \mathcal{O}_T$ . Therefore,  $L\eta_{\mathcal{I}}(C \otimes_{\mathcal{O}_T}^L \mathcal{O}_T/\mathcal{I}^{k+1})$  is represented by the complex  $\tilde{D}^\bullet$  with

$$\tilde{D}^i = \{(a, b) \in D^i \mid d_{D^i}(a, b) \in \mathcal{I}D^{i+1}\} \otimes_{\mathcal{O}_T} \mathcal{I}^i.$$

More explicitly, we have

$$d_{D^i}(a, b) = (d_{C^i}(a) + b, (-1)^i d_{C^{i+1}}(b)).$$

Since  $b \in C^{i+1} \otimes \mathcal{I}^{k+1} \subset \mathcal{I}C^{i+1}$ , one deduces that

$$\begin{aligned} \tilde{D}^i &= \{(a, b) \in C^i \oplus C^{i+1} \otimes_{\mathcal{O}_T} \mathcal{I}^{k+1} \mid d_{C^i}(a) \in \mathcal{I}C^{i+1} \text{ and } d_{C^{i+1}}(b) \in \mathcal{I}C^{i+1} \otimes_{\mathcal{O}_T} \mathcal{I}^{k+1}\} \otimes \mathcal{I}^i \\ &= \{a \in C^i \mid d_{C^i}(a) \in \mathcal{I}C^{i+1}\} \otimes \mathcal{I}^i \oplus \{b \in C^{i+1} \otimes_{\mathcal{O}_T} \mathcal{I}^k \mid d_{C^{i+1}}(b) \in \mathcal{I}C^{i+1} \otimes_{\mathcal{O}_T} \mathcal{I}^k\} \otimes_{\mathcal{O}_T} \mathcal{I}^{i+1} \\ &= (\eta_{\mathcal{I}}C^\bullet)^i \oplus (\eta_{\mathcal{I}}C^\bullet)^{i+1} \otimes_{\mathcal{O}_T} \mathcal{I}^k. \end{aligned}$$

It is straightforward to check that the differentials of  $\tilde{D}^\bullet$  are those arising from the cone of  $\eta_{\mathcal{I}}C^\bullet \otimes_{\mathcal{O}_T} \mathcal{I}^k \rightarrow \eta_{\mathcal{I}}C^\bullet$ , and that the resulting quasi-isomorphism

$$L\eta_{\mathcal{I}}C \otimes_{\mathcal{O}_T}^L \mathcal{O}_T/\mathcal{I}^k \cong L\eta_{\mathcal{I}}(C \otimes_{\mathcal{O}_T}^L \mathcal{O}_T/\mathcal{I}^{k+1})$$

is the one induced by the map (2.5).  $\square$

**2.4. Solid locally analytic representations.** Throughout this paper we will use the theory of solid locally analytic representations of [RJRC22, RJRC23]. In this section we briefly recall some of the main definitions and properties that will be needed later.

Let  $\text{Solid}$  be the abelian category of solid abelian groups and let  $\otimes_{\square}$  be its solid tensor product. for a ring  $R \in \text{Solid}$  we let  $\text{Solid}_R$  be the abelian category of solid  $R$ -modules. We write  $\mathcal{D}(R)$  for the derived  $\infty$ -category of  $\text{Solid}_R$ . Let  $G$  be a compact  $p$ -adic Lie group and let  $\mathbb{Z}_{p, \square}[G] = \varprojlim_H \mathbb{Z}_p[G/H]$  be the free solid  $\mathbb{Z}_p$ -algebra generated by  $G$ ; it coincides with the Iwasawa algebra of  $G$  with coefficients in  $\mathbb{Z}_p$ . We set  $\mathbb{Q}_{p, \square}[G] = \mathbb{Z}_{p, \square}[G][\frac{1}{p}]$ . The group  $G$  has a space  $C^{la}(G, \mathbb{Q}_p)$  of locally analytic functions, it can be written as the filtered colimit

$$C^{la}(G, \mathbb{Q}_p) = \varinjlim_{h \rightarrow \infty} C^h(G, \mathbb{Q}_p)$$

where  $C^h(G, \mathbb{Q}_p) = \mathcal{O}(\mathbb{G}^{(h)})$  is the affinoid algebra of a decreasing sequence of affinoid groups over  $\mathbb{Q}_p$

$$G \subset \dots \subset \mathbb{G}^{(h+1)} \subset \mathbb{G}^{(h)} \subset \dots$$

with  $\varprojlim_h \mathbb{G}^{(h)} = G$ .

Given  $C \in \mathcal{D}(\mathbb{Q}_{p, \square}[G])$  a derived solid  $G$ -representation, its (derived) locally analytic vectors [RJRC23, Definition 3.1.4] is the solid  $G$ -representation

$$C^{RG-la} := R\Gamma(G, C \otimes_{\mathbb{Q}_{p, \square}}^L C^{la}(G, \mathbb{Q}_p)_{\star_1})$$

where

- $C^{la}(G, \mathbb{Q}_p)_{\star_1}$  is endowed with the left regular  $G$ -action.
- The tensor  $C \otimes_{\mathbb{Q}_{p, \square}}^L C^{la}(G, \mathbb{Q}_p)_{\star_1}$  is endowed with the diagonal  $G$ -action.
- The  $G$ -action on  $C^{RG-la}$  arises from the right regular action on  $C^{la}(G, \mathbb{Q}_p)$ .



We say that  $C$  is (derived) locally analytic if the natural map

$$C^{RG-la} \rightarrow C$$

is an equivalence. We let  $\text{Rep}_{\mathbb{Q}_p, \square}^{la}(G) \subset \mathcal{D}(\mathbb{Q}_p, \square[G])$  be the full subcategory of locally analytic representations. This category enjoys the following properties:

- $\text{Rep}_{\mathbb{Q}_p, \square}^{la}(G)$  is stable under colimits and  $\mathbb{Q}_p, \square$ -linear tensor products in  $\mathcal{D}(\mathbb{Q}_p, \square[G])$  [RJRC23, Proposition 3.2.3].
- The  $t$ -structure of  $\mathcal{D}(\mathbb{Q}_p, \square[G])$  induces a  $t$ -structure on  $\text{Rep}_{\mathbb{Q}_p, \square}^{la}(G)$  [RJRC23, Proposition 3.2.5]. Moreover,  $\text{Rep}_{\mathbb{Q}_p, \square}^{la}(G)$  is the derived category of its heart [RJRC23, Proposition 3.2.6].
- The functor  $C \mapsto C^{RG-la}$  is the right adjoint of the inclusion  $\text{Rep}_{\mathbb{Q}_p, \square}^{la}(G) \subset \mathcal{D}(\mathbb{Q}_p, \square[G])$  [RJRC23, Corollary 3.2.7].

The key lemma that we will use in this paper is the following criterion of locally analyticity:

**Lemma 2.4.1.** *Let  $C \in \mathcal{D}_{\geq 0}(\mathbb{Z}_p, \square[G])$  be a connective derived  $p$ -adically complete solid representation of  $G$ . Suppose that there is an open compact subgroup  $G_0 \subset G$ , and a finite extension  $\mathcal{O}_K/\mathbb{Z}_p$  with pseudo-uniformizer  $\pi$ , such that for all  $g \in G_0$  the map*

$$1 - g : V \otimes_{\mathbb{Z}_p, \square}^L \mathcal{O}_K/\pi \rightarrow V \otimes_{\mathbb{Z}_p, \square}^L \mathcal{O}_K/\pi$$

*is homotopic to zero as  $\mathcal{O}_K/\pi$ -module. Then  $V[\frac{1}{p}]$  is a locally analytic representation of  $G$ .*

*Proof.* We can assume without loss of generality that  $G$  is a uniform pro- $p$ -group. By [RJRC23, Proposition 3.3.2] to show that  $V[\frac{1}{p}]$  is  $G$ -locally analytic, it suffices to show that for all  $g \in G$  it is  $\Gamma_g = g^{\mathbb{Z}_p}$ -locally analytic. Then, we can assume that  $G \cong \mathbb{Z}_p$ . The lemma follows from the same argument of [RJRC23, Proposition 3.3.3] applied to  $V \otimes_{\mathbb{Z}_p} \mathcal{O}_K$  and  $\pi$  instead of  $V$  and  $p$  respectively.  $\square$

With this criteria one can show that actions of  $p$ -adic Lie groups on rigid spaces are always locally analytic:

**Corollary 2.4.2.** *Let  $K$  be a complete non-archimedean field of characteristic zero. Let  $A$  be a Tate algebra of finite type over  $K$  and  $G$  a compact  $p$ -adic Lie group acting continuously on  $A$ . Then  $A$  is a locally analytic representation of  $G$ .*

*Proof.* Let  $A_0 \subset A$  be a ring of definition of  $A$ , we can suppose without loss of generality that  $A_0$  is stable under the action of  $G$  and that  $A_0$  is topologically generated over  $\mathcal{O}_K$  by finitely many variables  $T_1, \dots, T_n$ . Thus, since the action of  $G$  on  $A_0/p$  is smooth, there is some open subgroup  $G_0 \subset G$  leaving the variables  $T_i$  fixed. But then  $G_0$  acts trivially on  $A_0/p$  and by Lemma 2.4.1  $A$  is a locally analytic representation of  $G$ .  $\square$

**2.5. Equivariant sheaves over flag varieties.** Let  $\mathbf{G}$  be a reductive group over  $\mathbb{Q}_p$  and let  $\mu$  be a conjugacy class of cocharacters of  $G$  with field of definition  $E$ . We denote by  $\text{FL}_{\mathbf{G}, \mu, E}$  the flag variety over  $E$  parametrizing decreasing  $\mu$ -filtrations on  $\mathbf{G}$ -representations seen as an algebraic variety, we let  $\mathcal{F}\ell_{\mathbf{G}, \mu, E}$  denote its analytification as an adic space over  $\text{Spa}(E, \mathcal{O}_E)$  as in [Hub96]. Note that  $\text{FL}_{\mathbf{G}, \mu^{-1}, E}$  is also the flag variety parametrizing increasing  $\mu$ -filtrations.

Let  $C/E$  be a complete algebraically closed non-archimedean field and let us write  $\text{FL}_{\mathbf{G}, \mu}$  and  $\mathcal{F}\ell_{\mathbf{G}, \mu}$  for the base change of the flag varieties to  $C$ . We fix a cocharacter  $\mu : \mathbb{G}_m \rightarrow \mathbf{G}_C$  so that  $\text{FL}_{\mathbf{G}, \mu} \cong \mathbf{G}_C/\mathbf{P}_\mu$  where  $\mathbf{P}_\mu \subset \mathbf{G}_C$  is the parabolic subgroup parametrizing decreasing  $\mu$ -filtrations. We let  $\mathbf{N}_\mu \subset \mathbf{P}_\mu$  be its unipotent radical and let  $\mathbf{M}_\mu$  be the Levi subgroup, i.e. the centralizer of  $\mu$  in  $\mathbf{G}_C$ . We have a semi-direct product decomposition  $\mathbf{P}_\mu = \mathbf{N}_\mu \rtimes \mathbf{M}_\mu$ .

Set  $* = \text{Spec } C$ . We have an isomorphism of Artin stacks

$$[1] : */\mathbf{P}_\mu \xrightarrow{\sim} \mathbf{G}_C \backslash (\mathbf{G}_C/\mathbf{P}_\mu) = \mathbf{G}_C \backslash \text{FL}_{\mathbf{G}, \mu}.$$

Therefore, pullback along [1] gives rise to an equivalence of quasi-coherent sheaves on the stacks. The previous translates in the classical equivalence of representation categories:

$$[1]^* : \mathbf{G} - \text{QCoh}(\text{FL}_{\mathbf{G}, \mu}) \xrightarrow{\sim} \text{Rep}_C^{\text{alg}} \mathbf{P}_\mu \tag{2.6}$$

from  $\mathbf{G}$ -equivariant quasi-coherent sheaves on  $\mathrm{FL}_{\mathbf{G},\mu}$  and algebraic  $C$ -linear representations of  $\mathbf{P}_\mu$ . We write  $\mathcal{W}_{\mathbf{G},\mu}$  for the inverse of (2.6).

Next, we introduce some notation appearing in the localization theory of Beilinson-Bernstein [BB81]. Let  $\mathfrak{g} = \mathrm{Lie} \mathbf{G}$  be the Lie algebra of  $\mathbf{G}$  over  $\mathbb{Q}_p$  and let  $\mathfrak{g}_C$  be its base change to  $C$ . Let  $\mathfrak{p}_\mu, \mathfrak{n}_\mu$  and  $\mathfrak{m}_\mu$  be the Lie algebras of  $\mathbf{P}_\mu, \mathbf{N}_\mu$  and  $\mathbf{M}_\mu$  respectively. We let  $\mathfrak{g}_\mu^0 = \mathcal{O}_{\mathrm{FL}_{\mathbf{G},\mu}} \otimes_{\mathbb{Q}_p} \mathfrak{g}$ , and let  $\mathfrak{p}_\mu^0, \mathfrak{n}_\mu^0$  and  $\mathfrak{m}_\mu^0$  be the  $\mathbf{G}$ -equivariant sheaves over  $\mathrm{FL}_{\mathbf{G},\mu}$  corresponding to the adjoint action of  $\mathbf{P}_\mu$  via (2.6). Note that we have inclusions of  $\mathbf{G}$ -equivariant sheaves  $\mathfrak{n}_\mu^0 \subset \mathfrak{p}_\mu^0 \subset \mathfrak{g}_\mu^0$  and an isomorphism  $\mathfrak{p}_\mu^0/\mathfrak{n}_\mu^0 = \mathfrak{m}_\mu^0$ . The action of  $\mathbf{G}$  on  $\mathrm{FL}_{\mathbf{G},\mu}$  can be differentiated to a  $\mathbf{G}$ -equivariant  $\mathcal{O}_{\mathrm{FL}_{\mathbf{G},\mu}}$ -linear map

$$\alpha : \mathfrak{g}_\mu^0 \rightarrow \mathcal{T}_{\mathrm{FL}_{\mathbf{G},\mu}} \quad (2.7)$$

where  $\mathcal{T}_{\mathrm{FL}_{\mathbf{G},\mu}}$  is the tangent space of  $\mathrm{FL}_{\mathbf{G},\mu}$ . The map (2.7) induces an isomorphism

$$\bar{\alpha} : \mathfrak{g}_\mu^0/\mathfrak{p}_\mu^0 \xrightarrow{\sim} \mathcal{T}_{\mathrm{FL}_{\mathbf{G},\mu}}. \quad (2.8)$$

Similarly, let  $\pi_{\mathbf{M}_\mu} : \mathrm{FL}_{\mathbf{G},\mu}^+ \rightarrow \mathrm{FL}_{\mathbf{G},\mu}$  be the natural  $\mathbf{M}_\mu$ -torsor over  $\mathrm{FL}_{\mathbf{G},\mu}$  given by  $\mathrm{FL}_{\mathbf{G},\mu}^+ = \mathbf{G}_E/\mathbf{N}_\mu$ . The action of  $\mathbf{G}$  induces a  $\mathbf{G} \times \mathbf{M}_\mu$ -equivariant  $\mathcal{O}_{\mathrm{FL}_{\mathbf{G},\mu}^+}$ -linear map

$$\alpha^+ : \mathcal{O}_{\mathrm{FL}_{\mathbf{G},\mu}^+} \otimes_{\mathbb{Q}_p} \mathfrak{g} \rightarrow \mathcal{T}_{\mathrm{FL}_{\mathbf{G},\mu}^+}$$

with  $\mathcal{T}_{\mathrm{FL}_{\mathbf{G},\mu}^+}$  the tangent space of  $\mathrm{FL}_{\mathbf{G},\mu}^+$ . Taking pushforward along  $\pi_{\mathbf{M}_\mu}$  and  $\mathbf{M}_\mu$ -invariants, we get a  $\mathbf{G}$ -equivariant map

$$\alpha^+ : \mathfrak{g}_\mu^0 \rightarrow (\pi_{\mathbf{M}_\mu,*}(\mathcal{T}_{\mathrm{FL}_{\mathbf{G},\mu}^+}))^{\mathbf{M}_\mu}$$

that induces an isomorphism

$$\bar{\alpha}^+ : \mathfrak{g}_\mu^0/\mathfrak{n}_\mu^0 \xrightarrow{\sim} (\pi_{\mathbf{M}_\mu,*}(\mathcal{T}_{\mathrm{FL}_{\mathbf{G},\mu}^+}))^{\mathbf{M}_\mu}.$$

*Remark 2.5.1.* (1) In order to construct the equivalence (2.6) it suffices to consider a base change to  $F/E$  such that the conjugacy class  $\mu$  admits a representative. Then the groups  $\mathbf{P}_\mu, \mathbf{N}_\mu$  and  $\mathbf{M}_\mu$  are defined over  $F$ .

(2) The Lie algebroids  $\mathfrak{g}_\mu^0, \mathfrak{p}_\mu, \mathfrak{n}_\mu$  and  $\mathfrak{m}_\mu^0$  as well as the anchor map (2.7) admit natural descent to  $E$ . Indeed, the descent of the Lie algebroid  $\mathfrak{g}_\mu^0$  is nothing but  $\mathfrak{g}_{\mu,E}^0 = \mathcal{O}_{\mathrm{FL}_{\mathbf{G},\mu,E}} \otimes_{\mathbb{Q}_p} \mathfrak{g}$ . One has a sub Lie algebroid  $\mathfrak{g}_E^{\mathrm{der},0} \subset \mathfrak{g}_{\mu,E}^0$  induced by the derived Lie algebra  $\mathfrak{g}^{\mathrm{der}} \subset \mathfrak{g}$ . Since  $\mathbf{G}$  acts on  $\mathcal{F}\ell_{\mathbf{G},\mu,E}$  one has an anchor map by taking derivations

$$\mathfrak{g}_E^{\mathrm{der},0} \subset \mathfrak{g}_{\mu,E}^0 \rightarrow \mathcal{T}_{\mathrm{FL}_{\mathbf{G},\mu,E}}$$

with kernels  $\mathfrak{p}_{\mu,E}^{\mathrm{der},0}$  and  $\mathfrak{p}_{\mu,E}^0$  respectively. One can then define  $\mathfrak{n}_{\mu,E}^0$  as the unipotent radical of  $\mathfrak{p}_E^{\mathrm{der},0} \subset \mathfrak{g}_E^{\mathrm{der},0}$  and  $\mathfrak{m}_{\mu,E}^0 = \mathfrak{p}_{\mu,E}^0/\mathfrak{n}_{\mu,E}^0$  (the reason to take the derived Lie algebra  $\mathfrak{g}^{\mathrm{der}}$  is that  $\mathfrak{g}$  cannot distinguish the Lie algebra of a unipotent group and a torus).

We finish by introducing some notation that will be relevant in Section 5. Given the cocharacter  $\mu$  of  $\mathbf{G}_C$  we also have an opposite parabolic subgroup parametrizing increasing  $\mu$ -filtrations. It is equivalently obtained as  $\mathbf{P}_{\mu^{-1}}$ . Then, we have the following subgroups of  $\mathbf{G}_C$ :  $\mathbf{N}_{\mu^{-1}} \subset \mathbf{P}_{\mu^{-1}}$  with Levi quotient  $\mathbf{M}_{\mu^{-1}}$ . Note that  $\mathbf{M}_\mu = \mathbf{M}_{\mu^{-1}}$  as the centralizers of  $\mu$  and  $\mu^{-1}$  are the same, if the Levi subgroup is clear from the context we will write  $\mathbf{M}$  instead. We have another flag variety  $\mathrm{FL}_{\mathbf{G},\mu^{-1}}$  and the inverse of the equivalence (2.6) is written as  $\mathcal{W}_{\mathbf{G},\mu^{-1}}$ . To stress the difference between the Lie algebroids we shall write  $\mathfrak{g}_{\mu^{-1}}^0 := \mathcal{O}_{\mathrm{FL}_{\mathbf{G},\mu^{-1}}} \otimes_{\mathbb{Q}_p} \mathfrak{g}$ . We also have Lie algebroids over  $\mathrm{FL}_{\mathbf{G},\mu^{-1}}$  given by  $\mathfrak{n}_{\mu^{-1}}^0 \subset \mathfrak{p}_{\mu^{-1}}^0 \subset \mathfrak{g}_{\mu^{-1}}^0$  and  $\mathfrak{m}_{\mu^{-1}}^0 = \mathfrak{p}_{\mu^{-1}}^0/\mathfrak{n}_{\mu^{-1}}^0$ .

### 3. LOCAL SHIMURA VARIETIES

In this section we introduce local Shimura varieties following [SW20]. We first recall some facts about torsors on the Fargues-Fontaine curve, cf. [FS24, §III.4 and 5]. Then, we recall the definition of moduli spaces of shtukas of one leg from [SW20, Lecture XXIII] as well as the construction of the Grothendieck-Messing and Hodge-Tate period maps. Finally, we specialize the set up to local Shimura varieties and deduce a  $p$ -adic Riemann-Hilbert correspondence for automorphic proétale local systems. This last result

is a direct consequence of the theory developed in [SW20], and we only reformulate it in the version that is more convenient for this paper.

Throughout this section we use the notation of [Sch22]. Let  $\mathrm{Spd}\mathbb{Z}_p$  denote the  $v$ -sheaf parametrizing untilts  $S^\sharp$  of objects  $S \in \mathrm{Perf}$ , we let  $\mathrm{Spd}\mathbb{Q}_p \subset \mathrm{Spd}\mathbb{Z}_p$  be the open subspace parametrizing untilts in characteristic zero. Given an analytic adic space  $X$  over  $\mathbb{Z}_p$  we let  $X^\diamond$  denote its diamond over  $\mathrm{Spd}\mathbb{Z}_p$ . We let  $k = \overline{\mathbb{F}}_p$  be an algebraic closure of  $\mathbb{F}_p$  and let  $\mathrm{Perf}_k$  be the category of perfectoid spaces over  $k$ . We let  $\check{\mathbb{Q}}_p = W(k)[\frac{1}{p}]$  be the completion of the maximal unramified extension of  $\mathbb{Q}_p$ . For  $E/\mathbb{Q}_p$  a finite extension we write  $\check{E} = E\check{\mathbb{Q}}_p$ . We let  $\sigma$  denote the Frobenius automorphism of  $k$  and  $\check{\mathbb{Q}}_p$ .

**3.1.  $\mathbf{G}$ -torsors over Fargues-Fontaine curves.** In this section we recall some facts about torsors over the Fargues-Fontaine curve that we will need throughout the paper. Let  $\mathbf{G}$  be a reductive group over  $\mathbb{Q}_p$  and let  $B(\mathbf{G})$  be the Kottwitz set of Frobenius-conjugacy classes of elements in  $\mathbf{G}(\check{\mathbb{Q}}_p)$  [Kot97]. Given  $b \in B(\mathbf{G})$  and  $S \in \mathrm{Perf}_k$  a perfectoid space we let  $\mathcal{E}_b$  denote the  $\mathbf{G}$ -torsor on  $\mathcal{X}_S$  obtained via descent from the trivial torsor  $\mathbf{G} \times \mathcal{Y}_{(0,\infty),S}^{\mathrm{FF}}$  with Frobenius  $b \times \varphi$  (in the definition of torsor we take the Tannakian point of view of [SW20, Appendix to Lecture XIX]). Let  $(\mathcal{E}_b^{\geq r})_{r \in \mathbb{Q}}$  be the Harder-Narasimhan filtration of  $\mathcal{E}_b$ . We take the following definition from [FS24, §5.1].

**Definition 3.1.1.** Let  $b \in B(\mathbf{G})$ . The automorphism group of  $\mathcal{E}_b$  is the  $v$ -sheaf on groups

$$\tilde{\mathbf{G}}_b = \underline{\mathrm{Aut}}_{\mathbf{G}}(\mathcal{E}_b) : (S \in \mathrm{Perf}_k) \mapsto \mathrm{Aut}_{\mathbf{G} \times \mathcal{X}_S^{\mathrm{FF}}}(\mathcal{E}_b|_{\mathcal{X}_S^{\mathrm{FF}}}).$$

Let  $\mathbf{G}_b$  be the reductive group over  $\mathbb{Q}_p$  mapping a ring  $R$  to

$$\mathbf{G}_b(R) = \{g \in \mathbf{G}(R \otimes_{\mathbb{Q}_p} \check{\mathbb{Q}}_p) | gb = b\sigma(g)\}.$$

By [Kot97, §3.3]  $G_b = \mathbf{G}_b(\mathbb{Q}_p)$  is the automorphism group associated to the  $\mathbf{G}$ -isocrystal attached to  $b$ . We have the following structure theorem for the group  $\tilde{\mathbf{G}}_b$ .

**Proposition 3.1.2** ([FS24, Proposition III.5.1]). *One has*

$$\tilde{\mathbf{G}}_b = \tilde{G}_b^{>0} \rtimes G_b$$

where  $\tilde{G}_b^{>0}$  is the subgroup of unipotent automorphisms with respect to the Harder-Narasimhan filtration of  $\mathcal{E}_b$ . In particular, if  $b$  is basic,  $\tilde{G}_b = G_b$  is the  $\mathbb{Q}_p$ -valued points of a pure inner form of  $\mathbf{G}$ .

In order to construct the period maps we need to introduce the  $B_{\mathrm{dR}}^+$ -affine Grassmannian.

**Definition 3.1.3** ([SW20, Definition 19.1.1]). Let  $S \in \mathrm{Perf}_k / \mathrm{Spd}E$  be an affinoid perfectoid with untilt  $S^\sharp$  over  $\check{E}$ . Given  $H$  an algebraic variety over  $E$  we shall write  $L^+H$  for the  $v$ -sheafification of the presheaf

$$S \mapsto H(\mathbb{B}_{\mathrm{dR}}^+(\mathcal{O}(S^\sharp))).$$

Similarly we define  $LH$  to be the  $v$ -sheafification of

$$S \mapsto H(\mathbb{B}_{\mathrm{dR}}(\mathcal{O}(S^\sharp))).$$

The  $B_{\mathrm{dR}}^+$ -affine Grassmannian of  $\mathbf{G}$  is the  $v$ -sheaf over  $\mathrm{Spd}\mathbb{Q}_p$  given by the quotient of groups

$$\mathrm{Gr}_{\mathbf{G}} = L\mathbf{G}/L^+\mathbf{G}.$$

We shall write  $\mathrm{Gr}_{\mathbf{G},\check{E}}$  for the base change of  $\mathrm{Gr}_{\mathbf{G}}$  from  $\mathrm{Spd}\mathbb{Q}_p$  to  $\mathrm{Spd}E$ .

Let  $S \in \mathrm{Perf}_k$  be a perfectoid and let  $S^\sharp$  be an untilt over  $\check{\mathbb{Q}}_p$ . By [FS24, Proposition II.1.18] the map

$$\iota : S^\sharp \rightarrow \mathcal{X}_S^{\mathrm{FF}}$$

is an effective Cartier divisor. The pullback of  $\mathcal{E}_b$  to the completion  $\mathcal{X}_S^{\mathrm{FF},\wedge,\iota}$  of  $\mathcal{X}_S^{\mathrm{FF}}$  at  $\iota$  is a trivial  $\mathbf{G}$ -torsor. The automorphism group of the trivial  $\mathbf{G}$ -torsor over  $\mathcal{X}_S^{\mathrm{FF},\wedge,\iota}$  is then equal to  $L^+\mathbf{G}(S)$ . Thus, pullback along the formal completion gives rise to a group homomorphism of  $v$ -sheaves

$$\tilde{\mathbf{G}}_b \times \mathrm{Spd}\mathbb{Q}_p \rightarrow L^+\mathbf{G}. \tag{3.1}$$

**3.2. Moduli space of shtukas of one leg.** In the following section we recall the definition of moduli space of shtukas of one leg and the construction of the Grothendieck-Messing and Hodge-Tate period maps. We shall follow [SW20, §23.3].

Let  $(\mathbf{G}, b, \mu)$  be a local shtuka datum of one leg, namely, a triple consisting on a reductive group  $\mathbf{G}$  over  $\mathbb{Q}_p$ , an element  $b \in B(\mathbf{G})$ , and a conjugacy class of cocharacters  $\mu : \mathbb{G}_m \rightarrow \mathbf{G}_{\overline{\mathbb{Q}_p}}$ . Let  $E$  be the field of definition of  $\mu$ . Recall the equivalent definition of the moduli space of shtukas of one leg from [SW20, Proposition 23.3.1].

**Definition 3.2.1.** Let  $K \subset \mathbf{G}(\mathbb{Q}_p)$  be a compact open subgroup. The moduli space  $\text{Sht}_{\mathbf{G}, b, \mu, K}$  of stukas associated to  $(\mathbf{G}, b, \mu)$  at level  $K$  is the presheaf on  $\text{Perf}_k$  mapping  $S \in \text{Perf}_k$  to the isomorphism classes of quadruples  $(S^\sharp, \mathcal{E}, \alpha, \mathbb{P})$  where

- $S^\sharp$  is an untilt of  $S$  over  $\check{E}$ ,
- $\mathcal{E}$  is a  $\mathbf{G}$ -torsor on  $\mathcal{X}_S^{\text{FF}}$ , which is trivial at every geometric point of  $S$ ,
- $\alpha$  is an isomorphism of  $\mathbf{G}$ -torsors

$$\alpha : \mathcal{E}|_{\mathcal{X}_S^{\text{FF}} \setminus S^\sharp} \xrightarrow{\sim} \mathcal{E}_b|_{\mathcal{X}_S^{\text{FF}} \setminus S^\sharp},$$

which is meromorphic at  $S^\sharp$  and bounded by  $\mu$ , and finally

- $\mathbb{P}$  is a  $K$ -lattice in the proétale  $\mathbf{G}(\mathbb{Q}_p)$ -torsor corresponding to  $\mathcal{E}$  via [SW20, Theorem 22.5.2].

By [SW20, Theorem 23.1.4] the spaces  $\text{Sht}_{\mathbf{G}, b, \mu, K}$  are diamonds living over  $\text{Spd } \check{E}$ . We also define the moduli space of shtukas at infinite level.

**Definition 3.2.2.** Let  $\text{Sht}_{\mathbf{G}, b, \mu, \infty} = \varprojlim_{K \subset \mathbf{G}(\mathbb{Q}_p)} \text{Sht}_{\mathbf{G}, b, \mu, K}$  be the infinite level moduli space of shtukas.

By construction,  $\text{Sht}_{\mathbf{G}, b, \mu, \infty}$  is the presheaf on  $\text{Perf}_k$  parametrizing tuples  $(S^\sharp, \alpha)$  where

- $S^\sharp$  is an untilt of  $S$  over  $\check{E}$ .
- $\alpha$  is an isomorphism of  $\mathbf{G}$ -torsors

$$\alpha : \mathcal{E}_1|_{\mathcal{X}_S^{\text{FF}} \setminus S^\sharp} \xrightarrow{\sim} \mathcal{E}_b|_{\mathcal{X}_S^{\text{FF}} \setminus S^\sharp}$$

which is meromorphic at  $S^\sharp$  and bounded by  $\mu$ .

Let  $S \in \text{Perf}_k$  be a perfectoid space and let  $(S^\sharp, \alpha)$  be an  $S$ -point of  $\text{Sht}_{\mathbf{G}, b, \mu, \infty}$ . Let  $\iota : S^\sharp \rightarrow \mathcal{X}_S^{\text{FF}}$  be the closed Cartier divisor defined by the untilt. The pullbacks of  $\mathcal{E}_1$  and  $\mathcal{E}_b$  to the formal completion  $\mathcal{X}_S^{\text{FF}, \wedge, \iota}$  at  $\iota$  are trivial  $\mathbf{G}$ -torsors. Therefore, the modification  $\alpha$  is defined by an element  $g_\alpha \in L\mathbf{G}(S) = \mathbb{B}_{\text{dR}}(S^\sharp)$ . The automorphisms  $\mathbf{G}(\mathbb{Q}_p)$  of  $\mathcal{E}_1$  act on  $g_\alpha$  by right multiplication while the automorphisms  $\tilde{G}_b$  of  $\mathcal{E}_b$  act by left multiplication. Therefore, we have two maps to the affine  $B_{\text{dR}}$ -grassmannian

$$\begin{array}{ccc} & \text{Sht}_{\mathbf{G}, b, \mu, \infty} & \\ \swarrow \pi_{\text{GM}} & & \searrow \pi_{\text{HT}} \\ \text{LG}/L^+\mathbf{G} = \text{Gr}_{\mathbf{G}, \check{E}} & & \text{Gr}_{\mathbf{G}, \check{E}} = L^+\mathbf{G} \setminus \text{LG} \end{array} \quad (3.2)$$

by taking a left or right coset respectively. In particular, since  $\alpha$  is bounded by  $\mu$  by hypothesis, the diagram (3.2) actually restricts to

$$\begin{array}{ccc} & \text{Sht}_{\mathbf{G}, b, \mu, \infty} & \\ \swarrow \pi_{\text{GM}} & & \searrow \pi_{\text{HT}} \\ \text{Gr}_{\mathbf{G}, \check{E}, \leq \mu} & & \text{Gr}_{\mathbf{G}, \check{E}, \leq \mu^{-1}} \end{array} \quad (3.3)$$

When  $b$  is basic we have a duality for the diagram (3.3), cf. [SW20, Corollary 23.3.2].

**Proposition 3.2.3.** Let  $(\mathbf{G}, b, \mu)$  be a local shtuka datum with  $b$  basic. Define a shtuka datum  $(\check{\mathbf{G}}, \check{b}, \check{\mu})$  via  $\check{\mathbf{G}} = \mathbf{G}_b$ ,  $\check{b} = b^{-1} \in \mathbf{G}_b(\check{\mathbb{Q}_p}) = \mathbf{G}(\check{\mathbb{Q}_p})$  and  $\check{\mu} = \mu^{-1}$  under the identification  $\mathbf{G}_{\overline{\mathbb{Q}_p}} \cong \check{\mathbf{G}}_{\overline{\mathbb{Q}_p}}$ . Then there is a natural  $\mathbf{G}(\mathbb{Q}_p) \times \check{\mathbf{G}}(\mathbb{Q}_p)$ -equivariant isomorphism

$$\text{Sht}_{\mathbf{G}, b, \mu, \infty} \cong \text{Sht}_{\check{\mathbf{G}}, \check{b}, \check{\mu}, \infty} \quad (3.4)$$

interchanging the maps  $\pi_{\text{GM}}$  and  $\pi_{\text{HT}}$  of (3.3).

*Proof.* Let  $S \in \text{Perf}_k$  and  $(S^\sharp, \alpha)$  an  $S$ -point of  $\text{Sht}_{\mathbf{G}, b, \mu, \infty}$ . The equivariant isomorphism is [SW20, Corollary 23.3.2]. It is given by mapping a modification

$$\alpha : \mathcal{E}_1|_{\mathcal{X}_S^{\text{FF}} \setminus S^\sharp} \xrightarrow{\sim} \mathcal{E}_b|_{\mathcal{X}_S^{\text{FF}} \setminus S^\sharp}$$

to the modification of  $\mathbf{G}_b$ -torsors

$$\check{\alpha} : \mathcal{F}_1|_{\mathcal{X}_S^{\text{FF}} \setminus S^\sharp} \xrightarrow{\sim} \mathcal{F}_b|_{\mathcal{X}_S^{\text{FF}} \setminus S^\sharp}$$

obtained by mapping a  $\mathbf{G}$ -torsor  $\mathcal{E}$  to the  $\mathbf{G}_b$ -torsor  $\mathcal{F} = \text{Aut}_{\mathbf{G}}(\mathcal{E}, \mathcal{E}_b)$ . Then, since the pullback of the torsors  $\mathcal{E}_1$  and  $\mathcal{E}_2$  to  $\mathbb{B}_{\text{dR}}(S^\sharp)$  are trivial, the map  $\check{\alpha}$  seen as an object in  $\mathbf{G}(\mathbb{B}_{\text{dR}}(S^\sharp)) \cong \check{\mathbf{G}}(\mathbb{B}_{\text{dR}}(S^\sharp))$  is just the inverse of the map  $\alpha$ , proving that the period morphisms  $\pi_{\text{GM}}$  and  $\pi_{\text{HT}}$  are exchanged.  $\square$

**3.3. Local Shimura varieties.** Recall the definition of a local Shimura datum [SW20, Definition 24.1.1]

**Definition 3.3.1.** A local Shimura datum is a triple  $(\mathbf{G}, b, \mu)$  consisting of a reductive group  $\mathbf{G}$  over  $\mathbb{Q}_p$ , a conjugacy class  $\mu$  of minuscule cocharacters  $\mathbb{G}_m \rightarrow \mathbf{G}_{\mathbb{Q}_p}$ , and an element  $b \in B(\mathbf{G}, \mu^{-1})$ .

Let  $(\mathbf{G}, b, \mu)$  be a local Shimura datum and let  $E$  be the field of definition of  $\mu$ . We keep the representation theory notation of Section 2.5. Let  $K \subset \mathbf{G}(\mathbb{Q}_p)$  be an open compact subgroup and consider  $\text{Sht}_{\mathbf{G}, b, \mu, K}$  the moduli space of shtukas associated to  $(\mathbf{G}, b, \mu)$  at level  $K$ . By [SW20, Proposition 23.3.3] the period map

$$\pi_{\text{GM}} : \text{Sht}_{\mathbf{G}, b, \mu, K} \rightarrow \text{Gr}_{\mathbf{G}, \check{E}, \leq \mu}$$

is étale. On the other hand, since  $\mu$  is minuscule, the Bialynicki-Birula map

$$\pi_\mu : \text{Gr}_{\mathbf{G}, \check{E}, \leq \mu} = \text{Gr}_{\mathbf{G}, \check{E}, \mu} \rightarrow \mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^\diamond$$

is an isomorphism [SW20, Proposition 19.4.2]. This produces an étale map

$$\pi_{\text{GM}} : \text{Sht}_{\mathbf{G}, b, \mu, K} \rightarrow \mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^\diamond. \quad (3.5)$$

**Definition 3.3.2.** For  $K \subset \mathbf{G}(\mathbb{Q}_p)$  let  $\mathcal{M}_{\mathbf{G}, b, \mu, K}$  be the unique smooth rigid space over  $\check{E}$  endowed with an étale map  $\mathcal{M}_{\mathbf{G}, b, \mu, K} \rightarrow \mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^\diamond$  such that

$$\text{Sht}_{\mathbf{G}, b, \mu, K} \cong \mathcal{M}_{\mathbf{G}, b, \mu, K}^\diamond$$

as diamonds over  $\mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^\diamond$ . We shall write  $\mathcal{M}_{\mathbf{G}, b, \mu, \infty}^\diamond = \varprojlim_K \mathcal{M}_{\mathbf{G}, b, \mu, K}^\diamond$  for the infinite level Shimura variety.

By (3.2) we get a  $\mathbf{G}(\mathbb{Q}_p) \times \check{G}_b$ -equivariant diagram of period maps

$$\begin{array}{ccc} & \mathcal{M}_{\mathbf{G}, b, \mu, \infty}^\diamond & \\ \pi_{\text{GM}} \swarrow & & \searrow \pi_{\text{HT}} \\ \mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^\diamond & & \mathcal{F}\ell_{\mathbf{G}, \mu^{-1}, \check{E}}^\diamond \end{array} \quad (3.6)$$

In the rest of the section we will translate the diagram (3.6) in terms of  $p$ -adic Hodge theory of Shimura varieties. More precisely, we shall deduce a Riemann-Hilbert correspondence for proétale local systems arising from algebraic  $\mathbf{G}$ -representations.

Let  $\mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^a \subset \mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^\diamond$  be the admissible locus of the flag variety. By [SW20, Corollary 23.5.3] the map  $\pi_{\text{GM}}$  factors through  $\mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^a$  and the map

$$\pi_{\text{GM}} : \mathcal{M}_{\mathbf{G}, b, \mu, \infty}^\diamond \rightarrow \mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}}^a$$

is a proétale  $\mathbf{G}(\mathbb{Q}_p)$ -torsor.

**Definition 3.3.3.** For  $K \subset \mathbf{G}(\mathbb{Q}_p)$  a closed subgroup we denote

$$\mathcal{M}_{\mathbf{G}, b, \mu, K}^\diamond := \mathcal{M}_{\mathbf{G}, b, \mu, \infty}^\diamond / K.$$

Let  $V$  be a  $p$ -adically complete or  $\mathbb{Q}_p$ -Banach continuous representation of  $K$ . We let  $\mathcal{F}_V$  be the  $v$ -sheaf on  $\mathcal{M}_{\mathbf{G},b,\mu,K}^\diamond$  obtained via descent from the constant  $K$ -equivariant sheaf  $\underline{V}$  on  $\mathcal{M}_{\mathbf{G},\mu,\infty}^\diamond$  with

$$\underline{V}(S) = \text{Cont}(|S|, V)$$

for  $S \in \mathcal{M}_{\mathbf{G},b,\mu,\infty,v}^\diamond$  affinoid perfectoid. For  $V = \varinjlim_i V_i$  an ind-system of  $p$ -adically complete or Banach representations we define  $\mathcal{F}_V := \varinjlim_i \mathcal{F}_{V_i}$ .

We let  $E'$  be a finite extension over  $E$  over which  $\mathbf{G}$  is split and consider the pullback of the local Shimura varieties and flag varieties to  $\check{E}'$ . Furthermore, we fix a Hodge cocharacter  $\mu : \mathbb{G}_{m,E'} \rightarrow \mathbf{G}_{E'}$  which determines parabolic subgroups  $\mathbf{P}_\mu$  and  $\mathbf{P}_{\mu^{-1}}$  of  $\mathbf{G}_{E'}$ , as well as their unipotent radicals  $\mathbf{N}_\mu$  and  $\mathbf{N}_{\mu^{-1}}$ , and the Levi subgroup  $\mathbf{M}_\mu = \mathbf{M}_{\mu^{-1}} = \mathbf{M}^1$ .

Let  $V \in \text{Rep}_{\mathbb{Q}_p}^{\text{alg}}(\mathbf{G})$  be a  $\mathbb{Q}_p$ -linear algebraic representation of  $\mathbf{G}$ . Let  $V_{\text{dR}}$  be the  $\mathbf{G}$ -equivariant flat connection over  $\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}'}^a$  given by

$$V_{\text{dR}} = \mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}'}^a} \otimes_{\mathbb{Q}_p} V$$

with Hodge filtration induced by the  $\mathbf{P}_\mu$ -filtration of  $V \otimes_{\mathbb{Q}_p} E'$  via the functor  $\mathcal{W}_{\mathbf{G},\mu}$  of (2.6), see Remark 2.5.1 (1). By an abuse of notation we will also write  $V_{\text{dR}}$  for the restriction to the admissible locus.

Let  $\mathcal{F}_V$  be the  $v$ -sheaf over  $\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}'}^a$  associated to the  $\mathbf{G}(\mathbb{Q}_p)$ -representation  $V$  via Definition 3.3.3. Let  $V_{\text{proét}}$  be the restriction of  $\mathcal{F}_V$  to a sheaf on the proétale site  $\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}',\text{proét}}^a$  of [Sch13]. We have the following Riemann-Hilbert correspondence for local Shimura varieties.

**Proposition 3.3.4.** *The proétale local system  $V_{\text{proét}}$  is de Rham in the sense of [Sch13, Definition 8.3] with associated filtered flat connection  $V_{\text{dR}}$ <sup>2</sup>. More precisely, we have a  $\tilde{G}_b$ -equivariant map of filtered  $\mathbb{B}_{\text{dR}}$ -sheaves on  $\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}',\text{proét}}^a$*

$$V_{\text{proét}} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}} \cong \underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}, \quad (3.7)$$

where the  $\mathbb{B}_{\text{dR}}^+$ -filtration in the left hand side is the trivial one, and the filtration in the right hand side is given by

$$\text{Fil}^i(\underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}) := (\text{Fil}^i(V_{\text{dR}} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}'}^a}} \mathcal{O}_{\mathbb{B}_{\text{dR}}})^{\nabla=0}).$$

The action of  $\tilde{G}_b$  is trivial on  $V_{\text{proét}}$  in the left hand side and it factors through  $\tilde{G}_b \rightarrow L\mathbf{G}$  and the natural action on  $\underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}$  in the right hand side.

*Remark 3.3.5.* By [SW20, Corollary 17.1.9], for a smooth rigid variety  $X$  there is no distinction between filtered  $\mathbb{B}_{\text{dR}}^+$ -vector bundles on the proétale site  $X_{\text{proét}}$  of [Sch13] or filtered  $\mathbb{B}_{\text{dR}}^+$ -vector bundles on  $X_v$ . Thus, the equivariant isomorphism (3.7) can also be stated as a  $\tilde{G}_b$ -equivariant isomorphism of filtered  $\mathbb{B}_{\text{dR}}$  sheaves on the  $v$ -site

$$\mathcal{F}_V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}} \cong \underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}.$$

*Proof of Proposition 3.3.4.* Let us denote  $X = \mathcal{F}\ell_{\mathbf{G},\mu,\check{E}',\text{proét}}^a$ . Since  $V_{\text{dR}}$  has horizontal sections  $V$ , we have an isomorphism

$$\underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}} = (V_{\text{dR}} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbb{B}_{\text{dR}}})^{\nabla=0}$$

in  $X_{\text{proét}}$ . By [Sch13, Theorem 7.6]  $\mathbb{M}' := (\text{Fil}^0(V_{\text{dR}} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbb{B}_{\text{dR}}}))^{\nabla=0}$  is a  $\mathbb{B}_{\text{dR}}^+$ -lattice of  $\underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}$  in the proétale site of  $X$ . By [SW20, Corollary 17.1.9] we can view  $\mathbb{M}'$  as a  $\mathbb{B}_{\text{dR}}^+$ -lattice in the  $v$ -site of  $X$ . Thus, we will view  $\underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}$  as a filtered  $\mathbb{B}_{\text{dR}}^+$ -module in the  $v$ -site with  $\text{Fil}^i(\underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}) = \xi^i \mathbb{M}'$  for  $\xi$  a local generator of the kernel of  $\theta : \mathbb{B}_{\text{dR}}^+ \rightarrow \hat{\mathcal{O}}$ .

<sup>1</sup>Taking this finite extension is unnecessary for the forthcoming discussion but it allows us to use the dictionary between representations of the chosen parabolic  $\mathbf{P}_\mu$  and  $\mathbf{G}$ -equivariant quasi-coherent sheaves on the flag variety of Section 2.5. We left to the reader the cocharacter-free formulation of the statements in terms of filtered  $\mathbf{G}$ -representations.

<sup>2</sup>Strictly speaking this notion is only defined for lisse  $\mathbb{Z}_p$ -local systems and  $V_{\text{proét}}$  over  $\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}'}^a$  is not of this form. However, it becomes lisse after pulling back to any finite level local Shimura variety.

Now let  $S \in \text{Perf}_k$  and take  $(S^\sharp, \alpha)$  an  $S$ -point of  $\mathcal{M}_{\mathbf{G}, b, \mu, \infty, \check{E}'}^\diamond$ . Given  $V \in \text{Rep}_{\mathbb{Q}_p}^{\text{alg}} \mathbf{G}$  an algebraic representation, let  $\mathcal{V}_1$  and  $\mathcal{V}_b$  be the  $\mathbf{G}$ -equivariant vector bundles over  $\mathcal{X}_S^{\text{FF}}$  defined by the torsors  $\mathcal{E}_1$  and  $\mathcal{E}_b$  respectively. By definition of the modification we have a  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant isomorphism

$$(\mathcal{F}_V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}})(S^\sharp) = \mathcal{V}_1 \otimes_{\mathcal{O}_{\mathcal{X}_S^{\text{FF}}}} \mathbb{B}_{\text{dR}}(S^\sharp) \cong \mathcal{V}_b \otimes_{\mathcal{O}_{\mathcal{X}_S^{\text{FF}}}} \mathbb{B}_{\text{dR}}(S^\sharp) = \underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}(S^\sharp)$$

where  $\mathbf{G}(\mathbb{Q}_p)$  acts trivially on  $\underline{V}$  and via the projection  $\mathbf{G}(\mathbb{Q}_p) \rightarrow L\mathbf{G}$  on  $\mathcal{F}_V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}$ , and  $\tilde{G}_b$  acts trivially on  $\mathcal{F}_V$  and via the projection  $\tilde{G}_b \rightarrow L\mathbf{G}$  on  $\underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}$ . This produces the desired equivariant isomorphism (3.7). The fact that the isomorphism (3.7) is compatible with the filtration follows from the definition of the Bialynicki-Birula map and [SW20, Proposition 19.4.2].  $\square$

Given  $V \in \text{Rep}_{\mathbb{Q}_p}^{\text{alg}} \mathbf{G}$  let  $\mathbb{M} = \mathcal{F}_V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}^+$  and let  $\mathbb{M}_0 = \underline{V} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}}^+$ . The Hodge-Tate filtration of  $\mathcal{F}_V \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}$  is given by

$$\text{Fil}_n(\mathcal{F}_V \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}) = (\mathbb{M} \cap \text{Fil}^{-i}\mathbb{M}_0) / (\text{Fil}^1\mathbb{M} \cap \text{Fil}^{-i}\mathbb{M}_0).$$

Let  $\mathcal{W}_{\mathbf{G}, \mu^{-1}}$  be the inverse of the functor (2.6) for the cocharacter  $\mu^{-1}$ . As corollary we deduce that the Hodge-Tate period map encodes the Hodge-Tate filtration.

**Corollary 3.3.6.** *There is a  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant isomorphism of  $\hat{\mathcal{O}}$ -modules over  $\mathcal{M}_{\mathbf{G}, b, \mu, \infty, \check{E}'}^\diamond$*

$$\text{Fil}_n(\mathcal{F}_V \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}) \cong \pi_{\text{HT}}^*(\mathcal{W}_{\mathbf{G}, \mu^{-1}}(\text{Fil}_n(V \otimes_{\mathbb{Q}_p} E'))),$$

where  $\text{Fil}_\bullet(V \otimes_{\mathbb{Q}_p} E')$  is the (increasing)  $\mathbf{P}_{\mu^{-1}}$ -filtration of  $V \otimes_{\mathbb{Q}_p} E'$ .

*Proof.* This is a consequence of Proposition 3.3.4, the definition of the Bialynicki-Birula map and [SW20, Proposition 19.4.2]  $\square$

Moreover, taking graded pieces in Corollary 3.3.6 one deduces the isomorphism of  $\mathbf{M}$ -torsors on the infinite level Shimura variety, cf. [CS17, Theorem 2.1.3].

**Corollary 3.3.7.** *Let  $W \in \text{Rep}_{E'}^{\text{alg}} \mathbf{M}$  be an irreducible algebraic representation of the Levi subgroup. There is a natural  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant  $\otimes$ -isomorphism of  $\hat{\mathcal{O}}$ -modules over  $\mathcal{M}_{\mathbf{G}, b, \mu, \infty, \check{E}'}^\diamond$*

$$\pi_{\text{HT}}^*(\mathcal{W}_{\mathbf{G}, \mu^{-1}}(W)) \cong \pi_{\text{GM}}^*(\mathcal{W}_{\mathbf{G}, \mu}(W)) \otimes_{\hat{\mathcal{O}}} \hat{\mathcal{O}}(-\mu(W))$$

where  $\mu(W) \in \mathbb{Z}$  is the weight of  $W$  with respect to  $\mu$ . In particular, if  $\mathbf{M}_{\mu, \text{GM}}$  and  $\mathbf{M}_{\mu^{-1}, \text{HT}}$  denote the natural  $\mathbf{M}$ -torsors living over  $\mathcal{F}\ell_{\mathbf{G}, \mu, \check{E}'}$  and  $\mathcal{F}\ell_{\mathbf{G}, \mu^{-1}, \check{E}'}$  respectively (see Section 2.5), we have a  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant isomorphism of  $\mathbf{M}$ -torsors over the ringed site  $(\mathcal{M}_{\mathbf{G}, b, \mu, \infty, \check{E}', v}^\diamond, \hat{\mathcal{O}})$

$$\pi_{\text{HT}}^*(\mathbf{M}_{\mu^{-1}, \text{HT}}) \cong \pi_{\text{GM}}^*(\mathbf{M}_{\mu, \text{GM}}) \times^{\mathbb{G}_m, \mu} \mathbb{G}_m(-1),$$

where  $\mathbb{G}_m$  injects into the center of  $\mathbf{M}$  via  $\mu$ , and  $\mathbb{G}_m(-1)$  is the  $\mathbb{G}_m$ -torsor of trivializations of the Tate twist  $\hat{\mathcal{O}}(-1)$ .

*Proof.* This follows after taking graded pieces of the isomorphisms in Corollary 3.3.6 and [Sch13, Proposition 7.9], see also [RC24b, Theorem 4.2.1].  $\square$

#### 4. GEOMETRIC SEN OPERATORS OF LOCAL SHIMURA VARIETIES

In this section we compute the geometric Sen operator of local Shimura varieties, proving the local analogue of [RC24b, Theorem 5.2.5]. We keep the notation of Section 3.3, namely we let  $(\mathbf{G}, b, \mu)$  be a local Shimura datum with reflex field  $E/\mathbb{Q}_p$ . We let  $E'/E$  be a finite extension of  $E$  over which  $\mathbf{G}$  is split and fix a representative of the Hodge-cocharacter  $\mu : \mathbb{G}_{m, E'} \rightarrow \mathbf{G}_{E'}$ . For  $K \subset \mathbf{G}(\mathbb{Q}_p)$  compact open subgroup we let  $\mathcal{M}_{\mathbf{G}, b, \mu, K}$  be the local Shimura variety over  $\check{E}$  at level  $K$ , we denote by  $\mathcal{O}_{\mathcal{M}}$  its structural sheaf as a rigid space, and let  $\Omega_{\mathcal{M}}^1$  be its cotangent bundle.

**4.1. The Kodaira-Spencer map.** In the next paragraph we make explicit the Kodaira-Spencer isomorphism for local Shimura varieties in terms of representation theory over the flag variety. We follow [RC24b, Proposition 5.1.3].

Let  $\mathfrak{g}$  be the adjoint representation of  $\mathbf{G}$  over  $\mathbb{Q}_p$ , and let  $\mathfrak{g}_{\mathrm{dR}}^{\vee} = \mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu,E'}} \otimes_{\mathbb{Q}_p} \mathfrak{g}^{\vee} = \mathfrak{g}_{\mu}^{0,\vee}$  be its associated  $\mathbf{G}$ -equivariant vector bundle with flat connection over  $\mathcal{F}\ell_{\mathbf{G},\mu,E'}$ . Since  $\mu$  is minuscule,  $\mathfrak{g}_{\mathrm{dR}}^{\vee}$  has Hodge filtration concentrated in degrees  $[-1, 1]$  given by

$$(\mathfrak{g}_{\mathrm{dR}}/\mathfrak{p}_{\mu}^0)^{\vee} \subset \mathfrak{p}_{\mu}^{0,\vee} \subset \mathfrak{g}_{\mathrm{dR}}^{\vee}$$

such that

$$\mathrm{gr}^i \mathfrak{g}_{\mathrm{dR}}^{\vee} = \begin{cases} (\mathfrak{g}_{\mathrm{dR}}/\mathfrak{p}_{\mu}^0)^{\vee} & \text{if } i = 1, \\ \mathfrak{m}_{\mu}^{0,\vee} & \text{if } i = 0, \\ \mathfrak{n}_{\mu}^{0,\vee} & \text{if } i = -1. \end{cases}$$

Then, the flat connection  $\nabla : \mathfrak{g}_{\mathrm{dR}}^{\vee} \rightarrow \mathfrak{g}_{\mathrm{dR}}^{\vee} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu,E'}}} \Omega_{\mathcal{F}\ell_{\mathbf{G},\mu,E'}}^1$  induces a map in  $\mathrm{gr}^1$ -pieces

$$(\mathfrak{g}_{\mathrm{dR}}/\mathfrak{p}_{\mu}^0)^{\vee} \rightarrow \mathfrak{m}_{\mu}^{0,\vee} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu,E'}}} \Omega_{\mathcal{F}\ell_{\mathbf{G},\mu,E'}}^1.$$

Taking adjoints we get a  $\mathbf{G}$ -equivariant map

$$\widetilde{\mathrm{KS}} : (\mathfrak{g}_{\mathrm{dR}}/\mathfrak{p}_{\mu}^0)^{\vee} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu,E'}}} \mathfrak{m}_{\mu}^0 \rightarrow \Omega_{\mathcal{F}\ell_{\mathbf{G},\mu}}^1. \quad (4.1)$$

Looking at the fiber at  $[1] \in \mathcal{F}\ell_{\mathbf{G},\mu,E'}$  the map (4.1) is nothing but the natural adjoint action of  $\mathfrak{m}_{\mu}$  on  $(\mathfrak{g}_{E'}/\mathfrak{p}_{\mu})^{\vee}$  with  $\mathfrak{g}_{E'} = \mathfrak{g} \otimes_{\mathbb{Q}_p} E'$ :

$$(\mathfrak{g}_{E'}/\mathfrak{p}_{\mu})^{\vee} \otimes_{E'} \mathfrak{m}_{\mu} \xrightarrow{\mathrm{ad}} (\mathfrak{g}_{E'}/\mathfrak{p}_{\mu})^{\vee} \cong \Omega_{\mathcal{F}\ell_{\mathbf{G},\mu,E'}}^1|_{[1]}.$$

Therefore, the map  $\widetilde{\mathrm{KS}}$  induces the  $\mathbf{G}$ -equivariant Kodaira-Spencer isomorphism over the flag variety

$$\mathrm{KS} : (\mathfrak{g}_{\mathrm{dR}}/\mathfrak{p}_{\mu}^0)^{\vee} \xrightarrow{\sim} \Omega_{\mathcal{F}\ell_{\mathbf{G},\mu,E'}}^1. \quad (4.2)$$

which is the inverse of the dual of the anchor map  $\bar{\alpha}$  of (2.8). Note that in particular  $\mathrm{KS}$  is already defined over  $E$  as the anchor map is so, see Remark 2.5.1 (2).

We deduce the following proposition.

**Proposition 4.1.1.** *Let  $K \subset \mathbf{G}(\mathbb{Q}_p)$  be an open compact subgroup. The Kodaira-Spencer map of  $\mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}'}$*

$$\widetilde{\mathrm{KS}} : \mathrm{gr}^1(\mathfrak{g}_{\mathrm{dR}}^{\vee}) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{gr}^0(\mathfrak{g}_{\mathrm{dR}}) \rightarrow \Omega_{\mathcal{M}}^1$$

constructed in analogue fashion as (4.1) factors through an isomorphism

$$\mathrm{KS} : \mathrm{gr}^1(\mathfrak{g}_{\mathrm{dR}}^{\vee}) \xrightarrow{\sim} \Omega_{\mathcal{M}}^1$$

which is nothing but the pullback along  $\mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}'} \rightarrow \mathcal{F}\ell_{\mathbf{G},\mu,\check{E}'}$  of the Kodaira-Spencer isomorphism (4.2).

*Proof.* This follows from the Kodaira-Spencer isomorphism (4.2) and the fact that the map  $\mathcal{M}_{\mathbf{G},b,\mu,K} \rightarrow \mathcal{F}\ell_{\mathbf{G},\mu,\check{E}}$  is étale, namely, the filtered vector bundle with flat connection  $\mathfrak{g}_{\mathrm{dR}}^{\vee}$  over  $\mathcal{M}_{\mathbf{G},b,\mu,K}$  is the pullback of the analogue filtered vector bundle with flat connection over the flag variety.  $\square$

We finish this section by rewriting the Kodaira-Spencer map in the form that will be used in the paper. Let  $K \subset \mathbf{G}(\mathbb{Q}_p)$  be a compact open subgroup. By Proposition 3.3.4 the local system  $\mathcal{F}_{\mathfrak{g}^{\vee}}$  on the admissible locus  $\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}'}$  (see Definition 3.3.3) is de Rham with associated filtered flat connection  $\mathfrak{g}_{\mathrm{dR}}^{\vee}$ . Then, Corollaries 3.3.6 and 3.3.7 give rise  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant  $\widehat{\mathcal{O}}$ -sheaves on  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}'}$

$$\pi_{\mathrm{HT}}^*(\mathfrak{n}_{\mu^{-1}}^{0,\vee}) \cong \mathrm{gr}_1(\mathcal{F}_{\mathfrak{g}^{\vee}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}) \cong \mathrm{gr}^1(\mathfrak{g}_{\mathrm{dR}}^{\vee}) \otimes_{\mathcal{O}_{\mathcal{M}}} \widehat{\mathcal{O}}(-1) \cong \pi_{\mathrm{GM}}^*((\mathfrak{g}_{\mathrm{dR}}/\mathfrak{p}_{\mu}^{\vee})^{\vee})(-1). \quad (4.3)$$

Composing (4.2) and (4.3) we get the following incarnation of the Kodaira-Spencer map

$$\mathrm{KS}' : \pi_{\mathrm{HT}}^*(\mathfrak{n}_{\mu^{-1}}^{0,\vee}) \xrightarrow{\sim} \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \widehat{\mathcal{O}}(-1) \quad (4.4)$$

as  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant  $\widehat{\mathcal{O}}$ -sheaves on  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}',v}$ .



**4.2. Pullbacks of equivariant vector bundles along  $\pi_{\text{HT}}$ .** In this section we compute the Faltings extension of the local Shimura varieties in terms of the representation theory of the Hodge-Tate flag variety. For this computation we need to introduce some sheaves. We keep the representation theory notation of Section 2.5.

Let  $\mathcal{O}_{\text{dR}}^+$  and  $\mathcal{O}_{\text{dR}}$  be the big proétale de Rham sheaves of  $\mathcal{F}\ell_{\mathbf{G},\mu,\check{E}'}^a$  as in [Sch13]. Let  $\text{gr}^1 \mathcal{O}_{\text{dR}}^+$  be the Faltings extension, it is an  $\widehat{\mathcal{O}}$ -vector bundle in the proétale site and so it defines naturally a  $v$ -vector bundle that we denote in the same way. We can write

$$\text{gr}^0 \mathcal{O}_{\text{dR}} = \text{Sym}_{\widehat{\mathcal{O}}}(\text{gr}^1 \mathcal{O}_{\text{dR}}^+(-1))/(1 - e(1))$$

where 1 is the unit in the symmetric algebra, and where  $e : \widehat{\mathcal{O}} \rightarrow \text{gr}^1 \mathcal{O}_{\text{dR}}^+(-1)$  is the natural map. Therefore, we can see  $\text{gr}^0 \mathcal{O}_{\text{dR}}$  as a  $v$ -sheaf which is a filtered colimit of  $v$ -vector bundles. See also [RJRC22, Remark 2.1.2].

Let  $\mathbf{N}_{\mu^{-1}} \subset \mathbf{P}_{\mu^{-1}}$  be the unipotent radical of the opposite to the standard parabolic and let  $\mathcal{O}(\mathbf{N}_{\mu^{-1}})$  be its space of algebraic functions. We endow  $\mathcal{O}(\mathbf{N}_{\mu^{-1}})$  with the unique action of  $\mathbf{P}_{\mu} = \mathbf{N}_{\mu^{-1}} \rtimes \mathbf{M}_{\mu^{-1}}$  such that

- The restriction to  $\mathbf{N}_{\mu^{-1}}$  is the left regular action, i.e

$$(n_1 \cdot f)(n_2) = f(n_1^{-1}n_2)$$

for  $n_1, n_2 \in \mathbf{N}_{\mu^{-1}}$  and  $f \in \mathcal{O}(\mathbf{N}_{\mu^{-1}})$ .

- The restriction to  $\mathbf{M}_{\mu^{-1}}$  is the adjoint action, i.e.

$$(m \cdot f)(n) = f(m^{-1}nm)$$

for  $m \in \mathbf{M}_{\mu^{-1}}$ ,  $n \in \mathbf{N}_{\mu^{-1}}$  and  $f \in \mathcal{O}(\mathbf{N}_{\mu^{-1}})$ .

By [RC24b, Proposition 3.3.1] the algebra  $\mathcal{O}(\mathbf{N}_{\mu^{-1}})$  has an increasing  $\mathbf{P}_{\mu^{-1}}$ -filtration  $\mathcal{O}(\mathbf{N}_{\mu^{-1}})^{\leq n}$  with graded pieces  $\text{gr}_n(\mathcal{O}(\mathbf{N}_{\mu^{-1}})) \cong \text{Sym}_{E'}^n \mathfrak{n}_{\mu^{-1}}^{\vee}$ .

Recall that for an algebraic  $\mathbf{P}_{\mu^{-1}}$ -representation  $W$  we let  $\mathcal{W}_{\mathbf{G},\mu^{-1}}(W)$  denote the  $\mathbf{G}$ -equivariant quasi-coherent sheaf over  $\mathcal{F}\ell_{\mathbf{G},\mu^{-1},E'}$  associated to  $W$  via (2.6). We have the following theorem.

**Theorem 4.2.1.** *There is a natural  $\mathbf{G}(\mathbb{Q}_p) \times \widetilde{G}_b$ -equivariant isomorphism of  $\widehat{\mathcal{O}}$ -algebras over  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}',v}^{\diamond}$*

$$\text{gr}^0(\mathcal{O}_{\text{dR}}) \cong \pi_{\text{HT}}^*(\mathcal{W}_{\mathbf{G},\mu^{-1}}(\mathcal{O}(\mathbf{N}_{\mu^{-1}}))).$$

More precisely, we have a  $\mathbf{G}(\mathbb{Q}_p) \times \widetilde{G}_b$ -equivariant isomorphism of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{\mathcal{O}} & \longrightarrow & \pi_{\text{HT}}^*(\mathcal{W}_{\mathbf{G},\mu^{-1}}(\mathcal{O}(\mathbf{N}_{\mu^{-1}}))) & \longrightarrow & \pi_{\text{HT}}^*(\mathfrak{n}_{\mu^{-1}}^{0,\vee}) \longrightarrow 0 \\ & & \text{id} \downarrow & & \alpha \downarrow & & \downarrow -\text{KS}' \\ 0 & \longrightarrow & \widehat{\mathcal{O}} & \longrightarrow & \text{gr}^1 \mathcal{O}_{\text{dR}}^+(-1) & \longrightarrow & \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \widehat{\mathcal{O}}(-1) \longrightarrow 0 \end{array}$$

where  $\text{KS}'$  is the Kodaira-Spencer isomorphism of (4.4).

*Proof.* The proof follows exactly the same lines of the proof of [RC24b, Theorem 5.1.4] where the key inputs are the Riemann-Hilbert correspondence of Proposition 3.3.4 and the Kodaira-Spencer isomorphism (4.4). Note that in *loc. cit.* we denoted  $\mathcal{O}_{\text{Clog},Sh} = \text{gr}^0(\mathcal{O}_{\text{dR}})$ , and we have identified  $\mathfrak{g} \cong \mathfrak{g}^{\vee}$ ,  $\mathfrak{m}_{\mu^{-1}} \cong \mathfrak{m}_{\mu^{-1}}^{\vee}$  and  $\mathfrak{n}_{\mu^{-1}}^{\vee} \cong \mathfrak{g}_{E'}/\mathfrak{p}_{\mu^{-1}}$  via the Killing form of the derived Lie algebra of  $\mathfrak{g}$ <sup>3</sup>.  $\square$

**4.3. Computation of the geometric Sen operators.** We finish this section with the computation of the geometric Sen operators. By [RC23, Theorem 3.3.4], for any compact open subgroup  $K \subset \mathbf{G}(\mathbb{Q}_p)$ , there is a natural geometric Sen operator

$$\theta_{\mathcal{M}} : \mathcal{F}_{\mathfrak{g}^{\vee}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}} \rightarrow \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \widehat{\mathcal{O}}(-1) \tag{4.5}$$

seen as a morphism of  $\widehat{\mathcal{O}}$ -vector bundles over  $\mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}',v}$ . We have the following theorem.

<sup>3</sup>Strictly speaking one has to use the Killing form of the derived group  $\mathfrak{g}^{\text{der}}$  to obtain the self duality.

**Theorem 4.3.1.** *The geometric Sen operator (4.5) is the descent along the  $K$ -torsor  $\pi_K : \mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}'}^\diamond \rightarrow \mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}'}^\diamond$  of the  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant map of  $\widehat{\mathcal{O}}$ -vector bundles obtained by pulling back along  $\pi_{\text{HT}}$  the map*

$$\mathfrak{g}_{\mu^{-1}}^{0,\vee} \rightarrow \mathfrak{n}_{\mu^{-1}}^{0,\vee}$$

over  $\mathcal{F}\ell_{\mathbf{G},\mu^{-1},\check{E}'}$ , where  $\pi_{\text{HT}}^*(\mathfrak{n}_{\mu^{-1}}^{0,\vee}) \cong \Omega_{\mathcal{M}}^1 \otimes_{\mathcal{O}_{\mathcal{M}}} \widehat{\mathcal{O}}(-1)$  is the Kodaira-Spencer isomorphism (4.4).

*Proof.* The proof is the same as the one of [RC24b, Theorem 5.2.5] where Theorem 4.2.1 replaces [RC24b, Theorem 5.1.4].  $\square$

**Corollary 4.3.2.** *The geometric Sen operator (4.5) is a  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant map of  $\widehat{\mathcal{O}}$ -sheaves over  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,\check{E}'}^\diamond$ .*

*Proof.* This follows from the fact that the pullback along  $\pi_{\text{HT}}$  of the map  $\mathfrak{g}_{\mu^{-1}}^0 \rightarrow \mathfrak{n}_{\mu^{-1}}^{0,\vee}$  is  $\mathbf{G}(\mathbb{Q}_p) \times \tilde{G}_b$ -equivariant.  $\square$

We finish this section with the vanishing of higher locally analytic vectors for the sheaf  $\widehat{\mathcal{O}}$  and the computation of its arithmetic Sen operator. Let  $C/E'$  be the  $p$ -adic completion of an algebraic closure. Let  $K \subset \mathbf{G}(\mathbb{Q}_p)$  be a compact open subgroup, and let  $\mathcal{V}_K = C^{la}(K, \mathbb{Q}_p)_{\star_1}$  be the left regular locally analytic representation of  $K$ . Consider the  $v$ -sheaf  $\mathcal{F}_{\mathcal{V}_K}$  over  $\mathcal{M}_{\mathbf{G},b,\mu,K,\check{E}}$  of Definition 3.3.3 which is a filtered colimit of Banach  $\mathbb{Q}_p$ -linear  $v$ -sheaves.

**Theorem 4.3.3.** *Let  $U \subset \mathcal{M}_{\mathbf{G},b,\mu,K,C}$  be an open affinoid subspace admitting an étale map to a product of tori  $\mathbb{T}_C^d$  that factors as a finite composition of rational localizations and finite étale maps. Let  $U_\infty \subset \mathcal{M}_{\mathbf{G},b,\mu,\infty,C}^\diamond$  be the pullback of  $U$ , then*

$$R\Gamma_v(U, \widehat{\mathcal{O}} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}_{\mathcal{V}_K}) = \widehat{\mathcal{O}}(U_\infty)^{G-la} \quad (4.6)$$

sits in degree 0 and is equal to the locally analytic vectors of  $\widehat{\mathcal{O}}(U_\infty)$ . Here the completed tensor product is a filtered colimit of  $p$ -completed tensor products obtained by writing  $\mathcal{F}_{\mathcal{V}_K}$  as a colimit of Banach sheaves, it coincides with the solid tensor product of [AM24, Section 4.1].

Furthermore, the action of  $\mathfrak{g}_{\mu^{-1},C}^0 = \mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu^{-1},C}} \otimes_{\mathbb{Q}_p} \mathfrak{g}$  on  $\widehat{\mathcal{O}}(U_\infty)^{G-la}$  by derivations kills  $\mathfrak{n}_{\mu^{-1},C}^0$ . In particular we have an horizontal action of  $\mathfrak{m}_{\mu^{-1},C}^0$  on  $\widehat{\mathcal{O}}(U_\infty)^{G-la}$ . Moreover, the space  $\widehat{\mathcal{O}}(U_\infty)^{G-la}$  has an arithmetic Sen operator as in [RC24b, Theorem 6.3.5] given by the opposite of the derivative of the Hodge cocharacter  $-\theta_\mu = \theta_{\mu^{-1}} \in \mathfrak{m}_{\mu^{-1},C}^0$ .

*Proof.* The equivalence of (4.6) follows from the same proof of Proposition 6.2.8 (1) in [RC24b]. The vanishing of the action of  $\mathfrak{n}_{\mu^{-1}}^0$  on  $\widehat{\mathcal{O}}(U_\infty)^{G-la}$  is Corollary 6.2.12 of *loc. cit.*. Finally, the existence and computation of the arithmetic Sen operator is Theorem 6.3.5 of *loc. cit.*. Note that in [RC24b] the statement of the theorem involves proétale cohomology and not  $v$ -cohomology, these two are naturally the same thanks to the  $v$ -decent results of [AM24, Theorem 5.6].  $\square$

## 5. LOCALLY ANALYTIC VECTORS AT INFINITE LEVEL

In this last section we show the main results of this paper. First, in Section 5.1 we study the locally analytic vectors of period sheaves at infinite level local Shimura varieties. In particular, we prove that when  $b$  is basic the locally analytic vectors are independent of the two towers of local Shimura varieties (Corollary 5.1.9), generalizing a result of Pan for the Lubin-Tate tower [Pan22b, Corollary 5.3.9]. Then, in Section 5.2 we prove that, for  $b$  basic, the colimit of compactly supported de Rham cohomologies as the level goes to 1 are independent of the two towers (Theorem 5.2.2), this result has also been independently obtained by Guido Bosco, Wiesława Nizioł and the first author. Finally, in Section 5.3 we prove that the  $p$ -adic Jacquet-Langlands functor of Scholze [Sch18] for the Lubin-Tate tower is compatible with the passage to locally analytic vectors (Theorem 5.3.6).

**5.1. Locally analytic vectors of towers of rigid spaces.** Let us fix  $K$  a perfectoid field in characteristic zero with tilt  $K^b$ , we let  $\varpi \in K^b$  be a pseudo-uniformizer with  $|\varpi|_{K^b} = |p|_K$ . Throughout this section we suppose that  $K$  contains all  $p$ -th power roots of unit. For all  $b$  rational we shall take  $p^b \in K$  an element with  $|p^b|_K = |p|_K^b$  whenever it exists (similarly for  $\varpi \in K^b$ ). Let  $G$  and  $H$  be two compact  $p$ -adic Lie groups and  $X$  a smooth qcqs rigid space over  $K$  endowed with an action of  $H$ . Suppose we are given with an  $H$ -equivariant proétale  $G$ -torsor  $\tilde{X} \rightarrow X^\diamond$  seen as a diamond over  $\mathrm{Spd}(K)$ , so that  $\tilde{X}$  has a commuting action of  $G \times H$ .

Let  $I = [s, r] \subset (0, \infty)$  be a compact interval with rational ends and let  $\mathbb{B}_I$  be the period sheaf on the  $v$ -site  $X_v^\diamond$  as in Definition 2.1.3. We can write  $\mathbb{B}_I = \mathbb{B}_I^+[\frac{1}{[\varpi]}]$  with  $\mathbb{B}_I^+$  a  $[\varpi]$ -adically complete sheaf. Set  $\mathbb{B}_{I,b}^+ = \mathbb{B}_I^+ / [\varpi]^b$ . Consider the solid  $\mathbb{Q}_p$ -vector space  $R\Gamma_v(\tilde{X}, \mathbb{B}_I)$  with solid structure induced from the presentation

$$R\Gamma_v(\tilde{X}, \mathbb{B}_I) = R\varprojlim_b R\Gamma_v(\tilde{X}, \mathbb{B}_{I,b}^+) \left[ \frac{1}{[\varpi]} \right]$$

where  $R\Gamma_v(\tilde{X}, \mathbb{B}_{I,b}^+)$  is a discrete  $\mathbb{B}_{I,b}^+(K^b, K^{+,b})$ -complex, equal to the étale cohomology

$$R\Gamma_v(\tilde{X}, \mathbb{B}_{I,b}^+) = R\Gamma_{\text{ét}}(\tilde{X}, \mathbb{B}_{I,b}^+)$$

by [Sch22, Proposition 14.7]. Equivalently, it is the pushforward along  $\tilde{X} \rightarrow \mathrm{Spd} K$  of  $\mathbb{B}_I$  seen as a solid sheaf as in [AM24, §4].

The action of  $G \times H$  on  $\tilde{X}$  gives rise to the structure of an almost smooth  $G \times H$  representation on  $R\Gamma_v(\tilde{X}, \mathbb{B}_{I,b}^+)$  by Proposition 2.2.3 which then can be seen as an almost module over  $\mathbb{B}_{[0,r],b}^+(K^b, K^{+,b}) \square [G \times H]$  via Remark 2.2.4. Then, after taking limits and colimits, the solid  $\mathbb{Q}_p$ -vector space  $R\Gamma_v(\tilde{X}, \mathbb{B}_I)$  has a natural action of  $G \times H$ , and so it gives rise an object in the derived category  $\mathcal{D}(\mathbb{Q}_p \square [G \times H])$  of solid  $\mathbb{Q}_p$ -linear  $G \times H$ -representations (even an object in the derived  $\infty$ -category of semilinear solid representations of  $G \times H$  over the Huber pair  $(\mathbb{B}_I(K^b), \mathbb{B}_I^+(K^{+,b}))$ , namely the  $\infty$ -category  $\mathcal{D}((\mathbb{B}_I(K^b), \mathbb{B}_I^+(K^{+,b})) \square [G \times H])$ ). We want to prove the following theorem:

**Theorem 5.1.1.** *The natural map*

$$R\Gamma_v(\tilde{X}, \mathbb{B}_I)^{R(G \times H) - la} \xrightarrow{\sim} R\Gamma_v(\tilde{X}, \mathbb{B}_I)^{RG - la}$$

*from  $G \times H$ -locally analytic vectors to  $G$ -locally analytic vectors is an equivalence.*

*Proof.* We can assume without loss of generality that both  $G$  and  $H$  are uniform pro- $p$ -groups. We shall consider almost mathematics with respect to  $([\varpi]^{1/p^n})_n$ .

The strategy to prove Theorem 5.1.1 is to apply the locally analytic criterion of Lemma 2.4.1 for the group  $H$  for suitable “lattices” of  $R\Gamma_v(\tilde{X}, \mathbb{B}_I)^{RG - la}$ . We employ this strategy in different steps. We first make some formal reductions.

**Lemma 5.1.2.** *Suppose that Theorem 5.1.1 holds for smooth affinoid rigid spaces  $Y$  admitting toric coordinates  $\psi : Y \rightarrow \mathbb{T}_K^d$ . Then it holds for general qcqs smooth rigid space  $X$ .*

*Proof.* We first show that Theorem 5.1.1 holds for a quasi-compact and separated rigid space  $X$ . Let  $\{X_i\}_{i=1}^k$  be an affinoid cover of  $X$  by subspaces admitting toric charts and let  $\{X_J\}_{J \subset \{1, \dots, d\}}$  be the poset of finite intersections of the  $X_i$ . Any finite intersection  $X_J$  is then affinoid and admits a toric chart. For all  $J$  let us write  $\tilde{X}_J = X_J \times_X \tilde{X}$ . Thus, we have that

$$R\Gamma_v(\tilde{X}, \mathbb{B}_I) = \varprojlim_{J \subset \{1, \dots, d\}} R\Gamma_v(\tilde{X}_J, \mathbb{B}_I).$$

The claim follows since the functor of locally analytic vectors commutes with finite limits (being an exact functor of stable  $\infty$ -categories). Now, for general  $X$  qcqs, we argue as before by taking a finite affinoid cover  $\{X_i\}_{i=1}^n$ , and noticing that any finite intersection  $X_J$  is a quasi-compact separated rigid space.  $\square$

From now on we suppose that  $X$  has toric coordinates  $\psi : X \rightarrow \mathbb{T}_K^d$ .

5.1.1. *Modifying the locally analytic functions.* First, we can write the left regular representation  $C^{la}(G, \mathbb{Q}_p)_{\star_1} = \varinjlim_{h \rightarrow \infty} V_h$  as a colimit of analytic Banach representations of  $G$  with  $V_h = C^h(G, \mathbb{Q}_p)_{\star_1}$  a suitable space of  $h$ -analytic functions endowed with a left regular action. We can fix compatible  $\mathbb{Z}_p$ -lattices  $V_h^+ \subset V_h$  and define the  $p$ -adically complete  $v$ -sheaf  $\mathcal{F}_{V_h^+}$  over  $X_v$  obtained by proétale descent along the  $G$ -torsor  $\tilde{X} \rightarrow X^\diamond$ , see Definition 3.3.3. We set  $\mathcal{F}_{V_h} = \mathcal{F}_{V_h^+}[\frac{1}{p}]$ .

**Lemma 5.1.3.** *There is a natural  $G \times H$ -equivariant isomorphism of solid abelian groups*

$$R\Gamma_v(\tilde{X}, \mathbb{B}_I)^{RG-la} = \varinjlim_{h \rightarrow \infty} R\Gamma_v(X, \mathbb{B}_I^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_{V_h^+})[\frac{1}{[\varpi]}],$$

where the completed tensor in the RHS term is a  $p$ -adically complete tensor product.

*Proof.* Recall that both  $p$  and  $[\varpi]$  are pseudo-uniformizers in  $\mathbb{B}_I$ . Since  $\tilde{X}$  is qcqs we can write

$$R\Gamma_v(\tilde{X}, \mathbb{B}_I) = R\Gamma_v(\tilde{X}, \mathbb{B}_I^+)[\frac{1}{[\varpi]}]$$

Thus, since  $G$  is compact, we get that

$$\begin{aligned} R\Gamma_v(\tilde{X}, \mathbb{B}_I)^{RG-la} &= R\Gamma(G, R\Gamma_v(\tilde{X}, \mathbb{B}_I) \otimes_{\mathbb{Q}_p, \square}^L C^{la}(G, \mathbb{Q}_p)_{\star_1}) \\ &= \varinjlim_h R\Gamma(G, R\Gamma_v(\tilde{X}, \mathbb{B}_I^+) \otimes_{\mathbb{Z}_p, \square}^L V_h^+)[\frac{1}{[\varpi]}] \end{aligned}$$

where the spaces  $C^{la}(G, \mathbb{Q}_p)_{\star_1}$  and  $V_h$  have the left regular action of  $G$ , the first equality is solid group cohomology as in [RJC22, Definition 5.1 (1)], and in the second equality we use that the trivial representation is a compact  $\mathbb{Z}_p, \square[G]$ -module thanks to the Lazard resolution which exists since  $G$  is an uniform pro- $p$ -group.

Now, since both  $V_h^+$  and  $R\Gamma_v(\tilde{X}, \mathbb{B}_I^+)$  are almost bounded to the right and derived  $p$ -complete, we have by [Man22b, Proposition 2.12.10 (i)] that the solid tensor product  $R\Gamma_v(\tilde{X}, \mathbb{B}_I^+) \otimes_{\mathbb{Z}_p, \square}^L V_h^+$  is derived  $p$ -complete and almost equal to

$$R \varprojlim_{k,s} (R\Gamma_v(\tilde{X}, \mathbb{B}_I^+ / [\varpi]^k) \otimes_{\mathbb{Z}/p^s}^L V_h^+ / p^s) = R \varprojlim_{k,s} (R\Gamma_v(\tilde{X}, \mathbb{B}_I^+ / [\varpi]^k) \otimes_{\mathbb{Z}/p^s}^L \mathcal{F}_{V_h^+} / p^s) \quad (5.1)$$

where in the second term we use that  $X$  is qcqs and that  $V^+ / p^s$  is a discrete  $\mathbb{Z}/p^s$ -module. We get that

$$\begin{aligned} R\Gamma_v(\tilde{X}, \mathbb{B}_I)^{RG-la} &= \varinjlim_h \left( R \varprojlim_{k,s} R\Gamma(G, R\Gamma_v(\tilde{X}, \mathbb{B}_I^+ / [\varpi]^k) \otimes_{\mathbb{Z}/p^s}^L \mathcal{F}_{V_h^+} / p^s) \right) [\frac{1}{[\varpi]}] \\ &= \varinjlim_h \left( R \varprojlim_{k,s} R\Gamma^{sm}(G, R\Gamma_v(\tilde{X}, \mathbb{B}_I^+ / [\varpi]^k) \otimes_{\mathbb{Z}/p^s}^L \mathcal{F}_{V_h^+} / p^s) \right) [\frac{1}{[\varpi]}] \\ &= \varinjlim_h \left( R \varprojlim_{k,s} R\Gamma_v(X, \mathbb{B}_I^+ / [\varpi]^k) \otimes_{\mathbb{Z}/p^s}^L \mathcal{F}_{V_h^+} / p^s \right) [\frac{1}{[\varpi]}] \\ &= \varinjlim_h R\Gamma_v(X, \mathbb{B}_I^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_{V_h^+})[\frac{1}{[\varpi]}] \end{aligned} \quad (5.2)$$

where in the first equality we use (5.1), the fact group cohomology for  $G$  commutes with limits, and that is also commutes with colimits since  $G$  is uniform pro- $p$ . The second equality we use Proposition 2.2.3 and [Man22b, Remark 3.4.12] to compare solid and smooth cohomology. In the third equality we also use Proposition 2.2.3 and the fact that  $v$ -cohomology is the pushforward along the map  $X^\diamond = \tilde{X}/G \rightarrow (\mathrm{Spd} K)/G \rightarrow \mathrm{Spd} K$ . In the last equality we use that cohomology commutes with derived limits of sheaves and that  $\mathbb{B}_I^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_{V_h^+}$  is derived  $([\varpi], p)$ -complete. This proves the lemma.  $\square$

The objects  $R\Gamma_v(X, \mathbb{B}_I^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_{V_h^+})$  are  $\varpi$ -adically complete solid  $G \times H$ -representations over  $\mathbb{B}_I^+(K)$ , where the  $H$ -action arises from the action on  $X$  and the  $G$ -action is induced by the right regular action on  $V_h^+$ . To prove Theorem 5.1.1 it suffices to show the following:

**Dévisage 1.** *The solid  $H$ -representation*

$$R\Gamma_v(X, \mathbb{B}_I^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_{V_h^+}) \left[ \frac{1}{[\varpi]} \right] \quad (5.3)$$

is locally analytic.

To prove Dévisage 1, we need to modify a little bit more the lattices  $R\Gamma_v(X, \mathbb{B}_I^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_{V_h^+})$ .

5.1.2. *Modifying the sheaf of periods.* Let us write  $I = [s, r] \subset (0, \infty)$ . Consider the  $[\varpi]$ -adically complete complex

$$\widetilde{\mathbb{B}}_I^+ := [\mathbb{B}_{[0,r]}^+ \langle T \rangle \xrightarrow{p\Gamma - [\varpi]^{1/s}} \mathbb{B}_{[0,r]}^+ \langle T \rangle] \quad (5.4)$$

and take the lattice of (5.3) given by

$$R\Gamma_v(X, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+}). \quad (5.5)$$

5.1.3. *Modifying the level.* For  $G_0 \subset G$  an open compact normal subgroup let  $X_{G_0} = \widetilde{X}/G_0$ . Then

$$R\Gamma_v(X, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+}) \left[ \frac{1}{[\varpi]} \right]$$

is just a finite colimit (given by the invariants of  $G/G_0$ ) of

$$R\Gamma_v(X_{G_0}, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+}) \left[ \frac{1}{[\varpi]} \right].$$

Since the category of solid locally analytic representations is stable under colimits by [RJRC23, Proposition 3.2.3], to show Dévisage 1 it suffices to prove the following statement:

**Dévisage 2.** *There is an open compact subgroup  $G_0 \subset G$  such that*

$$R\Gamma_v(X_{G_0}, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+}) \left[ \frac{1}{[\varpi]} \right] \quad (5.6)$$

is a locally analytic representation of  $H$ .

We shall take  $G_0$  such that the  $G_0$ -module  $V_h^+/p^b$  is isomorphic to the trivial representation  $\bigoplus_I \mathbb{Z}/p^b$  for some fixed  $b$  that we shall choose in Section 5.1.5.

5.1.4. *Applying the décalage functor.* Finally, it is well known that the proétale cohomology of  $\widehat{\mathcal{O}}^+$  has some junk torsion (eg. see [BMS18]), this makes difficult to apply the analyticity criterion of Lemma 2.4.1 to the lattice of Dévisage 2. A way to solve this problem is to modify the lattice a little bit by applying a décalage functor. First, we need to guarantee that the décalage functor preserves the structure of a solid representation, for this it suffices to see that the category of solid representations of a profinite group  $\Pi$  can be obtained as the derived category of abelian sheaves on a ringed topos. This is a consequence of the next lemma:

**Lemma 5.1.4** ([Man22a, Lemma 10.3]). *Let  $\Pi$  be a profinite group, and  $\Lambda$  a nuclear  $\mathbb{Z}_p$ -algebra. Then there is a natural equivalence of  $\infty$ -categories*

$$\mathcal{D}_{\square}(*/\Pi, \Lambda) \cong \mathcal{D}_{\square}(\Lambda, \mathbb{Z}_p)^{BG} = \mathcal{D}(\Lambda_{\square}[G])$$

between solid sheaves on the proétale site of the  $v$ -stack  $*/\Pi$  with  $\mathbb{Z}_p$ -coefficients, and the category of  $\Lambda$ -linear solid representations of  $\Pi$ .

*Proof.* The result in *loc. cit.* is only stated for  $\ell \neq p$ , let us see that this is actually not necessary. Indeed, the proétale site of  $*/G$  is independent of the prime  $p$  used in the definition of  $v$ -stacks where  $*/G$  is considered. So we could have taken  $*/G$  as an object in  $v$ -stacks for perfectoid spaces in characteristic  $\ell \neq p$  and still get the same conclusion.  $\square$

By taking  $\Lambda = \mathbb{B}_I^+(K^b)$  we can apply the décalage functor  $L\eta_{[\varpi]^\varepsilon}$  for any rational  $\varepsilon$  to  $\Lambda$ -linear solid representations of  $\Pi$ .

For  $\varepsilon > 0$  rational, to be determined in Section 5.1.5, we shall consider the following lattice of (5.6)

$$L\eta_{[\varpi]^\varepsilon} R\Gamma_v(X_{G_0}, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+}). \quad (5.7)$$

In this way, to show Dévisage 2 it suffices to prove the following:

**Dévisage 3.** *There exists  $G_0$  and  $\varepsilon > 0$  such that the  $[\varpi]$ -complete lattice*

$$L\eta_{[\varpi]^\varepsilon} R\Gamma_v(X_{G_0}, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+})$$

*satisfies the hypothesis of Lemma 2.4.1 for the action of  $H$ .*

5.1.5. *Reduction to  $\mathcal{O}_X^+/p^{1/r}$ .* We finally perform the last dévisage in the proof of Theorem 5.1.1. Let us write  $I = [s, r] \subset (0, \infty)$  so that  $|\varpi|^{1/s} \leq |p| \leq |\varpi|^{1/r}$ . Thus, we shall make the following choices:

- i. We take  $G_0$  such that the action of  $G_0$  on  $V_h^+ \otimes_{\mathbb{Z}_p} \mathbb{B}_{[0,r]}^+(K^b)/[\varpi]^{1/s}$  is trivial. In particular, as  $V_h^+$  is a torsion free  $p$ -adically complete  $\mathbb{Z}_p$ -module, the  $G_0$ -representation  $V_h^+ \otimes_{\mathbb{Z}_p} \mathbb{B}_{[0,r]}^+(K^b)/[\varpi]^{1/s}$  is isomorphic to a direct sum of copies of  $\mathbb{B}_{[0,r]}^+(K^b)/[\varpi]^{1/s}$ .
- ii. We take  $\varepsilon < 1/r$ .

**Step 1.** We first have to guarantee that  $L\eta_{[\varpi]^\varepsilon} R\Gamma_v(X_{G_0}, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+})$  is  $[\varpi]$ -adically complete and bounded to the right. The first claim follows from [BMS18, Lemma 6.20] and the fact that  $R\Gamma_v(X_{G_0}, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+})$  is  $\varpi$ -adically complete. To see that it is bounded to the right, by  $[\varpi]$ -adically completeness and since the décalage functor kills  $[\varpi]^\varepsilon$ -torsion, it suffices to see that

$$R\Gamma_v(X_{G_0}, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+})/{}^{\mathbb{L}}[\varpi]^{1/r}$$

is almost bounded to the right. By (5.4), the complex  $\widetilde{\mathbb{B}}_I^+$  is constructed with terms given by  $\mathbb{B}_{[0,r]}^+\langle T \rangle$ . Thus, it suffices to show that

$$R\Gamma_v(X_{G_0}, (\mathbb{B}_{[0,r]}^+ / [\varpi]^{1/r}) \otimes_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+})$$

is bounded to the right. By the choice of  $G_0$ , we know that  $\mathbb{B}_{[0,r]}^+ / [\varpi]^{1/r} \otimes_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+}$  is isomorphic to a direct sum of copies of  $\mathbb{B}_{[0,r]}^+ / [\varpi]^{1/r}$  which by Lemma 2.1.4 (2) is almost isomorphic to a polynomial algebra

$$\mathbb{B}_{[0,r]}^+ / [\varpi]^{1/r} \cong^a \mathcal{O}_X^+ / \varpi^{1/r}[S]. \quad (5.8)$$

Hence, we are reduced to see that

$$R\Gamma_v(X_{G_0}, \mathcal{O}_X^+ / p^{1/r})$$

is almost bounded to the right, which is clear as  $X_{G_0}$  is an affinoid smooth rigid space.

**Step 2.** Since  $[\varpi]$  and  $p$  are both pseudo-uniformizers of  $\mathbb{B}_I$ , if the hypothesis of Lemma 2.4.1 holds for a power  $[\varpi]^\delta$  of  $[\varpi]$  then, after base change by a sufficiently ramified extension  $K$  of  $\mathbb{Q}_p$ , the hypothesis will hold for the pseudo-uniformizer of  $\pi$  of  $K$ . Indeed, we just need to pick  $\pi$  such that  $|\pi| = |p|^{\delta r} \leq |\varpi|^\delta$ .

**Step 3.** We will show that there is an open compact subgroup  $H_0 \subset H$  such that for all  $h \in H_0$  the map  $1 - h$  on

$$L\eta_{[\varpi]^\varepsilon} R\Gamma_v(X_{G_0}, \widetilde{\mathbb{B}}_I^+ \widehat{\otimes}_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+})/{}^{\mathbb{L}}[\varpi]^{1/r-\varepsilon} \quad (5.9)$$

is homotopic to zero as  $\mathbb{B}_{[0,r]}^+(K^b, K^{+,b})/[\varpi]^{1/r-\varepsilon}$ -module.

By Steps 1 and 2 and Lemma 2.4.1 we will obtain that the  $H$ -representation (5.6) is locally analytic proving Theorem 5.1.1.

By Lemma 2.3.3 the object (5.9) is equivalent to

$$L\eta_{[\varpi]^\varepsilon} R\Gamma_v(X_{G_0}, (\mathbb{B}_I^+ / {}^{\mathbb{L}}[\varpi]^{1/r}) \otimes_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+}). \quad (5.10)$$

By the choice of  $G_0$ , the proétale sheaf  $(\widetilde{\mathbb{B}}_I^+/\mathbb{L}[\varpi]^{1/r}) \otimes_{\mathbb{Z}_p}^L \mathcal{F}_{V_h^+}$  is isomorphic to a direct sum of copies of  $\widetilde{\mathbb{B}}_I^+/\mathbb{L}[\varpi]^{1/r}$ . Therefore, since the décalage functor commutes with direct sums [BMS18, Corollary 6.5], it suffices to show that for all  $h \in H_0$  the operator  $1 - h$  is homotopically equivalent to zero when acting on

$$L\eta_{[\varpi]^\varepsilon} R\Gamma_v(X_{G_0}, (\widetilde{\mathbb{B}}_I^+/\mathbb{L}[\varpi]^{1/r})).$$

By the definition of  $\widetilde{\mathbb{B}}_I^+$  in (5.4), and since  $p$  and  $[\varpi]^{1/r}$  are divisible by  $[\varpi]^{1/r}$  in  $\mathbb{B}_{[0,r]}^+$ , we have that

$$\widetilde{\mathbb{B}}_I^+/\mathbb{L}[\varpi]^{1/r} = \mathbb{B}_{[0,r]}^+/\mathbb{L}[\varpi]^{1/r}[T][1] \oplus \mathbb{B}_{[0,r]}^+/\mathbb{L}[\varpi]^{1/r}[T].$$

Finally, by (5.8) and since  $L\eta_{[\varpi]^\varepsilon}$  preserves shifts and direct sums, we are reduced to show the following dévissage:

**Dévissage 4.** *There is an open compact subgroup  $H_0 \subset H$  such that for all  $h \in H_0$  the operator  $1 - h$  on*

$$L\eta_{p^\varepsilon} R\Gamma_v(X_{G_0}, \mathcal{O}_X^+/p^{1/r})$$

*is homotopic to zero as  $\mathcal{O}_K/p^{1/r}$ -module.*

5.1.6. *Final computation.* We now prove Dévissage 4. We can assume without loss of generality that  $G_0 = G$  and so  $X_{G_0} = X$ . By Lemma 5.1.2 we can also assume that  $X$  has toric coordinates  $\Psi : X \rightarrow \mathbb{T}_K^d$ . Let  $\mathbb{T}_{K,\infty}^d$  be the perfectoid torus and let  $\Gamma = \mathbb{Z}_p(1)^d$  the Galois group of  $\mathbb{T}_{K,\infty}^d$  over  $\mathbb{T}_K^d$ . For  $n \in \mathbb{N}$  we let  $\mathbb{T}_{K,n}^d = \mathbb{T}_{K,\infty}^d/\Gamma^{p^n}$  and  $X_n = X \times_{\mathbb{T}_K^d} \mathbb{T}_{K,n}^d$ , similarly we let  $X_\infty = X \times_{\mathbb{T}_K^d} \mathbb{T}_{K,\infty}^d$ . By [RC23, Proposition 3.2.3] the pair  $(\mathcal{O}(X_\infty), \Gamma)$  is a strongly decomposable Sen theory in the sense of [RC23, Definition 2.2.6]. In particular, we have Sen traces  $\mathrm{Tr}_n : \mathcal{O}(X_\infty) \rightarrow \mathcal{O}(X_n)$  with kernel  $C_n$ . Thus, by letting  $C_n^+ = C_n \cap \mathcal{O}(X_n)^+$ , there is some  $n \gg 0$  such that the cokernel of the map

$$C_n^+ \oplus \mathcal{O}^+(X_n) \rightarrow \mathcal{O}^+(X_\infty)$$

as well as the group cohomology  $R\Gamma(\Gamma, C_n^+)$  are killed by  $p^\varepsilon$ . Since we have an almost equivalence

$$R\Gamma(\Gamma, \mathcal{O}^+(X_\infty)/p^{1/r}) \cong^a R\Gamma_v(X, \mathcal{O}_X^+/p^{1/r}),$$

there exists some  $n \gg 0$  depending on  $\varepsilon$  such that we have an equivalence

$$L\eta_{p^\varepsilon}(R(\Gamma, \mathcal{O}^+(X_n)/p^{1/r})) \cong L\eta_{p^\varepsilon} R\Gamma_v(X, \mathcal{O}_X^+/p^{1/r}) \quad (5.11)$$

of  $\mathcal{O}_K/p^{1/r}$ -complexes. Let us now justify that (5.11) can be promoted to an equivalence of smooth  $H_0$ -representations for some  $H_0 \subset H$  small enough. Indeed, by [Sch18, Lemma 2.3] there is an open subgroup  $H_0 \subset H$  such that the action of  $H_0$  can be lifted from  $X$  to an action  $\sigma$  on  $X_n$ . Moreover, since for all  $\gamma \in \Gamma/\Gamma^{p^n}$  the conjugation  $\gamma^{-1} \circ \sigma \circ \gamma$  is another lift of the action of  $H_0$  to  $X_n$ , by refining  $H_0$  and using [Sch18, Lemma 2.3] again we can suppose that both the actions of  $H_0$  and  $\Gamma/\Gamma^{p^n}$  on  $X_n$  commute. This shows that the map (5.11) can be upgraded to a map of smooth  $H_0$ -representations as wanted.

But then, by Corollary 2.4.2, we can shrink  $H_0$  so that it acts trivially on  $\mathcal{O}^+(X_n)/p^{1/r}$ . Thus, for  $h \in H_0$  the action of  $1 - h$  on the left hand side of (5.11) is homotopic to zero as  $\mathcal{O}_K/p^{1/r}$ -module finishing the proof of Theorem 5.1.1.  $\square$

*Remark 5.1.5.* Theorem 5.1.1 will hold for a much larger class of  $v$ -sheaves or complexes  $\mathcal{F}$  following essentially the same proof. For example it holds under the following conditions which hold for  $\mathbb{B}_I$  and  $\widehat{\mathcal{O}}$ -vector bundles:

- The complex  $\mathcal{F}$  is of the form  $\mathcal{F} = \mathcal{F}^\circ[\frac{1}{p}]$  with  $\mathcal{F}^\circ$  a connective derived  $p$ -complete sheaf such that  $\mathcal{F}^\circ/\mathbb{L}p$  arises from the étale site of  $X$  (in particular  $\mathcal{F}$  is a solid sheaf as in [AM24, §4]).
- There is some  $b > 0$  such that  $\mathcal{F}^\circ/\mathbb{L}p^b$  is a retract of a  $v$ -sheaf of the form  $\widehat{\mathcal{O}}^+/p^b \otimes_{\mathbb{Z}_p}^L V$  with  $V$  a complex of  $\mathbb{Z}_p$ -modules.

Indeed, one has to prove Dévissage 3 for the cohomology  $L\eta_{p^\varepsilon} R\Gamma_v(X_{G_0}, \mathcal{F}^+)$ . But since  $X_{G_0}$  is qcqs and the décalage operator sends retracts to retracts, the criterion of Lemma 2.4.1 holds for this lattice if it does for

$$L\eta_{p^\varepsilon} R\Gamma_v(X_{G_0}, \widehat{\mathcal{O}}^+/p^b \otimes_{\mathbb{Z}_p}^L V) = L\eta_{p^\varepsilon} [R\Gamma_v(X_{G_0}, \widehat{\mathcal{O}}^+/p^b) \otimes_{\mathbb{Z}_p}^L V]$$

Using the symmetric monoidality of the décalage operator [BMS18, Proposition 6.8] one is reduced to Dévisage 4.

An immediate corollary of Theorem 5.1.1 is that the cohomology groups of qcqs smooth rigid spaces endowed with an action of a  $p$ -adic Lie group is locally analytic.

**Corollary 5.1.6.** *Let  $X$  be a qcqs smooth rigid space over a perfectoid field  $K$  admitting all  $p$ -th power roots of unit. Suppose that  $X$  is endowed with a continuous action of a  $p$ -adic Lie group  $H$ . Then for any compact interval  $I \subset (0, \infty)$  the  $v$ -cohomology*

$$R\Gamma_v(X, \mathbb{B}_I)$$

is a solid locally analytic  $H$ -representation.

*Proof.* This is a particular case of Theorem 5.1.1 where  $G = 1$ .  $\square$

5.1.7. *Conclusion for local Shimura varieties.* In this paragraph we apply Theorem 5.1.1 to local Shimura varieties. Recall that  $(\mathbf{G}, b, \mu)$  denotes a local Shimura datum, and for  $K \subset \mathbf{G}(\mathbb{Q}_p)$  a compact open subgroup we have the local Shimura variety  $\mathcal{M}_{\mathbf{G}, b, \mu, K}$  of level  $K$ . We also let  $\mathcal{M}_{\mathbf{G}, b, \mu, \infty}^\diamond$  be the infinite level local Shimura variety seen as a diamond over  $\mathrm{Spd} \check{E}$ . We fix  $C/\check{E}$  be a complete algebraically closed extension, and consider the base change of local Shimura varieties to  $C$ .

**Definition 5.1.7.** We let  $\widehat{\mathcal{O}}_{\mathcal{M}}$  be the restriction of the  $v$ -sheaf  $\widehat{\mathcal{O}}$  to the topological space  $|\mathcal{M}_{\mathbf{G}, b, \mu, \infty, C}|$  and let  $\mathcal{O}_{\mathcal{M}}^{G-la} \subset \widehat{\mathcal{O}}_{\mathcal{M}}$  be the subsheaf mapping a qcqs open subspace  $U_\infty \subset \mathcal{M}_{\mathbf{G}, b, \mu, \infty, C}^\diamond$  to the space of  $G = \mathbf{G}(\mathbb{Q}_p)$ -locally analytic vectors

$$\mathcal{O}_{\mathcal{M}}^{G-la}(\check{U}_\infty) = \widehat{\mathcal{O}}_{\mathcal{M}}(U_\infty)^{K_{U_\infty} - la}$$

where  $K_{U_\infty} \subset \mathbf{G}(\mathbb{Q}_p)$  is the stabilizer of  $U_\infty$ . If  $G$  is clear from the context we write  $\mathcal{O}_{\mathcal{M}}^{la}$  instead of  $\mathcal{O}_{\mathcal{M}}^{G-la}$ .

*Remark 5.1.8.* The fact that  $\mathcal{O}_{\mathcal{M}}^{la}$  is a sheaf follows from [RC24b, Lemma 6.2.2].

We obtain a generalization of a theorem of Lue Pan for the Lubin-Tate space, see [Pan22b, Corollary 5.3.9].

**Corollary 5.1.9.** *For any  $p$ -adic Lie group  $H \subset \check{G}_b$  and any qcqs open subspace  $U_\infty \subset \mathcal{M}_{\mathbf{G}, b, \mu, \infty, C}^\diamond$  the natural map*

$$\mathcal{O}_{\mathcal{M}}^{G-la}(U_\infty)^{RH-la} \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}}^{G-la}(U_\infty)$$

is an equivalence. In particular, if  $b$  is basic we have an equality of subsheaves of  $\widehat{\mathcal{O}}_{\mathcal{M}}$

$$\mathcal{O}_{\mathcal{M}}^{G-la} = \mathcal{O}_{\mathcal{M}}^{G_b-la}.$$

More generally, for  $b$  basic and  $I \subset (0, \infty)$  a compact interval, we have an equivalence of derived solid locally analytic representations of  $G \times G_b$

$$R\Gamma_v(U_\infty, \mathbb{B}_I)^{RG_b-la} \xrightarrow{\sim} R\Gamma_v(U_\infty, \mathbb{B}_I)^{RG \times G_b-la} \xleftarrow{\sim} R\Gamma_v(U_\infty, \mathbb{B}_I)^{RG-la}.$$

*Proof.* The first claim follows from Theorem 5.1.1 and Remark 5.1.5, and the fact that  $\widehat{\mathcal{O}}_{\mathcal{M}}$  has no higher locally analytic vectors by Theorem 4.3.3. The claim when  $b$  is basic follows also from Theorem 5.1.1 and the fact that for  $K_b \subset G_b$  a compact open subgroup, the quotient  $\mathcal{M}_{\mathbf{G}, b, \mu, \infty}^\diamond / K_b \cong \mathcal{M}_{\check{\mathbf{G}}, \check{b}, \check{\mu}, K_b}^\diamond$  is the diamond attached to a local Shimura variety of level  $K_b$  for the dual Shimura datum  $(\check{\mathbf{G}}, \check{b}, \check{\mu})$ .  $\square$

In the following we shall write  $\mathcal{F}l_{\mathbf{G}, \mu}$  and  $\mathcal{F}l_{\mathbf{G}, \mu^{-1}}$  for the base change to  $(C, \mathcal{O}_C)$  of the flag varieties  $\mathcal{F}l_{\mathbf{G}, \mu, E}$  and  $\mathcal{F}l_{\mathbf{G}, \mu^{-1}, E}$  respectively. For  $b$  basic, we write  $\mathcal{O}_{\mathcal{M}}^{la}$  for  $\mathcal{O}_{\mathcal{M}}^{G-la}$ , by Corollary 5.1.9 there is no ambiguity in the locally analytic vectors for the group  $G$  or  $G_b$ . Let  $\mathfrak{g}_{\mu^{-1}}^0 = \mathrm{Lie} \mathbf{G}(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{F}l_{\mathbf{G}, \mu^{-1}}}$  and  $\mathfrak{g}_\mu^0 = \mathrm{Lie} G_b \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{F}l_{\mathbf{G}, \mu}}$  be the Lie algebroids over the Hodge-Tate and Grothendieck-Messing flag varieties respectively. Let  $\mathfrak{n}_\mu^0 \subset \mathfrak{p}_\mu^0 \subset \mathfrak{g}_\mu^0$  be the natural filtration on  $\mathcal{F}l_{\mathbf{G}, \mu}$  with Levi quotient  $\mathfrak{m}_{\mu^{-1}}^0$  (resp. for  $\mathfrak{n}_{\mu^{-1}}^0 \subset \mathfrak{p}_{\mu^{-1}}^0 \subset \mathfrak{g}_{\mu^{-1}}^0$  over  $\mathcal{F}l_{\mathbf{G}, \mu^{-1}}$  with Levi quotient  $\mathfrak{m}_\mu^0$ ). They arise from the  $\mathbf{P}_\mu$ -equivariant filtration  $\mathfrak{n}_\mu \subset \mathfrak{p}_\mu \subset \mathfrak{g}_C$  and Levi quotient  $\mathfrak{m}_\mu := \mathfrak{p}_\mu / \mathfrak{n}_\mu$  (resp. for the  $\mathbf{P}_{\mu^{-1}}$ -equivariant filtration  $\mathfrak{n}_{\mu^{-1}} \subset \mathfrak{p}_{\mu^{-1}} \subset \mathfrak{g}_C$  and Levi quotient  $\mathfrak{m}_{\mu^{-1}} = \mathfrak{p}_{\mu^{-1}} / \mathfrak{n}_{\mu^{-1}}$ ). We identify the pullback of  $\mathfrak{m}_\mu^0$  and  $\mathfrak{m}_{\mu^{-1}}^0$  to  $\mathcal{O}_{\mathcal{M}}^{la}$  via Corollary 3.3.7



(after taking locally analytic vectors) and denote it  $\mathfrak{m}^{0,la}$ . The following is the generalization of [Pan22b, Corollary 5.3.13].

**Theorem 5.1.10.** *The actions of  $\mathfrak{n}_\mu^0$  and  $\mathfrak{n}_{\mu-1}^0$  on  $\mathcal{O}_M^{la}$  vanish. Furthermore, the actions of  $\mathfrak{m}_\mu^0$  and  $\mathfrak{m}_{\mu-1}^0$  on  $\mathcal{O}_M^{la}$  by derivations are identified via the pullback*

$$\mathfrak{m}_{\mu-1}^0 \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}} \mathcal{O}_M^{la} = \mathfrak{m}^{0,la} = \mathcal{O}_M^{la} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu}}} \mathfrak{m}_\mu^0.$$

In particular, the central character of the actions of  $\mathfrak{m}_\mu^0$  and  $\mathfrak{m}_{\mu-1}^0$  on  $\mathcal{O}_M^{la}$  agree under the natural isomorphism of the center of the enveloping algebras  $\mathcal{Z}(\mathfrak{m}_\mu)_C \cong \mathcal{Z}(\mathfrak{m}_{\mu-1})_C$ .

*Proof.* The vanishing for the action of the geometric Sen operators follows from Theorem 4.3.3. We now prove the relation between the horizontal actions. In the following we forget about the action of the Galois group of  $E$  and fix a trivialization of the Tate twist  $\mathbb{Z}_p(1) \cong \mathbb{Z}_p$  obtained by fixing a sequence of  $p$ -th power roots of unit  $(\zeta_{p^n})_n$ . In the following all completed tensor products are solid.

Let  $\mathcal{M}_{\infty,C}^{la}$  be the ringed space whose underlying topological space is  $|\mathcal{M}_{\mathbf{G},b,\mu,\infty,C}|$  and sheaf of functions given by the algebra  $\mathcal{O}_M^{la}$ . We have locally analytic Hodge-Tate period maps

$$\begin{array}{ccc} & \mathcal{M}_{\infty,C}^{la} & \\ \pi_{\mathbf{G}\mathbf{M}}^{la} \swarrow & & \searrow \pi_{\mathbf{HT}}^{la} \\ \mathcal{F}\ell_{\mathbf{G},\mu} & & \mathcal{F}\ell_{\mathbf{G},\mu-1}. \end{array}$$

Let  $W$  be a representation of the Levi  $\mathbf{M}(= \mathbf{M}_\mu = \mathbf{M}_{\mu-1})$ , taking locally analytic vectors in Corollary 3.3.7 we get  $\mathbf{G}(\mathbb{Q}_p) \times G_b$ -equivariant isomorphisms of vector bundles over  $\mathcal{M}_{\infty,C}^{la}$

$$\mathcal{W}_{\mathbf{G},\mu-1}(W) \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}} \mathcal{O}_M^{la} = \mathcal{O}_M^{la} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu}}} \mathcal{W}_{\mathbf{G},\mu}(W). \quad (5.12)$$

Let  $\mathbf{M}_{\mu,\mathbf{GM}} \rightarrow \mathcal{F}\ell_{\mathbf{G},\mu}$  and  $\mathbf{M}_{\mu-1,\mathbf{HT}} \rightarrow \mathcal{F}\ell_{\mathbf{G},\mu-1}$  be the natural  $\mathbf{M}$ -torsors, the equation (5.12) gives rise to a natural isomorphism of  $\mathbf{M}$ -torsors over  $\mathcal{M}_{\infty,C}^{la}$

$$\pi_{\mathbf{HT}}^{la,*}(\mathbf{M}_{\mu-1,\mathbf{HT}}) \cong \pi_{\mathbf{GM}}^{la,*}(\mathbf{M}_{\mu,\mathbf{GM}}).$$

Thus, if  $\mathbf{M}_{\mu,\mathbf{GM}}^{an}$  and  $\mathbf{M}_{\mu-1,\mathbf{HT}}^{an}$  denote the analytification of the algebraic torsors over the flag varieties, the period maps refined to a mixed period map

$$\pi_{\mathbf{GM},\mathbf{HT}}^{la} : \mathcal{M}_{\infty,C}^{la} \rightarrow \mathbf{M}_{\mu,\mathbf{GM}}^{an} \times^{\mathbf{M}^{an}} \mathbf{M}_{\mu-1,\mathbf{HT}}^{an}.$$

Note that the Lie algebra  $\mathfrak{g} \times \mathfrak{g}_b$  acts on  $\mathbf{M}_{\mu,\mathbf{GM}}^{an} \times^{\mathbf{M}^{an}} \mathbf{M}_{\mu-1,\mathbf{HT}}^{an}$  by derivations, and by construction both horizontal actions  $\mathfrak{m}_\mu^0$  and  $\mathfrak{m}_{\mu-1}^0$  are identified after pullback (similarly for the infinitesimal actions of  $\mathcal{Z}(\mathfrak{m}_\mu)_C$  and  $\mathcal{Z}(\mathfrak{m}_{\mu-1})_C$ ). Therefore, in order to show the theorem it suffices to show that the map of rings

$$\pi_{\mathbf{GM},\mathbf{HT}}^{la,-1}(\mathcal{O}_{\mathbf{M}_{\mu,\mathbf{GM}}^{an} \times^{\mathbf{M}^{an}} \mathbf{M}_{\mu-1,\mathbf{HT}}^{an}}) \rightarrow \mathcal{O}_M^{la}$$

is dense in a suitable sense. To make this precise, for any compact open subgroup  $K_p \subset \mathbf{G}(\mathbb{Q}_p)$  consider the  $K_p$ -equivariant sheaf over  $\mathcal{F}\ell_{\mathbf{G},\mu-1}$  given by  $C^{la}(K_p, \mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}})$  and consider the colimit

$$C^{la}(\mathfrak{g}_{\mu-1}^0, \mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}) := \varinjlim_{K_p} C^{la}(K_p, \mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}).$$

Define

$$C^{la}(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}) = C^{la}(\mathfrak{g}_{\mu-1}^0, \mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}})^{\mathfrak{n}_{\mu-1}^0, \star_1 = 0}$$

to be the invariant subspace of  $\mathfrak{n}_{\mu-1}^0$ -horizontal sections for the left regular action. Let  $\mathcal{O}_M^{\mathbf{G}(\mathbb{Q}_p)-sm} \subset \mathcal{O}_M^{la}$  be the subalgebra of  $\mathbf{G}(\mathbb{Q}_p)$ -smooth sections, equal to the colimit of the structural sheaves of the finite level local Shimura varieties  $\mathcal{M}_{\mathbf{G},b,\mu,K_p,C}$ . By the proof of [RC24b, Proposition 6.2.8] (more precisely, Lemma 6.2.9), the pullback to  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,C,v}$

$$C^{la}(\mathfrak{g}_{\mu-1}^0, \widehat{\mathcal{O}}) = C^{la}(\mathfrak{g}_{\mu-1}^0, \mathcal{O}_{\mathcal{F}\ell}) \widehat{\otimes}_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}} \widehat{\mathcal{O}}$$

is a filtered colimit of ON Banach  $\widehat{\mathcal{O}}$ -modules which are relatively locally analytic in the sense of [RC23, Definition 1.0.1] (resp. for the pullback of  $C^{la}(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \widehat{\mathcal{O}})$ ). Moreover, the  $\mathbf{G}(\mathbb{Q}_p)$ -smooth vectors of  $C^{la}(\mathfrak{g}_{\mu-1}^0, \widehat{\mathcal{O}})$  (more precisely, of its restriction  $C^{la}(\mathfrak{g}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}})$  to a sheaf on the topological space  $|\mathcal{M}_{\mathbf{G}, b, \mu, \infty, C}|$ ) are well defined (by writing the module as a colimit of  $K_p$ -representations as  $K_p \rightarrow 1$ ), and by construction they consist on the algebra  $\mathcal{O}_{\mathcal{M}}^{la}$ . By [RC23, Theorem 3.3.2 (2)] and the computation of the geometric Sen operators in Theorem 4.3.1, we have that

$$\begin{aligned} \mathcal{O}_{\mathcal{M}}^{la} &= C^{la}(\mathfrak{g}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}})^{\mathbf{G}(\mathbb{Q}_p)-sm} \\ &= C^{la}(\mathfrak{g}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}})^{\mathbf{G}(\mathbb{Q}_p)-sm, \mathfrak{n}_{\mu-1}^0, *_1=0} \\ &= C^{la}(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}})^{\mathbf{G}(\mathbb{Q}_p)-sm}. \end{aligned}$$

But now  $C^{la}(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}})$  has trivial geometric Sen action, then [RC23, Theorem 3.3.2 (3)] implies that the orbit map (equivariant for the right regular action in the RHS)

$$\mathcal{O}_{\mathcal{M}}^{la} \rightarrow C^{la}(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}})$$

extends to a  $\mathfrak{g}$ -equivariant and  $\widehat{\mathcal{O}}_{\mathcal{M}}$ -linear isomorphism

$$\mathcal{O}_{\mathcal{M}}^{la} \widehat{\otimes}_{\mathcal{O}_{\mathcal{M}}^{\mathbf{G}(\mathbb{Q}_p)-sm}} \widehat{\mathcal{O}}_{\mathcal{M}} \xrightarrow{\sim} C^{la}(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}}) \quad (5.13)$$

where the action of  $\mathfrak{g}$  on the right hand side term is via the right regular action.

Let us write  $X = \mathbf{M}_{\mu, \text{GM}}^{an} \times^{\mathbf{M}^{an}} \mathbf{M}_{\mu-1, \text{HT}}^{an}$ . Over  $X$  we also have the Lie algebroid  $\mathfrak{g}_{\mu-1, X}^0/\mathfrak{n}_{\mu-1, X}^0$  which is nothing but the relative tangent space of  $X$  over  $\mathcal{F}\ell_{\mathbf{G}, \mu}$  and we have an isomorphism

$$C^{la}(\mathfrak{g}_{\mu-1, X}^0/\mathfrak{n}_{\mu-1, X}^0, \mathcal{O}_X) \widehat{\otimes}_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{M}} = C^{la}(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}})$$

equivariant for infinitesimal action of  $\mathbf{G}(\mathbb{Q}_p)$  for the left and right regular actions, and the action on the coefficients (by writing this sheaf as colimit of Banach sheaves where the actions integrate to compact open subgroups). Thus, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{M}}^{la} & \longrightarrow & C^{la}(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \widehat{\mathcal{O}}_{\mathcal{M}}) \\ \uparrow & & \uparrow \\ \pi_{\text{GM, HT}}^{la, -1}(\mathcal{O}_X) & \longrightarrow & \pi_{\text{GM, HT}}^{la, -1}(C^{la}(\mathfrak{g}_{\mu-1, X}^0/\mathfrak{n}_{\mu-1, X}^0, \mathcal{O}_X)) \end{array}$$

where the horizontal maps are the orbit maps and the vertical maps are the natural inclusions. This diagram together with (5.13) show that both horizontal Levi actions of  $\mathfrak{m}_{\mu}^0$  and  $\mathfrak{m}_{\mu-1}^0$  agree on  $\mathcal{O}_{\mathcal{M}}^{la}$  are they do over  $\mathcal{O}_X$  and they transform in the natural left regular action of

$$\mathfrak{m}_{\mu-1}^0 \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G}, \mu-1}}} \mathcal{O}_X = \mathfrak{m}_X^0 = \mathcal{O}_X \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G}, \mu}}} \mathfrak{m}_{\mu}^0$$

on  $C^{la}(\mathfrak{g}_{\mu-1, X}^0/\mathfrak{n}_{\mu-1, X}^0, \mathcal{O}_X)$ . This ends the proof of the theorem.  $\square$

**5.2. De Rham cohomology of the two towers.** In this section we show that the sheaf  $\mathcal{O}_{\mathcal{M}}^{la}$  of Definition 5.1.7 produces an isomorphism between the de Rham cohomology (with compact supports) of the two towers for a duality of local Shimura varieties. Similar results have been obtained independently by Bosco-Dospinescu-Niziol. In order to state the theorem, we keep the notation prior Theorem 5.1.10. Let  $X = \mathbf{M}_{\mu, \text{GM}}^{an} \times^{\mathbf{M}^{an}} \mathbf{M}_{\mu-1, \text{HT}}^{an}$  and consider the mixed Lie algebroid over  $\mathcal{O}_{\mathcal{M}}^{la}$

$$\mathcal{T}^{la} = \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{M}}^{la}$$

obtained as the pullback of the tangent space of  $X$  via the map  $\pi_{\text{GM, HT}}^{la}$ . By Theorem 5.1.10 this Lie algebroid acts by derivations on  $\mathcal{O}_{\mathcal{M}}^{la}$ , compatible with the derivations on  $X$ . Indeed, let us write by  $\mathfrak{n}_{\mu-1}^{0, la} \subset$

$\mathfrak{p}_{\mu^{-1}}^{0,la} \subset \mathfrak{g}_{\mu^{-1}}^{0,la}$  the base change of  $\mathfrak{n}_{\mu^{-1}}^0 \subset \mathfrak{p}_{\mu^{-1}}^0 \subset \mathfrak{g}_{\mu^{-1}}^0$  from  $\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu^{-1}}}$  to  $\mathcal{O}_{\mathcal{M}}^{la}$ . Similarly, let  $\mathfrak{n}_{\mu}^{0,la} \subset \mathfrak{p}_{\mu}^{0,la} \subset \mathfrak{g}_{\mu}^{0,la}$  be the base change of  $\mathfrak{n}_{\mu}^0 \subset \mathfrak{p}_{\mu}^0 \subset \mathfrak{g}_{\mu}^0$  from  $\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu}}$  to  $\mathcal{O}_{\mathcal{M}}^{la}$ . Theorem 5.1.10 also provides an isomorphism

$$\mathfrak{m}_{\mu^{-1}}^0 \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu^{-1}}}} \mathcal{O}_{\mathcal{M}}^{la} \cong \mathfrak{m}^{0,la} \cong \mathcal{O}_{\mathcal{M}}^{la} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu}}} \mathfrak{m}_{\mu}^0.$$

Then,  $\mathcal{T}^{la}$  is the quotient of  $\mathfrak{g}_{\mu}^{0,la} \oplus \mathfrak{g}_{\mu^{-1}}^{0,la}$  by the Lie algebroid  $\tilde{\mathfrak{p}}^{0,la}$  sitting in the cartesian square

$$\begin{array}{ccc} \tilde{\mathfrak{p}}^{0,la} & \longrightarrow & \mathfrak{m}^{0,la} \\ \downarrow & & \downarrow (\iota, -\iota) \\ \mathfrak{p}_{\mu}^{0,la} \oplus \mathfrak{p}_{\mu^{-1}}^{0,la} & \longrightarrow & \mathfrak{m}_{\mu}^{0,la} \oplus \mathfrak{m}_{\mu^{-1}}^{0,la} \end{array}$$

where  $(\iota, -\iota)$  is the anti-diagonal map. Since  $\tilde{\mathfrak{p}}^{0,la}$  acts trivially on  $\mathcal{O}_{\mathcal{M}}^{la}$ , the action by derivations of  $\mathfrak{g}_{\mu}^{0,la} \oplus \mathfrak{g}_{\mu^{-1}}^{0,la}$  on  $\mathcal{O}_{\mathcal{M}}^{la}$  descends to  $\mathcal{T}^{la}$ .

*Remark 5.2.1.* With some additional effort one can prove that  $\mathcal{O}_{\mathcal{M}}^{la}$  is formally smooth over  $C$  and that its tangent space is given by  $\mathcal{T}^{la}$  but we will not need this fact for the applications in this paper.

We have the following theorem

**Theorem 5.2.2.** *There are natural  $\mathbf{G}(\mathbb{Q}_p) \times G_b$ -equivariant isomorphisms of de Rham complexes over the topological space  $|\mathcal{M}_{\mathbf{G},b,\mu,\infty,C}|$*

$$DR(\mathcal{O}_{\mathcal{M}}^{\mathbf{G}(\mathbb{Q}_p)-sm}) \cong R\Gamma(\mathcal{T}^{la}, \mathcal{O}_{\mathcal{M}}^{la}) \cong DR(\mathcal{O}_{\mathcal{M}}^{G_b-sm}), \quad (5.14)$$

where:

- (1)  $DR(\mathcal{O}_{\mathcal{M}}^{\mathbf{G}(\mathbb{Q}_p)-sm})$  is the de Rham complex of the colimit of structural sheaves of the finite level local Shimura varieties  $\mathcal{M}_{\mathbf{G},b,\mu,K_p}$  with  $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ .
- (2)  $DR(\mathcal{O}_{\mathcal{M}}^{G_b-sm})$  is the de Rham complex of the colimit of structural sheaves of the finite level local dual Shimura varieties  $\mathcal{M}_{\check{\mathbf{G}},\check{b},\check{\mu},K_{b,p}}$  with  $K_{b,p} \subset \check{\mathbf{G}}(\mathbb{Q}_p) = G_b$ .
- (3)  $R\Gamma(\mathcal{T}^{la}, \mathcal{O}_{\mathcal{M}}^{la})$  is the de Rham cohomology of  $\mathcal{O}_{\mathcal{M}}^{la}$  with respect to the action of the Lie algebroid  $\mathcal{T}^{la}$  acting by derivations.

In particular, we have a natural  $\mathbf{G}(\mathbb{Q}_p) \times G_b$ -equivariant isomorphism of de Rham cohomologies with compact supports

$$\varinjlim_{K_p} H_{dR,c}^i(\mathcal{M}_{\mathbf{G},b,\mu,K_p,C}) \cong \varinjlim_{K_{b,p}} H_{dR,c}^i(\mathcal{M}_{\check{\mathbf{G}},\check{b},\check{\mu},K_{b,p},C}).$$

*Remark 5.2.3.* One can use the theory of the analytic de Rham stack of [RC24a] to prove Theorem 5.2.2. Indeed, as it was explained by Scholze to the second author, the formation of the analytic de Rham stack descends to (a suitable notion of) diamonds and, at least for what cohomology concerns, commutes with cofiltered limits of qcqs maps. Thus, Theorem 5.2.2 should be thought as an evidence to the fact that one has an equivalence of analytic de Rham stacks

$$\varprojlim_{K_p} \mathcal{M}_{\mathbf{G},b,\mu,K_p}^{dR} \cong \mathcal{M}_{\mathbf{G},b,\mu,\infty}^{dR} \cong \varprojlim_{K_{b,p}} \mathcal{M}_{\check{\mathbf{G}},\check{b},\check{\mu},K_{b,p}}^{dR}.$$

After taking quotients by the smooth groups  $\mathbf{G}(\mathbb{Q}_p)^{sm}$  and  $\check{\mathbf{G}}(\mathbb{Q}_p)^{sm}$  such an equivalence would also prove that one has an equivalence of analytic stacks

$$\mathcal{F}\ell_{\mathbf{G}(\mathbb{Q}_p),\mu}^{a,dR} / \check{\mathbf{G}}(\mathbb{Q}_p)^{sm} = \mathcal{F}\ell_{\mathbf{G}(\mathbb{Q}_p),\mu^{-1}}^{a,dR} / \mathbf{G}(\mathbb{Q}_p)^{sm}$$

between the analytic de Rham stacks of the quotients of the admissible locus of the flag varieties. This gives rise to a ‘‘Jacquet-Langlands equivalence’’ of equivariant analytic  $D$ -modules. We shall not prove this fact in this paper, instead we will give a first shadow of this compatibility of analytic de Rham stacks in the locally analytic Jacquet-Langlands functor for the Lubin-Tate tower in Section 5.3.

*Proof of Theorem 5.2.2.* Let  $\mathcal{T}_\mu$  and  $\mathcal{T}_{\mu-1}$  be the tangent spaces of  $\mathcal{F}\ell_{\mathbf{G},\mu}$  and  $\mathcal{F}\ell_{\mathbf{G},\mu-1}$  respectively. We have identifications  $\mathcal{T}_\mu = \mathfrak{g}_\mu^0/\mathfrak{p}_\mu^0$  and  $\mathcal{T}_{\mu-1} = \mathfrak{g}_{\mu-1}^0/\mathfrak{p}_{\mu-1}^0$  via the anchor map (2.7). By construction of the Lie algebroid  $\mathcal{T}^{la}$ , we have a short exact sequence

$$0 \rightarrow \mathfrak{m}^{0,la} \rightarrow \mathcal{T}^{la} \rightarrow \mathfrak{g}_\mu^{0,la}/\mathfrak{p}_\mu^{0,la} \oplus \mathfrak{g}_{\mu-1}^{0,la}/\mathfrak{p}_{\mu-1}^{0,la} \rightarrow 0.$$

The pullback along the inclusion of  $\mathfrak{g}_\mu^{0,la}/\mathfrak{p}_\mu^{0,la}$  in the direct sum corresponds to the Lie algebroid  $\mathfrak{g}_\mu^{0,la}/\mathfrak{n}_\mu^{0,la}$  (similarly the pullback for the inclusion of  $\mathfrak{g}_{\mu-1}^{0,la}/\mathfrak{p}_{\mu-1}^{0,la}$  is  $\mathfrak{g}_{\mu-1}^{0,la}/\mathfrak{n}_{\mu-1}^{0,la}$ ). Thus, we can write the  $\mathcal{T}^{la}$ -de Rham complex as the composite

$$R\Gamma(\mathfrak{g}_{\mu-1}^{0,la}/\mathfrak{p}_{\mu-1}^{0,la}, R\Gamma(\mathfrak{g}_\mu^{0,la}/\mathfrak{n}_\mu^{0,la}, \mathcal{O}_{\mathcal{M}}^{la})) \cong R\Gamma(\mathcal{T}^{la}, \mathcal{O}_{\mathcal{M}}^{la}) \cong R\Gamma(\mathfrak{g}_\mu^{0,la}/\mathfrak{p}_\mu^{0,la}, R\Gamma(\mathfrak{g}_{\mu-1}^{0,la}/\mathfrak{n}_{\mu-1}^{0,la}, \mathcal{O}_{\mathcal{M}}^{la})).$$

Therefore, in order to prove the quasi-isomorphisms (5.14) it suffices to show the two following facts:

- (1) The natural map  $\mathcal{O}_{\mathcal{M}}^{\mathbf{G}(\mathbb{Q}_p)-sm} \rightarrow R\Gamma(\mathfrak{g}_{\mu-1}^0/\mathfrak{n}_{\mu-1}^0, \mathcal{O}_{\mathcal{M}}^{la})$  is an equivalence.
- (2) The natural map  $\mathcal{O}_{\mathcal{M}}^{\mathbf{G}(\mathbb{Q}_p)-sm} \rightarrow R\Gamma(\mathfrak{g}_\mu^0/\mathfrak{n}_\mu^0, \mathcal{O}_{\mathcal{M}}^{la})$  is an equivalence.

These claims are symmetric with respect to the period maps, so it suffices to prove the first.

Let us write  $\mathcal{M}_\infty$  for the infinite level Shimura variety and  $\mathcal{M}_{K_p} = \mathcal{M}_\infty/K_p$  for its quotient by an open compact subgroup. Let  $\mathrm{gr}^0(\mathcal{O}_{\mathbb{B}_{\mathrm{dR}}})$  be the Hodge-Tate proétale sheaf of  $\mathcal{M}_{K_p}$  that appeared in Theorem 4.2.1, let  $\mathbf{N}_{\mu-1} \subset \mathbf{P}_{\mu-1}$  be the unipotent radical and let  $\mathcal{O}(\mathbf{N}_{\mu-1})$  be the space of algebraic functions of  $\mathbf{N}_{\mu-1}$  endowed with the natural action of  $\mathbf{P}_{\mu-1}$  as in Section 4.2. By Theorem 4.2.1 we have that

$$\mathrm{gr}^0(\mathcal{O}_{\mathbb{B}_{\mathrm{dR}}}) = \pi_{\mathrm{HT}}^* \mathcal{W}_{\mathbf{G},\mu-1}(\mathcal{O}(\mathbf{N}_{\mu-1})). \quad (5.15)$$

Let  $\nu_{K_p} : \mathcal{M}_{K_p,C,\mathrm{proét}} \rightarrow \mathcal{M}_{K_p,C,\mathrm{an}}$  be the projection of sites, then by [Sch13, Proposition 6.16] one has that

$$R\nu_{K_p,*} \mathrm{gr}^0(\mathcal{O}_{\mathbb{B}_{\mathrm{dR}},\mathcal{M}}) = \mathcal{O}_{\mathcal{M}_{K_p,C}}.$$

On the other hand, by taking locally analytic vectors for the action of  $\mathbf{G}(\mathbb{Q}_p)$ , by the vanishing of higher locally analytic vectors of  $\widehat{\mathcal{O}}_{\mathcal{M}}$  of Theorem 4.3.3, and the group cohomology comparisons of [RJRC23, Theorem 6.3.4], one deduces that for  $V \subset \mathcal{M}_{K_p}$  open affinoid and  $V_\infty = \mathcal{M}_\infty \times_{\mathcal{M}_{K_p}} V$  one has

$$\begin{aligned} \mathcal{O}_{\mathcal{M}_{K_p,C}}(V) &= R\Gamma_{\mathrm{proét}}(V, \mathrm{gr}^0(\mathcal{O}_{\mathbb{B}_{\mathrm{dR}},\mathcal{M}})) \\ &= R\Gamma(K_p, R\Gamma_{\mathrm{proét}}(V_\infty, \mathrm{gr}^0(\mathcal{O}_{\mathbb{B}_{\mathrm{dR}},\mathcal{M}}))) \\ &= R\Gamma(K_p, R\Gamma_{\mathrm{proét}}(V_\infty, \mathrm{gr}^0(\mathcal{O}_{\mathbb{B}_{\mathrm{dR}},\mathcal{M}}))^{RK_p^{-la}}) \\ &= R\Gamma(K_p, R\Gamma_{\mathrm{proét}}(V_\infty, \widehat{\mathcal{O}}_{\mathcal{M}})^{RK_p^{-la}} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}} \mathcal{W}_{\mathbf{G},\mu-1}(\mathcal{O}(\mathbf{N}_{\mu-1}))) \\ &= R\Gamma(K_p, \mathcal{O}_{\mathcal{M}}^{la}(V_\infty) \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}} \mathcal{W}_{\mathbf{G},\mu-1}(\mathcal{O}(\mathbf{N}_{\mu-1}))) \\ &= R\Gamma^{sm}(K_p, R\Gamma(\mathfrak{g}, \mathcal{O}_{\mathcal{M}}^{la}(V_\infty) \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}} \mathcal{W}_{\mathbf{G},\mu-1}(\mathcal{O}(\mathbf{N}_{\mu-1})))) \end{aligned}$$

where the second equality is decent along the  $K_p$ -torsor  $V_\infty \rightarrow V$ . The third equality is the comparison between solid and locally analytic group cohomology of [RJRC23, Theorem 6.3.4]. The fourth equality follows from the projection formula of locally analytic vectors [RJRC23, Corollary 3.1.15 (3)] and the isomorphism (5.15). The fifth equality follows from the vanishing of higher locally analytic vectors of Theorem 4.3.3. Finally, the sixth equality is the Lie algebra/smooth vs locally analytic cohomology comparison of Theorem [RJRC23, Theorem 6.3.4].

Taking colimits as  $K_p \rightarrow 1$ , we deduce that

$$\mathcal{O}_{\mathcal{M}}^{\mathbf{G}(\mathbb{Q}_p)-sm} = \lim_{\substack{\longrightarrow \\ K_p}} R\nu_{K_p,*} \mathrm{gr}^0(\mathcal{O}_{\mathbb{B}_{\mathrm{dR}}}) = R\Gamma(\mathfrak{g}, \mathcal{O}_{\mathcal{M}}^{la} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}} \mathcal{W}_{\mathbf{G},\mu-1}(\mathcal{O}(\mathbf{N}_{\mu-1}))). \quad (5.16)$$

But since  $\mathbf{N}_{\mu-1}$  is an affine space we know that

$$R\Gamma(\mathfrak{n}_{\mu-1}, \mathcal{O}(\mathbf{N}_{\mu-1})_C) = C,$$

taking the associated equivariant vector bundles over the flag variety and taking pullbacks along  $\pi_{\mathrm{HT}}$  one deduces that

$$R\Gamma(\mathfrak{n}_{\mu-1}^{0,la}, \mathcal{O}_{\mathcal{M}}^{la} \otimes_{\mathcal{O}_{\mathcal{F}\ell_{\mathbf{G},\mu-1}}} \mathcal{W}_{\mathbf{G},\mu-1}(\mathcal{O}(\mathbf{N}_{\mu-1}))) = \mathcal{O}_{\mathcal{M}}^{la}.$$

Combining this with (5.16), and by computing  $\mathfrak{g}_{\mu^{-1}}^0$ -lie algebra cohomology in two steps, one gets that

$$\begin{aligned} \mathcal{O}_{\mathcal{M}}^{\mathbf{G}(\mathbb{Q}_p)-sm} &= R\Gamma(\mathfrak{g}_{\mu^{-1}}^{0,la}/\mathfrak{n}_{\mu^{-1}}^0, R\Gamma(\mathfrak{n}_{\mu^{-1}}^{0,la}, \mathcal{O}_{\mathcal{M}}^{la} \otimes_{\mathcal{O}_{\mathcal{F}^{\ell}_{\mathbf{G},\mu^{-1}}}} \mathcal{W}_{\mathbf{G},\mu^{-1}}(\mathcal{O}(\mathbf{N}_{\mu^{-1}}))) \\ &= R\Gamma(\mathfrak{g}_{\mu^{-1}}^{0,la}/\mathfrak{n}_{\mu^{-1}}^0, \mathcal{O}_{\mathcal{M}}^{la}) \end{aligned}$$

proving what we wanted.

The claim about the cohomology comparisons for the de Rham cohomology with compact supports follows for example by using the definition of compactly supported de Rham cohomology arising from the six functor formalism of analytic  $D$ -modules of [RC24a]. One can also argue by using the adhoc definition of [GK00]. Indeed, the compactly supported cohomology of the de Rham complex of *loc. cit.* is nothing but the compactly supported cohomology of the de Rham complex seen as a sheaf on the underlying Berkovich space of  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,C}$ . To see that this cohomology with compact supports is well defined one can argue as follows: the map  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,C} \rightarrow \mathcal{M}_{\mathbf{G},b,\mu,K_p,C}$  gives rise to a  $K_p$ -torsor of Berkovich spaces

$$\mathcal{M}_{\mathbf{G},b,\mu,\infty,C}^B \rightarrow \mathcal{M}_{\mathbf{G},b,\mu,K_p,C}^B. \quad (5.17)$$

The space  $\mathcal{M}_{\mathbf{G},b,\mu,K_p,C}^B$  is a locally finite dimensional Hausdorff space (being the Berkovich space of a rigid space) and [HM24, Theorem 4.8.9 (i)] implies that  $\mathcal{M}_{\mathbf{G},b,\mu,K_p,C}^B$  has a well define functor of cohomology with compact supports for sheaves over  $\mathbb{Q}_p$  (in the language of *loc. cit.* it is  $\mathbb{Q}_p$ -fine). Since (5.17) is represented in profinite sets,  $\mathcal{M}_{\mathbf{G},b,\mu,\infty,C}^B$  is also a  $\mathbb{Q}_p$ -fine map (this follows from [HM24, Theorem 3.4.11 (ii)] since any maps between profinite sets is  $\mathbb{Q}_p$ -fine by construction, see Section 3.5.16 in *loc. cit.*), i.e. it has a well defined functor of cohomology with compact supports.  $\square$

**5.3. The Jacquet-Langlands functor for admissible locally analytic representations.** In this last section we recall the definition of the Jacquet-Langlands functor of [Sch18] for admissible Banach representations. We then proof that this functor is compatible with the passage to locally analytic vectors.

**5.3.1. Scholze's Jacquet-Langlands functor.** Let  $n \geq 1$  be an integer and  $F/\mathbb{Q}_p$  a finite extension with ring of integers  $\mathcal{O} \subset F$  and  $\varpi \in \mathcal{O}$  a uniformizer. Let  $\mathbb{F} = \mathbb{F}_q$  be the residue field of  $\mathcal{O}$ . Consider the group  $\mathbf{GL}_{n,F}$ ,  $\mu$  the cocharacter given by  $(1, 0, \dots, 0)$  with  $n - 1$  occurrences of 0, and  $b$  corresponds to a formal  $\mathcal{O}$ -module  $\mathbb{X}_b$  over  $\overline{\mathbb{F}}$  of dimension 1 and  $F$ -height  $n$ . Let  $D$  be the division algebra over  $F$  of invariant  $1/n$ , we have  $\tilde{G}_b = D^\times$ . Finally, we fix  $\mathbb{C}_p/F$  the  $p$ -completion of an algebraic closure of  $F$ .

**Definition 5.3.1.** We let  $\text{Def}_{\mathbb{X}}$  be the functor on formal schemes over  $\check{\mathcal{O}}$  sending  $S$  to the set of isomorphism classes of pairs  $(X, \rho)$ , where  $X/S$  is a formal  $F$ -module, and  $\rho : X \times_S \overline{S} \xrightarrow{\sim} \mathbb{X} \times_{\overline{\mathbb{F}}} \overline{S}$  is a quasi-isogeny of formal  $\mathcal{O}_F$ -modules, where  $\overline{S} = S \times_{\text{Spf } \check{\mathcal{O}}} \text{Spec } \overline{\mathbb{F}}$ .

By [RZ96] the functor  $\text{Def}_{\mathbb{X}}$  is representable by a formal scheme  $\mathfrak{M}_{\mathbb{X}}$  over  $\text{Spf } \check{\mathcal{O}}$ , which is formally smooth and locally formally of finite type. We let  $\mathcal{M}_{\mathbb{X}}$  denote the generic fiber of  $\mathfrak{M}_{\mathbb{X}}$  as a rigid space.

**Theorem 5.3.2** ([SW20, Corollary 24.3.5]). *There is a natural equivalence of diamonds  $\mathcal{M}_{\mathbb{X}}^{\diamond} \cong \mathcal{M}_{\mathbf{GL}_{n,F},b,\mu,K}^{\diamond}$  with  $K = \mathbf{GL}_n(\mathcal{O})$ .*

Let  $\mathcal{M}_{\infty}^{\diamond} = \mathcal{M}_{\mathbf{GL}_{n,F},b,\mu,\infty}^{\diamond}$ . In this situation the  $\mathbf{GL}_n(F) \times D^\times$ -equivariant period maps (3.6) restrict to a diagram

$$\begin{array}{ccc} & \mathcal{M}_{\infty}^{\diamond} & \\ \pi_{\text{GM}} \swarrow & & \searrow \pi_{\text{HT}} \\ \mathbb{P}_{\check{F}}^{n-1} & & \Omega_{\check{F}}^{\vee} \end{array}$$

where

- $\pi_{\text{GM}}$  is a proétale  $\mathbf{GL}_n(F)$ -torsor and  $H$  acts on  $\mathbb{P}_{\check{F}}^{n-1}$  via the natural inclusion of the map  $H^\times \subset \mathbf{GL}_n(\check{F})$ .
- $\pi_{\text{HT}}$  is a proétale  $D^\times$ -torsor and  $\Omega_{\check{F}}^{\vee} \subset \mathbb{P}_{\check{F}}^{n-1}$  is the  $\mathbf{GL}_n(F)$ -stable open Drinfeld space obtained by removing all  $F$ -rational hyperplanes.

Thus, we have an equivalence of  $v$ -stacks

$$[\mathbb{P}_{\check{F}}^{n-1}/D^\times] \cong [\Omega_{\check{F}}/\mathbf{GL}_n(F)].$$

The Jacquet-Langlands functor is defined as follows.

**Definition 5.3.3.** Let  $\pi$  be a  $p$ -power torsion admissible representation of  $\mathbf{GL}_n(F)$  over  $\mathbb{Z}_p$  and let  $\mathcal{F}_\pi$  be the étale sheaf over  $\mathbb{P}_{\check{F}}^{n-1}$  obtained by descent along  $\pi_{\text{GM}}$ . The Jacquet-Langlands functor  $\mathcal{JL}$  is the functor mapping such  $\pi$  to the complex of smooth  $D^\times$ -representations

$$\mathcal{JL}(\pi) = R\Gamma_{\text{ét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\pi).$$

**Theorem 5.3.4** ([Sch18, Theorem 1.1]). *Let  $\pi$  be a  $p$ -power torsion admissible representation of  $\mathbf{GL}_n(F)$  over  $\mathbb{Z}_p$ , then  $\mathcal{JL}(\pi)$  is a complex of admissible representations of  $D^\times$ . In other words, for all  $i \in \mathbb{Z}$  the cohomology*

$$\mathcal{JL}^i(\pi) = H_{\text{ét}}^i(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\pi)$$

is an admissible representation of  $H$ .

For convenience we shall consider the  $p$ -completed analogue of Theorem 5.3.4. Let  $\pi$  be a  $p$ -adically complete admissible representation of  $\mathbf{GL}_n(F)$  over  $\mathbb{Z}_p$ , we shall write by  $\mathcal{F}_\pi$  the pro-étale sheaf over  $\mathbb{P}_{\check{F}}^{n-1}$  given by the limit of étale sheaves  $\mathcal{F}_\pi = \varprojlim_s \mathcal{F}_{\pi/p^s}$ . Finally, we denote by  $\mathcal{JL}(\pi)$  the  $p$ -adically complete  $D^\times$ -representation

$$\mathcal{JL}(\pi) := R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\pi) = R\varprojlim_s R\Gamma_{\text{ét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi/p^s}).$$

**Corollary 5.3.5.** *Let  $\pi$  be a  $p$ -adically complete admissible representation of  $\mathbf{GL}_n(F)$ , then  $\mathcal{JL}(\pi)$  is a complex of  $p$ -adically complete admissible representations of  $D^\times$ . In other words, the cohomology groups*

$$\mathcal{JL}^i(\pi) = H_{\text{proét}}^i(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\pi)$$

are  $p$ -adically complete admissible representations of  $D^\times$ . Moreover, we have that

$$\mathcal{JL}^i(\pi) = \varprojlim_s \mathcal{JL}^i(\pi/p^s).$$

*Proof.* Let  $K_D \subset D^\times$  be a compact open subgroup which we assume to be a uniform pro- $p$ -group. Taking  $\mathbb{Z}_p$ -duals the complex  $\mathcal{JL}(\pi)^\vee = R\text{Hom}(\mathcal{JL}(\pi), \mathbb{Z}_p)$  is a  $p$ -adically complete module over the Iwasawa algebra  $\mathbb{Z}_{p,\square}[K_D]$  whose reduction modulo  $p$  is a perfect  $\mathbb{F}_{p,\square}[K_D]$ -complex by Theorem 5.3.4, this implies that  $\mathcal{JL}(\pi)^\vee$  is itself a perfect complex of  $\mathbb{Z}_{p,\square}[K_D]$ -modules and so  $\mathcal{JL}(\pi)$  can be represented by a complex of admissible representations of  $H$ . The rest of the statements are classical and left to the reader, see for example [Eme06, Proposition 1.2.12].  $\square$

**5.3.2. Locally analytic Jacquet-Langlands functor.** Next we show that the Jacquet-Langlands functor of Definition 5.3.3 is compatible with locally analytic vectors. Let  $\Pi$  be an admissible locally analytic representation of  $\mathbf{GL}_n(F)$  over  $\mathbb{Q}_p$ , we let  $\mathcal{F}_\Pi$  be the proétale sheaf over  $\mathbb{P}_{\check{F}}^{n-1}$  whose  $S$ -points for an affinoid perfectoid  $S \rightarrow \mathbb{P}_{\check{F}}^{n-1,\diamond}$  are given by

$$\mathcal{F}(\Pi) = (C(|\mathcal{M}_\infty \times_{\mathbb{P}_{\check{F}}^{n-1,\diamond}} S|, \mathbb{Q}_p) \widehat{\otimes}_{\mathbb{Q}_p} \Pi)^{\mathbf{GL}_n(F)}$$

where

- For a perfectoid space  $X$  the algebra  $C(|X|, \mathbb{Q}_p)$  is the space of continuous functions from  $|X|$  to  $\mathbb{Q}_p$ .
- The completed tensor product is a tensor product of LB representations (equivalently a solid tensor product).
- The group  $\mathbf{GL}_n(F)$  acts via the diagonal action.

This is the same as the proétale solid sheaf on  $\mathbb{P}_{\check{F}}^{n-1}$  obtained by descent from the constant sheaf on  $\mathcal{M}_\infty$  via [AM24, Corollary 4.5].

**Theorem 5.3.6.** *Let  $\pi$  be a  $p$ -adically complete admissible representation and  $\Pi = (\pi[\frac{1}{p}])^{\mathbf{GL}_n(F)-la}$  its LB subrepresentation of locally analytic vectors. Then there is a natural equivalence*

$$(\mathcal{JL}(\pi)[\frac{1}{p}])^{RD^\times-la} \cong R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\Pi) \quad (5.18)$$

where the left hand side is the complex of derived  $D^\times$ -analytic vectors of the solid  $D^\times$ -representation  $\mathcal{JL}(\pi)[\frac{1}{p}]$ . Moreover, for all  $i \in \mathbb{Z}$  we have an isomorphism of locally analytic admissible  $D^\times$ -representations

$$(\mathcal{JL}^i(\pi)[\frac{1}{p}])^{D^\times-la} \cong H_{\text{proét}}^i(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\Pi). \quad (5.19)$$

*Proof.* In the following proof we work with the derived  $\infty$ -categories of solid sheaves of diamonds as in [AM24, §4].

*Step 0.* The equivalence in Eq. (5.19) follows from Eq. (5.18). Indeed, the object  $\mathcal{JL}(\pi)[\frac{1}{p}]$  is a complex with cohomologies given by admissible Banach representations of  $D^\times$ . By [RJRC22, Proposition 4.48] (see also [RC24b, Proposition 2.3.1]) the higher locally analytic vectors of a Banach admissible representation vanish, then by the spectral sequence of [RC24b, Theorem 1.5] one deduces that

$$H^i((\mathcal{JL}(\pi)[\frac{1}{p}])^{RD^\times-la}) = (\mathcal{JL}^i(\pi)[\frac{1}{p}])^{D^\times-la}.$$

*Step 1.* We first reinterpret the problem using the period sheaves. By [FS24, Proposition II.2.5] we have a short exact sequence of proétale sheaves

$$0 \rightarrow \mathbb{Q}_p \rightarrow \mathbb{B}_{[1,p]} \xrightarrow{\varphi-1} \mathbb{B}_{[1,1]} \rightarrow 0.$$

Taking solid (eq.  $p$ -complete in this case) tensor products with the sheaf  $\mathcal{F}_\pi$  we get a short exact sequence

$$0 \rightarrow \mathcal{F}_\pi[\frac{1}{p}] \rightarrow \mathbb{B}_{[1,p]} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_\pi \rightarrow \mathbb{B}_{[1,1]} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_\pi \rightarrow 0.$$

Taking proétale cohomology we get an exact triangle

$$\mathcal{JL}(\pi)[\frac{1}{p}] \rightarrow R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathbb{B}_{[1,p]} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}_\pi) \rightarrow R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathbb{B}_{[1,1]} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}_\pi) \xrightarrow{\pm}.$$

On the other hand, taking LB-completed tensor products we get a short exact sequence of proétale sheaves

$$0 \rightarrow \mathcal{F}_\Pi \rightarrow \mathbb{B}_{[1,p]} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}_\Pi \rightarrow \mathbb{B}_{[1,1]} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}_\Pi \rightarrow 0.$$

Therefore, to prove the theorem it suffices to show that for all  $I \subset (0, \infty)$  compact interval, we have a natural equivalence of representations of  $D^\times$

$$R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathbb{B}_I \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_\pi)^{RD^\times-la} \cong R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathbb{B}_I \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_\Pi). \quad (5.20)$$

*Step 2.* We now reduce the proof of (5.20) to affinoid subspaces of  $\mathbb{P}_{\mathbb{C}_p}^{n-1}$ . Let  $\nu : \mathbb{P}_{\mathbb{C}_p, \text{proét}}^{n-1} \rightarrow \mathbb{P}_{\mathbb{C}_p, \text{an}}^{n-1}$  be the projection of sites and let  $\mathfrak{U} = \{U_i\}_{i \in I}$  be a finite rational open cover of  $\mathbb{P}_{\mathbb{C}_p}^{n-1}$ . Then, for any proétale sheaf  $\mathcal{F}$  over  $\mathbb{P}_{\mathbb{C}_p}^{n-1}$  we have equivalences of complexes

$$R\check{\Gamma}_{\text{an}}(\mathfrak{U}, R\nu_* \mathcal{F}) \cong R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}).$$

functorial on  $\mathcal{F}$ , where the left hand side is the Čech cohomology given by

$$R\check{\Gamma}_{\text{an}}(\mathfrak{U}, R\nu_* \mathcal{F}) = \varprojlim_{V \in \text{Int}(\mathfrak{U})} R\Gamma_{\text{an}}(V, R\nu_* \mathcal{F})$$

with  $\text{Int}(\mathfrak{U})$  the poset of finite intersections of elements in  $\mathfrak{U}$ .

Therefore, in order to show (5.20) it suffices to prove that for  $U \subset \mathbb{P}_{\mathbb{C}_p}^{n-1}$  a rational open subspace we have a natural equivalence

$$R\Gamma_{\text{proét}}(U, \mathbb{B}_I \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_\pi)^{RH-la} \cong R\Gamma_{\text{proét}}(U, \mathbb{B}_I \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_\Pi). \quad (5.21)$$

*Step 3.* Finally, we prove (5.21). We can assume without loss of generality that  $U$  is a rational subspace admitting a section  $U \subset \mathcal{M}_{\mathbb{X}, \mathbb{C}_p}$ . Let  $K_{D,U} \subset D^\times$  be a compact open subgroup stabilizing  $U$  and let  $U_\infty = \mathcal{M}_\infty \times_{\mathcal{M}_{\mathbb{X}}} U$ . Then, we have that

$$\begin{aligned}
R\Gamma_{\text{proét}}(U, \mathbb{B}_I \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_\pi)^{RD^\times - la} &\cong R\Gamma(K_{D,U}, R\Gamma(\mathbf{GL}_n(\mathcal{O}), R\Gamma_{\text{proét}}(U_\infty, \mathbb{B}_I \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{F}_\pi) \widehat{\otimes}_{\mathbb{Q}_p}^L C^{la}(K_{D,U}, \mathbb{Q}_p)) \\
&\cong R\Gamma(K_{D,U} \times \mathbf{GL}_n(\mathcal{O}), R\Gamma_{\text{proét}}(U_\infty, \mathbb{B}_I) \widehat{\otimes}_{\mathbb{Z}_p}^L \pi \widehat{\otimes}_{\mathbb{Q}_p}^L C^{la}(K_{D,U}, \mathbb{Q}_p)) \\
&\cong R\Gamma(\mathbf{GL}_n(\mathcal{O}), R\Gamma_{\text{proét}}(U_\infty, \mathbb{B}_I)^{RK_H - la} \widehat{\otimes}_{\mathbb{Z}_p}^L \pi) \\
&\cong R\Gamma(\mathbf{GL}_n(\mathcal{O}), R\Gamma_{\text{proét}}(U_\infty, \mathbb{B}_I)^{RG - la} \widehat{\otimes}_{\mathbb{Z}_p}^L \pi) \\
&\cong R\Gamma(\mathbf{GL}_n(\mathcal{O}), R\Gamma_{\text{proét}}(U_\infty, \mathbb{B}_I)^{RG - la} \widehat{\otimes}_{\mathbb{Q}_p}^L \Pi) \\
&\cong R\Gamma(\mathbf{GL}_n(\mathcal{O}), R\Gamma_{\text{proét}}(U_\infty, \mathbb{B}_I) \widehat{\otimes}_{\mathbb{Q}_p}^L \Pi) \\
&\cong R\Gamma_{\text{proét}}(U, \mathcal{F}_\Pi).
\end{aligned}$$

In the first equivalence we use descent along the  $\mathbf{GL}_n(\mathcal{O})$ -torsor  $U_\infty \rightarrow U$  and write explicitly the definition of  $K_{D,U}$ -locally analytic vectors. The second equivalence is clear as  $U_\infty$  is qcqs and  $\pi$  is a Banach space, namely this follows from the analogue computations of the equation (5.2) in the proof of Lemma 5.1.3. The third equivalence follows from projection formula of locally analytic vectors [RJRC23, Corollary 3.1.15 (3)] and the fact that  $\pi$  is a trivial  $K_{D,U}$ -representation. The fourth equivalence is Corollary 5.1.9. The fifth equivalence follows from the projection formula of locally analytic vectors and the fact that  $(\pi)[\frac{1}{p}]^{RG - la} = \Pi$  as  $\pi$  is an admissible representation. The sixth equivalence is the projection formula again. The last equivalence is descent along the torsor  $U_\infty \rightarrow U$ . This finishes the proof of the theorem.  $\square$

As a corollary we can prove that the Jacquet-Langlands functor for Banach admissible locally analytic representations preserves central characters.

**Corollary 5.3.7.** *Let  $\pi$  be an admissible Banach representation of  $\mathbf{GL}_n(L)$  over  $\mathbb{Q}_p$  and suppose that  $\Pi = \pi^{\mathbf{GL}_n(L) - la}$  has central character  $\chi$ . Then, for all  $i \in \mathbb{Z}$ , the locally analytic  $D^\times$ -representation  $\mathcal{JL}^i(\pi)^{D^\times - la}$  has central character  $\chi$  under the natural identification  $\mathcal{Z}(\text{Lie } D^\times) \cong \mathcal{Z}(\text{Lie } G)$ .*

*Proof.* The statement can be proven after base change to  $C$ . By [Sch18, Theorem 3.2] we have a natural equivalence

$$\mathcal{JL}(\pi) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p = R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\pi \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}).$$

Then, by (5.20) we deduce an  $D^\times$ -equivariant equivalence

$$(\mathcal{JL}(\pi) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{RD^\times - la} \cong R\Gamma_{\text{proét}}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_\Pi),$$

thus it suffices to show that the RHS term has central character given by  $\chi$ . By picking a suitable affinoid cover  $\{U_i\}_i$  of  $\mathbb{P}_{\mathbb{C}_p}^{n-1}$  as in Steps 2 and 3 of the proof of Theorem 5.3.6, we are reduced to show that for any small enough open affinoid  $U \subset \mathcal{M}_{\mathbb{X}}$  with stabilizer  $K_{D,U} \subset D^\times$ , the central character of  $R\Gamma_{\text{proét}}(U, \mathcal{F}_\Pi \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}})$  for the action of  $K_{D,U}$  is  $\chi$ . Let  $U_\infty \subset \mathcal{M}_\infty$  be the pullback of  $U$  to infinite level, by Step 3 of the proof of Theorem 5.3.6 we have that

$$R\Gamma_{\text{proét}}(U, \mathcal{F}_\Pi \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}) \cong R\Gamma(\mathbf{GL}_n(\mathcal{O}), R\Gamma_{\text{proét}}(U_\infty, \widehat{\mathcal{O}})^{RK_{D,U} - la} \widehat{\otimes}_{\mathbb{Q}_p} \Pi),$$

but by taking  $U$  small enough, the vanishing of higher locally analytic vectors of Theorem 4.3.3 implies that

$$R\Gamma_{\text{proét}}(U, \mathcal{F}_\Pi \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}) \cong R\Gamma(\mathbf{GL}_n(\mathcal{O}), \mathcal{O}_{\mathcal{M}}^{la}(U_\infty) \widehat{\otimes}_{\mathbb{Q}_p} \Pi).$$

The corollary follows from the identification of the central horizontal actions  $\mathcal{Z}(\mathfrak{m}_\mu)_{\mathbb{C}_p} \cong \mathcal{Z}(\mathfrak{m}_{\mu-1})_{\mathbb{C}_p}$  on  $\mathcal{O}_{\mathcal{M}}^{la}(U_\infty)$  of Theorem 4.3.3 and the fact that the central actions of  $\mathcal{Z}(\text{Lie } D^\times) \cong \mathcal{Z}(\text{Lie } G)$  factor through the horizontal actions.  $\square$



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