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Kinetic theory and moment models of electrons in a reactive weakly-ionized non-equilibrium plasma

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Abstract

We study the electrons in a multi-component weakly-ionized plasma with an external electric field under conditions that are far from thermodynamic equilibrium, representative of a gas discharge plasma. Our starting point is the generalized Boltzmann equation with elastic, inelastic and reactive collisions. We perform a dimensional analysis of the equation and an asymptotic analysis of the collision operators for small electron-to-atom mass ratios and small ionization levels. The dimensional analysis leads to a diffusive scaling for the electron transport. We perform a Hilbert expansion of the electron distribution function that, in the asymptotic limit, results in a reduced model characterized by a spherically symmetric distribution function in the velocity space with a small anisotropic perturbation. We show that the spherical-harmonics expansion model, widely used in low-temperature plasmas, is a particular case of our approach. We approximate the solution of our kinetic model with a truncated moment hierarchy. Finally, we study the moment problem for a particular case: a Langevin collision (equivalent to Maxwell molecules) for the electron-gas elastic collisions. The resulting Stieltjes moment problem leads to an advection-diffusion-reaction system of equations that is approximated with two different closures: the quadrature method of moments and a Hermitian moment closure. A special focus is given along the derivations and approximations to the notion of entropy dissipation.

1 Introduction

Modeling the transport of electrons in plasmas is fundamental in order to understand the thermal, chemical, electrical and radiation properties of the plasma. When the plasma is weakly ionized, the density of the charged species is only a small fraction of the density of neutral species (atoms and molecules). In this case, the charged species collide more often with the neutral particles than among themselves. Due to the small number of electron-electron collisions and the presence of inelastic collisions involving electrons and neutral species, the electron distribution functions are often far from the thermodynamic equilibrium in weakly-ionized plasmas. These plasmas are widely used in industry (often referred to as gas discharge plasmas or low-temperature plasmas), e.g., for material processing for electronics [1, 2] or electric propulsion thrusters [3]. Under these non-equilibrium conditions, the electron transport processes can largely differ from the framework of the traditional hydrodynamic description [4, 5].

1.1 Physical considerations

Multi-component plasma

When studying the transport of electrons in multi-component plasmas, the collision terms of the electron kinetic equation can be largely simplified by taking into account the smallness of the square root of the

electron-to-heavy (ions, atoms, and molecules) mass ratio. The simplest of these collision operators corresponds to the Lorentz gas model [6] where the elastic electron-heavy collisions are modeled as electrons colliding with infinitely heavy and stationary species. Braginskii [7] first used the Lorentz gas model as a starting point to simplify the Landau Fokker-Planck collision operator for electron-ion collisions. Close to the thermodynamic equilibrium, following the Chapman-Enskog method [8], the expansion of the electron-heavy collision operator in powers of the square root of the electron-to-heavy mass ratio was used by several authors in order to compute the electron transport properties in multi-component plasmas. Chmielewski and Ferziger [9, 10] computed the transport properties in partially-ionized plasmas with the approximation of considering the heavy distribution functions as delta distributions. Due to the mass disparity, the hydrodynamic model is a two-temperature model, i.e., where the temperature of heavy species and electrons are different. Devoto [11] proposed a procedure to decouple the electron and heavy transport systems based on the smallness of the electron-to-heavy mass ratio. Petit and Darrozes [12] showed that the consistent manner to utilize the expansion of the electron-heavy collision operator into a perturbative method as the Chapman-Enskog expansion was by assuming that the Knudsen number and the electron-to-heavy mass ratio were related through a dimensional analysis of the kinetic equation. This idea was utilized in fully-ionized plasmas by Degond and Lucquin-Desreux [13], in partially-ionized plasmas by Magin and Degrez [14] and in the presence of a magnetic field later by Graille et al. [15].

Partially-ionized plasma

In the case of partially-ionized plasmas, the Boltzmann kinetic equation is generalized in order to include the inelastic collisions with atoms and molecules and the chemical reactions. The kinetic equation for polyatomic mixtures was proposed by Wang-Chang and Uhlenbeck [16], following a semi-classical approach, where the collision cross-sections are averaged by degeneracies. Alternatively, the chemical reactions were modeled in the kinetic equation by Ludwig and Heil [17], Alexeev et al. [18], and Ern and Giovangigli [19]. This generalized Boltzmann equation was studied by Orlac'h et al. [20] in order to extend the work of Graille et al. [15] for partially-ionized reacting plasmas as well as Zhdanov and Stepanenko [21, 22], both references studying conditions close to thermodynamic equilibrium in weakly non-thermal plasmas, i.e. the difference between heavy and electron temperatures is of the order of the heavy temperature.

Non-equilibrium conditions

Far from thermodynamic equilibrium, collisional weakly-ionized plasmas are often described by the spherical-harmonics expansion (SHE) kinetic model. In this framework, the electron distribution function is expanded into spherical harmonics, usually considering the first two terms of this expansion. With this ansatz, an angular moment hierarchy is solved, where the first equation corresponds to the isotropic distribution function in the velocity space. This model with slight variations is also used in electron swarms [23, 24] and electrons in semiconductors [25, 26]. In gas discharges, the early works by Davydov [27] and Allis [28] are based on the Lorentz gas approximation for the kinetic equation without a rigorous link to the full Boltzmann operator nor the generalization of the Boltzmann equation for reactive mixtures (that was proposed posterior to these publications). The derivation of this SHE model was comprehensively explained later by Shkarofsky et al. [29] and more recently by Robson et al. [30] and Colonna et al. [31] as well as its spatially-homogeneous numerical implementation by Hagelaar and Pitchford [32] is widely used to compute the electron transport coefficients in hydrodynamic drift-diffusion models. However, the relation between the SHE model for gas discharges and the generalized Boltzmann equation with a dimensional and asymptotic analysis was studied in a less systematic way than in Chapman-Enskog models for plasmas. Choquet et al. [33] have developed a dimensional analysis of the electron kinetic equation that leads to a diffusive scaling that is compatible with the SHE kinetic model without including the excitation of the atoms.

1.2 Present contribution

In this paper, we follow an approach similar to Choquet et al. [33]: One first important result is the dimensional and asymptotic analysis of the generalized Boltzmann equation for electrons in a weakly-ionized plasma, including the inelastic and chemical (ionization) reactions. This dimensional analysis leads to a diffusive scaling for the electron transport. Then, we perform an expansion of Hilbert type in orders of the square root of the electron-gas mass ratio. The asymptotic limit yields that the zero-th order electron distribution function is isotropic in velocity space, and the first-order perturbation follows a Fredholm integral equation. The SHE model is a particular case of our asymptotic model when the spherical harmonics ansatz is introduced and angular moments are taken. Eventually, we propose a velocity-moment hierarchy to approximate the solution of the resulting kinetic model.

The advantage of velocity-moment closures is the reduction of the dimensionality of the problem as compared to the kinetic equation (6 dimensions) or the SHE-kinetic equation (4 dimensions). This explains the popularity of the drift-diffusion models based on the local-field approximation (LFA [34]) or the local mean-energy approximation (LMEA [35]) in the low-temperature plasma modeling. However, these models often oversimplify the electron transport processes due to the truncation of the moment hierarchy at the density equation (LFA) or the energy conservation equation (LMEA). For this reason, a higher-order moment hierarchy is a sound alternative that can increase the precision of the macroscopic approximation. In the literature, various velocity-moment approaches have been proposed (see e.g. the reviews in [36, 37] and references therein). Due to the complexity of the model, we focus on some of the simplest ones: the quadrature-based methods and Grad's methods. The quadrature method of moments (QMOM [38], see also the review [39] and its application to collisionless plasmas [40]) exploits the theory of orthogonal polynomials (see e.g. [41, 42]) and benefits from well-established algorithms [43, 44] for the moment inversion problem, i.e., expressing the parameters of the approximation from the moments. Grad's method [45] consists in a classical Hermite polynomial expansion of the distribution function. It was exhaustively studied more recently by numerous authors, see e.g. [46, 47, 48, 36, 49] for rarefied gases [50, 51] for fully-ionized plasmas and [52, 53] for partially-ionized plasmas. Various alternative approaches were proposed to overcome either mathematical drawbacks or to improve the approximation globally or in specific physical regimes. Among those, the entropy minimization closure has been popularized for rarefied gases [54], and recently applied for magnetized plasmas [55]. However, its application is more complicated in our context than the other two due to the presence of inelastic and ionization collisions (which requires the integration of the maximum-entropy distribution function in order to compute the collision rates for the calculation of transport properties). In this paper, we consider a simplified collision kernel that corresponds to a Langevin collision type for the electron-neutral elastic collisions. Nevertheless, we stress that the asymptotic analysis and the resulting kinetic model are general to any collision cross-section.

The paper is organized as follows: First, we describe the generalized Boltzmann equation for electrons in a partially-ionized plasma, the dimensional analysis, and the expansion of the electron-heavy collision operators. We present an expansion of the elastic collisions that slightly differs from Graille et al. [15] in the second order term, but that is consistent with Davydov's expression of the collision operator [27] in the SHE theory. However, unlike Davydov, this term is obtained through a rigorous expansion of the scaled full Boltzmann operator. Second, we present a Hilbert expansion method that retrieves a coupled system of zero-th and first-order kinetic equations as well as a macroscopic moment hierarchy composed by the non-linear transport equations for the even moments and a transport system for the odd moments. Third, we present the moment hierarchy closure for a particular type of electron-neutral collisions (Langevin collisions), while keeping arbitrary cross-sections for the other collision types. Then, we define the moment system that corresponds to a Stieltjes type. For this moment problem, we present two different example closure models, respectively based on QMOM and on a Hermitian expansion. Finally, we finish with a discussion of the results and conclusions.

2 Electron kinetic equation in a weakly-ionized reacting atomic gas plasma

2.1 Assumptions

In this work, we consider a physical regime that is representative of weakly-ionized gas discharge plasmas in noble gases (similar to the regime considered in [28, 29, 32]). These read as follows:

1. The electron kinetic equation is based on classical mechanics, i.e., the mean distance between particles is larger than the thermal de Broglie wavelength, and the square of the ratio of the electron thermal velocity to the speed of light is small.
2. We consider the kinetic equation of electrons in a weakly-ionized plasma composed of gas atoms, ions, and electrons. The dominant collisional processes of such a plasma satisfy the following assumptions:
 - (a) The electron scales are largely separated from those of the heavy species, and we only consider the evolution of the state of electrons. The other species, i.e., the gas atoms and the ions, are considered as a background at rest with temperatures different and smaller than the one of the electrons. As shown by Choquet et al. [33], the separation of scales between electrons and the heavy species allows to study the transport properties of electrons decoupled from these of the heavy species, at least in the first two orders of accuracy.
 - (b) The elastic collisions are modeled by a Boltzmann operator as binary encounters: the gas is dilute (the mean distance between particles is larger than the particle interaction distance) and the plasma parameter is large (the Coulomb interactions can be modelled by means of a Coulomb potential screened at the Debye length).
 - (c) The inelastic collisions are modeled with a Wang Chang-Uhlenbeck operator [16]. We assume that the density of the excited atom species is much smaller than the density of the atoms at the ground state (as often at low and intermediate pressure in gas discharges). The excited states of the atoms are not tracked, considering that all the excitations occur from a non-degenerate ground state. Similarly, ionic excited states are not considered in this work.
 - (d) The ionization is created by an electron impact with a gas atom in the ground state. The ionization reactive collision operator is modeled with an Alexeev-Giovangigli collision operator [18, 19]. Under the considered conditions, the ionization collisions are a slow collisional process and far from chemical equilibrium.
3. We assume that the electrons are impacted by an electrostatic field, and we neglect the induced magnetic field (the magnetic energy is much smaller than the plasma thermal energy). We also neglect the displacement current (the kinetic time is slower than the characteristic time of propagation of electromagnetic waves).
4. We are interested in the following physical regime, which is representative of the bulk of a gas discharge (see e.g. [28, 32, 29]):
 - (a) The plasma is weakly-ionized, i.e., the density of the charged species is a small fraction of the density of the atoms. For this reason, the dominant collision mechanism is the electron-gas atom elastic collisions.
 - (b) The characteristic macroscopic time is chosen to be the transit time of a gas acoustic wave, which is much larger than the electron kinetic time.
 - (c) The characteristic macroscopic length is much larger than the electron-gas atom elastic collision mean free path.
 - (d) The characteristic electric energy is of the order of the electron mean energy (e.g., the electrostatic field is an ambipolar field created by the plasma).

5. We represent the bulk of the plasma, and hence we neglect the effect of the sheath that can form close to the boundaries.

2.2 Generalized Boltzmann equation

Using the assumptions presented above, the electron kinetic equation reads as follows:

$$\frac{\partial f_{\epsilon}}{\partial t} + \mathbf{v} \cdot \nabla f_{\epsilon} - \frac{e\mathbf{E}}{m_{\epsilon}} \cdot \nabla_{\mathbf{v}} f_{\epsilon} = C(f_{\epsilon}). \quad (1)$$

Here, $f_{\epsilon}(\mathbf{x}, \mathbf{v}, t)$ is the electron distribution function, \mathbf{E} is the electric field, m_{ϵ} is the electron mass, and e is the elementary charge. The electric field is the sum of the external electric field and the electrostatic field created by the plasma charged species, solution of Gauss' law. In the following, all the bold quantities such as \mathbf{v} refer to vectors, while their regular font one $v = \|\mathbf{v}\|$ refers to their Euclidean norm. In (1), the right-hand-side term corresponds to the rate at which the distribution function changes due to collisional processes. We consider elastic collisions with electrons, ions, and atoms, inelastic collisions with the atoms, and ionization collisions. This reads

$$C(f_{\epsilon}) = \sum_{\alpha=\epsilon, i, g} C_{\epsilon, \alpha}^{el}(f_{\epsilon}, f_{\alpha}) + C_{\epsilon, g}^{inel}(f_{\epsilon}, f_{g}) + C_{\epsilon, g}^{iz}(f_{\epsilon}, f_{g}, f_i), \quad (2)$$

where the subscripts ϵ , i and g refer to electrons, ions and gas atoms, respectively.

2.2.1 Elastic collisions

The elastic collisions of an electron and a species $\alpha \in \{\epsilon, g, i\}$ conserve the momentum and energy,

$$m_{\epsilon}\mathbf{v}_{\epsilon} + m_{\alpha}\mathbf{v}_{\alpha} = m_{\epsilon}\mathbf{v}'_{\epsilon} + m_{\alpha}\mathbf{v}'_{\alpha} \quad \text{and} \quad \frac{1}{2}m_{\epsilon}v_{\epsilon}^2 + \frac{1}{2}m_{\alpha}v_{\alpha}^2 = \frac{1}{2}m_{\epsilon}v_{\epsilon}'^2 + \frac{1}{2}m_{\alpha}v_{\alpha}'^2, \quad (3)$$

where we use the prime to denote the post-collisional velocities. The Boltzmann operator reads

$$C_{\epsilon, \alpha}^{el}(f_{\epsilon}, f_{\alpha}) = \int_{\mathbb{R}^3 \times \mathcal{S}^2} (f'_{\epsilon}f'_{\alpha} - f_{\epsilon}f_{\alpha}) g_{\epsilon\alpha} \sigma_{\epsilon\alpha}(g_{\epsilon\alpha}, \boldsymbol{\omega} \cdot \mathbf{e}) d\boldsymbol{\omega} d\mathbf{v}_{\alpha}, \quad (4)$$

where $f'_{\alpha} = f_{\alpha}(\mathbf{x}, \mathbf{v}'_{\alpha}, t)$. The unit vectors in the direction of the pre-collisional and post-collisional relative velocities are denoted $\mathbf{e} = (\mathbf{v}_{\epsilon} - \mathbf{v}_{\alpha})/\|\mathbf{v}_{\epsilon} - \mathbf{v}_{\alpha}\|$ and $\boldsymbol{\omega} = (\mathbf{v}'_{\epsilon} - \mathbf{v}'_{\alpha})/\|\mathbf{v}'_{\epsilon} - \mathbf{v}'_{\alpha}\|$ and the norm of the relative velocity between the colliding particles is $g_{\epsilon\alpha} = \|\mathbf{v}_{\epsilon} - \mathbf{v}_{\alpha}\|$. Finally, the differential cross-section of the collision between an electron and a species α is denoted $\sigma_{\epsilon\alpha}(g_{\epsilon\alpha}, \boldsymbol{\omega} \cdot \mathbf{e})$ and depends only on $g_{\epsilon\alpha}$ and the scattering angle, $\arccos(\boldsymbol{\omega} \cdot \mathbf{e})$.

2.2.2 Inelastic collisions

For the inelastic collisions, we use in this work the Wang Chang-Uhlenbeck operator [16]. We consider the following collision mechanism, where the atom quantum state changes from k to l energy shell after the collision with the electron. The conservation of momentum and energy reads

$$m_{\epsilon}\mathbf{v}_{\epsilon} + m_{g}\mathbf{v}_{g} = m_{\epsilon}\mathbf{v}'_{\epsilon} + m_{g}\mathbf{v}'_{g}, \quad \frac{1}{2}m_{\epsilon}v_{\epsilon}^2 + \frac{1}{2}m_{g}v_{g}^2 + e\phi_k^* = \frac{1}{2}m_{\epsilon}v_{\epsilon}'^2 + \frac{1}{2}m_{g}v_{g}'^2 + e\phi_l^*, \quad (5)$$

where ϕ_k^* and ϕ_l^* are the pre- and post-collisional internal energies of the atom, in eV. The collision operator reads

$$\begin{aligned} C_{\epsilon, g}^{inel}(f_{\epsilon}, f_{g}) &= \sum_{k \in \mathcal{Q}_g} C_{\epsilon, g}^{k, inel}(f_{\epsilon}, f_{g}) \\ &= \sum_{\substack{k, l \in \mathcal{Q}_g \\ k \neq l}} \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} \left(\frac{a_k}{a_l} f'_{\epsilon} f'_{g, l} - f_{\epsilon} f_{g, k} \right) g_{\epsilon g} \sigma_{\epsilon g}^{k \rightarrow l}(g_{\epsilon g}, \boldsymbol{\omega} \cdot \mathbf{e}) d\boldsymbol{\omega} d\mathbf{v}_g. \end{aligned} \quad (6)$$

Here, we denote by \mathcal{Q}_g the indexing set of quantum internal energy states of the gas atom, a_l and a_k are the degeneracy of the l -th and k -th quantum states, and $f_{g,l}(\mathbf{x}, \mathbf{v}_g, t)$, resp. with a prime, designates the distribution function of atoms with an index l corresponding to the quantum energy shell. With this notation, we denote $f_g = \sum_{l \in \mathcal{Q}_g} f_{g,l}$. Finally, $\sigma_{\epsilon g}^{k \rightarrow l}(g_{\epsilon g}, \boldsymbol{\omega} \cdot \mathbf{e})$ is the differential cross-section of the inelastic collision that

excites the atom from the state k to the state l . It depends again on the relative velocity norm $g_{\epsilon g} = \|\mathbf{v}_\epsilon - \mathbf{v}_g\|$ and on the scattering angle $\arccos(\boldsymbol{\omega} \cdot \mathbf{e})$ with $\mathbf{e} = (\mathbf{v}_\epsilon - \mathbf{v}_\alpha) / \|\mathbf{v}_\epsilon - \mathbf{v}_\alpha\|$ and $\boldsymbol{\omega} = (\mathbf{v}'_\epsilon - \mathbf{v}'_\alpha) / \|\mathbf{v}'_\epsilon - \mathbf{v}'_\alpha\|$. Using the energy and momentum conservation properties (5), one verifies that the norm of the pre- and post-collisional relative velocity are related through (see also, e.g., [56])

$$\frac{(g_{\epsilon g})^2 - (g'_{\epsilon g})^2}{2} = \frac{e(\phi_k^* - \phi_l^*)}{\mu_{\epsilon g}},$$

where $\mu_{\epsilon g} = m_\epsilon m_g / (m_\epsilon + m_g)$.

2.2.3 Ionization

The electron-impact ionization reaction $\epsilon + g \rightleftharpoons \epsilon_1 + \epsilon_2 + i$ has the following conservation laws,

$$\begin{aligned} m_\epsilon \mathbf{v}_\epsilon + m_g \mathbf{v}_g &= m_\epsilon \mathbf{v}'_{\epsilon_1} + m_\epsilon \mathbf{v}'_{\epsilon_2} + m_i \mathbf{v}'_i, \\ \frac{1}{2} m_\epsilon v_\epsilon^2 + \frac{1}{2} m_g v_g^2 &= \frac{1}{2} m_\epsilon v_{\epsilon_1}^2 + \frac{1}{2} m_\epsilon v_{\epsilon_2}^2 + \frac{1}{2} m_i v_i^2 + e\phi_{iz}^*. \end{aligned} \quad (7)$$

The reactive collision operator, as explained in [57, 58, 18], reads,

$$\begin{aligned} C_{\epsilon, g}^{iz}(f_\epsilon, f_g, f_i) &= \int_{(\mathbb{R}^3)^4} \left(f'_i f'_{\epsilon_1} f'_{\epsilon_2} \frac{\beta_\epsilon^2 \beta_i}{\beta_g \beta_\epsilon} - f_g f_\epsilon \right) W_{g\epsilon}^{ie\epsilon} d\mathbf{v}_g d\mathbf{v}'_i d\mathbf{v}'_{\epsilon_1} d\mathbf{v}'_{\epsilon_2} \\ &+ 2 \int_{(\mathbb{R}^3)^4} \left(f'_g f'_{\epsilon_1} \frac{\beta_g \beta_\epsilon}{\beta_\epsilon^2 \beta_i} - f_i f_\epsilon f_{\epsilon_2} \right) W_{i\epsilon\epsilon}^{g\epsilon} d\mathbf{v}'_g d\mathbf{v}'_{\epsilon_1} d\mathbf{v}_i d\mathbf{v}_{\epsilon_2}, \end{aligned} \quad (8)$$

where $\beta_\alpha = (h_P / m_\alpha)^3$ is the statistical weight of the particle $\alpha \in \{\epsilon, g, i\}$ where h_P is Planck's constant and $W_{g\epsilon}^{ie\epsilon}$ and $W_{i\epsilon\epsilon}^{g\epsilon}$ are respectively the ionization and recombination transition probabilities.

Remark that f_ϵ , f_{ϵ_1} and f_{ϵ_2} refer to the same distribution function of electrons, but evaluated at different values of velocity variable, e.g., $f'_{\epsilon_1} = f_\epsilon(\mathbf{x}, \mathbf{v}'_{\epsilon_1}, t)$ and the two electrons involved after collisions in case of ionization or before in case of recombination are distinguished by adding a number to their index.

2.2.4 Kinetic entropy production

Define the Boltzmann kinetic entropy

$$\mathcal{S}_\epsilon^{kin} = \int_{\mathbb{R}^3} \eta(f_\epsilon) d\mathbf{v}_\epsilon \quad \text{with} \quad \eta(f_\epsilon) = -k_B f_\epsilon (\log(\beta_\epsilon f_\epsilon) - 1), \quad (9)$$

with the Boltzmann constant k_B . The production of entropy due to collisions is

$$E_\alpha^{coll} = \int_{\mathbb{R}^3} \eta'(f_\alpha) C_\alpha^{coll} d\mathbf{v}_\alpha, \quad (10)$$

where the subscript refers to the considered particle equation and the superscript refers to the type of collision (and the particles involved when necessary).

The electron-electron elastic collisions is known to dissipate Boltzmann kinetic entropy, in the sense that the following production term is non-negative,

$$E_\epsilon^{\epsilon, \epsilon, el} = \int_{\mathbb{R}^3} \eta'(f_\epsilon) C_{\epsilon, \epsilon}^{el}(f_\epsilon, f_\epsilon) d\mathbf{v}_\epsilon \geq 0. \quad (11)$$

For all the interspecies collisions, the cross-sections are assumed to satisfy the micro-reversibility principle [59] (sometimes referred to as detailed balance) which yields respectively

$$g_{\epsilon\alpha}\sigma_{\epsilon\alpha}(g_{\epsilon,\alpha}, \boldsymbol{\omega} \cdot \boldsymbol{e})d\boldsymbol{\omega}d\bar{\boldsymbol{v}}_\alpha d\bar{\boldsymbol{v}}_\epsilon = g'_{\epsilon\alpha}\sigma_{\epsilon\alpha}(g'_{\epsilon,\alpha}, \boldsymbol{\omega}' \cdot \boldsymbol{e}')d\boldsymbol{\omega}'d\bar{\boldsymbol{v}}'_\alpha d\bar{\boldsymbol{v}}'_\epsilon, \quad (12a)$$

$$a_k g_{\epsilon\mathfrak{g}}\sigma_{\epsilon\mathfrak{g}}^{k \rightarrow l}(g_{\epsilon\mathfrak{g}}, \boldsymbol{\omega} \cdot \boldsymbol{e})d\boldsymbol{\omega}d\bar{\boldsymbol{v}}_\mathfrak{g}d\bar{\boldsymbol{v}}_\epsilon = a_l g'_{\epsilon\mathfrak{g}}\sigma_{\epsilon\mathfrak{g}}^{l \rightarrow k}(g'_{\epsilon\mathfrak{g}}, \boldsymbol{\omega}' \cdot \boldsymbol{e}')d\boldsymbol{\omega}'d\bar{\boldsymbol{v}}'_\mathfrak{g}d\bar{\boldsymbol{v}}'_\epsilon, \quad (12b)$$

$$\beta_\mathfrak{g}\beta_\epsilon W_{\text{icc}}^{\mathfrak{g}\epsilon} = \beta_\epsilon^2\beta_\mathfrak{i}W_{\text{icc}}^{\mathfrak{i}\epsilon} = \beta_\mathfrak{g}\beta_\epsilon W_{\text{icc}}^{\epsilon\mathfrak{g}} = \beta_\epsilon^2\beta_\mathfrak{i}W_{\text{icc}}^{\epsilon\mathfrak{i}}. \quad (12c)$$

Under these conditions, the entropy production terms read respectively (we refer e.g. to [60, 56, 19, 58] for details)

$$E_\epsilon^{\epsilon,\alpha,el} + E_\alpha^{\alpha,\epsilon,el} = \frac{k_B}{4} \int \Omega(f'_\epsilon f'_\alpha, f_\epsilon f_\alpha) g_{\epsilon\alpha}\sigma_{\epsilon\alpha}^{el}(g_{\epsilon\alpha}, \boldsymbol{\omega} \cdot \boldsymbol{e})d\boldsymbol{\omega}d\boldsymbol{v}_\epsilon d\boldsymbol{v}_\alpha \geq 0, \quad \text{for } \alpha \in \{\mathfrak{i}, \mathfrak{g}\}, \quad (13a)$$

$$E_\epsilon^{\mathfrak{s},\mathfrak{g},inel} + E_\mathfrak{g}^{\mathfrak{s},\epsilon,inel} = \frac{k_B}{4} \int \sum_{k,l \in \mathcal{Q}_\mathfrak{g}} \Omega\left(\frac{f'_\epsilon f'_{\mathfrak{g},l}}{a_l}, \frac{f_\epsilon f_{\mathfrak{g},k}}{a_k}\right) \frac{g_{\epsilon\mathfrak{g}}}{a_k} \sigma_{\epsilon\mathfrak{g}}^{k \rightarrow l}(g_{\epsilon\mathfrak{g}}, \boldsymbol{\omega} \cdot \boldsymbol{e})d\boldsymbol{\omega}d\boldsymbol{v}_\epsilon d\boldsymbol{v}_\mathfrak{g} \geq 0, \quad (13b)$$

$$E_\epsilon^{iz} + E_\mathfrak{g}^{iz} + E_\mathfrak{i}^{iz} = \frac{k_B}{4} \int \Omega\left(\frac{f'_{\epsilon_1} f'_{\epsilon_2} f'_\mathfrak{i}}{\beta_\epsilon^2 \beta_\mathfrak{i}}, \frac{f_\epsilon f_\mathfrak{g}}{\beta_\mathfrak{g} \beta_\epsilon}\right) \frac{g_{\epsilon\mathfrak{g}}}{\beta_\epsilon \beta_\mathfrak{g}} W_{\text{icc}}^{\mathfrak{i}\epsilon} d\boldsymbol{v}_\epsilon d\boldsymbol{v}_\mathfrak{g} d\boldsymbol{v}'_\mathfrak{i} d\boldsymbol{v}'_{\epsilon_1} d\boldsymbol{v}'_{\epsilon_2} \geq 0, \quad (13c)$$

using the notation $\Omega(x, y) = \log(x/y)(x - y)$. Remark that, since only the electron equation is considered in this work, all the term $E_\mathfrak{g}^{coll}$ and $E_\mathfrak{i}^{coll}$ account for the entropy production with the ‘‘background’’. A priori, they cannot be neglected in the entropy balance. However, in the next section, we exploit the assumptions of Section 2.1 in order to simplify those formulae in the considered limit.

2.3 Scaled equations and expansion of the collision operators

We now rescale the equations in order to isolate the leading effects under the considered regime. In this part, we relate the non-dimensional parameter to each others and reduce them to two small parameters: the electron-to-gas density ratio (also referred to as ionization level or ionization fraction) and the square root of the electron-to-gas mass ratio, defined by

$$\delta = \frac{n_\epsilon^0}{n_\mathfrak{g}^0}, \quad \varepsilon = \sqrt{\frac{m_\epsilon}{m_\mathfrak{g}}}, \quad (14)$$

where n_α^0 are the reference density of α and m_α are the mass of a particle α . Following the hypothesis (4 a), we assume in the following

$$\delta \ll \varepsilon \ll 1.$$

Remark 1. In practice, in the considered regime, the parameter δ has a numerical value close to ε^2 . In the following, the term δ is only used to reduce the collision operators to their leading order in δ (the rest being negligible anyway), while the parameter ε is used to perform an asymptotic expansion. The only difference in the resulting model between neglecting all the terms $\mathcal{O}(\delta)$ or considering that $\delta = \varepsilon^2$ is the presence or not of the electron-electron collision operator, but the treatment of this term presents little difficulty in our work.

2.3.1 Scaling of the kinetic equation

Following a similar procedure as this proposed by Graille et al. [15], we consider the following normalized kinetic equation:

$$\frac{\partial \bar{f}_\epsilon}{\partial \bar{t}} + \frac{1}{\text{St}} \bar{\boldsymbol{v}} \cdot \bar{\nabla} \bar{f}_\epsilon - \frac{\text{El}}{\text{St Kn}_{\epsilon\mathfrak{g}}^{el}} \bar{\boldsymbol{E}} \cdot \nabla_{\bar{\boldsymbol{v}}} \bar{f}_\epsilon = \frac{1}{\text{St Kn}_{\epsilon\mathfrak{g}}^{el}} \bar{C}(\bar{f}_\epsilon). \quad (15)$$

Here, $\text{St} = L^0/(V_{th_\epsilon}^0 t^0)$ is the Strouhal number, $\text{Kn}_{\epsilon\mathfrak{g}}^{el} = 1/(n_\mathfrak{g}^0 \sigma_{\epsilon\mathfrak{g}}^0 L^0)$ is the electron-gas atom elastic collisions Knudsen number, and $\text{El} = eE^0 \tau_{\epsilon\mathfrak{g}}^0/(m_\epsilon V_{th_\epsilon}^0)$ is the electric field parameter with the characteristic time

between electron-gas atom elastic collisions as $\tau_{\text{eg}}^0 = (n_{\text{g}}^0 V_{th_{\text{e}}}^0 \sigma_{\text{eg}}^0)^{-1}$. The electric field parameter represents the ratio between the characteristic electric field force exerted to an electron and the characteristic electron-gas collision drag force. Note that we have used the electron-gas atom elastic collisions to define the Knudsen number and the electric field parameter since, for most low-temperature weakly-ionized plasmas, this is the dominant collision mechanism in the electron kinetic equation (Assumption (4a)). The normalized variables are

$$\bar{t} = \frac{t}{t^0}, \quad \bar{\mathbf{x}} = \frac{\mathbf{x}}{L^0}, \quad \bar{f}_{\text{e}} = f_{\text{e}} \frac{(V_{th_{\text{e}}}^0)^3}{n_{\text{e}}^0}, \quad \bar{C}(\bar{f}_{\text{e}}) = C(f_{\text{e}}) \frac{(V_{th_{\text{e}}}^0)^2}{n_{\text{g}}^0 n_{\text{e}}^0 \sigma_{\text{eg}}^0},$$

$$\bar{\mathbf{E}} = \frac{\mathbf{E}}{E^0} \quad \text{and} \quad \bar{\mathbf{v}} = \frac{\mathbf{v}}{V_{th_{\text{e}}}^0},$$

with $V_{th_{\text{e}}}^0 = (eT_{\text{e}}^0/m_{\text{e}})^{1/2}$ and where the electron temperature is given in eV. The symbols with the superscript 0 correspond to the characteristic quantities of the problem.

We rewrite and specify the physical regime presented in the Assumptions (4 b-d) in mathematical terms with

$$t^0 = \frac{L^0}{V_{th_{\text{g}}}^0}, \quad L^0 = \frac{l_{\text{eg}}^0}{\varepsilon}, \quad E^0 L^0 = T_{\text{e}}^0, \quad (16)$$

where the electron-gas characteristic collision mean free path is

$$l_{\text{eg}}^0 = (n_{\text{g}}^0 \sigma_{\text{eg}}^0)^{-1}.$$

Also, we stress that in an atomic gas, the mass of the atom and the ions is nearly the same. The studied regime is similar to this of Graille et al. [15]. The main difference is that in our work we distinguish between the Knudsen number of the different collisional processes, whereas in Ref. [15] the Knudsen of all the interactions is assumed to be the same.

In the considered regime, the non-dimensional numbers read

$$\text{St} = \kappa \varepsilon, \quad \text{El} = \text{Kn}_{\text{eg}}^{\text{el}} = \varepsilon.$$

Here, we denote by $\kappa = \sqrt{T_{\text{e}}^0/T_{\text{h}}^0}$ the square root of the heavy species-to-electron temperature ratio, where T_{h}^0 is the characteristic temperature of the heavy species (ions and neutrals). Note that in low-temperature plasmas the ion and gas temperatures are usually of the same order, hence $T_{\text{h}}^0 = T_{\text{g}}^0 = T_{\text{i}}^0$. In this paper, we consider only leading order terms of the parameter κ , although this parameter can also be small in low-temperature plasmas. Nevertheless, we consider that $\varepsilon \ll \kappa$.

As a result, we obtain the following scaled equation,

$$\frac{\partial \bar{f}_{\text{e}}}{\partial \bar{t}} + \frac{1}{\varepsilon} \bar{\mathbf{v}} \cdot \bar{\nabla} \bar{f}_{\text{e}} - \frac{1}{\varepsilon} \bar{\mathbf{E}} \cdot \nabla_{\bar{\mathbf{v}}} \bar{f}_{\text{e}} = \frac{1}{\varepsilon^2} \bar{C}(\bar{f}_{\text{e}}). \quad (17)$$

2.3.2 Scaling of the collisional processes

As mentioned above, in this work, we distinguish between the Knudsen number of the different collisional processes. As a result, the normalized collision operator reads,

$$\bar{C}(\bar{f}_{\text{e}}) = \sum_{\alpha \in \{\text{e}, \text{i}, \text{g}\}} \left(\frac{\text{Kn}_{\text{eg}}^{\text{el}}}{\text{Kn}_{\text{e}\alpha}^{\text{el}}} \right) \bar{C}_{\text{e},\alpha}^{\text{el}}(\bar{f}_{\text{e}}, \bar{f}_{\alpha}) + \left(\frac{\text{Kn}_{\text{eg}}^{\text{el}}}{\text{Kn}_{\text{eg}}^{\text{inel}}} \right) \bar{C}_{\text{e},\text{g}}^{\text{inel}}(\bar{f}_{\text{e}}, \bar{f}_{\text{g}}) + \left(\frac{\text{Kn}_{\text{eg}}^{\text{el}}}{\text{Kn}_{\text{eg}}^{\text{iz}}} \right) \bar{C}_{\text{e},\text{g}}^{\text{iz}}(\bar{f}_{\text{e}}, \bar{f}_{\text{g}}, \bar{f}_{\text{i}}). \quad (18)$$

Here, the different Knudsen numbers are defined as $\text{Kn}_{\text{e}\alpha}^{\text{coll}} = (n_{\alpha}^0 \sigma_{\text{e}\alpha}^{\text{coll},0} L^0)^{-1}$.

In noble gases, we can assume that the Knudsen number of the excitation and the ionization collisions are of the same order. This choice is motivated by the fact that the inelastic and ionization collisions have an

energy threshold, unlike the elastic collisions. In gas discharge plasmas of noble gases, the energy threshold of both the ionization and the first excitation potential is large enough compared to the electron mean energy to produce inelastic and ionization collision rates that are much smaller than the electron-gas elastic collision rate (c.f., Figure 3.16 from Lieberman and Lichtenberg [1] for the collision rates in an argon plasma). As a result, we assume the following scaling:

$$\frac{\sigma_{\mathbf{e}\mathbf{g}}^{inel,0}}{\sigma_{\mathbf{e}\mathbf{g}}^{iz,0}} = \mathcal{O}(1) \quad \text{and} \quad \frac{\text{Kn}_{\mathbf{e}\mathbf{g}}^{el}}{\text{Kn}_{\mathbf{e}\mathbf{g}}^{inel}} = \frac{\sigma_{\mathbf{e}\mathbf{g}}^{inel,0}}{\sigma_{\mathbf{e}\mathbf{g}}^{el,0}} = \mathcal{O}(\varepsilon^2).$$

Concerning the Coulomb collisions, the cross-section is usually much larger than the electron-neutral cross-section, in particular at low electron energies. In the weakly-ionized regime, the effect of the Coulomb collisions is very small compared to the electron-gas collisions due to the low ionization level (i.e., the low electron-gas density ratio) and the high electron temperature. In particular, in this paper, we consider the following regime:

$$\frac{\text{Kn}_{\mathbf{e}\mathbf{g}}^{el}}{\text{Kn}_{\mathbf{e}\mathbf{e}}^{el}} = \delta \frac{\sigma_{\mathbf{e}\mathbf{g}}^{el,0}}{\sigma_{\mathbf{e}\mathbf{e}}^{el,0}} = \mathcal{O}(\varepsilon^2).$$

In summary, we study the regime where the considered collisional processes scale as follows:

$$\frac{\text{Kn}_{\mathbf{e}\mathbf{g}}^{el}}{\text{Kn}_{\mathbf{e}\mathbf{g}}^{inel}} = \mathcal{O}(\varepsilon^2), \quad \frac{\text{Kn}_{\mathbf{e}\mathbf{g}}^{el}}{\text{Kn}_{\mathbf{e}\mathbf{e}}^{el}} = \mathcal{O}(\varepsilon^2), \quad \frac{\text{Kn}_{\mathbf{e}\mathbf{g}}^{el}}{\text{Kn}_{\mathbf{e}\mathbf{i}}^{el}} = \mathcal{O}(\varepsilon^2), \quad \frac{\text{Kn}_{\mathbf{e}\mathbf{g}}^{el}}{\text{Kn}_{\mathbf{e}\mathbf{g}}^{iz}} = \mathcal{O}(\varepsilon^2). \quad (19)$$

2.3.3 Expansion of the collision operators

In the collisions, different type of particles participate with disparate masses and species with different densities. As first proposed by Lorentz [6], we can exploit the mass disparity between electrons and the heavy species in order to expand the collision operators in different orders of the square root of the mass ratio. The following expansion of the collision operators at the kinetic level will simplify the computations in the next section.

Expansion of the electron-heavy species elastic collision operator In this work, we use the expansion of the elastic collision operator in terms of the electron-to-heavy mass ration, as proposed by Fernández de la Mora et al. [61] and Graille et al. [15]. As there are some differences between the two references, we present an overall proof here and the computational details are left in A.

Proposition 1 ([61]). *The operator for the electron elastic collisions with a heavy (atom or ion) species expands as follows:*

$$\bar{C}_{\mathbf{e},\alpha}^{el} = \bar{C}_{\mathbf{e},\alpha}^{0,el} + \varepsilon \bar{C}_{\mathbf{e},\alpha}^{1,el} + \varepsilon^2 \bar{C}_{\mathbf{e},\alpha}^{2,el} + \mathcal{O}(\varepsilon^3) \quad \text{for } \alpha \in \{\mathbf{i}, \mathbf{g}\}. \quad (20)$$

These first three orders of $\bar{C}_{\mathbf{e},\alpha}^{el}$ depend on \bar{f}_α only through its first three moments, i.e. heavy species density, mean velocity and energy tensors: for $\alpha \in \{\mathbf{i}, \mathbf{g}\}$

$$\bar{n}_\alpha = \int_{\mathbb{R}^3} \bar{f}_\alpha d\bar{\mathbf{v}}_\alpha, \quad \bar{\mathbf{u}}_\alpha = \frac{1}{n_\alpha} \int_{\mathbb{R}^3} \bar{\mathbf{v}}_\alpha \bar{f}_\alpha d\bar{\mathbf{v}}_\alpha, \quad \bar{\mathbf{u}}_\alpha \otimes \bar{\mathbf{u}}_\alpha + \bar{\mathbf{T}}_\alpha = \frac{1}{n_\alpha} \int_{\mathbb{R}^3} \bar{\mathbf{v}}_\alpha \otimes \bar{\mathbf{v}}_\alpha \bar{f}_\alpha d\bar{\mathbf{v}}_\alpha. \quad (21)$$

Denoting for simplicity $\tilde{f}_\mathbf{e} \equiv \tilde{f}_\mathbf{e}(\mathbf{x}, \tilde{\mathbf{v}}_\mathbf{e}, t)$ the electron distribution function at the zero-th order post-collisional velocity with the zero-th order electron post-collision velocity as $\tilde{\mathbf{v}}_\mathbf{e} = \bar{\mathbf{v}}_\mathbf{e} \boldsymbol{\omega}$ with a variable $\boldsymbol{\omega} \in \mathcal{S}^2$, the collision operator at different orders read:

$$\bar{C}_{\mathbf{e},\alpha}^{0,el}(\bar{f}_\mathbf{e}, \bar{f}_\alpha) = \int_{\mathcal{S}^2} (\tilde{f}_\mathbf{e} - \bar{f}_\mathbf{e}) \bar{v}_{\mathbf{e}\alpha} d\boldsymbol{\omega} \quad (22a)$$

$$\bar{C}_{\mathbf{e},\alpha}^{1,el}(\bar{f}_\mathbf{e}, \bar{f}_\alpha) = \bar{\mathbf{u}}_\alpha \cdot \left(\int_{\mathcal{S}^2} (\nabla_{\tilde{\mathbf{v}}_\mathbf{e}} \tilde{f}_\mathbf{e} - \nabla_{\bar{\mathbf{v}}_\mathbf{e}} \bar{f}_\mathbf{e}) \bar{v}_{\mathbf{e}\alpha} d\boldsymbol{\omega} - \nabla_{\bar{\mathbf{v}}_\mathbf{e}} \int_{\mathcal{S}^2} (\tilde{f}_\mathbf{e} - \bar{f}_\mathbf{e}) \bar{v}_{\mathbf{e}\alpha} d\boldsymbol{\omega} \right) \quad (22b)$$

$$\bar{C}_{\mathbf{e},\alpha}^{2,el}(\bar{f}_\mathbf{e}, \bar{f}_\alpha) = \bar{C}_{n_\alpha}^{2,el}(\bar{f}_\mathbf{e}) + (\bar{\mathbf{u}}_\alpha \otimes \bar{\mathbf{u}}_\alpha + \bar{\mathbf{T}}_\alpha) : \bar{C}_{\mathbf{T}_\alpha}^{2,el}(\bar{f}_\mathbf{e}) \quad (22c)$$

and with the operators

$$\bar{C}_{n_\alpha}^{2,el}(\bar{f}_\epsilon) = \nabla_{\bar{\mathbf{v}}} \cdot \int_{S^2} (\mathbf{v}_\epsilon - v_\epsilon \boldsymbol{\omega}) \tilde{f}_\epsilon \bar{v}_{\epsilon\alpha} d\boldsymbol{\omega} \quad (23a)$$

$$\begin{aligned} \bar{C}_{T_\alpha}^{2,el}(\bar{f}_\epsilon) &= \nabla_{\bar{\mathbf{v}}} \otimes \left(\int_{S^2} (\nabla_{\bar{\mathbf{v}}} \bar{f}_\epsilon - \nabla_{\bar{\mathbf{v}}} \tilde{f}_\epsilon) \bar{v}_{\epsilon\alpha} d\boldsymbol{\omega} + \frac{1}{2} \nabla_{\bar{\mathbf{v}}} \int_{S^2} (\tilde{f}_\epsilon - \bar{f}_\epsilon) \bar{v}_{\epsilon\alpha} d\boldsymbol{\omega} \right) \\ &\quad - \frac{1}{2} \int_{S^2} (\nabla_{\bar{\mathbf{v}}} \otimes \nabla_{\bar{\mathbf{v}}} \tilde{f}_\epsilon - \nabla_{\bar{\mathbf{v}}} \otimes \nabla_{\bar{\mathbf{v}}} \bar{f}_\epsilon) \bar{v}_{\epsilon\alpha} d\boldsymbol{\omega}. \end{aligned} \quad (23b)$$

The dependence on \bar{f}_α appears also through the collision frequency which is defined as

$$\bar{v}_{\epsilon\alpha} = \bar{n}_\alpha \bar{v}_\epsilon \bar{\sigma}_{\epsilon\alpha}(\bar{\mathbf{v}}_\epsilon, \boldsymbol{\omega} \cdot \hat{\mathbf{v}}_\epsilon)$$

with the unit vector in the direction of the electron pre-collision velocity as $\hat{\mathbf{v}}_\epsilon = \bar{\mathbf{v}}_\epsilon / \bar{v}_\epsilon$.

Proof. We start the proof by normalizing the collision operator of (4), as follows,

$$\bar{C}_{\epsilon,\alpha}^{el}(\bar{f}_\epsilon, \bar{f}_\alpha) = \int_{\mathbb{R}^3} \int_{S^2} (\bar{f}'_\epsilon \bar{f}'_\alpha - \bar{f}_\epsilon \bar{f}_\alpha) \bar{g}_{\epsilon\alpha} \bar{\sigma}_{\epsilon\alpha}(\bar{g}_{\epsilon\alpha}, \boldsymbol{\omega} \cdot \mathbf{e}) d\boldsymbol{\omega} d\bar{\mathbf{v}}_\alpha, \quad (24)$$

where $\bar{g}_{\epsilon\alpha} = g_{\epsilon\alpha} / V_{th_\epsilon}^0$ and the normalized distribution function $\bar{f}_\alpha = (V_{th_\alpha}^0)^3 f_\alpha / n_\alpha^0$.

Similarly, we normalize the momentum and energy conservation relations of the elastic collisions (3) by defining the heavy-species normalized velocity as $\bar{\mathbf{v}}_\alpha = \mathbf{v}_\alpha / V_{th_\alpha}^0$, with $V_{th_\alpha}^0 = (\epsilon T_\alpha^0 / m_\alpha)^{1/2}$ where the temperature is measured in eV. Note that the characteristic velocity of the heavy particle is much smaller than this of the electron because of the disparity of masses, i.e., $V_{th_\alpha}^0 = \epsilon V_{th_\epsilon}^0$. As the elastic collision conserves the kinetic energy, the norm of the relative velocity between the particles is conserved. As a result, the post-collisional relative velocity is a rotation of the pre-collisional relative velocity, as follows, $\bar{\mathbf{g}}' = \boldsymbol{\Omega} \bar{\mathbf{g}}$, where $\boldsymbol{\Omega}$ is the rotation matrix. The non-dimensional post-collisional velocities are obtained by using the normalized relations in (3) and the rotation matrix introduced above, as follows,

$$\bar{\mathbf{v}}'_\epsilon = \boldsymbol{\Omega} \bar{\mathbf{v}}_\epsilon - \epsilon (\boldsymbol{\Omega} \bar{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) - \epsilon^2 (\boldsymbol{\Omega} \bar{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) + \mathcal{O}(\epsilon^3), \quad (25a)$$

$$\bar{\mathbf{v}}'_\alpha = \bar{\mathbf{v}}_\alpha - \epsilon (\boldsymbol{\Omega} \bar{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) + \epsilon^2 (\boldsymbol{\Omega} \bar{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) + \mathcal{O}(\epsilon^3) \quad \text{for } \alpha \in \{\mathbf{i}, \mathbf{g}\}. \quad (25b)$$

We introduce the notation $\tilde{\mathbf{v}}_\epsilon = \boldsymbol{\Omega} \bar{\mathbf{v}}_\epsilon$ and $\tilde{\mathbf{v}}_\alpha = \boldsymbol{\Omega} \bar{\mathbf{v}}_\alpha$ to refer to the rotated pre-collisional velocity vectors. With the previous notation and with the expansion of the post-collisional velocities of (25), we can expand the distribution function at the post-collisional velocity in series of ϵ , as follows,

$$\bar{f}_\epsilon(\bar{\mathbf{v}}'_\epsilon) = \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) - \epsilon (\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) \cdot \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) \quad (26a)$$

$$- \epsilon^2 \left[(\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) - \frac{1}{2} (\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) \otimes (\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) : \nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) \right],$$

$$\bar{f}_\alpha(\bar{\mathbf{v}}'_\alpha) = \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) - \epsilon (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\bar{\mathbf{v}}_\alpha} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) \quad (26b)$$

$$+ \epsilon^2 \left[(\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) \cdot \nabla_{\bar{\mathbf{v}}_\alpha} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) + \frac{1}{2} (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \otimes (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) : \nabla_{\bar{\mathbf{v}}_\alpha} \otimes \nabla_{\bar{\mathbf{v}}_\alpha} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) \right],$$

We define the collision kernel as $S(\bar{\mathbf{g}}') := \bar{g}_{\epsilon\alpha} \bar{\sigma}_{\epsilon\alpha}(\bar{g}_{\epsilon\alpha}, \boldsymbol{\omega} \cdot \mathbf{e})$, which is a function of $\bar{\mathbf{g}}'$ as it is a function of the norm of the relative velocity (which is conserved) and the scattering angle of the relative velocity. We can expand the collision kernel in powers of ϵ . To do so, we write the post-collisional relative vector as $\bar{\mathbf{g}}' = \boldsymbol{\Omega} \bar{\mathbf{g}} = \tilde{\mathbf{v}}_\epsilon - \epsilon \tilde{\mathbf{v}}_\alpha$. As a result, the expansion of the collision kernel reads,

$$S(\bar{\mathbf{g}}') = S(\tilde{\mathbf{v}}_\epsilon) - \epsilon \tilde{\mathbf{v}}_\alpha \cdot \nabla_{\tilde{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) + \frac{1}{2} \epsilon^2 (\tilde{\mathbf{v}}_\alpha \otimes \tilde{\mathbf{v}}_\alpha) : \nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) + \mathcal{O}(\epsilon^3). \quad (27)$$

Remark that, using the fact that $\boldsymbol{\Omega}\boldsymbol{\Omega}^T = \mathbb{I}$, then the differential operators rewrite

$$\tilde{\mathbf{v}}_\alpha \cdot \nabla_{\tilde{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) = \bar{\mathbf{v}}_\alpha \cdot \nabla_{\bar{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon), \quad (\tilde{\mathbf{v}}_\alpha \otimes \tilde{\mathbf{v}}_\alpha) : \nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) = (\bar{\mathbf{v}}_\alpha \otimes \bar{\mathbf{v}}_\alpha) : \nabla_{\bar{\mathbf{v}}} \otimes \nabla_{\bar{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon).$$

Injecting the expansions (26-27) into the normalized operator of (24) eventually yields (20) with the operators of (22-23). As there are some small differences in the results of [15] and [61], details on these calculations are given, but in A for the sake of conciseness. \square

This elastic operator can be simplified under relevant hypothesis used in the next sections.

Corollary 1. *Suppose that $\bar{f}_\epsilon = \bar{f}_\epsilon^0$ and $\bar{f}_\alpha = \bar{f}_\alpha^0$ are isotropic in velocity, then $\bar{C}_{\epsilon,\alpha}^{0,el} = 0 = \bar{C}_{\epsilon,\alpha}^{1,el}$ and the elastic collision operator simplifies into*

$$\bar{C}_{\epsilon,\alpha}^{el} = \varepsilon^2 \bar{C}_{\epsilon,\alpha}^{2,el} + \mathcal{O}(\varepsilon^3), \quad \bar{C}_{\epsilon,\alpha}^{2,el}(\bar{f}_\epsilon^0, \bar{f}_\alpha^0) = \frac{1}{\bar{v}_\epsilon^2} \frac{\partial}{\partial \bar{v}_\epsilon} \left[\bar{n}_\alpha \bar{Q}_{\epsilon\alpha}^1 \bar{v}_\epsilon^3 \left(\bar{v}_\epsilon \bar{f}_\epsilon^0 + \bar{T}_\alpha \frac{\partial \bar{f}_\epsilon^0}{\partial \bar{v}_\epsilon} \right) \right], \quad (28)$$

for $\alpha \in \{\mathbf{i}, \mathbf{g}\}$. Here, the momentum transfer cross-section is defined as

$$\bar{Q}_{\epsilon\alpha}^1(\bar{v}_\epsilon) = \int_{S^2} (1 - \boldsymbol{\omega} \cdot \hat{\mathbf{v}}_\epsilon) \bar{\sigma}_{\epsilon\alpha}(\bar{v}_\epsilon, \boldsymbol{\omega} \cdot \hat{\mathbf{v}}_\epsilon) d\boldsymbol{\omega}. \quad (29)$$

Proof. First, note that for isotropic heavy distributions, from (21), we have

$$\bar{\mathbf{u}}_\alpha = 0 \quad \text{and} \quad \bar{\mathbf{T}}_\alpha = \bar{T}_\alpha \mathbb{I}, \quad (30)$$

where \mathbb{I} is the identity matrix. Similarly, an isotropic electron distribution implies $\tilde{f}_\epsilon^0 = \bar{f}_\epsilon^0$ and together with (30), the collision operators of (22) and (23) at different orders simplify as follows

$$\bar{C}_{\epsilon,\alpha}^{0,el}(\bar{f}_\epsilon^0, \bar{f}_\alpha^0) = 0 = \bar{C}_{\epsilon,\alpha}^{1,el}(\bar{f}_\epsilon^0, \bar{f}_\alpha^0) \quad (31a)$$

$$\begin{aligned} \bar{C}_{\epsilon,\alpha}^{2,el}(\bar{f}_\epsilon^0, \bar{f}_\alpha^0) &= \nabla_{\bar{\mathbf{v}}} \cdot \int_{S^2} (\mathbf{v}_\epsilon - v_\epsilon \boldsymbol{\omega}) \bar{v}_{\epsilon\alpha} \bar{f}_\epsilon^0 d\boldsymbol{\omega} \\ &\quad + \bar{T}_\alpha \nabla_{\bar{\mathbf{v}}} \cdot \left(\int_{S^2} \bar{v}_{\epsilon\alpha} (\nabla_{\bar{\mathbf{v}}} \bar{f}_\epsilon^0 - \nabla_{\bar{\mathbf{v}}} \bar{f}_\epsilon^0) d\boldsymbol{\omega} \right). \end{aligned} \quad (31b)$$

To obtain (31b), we have used $\mathbb{I} : \nabla_{\bar{\mathbf{v}}} \otimes \mathbf{A}(\bar{\mathbf{v}}_\epsilon) = \nabla_{\bar{\mathbf{v}}} \cdot \mathbf{A}(\bar{\mathbf{v}}_\epsilon)$ and $(\nabla_{\bar{\mathbf{v}}} \cdot \nabla_{\bar{\mathbf{v}}} \bar{f}_\epsilon^0 - \nabla_{\bar{\mathbf{v}}} \cdot \nabla_{\bar{\mathbf{v}}} \bar{f}_\epsilon^0) = 0$.

The first term of (31b) can be simplified for isotropic distribution functions. By using $\bar{v}_{\epsilon\alpha} = \bar{n}_\alpha \bar{v}_\epsilon \bar{\sigma}_{\epsilon\alpha}$, we have,

$$\begin{aligned} \nabla_{\bar{\mathbf{v}}} \cdot \int_{S^2} (\mathbf{v}_\epsilon - v_\epsilon \boldsymbol{\omega}) \bar{n}_\alpha \bar{v}_\epsilon \bar{\sigma}_{\epsilon\alpha} \bar{f}_\epsilon^0 d\boldsymbol{\omega} &= \nabla_{\bar{\mathbf{v}}} \cdot \int_{S^2} \bar{n}_\alpha \bar{v}_\epsilon^2 \bar{\sigma}_{\epsilon\alpha} \bar{f}_\epsilon^0 \mathbf{v}_\epsilon \hat{\mathbf{v}}_\epsilon (1 - \hat{\mathbf{v}}_\epsilon \cdot \boldsymbol{\omega}) d\boldsymbol{\omega} \\ &= \nabla_{\bar{\mathbf{v}}} \cdot \left(\bar{n}_\alpha \bar{Q}_{\epsilon\alpha}^1 \bar{v}_\epsilon^2 \bar{f}_\epsilon^0 \hat{\mathbf{v}}_\epsilon \right) \\ &= \frac{\partial}{\partial \bar{v}_\epsilon} \left(\bar{n}_\alpha \bar{Q}_{\epsilon\alpha}^1 \bar{v}_\epsilon^2 \bar{f}_\epsilon^0 \right) + 2 \bar{n}_\alpha \bar{Q}_{\epsilon\alpha}^1 \bar{v}_\epsilon \bar{f}_\epsilon^0 \\ &= \frac{1}{\bar{v}_\epsilon^2} \frac{\partial}{\partial \bar{v}_\epsilon} \left(\bar{n}_\alpha \bar{Q}_{\epsilon\alpha}^1 \bar{v}_\epsilon^4 \bar{f}_\epsilon^0 \right). \end{aligned}$$

In the last equation, we have used the definition of momentum transfer cross-section of (29) and the following equality. Let us consider $A(\bar{v}_\epsilon)$ a scalar function of the norm of the velocity, we have

$$\begin{aligned} \nabla_{\bar{\mathbf{v}}} \cdot (A(\bar{v}_\epsilon) \hat{\mathbf{v}}_\epsilon) &= \frac{\partial A(\bar{v}_\epsilon)}{\partial \bar{v}_\epsilon} (\hat{\mathbf{v}}_\epsilon \cdot \hat{\mathbf{v}}_\epsilon) + A(\bar{v}_\epsilon) \nabla_{\bar{\mathbf{v}}} \cdot \left(\frac{\hat{\mathbf{v}}_\epsilon}{\bar{v}_\epsilon} \right) \\ &= \frac{\partial A(\bar{v}_\epsilon)}{\partial \bar{v}_\epsilon} + A(\bar{v}_\epsilon) \left(\frac{\nabla_{\bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}_\epsilon}{\bar{v}_\epsilon} - \frac{\hat{\mathbf{v}}_\epsilon \cdot \hat{\mathbf{v}}_\epsilon}{\bar{v}_\epsilon} \right) = \frac{\partial A(\bar{v}_\epsilon)}{\partial \bar{v}_\epsilon} + 2 \frac{A(\bar{v}_\epsilon)}{\bar{v}_\epsilon} \\ &= \frac{1}{\bar{v}_\epsilon^2} \frac{\partial}{\partial \bar{v}_\epsilon} (A(\bar{v}_\epsilon) \bar{v}_\epsilon^2). \end{aligned} \quad (32)$$

The second term of (31b), by using (32), simplifies as follows,

$$\begin{aligned} \nabla_{\bar{v}} \cdot \left(\int_{S^2} \bar{v}_{\epsilon\alpha} \left(\nabla_{\bar{v}} \bar{f}_{\epsilon}^0 - \nabla_{\bar{v}} \bar{f}_{\epsilon}^0 \right) d\omega \right) &= \nabla_{\bar{v}} \cdot \left(\bar{n}_{\alpha} \bar{Q}_{\epsilon\alpha}^1 \bar{v}_{\epsilon} \frac{\partial \bar{f}_{\epsilon}^0}{\partial \bar{v}_{\epsilon}} \hat{v}_{\epsilon} \right) \\ &= \frac{1}{\bar{v}_{\epsilon}^2} \frac{\partial}{\partial \bar{v}_{\epsilon}} \left(\bar{n}_{\alpha} \bar{Q}_{\epsilon\alpha}^1 \bar{v}_{\epsilon}^3 \frac{\partial \bar{f}_{\epsilon}^0}{\partial \bar{v}_{\epsilon}} \right). \end{aligned}$$

□

Remark 2. • The electron-heavy elastic collision operator for isotropic distribution functions of (28) is the same as this proposed by Davydov [27] and used by Allis [28] in the SHE theory. We note that here it has been obtained by a rigorous expansion of the Boltzmann collision operator in terms of the mass ratio, whereas to the best of our knowledge, the previous work modifies a Lorentz operator by introducing the second term heuristically in order to account for the finite temperature of the heavy species. The previous corollary confirms that the operator proposed by Allis and used in the spherical harmonics expansion theory [28, 29, 32] is consistent with a rigorous expansion of the Boltzmann operator.

- The solution of $\bar{C}_{\epsilon,\alpha}^{el} \left(\bar{f}_{\epsilon}^0, \bar{f}_{\alpha}^0 \right) = 0$ is a Maxwellian at the heavy species temperature \bar{T}_{α} , as noted by Allis [28].
- When considering electron-electron collisions, the masses of the two electrons are identical and no mass ratio ε appear in the computation. This simply yields that this collision operator is its own zero-th order

$$\bar{C}_{\epsilon,\epsilon}^{el} = \bar{C}_{\epsilon,\epsilon}^{0,el}.$$

Inelastic collisions between electrons and heavy species The inelastic collision operator can be reduced by reformulating the hypotheses of the previous section. For our Hilbert expansion, only the zero-th order terms of the inelastic collision operator is required. Following Assumption (2c), we suppose that the following density ratio

$$\frac{\bar{n}_{\mathbf{g},k}}{\bar{n}_{\mathbf{g},0}} = \frac{\int_{\mathbb{R}^3} \bar{f}_{\mathbf{g},k}(\mathbf{x}, \mathbf{v}_{\mathbf{g}}, t) d\mathbf{v}_{\mathbf{g}}}{\int_{\mathbb{R}^3} \bar{f}_{\mathbf{g},0}(\mathbf{x}, \mathbf{v}_{\mathbf{g}}, t) d\mathbf{v}_{\mathbf{g}}} = \mathcal{O}(\delta) \quad \text{for all } k \neq 0, \quad (33)$$

such that we can write $\bar{f}_{\mathbf{g}} = \bar{f}_{\mathbf{g},0} + \mathcal{O}(\delta)$. In the following, we neglect the excited atoms and all the contributions in $\mathcal{O}(\delta)$, and the expansion is only performed with respect to ε .

For the non-dimensional analysis, the operator of (6) is normalized: we use the same reference quantities for the gas as in the elastic collision operator and the atom internal energy (in eV) is normalized to the electron reference temperature, i.e. $\bar{\phi}_l^* = \phi_l^*/T_{\epsilon}^0$ for $l \in \mathcal{Q}_{\mathbf{g}}$. The normalized operator with the above-mentioned assumptions yields:

$$\bar{C}_{\epsilon,\mathbf{g}}^{inel}(\bar{f}_{\epsilon}, \bar{f}_{\mathbf{g}}) = \sum_{l \in \mathcal{Q}_{\mathbf{g}}^*} \int_{\mathbb{R}^3} \int_{S^2} \bar{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow l} (\|\bar{v}_{\epsilon} - \varepsilon \bar{v}_{\mathbf{g}}\|, \omega \cdot \mathbf{e}) \|\bar{v}_{\epsilon} - \varepsilon \bar{v}_{\mathbf{g}}\| \left(\bar{f}_{\epsilon}' \bar{f}_{\mathbf{g},0}' - \bar{f}_{\epsilon} \bar{f}_{\mathbf{g},0} \right) d\omega d\bar{v}_{\mathbf{g}} + \mathcal{O}(\delta). \quad (34)$$

Note that the terms of non-zero order in ε correspond to the super-elastic collisions (de-excitation collisions) and transitions between two excited levels different from the ground state. The zero-th order expansion of this operator is given in the following proposition.

Proposition 2. *The collision operator of an inelastic collision between an electron and an atom at a non-degenerate ground state can be expanded in terms of the electron-to-atom mass ratio as follows:*

$$\bar{C}_{\mathbf{e},\mathbf{g}}^{inel}(\bar{f}_{\mathbf{e}}, \bar{f}_{\mathbf{g}}) = \bar{C}_{\mathbf{e},\mathbf{g}}^{0,inel}(\bar{f}_{\mathbf{e}}, \bar{f}_{\mathbf{g}}) + \mathcal{O}(\varepsilon + \delta), \quad (35)$$

where the zero-th order operator reads:

$$\bar{C}_{\mathbf{e},\mathbf{g}}^{0,inel}(\bar{f}_{\mathbf{e}}, \bar{f}_{\mathbf{g}}) = \bar{n}_{\mathbf{g}} \sum_{l \in \mathcal{Q}_{\mathbf{g}}^*} \int_{S^2} \bar{\sigma}_{\mathbf{e},\mathbf{g}}^{0 \rightarrow l}(\bar{v}_{\mathbf{e}}, \boldsymbol{\omega} \cdot \hat{\mathbf{v}}_{\mathbf{e}}) \bar{v}_{\mathbf{e}} [\bar{f}_{\mathbf{e}}(\tilde{\mathbf{v}}_{\mathbf{e}}^{l,inel}) - \bar{f}_{\mathbf{e}}(\bar{v}_{\mathbf{e}})] d\boldsymbol{\omega}, \quad (36)$$

with the zero-th order post-collisional velocity $\tilde{\mathbf{v}}_{\mathbf{e}}^{l,inel} = \sqrt{(\bar{v}_{\mathbf{e}}^2 - 2\bar{\phi}_l^*)} \boldsymbol{\omega}$.

Proof. From the conservation laws in an inelastic collision (5), we can derive the non-dimensional post-collisional velocities, as follows:

$$\bar{\mathbf{v}}_{\mathbf{e}}' = \frac{\varepsilon^2}{\varepsilon^2 + 1} \bar{\mathbf{v}}_{\mathbf{e}} + \frac{\varepsilon}{\varepsilon^2 + 1} \bar{\mathbf{v}}_{\mathbf{g}} + \frac{\|\bar{\mathbf{v}}_{\mathbf{e}}' - \varepsilon \bar{\mathbf{v}}_{\mathbf{g}}'\|}{\varepsilon^2 + 1} \boldsymbol{\omega}, \quad (37a)$$

$$\bar{\mathbf{v}}_{\mathbf{g}}' = \frac{\varepsilon}{\varepsilon^2 + 1} \bar{\mathbf{v}}_{\mathbf{e}} + \frac{1}{\varepsilon^2 + 1} \bar{\mathbf{v}}_{\mathbf{g}} - \varepsilon \frac{\|\bar{\mathbf{v}}_{\mathbf{e}}' - \varepsilon \bar{\mathbf{v}}_{\mathbf{g}}'\|}{\varepsilon^2 + 1} \boldsymbol{\omega}, \quad (37b)$$

As a result, the zero-th order post-collisional velocities read

$$\bar{\mathbf{v}}_{\mathbf{e}}' = \bar{v}_{\mathbf{e}}' \boldsymbol{\omega} + \mathcal{O}(\varepsilon), \quad \bar{\mathbf{v}}_{\mathbf{g}}' = \bar{\mathbf{v}}_{\mathbf{g}} + \mathcal{O}(\varepsilon). \quad (38)$$

with these relations, we can obtain the zero-th order energy conservation, as follows,

$$\bar{v}_{\mathbf{e}}'^2 = \bar{v}_{\mathbf{e}}'^2 + 2\bar{\phi}_l^* + \mathcal{O}(\varepsilon) \quad (39)$$

where in the above expression we have considered that the internal energy of the atom before the collision is zero (i.e., at the ground state). As a result, the distribution functions of the restitution collision read

$$\bar{f}_{\mathbf{e}}(\bar{\mathbf{v}}_{\mathbf{e}}') = \bar{f}_{\mathbf{e}}(\tilde{\mathbf{v}}_{\mathbf{e}}^{l,inel}) + \mathcal{O}(\varepsilon), \quad \bar{f}_{\mathbf{g}}(\bar{\mathbf{v}}_{\mathbf{g}}') = \bar{f}_{\mathbf{g}}(\bar{\mathbf{v}}_{\mathbf{g}}) + \mathcal{O}(\varepsilon). \quad (40)$$

By introducing these zero-th order distribution functions into the normalized operator of (34) and neglecting terms of $\mathcal{O}(\varepsilon)$ and higher, we obtain the operator of (36). \square

Ionization collisions As in the case of inelastic collisions, under the studied weakly-ionized conditions, we only need the ionization operator at the zero-th order. The normalized collision operator reads:

$$\begin{aligned} \bar{C}_{\mathbf{e},\mathbf{g}}^{iz}(\bar{f}_{\mathbf{e}}, \bar{f}_{\mathbf{g}}, \bar{f}_{\mathbf{i}}) = & \int_{(\mathbb{R}^3)^4} \left(\bar{f}_{\mathbf{i}}' \bar{f}_{\mathbf{e}_1}' \bar{f}_{\mathbf{e}_2}' \frac{n_{\mathbf{e}}^0}{n_{\mathbf{g}}^0} \frac{n_{\mathbf{e}}^0}{Q_{\mathbf{e}}^0} - \bar{f}_{\mathbf{g}} \bar{f}_{\mathbf{e}} \right) \bar{W}_{\mathbf{g}\mathbf{e}}^{ice} d\bar{\mathbf{v}}_{\mathbf{g}} d\bar{\mathbf{v}}_i' d\bar{\mathbf{v}}_{\mathbf{e}_1}' d\bar{\mathbf{v}}_{\mathbf{e}_2}' \\ & - 2 \int_{(\mathbb{R}^3)^4} \left(\bar{f}_{\mathbf{i}} \bar{f}_{\mathbf{e}} \bar{f}_{\mathbf{e}_2} \frac{n_{\mathbf{e}}^0}{n_{\mathbf{g}}^0} \frac{n_{\mathbf{e}}^0}{Q_{\mathbf{e}}^0} - \bar{f}_{\mathbf{g}}' \bar{f}_{\mathbf{e}_1}' \right) \bar{W}_{\mathbf{g}\mathbf{e}}^{ice} d\bar{\mathbf{v}}_i' d\bar{\mathbf{v}}_{\mathbf{e}_1}' d\bar{\mathbf{v}}_i d\bar{\mathbf{v}}_{\mathbf{e}_2}, \end{aligned} \quad (41)$$

where $Q_{\mathbf{e}}^0 = (m_{\mathbf{e}} e T_{\mathbf{e}}^0 / h_{\mathbf{P}}^2)^{3/2}$ is a quantity proportional to the electron translational partition function.

Note that we have considered that the characteristic density of ions and electrons are equal due to the quasi-neutrality. As stated in Assumption (2 d), the studied conditions are far from chemical equilibrium. As a result, we can deduce that the forward (ionization) reaction is much faster than the backward (recombination), especially the terms proportional to

$$\frac{n_{\mathbf{e}}^0}{n_{\mathbf{g}}^0} \frac{n_{\mathbf{e}}^0}{Q_{\mathbf{e}}^0} = \mathcal{O}(\delta) \quad (42)$$

are negligible as Assumption (1) implies $n_{\mathbf{e}}^0 / Q_{\mathbf{e}}^0 \ll 1$ and $n_{\mathbf{e}}^0 / n_{\mathbf{g}}^0 = \delta$. As a result, we obtain that the normalized operator simplifies into

$$\bar{C}_{\mathbf{e},\mathbf{g}}^{iz}(\bar{f}_{\mathbf{e}}, \bar{f}_{\mathbf{g}}, \bar{f}_{\mathbf{i}}) = \int_{(\mathbb{R}^3)^4} (2\bar{f}_{\mathbf{g}}' \bar{f}_{\mathbf{e}_1}' - \bar{f}_{\mathbf{g}} \bar{f}_{\mathbf{e}}) \bar{W}_{\mathbf{g}\mathbf{e}}^{ice} d\bar{\mathbf{v}}_{\mathbf{g}} d\bar{\mathbf{v}}_i' d\bar{\mathbf{v}}_{\mathbf{e}_1}' d\bar{\mathbf{v}}_{\mathbf{e}_2}' + \mathcal{O}(\delta). \quad (43)$$

Remark 3. Formally, the ionization transition probability depends on the electron-neutral relative velocity, and the electrons relative velocity and the electrons-ion relative velocity. However, in practice, the available cross-section data do not contain this information. In most cases, the scattering of the electrons is considered to be isotropic and the only information of the cross-section is the dependence on the electron pre-collisional energy. For this reason, as proposed by Alexeev et al. [18], we parameterize the transition probability as follows,

$$\bar{W}_{\mathbf{g}\mathbf{e}}^{\text{ie}\mathbf{e}} d\bar{\mathbf{v}}'_i d\bar{\mathbf{v}}'_{\mathbf{e}_1} d\bar{\mathbf{v}}'_{\mathbf{e}_2} = \bar{g}_{\mathbf{e}\mathbf{g}} d\bar{\sigma}^{iz} \quad (44)$$

where $\bar{\sigma}^{iz}(\bar{g}_{\mathbf{e}\mathbf{g}})$ is a function of the electron-gas relative velocity.

As done with the previous operators, the latter operator can be expanded in terms of the electron-to-heavy mass ratio. The expansion is given in the following proposition.

Proposition 3. *The collision operator of an electron-impact ionization collision can be expanded in terms of the electron-to-atom mass ratio as follows*

$$\bar{C}_{\mathbf{e},\mathbf{g}}^{iz}(\bar{f}_{\mathbf{e}}, \bar{f}_{\mathbf{g}}, \bar{f}_i) = \bar{C}_{\mathbf{e},\mathbf{g}}^{0,iz}(\bar{f}_{\mathbf{e}}) + \mathcal{O}(\varepsilon + \delta), \quad (45)$$

where the zero-th order operator reads:

$$\bar{C}_{\mathbf{e},\mathbf{g}}^{0,iz}(\bar{f}_{\mathbf{e}}) = \bar{n}_{\mathbf{g}} \int (2\bar{f}'_{\mathbf{e}_1} - \bar{f}_{\mathbf{e}}) \bar{v}_{\mathbf{e}} d\bar{\sigma}^{0,iz}. \quad (46)$$

where the zero-th order cross-section is $\bar{\sigma}^{0,iz} = \bar{\sigma}^{iz}(\bar{v}_{\mathbf{e}})$ (see computations in C.2), and the electron velocity of the restitution collision is computed as

$$\bar{\mathbf{v}}_{\mathbf{e}_1} = -\bar{g}_{\text{ie}\mathbf{e}}\boldsymbol{\omega}_{\text{ie}\mathbf{e}} + \frac{1}{2}\bar{g}_{\mathbf{e}_1\mathbf{e}_2}\boldsymbol{\omega}_{\mathbf{e}_1\mathbf{e}_2} \quad \text{with} \quad \bar{g}_{\mathbf{e}_1\mathbf{e}_2}^2 = 2\bar{v}_{\mathbf{e}} - 4\bar{\phi}_{iz}^* - 4\bar{g}_{\text{ie}\mathbf{e}}^2. \quad (47)$$

Proof. We start the proof by writing the post-collisional and pre-collisional velocities in terms of Jacobi variables and considering the zero-th order terms. As proposed by Alexeev et al. [18], we consider the post-collisional electron-electron as a subsystem

$$\begin{aligned} \bar{\mathbf{v}}_{\mathbf{g}} &= \bar{\mathbf{G}} - \frac{\varepsilon}{\varepsilon^2 + 1} \bar{g}_{\mathbf{e}\mathbf{g}} = \bar{\mathbf{G}} + \mathcal{O}(\varepsilon), & \bar{\mathbf{v}}_{\mathbf{e}} &= \varepsilon \bar{\mathbf{G}} + \frac{1}{\varepsilon^2 + 1} \bar{g}_{\mathbf{e}\mathbf{g}} = \bar{g}_{\mathbf{e}\mathbf{g}} + \mathcal{O}(\varepsilon), \\ \bar{\mathbf{v}}_i &= \bar{\mathbf{G}} + \frac{2\varepsilon}{\varepsilon^2 + 1} \bar{g}_{\text{ie}\mathbf{e}} = \bar{\mathbf{G}} + \mathcal{O}(\varepsilon), \\ \bar{\mathbf{v}}_{\mathbf{e}_1} &= \varepsilon \bar{\mathbf{G}} - \frac{1 - \varepsilon^2}{\varepsilon^2 + 1} \bar{g}_{\text{ie}\mathbf{e}} + \frac{1}{2} \bar{g}_{\mathbf{e}_1\mathbf{e}_2} = -\bar{g}_{\text{ie}\mathbf{e}} + \frac{1}{2} \bar{g}_{\mathbf{e}_1\mathbf{e}_2} + \mathcal{O}(\varepsilon) \\ \bar{\mathbf{v}}_{\mathbf{e}_2} &= \varepsilon \bar{\mathbf{G}} - \frac{1 - \varepsilon^2}{\varepsilon^2 + 1} \bar{g}_{\text{ie}\mathbf{e}} - \frac{1}{2} \bar{g}_{\mathbf{e}_1\mathbf{e}_2} = -\bar{g}_{\text{ie}\mathbf{e}} - \frac{1}{2} \bar{g}_{\mathbf{e}_1\mathbf{e}_2} + \mathcal{O}(\varepsilon), \end{aligned}$$

where the Jacobi variables are the relative velocities between the particles are defined as $\mathbf{g}_{\mathbf{e}\mathbf{g}} = \mathbf{v}_{\mathbf{e}} - \mathbf{v}_{\mathbf{g}}$, $\mathbf{g}_{\text{ie}\mathbf{e}_1} = \mathbf{v}_i - (\mathbf{v}_{\mathbf{e}_1} + \mathbf{v}_{\mathbf{e}_2})/2$, and $\mathbf{g}_{\mathbf{e}_1\mathbf{e}_2} = \mathbf{v}_{\mathbf{e}_1} - \mathbf{v}_{\mathbf{e}_2}$, normalized as $V_{th_{\mathbf{g}}}^0 \bar{\mathbf{G}} = (m_{\mathbf{g}} \mathbf{v}_{\mathbf{g}} + m_{\mathbf{e}} \mathbf{v}_{\mathbf{e}})/(m_{\mathbf{g}} + m_{\mathbf{e}})$, $V_{th_{\mathbf{e}}}^0 \bar{g}_{\text{ie}\mathbf{e}} = \mathbf{g}_{\text{ie}\mathbf{e}}$ and $V_{th_{\mathbf{e}}}^0 \bar{g}_{\mathbf{e}_1\mathbf{e}_2} = \mathbf{g}_{\mathbf{e}_1\mathbf{e}_2}$. As a result, the zero-th order normalized momentum and energy conservation relations read

$$\bar{\mathbf{v}}_{\mathbf{g}} = \bar{\mathbf{v}}_i + \mathcal{O}(\varepsilon) \quad \text{and} \quad \frac{1}{2} \bar{v}_{\mathbf{e}} = \bar{g}_{\text{ie}\mathbf{e}}^2 + \frac{1}{4} \bar{g}_{\mathbf{e}_1\mathbf{e}_2}^2 + \bar{\phi}_{iz}^* + \mathcal{O}(\varepsilon). \quad (48)$$

Consequently, we can see that the zero-th order distribution functions of the restitution collision read:

$$\bar{f}_{\mathbf{e}}(\bar{\mathbf{v}}_{\mathbf{e}_1}) = \bar{f}_{\mathbf{e}} \left(-\bar{g}_{\text{ie}\mathbf{e}} + \frac{1}{2} \bar{g}_{\mathbf{e}_1\mathbf{e}_2} \right) + \mathcal{O}(\varepsilon), \quad (49a)$$

$$\bar{f}_{\mathbf{g}}(\bar{\mathbf{v}}'_{\mathbf{g}}) = \bar{f}_{\mathbf{g}}(\bar{\mathbf{v}}_{\mathbf{g}}) + \mathcal{O}(\varepsilon). \quad (49b)$$

Finally, we finish the proof by considering the zero-th order cross-section, $\bar{\sigma}^{iz} = \bar{\sigma}^{iz}(\bar{v}_{\mathbf{e}}) + \mathcal{O}(\varepsilon)$. \square

Remark 4. In practice, the available cross-sectional data do not contain the information about the angular dependence of the scattered electrons and the relative velocity. Often, an additional model is required in order to integrate the collisional operator. The two common choices [32] in the literature are the zero-sharing model (where one of the electrons after the ionization takes all the energy) and the equal-sharing model (the energy is equally distributed between the two electrons after the ionization), both cases with isotropic scattering in the deflection angles.

3 The Hilbert expansion

In the following, only the electron equation and normalized quantities are looked at. For simplicity, we abuse notation in the remaining sections and the dependencies with respect to f_g and f_i as well as all the indices ϵ and the normalization bar are dropped, except where ambiguity remains.

3.1 The overall expansion

The present method relies on the postulate that the normalized electron distribution function \bar{f} expands in order of the small parameter ε , as follows,

$$\bar{f} = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \mathcal{O}(\varepsilon^3). \quad (50)$$

Additional hypotheses on the different orders f^i of the perturbation are performed below when appropriate and one needs to verify a posteriori that such an expansion is valid. Then, injecting it in (17) reads

$$\varepsilon^{-1} \mathcal{D}^{-1} + \varepsilon^0 \mathcal{D}^0 + \mathcal{O}(\varepsilon) = \varepsilon^{-2} C^{-2} + \varepsilon^{-1} C^{-1} + \varepsilon^0 C^0 + \mathcal{O}(\varepsilon),$$

with terms of different order in ε . Balancing the first orders of ε on both sides reads

$$\text{Order } \varepsilon^{-2}: \quad 0 = C^{-2}, \quad (51a)$$

$$\text{Order } \varepsilon^{-1}: \quad \mathcal{D}^{-1} = C^{-1}, \quad (51b)$$

$$\text{Order } \varepsilon^0: \quad \mathcal{D}^0 = C^0, \quad (51c)$$

where the different orders of the streaming operators in (51) read

$$\mathcal{D}^{-1}(f^0) = \mathbf{v} \cdot \nabla f^0 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f^0, \quad (52a)$$

$$\mathcal{D}^0(f^0, f^1) = \frac{\partial f^0}{\partial t} + \mathbf{v} \cdot \nabla f^1 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f^1, \quad (52b)$$

and the orders of the collision operators read

$$C^{-2}(f^0) = C_{\epsilon, g}^{0, el}(f^0) \quad (53a)$$

$$C^{-1}(f^1) = C_{\epsilon, g}^{0, el}(f^1) + C_{\epsilon, g}^{1, el}(f^0) \quad (53b)$$

$$\begin{aligned} C^0(f^0, f^1, f^2) &= C_{\epsilon, g}^{0, el}(f^2) + C_{\epsilon, g}^{1, el}(f^1) + C_{\epsilon, g}^{2, el}(f^0) \\ &\quad + C_{\epsilon, \epsilon}^{el}(f^0) + C_{\epsilon, i}^{0, el}(f^0) + C_{\epsilon, g}^{0, inel}(f^0) + C_{\epsilon, g}^{0, iz}(f^0). \end{aligned} \quad (53c)$$

Following Remark 1, we have used the value $\delta = \varepsilon^2$ here to keep the terms $C_{\epsilon, \epsilon}^{el}$ and $C_{\epsilon, i}^{0, el}$ in (53c). Considering $\delta \ll \varepsilon$ would simply consist in neglecting those terms here. As illustrated below, they actually have little impact, and their treatment in our work is simple. In the following, we simplify those operators by considering a zero bulk velocity $\mathbf{u}_g = 0$ (Assumption (2a)). Using (22b), this implies

$$C_{\epsilon, g}^{1, el}(\bar{f}) = 0.$$

Other simplifications will be shown and used in the next paragraphs.

3.2 Order ε^{-2} : electron isotropic distribution function

The lowest order term of (51), i.e. the terms in $\mathcal{O}(\varepsilon^{-2})$, provide

$$C_{\varepsilon, \mathfrak{g}}^{0, el}(f^0) = 0, \quad (54)$$

or equivalently $f^0 \in Ker(C_{\varepsilon, \mathfrak{g}}^{0, el})$. From (22a), one verifies that $Ker(C_{\varepsilon, \mathfrak{g}}^{0, el})$ is composed of the isotropic distribution (see also e.g. Corollary 3.4 of Graille et al. [15]), i.e. by abusing notations

$$Ker(C_{\varepsilon, \mathfrak{g}}^{0, el}) = \left\{ f \quad \text{s.t.} \quad f(v\boldsymbol{\omega}) = \frac{1}{4\pi} \int_{\mathcal{S}^2} f(v\boldsymbol{\omega}) d\boldsymbol{\omega} = f(v) \right\}, \quad \text{and} \quad f^0(\mathbf{v}) = f^0(v), \quad (55)$$

where $v = \|\mathbf{v}\|$. Especially, the anisotropic variations of f are an order of ε smaller than the isotropic one. Note that this observation is the same that motivates the two-term Boltzmann [28, 29, 32].

Also, one verifies the zero-th order ion-electron collision operator vanishes for an isotropic electron distribution function (See Corollary 1)

$$C_{\varepsilon, i}^{0, el}(f^0) = 0.$$

Remark 5. The equilibrium distribution function are not Maxwellian distributions at zero-th order, as usually done in the Chapman-Enskog method. As noted above, this is the main difference of our physical regime as this of Graille et al. [15], where all the Knudsen number of all elastic collision processes are the same.

3.3 Order ε^{-1} : Fredholm equation for the first-order distribution function

The first order of the Hilbert expansion corresponds to the following equation,

$$\mathcal{O}(\varepsilon^{-1}) : \quad \mathcal{D}^{-1}(f^0) = \mathbf{v} \cdot \nabla f^0 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f^0 = C_{\varepsilon, \mathfrak{g}}^{0, el}(f^1). \quad (56a)$$

The collision term is an integral operator of the form:

$$C_{\varepsilon, \mathfrak{g}}^{0, el}(f^1) = n_{\mathfrak{g}} v \left(\int_{\mathcal{S}^2} \sigma_{\varepsilon, \mathfrak{g}}(v, \boldsymbol{\omega} \cdot \hat{\mathbf{v}}_{\varepsilon}) f^1(v\boldsymbol{\omega}) d\boldsymbol{\omega} - f^1(\mathbf{v}) \int_{\mathcal{S}^2} \sigma_{\varepsilon, \mathfrak{g}}(v, \boldsymbol{\omega} \cdot \hat{\mathbf{v}}_{\varepsilon}) d\boldsymbol{\omega} \right). \quad (56b)$$

For a given f^0 , Eq. (56a) at all $\mathbf{v} \in \mathbb{R}^{*,+}$ yields a linear inhomogeneous Fredholm integral equation over the function $\boldsymbol{\omega} \mapsto f^1(\mathbf{x}, v\boldsymbol{\omega}, t)$ in $L^2(\mathcal{S}^2)$. Especially, if the kernel $\sigma_{\varepsilon, \mathfrak{g}}(v, \cdot)$ is square-integrable for all $v \in \mathbb{R}^{*,+}$, then the operator (56b) satisfies Hilbert-Schmidt property. It is therefore continuous and compact, and (56a) follows Fredholm alternatives. Eq. (56a) has a solution only if f^0 follows an equation of the form

$$\Pi_{Ran(C_{\varepsilon, \mathfrak{g}}^{0, el})^\perp} [\mathbf{v} \cdot \nabla f^0 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f^0] = 0, \quad (57)$$

where $\Pi_{Ran(C_{\varepsilon, \mathfrak{g}}^{0, el})^\perp}$ is the projector onto the orthogonal complement of the range of the linear operator $C_{\varepsilon, \mathfrak{g}}^{0, el}$. One verifies that $Ran(C_{\varepsilon, \mathfrak{g}}^{0, el})^\perp = Ker(C_{\varepsilon, \mathfrak{g}}^{0, el})$ is the set of isotropic distributions, and the projector yields the averaging operator

$$(\Pi_{Ran(C_{\varepsilon, \mathfrak{g}}^{0, el})^\perp} f)(v\boldsymbol{\omega}) = \frac{1}{4\pi} \int_{\mathcal{S}^2} f(v\boldsymbol{\omega}) d\boldsymbol{\omega}, \quad (58)$$

such that (56a) is trivial as long as f^0 satisfies (54). We refer e.g. to [62, 63, 64] for further discussions on Fredholm alternative and its applications.

Now, under this condition, the solution f^1 to (56a) is unique up to an additional function in $Ker(C_{\varepsilon, \mathfrak{g}}^{0, el})$. In order to choose among those solutions, we add the following hypothesis:

Hypothesis: The zero-th order is the isotropic part of the solution f , i.e.

$$f^0 = \Pi_{Ran(C_{\varepsilon, \mathfrak{g}}^{0, el})^\perp} f. \quad (59)$$

This hypothesis especially implies that $\Pi_{\text{Ran}(C_{\epsilon, \mathbf{g}}^{0, el})^\perp} f^1 = 0$ and provides uniqueness of such a solution, as usually done with a Chapman-Enskog expansion. Remark though that this hypothesis is stronger than necessary for this purpose, because it impacts the higher orders f^i for $i > 1$. The reason for this choice is explained in the next paragraph (for the higher order terms) and in the next section (for the moment approximation). Remark though that the Hilbert expansion (50) is only formal and non-unique, and we can perform additional assumptions as long as they ultimately provide a well-defined (unique and converging) series (50).

3.4 Order ε^0 : Evolution equation for the zero-th order distribution function

Eq. (51) at order ε^0 leads to the following equation,

$$\begin{aligned} \mathcal{O}(\varepsilon^0) : \quad \frac{\partial f^0}{\partial t} + \mathbf{v} \cdot \nabla f^1 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f^1 &= C_{\epsilon, \mathbf{g}}^{0, el}(f^2) + C_{\epsilon, \mathbf{g}}^{2, el}(f^0) \\ &+ C_{\epsilon, \epsilon}^{el}(f^0, f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\epsilon, \mathbf{g}_k}^{0, inel}(f^0) + C_{\epsilon, \mathbf{g}}^{0, iz}(f^0). \end{aligned} \quad (60)$$

Using again Fredholm alternative, this equation possesses a solution only if

$$\begin{aligned} \Pi_{\text{Ran}(C_{\epsilon, \mathbf{g}}^{0, el})^\perp} \left[\frac{\partial f^0}{\partial t} + \mathbf{v} \cdot \nabla f^1 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f^1 \right. \\ \left. - C_{\epsilon, \mathbf{g}}^{2, el}(f^0) - C_{\epsilon, \epsilon}^{el}(f^0, f^0) - \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\epsilon, \mathbf{g}_k}^{0, inel}(f^0) - C_{\epsilon, \mathbf{g}}^{0, iz}(f^0) \right] = 0. \end{aligned} \quad (61)$$

The solution f^2 is again unique up to an additional function in $\text{Ker}(C_{\epsilon, \mathbf{g}}^{0, el})$, and it turns unique under the hypothesis of the previous section, which implies here that

$$\Pi_{\text{Ran}(C_{\epsilon, \mathbf{g}}^{0, el})^\perp} f^2 = 0. \quad (62)$$

3.5 Relation to the spherical-harmonics expansion kinetic model

The asymptotic kinetic model obtained in (56) and (60) is closely related to the spherical-harmonics expansion model used in gas discharges [27, 28, 29, 32]. The advantages of the present asymptotic theory compared to those are twofold: (1) it highlights the range of applicability of the model through a rigorous derivation (as discussed in the dimensional analysis of Section 2.3) and (2) it expands the collision operators consistently.

Indeed, the hypothesis shown in (59), implies that f^0 is exactly the isotropic part of the distribution function and f^i for $i \geq 1$ do not contribute to the isotropic part of the distribution function, as in the spherical-harmonics expansion (SHE) ansatz. Note that other choices are possible, such as the model proposed by Choquet et al. [33], where f^1 contains an arbitrary isotropic function. Nevertheless, our choice, done in analogy to Enskog's method ([8]) for the solution of f^1 to be unique, provides further advantages for the derivation of a macroscopic model discussed in the next sections.

Note that without (59), the model (56) is more general than the two-term SHE expansion, as no assumption on the form of f^1 has been taken, apart from the one deduced of the asymptotic limit. In fact, the two-term SHE model is only a particular case of the present kinetic model:

Proposition 4. *Approximating the first-order distribution function f^1 by the first spherical harmonics, i.e. writing*

$$f^1(\mathbf{x}, \mathbf{v}, t) = \hat{\mathbf{v}} \cdot \mathbf{f}^{1, SHE}(\mathbf{x}, \mathbf{v}, t), \quad \text{with} \quad \hat{\mathbf{v}} = \frac{\mathbf{v}}{v} \quad (63)$$

and neglecting the electron-electron collisions yields the following two-term spherical harmonics expansion equations (obtained from (60) and (56)):

$$\frac{\partial f^0}{\partial t} + \nabla \cdot \left(\frac{v}{3} \mathbf{f}^{1,SHE} \right) - \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \frac{\mathbf{E}}{3} \cdot \mathbf{f}^{1,SHE} \right) = C_{\epsilon, \mathbf{g}}^{2,el}(f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\epsilon, \mathbf{g}_k}^{0,inel}(f^0) + C_{\epsilon, \mathbf{g}}^{0,iz}(f^0), \quad (64a)$$

$$\nabla (v f^0) - \frac{\partial}{\partial v} (\mathbf{E} f_0) = -\nu_{\epsilon \mathbf{g}_1} \mathbf{f}^{1,SHE}, \quad (64b)$$

with the momentum transfer frequency reads $\nu_{\epsilon \mathbf{g}_1}(v) = n_{\mathbf{g}} v \int_{S^2} (1 - \hat{\mathbf{v}} \cdot \boldsymbol{\omega}) \bar{\sigma}_{\epsilon \mathbf{g}} d\boldsymbol{\omega}$.

Proof. Eq. (64a) is obtained by integrating (60) over the unit sphere of the velocity space, i.e. taking the zero-th order angular moment:

$$\int_{S^2} \mathcal{D}^0(f^0, \mathbf{f}^{1,SHE}) d\hat{\mathbf{v}} = \int_{S^2} C_{\epsilon, \mathbf{g}}^{0,el}(f^2) + C_{\epsilon, \mathbf{g}}^{2,el}(f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\epsilon, \mathbf{g}_k}^{0,inel}(f^0) + C_{\epsilon, \mathbf{g}}^{0,iz}(f^0) d\hat{\mathbf{v}}. \quad (65)$$

On the right-hand side, the first term vanishes due to (62) and the others are isotropic, such that their integration is trivial. Concerning the integration of the streaming operator with the ansatz (63), we refer the reader to [28, 29], where the calculations are detailed.

Eq. (64b) is obtained by taking the first-order angular moment of (56a):

$$\int_{S^2} \hat{\mathbf{v}} \mathcal{D}^{-1}(f^0) d\hat{\mathbf{v}} = \int_{S^2} \hat{\mathbf{v}} C_{\epsilon, \mathbf{g}}^{0,el}(\mathbf{f}^{1,SHE}) d\hat{\mathbf{v}} \quad (66)$$

As before, the integration of the streaming operator (63) can be found in [28, 29]. The integration of the collision operator reads, using the reciprocity relation,

$$\int_{S^2} \int_{S^2} \hat{\mathbf{v}} \nu_{\epsilon \mathbf{g}}(v, \hat{\mathbf{v}} \cdot \boldsymbol{\omega}) [f^1(v\boldsymbol{\omega}) - f^1(\mathbf{v})] d\boldsymbol{\omega} d\hat{\mathbf{v}} = \int_{S^2} \int_{S^2} (\boldsymbol{\omega} - \hat{\mathbf{v}}) \nu_{\epsilon \mathbf{g}}(v, \hat{\mathbf{v}} \cdot \boldsymbol{\omega}) f^1(\mathbf{v}) d\boldsymbol{\omega} d\hat{\mathbf{v}}. \quad (67)$$

Remarking that the integral

$$\int_{S^2} (\boldsymbol{\omega} - \hat{\mathbf{v}}) \nu_{\epsilon \mathbf{g}}(v, \hat{\mathbf{v}} \cdot \boldsymbol{\omega}) d\boldsymbol{\omega} = -\hat{\mathbf{v}} \int_{S^2} (1 - \hat{\mathbf{v}} \cdot \boldsymbol{\omega}) \nu_{\epsilon \mathbf{g}}(v, \hat{\mathbf{v}} \cdot \boldsymbol{\omega}) d\boldsymbol{\omega},$$

and injecting the ansatz (63) into (67), one obtains,

$$-\nu_{\epsilon \mathbf{g}_1}(v) \int_{S^2} (\hat{\mathbf{v}} \otimes \hat{\mathbf{v}}) \mathbf{f}^{1,SHE}(v) d\hat{\mathbf{v}} = -\frac{4\pi}{3} \nu_{\epsilon \mathbf{g}_1} \mathbf{f}^{1,SHE}, \quad (68)$$

which leads to the term in the right-hand-side of (64b). Note that the factor $4\pi/3$ simplifies with the left-hand-side terms. \square

3.6 Electron kinetic entropy balance equation

In order to identify formally the leading terms in the entropy balance equation, we normalize:

$$\begin{aligned} f &= \delta (\underline{f} + \mathcal{O}(\delta)) & \text{with} & & \underline{f} &= \underline{f}^0 + \varepsilon \underline{f}^1 + \varepsilon^2 \underline{f}^2 + \mathcal{O}(\varepsilon^3), \\ f_{\mathbf{g}} &= M_{\mathbf{g}} + \mathcal{O}(\delta), \\ f_{\mathbf{i}} &= \delta (M_{\mathbf{i}} + \mathcal{O}(\delta)), \end{aligned}$$

where M_{α} denotes a Maxwellian. The scaling by δ simply results from the weakly-ionized hypothesis, and the Maxwellian leading terms from the hypothesis that the heavy particles are at thermodynamic equilibrium in the present reference. Remembering that $\delta \ll \varepsilon$, we focus on the leading terms in δ and consider the expansion in ε .

Proposition 5. *The electron system standalone (without considering the evolution of ions and neutral species) admits a kinetic entropy balance equation:*

$$\frac{\partial}{\partial t} \left(\int_{\mathbb{R}^3} \eta(\underline{f}^0) d\mathbf{v} \right) + \nabla \cdot \left(\int_{\mathbb{R}^3} \mathbf{v} \eta'(\underline{f}^0) \underline{f}^1 d\mathbf{v} \right) = E_{\mathbf{e},0}^{\mathbf{e},\mathbf{g},el} + E_{\mathbf{e},2}^{\mathbf{e},\mathbf{g},el} + E_{\mathbf{e},0}^{\mathbf{e},\mathbf{g},inel} + E_{\mathbf{e},0}^{iz} + \mathcal{O}(\delta + \varepsilon), \quad (69a)$$

where the functional $\eta(f) = -f(\log f - 1)$ is the Boltzmann entropy and the production terms yield

$$E_{\mathbf{e},0}^{\mathbf{e},\mathbf{g},el} = \int_{\mathbb{R}^3} \eta''(\underline{f}^0) \underline{f}^1 C_{\mathbf{e},\mathbf{g}}^{0,el}(\underline{f}^1) d\mathbf{v}, \quad E_{\mathbf{e},2}^{\mathbf{e},\mathbf{g},el} = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C_{\mathbf{e},\mathbf{g}}^{2,el}(\underline{f}^0) d\mathbf{v}, \quad (69b)$$

$$E_{\mathbf{e},0}^{\mathbf{e},\mathbf{g},inel} = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C_{\mathbf{e},\mathbf{g}}^{0,inel}(\underline{f}^0) d\mathbf{v}, \quad E_{\mathbf{e},0}^{iz} = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C_{\mathbf{e},\mathbf{g}}^{0,iz}(\underline{f}^0) d\mathbf{v}, \quad (69c)$$

and $E_{\mathbf{e},0}^{\mathbf{e},\mathbf{g},el} \geq 0$ with equality only if $f^1 = 0$.

Assuming instead that $\delta = \varepsilon^2$ would provide the additional term in (69a) $E_{\mathbf{e}}^{\mathbf{e},\mathbf{e},el} \geq 0$ computed in (11) that is already signed.

Proof. Formally, the normalized Boltzmann entropy, as defined in (9), can also be expanded in orders of ε :

$$\eta'(f) = \eta' \left(\sum_{i=0}^{\infty} \varepsilon^i \underline{f}^i \right) = \eta'(\underline{f}^0) + \varepsilon \eta''(\underline{f}^0) \underline{f}^1 + \varepsilon^2 \left(\eta'''(\underline{f}^0) \underline{f}^2 + \eta^{(3)}(\underline{f}^0) \frac{(\underline{f}^1)^2}{2} \right) + \mathcal{O}(\varepsilon^3 + \delta).$$

Then, multiplying the scaled kinetic equation (17) by $\delta^{-1} \eta'(f)$ and integrating yields the scaled kinetic entropy equation

$$\varepsilon^{-1} \mathcal{D}^{-1,E} + \mathcal{D}^{0,E} + \mathcal{O}(\varepsilon) = \varepsilon^{-2} \mathcal{C}^{-2,E} + \mathcal{C}^{-1,E} + \mathcal{C}^{0,E} + \mathcal{O}(\varepsilon) \quad (70)$$

with

$$\mathcal{D}^{-1,E} = \nabla \cdot \int_{\mathbb{R}^3} \mathbf{v} \eta(\underline{f}^0) d\mathbf{v},$$

$$\mathcal{D}^{0,E} = \frac{\partial}{\partial t} \left(\int_{\mathbb{R}^3} \eta(\underline{f}^0) d\mathbf{v} \right) + \nabla \cdot \left(\int_{\mathbb{R}^3} \mathbf{v} \eta'(\underline{f}^0) \underline{f}^1 d\mathbf{v} \right),$$

$$\mathcal{C}^{-2,E} = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C^{-2}(\underline{f}^0) d\mathbf{v},$$

$$\mathcal{C}^{-1,E} = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C^{-1}(\underline{f}^0, \underline{f}^1) + \eta''(\underline{f}^0) \underline{f}^1 C^{-2}(\underline{f}^0) d\mathbf{v},$$

$$\mathcal{C}^{0,E} = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C^0(\underline{f}^0, \underline{f}^1, \underline{f}^2) + \eta''(\underline{f}^0) \underline{f}^1 C^{-1}(\underline{f}^0, \underline{f}^1) + \left(\eta'''(\underline{f}^0) \underline{f}^2 + \eta^{(3)}(\underline{f}^0) \frac{(\underline{f}^1)^2}{2} \right) C^{-2}(\underline{f}^0) d\mathbf{v}.$$

At the different orders, this reduces into:

- At the order ε^{-2} , this rewrites:

$$\text{Order } \varepsilon^{-2}: \quad 0 = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C^{-2}(\underline{f}^0) d\mathbf{v} = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C_{\mathbf{e},\mathbf{g}}^{0,el}(\underline{f}^0) d\mathbf{v}, \quad (71)$$

which is therefore a non-negative entropy source at the first non-zero order. Remark that $C^{-2} = 0$ according to (51), and this equation is trivial.

- At the order ε^{-1} , this rewrites

$$\text{Order } \varepsilon^{-1}: \quad \mathcal{D}^{-1,H} = \int_{\mathbb{R}^3} \eta'(\underline{f}^0) C_{\mathbf{e},\mathbf{g}}^{0,el}(\underline{f}^1) + \eta''(\underline{f}^0) \underline{f}^1 C_{\mathbf{e},\mathbf{g}}^{0,el}(\underline{f}^0) d\mathbf{v} = 0 \quad (72)$$

By reproducing the computation in (76) and using that $\eta(\underline{f}^0)$ is isotropic, one retrieves that the first term is zero and by using that $C_{\mathbf{e},\mathbf{g}}^{0,el}(\underline{f}^0)$ the second term is also zero. As a result, at this order, the entropy production term is therefore also non-negative.

- At order ε^0 , using (51), (53b-22b) and (62), leads to the simplification:

$$\begin{aligned} \text{Order } \varepsilon^0: \quad \mathcal{D}^{0,H} &= \int_{\mathbb{R}^3} \eta''(\underline{f}^0) \underline{f}^1 C_{\varepsilon,\mathbf{g}}^{0,el}(\underline{f}^1) \\ &\quad + \eta'(\underline{f}^0) (C_{\varepsilon,\mathbf{g}}^{2,el}(\underline{f}^0) + C_{\varepsilon,\varepsilon}^{el}(\underline{f}^0) + C_{\varepsilon,\mathbf{g}}^{0,inel}(\underline{f}^0) + C_{\varepsilon,\mathbf{g}}^{0,iz}(\underline{f}^0)) d\mathbf{v}. \end{aligned}$$

For the first term, assuming the positivity of \underline{f}^0 (therefore $\eta''(\underline{f}^0)$ is negative) and that $\nu_{\varepsilon,\mathbf{g}}$ is square-integrable, then a Cauchy-Schwartz inequality provides its non-negative:

$$\begin{aligned} \int_{\mathbb{R}^3} \eta''(\underline{f}^0) \underline{f}^1 C_{\varepsilon,\mathbf{g}}^{0,el}(\underline{f}^1) d\mathbf{v} &= \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} \eta''(\underline{f}^0(\mathbf{v})) \underline{f}^1(\mathbf{v}) \nu_{\varepsilon,\mathbf{g}}(\mathbf{v} \cdot \boldsymbol{\omega}) (f^1(\mathbf{v}\boldsymbol{\omega}) - f^1(\mathbf{v})) d\boldsymbol{\omega} d\mathbf{v} \\ &\geq \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} \eta''(\underline{f}^0(\mathbf{v})) \nu_{\varepsilon,\mathbf{g}}(\mathbf{v} \cdot \boldsymbol{\omega}) \frac{f^1(\mathbf{v})^2}{2} d\boldsymbol{\omega} d\mathbf{v} \geq 0. \end{aligned}$$

The quadratic term $\delta^{-1} C_{\varepsilon,\varepsilon}^{el}(\underline{f}, \underline{f}) = \mathcal{O}(\delta)$ is one order higher in δ than the linear terms and is therefore neglected. Remark though that the entropy production term arising from these collisions is always signed. Similarly, the electron-ions elastic collisions are one order higher in δ . □

When we consider the part of the collisions that dissipate the electron energy at order $\mathcal{O}(\varepsilon^0)$, no H-theorem for the electrons standalone can be derived at this order, neither the entropy dissipation nor equilibrium representation. In the case of the inelastic (excitation and ionization) collisions, the breakdown of the H-theorem is caused by the violation of detailed balance in dissipative collisions (as the backward collision, i.e., the super-elastic and recombination collisions are not considered at this order). In the electron-gas elastic collisions, this is because the second-order term is a correction to the Lorentz gas collision operator that accounts for the recoil of the neutral particle during the electron-gas collision, i.e., a part of the electron energy is given to the gas particle. The breakdown of the Boltzmann H-theorem is known in the case of systems with dissipative collisions, e.g., granular gases [65]. Note that in weakly-ionized plasmas such as discharges, there is usually an external source of energy that is given to the electron and that do not affect much the rest of species (e.g., radio-frequency electromagnetic fields) that counterbalances the loss of energy of the inelastic processes, which allows maintaining the system out of thermodynamic equilibrium. If the electron energy decreases to values close to the thermodynamic and chemical equilibrium, an H-theorem can be found again, but these conditions are not considered by our model.

Proposition 6. *The sum of the entropy production terms in Proposition 5 with their counterparts is signed:*

$$\begin{aligned} E_{\varepsilon,2}^{\varepsilon,\mathbf{g},el} + \int_{\mathbb{R}^3} \eta'(M_{\mathbf{g}}) C_{\mathbf{g},\varepsilon}^{2,el}(\underline{f}^0) d\mathbf{v} + \mathcal{O}(\delta + \varepsilon) &\geq 0, \\ E_{\varepsilon,0}^{\varepsilon,\mathbf{g},inel} + \int_{\mathbb{R}^3} \eta'(M_{\mathbf{g}}) C_{\varepsilon,\mathbf{g}}^{0,inel}(\underline{f}^0) d\mathbf{v} + \mathcal{O}(\delta + \varepsilon) &\geq 0, \\ E_{\varepsilon,0}^{iz} + \int_{\mathbb{R}^3} \eta'(M_{\mathbf{g}}) C_{\mathbf{g}}^{iz}(\underline{f}^0) d\mathbf{v} + \mathcal{O}(\delta + \varepsilon) &\geq 0. \end{aligned}$$

Proof. The inequality follows from (13). The expansion of the terms E_{ε}^{coll} were performed in Proposition 5. Computing

$$\begin{aligned} E_{\mathbf{g}}^{\mathbf{g},\varepsilon,el} &= \int_{\mathbb{R}^3} \eta'(M_{\mathbf{g}}) (C_{\mathbf{g},\varepsilon}^{0,el}(\underline{f}^0) + \varepsilon (C_{\mathbf{g},\varepsilon}^{0,el}(\underline{f}^1) + C_{\mathbf{g},\varepsilon}^{1,el}(\underline{f}^0)) \\ &\quad + \varepsilon^2 (C_{\mathbf{g},\varepsilon}^{0,el}(\underline{f}^2) + C_{\mathbf{g},\varepsilon}^{1,el}(\underline{f}^1) + C_{\mathbf{g},\varepsilon}^{2,el}(\underline{f}^0))) d\mathbf{v} + \mathcal{O}(\delta + \varepsilon^3). \end{aligned}$$

Under the hypothesis that $M_{\mathbf{g}}$ is centered around a zero velocity (then isotropic), one verifies that $C_{\mathbf{g},\varepsilon}^{0,el}(\underline{f}^0) = 0 = C_{\mathbf{g},\varepsilon}^{1,el}(\underline{f}^i)$ for $i = 0, 1$ and the integrals $\int \eta'(M_{\mathbf{g}}) C_{\mathbf{g},\varepsilon}^{0,el}(\underline{f}^i) = 0$ for $i = 1, 2$. This provides the first inequality.

Similarly, computing the leading term for the inelastic collisions and for the ionization one (and neglecting the higher order terms in δ and ε) provides the second and third inequalities. □

4 Moment hierarchy

In order to approximate the solution of this system of equations, we perform a truncated moment approximation for every order in ε of the previous kinetic equation. This moment approximation is understood as a Galerkin approximation using polynomial sets of test functions, rather than a proper hydrodynamic regime. In this section, we describe the equations resulting from the projection onto these test functions, while choices of approximations of the solution are described in the next one.

4.1 Order ε^{-2} : electron isotropic distribution function

The zero-th order distribution function is isotropic. In the moment framework, it is represented by the infinite set of even-order scalar velocity moments defined by the following formula.

Definition 4.1. • The space \mathcal{I} of scalar monomials of the norm of the velocity

$$\mathcal{I} = \text{Span}(\psi^{2l}), \quad \text{where} \quad \psi^{2l} = v^{2l} \quad \text{for} \quad l \geq 0 \quad (73)$$

- The scalar product is denoted

$$\langle\langle \xi, \chi \rangle\rangle = \int_{\mathbb{R}^3} \xi \chi d\mathbf{v}. \quad (74)$$

- The macroscopic scalar quantities are a set of moments of the distribution function with respect to the scalars ψ^{2l} : for all $l \in \mathbb{N}$, we define

$$p^{(2l)}(\bar{f}) = \langle\langle \psi^{2l}, \bar{f} \rangle\rangle. \quad (75)$$

As always, the zero-th and second order moments are connected to the density $p^{(0)} = n$ and the pressure or temperature $p^{(2)} = 3p = 3nT$.

Property 1. • The collision operator $C_{\mathbf{e},\mathbf{g}}^{0,el}(\bar{f})$ is orthogonal to the space \mathcal{I} , i.e. for all $l \in \mathbb{N}$

$$p^{(2l)}(C_{\mathbf{e},\mathbf{g}}^{0,el}(\bar{f})) = 0. \quad (76)$$

- Following hypothesis (59), the macroscopic scalar quantities are the moments of the equilibrium distribution, i.e. for all $l \in \mathbb{N}$

$$p^{(2l)}(\bar{f}) = p^{(2l)}(f^0). \quad (77)$$

Proof. • The projection of the zero-th order electron-neutral collision operator reads

$$\begin{aligned} p^{(2l)}(C_{\mathbf{e},\mathbf{g}}^{0,el}(\bar{f})) &= \langle\langle \psi^{2l}, C_{\mathbf{e},\mathbf{g}}^{0,el}(\bar{f}) \rangle\rangle \\ &= \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} \psi^{2l}(v) \nu_{\mathbf{e},\mathbf{g}}(v, \hat{\mathbf{v}}_{\mathbf{e}} \cdot \boldsymbol{\omega}) [\bar{f}(\mathbf{x}, v\boldsymbol{\omega}, t) - \bar{f}(\mathbf{x}, \mathbf{v}, t)] d\boldsymbol{\omega} dv \\ &= \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} (\psi^{2l}(\|v\boldsymbol{\omega}\|) - \psi^{2l}(v)) \nu_{\mathbf{e},\mathbf{g}}(v, \hat{\mathbf{v}}_{\mathbf{e}} \cdot \boldsymbol{\omega}) \bar{f}(\mathbf{v}) d\boldsymbol{\omega} dv = 0. \end{aligned}$$

The operator vanishes as $\psi^{2l}(v)$ is a function of the norm of the velocity and $\|v\boldsymbol{\omega}\| = \|\mathbf{v}\|$, hence, $\psi^{2l}(v) = \psi^{2l}(\|v\boldsymbol{\omega}\|)$.

- The functions ψ^{2l} are all isotropic, and therefore

$$p^{(2l)}(f^i) = \int_{\mathbb{R}^+} \int_{\mathcal{S}^2} \psi^{2l}(v) f^i(v\boldsymbol{\omega}) d\boldsymbol{\omega} dv = \int_{\mathbb{R}^+} \psi^{2l}(v) \int_{\mathcal{S}^2} f^i(v\boldsymbol{\omega}) d\boldsymbol{\omega} dv = 0, \quad (78)$$

or equivalently the different perturbation orders $f^i \in \mathcal{I}^\perp$ for all $i > 0$ are in the orthogonal complement of \mathcal{I} . □

Note that the hypothesis (59) is performed for most Chapman-Enskog expansions ([8, 45, 66]). There is a great advantage of this choice (as explained in the references above), as f^0 can be fully determined by the scalar moments, and therefore the equation of f^1 as well, as will be shown in the next section. Note that it would not be the case if f^1 contains an isotropic component and contributes to the scalar moments.

4.2 Order ε^{-1} : flux vector transport relations

With the first-order perturbation distribution function, we can define the macroscopic fluxes in a reference frame at rest as well as the fluxes in the reference frame moving at the electron velocity, as described in the following definition.

Definition 4.2. The macroscopic fluxes in the reference frame at rest can be computed as the vectorial odd order moments of a distribution function, i.e. by projecting it onto the space of vectors $\boldsymbol{\psi}^{2l+1}$ defined as follows: for all $l \in \mathbb{N}$

$$\boldsymbol{\psi}^{2l+1}(\mathbf{v}) = \psi^{2l}(v)\mathbf{v}, \quad \boldsymbol{\Gamma}^{(2l+1)}(\bar{f}) = \langle\langle \boldsymbol{\psi}^{2l+1}, \bar{f} \rangle\rangle. \quad (79)$$

Remark again that the orders $\boldsymbol{\Gamma}^{(1)}(\bar{f}) \equiv n\mathbf{u}$ and $\boldsymbol{\Gamma}^{(3)}(\bar{f})$ correspond as always to the momentum and the heat flux.

Extracting the scalar moments of (56a) provides no further information. Indeed, this yields

$$\nabla \cdot \boldsymbol{\Gamma}^{(2l+1)}(f^0) + \frac{2l+3}{3}\mathbf{E} \cdot \boldsymbol{\Gamma}^{(2l-1)}(f^0) = p^{(2l)}(C_{\mathbf{e},\mathbf{g}}^{0,el}(f^1)),$$

and both side of this equation equals zero: The left-side is zero because the odd order moments $\boldsymbol{\Gamma}^{(2l+1)}(f^0)$ of f^0 are, see (55); the right-side is zero due to the orthogonality of $C_{\mathbf{e},\mathbf{g}}^{0,el}(f)$ with \mathcal{I} , see (76).

Though, Eq. (56a) can be used to derive the equations for the transport fluxes, i.e. relations between the transport fluxes and the gradients of the zero-th macroscopic variables, as shown below.

Proposition 7. *We can obtain relations for the transport fluxes by projecting Eq. (56a) into the space of the vectors $\boldsymbol{\psi}^{2l}\mathbf{v}$, for $l \in \mathbb{N}$. The relations read for $l \in \mathbb{N}$*

$$\frac{1}{3}\nabla p^{(2l+2)}(f^0) + \frac{2l+3}{3}p^{(2l)}(f^0)\mathbf{E} = \boldsymbol{\Gamma}^{(2l+1)}(C_{\mathbf{e},\mathbf{g}}^{0,el}(f^1)). \quad (80)$$

Proof. Eq. (56a) projected onto the vectorial functions $\boldsymbol{\psi}^{2l+1} \equiv \psi^{2l}(v)\mathbf{v}$ reads

$$\langle\langle \boldsymbol{\psi}^{2l+1}, \mathcal{D}^{-1}(f^0) \rangle\rangle = \langle\langle \boldsymbol{\psi}^{2l+1}, C_{\mathbf{e},\mathbf{g}}^{0,el}(f^1) \rangle\rangle, \quad \text{for } l \in \mathbb{N}. \quad (81)$$

The projection of the streaming operator can be decomposed into two components, as follows:

$$\langle\langle \boldsymbol{\psi}^{2l+1}, \mathcal{D}^{-1}(f^0) \rangle\rangle = \langle\langle \boldsymbol{\psi}^{2l+1}, \mathbf{v} \cdot \nabla f^0 \rangle\rangle - \langle\langle \boldsymbol{\psi}^{2l+1}, \mathbf{E} \cdot \nabla_{\mathbf{v}} f^0 \rangle\rangle \quad \text{for } l \in \mathbb{N}. \quad (82)$$

The first term can be computed as follows:

$$\langle\langle \boldsymbol{\psi}^{2l+1}, \mathbf{v} \cdot \nabla f^0 \rangle\rangle = \nabla \cdot \int_{\mathbb{R}^3} \psi^{2l}(v)\mathbf{v} \otimes \mathbf{v} f^0 d\mathbf{v} = \frac{1}{3}\nabla \int_{\mathbb{R}^3} \psi^{2l}(v)v^2 f^0 d\mathbf{v}, \quad (83)$$

where we have used the parity of f^0 and the fact that $\langle\langle v_i^2, f^0 \rangle\rangle = \langle\langle v_j^2, f^0 \rangle\rangle$ for all i, j such that $\langle\langle \frac{1}{3}v^2\mathbb{I} - \mathbf{v} \otimes \mathbf{v}, f^0 \rangle\rangle = 0$.

To reduce (83), we remark that:

$$\psi^{2l}(v)v^2 = \psi^{2(l+1)}. \quad (84)$$

By injecting this into (83) and by using the definitions of the zero-th order moments (79), we obtain the gradient terms of (80).

For the electric field terms, we use the following identity,

$$\langle\langle \boldsymbol{\psi}^{2l+1}, \mathbf{E} \cdot \nabla_{\mathbf{v}} f^0 \rangle\rangle = -\langle\langle f^0, \mathbf{E} \cdot \nabla_{\mathbf{v}} \boldsymbol{\psi}^{2l+1} \rangle\rangle. \quad (85)$$

This is a consequence of f^0 being zero at $\mathbf{v} \rightarrow \infty$. After some algebra, we can compute the following relations

$$\nabla_{\mathbf{v}} \boldsymbol{\psi}^{2l+1} = 2lv^{2(l-1)} \mathbf{v} \otimes \mathbf{v} + v^{2l} \mathbb{I}. \quad (86)$$

As a result, we obtain

$$\langle\langle f^0, \mathbf{E} \cdot \nabla_{\mathbf{v}} \boldsymbol{\psi}^{2l+1} \rangle\rangle = \frac{2l+3}{3} p^{(2l)}(f^0) \mathbf{E}. \quad (87)$$

by using again $\langle\langle \frac{1}{3}v^2 \mathbb{I} - \mathbf{v} \otimes \mathbf{v}, f^0 \rangle\rangle = 0$. With this result, we inject into (81) and we obtain the electric field terms in (80). \square

4.3 Order ε^0 : Electron moment equations

Eventually, at order ε^0 , we obtain the following system of moment equations out of (60).

Proposition 8. *With the definition 4.2 of the fluxes, the electron particle density, energy and l -th order moment balance equations read*

Even orders:

$$\begin{aligned} \frac{\partial}{\partial t} p^{(2l)}(f^0) + \nabla \cdot \boldsymbol{\Gamma}^{(2l+1)}(f^1) + 2l \boldsymbol{\Gamma}^{(2l-1)}(f^1) \cdot \mathbf{E} \\ = p^{(2l)} \left(C_{\mathbf{e}, \mathbf{g}}^{2, el}(f^0) + C_{\mathbf{e}, \mathbf{e}}^{el}(f^0, f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\mathbf{e}, \mathbf{g}k}^{0, inel}(f^0) + C_{\mathbf{e}, \mathbf{g}}^{0, iz}(f^0) \right), \end{aligned} \quad (88a)$$

Odd orders:

$$\begin{aligned} \nabla \cdot \boldsymbol{\mathfrak{M}}^{2l+2}(f^1) + 2l \boldsymbol{\mathfrak{M}}^{2l}(f^1) \cdot \mathbf{E} \\ = \boldsymbol{\Gamma}^{(2l+1)} \left(C_{\mathbf{e}, \mathbf{g}}^{0, el}(f^2) + C_{\mathbf{e}, \mathbf{g}}^{2, el}(f^0) + C_{\mathbf{e}, \mathbf{e}}^{el}(f^0, f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\mathbf{e}, \mathbf{g}k}^{0, inel}(f^0) + C_{\mathbf{e}, \mathbf{g}}^{0, iz}(f^0) \right), \end{aligned} \quad (88b)$$

Here, the moment matrices $\boldsymbol{\mathfrak{M}}^{2l+2}(f) = \langle\langle \boldsymbol{\psi}^{2l} \mathbf{v} \mathbf{v}^T, f \rangle\rangle$ for $l > 1$ and, by abuse of notations, denoting $2l \boldsymbol{\Gamma}^{(2l-1)}(f) = 0$ and $2l \boldsymbol{\mathfrak{M}}^{2l}(f) = 0$ when $l = 0$.

Proof. Projecting (60) onto $\boldsymbol{\psi}^{2l}$ and $\boldsymbol{\psi}^{2l+1}$. For the left-hand-side of the equations, we use the following identity

$$\langle\langle \boldsymbol{\psi}^{2l}, \mathcal{D}^0(f^0) \rangle\rangle = \langle\langle \boldsymbol{\psi}^{2l}, \frac{\partial f^0}{\partial t} \rangle\rangle + \langle\langle \boldsymbol{\psi}^{2l}, \nabla \cdot (\mathbf{v} f^1) \rangle\rangle + \langle\langle f^1, \mathbf{E} \cdot \nabla_{\mathbf{v}} \boldsymbol{\psi}^{2l} \rangle\rangle, \quad (89a)$$

$$\langle\langle \boldsymbol{\psi}^{2l+1}, \mathcal{D}^0(f^0) \rangle\rangle = \langle\langle \boldsymbol{\psi}^{2l+1}, \frac{\partial f^0}{\partial t} \rangle\rangle + \langle\langle \boldsymbol{\psi}^{2l+1}, \nabla \cdot (\mathbf{v} f^1) \rangle\rangle + \langle\langle f^1, \mathbf{E} \cdot \nabla_{\mathbf{v}} \boldsymbol{\psi}^{2l+1} \rangle\rangle. \quad (89b)$$

Using that f^0 is isotropic (55) (hence, $\langle\langle \boldsymbol{\psi}^{2l+1}, f^0 \rangle\rangle = 0$), the orthogonality of f^1 with \mathcal{I} (78) and after some algebra, we obtain the left-hand-side of (88). For the right-hand-side, we have only used the orthogonality of $C_{\mathbf{e}, \mathbf{g}}^{0, el}(f)$ with \mathcal{I} . \square

Remark 6. • The second order perturbation f^2 does not contribute to the even scalar moment equations at the order $\mathcal{O}(\varepsilon^0)$.

- The matrices $\boldsymbol{\mathfrak{M}}^{2l+2}(f^1)$ are traceless due to (79), but they can be non-zero.

4.4 The unclosed moment system

Eventually, we obtain the following coupled system of equations on the moments of f^0 and of the first order perturbation f^1 : for $l \in \mathbb{N}$

$$\frac{1}{3}\nabla p^{(2l+2)}(f^0) + \frac{2l+3}{3}p^{(2l)}(f^0)\mathbf{E} = \mathbf{\Gamma}^{(2l+1)}(C_{\mathbf{e},\mathbf{g}}^{0,el}(f^1)), \quad (90a)$$

$$\begin{aligned} \frac{\partial}{\partial t}p^{(2l)}(f^0) + \nabla \cdot \mathbf{\Gamma}^{(2l+1)}(f^1) + 2l\mathbf{\Gamma}^{(2l-1)}(f^1) \cdot \mathbf{E} \\ = p^{(2l)} \left(C_{\mathbf{e},\mathbf{g}}^{2,el}(f^0) + C_{\mathbf{e},\mathbf{e}}^{el}(f^0, f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\mathbf{e},\mathbf{g}_k}^{0,inel}(f^0) + C_{\mathbf{e},\mathbf{g}}^{0,iz}(f^0) \right). \end{aligned} \quad (90b)$$

This system is not closed and in the next section, we study closure relations for this system to be solvable. Remark that the first even moment equations reduce into the particle and electron energy conservation equations (as solved by the LMEA approximation),

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = \langle\langle 1, C_{\mathbf{e},\mathbf{g}}^{0,iz}(f^0) \rangle\rangle, \quad (91a)$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \frac{1}{2}\nabla \cdot (\mathbf{\Gamma}^{(3)}(f^1)) + n\mathbf{u} \cdot \mathbf{E} = \langle\langle \frac{v^2}{2}, C_{\mathbf{e},\mathbf{g}}^{2,el}(f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\mathbf{e},\mathbf{g}_k}^{0,inel}(f^0) + C_{\mathbf{e},\mathbf{g}}^{0,iz}(f^0) \rangle\rangle, \quad (91b)$$

by remarking that the inelastic collision conserves the number of particles, i.e. $\langle\langle 1, C_{\mathbf{e},\mathbf{g}_k}^{0,inel} \rangle\rangle = 0$, and that the electron-electron collision operator is orthogonal to ψ^{2l} for $l \in \{1, 2\}$, i.e.

$$\langle\langle \psi^{2l}, C_{\mathbf{e},\mathbf{e}}^{el}(\bar{f}, \bar{f}) \rangle\rangle = 0 \quad \text{for } l \in \{1, 2\}.$$

This can be proven as the particle number and the energy is conserved in the elastic collision. Note that this is not the case for the fourth-order moment, i.e. $l = 2$.

The system (90) can potentially be supplemented with an equation on the second order perturbation f^2

$$\begin{aligned} \nabla \cdot \mathfrak{M}^{2l+2}(f^1) + 2l\mathfrak{M}^{2l}(f^1) \cdot \mathbf{E} \\ = \mathbf{\Gamma}^{(2l+1)} \left(C_{\mathbf{e},\mathbf{g}}^{0,el}(f^2) + C_{\mathbf{e},\mathbf{g}}^{2,el}(f^0) + C_{\mathbf{e},\mathbf{e}}^{el}(f^0, f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\mathbf{e},\mathbf{g}_k}^{0,inel}(f^0) + C_{\mathbf{e},\mathbf{g}}^{0,iz}(f^0) \right). \end{aligned} \quad (92)$$

But studying this equation would require further assumptions or approximations on f^2 . The first odd vectorial equation reduces to

$$\nabla \cdot \mathfrak{M}^2(f^1) = \langle\langle v, C_{\mathbf{e},\mathbf{g}}^{0,el}(f^2) + C_{\mathbf{e},\mathbf{g}}^{2,el}(f^0) + C_{\mathbf{e},\mathbf{e}}^{el}(f^0, f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\mathbf{e},\mathbf{g}_k}^{0,inel}(f^0) + C_{\mathbf{e},\mathbf{g}}^{0,iz}(f^0) \rangle\rangle.$$

Eventually, in order to approximate numerically the solution of the system (90). In this purpose, the following strategy is followed in the next two sections:

- Under assumptions on the collision operators, the operator $\mathbf{\Gamma}^{(2l+1)}(C_{\mathbf{e},\mathbf{g}}^{0,el}(f^1))$ in (90a) is inverted in order to express f^1 (or at least its moments) as a function of f^0 . This leads to a moment system over f^0 only.
- Moment closures are studied to approximate the solution of the resulting system. The moment problem ultimately yields a Stieltjes moment problem [67, 68] in order to reconstruct the function f^0 of the scalar variable $|v|^2 \in \mathbb{R}^+$ based on its moments and close all remaining terms using this reconstruction.

Remark that those two sections present approximations performed to arrive to a final system in closed form, but those do not rely on a physical interpretation and the quality of such approximation can be improved.

5 Inversion of the collision operator

In this section, we aim at expressing the odd flux moments $\Gamma^{(2n+1)}(f^1)$ as a function of f^0 . For simplicity, we add the following assumption:

Hypothesis: The collision kernel depends only on the deflection angle, and the formula (22a) simplifies into

$$C_{\epsilon, \mathfrak{g}}^{0, el}(\bar{f})(\mathbf{v}) = \int_{\mathcal{S}^2} \nu_{\epsilon \mathfrak{g}}(\boldsymbol{\omega} \cdot \hat{\mathbf{v}}_{\epsilon}) (\bar{f}(v\boldsymbol{\omega}) - \bar{f}(\mathbf{v})) d\boldsymbol{\omega}. \quad (93)$$

This corresponds to the case of a Langevin collision or Maxwell molecules.

Proposition 9. *Under this assumption, the term $\Gamma^{(2l+1)}(C_{\epsilon, \mathfrak{g}}^{0, el}(f^1))$ simplifies into*

$$\Gamma^{(2l+1)}(C_{\epsilon, \mathfrak{g}}^{0, el}(f^1)) = -\nu_{\epsilon \mathfrak{g}_1} \Gamma^{(2l+1)}(f^1), \quad (94)$$

where the momentum transfer frequency reads $\nu_{\epsilon \mathfrak{g}_1} = n_{\mathfrak{g}} v \int_{\mathcal{S}^2} (1 - \hat{\mathbf{v}}_{\epsilon} \cdot \boldsymbol{\omega}) \bar{\sigma}_{\epsilon \mathfrak{g}} d\boldsymbol{\omega}$.

Proof. As in (76), one computes:

$$\begin{aligned} \langle\langle \psi^{2l+1}, C_{\epsilon, \mathfrak{g}}^{0, el}(f^1) \rangle\rangle &= \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} \mathbf{v} \psi^{2l}(v) \nu_{\epsilon \mathfrak{g}}(\hat{\mathbf{v}}_{\epsilon} \cdot \boldsymbol{\omega}) [f^1(\mathbf{x}, v\boldsymbol{\omega}, t) - f^1(\mathbf{x}, \mathbf{v}, t)] d\boldsymbol{\omega} dv \\ &= \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} (v\boldsymbol{\omega} - \mathbf{v}) \psi^{2l}(v) \nu_{\epsilon \mathfrak{g}}(\hat{\mathbf{v}}_{\epsilon} \cdot \boldsymbol{\omega}) f^1(\mathbf{v}) d\boldsymbol{\omega} dv. \end{aligned}$$

Remarking that the integral

$$\int_{\mathcal{S}^2} (v\boldsymbol{\omega} - \mathbf{v}) \nu_{\epsilon \mathfrak{g}}(\hat{\mathbf{v}}_{\epsilon} \cdot \boldsymbol{\omega}) d\boldsymbol{\omega} = -\mathbf{v} \int_{\mathcal{S}^2} (1 - \hat{\mathbf{v}}_{\epsilon} \cdot \boldsymbol{\omega}) \nu_{\epsilon \mathfrak{g}}(\hat{\mathbf{v}}_{\epsilon} \cdot \boldsymbol{\omega}) d\boldsymbol{\omega} := -\nu_{\epsilon \mathfrak{g}_1} \mathbf{v}$$

yields the result.

$$\langle\langle \psi^{2l+1}, C_{\epsilon, \mathfrak{g}}^{0, el}(f^1) \rangle\rangle = -\nu_{\epsilon \mathfrak{g}_1} \int_{\mathbb{R}^3} \psi^{2l}(v) \mathbf{v} f^1(\mathbf{v}) d\mathbf{v} = -\nu_{\epsilon \mathfrak{g}_1} \Gamma^{(2l+1)}(f^1) \quad (95)$$

□

In the following, by abuse of notation, we denote the constant momentum transfer collision frequency (i.e., with Hypothesis (93)) as $\nu_{\epsilon \mathfrak{g}}$. Then, with (90a), we obtain

$$\Gamma^{(2l+1)}(f^1) = -\frac{1}{3\nu_{\epsilon \mathfrak{g}}} \left(\nabla p^{(2l+2)}(f^0) + (2l+3)p^{(2l)}(f^0) \mathbf{E} \right). \quad (96)$$

Injecting this in the moment system (90) yields

$$\begin{aligned} \frac{\partial}{\partial t} p^{(2l)}(f^0) - \frac{1}{3\nu_{\epsilon \mathfrak{g}}} \left(\Delta p^{(2l+2)}(f^0) + (4l+3) \nabla p^{(2l)}(f^0) \cdot \mathbf{E} \right. \\ \left. + (2l+3)p^{(2l)}(f^0) \nabla \cdot \mathbf{E} + 2l(2l+1)p^{(2l-2)}(f^0) \|\mathbf{E}\|^2 \right) \\ = p^{(2l)} \left(C_{\epsilon, \mathfrak{g}}^{2, el}(f^0) + C_{\epsilon, \epsilon}^{el}(f^0, f^0) + \sum_{k \in \mathcal{Q}_{\mathfrak{g}}} C_{\epsilon, \mathfrak{g}_k}^{0, inel}(f^0) + C_{\epsilon, \mathfrak{g}}^{0, iz}(f^0) \right). \end{aligned} \quad (97)$$

which depends only on f^0 .

Remark 7. Without imposing (93), one could express $\Gamma^{(2l+1)}(C_{\epsilon, \mathbf{g}}^{0, el}(f^1))$ as a function of $\Gamma^{(2l+1)}(f^1)$ by imposing that f^1 has a particular form. This is performed in the next section to construct the closure relation for f^0 . But this relation afterward needs to be inverted, which is not trivial in the general case.

There remains to close the system (97), i.e. to express the higher order moment $p^{(2L+2)}(f^0)$ and the collision operators as functions of the moments $p^{(2l)}(f^0)$ for $l = 0, \dots, N$.

The system (97) with such a closure can be rewritten under the generic form

$$\frac{\partial}{\partial t} \mathbf{U} + \Delta \mathbf{D}(\mathbf{U}) + \nabla \cdot (\mathbf{A}\mathbf{U}) + \mathbf{L}\mathbf{U} = \mathbf{S}(\mathbf{U})$$

where the unknown $\mathbf{U} = (p^{(0)}(f^0), p^{(2)}(f^0), \dots, p^{(2l)}(f^0))$ is a vector of scalar moments, with linear diagonal advection and bi-diagonal source terms \mathbf{A}_i and \mathbf{L} (due to the x - and v -derivatives at the kinetic level) defined by

$$\mathbf{A}_{i,l,m} = \frac{-\mathbf{E}_i(4l+3)}{3\nu_{\epsilon \mathbf{g}}} \delta_{l,m}, \quad \mathbf{L}_{l,m} = \frac{-1}{3\nu_{\epsilon \mathbf{g}}} ((2l+3)\nabla \cdot \mathbf{E} \delta_{l,m} + 2l(2l+1)\|\mathbf{E}\|^2 \delta_{l+1,m}). \quad (98)$$

The only non-linearity may appear in the collision term $\mathbf{S}(\mathbf{U})$ on the right-hand-side and in the last term of the diffusion operator $\mathbf{D}(\mathbf{U})_N$ depending on the chosen closure. They yield respectively

$$\mathbf{D}(\mathbf{U})_l = \frac{-p^{(2l+2)}(f^0)}{3\nu_{\epsilon \mathbf{g}}}, \quad (99a)$$

$$\mathbf{S}(\mathbf{U})_l = p^{(2l)} \left(C_{\epsilon, \mathbf{g}}^{2, el}(f^0) + C_{\epsilon, \mathbf{e}}^{el}(f^0, f^0) + \sum_{k \in \mathcal{Q}_{\mathbf{g}}} C_{\epsilon, \mathbf{g}_k}^{0, inel}(f^0) + C_{\epsilon, \mathbf{g}}^{0, iz}(f^0) \right). \quad (99b)$$

The diffusive term is often (formally) rewritten under the form

$$\Delta \mathbf{D}(\mathbf{U}) = \nabla \cdot (\mathbf{B}(\mathbf{U})\nabla \mathbf{U}),$$

$$\mathbf{B}(\mathbf{U}) = \frac{\partial \mathbf{D}}{\partial \mathbf{U}}(\mathbf{U}) = \frac{-1}{3\nu_{\epsilon \mathbf{g}}} \begin{pmatrix} 0 & 1 & 0 & \dots & & 0 \\ 0 & 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 0 \\ 0 & \dots & & 0 & 0 & 1 \\ B_{N0} & B_{N1} & \dots & & B_{NN-1} & B_{NN} \end{pmatrix},$$

where the last row depends on the chosen closure.

Contrarily to common moment models, the non-linear closure appears in the diffusion operator while the transport operator remains linear. Therefore, the diffusion matrix $\mathbf{B} = \partial \mathbf{D} / \partial \mathbf{U}$ needs to be non-positive, but the definiteness is a priori not required. And as always, the symmetrizability of the Jacobian $\mathbf{B} = \partial \mathbf{D} / \partial \mathbf{U}$ is equivalent to the existence of a strictly convex entropy following Kawashima theory (see [69, 70]). The two examples of closure in the next section are not (at least globally) symmetrizable but present other notions of stability, especially through the non-positivity of the diffusion matrix \mathbf{B} . The study of such non-linear parabolic equation with specific closures still needs to be addressed, but this is beyond the scope of the present paper.

6 Stieltjes moment closures

We stress that the moment hierarchy of the previous section is general for electron-gas Langevin elastic collisions. However, it requires a closure in order to compute the collision integrals as well as to compute the flux of the last moment of a truncated hierarchy.

In order to close the system, the common strategy consists in solving a moment problem that takes the form:

$$\text{For } \mathbf{p} \in \mathcal{R} \subset \mathbb{R}^{L+1}, \text{ find } f_R \text{ s.t. } \int_{\mathbb{R}^3} \psi^{2l}(|\bar{v}|) f_R(\bar{v}) d\bar{v} = \mathbf{p}_l, \quad (100a)$$

which reduces to a Stieltjes moment problem:

$$\text{For } \mathbf{p} \in \mathcal{R} \subset \mathbb{R}^{L+1}, \text{ find } f_R \text{ s.t. } \int_{\mathbb{R}^+} v^{2l} f_R(v) dv = \frac{\mathbf{p}_l}{4\pi}. \quad (100b)$$

Then the remaining moment and the collision operators are rewritten by replacing f^0 by f_R in their expression. This yield a closed system of equations. Now there remains to choose a solution to the moment problem (100).

6.1 Quadrature closure

The quadrature closure consists, for $L + 1 = 2N$ even, in writing f_R as a sum of N Dirac measures, or equivalently to replace the integral in (100b) by a quadrature

$$f_R(v) = \sum_{i=1}^N m_i \delta_{v_i}(v) \quad \text{such that} \quad \mathbf{p}_l = \sum_{i=1}^N m_i v_i^{2l},$$

with non-negative quadrature points and weights $v_i, m_i \in \mathbb{R}^+$. These weights and points can be obtained as usually through the theory of orthogonal polynomials [42, 41] using Chebychev or Wheeler algorithm [44, 43].

The advantage of this closure yields in the discretization of the collision operator once the quadrature points and weights e_i, m_i are computed. They simply yield

$$p^{(2l)}(C_{\mathbf{e}, \mathbf{g}}^{2, el}(f_R)) = \sum_i m_i K_{\mathbf{e}, \mathbf{g}, i}^{2, el, (2l)}(v_i), \quad (101a)$$

$$p^{(2l)}(C_{\mathbf{e}, \mathbf{e}}^{el}(f_R, f_R)) = \sum_{i, j} m_i m_j K_{\mathbf{e}, \mathbf{e}, i, j}^{0, el, (2l)}(v_i, v_j), \quad (101b)$$

$$p^{(2l)}(C_{\mathbf{e}, \mathbf{g}}^{0, inel}(f_R)) = \sum_i m_i K_{\mathbf{e}, \mathbf{g}, i}^{0, inel, (2l)}(v_i), \quad (101c)$$

$$p^{(2l)}(C_{\mathbf{e}, \mathbf{g}}^{0, iz}(f_R)) = \sum_i m_i K_{\mathbf{e}, \mathbf{g}, i}^{0, iz, (2l)}(v_i). \quad (101d)$$

For the sake of readability, the computations are left in **B** together with the different values of K .

As the Jacobian of the flux in the common QMOM method, the diffusion matrix $\mathbf{B}(\mathbf{U})$ is similar to a block diagonal matrix with 2×2 Jordan blocks associated to non-positive eigenvalues $-v_i^2/(3\nu_{\mathbf{e}, \mathbf{g}})$. This is simply obtained by computing the Jacobians of \mathbf{U} and \mathbf{D} with respect to the parameters $\mathbf{p} = (m_1, v_1, \dots, v_L)$ and applying the chain rule, we refer e.g. to [71] for those computations. One remarks though that this closure does not provide symmetrizability, but the resulting diffusion matrix \mathbf{B} is non-positive.

6.1.1 Examples of truncated QMOM systems

We present two examples with analytical moment inversion. Note that, in general, QMOM requires a numerical obtention of the quadrature points and weights.

Two-moment system ($N = 1$) The considered moments are:

$$\mathbf{U} = \left(p^{(0)}, p^{(2)} \right)^T := (n, 3nT)^T,$$

where we have introduced the electron particle density n , and electron temperature T . The moment problem can be analytically inverted and the zero-th order distribution functions can be obtained, as follows,

$$f_{(2M)}^{0,QMOM}(v) = m_1 \delta_{v_1}(v) \quad \text{with} \quad m_1 = \frac{n}{4\pi} = \frac{p^{(0)}}{4\pi} \quad \text{and} \quad v_1 = \sqrt{3T} = \sqrt{\frac{p^{(2)}}{p^{(0)}}}. \quad (102)$$

The closure flux depends on the scalar fourth-order moment that can be computed from the first two moments, as follows,

$$p^{(4),QMOM} = m_1 v_1^4 = 9nT^2 = p^{(0)} \left(\frac{p^{(2)}}{p^{(0)}} \right)^2.$$

The 2M-model considers the balance equations for the density and energy. We write the equations in a form similar to the LMEA, as shown in [32], as follows,

$$\partial_t p^{(0)} + \nabla \cdot \mathbf{\Gamma}^{(1)} = C^{(0)}, \quad (103a)$$

$$\partial_t p^{(2)} + \nabla \cdot \mathbf{\Gamma}^{(3)} + 2\mathbf{\Gamma}^{(1)} \cdot \mathbf{E} = C^{(2)}. \quad (103b)$$

The sources for $l = 0, 1$ are of the form

$$C^{(2l)} = m_1 \left(K_{\mathbf{e}\mathbf{g}}^{2,el,(2l)}(v_1) + m_1 K_{\mathbf{e}\mathbf{e}}^{0,el,(2l)}(v_1, v_1) + K_{\mathbf{e}\mathbf{g}}^{0,inel,(2l)}(v_1) + K_{\mathbf{e}\mathbf{g}}^{0,iz,(2l)}(v_1) \right), \quad (104)$$

with (m_1, v_1) given in (102), and with given functions $K_{\mathbf{e}\mathbf{g}}^{2,el,(2l)}$, $K_{\mathbf{e}\mathbf{e}}^{0,el,(2l)}$, $K_{\mathbf{e}\mathbf{g}}^{0,inel,(2l)}$ and $K_{\mathbf{e}\mathbf{g}}^{0,iz,(2l)}$. And the fluxes of the form

$$\mathbf{\Gamma}^{(1)} = n\mathbf{u} = -\nabla \left(D p^{(0)} \right) - \mu p^{(0)} \mathbf{E}, \quad (105a)$$

$$\mathbf{\Gamma}^{(3)} = -\nabla \left(D_{\epsilon}^{QMOM} p^{(2)} \right) - \mu_{\epsilon} p^{(2)} \mathbf{E}, \quad (105b)$$

where we introduce the diffusion coefficient D , the mobility μ , the energy diffusion coefficient D_{ϵ} , and the energy mobility μ_{ϵ} , that in the case of 2M-QMOM, reads

$$D = \frac{T}{\nu_{\mathbf{e}\mathbf{g}}} = \frac{p^{(2)}}{3\nu_{\mathbf{e}\mathbf{g}} p^{(0)}} = D_{\epsilon}^{QMOM}, \quad \mu = \frac{1}{\nu_{\mathbf{e}\mathbf{g}}}, \quad \mu_{\epsilon} = \frac{5}{3\nu_{\mathbf{e}\mathbf{g}}}.$$

We note that the density coefficients satisfy the Einstein relation $D/\mu = T$. This relation holds only near equilibrium for general cross sections, while it is always holds for Maxwell molecules [29], as retrieved by our model.

Four-moment system ($N = 3$) The considered moments are:

$$\mathbf{U} = \left(p^{(0)}, p^{(2)}, p^{(4)}, p^{(6)} \right)^T.$$

The moment problem can also be analytically inverted and the zero-th order distribution functions can be obtained

$$f_{(4M)}^{0,QMOM}(v) = m_1 \delta_{v_1}(v) + m_2 \delta_{v_2}(v),$$

where the weights and abscissas can be computed from the moments, as follows,

$$\Delta = \frac{p^{(0)} p^{(6)} - p^{(2)} p^{(4)}}{(p^{(2)})^2 - p^{(0)} p^{(4)}}, \quad \tilde{\Delta} = \frac{(p^{(4)})^2 - p^{(2)} p^{(6)}}{(p^{(2)})^2 - p^{(0)} p^{(4)}}, \quad (106a)$$

$$v_2^2 = \frac{-\Delta + \sqrt{\Delta^2 - 4\tilde{\Delta}}}{2}, \quad v_1^2 = \frac{-\Delta - \sqrt{\Delta^2 - 4\tilde{\Delta}}}{2}, \quad (106b)$$

$$m_2 = \frac{p^{(2)} - p^{(0)} v_1^2}{4\pi (v_2^2 - v_1^2)}, \quad m_1 = \frac{p^{(0)} v_2^2 - p^{(2)}}{4\pi (v_2^2 - v_1^2)}. \quad (106c)$$

With this distribution function, we compute the closure flux, as follows,

$$p_{(4M)}^{(8),QMOM} = m_1 v_1^8 + m_2 v_2^8 = \frac{p^{(0)}(p^{(6)})^2 - 2p^{(2)}p^{(4)}p^{(6)} + (p^{(4)})^3}{p^{(0)}p^{(4)} - (p^{(2)})^2}. \quad (107)$$

The truncated moment system of equations reads

$$\frac{\partial p^{(0)}}{\partial t} + \nabla \cdot \mathbf{\Gamma}^{(1)} = C^{(0)} \quad (108a)$$

$$\frac{\partial p^{(2)}}{\partial t} + \nabla \cdot \mathbf{\Gamma}^{(3)} + 2\mathbf{\Gamma}^{(1)} \cdot \mathbf{E} = C^{(2)} \quad (108b)$$

$$\frac{\partial p^{(4)}}{\partial t} + \nabla \cdot \mathbf{\Gamma}^{(5)} + 4\mathbf{\Gamma}^{(3)} \cdot \mathbf{E} = C^{(4)} \quad (108c)$$

$$\frac{\partial p^{(6)}}{\partial t} + \nabla \cdot \mathbf{\Gamma}^{(7)} + 6\mathbf{\Gamma}^{(5)} \cdot \mathbf{E} = C^{(6)} \quad (108d)$$

with the fluxes written as

$$\mathbf{\Gamma}^{(1)} = -\nabla \left(\frac{p^{(2)}}{3\nu_{\mathbf{e}_g}} \right) - \frac{p^{(0)}}{\nu_{\mathbf{e}_g}} \mathbf{E}, \quad (109a)$$

$$\mathbf{\Gamma}^{(3)} = -\nabla \left(\frac{p^{(4)}}{3\nu_{\mathbf{e}_g}} \right) - \frac{5}{3} \frac{p^{(2)}}{\nu_{\mathbf{e}_g}} \mathbf{E}, \quad (109b)$$

$$\mathbf{\Gamma}^{(5)} = -\nabla \left(\frac{p^{(6)}}{3\nu_{\mathbf{e}_g}} \right) - \frac{7}{3} \frac{p^{(4)}}{\nu_{\mathbf{e}_g}} \mathbf{E}, \quad (109c)$$

$$\mathbf{\Gamma}^{(7)} = -\nabla \left(\frac{p_{(4M)}^{(8),QMOM}}{3\nu_{\mathbf{e}_g}} \right) - 3 \frac{p^{(6)}}{\nu_{\mathbf{e}_g}} \mathbf{E}. \quad (109d)$$

The collisional terms $C^{(2l)}$ are computed with (101) and the weights and quadrature points with (106).

6.2 Hermitian moment closure

In the following, we propose one possible solution to the moment problem (100) based on an expansion of the zero-th order distribution function into Hermite polynomials (as proposed by Grad [45]). As a result, the zero-th order distribution is written as a truncated expansion of scalar Hermite polynomials, as follows,

$$f^0(\mathbf{x}, \mathcal{V}, t) = nT^{-3/2} \phi^0(\mathcal{V}) [1 + \chi(\mathbf{x}, \mathcal{V}, t)], \quad \phi^0(\mathcal{V}) = \frac{e^{-\frac{\mathcal{V}^2}{2}}}{(2\pi)^{3/2}}, \quad \mathcal{V} = T^{-1/2} \mathbf{v}, \quad (110)$$

where the function χ is computed as the following expansion (note that this expansion is substantially identical to a Sonine polynomial expansion):

$$\chi(\mathbf{x}, \mathcal{V}, t) = \sum_{l=2}^N h^{(2l)}(\mathbf{x}, t) H^{(2l)}(\mathcal{V}). \quad (111)$$

Here, N is the order of the truncation, i.e., $2N$ is the order of the highest-moment, and $h^{(2n)}$ are the scalar Hermite moments, computed as

$$nh^{(2l)}(\mathbf{x}, t) = \langle\langle H^{(2l)}, f^0 \rangle\rangle \quad \text{for } l \in \{2, \dots, N\}. \quad (112)$$

where $H^{(2l)}(\mathcal{V})$ are the scalar irreducible Hermite polynomials. We note that due to the orthogonality properties, $h^{(0)} = h^{(2)} = 0$. The formalism for the Hermite polynomials used in this work is taken from Balescu [50], and summarized in Appendix C.1.

With the ansatz used for the distribution function (110-111), we can compute the closure for the moment system. In the case of Maxwell molecules (97), we need $p^{(2N+2)}$ as a function of the previous moments, which reads,

$$p^{(2N+2)} = nT^{N+1}\alpha_{N+1} + \sum_{m=2}^N \sum_{l=2}^m \alpha_{lm} T^{N+1-l} p^{(2l)}, \quad (113)$$

with the density $n = p^{(0)}$ and the temperature $T = p^{(2)}/(3p^{(0)})$ and the coefficients α_{N+1} and α_{lm} are fractions and the expressions are given in (136) in Appendix C.1.

Similarly, in the system of (97), we need the collisional terms expressed as a function of the moments. With a Hermitian expansion of f^0 , we can compute analytically the collision terms as follows,

$$p^{(2l)}(C_{\mathbf{c},\mathbf{g}}^{2,el}(f^0)) = -2l\nu_{\mathbf{c},\mathbf{g}} \left\{ \left(1 - \frac{T_{\mathbf{g}}}{T}\right) p^{(2l)} + \frac{T_{\mathbf{g}}}{T} \left(p^{(2l)} - (2l+1)T p^{(2l-2)} \right) \right\} \quad \text{for } l \in \{1, \dots, N\}, \quad (114a)$$

$$p^{(2l)}(C_{\mathbf{c},\mathbf{c}}^{el}(f^0, f^0)) = -nT^l \nu_{\mathbf{c},\mathbf{c}}^{(2l)} \quad \text{for } l \in \{2, \dots, N\}, \quad (114b)$$

$$p^{(2l)}(C_{\mathbf{c},\mathbf{g}}^{0,inel}(f^0)) = n_{\mathbf{g}} n T^l \sum_{m \in \mathcal{Q}_{\mathbf{g}}^*} \sum_{r=0}^{l-1} \frac{2^l l! (-1)^{l-r}}{r!(l-r)!} K_{inel,m}^{(r)} \left(\frac{\phi_m^*}{T} \right)^{l-r} \quad \text{for } l \in \{1, \dots, N\}, \quad (114c)$$

$$p^{(2l)}(C_{\mathbf{c},\mathbf{g}}^{0,iz}(f^0)) = \begin{cases} n_{\mathbf{g}} n K_{iz}^{(0)} & \text{for } l = 0 \\ n_{\mathbf{g}} n T^l \left\{ a_{iz}^{(l)} K_{iz}^{(l)} + \sum_{r=0}^{l-1} a_{iz}^{(r)} K_{iz}^{(r)} \left(\frac{\phi_{iz}^*}{T} \right)^{l-r} \right\} & \text{for } l \in \{1, \dots, N\} \end{cases} \quad (114d)$$

We note that the Maxwell molecule model is only used for the electron-neutral elastic collisions and the rest of collisions have a non-trivial collision kernel. As a result, the rates and collision frequencies depend on the integration of the kernel and the zero-th distribution function, that are given in (139), (143), and (148). As done in the case of QMOM, for the sake of readability, we detail the derivation of these expressions in Appendix C.2.

Concerning the mathematical structure of the Hermitian moment closure, it suffers from the same problems as the original Grad's model [45]. Although the matrix \mathbf{A} , associated to the hyperbolic part of the equation, is indeed hyperbolic (it is diagonal with real eigenvalues), the diffusion matrix \mathbf{B} is not guaranteed, in general, to be non-positive in the Hermitian moment approximation. The matrix is non-positive only in the linearized regime in the vicinity of the thermodynamic equilibrium. This regime of validity, which depends on the number of moments, will be investigated in a future work.

6.2.1 Examples of truncated Hermitian moment systems

We present the same examples, as in the QMOM case. Note that the Hermitian closure allows for analytical solutions for arbitrary order that are given in Appendix C.

Two-moment system ($N = 1$) The considered moments are:

$$\mathbf{U} = \left(p^{(0)}, p^{(2)} \right)^T := (n, 3nT)^T.$$

In the 2M-Hermitian closure, the distribution function is simply a Maxwellian, as follows,

$$f_{(2M)}^{0,Herm}(v) = \frac{n}{(2\pi T)^{3/2}} e^{-\frac{v^2}{2T}}.$$

The closure flux depends on the scalar fourth-order moment that can be computed from the first two moments:

$$p^{(4),Herm} = 15nT^2 = \frac{15p^{(0)}}{9} \left(\frac{p^{(2)}}{p^{(0)}} \right)^2.$$

The system is (103) with the following fluxes:

$$\mathbf{\Gamma}^{(1)} = n\mathbf{u} = -\nabla \left(Dp^{(0)} \right) - \mu p^{(0)} \mathbf{E}, \quad (115a)$$

$$\mathbf{\Gamma}^{(3)} = -\nabla \left(D_\epsilon^{Herm} p^{(2)} \right) - \mu_\epsilon p^{(2)} \mathbf{E}, \quad (115b)$$

with the transport coefficients

$$D = \frac{T}{\nu_{\epsilon g}} = \frac{p^{(2)}}{3\nu_{\epsilon g} p^{(0)}}, \quad \mu = \frac{1}{\nu_{\epsilon g}}, \quad \mu_\epsilon = \frac{5}{3\nu_{\epsilon g}}, \quad D_\epsilon^{Herm} = \frac{5T}{3\nu_{\epsilon g}} = \frac{5p^{(2)}}{9\nu_{\epsilon g} p^{(0)}}.$$

The density and energy coefficients satisfy the Einstein relations $D/\mu = T = D_\epsilon/\mu_\epsilon$, as well as the relations for constant collision frequency (see [32]), i.e. $D_\epsilon/D = \mu_\epsilon/\mu = 5/3$. This shows that the Hermitian model is consistent with the previous thermodynamic theories at equilibrium with a constant collision frequency.

The collision terms are computed from (114), which reads in the 2M-case:

$$C^{(0)} = n_g n K_{iz}^{(0)},$$

$$C^{(2)} = -2\nu_{\epsilon g} \{3n(T - T_g)\} - 2n_g n K_{iz}^{(0)} \phi_{iz}^* - 2n_g n \sum_{m \in \mathcal{Q}_g^*} K_{inel,m}^{(0)} \phi_m^*,$$

where the rates $K_{iz}^{(0)}$ and $K_{inel,m}^{(0)}$ are functions of the temperature. Note that the collisional terms are these found in the literature with a Maxwellian distribution function with constant collision frequency (see e.g. [1]). The factor 2 in the second equation results from our definition of the second-order moment that is twice the mean energy.

Four-moment system ($N = 3$) The considered moments are:

$$\mathbf{U} = \left(p^{(0)}, p^{(2)}, p^{(4)}, p^{(6)} \right)^T$$

$$:= \left(n, 3nT, 15nT^2 \left(1 + \Delta^{(4)} \right), 105nT^3 \left(1 + \Delta^{(6)} \right) \right)^T,$$

where the normalized variables are

$$\Delta^{(4)} = \frac{p^{(4)} - 15nT^2}{15nT^2} \quad \text{and} \quad \Delta^{(6)} = \frac{p^{(6)} - 105nT^3}{105nT^3}, \quad (116)$$

that represent the standardized deviation from the Maxwellian of the higher-order moments. The assumed zero-th distribution function reads,

$$f_{(4M)}^{0,Herm}(v) = \frac{n}{(2\pi T)^{3/2}} e^{-\frac{v^2}{2T}} \left[1 + \chi \left(vT^{1/2} \right) \right] \quad \text{with}$$

$$\chi(\mathcal{V}) = \frac{\Delta^{(4)}}{8} (\mathcal{V}^4 - 10\mathcal{V}^2 + 15) - \frac{(3\Delta^{(4)} - \Delta^{(6)})}{48} (\mathcal{V}^6 - 21\mathcal{V}^4 + 105\mathcal{V}^2 - 105).$$

The closure flux with this distribution as a function of the considered moments reads

$$p_{(4M)}^{(8),Herm} = 945nT^4 \left(1 - 6\Delta^{(4)} + 4\Delta^{(6)} \right). \quad (117)$$

Finally, the truncated 4M-system of equations in the Hermitian closure reads (108), with the fluxes (109), and the closure (117).

The collisional source terms $C^{(2l)}$ can be computed from the recurrence formulae (114).

7 Discussions and conclusion

As a summary, we have derived a model for the standalone transport of electrons in a weakly-ionized plasma dominated by elastic electron-neutral interactions. With a Hilbert expansion and a dimensional analysis of the Boltzmann equation (including elastic, inelastic, and ionization collisions), we have obtained a reduced kinetic model consisting of an evolution equation of the isotropic part (in velocity space) of the distribution function and a Fredholm equation for the first-order distribution. The model is a generalization of the SHE model that is traditionally used for electrons in gas discharges. We have studied the entropy dissipation of the asymptotic model. Even though the full model does not necessarily lead to a non-negative entropy production (it does only when the other species are included in the study), we exhibited a part of the entropy production linked to the zero-th order electron-neutral elastic collision operator that is signed. We have constructed a hierarchy of moment equations obtained by taking scalar and vectorial velocity moments of the obtained kinetic model. With a particular collision kernel (Maxwell molecules), this model eventually takes the form of a system of balance laws with a non-linear diffusion, a linear advection and a linear source due to the electric field and a non-linear collision terms (due to the remaining quadratic electron-electron collisions). It is simple enough to be solved numerically, and numerical experiments together with application to physical problems are part of future work.

Several assumptions were made in order to simplify the model or its writing. Some of them can be lifted to reach more complex physics:

- The plasma heavy species (neutral atoms and molecules and ions) were considered as a background in this paper. Note that, as shown through the dimensional analysis by Choquet et al. in Section 9 of [33], the heavy species thermalize at order $\mathcal{O}(\varepsilon^{-1})$, while their evolution is described by the Euler equations at order $\mathcal{O}(\varepsilon^0)$. Similar conclusions were obtained by other authors [15, 20]. As a consequence, the results in this paper still hold when the electron model is coupled to a hydrodynamic description for the heavy species, because the heavy species distribution functions that were considered as a background in our work can eventually be determined with a hydrodynamic model. However, f^2 would be needed if the Navier-Stokes level is required for the heavy species, as presented in Graille et al. [15].
- Similarly, the electric field used in this model requires to be coupled to Gauss's law. The self-consistent resolution of the electric field that is created by the charged species within the plasma is fundamental in order to reproduce the plasma dynamics. Nevertheless, the present results still hold when considering Gauss's law for the electrostatic field, as long as the orders of magnitude of the field are these discussed in Assumption 4(d), i.e. $E^0 L^0 = T_e^0$. This also applies for external fields. Note that very large electric fields or fast evolving electric fields can produce large anisotropies in the electron distribution function that would be beyond our proposed scaling.
- The neutrals were considered at thermodynamic equilibrium with a zero mean velocity. If the equilibrium hypothesis is reasonable, one could expect the neutrals to have a non-zero velocity. Such a hypothesis could easily be lifted in this work, leading to additional linear terms at order 1 in ε . Also, one would expect the present equilibrium, or zero-th order, f^0 not to be isotropic, but to depend only on the norm $\|\mathbf{u}_g - \mathbf{v}\|$ instead of on the norm v .
- Maxwell molecules were used in order to invert the collision operator $C_{\text{eg}}^{0,el}$ easily. More physically relevant kernels can also be used. This would only lead either to a more complex formula than the one in (94), or one would need to solve the coupled system (90) instead of the reduced one (97).

Other hypotheses, common in the field of kinetic theory, would be more complex to reduce:

- The construction relies on the first orders of a Hilbert expansion (50). Such an expansion is a priori only formal, non-unique, and one may only assume that the series converges (the demonstration being out-of-reach). We have shown that the first orders of the present construction are stable. Remark that this is not sufficient in general, as the higher orders of the Chapman-Enskog expansion, after Navier-Stokes, lead to unstable Burnett and super-Burnett equations.

- Only the electron transport was considered in the present work, and the transport of neutrals and ions were discarded. Under weakly-ionized gas discharge conditions, the fast dynamics of the electrons is usually largely separated from the one of the heavy species. First, the impact of the plasma dynamics on the neutrals can be seen as a small perturbation. Even if they are responsible for the main collisional processes in the plasma, these collisions have a negligible impact on the dynamics of the neutrals. Furthermore, as suggested in the hypothesis, the heavy (atoms and ions) thermalize faster than the electrons, such that the ions can be considered at thermodynamic equilibrium (as they collide with the gas particles and, additionally, they are created by ionization at their temperature). For this reason, their transport can be captured by simpler and more common models. However, even if we ultimately obtain entropy dissipative moment models, no entropy dissipation at the kinetic level can be obtained without considering the transport of ions and atoms and their coupling with the electrons.

Concerning the potential application of these models, it is similar to the models based on the two-term Boltzmann approximation, which are widely used in low-temperature plasma discharges for plasma processing, both at the kinetic level and in the fluid (drift-diffusion models) level. Some examples of potential applications are given below. One example are atmospheric plasmas with applications to polymer etching, treatment and decontamination of food, water, and biological tissues [72], as well as plasma assisted ignition and combustion [73]. These plasmas can be described by the local-field approximation (see e.g. [74] for a review on the modeling). However, the local-field approximation is limited when the plasma interacts with the surfaces and hence the self-consistent inclusion of higher-order moments can potentially solve some of these limitations associated. Alternatively, at lower pressures, low-temperature plasmas are widely used for semi-conductor fabrication in the microelectronics industry. At lower pressures (from few tens of Torr to few milliTorrs), the local-field approximation becomes less reliable as compared to experiment and numerical simulations [75]. These pressures are investigated with the so-called non-local model of the two-term Boltzmann approximation [76, 77]. Nevertheless, as shown in Section 3.5, the two-term Boltzmann approximation is a particular case of our kinetic model and, therefore, the moment model can also tackle the abovementioned problems. Finally, the last application where these model can be applied is plasmas in electric propulsion such as gridded-ion thrusters [78] or Hall thrusters, although the inclusion a magnetic field should be introduced in the model, which would increase the complexity of the transport model [79, 80, 81].

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A Expansion of electron-heavy species collision operator

In this section, we detail the calculations of the proof to Proposition 1, in order to obtain the expansion of the collision operator in powers of ε . We inject the expansion of the distribution functions at the post-collisional velocity of (26) and the collision kernel (27) into the normalized collision operator (24). We note that the integration over the heavy species velocity space can be performed analytically, as follows,

$$\int_{\mathbb{R}^3} \left(\bar{f}'_{\epsilon} \bar{f}'_{\alpha} - \bar{f}_{\epsilon} \bar{f}_{\alpha} \right) S(\bar{g}') d\bar{\mathbf{v}}_{\alpha} = I_0 + \varepsilon I_1 + \varepsilon^2 I_2 + \mathcal{O}(\varepsilon^3). \quad (118)$$

The zero-th order integral reads

$$I_0 = \int_{\mathbb{R}^3} \bar{f}_{\alpha}(\bar{\mathbf{v}}_{\alpha}) [\bar{f}_{\epsilon}(\bar{\mathbf{v}}_{\epsilon}) - \bar{f}_{\epsilon}(\bar{\mathbf{v}}_{\epsilon})] S(\bar{\mathbf{v}}_{\epsilon}) d\bar{\mathbf{v}}_{\alpha} = \bar{n}_{\alpha} S(\bar{\mathbf{v}}_{\epsilon}) (\tilde{f}_{\epsilon} - \bar{f}_{\epsilon}), \quad (119)$$

with the distribution function at the rotated velocity that is denoted as $\tilde{f}_{\epsilon} \equiv \bar{f}_{\epsilon}(\bar{\mathbf{x}}, \bar{\mathbf{v}}_{\epsilon}, \bar{t})$.

The first-order integral reads

$$I_1 = I_1^0 + I_1^1 + I_1^2, \quad (120)$$

where

$$I_1^0 = - \int_{\mathbb{R}^3} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) (\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) \cdot \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) S(\tilde{\mathbf{v}}_\epsilon) d\bar{\mathbf{v}}_\alpha = \bar{n}_\alpha \bar{\mathbf{u}}_\alpha \cdot \left(\nabla_{\tilde{\mathbf{v}}} \tilde{f}_\epsilon - \nabla_{\bar{\mathbf{v}}} \tilde{f}_\epsilon \right) S(\tilde{\mathbf{v}}_\epsilon), \quad (121a)$$

$$\begin{aligned} I_1^1 &= - \int_{\mathbb{R}^3} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\bar{\mathbf{v}}} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) S(\tilde{\mathbf{v}}_\epsilon) d\bar{\mathbf{v}}_\alpha \\ &= \int_{\mathbb{R}^3} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) \nabla_{\bar{\mathbf{v}}} \cdot \left[\tilde{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) S(\tilde{\mathbf{v}}_\epsilon) \right] d\bar{\mathbf{v}}_\alpha = 0, \end{aligned} \quad (121b)$$

$$I_1^2 = - \int_{\mathbb{R}^3} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) \left(\tilde{f}_\epsilon - \bar{f}_\epsilon \right) (\bar{\mathbf{v}}_\alpha \cdot \nabla_{\bar{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon)) d\bar{\mathbf{v}}_\alpha = -\bar{n}_\alpha \bar{\mathbf{u}}_\alpha \cdot \nabla_{\bar{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) \left(\tilde{f}_\epsilon - \bar{f}_\epsilon \right). \quad (121c)$$

By grouping terms, we can re-write the first-order integral as

$$I_1 = I_1^0 + I_1^2 = \bar{n}_\alpha \bar{\mathbf{u}}_\alpha \cdot \left\{ \left(\nabla_{\tilde{\mathbf{v}}} \tilde{f}_\epsilon - \nabla_{\bar{\mathbf{v}}} \bar{f}_\epsilon \right) S(\tilde{\mathbf{v}}_\epsilon) - \nabla_{\bar{\mathbf{v}}} \left[S(\tilde{\mathbf{v}}_\epsilon) \left(\tilde{f}_\epsilon - \bar{f}_\epsilon \right) \right] \right\}. \quad (122)$$

The second-order integrals read

$$I_2 = I_2^0 + I_2^1 + I_2^2 + I_2^3 + I_2^4 + I_2^5 + I_2^6 + I_2^7, \quad (123)$$

with

$$I_2^0 = - \int_{\mathbb{R}^3} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) S(\tilde{\mathbf{v}}_\epsilon) d\bar{\mathbf{v}}_\alpha = -\bar{n}_\alpha S(\tilde{\mathbf{v}}_\epsilon) (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon),$$

$$\begin{aligned} I_2^1 &= \frac{1}{2} \int_{\mathbb{R}^3} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) (\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) \otimes (\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) : \nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) S(\tilde{\mathbf{v}}_\epsilon) d\bar{\mathbf{v}}_\alpha \\ &= \frac{1}{2} \bar{n}_\alpha (\bar{\mathbf{u}}_\alpha \otimes \bar{\mathbf{u}}_\alpha + \bar{\mathbf{T}}_\alpha) : \left[S(\tilde{\mathbf{v}}_\epsilon) \left(\nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} \tilde{f}_\epsilon - 2 \nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} \tilde{f}_\epsilon + \nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} \tilde{f}_\epsilon \right) \right], \end{aligned}$$

$$I_2^2 = \int_{\mathbb{R}^3} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) (\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) \cdot \nabla_{\bar{\mathbf{v}}} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) S(\tilde{\mathbf{v}}_\epsilon) d\bar{\mathbf{v}}_\alpha = -\bar{n}_\alpha S(\tilde{\mathbf{v}}_\epsilon) (\text{tr}(\bar{\boldsymbol{\Omega}}) - 3) \tilde{f}_\epsilon,$$

$$I_2^3 = \frac{1}{2} \int_{\mathbb{R}^3} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \otimes (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) : \nabla_{\bar{\mathbf{v}}} \otimes \nabla_{\bar{\mathbf{v}}} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) S(\tilde{\mathbf{v}}_\epsilon) d\bar{\mathbf{v}}_\alpha = 0,$$

$$\begin{aligned} I_2^4 &= \int_{\mathbb{R}^3} \left[(\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) \cdot \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) \right] \left[(\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\bar{\mathbf{v}}} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) \right] S(\tilde{\mathbf{v}}_\epsilon) d\bar{\mathbf{v}}_\alpha \\ &= 2 \bar{n}_\alpha S(\tilde{\mathbf{v}}_\epsilon) (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon, \end{aligned}$$

$$\begin{aligned} I_2^5 &= \int_{\mathbb{R}^3} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) \left[(\tilde{\mathbf{v}}_\alpha - \bar{\mathbf{v}}_\alpha) \cdot \nabla_{\tilde{\mathbf{v}}} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) \right] \left[\bar{\mathbf{v}}_\alpha \cdot \nabla_{\bar{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) \right] d\bar{\mathbf{v}}_\alpha \\ &= \bar{n}_\alpha (\bar{\mathbf{u}}_\alpha \otimes \bar{\mathbf{u}}_\alpha + \bar{\mathbf{T}}_\alpha) : \left[\left(\nabla_{\tilde{\mathbf{v}}} \tilde{f}_\epsilon - \nabla_{\bar{\mathbf{v}}} \tilde{f}_\epsilon \right) \otimes \nabla_{\bar{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) \right], \end{aligned}$$

$$\begin{aligned} I_2^6 &= \int_{\mathbb{R}^3} \bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) \left[(\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\bar{\mathbf{v}}} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) \right] \left[\bar{\mathbf{v}}_\alpha \cdot \nabla_{\bar{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) \right] d\bar{\mathbf{v}}_\alpha \\ &= -\bar{n}_\alpha (\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \cdot \nabla_{\bar{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) \tilde{f}_\epsilon, \end{aligned}$$

$$\begin{aligned} I_2^7 &= \frac{1}{2} \int_{\mathbb{R}^3} \bar{f}_\alpha(\bar{\mathbf{v}}_\alpha) \left(\bar{f}_\epsilon(\tilde{\mathbf{v}}_\epsilon) - \bar{f}_\epsilon(\bar{\mathbf{v}}_\epsilon) \right) \left[(\bar{\mathbf{v}}_\alpha \otimes \bar{\mathbf{v}}_\alpha) : \nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon) \right] d\bar{\mathbf{v}}_\alpha \\ &= \frac{1}{2} \left(\tilde{f}_\epsilon - \bar{f}_\epsilon \right) \bar{n}_\alpha (\bar{\mathbf{u}}_\alpha \otimes \bar{\mathbf{u}}_\alpha + \bar{\mathbf{T}}_\alpha) : \nabla_{\tilde{\mathbf{v}}} \otimes \nabla_{\tilde{\mathbf{v}}} S(\tilde{\mathbf{v}}_\epsilon). \end{aligned}$$

We can regroup terms as follows,

$$I_2^0 + I_2^2 + I_2^4 + I_2^6 = \nabla_{\tilde{\mathbf{v}}} \cdot \left[(\tilde{\mathbf{v}}_\epsilon - \bar{\mathbf{v}}_\epsilon) \bar{n}_\alpha S(\tilde{\mathbf{v}}_\epsilon) \tilde{f}_\epsilon \right]. \quad (125)$$

Similarly,

$$I_2^1 + I_2^5 + I_2^7 = \bar{n}_\alpha (\bar{\mathbf{u}}_\alpha \otimes \bar{\mathbf{u}}_\alpha + \bar{\mathbf{T}}_\alpha) : \left(\nabla_{\bar{\mathbf{v}}} \otimes \left\{ S(\tilde{\mathbf{v}}_\epsilon) \left[\tilde{f}_\epsilon - \bar{f}_\epsilon \right] \right\} + \frac{1}{2} \nabla_{\bar{\mathbf{v}}} \left[S(\tilde{\mathbf{v}}_\epsilon) \left(\tilde{f}_\epsilon - \bar{f}_\epsilon \right) \right] - \frac{1}{2} S(\tilde{\mathbf{v}}_\epsilon) \left(\nabla_{\bar{\mathbf{v}}} \otimes \nabla_{\bar{\mathbf{v}}} \tilde{f}_\epsilon - \nabla_{\bar{\mathbf{v}}} \otimes \nabla_{\bar{\mathbf{v}}} \bar{f}_\epsilon \right) \right). \quad (126)$$

Finally, in order to obtain the operators of (22-23), we need to integrate over the unit sphere defined by the vector $\boldsymbol{\omega}$. Note that in these operators we have defined the collision frequency, as follows,

$$\bar{v}_{\epsilon\alpha} := \bar{n}_\alpha S(\tilde{\mathbf{v}}_\epsilon) = \bar{n}_\alpha \bar{v}_\epsilon \bar{\sigma}_{\epsilon\alpha}(\bar{v}_\epsilon, \boldsymbol{\omega} \cdot \hat{\mathbf{v}}_\epsilon). \quad (127)$$

Here, we have used the fact that the norm of the velocity vector is conserved in the rotation, i.e., $\|\tilde{\mathbf{v}}_\epsilon\| = \bar{v}_\epsilon$ and we have defined the unit vectors $\hat{\mathbf{v}}_\epsilon = \tilde{\mathbf{v}}_\epsilon / \bar{v}_\epsilon$ and $\boldsymbol{\omega} = \tilde{\mathbf{v}}_\epsilon / \bar{v}_\epsilon$.

B Closing the collision integrals with QMOM

B.1 Elastic electron-heavy collisions

The coefficients read

$$K_{\mathbf{e}\mathbf{g},i}^{2,el,(2l)}(v_i) = 8(l-1)\pi n_{\mathbf{g}} v_i^{2l-1} \left[(2lT_{\mathbf{g}} - v_i^2) Q_{1,\mathbf{g}}(v_i) + T_{\mathbf{g}} v_i Q'_{1,\mathbf{g}}(v_i) \right].$$

Indeed, starting from (28), one computes

$$\begin{aligned} p^{(2l)}(\bar{C}_{\mathbf{e}\mathbf{g}}^{2,el}(f_R)) &= \int_{\mathbb{R}^3} \bar{v}^{2l-2} \partial_{\bar{v}} \left(n_{\mathbf{g}} Q_{1,\mathbf{g}}(\bar{v}) \bar{v}^3 (\bar{v} f_R + T_{\mathbf{g}} \partial_{\bar{v}} f_R) \right) \\ &= -8(l-1)\pi n_{\mathbf{g}} \int_{\mathbb{R}^+} \bar{v}^{2l} Q_{1,\mathbf{g}}(\bar{v}) (\bar{v} f_R + T_{\mathbf{g}} \partial_{\bar{v}} f_R) \\ &= 8(l-1)\pi n_{\mathbf{g}} \int_{\mathbb{R}^+} \bar{v}^{2l-1} \left((2lT_{\mathbf{g}} - \bar{v}^2) Q_{1,\mathbf{g}}(\bar{v}) + T_{\mathbf{g}} \bar{v} Q'_{1,\mathbf{g}}(\bar{v}) \right) f_R \\ &= 8(l-1)\pi n_{\mathbf{g}} \sum_i m_i v_i^{2l-1} \left((2lT_{\mathbf{g}} - v_i^2) Q_{1,\mathbf{g}}(v_i) + T_{\mathbf{g}} v_i Q'_{1,\mathbf{g}}(v_i) \right). \end{aligned}$$

B.2 Elastic electron-electron collisions

The coefficients read

$$\begin{aligned} K_{\mathbf{e}\mathbf{e},i,j}^{2,el,(2l)}(v_i, v_j) &= \int_{(\mathcal{S}^2)^3} \left| \frac{v_i \omega_1 + v_j \omega_2}{2} + \frac{v_i - v_j}{2} \omega_3 \right|^{2l} \\ &\quad \times \sigma_{\mathbf{e}\mathbf{e}} \left(|v_i \omega_1 - v_j \omega_2|, \omega_3 \cdot \frac{v_i \omega_1 - v_j \omega_2}{|v_i \omega_1 - v_j \omega_2|} \right) |v_i \omega_1 - v_j \omega_2| d\omega_1 d\omega_2 d\omega_3 \\ &\quad - v_i^{2l} \int_{(\mathcal{S}^2)^3} \sigma_{\mathbf{e}\mathbf{e}} \left(|v_i \omega_1 - v_j \omega_2|, \omega_3 \cdot \frac{v_i \omega_1 - v_j \omega_2}{|v_i \omega_1 - v_j \omega_2|} \right) |v_i \omega_1 - v_j \omega_2| d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (128)$$

First, the conservation of momentum and energy yields velocities $\mathbf{v}' = \frac{\mathbf{v} + \mathbf{v}_*}{2} + \frac{|\mathbf{v} - \mathbf{v}_*|}{2} \boldsymbol{\omega}$. Denote v' and v'_* their norm, then the considered integrals read

$$p^{(2l)}(\bar{C}_{\mathbf{e}\mathbf{e}}^{el}(f_R)) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} \bar{v}^{2l} \sigma_{\mathbf{e}\mathbf{e}} \left(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\omega} \cdot \frac{\mathbf{v} - \mathbf{v}_*}{|\mathbf{v} - \mathbf{v}_*|} \right) |\mathbf{v} - \mathbf{v}_*| [f_R(v'_*) f_R(v') - f_R(v_*) f_R(v)].$$

Using the micro-reversibility and the chosen ansatz, this simplifies into (101b) with the coefficients (128).

Especially, the conservation of mass and energy yield $K_{i,j}^{(0)} = 0 = K_{i,j}^{(2)}$, while the higher-order moments are a priori not preserved.

B.3 Inelastic collisions

For the third, the coefficients read

$$K_{\epsilon\alpha,i}^{0,inel,(2l)} = 16\pi^2 n_{\mathbf{g}} \sum_{k \in \mathcal{Q}_g^*} \left((v_i^2 + 2\phi_k^*)^l \tilde{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow k} \left(\sqrt{v_i^2 + 2\phi_k^*} \right) - v_i^{2l+1} \tilde{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow k}(v_i) \right).$$

Indeed, starting from (36), one computes

$$\begin{aligned} p^{(2l)}(\bar{C}_{\epsilon\mathbf{g}}^{0,inel}(f_R)) &= n_{\mathbf{g}} \int_{\mathbb{R}^3} v^{2l} \sum_{k \in \mathcal{Q}_g^*} \int_{\mathcal{S}^2} \bar{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow k}(v, \omega \cdot \hat{\mathbf{v}}_{\epsilon}) v \left[f_R \left(\sqrt{v^2 - 2\phi_k^*} \right) - f_R(v) \right] \\ &= 16\pi^2 n_{\mathbf{g}} \sum_{k \in \mathcal{Q}_g^*} \int_{\mathbb{R}^+} v^{2l+1} \tilde{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow k}(v) \left[f_R \left(\sqrt{v^2 - 2\phi_k^*} \right) - f_R(v) \right] \\ &= 16\pi^2 n_{\mathbf{g}} \sum_{k \in \mathcal{Q}_g^*} \left[\left(\int_{\mathbb{R}^+} (w^2 + 2\phi_k^*)^l \tilde{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow k} \left(\sqrt{w^2 + 2\phi_k^*} \right) f_R(w) \right) \right. \\ &\quad \left. - \left(\int_{\mathbb{R}^+} v^{2l+1} \tilde{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow k}(v) f_R(v) \right) \right] \\ &= 16\pi^2 n_{\mathbf{g}} \sum_i m_i \sum_{k \in \mathcal{Q}_g^*} \left((v_i^2 + 2\phi_k^*)^l \tilde{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow k} \left(\sqrt{v_i^2 + 2\phi_k^*} \right) - v_i^{2l+1} \tilde{\sigma}_{\epsilon\mathbf{g}}^{0 \rightarrow k}(v_i) \right). \end{aligned}$$

B.4 Ionization

Below, we use the following parametrization for the ionization cross section, as proposed by Alexeev [18]

$$W_{\mathbf{g}\epsilon}^{\text{ie}\epsilon} dv'_1 dv'_{\mathbf{e}_1} dv'_{\mathbf{e}_2} = \frac{\sigma_{\epsilon\mathbf{g}}^{\text{ie}\epsilon}}{\frac{16\pi^2}{3} \left(\frac{\mu_{\epsilon\mathbf{g}} g_{\epsilon\mathbf{g}}^2 - e\phi_{iz}^*}{\mu_{\text{ie}\epsilon}} \right)^{3/2}} g_{\epsilon\mathbf{g}} g_{\text{ie}\epsilon}^2 dg_{\text{ie}\epsilon} d\omega_{\text{ie}\epsilon} d\omega_{\mathbf{e}_1\mathbf{e}_2} \quad (129)$$

where $\mu_{\text{ie}\epsilon} = m_{\epsilon} m_i / (m_{\mathbf{g}} + m_{\epsilon})$, the relative velocities between the particles are defined as $\mathbf{g}_{\epsilon\mathbf{g}} = \mathbf{v}_{\epsilon} - \mathbf{v}_{\mathbf{g}}$, $\mathbf{g}_{\mathbf{e}_1\mathbf{e}_2} = \mathbf{v}_i - (\mathbf{v}_{\mathbf{e}_1} + \mathbf{v}_{\mathbf{e}_2})/2$, and $\mathbf{g}_{\mathbf{e}_1\mathbf{e}_2} = \mathbf{v}_{\mathbf{e}_1} - \mathbf{v}_{\mathbf{e}_2}$. The unit vector in the direction of the relative velocities are $\mathbf{e} = \mathbf{g}_{\epsilon\mathbf{g}}/g_{\epsilon\mathbf{g}}$, $\omega_{\text{ie}\epsilon} = \mathbf{g}_{\text{ie}\epsilon}/g_{\text{ie}\epsilon}$, and $\omega_{\mathbf{e}_1\mathbf{e}_2} = \mathbf{g}_{\mathbf{e}_1\mathbf{e}_2}/g_{\mathbf{e}_1\mathbf{e}_2}$. Following Graille et al. [58, 82], the cross-section $\sigma_{\epsilon\mathbf{g}}^{\text{ie}\epsilon}$ is commonly assumed to depend only on the parameters the norm relative velocities and the deflection angles, i.e. on $(g_{\epsilon\mathbf{g}}, g_{\mathbf{e}_1\mathbf{e}_2}, \omega_{\text{ie}\epsilon} \cdot \mathbf{e}, \omega_{\mathbf{e}_1\mathbf{e}_2} \cdot \mathbf{e}, \omega_{\text{ie}\epsilon} \cdot \omega_{\mathbf{e}_1\mathbf{e}_2})$.

Finally, for the ionizing collisions, the

$$\begin{aligned} K_{\epsilon\mathbf{g},i}^{0,iz,(2l)} &= \frac{3\bar{n}_{\mathbf{g}}}{4\pi} \int_{\mathcal{S}^2} \int_{\mathcal{S}^2} \left(2 \int_0^{\sqrt{\frac{v_{e_1}^{-1}(v_i)}{2} - \bar{\phi}_{iz}^*}} \frac{v_{e_1}^{-1}(v_i)^{2l+1} \sigma_{\epsilon\mathbf{g}}^{\text{ie}\epsilon,0}(v_{e_1}^{-1}(v_i))}{\left(\frac{1}{2}v_{e_1}^{-1}(v_i) - \bar{\phi}_{iz}^*\right)^{3/2}} \bar{g}_{\text{ie}\epsilon}^2 \frac{dv_{e_1}^{-1}}{dv}(v_i) d\bar{g}_{\text{ie}\epsilon} \right. \\ &\quad \left. - \int_0^{\sqrt{\frac{v_i}{2} - \bar{\phi}_{iz}^*}} \frac{v_i^{2l+1} \sigma_{\epsilon\mathbf{g}}^{\text{ie}\epsilon,0}(v_i)}{\left(\frac{1}{2}v_i - \bar{\phi}_{iz}^*\right)^{3/2}} \bar{g}_{\text{ie}\epsilon}^2 d\bar{g}_{\text{ie}\epsilon} \right) d\omega_{\text{ie}\epsilon} d\omega_{\mathbf{e}_1\mathbf{e}_2}. \end{aligned}$$

where $v_{e_1}^{-1}(v) = 2 \left((v + \bar{g}_{\text{ie}\epsilon} \omega_{\text{ie}\epsilon})^2 + \phi_{iz}^* + \bar{g}_{\text{ie}\epsilon} \right)$ such that $\frac{dv_{e_1}^{-1}}{dv}(v) = 4 |v + \bar{g}_{\text{ie}\epsilon} \omega_{\text{ie}\epsilon}|$.

Indeed, starting from (46), one computes

$$\begin{aligned} p^{(2l)}(\bar{C}_{\epsilon,\mathbf{g}}^{0,iz}(\bar{f}_{\epsilon})) &= \frac{3\bar{n}_{\mathbf{g}}}{4\pi} \int_{\mathbb{R}^+} \int_{\mathcal{S}^2} \int_{\mathcal{S}^2} \int_0^{\sqrt{\frac{1}{2}v - \bar{\phi}_{iz}^*}} \frac{v^{2l+1} \sigma_{\epsilon\mathbf{g}}^{\text{ie}\epsilon,0}}{\left(\frac{1}{2}v - \bar{\phi}_{iz}^*\right)^{3/2}} [2f_R(v_{e_1}(v)) - f_R(v)] \\ &\quad \times \bar{g}_{\text{ie}\epsilon}^2 d\bar{g}_{\text{ie}\epsilon} d\omega_{\text{ie}\epsilon} d\omega_{\mathbf{e}_1\mathbf{e}_2}, \end{aligned}$$

where $v_{e_1}(v) = \left| -g_{\text{icc}}\omega_{\text{icc}} + \sqrt{\frac{v}{2} - \bar{\phi}_{iz}^* - g_{\text{icc}}^2\omega_{\mathbf{e}_1\mathbf{e}_2}} \right|$. The second term simply reads

$$\begin{aligned}
& \int_{\mathbb{R}^+} \int_{\mathcal{S}^2 \times \mathcal{S}^2} \int_0^{\sqrt{\frac{1}{2}v - \bar{\phi}_{iz}^*}} \frac{v^{2l+1} \sigma_{\mathbf{e}\mathbf{g}}^{\text{icc},0}}{\left(\frac{1}{2}v - \bar{\phi}_{iz}^*\right)^{3/2}} f_R(v) \bar{g}_{\text{icc}}^2 d\bar{g}_{\text{icc}} d\omega_{\text{icc}} d\omega_{\mathbf{e}_1\mathbf{e}_2} \\
&= \int_{\mathcal{S}^2 \times \mathcal{S}^2} \int_{\mathbb{R}^+} \int_0^{+\infty} \frac{v^{2l+1} \sigma_{\mathbf{e}\mathbf{g}}^{\text{icc},0}}{\left(\frac{1}{2}v - \bar{\phi}_{iz}^*\right)^{3/2}} f_R(v) \bar{g}_{\text{icc}}^2 d\bar{g}_{\text{icc}} d\omega_{\text{icc}} d\omega_{\mathbf{e}_1\mathbf{e}_2} \\
&= \sum_i m_i \int_{\mathcal{S}^2 \times \mathcal{S}^2} \int_{\mathbb{R}^+} \mathbf{1}_{\mathbb{R}^+} \left(v_i - 2(\bar{g}_{\text{icc}}^2 + \bar{\phi}_{iz}^*) \right) \frac{v_i^{2l+1} \sigma_{\mathbf{e}\mathbf{g}}^{\text{icc},0}(v_i)}{\left(\frac{1}{2}v_i - \bar{\phi}_{iz}^*\right)^{3/2}} \bar{g}_{\text{icc}}^2 d\bar{g}_{\text{icc}} d\omega_{\text{icc}} d\omega_{\mathbf{e}_1\mathbf{e}_2} \\
&= \sum_i m_i \int_{\mathcal{S}^2 \times \mathcal{S}^2} \int_0^{\sqrt{\frac{v_i}{2} - \bar{\phi}_{iz}^*}} \frac{v_i^{2l+1} \sigma_{\mathbf{e}\mathbf{g}}^{\text{icc},0}(v_i)}{\left(\frac{1}{2}v_i - \bar{\phi}_{iz}^*\right)^{3/2}} \bar{g}_{\text{icc}}^2 d\bar{g}_{\text{icc}} d\omega_{\text{icc}} d\omega_{\mathbf{e}_1\mathbf{e}_2}
\end{aligned}$$

The first integral needs a change of variable on top of it and yields

$$\begin{aligned}
& \int_{\mathbb{R}^+} \int_{\mathcal{S}^2 \times \mathcal{S}^2} \int_0^{\sqrt{\frac{1}{2}v - \bar{\phi}_{iz}^*}} \frac{v^{2l+1} \sigma_{\mathbf{e}\mathbf{g}}^{\text{icc},0}}{\left(\frac{1}{2}v - \bar{\phi}_{iz}^*\right)^{3/2}} f_R(v_{e_1}(v)) \bar{g}_{\text{icc}}^2 d\bar{g}_{\text{icc}} d\omega_{\text{icc}} d\omega_{\mathbf{e}_1\mathbf{e}_2} \\
&= \sum_i m_i \int_{\mathcal{S}^2 \times \mathcal{S}^2} \int_0^{\sqrt{\frac{v_{e_1}^{-1}(v_i)}{2} - \bar{\phi}_{iz}^*}} \frac{v_{e_1}^{-1}(v_i)^{2l+1} \sigma_{\mathbf{e}\mathbf{g}}^{\text{icc},0}(v_{e_1}^{-1}(v_i))}{\left(\frac{1}{2}v_{e_1}^{-1}(v_i) - \bar{\phi}_{iz}^*\right)^{3/2}} \bar{g}_{\text{icc}}^2 \frac{dv_{e_1}^{-1}}{dv}(v_i) d\bar{g}_{\text{icc}} d\omega_{\text{icc}} d\omega_{\mathbf{e}_1\mathbf{e}_2}.
\end{aligned}$$

C Hermitian closure and collisional integrals

C.1 Irreducible Hermite polynomials: definitions and closure

As described by Balescu [50], the irreducible scalar Hermite polynomials are written as

$$H^{(2l)}(\mathcal{V}) = \beta_{2l} \sum_{m=0}^l (-1)^{m+l} \frac{l!}{m!(l-m)!} \frac{(2l+1)!!}{(2m+1)!!} \mathcal{V}^{2m} \quad \text{with} \quad \beta_{2l} = \left(\frac{1}{2^l l! (2l+1)!!} \right)^{1/2}, \quad (130)$$

which fulfill the orthogonality relation

$$\int_{\mathbb{R}^3} \phi^0(\mathcal{V}) H^{(2l)} H^{(2m)} d\mathcal{V} = \delta_{lm}. \quad (131)$$

We can invert the definition of (130) to obtain the monomial velocity moments as function of the Hermitian moments, as follows,

$$\mathcal{V}^{2l} = \sum_{m=0}^l \frac{1}{\beta_{2m}} \frac{l!}{m!(l-m)!} \frac{(2l+1)!!}{(2m+1)!!} H^{(2m)}. \quad (132)$$

With these relations, one can compute the relation between the (monomial) velocity moments $p^{(2l)}$ defined in (79), and the non-dimensional scalar Hermitian moments $h^{(2m)}$ (112) for a moment closure considering N moments, as follows,

$$p^{(2l)} = nT^l \left[(2l+1)!! + \sum_{m=2}^l \frac{1}{\beta_{2m}} \frac{l!}{m!(l-m)!} \frac{(2l+1)!!}{(2m+1)!!} h^{(2m)} \right]. \quad (133)$$

Note that the sum does not take into account the first two Hermite moments as they are $h^{(0)} = h^{(2)} = 0$.

Similarly, we can retrieve the inverse relation, which reads,

$$h^{(2l)} = \beta_{2l} \sum_{m=2}^l (-1)^{m+l} \frac{l!}{m!(l-m)!} \frac{(2l+1)!!}{(2m+1)!!} \left[\frac{p^{(2m)}}{nT^m} - (2m+1)!! \right]. \quad (134)$$

The system of moments is closed with $p^{(2N+2)}$ that needs to be computed as a function of $p^{(2l)}$ with $l \in \{0, 1, \dots, N\}$. With (133) and (134), we can write

$$p^{(2N+2)} = nT^{N+1} \left[(2N+3)!! + \sum_{m=2}^N \frac{1}{\beta_{2m}} \frac{(N+1)!}{m!(N+1-m)!} \frac{(2N+3)!!}{(2m+1)!!} h^{(2m)} \right] = nT^{N+1} \alpha_{N+1} + \sum_{m=2}^N \sum_{l=2}^m \alpha_{lm} T^{N+1-l} p^{(2l)}, \quad (135)$$

Where the coefficients α_{N+1} and α_{lm} are fractions that are obtained by using (134) and read

$$\alpha_{N+1} = \left\{ (2N+3)!! - \sum_{m=2}^N \sum_{l=2}^m \frac{(-1)^{m+l} (N+1)! (2N+3)!!}{l!(m-l)!(N+1-m)!} \right\}, \quad (136a)$$

$$\alpha_{lm} = \frac{(-1)^{m+l} (N+1)! (2N+3)!!}{l!(m-l)!(N+1-m)!(2l+1)!!}. \quad (136b)$$

C.2 Collisional terms in the Hermitian moment model

The Hermitian moment closure of the collisional terms, i.e., right-hand-side of (97), is given in the following propositions.

Proposition 10. *The electron-neutral elastic collision term in the moment equations with the Hermitian expansion of the zero-th order distribution function, given in (110), reads as follows*

$$p^{(2l)} (C_{\epsilon, \mathbf{g}}^{2,el}(f^0)) = -2l\nu_{\epsilon \mathbf{g}} \left\{ \left(1 - \frac{T_{\mathbf{g}}}{T} \right) p^{(2l)} + \frac{T_{\mathbf{g}}}{T} \left[p^{(2l)} - (2l+1)T p^{(2l-2)} \right] \right\} \quad (137)$$

for $l \in \{1, \dots, N\}$

and $p^{(0)} (C_{\epsilon, \mathbf{g}}^{2,el}(f^0)) = 0$.

Proof. For this proof, we use the result of Corollary 1 with Maxwell molecules, and we take the moment,

$$\begin{aligned} p^{(2l)} (C_{\epsilon, \mathbf{g}}^{2,el}(f^0)) &= \nu_{\epsilon \mathbf{g}} \int_{\mathbb{R}^3} v_{\epsilon}^{2l-2} \frac{\partial}{\partial v_{\epsilon}} \left\{ v_{\epsilon}^2 \left(v_{\epsilon} f^0 + T_{\mathbf{g}} \frac{\partial f^0}{\partial v_{\epsilon}} \right) \right\} dv_{\epsilon} \\ &= -2l\nu_{\epsilon \mathbf{g}} \int_{\mathbb{R}^3} v_{\epsilon}^{2l} \left(f^0 + \frac{T_{\mathbf{g}}}{v_{\epsilon}} \frac{\partial f^0}{\partial v_{\epsilon}} \right) dv_{\epsilon}. \end{aligned} \quad (138)$$

In order to obtain the previous relation, we have used spherical coordinates for the integration in the velocity space and the fact that $\int_0^{\infty} \frac{\partial}{\partial v_{\epsilon}} \left\{ v_{\epsilon}^{2l+2} \left(v_{\epsilon} f^0 + T_{\mathbf{g}} \frac{\partial f^0}{\partial v_{\epsilon}} \right) \right\} dv_{\epsilon} = 0$. By injecting the Hermite expansion of (110) into the previous integral and after some algebra, we obtain (137). Finally, to obtain (137), we have injected the Hermitian expansion into (138). For $l = 0$, the contribution is zero, as the elastic collision conserves the number of electrons. \square

Proposition 11. *The electron-neutral inelastic collision term in the moment equations with the Hermitian expansion of the zero-th order distribution function, for general cross-sections, reads as follows*

$$p^{(2l)}(C_{\mathbf{e},\mathbf{g}}^{0,inel}(f^0)) = n_{\mathbf{g}}nT^l \sum_{m \in \mathcal{Q}_{\mathbf{g}}^*} \sum_{r=0}^{l-1} \frac{2^l l! (-1)^{l-r}}{r!(l-r)!} K_{inel,m}^{(r)} \left(\frac{\phi_m^*}{T} \right)^{l-r} \quad (139)$$

for $l \in \{1, \dots, N\}$ where the rates $K_{inel,m}^{(r)}(T, p^{(4)}, \dots, p^{(2N)})$ are functions of the moments, and they are computed by integrating the cross-section over the norm of the electron velocity, as follows,

$$\bar{K}_{inel,m}^{(r)} = 4\pi \left(\frac{1}{2T} \right)^r \int_0^\infty v_{\mathbf{e}}^{2r+3} \frac{f^0}{n} Q_{\mathbf{e}\mathbf{g}}^{0 \rightarrow m} dv_{\mathbf{e}}$$

$$\text{with the total cross section} \quad Q_{\mathbf{e}\mathbf{g}}^{0 \rightarrow m}(\bar{v}_{\mathbf{e}}) = \int_{S^2} \sigma_{\mathbf{e}\mathbf{g}}^{0 \rightarrow m} d\omega.$$

Proof. The moment of the inelastic operator of the transition from the ground state to the m -th excited state reads:

$$p^{(2l)}(C_{\mathbf{e},\mathbf{g}}^{0,inel}(f^0)) = \int_{\mathbb{R}^3} \int_{S^2} v_{\mathbf{e}}^{2l} \sigma_{\mathbf{e}\mathbf{g}}^{0 \rightarrow m}(v_{\mathbf{e}}, \boldsymbol{\omega} \cdot \hat{v}_{\mathbf{e}}) v_{\mathbf{e}} [f^0(\bar{v}_{\mathbf{e}}^{m,inel}) - f^0(v_{\mathbf{e}})] d\boldsymbol{\omega} dv_{\mathbf{e}}. \quad (140)$$

By using the reciprocity relations, we can write the previous integral as

$$p^{(2l)}(C_{\mathbf{e},\mathbf{g}}^{0,inel}(f^0)) = \int_{\mathbb{R}^3} \int_{S^2} (\bar{v}_{\mathbf{e}}^{2l} - v_{\mathbf{e}}^{2l}) \sigma_{\mathbf{e}\mathbf{g}}^{0 \rightarrow m}(v_{\mathbf{e}}, \boldsymbol{\omega} \cdot \hat{v}_{\mathbf{e}}) v_{\mathbf{e}} f^0(v_{\mathbf{e}}) d\boldsymbol{\omega} dv_{\mathbf{e}}. \quad (141)$$

with the energy conservation at the zero-th order $\bar{v}_{\mathbf{e}}^2 = v_{\mathbf{e}}^2 - 2\phi_m^*$. With the binomial theorem and the energy conservation, one obtains the following relation

$$\bar{v}_{\mathbf{e}}^{2l} - v_{\mathbf{e}}^{2l} = \sum_{r=0}^{l-1} \binom{l}{r} v_{\mathbf{e}}^{2r} (-2\phi_m^*)^{l-r}. \quad (142)$$

By injecting the previous relation into (141), after some algebra, one retrieves (139). \square

Proposition 12. *The electron-neutral ionization collision term in the moment equations with the Hermitian expansion of the zero-th order distribution function, for general cross-sections, reads as follows*

$$p^{(2l)}(C_{\mathbf{e},\mathbf{g}}^{0,iz}(f^0)) = \begin{cases} n_{\mathbf{g}}nK_{iz}^{(0)} & \text{for } l = 0 \\ n_{\mathbf{g}}nT^l \left\{ a_{iz}^{(l)} K_{iz}^{(l)} + \sum_{r=0}^{l-1} a_{iz}^{(r)} K_{iz}^{(r)} \left(\frac{\phi_{iz}^*}{T} \right)^{l-r} \right\} & \text{for } l \in \{1, \dots, N\} \end{cases} \quad (143)$$

where the rates $K_{iz,m}^{(r)}(T, p^{(4)}, \dots, p^{(2N)})$ are functions of the moments, and they are computed by integrating the cross-section over the norm of the electron velocity, as follows,

$$\bar{K}_{iz,m}^{(r)} = 4\pi \left(\frac{1}{2T} \right)^r \int_0^\infty v_{\mathbf{e}}^{2r+3} \frac{f^0}{n} Q_{\mathbf{e}\mathbf{g}}^{iz} dv_{\mathbf{e}} \quad \text{with the total cross section} \quad Q_{\mathbf{e}\mathbf{g}}^{iz}(\bar{v}_{\mathbf{e}}) = \int d\sigma_{iz}^0. \quad (144)$$

The coefficients a_{iz} depend on the energy sharing between the two resulting electrons. In the case of equal-sharing and zero-sharing, the coefficients read:

$$a_{iz}^{(l)} = \begin{cases} (2 - 2^l) & \text{for equal-sharing} \\ 0 & \text{for zero-sharing} \end{cases} \quad (145a)$$

$$a_{iz}^{(r)} = \begin{cases} \frac{2^l l! (-1)^{l-r}}{r!(l-r)!} & \text{for equal-sharing} \\ \frac{2^l l! (-1)^{l-r}}{r!(l-r)!} & \text{for zero-sharing} \end{cases} \quad \text{for } r \in \{1, \dots, l-1\} \quad (145b)$$

Proof. The moment of the ionization operator reads:

$$p^{(2l)}(C_{\epsilon, \mathbf{g}}^{0, iz}(f^0)) = \int_{\mathbb{R}^3} v_{\epsilon}^{2l} (2f^0(v'_{\epsilon 1}) - f^0(v_{\epsilon})) Q_{\epsilon \mathbf{g}}^{iz}(v_{\epsilon}) v_{\epsilon} d\mathbf{v}_{\epsilon}. \quad (146)$$

By using the reciprocity relations, we can write the previous integral as

$$p^{(2l)}(C_{\epsilon, \mathbf{g}}^{0, inel}(f^0)) = \int_{\mathbb{R}^3} (2v_{\epsilon 1}'^{2l} - v_{\epsilon}^{2l}) Q_{\epsilon \mathbf{g}}^{iz}(v_{\epsilon}) v_{\epsilon} f^0(\mathbf{v}_{\epsilon}) d\mathbf{v}_{\epsilon}. \quad (147)$$

with the energy conservation at the zero-th order that reads, for the two electron-sharing models, as follows:

$$\begin{cases} 2v_{\epsilon 1}'^2 = v_{\epsilon}^2 - 2\phi_{iz}^* & \text{for equal-sharing} \\ v_{\epsilon 1}'^2 = v_{\epsilon}^2 - 2\phi_{iz}^* & \text{for zero-sharing} \end{cases}$$

With the binomial theorem and the energy conservation, one obtains the following relation

$$\begin{cases} 2v_{\epsilon 1}'^{2l} - v_{\epsilon}^{2l} = 2 \sum_{r=0}^l \binom{l}{r} \left(\frac{v_{\epsilon}^2}{2}\right)^r (-\phi_{iz}^*)^{l-r} - v_{\epsilon}^{2l} & \text{for equal-sharing} \\ v_{\epsilon 1}'^{2l} - v_{\epsilon}^{2l} = \sum_{r=0}^l \binom{l}{r} v_{\epsilon}^{2r} (-2\phi_{iz}^*)^{l-r} - v_{\epsilon}^{2l} & \text{for zero-sharing} \end{cases}$$

By injecting the previous relation into the moment of the operator and integrating, after some algebra, one retrieves (143). \square

Proposition 13. *The electron-electron elastic collision term in the moment equations with the Hermitian expansion of the zero-th order distribution function, given in (110), reads as follows*

$$p^{(2l)}(C_{\epsilon, \epsilon}^{el}(f^0, f^0)) = -nT^l \nu_{\epsilon \epsilon}^{(2l)} \quad \text{for } l \in \{2, \dots, N\} \quad (148)$$

where the frequencies $\nu_{\epsilon \epsilon}^{(ln)}(n, T, h^{(2)}, \dots, h^{(2N)})$ are functions of the moments, and they are computed by integrating the electron-electron cross-section over the angles and the electron velocity norm, as follows,

$$\begin{aligned} \bar{\nu}_{\epsilon \epsilon}^{(2l)} &= \frac{1}{nT^l} \sum_{r=0}^{\frac{l-1}{2}} \sum_{m=0}^{l-2r} \frac{l!}{2^m (2r)! m! (l-2r-m)!} \\ &\quad \times \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G^{2(l-2r-m)} g^{2m+2r+1} \mathcal{K}^{(n)} : \underbrace{\mathbf{G} \otimes \dots \otimes \mathbf{G}}_{2r \text{ times}} f_1^0 f_1^0 d\mathbf{G} d\mathbf{g} \end{aligned} \quad (149)$$

with the angular integral of the cross-section

$$\mathcal{K}^{(n)}(\mathbf{g}) = \int_{\mathcal{S}^2} \underbrace{(\mathbf{e} \otimes \dots \otimes \mathbf{e})}_{2r \text{ times}} - \underbrace{(\boldsymbol{\omega} \otimes \dots \otimes \boldsymbol{\omega})}_{2r \text{ times}} \sigma_{\epsilon \epsilon}(g, \boldsymbol{\omega}) d\boldsymbol{\omega} \quad (150)$$

where the unit vector in the direction of the relative velocity $\mathbf{e} = \mathbf{g}/g$ and the distribution functions that are written as function of the integration variables with the transformation

$$\mathbf{v}_{\epsilon} = \mathbf{G} + \frac{1}{2}\mathbf{g} \quad \text{and} \quad \mathbf{v}_{\epsilon 1} = \mathbf{G} - \frac{1}{2}\mathbf{g}. \quad (151)$$

Proof. By using the collision reciprocal relations, the moment of the electron-electron collision operator can be written as,

$$p^{(2l)}(C_{\epsilon, \epsilon}^{el}(f^0, f^0)) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathcal{S}^2} (v_{\epsilon}^{2l} + v_{\epsilon 1}^{2l} - v_{\epsilon}^{2l} - v_{\epsilon 1}^{2l}) g \sigma_{\epsilon \epsilon} f_1^0 f_1^0 d\boldsymbol{\omega} d\mathbf{v}_{\epsilon 1} \mathbf{v}_{\epsilon}, \quad (152)$$

The integral is done by using the center of mass variables, which, after using the energy and momentum conservation, allows for writing the electron velocities as follows

$$\mathbf{v}_e = \mathbf{G} + \frac{1}{2}\mathbf{g}, \quad \mathbf{v}_{e1} = \mathbf{G} - \frac{1}{2}\mathbf{g}, \quad \mathbf{v}'_e = \mathbf{G} + \frac{1}{2}\mathbf{g}', \quad \text{and} \quad \mathbf{v}'_{e1} = \mathbf{G} - \frac{1}{2}\mathbf{g}', \quad (153)$$

with $\mathbf{G} = \frac{1}{2}(\mathbf{v}_e + \mathbf{v}_{e1})$ the velocity of the center of mass and $\mathbf{g} = (\mathbf{v}_e - \mathbf{v}_{e1})$ the relative velocity. Note that the energy conservation yields $\|\mathbf{g}\| = \|\mathbf{g}'\|$. With these relations, we can write,

$$v_e^{2l} = \sum_{k=0}^l \binom{l}{k} \left(G^2 + \frac{1}{4}g^2\right)^k (\mathbf{G} \cdot \mathbf{g})^{l-k} \quad (154a)$$

$$v_{e1}^{2l} = \sum_{k=0}^l (-1)^{l-k} \binom{l}{k} \left(G^2 + \frac{1}{4}g^2\right)^k (\mathbf{G} \cdot \mathbf{g})^{l-k} \quad (154b)$$

$$v_e'^{2l} = \sum_{k=0}^l \binom{l}{k} \left(G^2 + \frac{1}{4}g^2\right)^k (\mathbf{G} \cdot \mathbf{g}')^{l-k} \quad (154c)$$

$$v_{e1}'^{2l} = \sum_{k=0}^l (-1)^{l-k} \binom{l}{k} \left(G^2 + \frac{1}{4}g^2\right)^k (\mathbf{G} \cdot \mathbf{g}')^{l-k} \quad (154d)$$

when $l - k$ is odd, some of the terms of the integral (152) cancel out. As a result, we keep only the terms where $l - k$ is even, as follows

$$v_e'^{2l} + v_{e1}'^{2l} - v_e^{2l} - v_{e1}^{2l} = \sum_{r=0}^{\frac{l-1}{2}} \binom{l}{l-2r} \left(G^2 + \frac{1}{4}g^2\right)^{l-2r} 2 \{(\mathbf{G} \cdot \mathbf{g}')^{2r} - (\mathbf{G} \cdot \mathbf{g})^{2r}\} \quad (155)$$

By using the binomial theorem,

$$\left(G^2 + \frac{1}{4}g^2\right)^{l-2r} = \sum_{m=0}^{l-2r} \binom{l-2r}{m} \frac{1}{2^{2m}} G^{2(l-2r-m)} g^{2m}. \quad (156)$$

Finally, to obtain (148), we use the energy conservation to write $\mathbf{g}' = g\boldsymbol{\omega}$, and hence

$$\{(\mathbf{G} \cdot \mathbf{g}')^{2r} - (\mathbf{G} \cdot \mathbf{g})^{2r}\} = g^{2r} \underbrace{(\mathbf{e} \otimes \dots \otimes \mathbf{e})}_{2r \text{ times}} - \underbrace{(\boldsymbol{\omega} \otimes \dots \otimes \boldsymbol{\omega})}_{2r \text{ times}} : \underbrace{(\mathbf{G} \otimes \dots \otimes \mathbf{G})}_{2r \text{ times}}. \quad (157)$$

After some algebra, and using the fact that the Jacobian of the transformation $(\mathbf{v}_e, \mathbf{v}_{e1}) \rightarrow (\mathbf{G}, \mathbf{g})$ is unity, one obtains the expression in (148). \square

Note that the integral of (148) can be analytically obtained as a function of the Chapman-Cowling Ω -integrals in the case of Hermitian expansions.

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