Observer design for hybrid systems with partially affine forms and known jump times: Applications to walking robots

Gia Quoc Bao Tran, Pauline Bernard, Ricardo G Sanfelice

Observer design for hybrid systems with partially affine forms and known jump times: Applications to walking robots ⋆

Gia Quoc Bao Tran [a], Pauline Bernard [a], Ricardo G. Sanfelice [b]

[a] *Centre Automatique et Systèmes (CAS), Mines Paris - PSL, Paris 75006, France*

[b] *Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064, USA*

Abstract

We propose novel observer designs for hybrid systems, with a certain nonlinear structure that is affine with respect to certain state components and with known jump times, based on decomposing the state into parts exhibiting different observability properties. We assume that the state of the hybrid system can be decomposed into two components. During flows, the first component is independent of the second one and is assumed to be instantaneously observable from the flow output. The second one is required to be either detectable or backward distinguishable via the combination of flows and jumps, from the jump output as well as a fictitious output describing how this second part impacts the first one at jumps. An observer is designed to estimate each component: a high-gain flow-based observer using the flow output estimates the first one and an LMI/KKL-based jump-based observer using an extended jump output estimates the second one. Global exponential convergence and stability of the estimation error in the original coordinates are proven using Lyapunov analysis. The proposed observers are exercised in the problem of estimating the state and uncertainties at impacts in a bipedal walking robot.

Key words: Hybrid system; high-gain observer; observability; detectability; Lyapunov analysis; robotics.

1 Introduction

1.1 Background

Hybrid systems, including switched or impulsive systems, appear in many applications, ranging from mechanical systems with impacts to electrical circuits, communication networks, and biological systems. The observation problem for subclasses of such systems has been widely studied but faces two main challenges: the intricate interconnection of continuous and discrete behaviors, and the dependence of the time domain of solutions on their unknown initial condition. The latter induces the time domains of the system and observer solutions to differ, making both design and analysis of convergence challenging [1]. That is why, very few general results exist apart from [2,3] relying on *gluing* functions, or [4] assuming the flow dynamics are differentially observable. On the other hand, in the favorable case where the solutions' jump times, namely the times at which discrete events appear, are known or detected, the observer's jumps can be triggered at the same time as the system's, leading to the system and observer solutions to share the same time domain. Yet, the complex interaction between continuous and discrete dynamics remains. Such systems include two important classes of hybrid systems:

- *Impulsive (possibly switched) systems*: These are continuous-time systems with discrete events such as switches or impulses occurring at specific time instants. Their observability/determinability is analyzed (possibly with algebraic certificates) in [5–9]. In [10], switching observers are designed using linear matrix inequalities (LMIs) for both continuous- and discrete-time linear systems assuming the full detectability of each mode, while the switching signal is completely known. The work [11] develops parallel

⋆ This paper extends our first results presented at the 22nd IFAC World Congress. Corresponding author G. Q. B. Tran. *Research supported by the French National Research Agency (ANR-23-CE48-0013-01). Research by R. G. Sanfelice partially supported by NSF Grants no. CNS-2039054 and CNS-2111688, by AFOSR Grants nos. FA9550-19-1-0169, FA9550-20-1-0238, FA9550-23-1-0145, and FA9550-23-1-0313, by AFRL Grant nos. FA8651-22-1-0017 and FA8651-23-1-0004, by ARO Grant no. W911NF-20-1-0253, and by DoD Grant no. W911NF-23-1-0158.

Email addresses:
`gia-quoc-bao.tran@minesparis.psl.eu` (Gia Quoc Bao Tran), `pauline.bernard@minesparis.psl.eu` (Pauline Bernard), `ricardo@ucsc.edu` (Ricardo G. Sanfelice).

observers estimating the observable part of each mode under pre-determined transitions of bounded varying intervals, relying on the *determinability* gained after a high enough number of switches. In the linear context, [9] suggests a change of coordinates based on the Kalman decomposition to extract the observable components of each individual mode, which are then estimated during each interval using a Luenberger observer;
- *Continuous-time systems with sampled measurements*: This is a particular class of hybrid systems with an identity jump map. Some observer designs consist in adapting existing continuous-time observers, usually with constraints on the maximal sampling period. For instance, [12] adapts the high-gain observer design into a continuous-discrete extended Kalman filter where exponential stability is achieved if the gain belongs to an interval that contracts when the constant sampling period increases; and [13] proposes an impulsive Luenberger observer for linear systems with quantized measurements that guarantees stability if the varying quantizer transition inter-arrival time is taken sufficiently small. LMI-based impulsive and sample-and-hold designs with varying sampling times are developed in [14,15] respectively, taking into account the Lipschitzness of the system nonlinearity. Moreover, a sample-and-hold observer is designed in [16] with a time-varying adaptation gain that is reset to 1 at each sampling instant and decreases in between sampling events. On the other hand, hybrid designs with correction at sampling times only have also been developed using LMIs, with both a polytopic approach [17] and a grid-based one [18].

Existing results typically assume at least one of the following properties:

- Observability of the flow dynamics or jump dynamics via Lyapunov/LMI-based sufficient conditions, such as [19] or [20, Section 3], but without constructive observability-based criteria to check their solvability;
- Observability of the full state during flows from the flow output only, leading to *flow-based* observers, essentially exploiting the continuous-time high-gain observer design and an average dwell time, i.e., *persistent flowing*; see, for example, [20, Section 4];
- Observability of the full state from the jump output only exploiting the combination of flows and jumps, leading to *jump-based* observers, essentially exploiting discrete-time observers on an equivalent discrete-time system sampled at jump times and exploiting *persistence of jumps*; see, for instance, [20, Section 5] or [21,17,18], including observers for continuous-time systems with sampled measurements;
- Observability exploiting the combination of flows and jumps, which is the scope of this paper.

In fact, the detectability and observability of hybrid systems have been extensively studied and defined in the literature. Roughly speaking, (asymptotic) detectability means that any complete solutions producing the same output must asymptotically converge to each other—it is thus an *incremental* notion—while observability means that the output determines uniquely the solution (possibly over a given time window). Usually, those properties constitute necessary conditions for the existence of an observer depending on its properties (convergence, stability, tunability, robustness, etc.)—see [22]. In [1], detectability is defined for general hybrid systems with nonlinear maps and unknown jump times in a way that makes this property necessary for the existence of an asymptotic observer. Re-parameterization of the solutions is therefore needed to compare them on a common hybrid time domain. In the context of known jump times, it is shown in [23] that detectability reduces to the incremental convergence of solutions with the same time domain and the same output. Stronger notions of detectability could be similarly defined when asking more of the observer, for instance, stability properties as in [24]. When the dynamics and output maps are linear, detectability (resp., observability) in turn reduces to *zero detectability* (resp., *zero observability*) [1] as considered in [9,25]. Also thanks to linearity in the maps, algebraic observability certificates are developed in [5–9,25]. Note that detectability/observability notions have also been developed for hybrid automata, i.e., systems with both continuous states and discrete, event-triggered ones (e.g., [26–28]), but using somewhat different vocabulary.

Indeed, in hybrid systems, state components may exhibit different types of observability properties, associated with the flow and/or jump output(s) or even hidden inside the flow-jump coupling as in this paper. Such information can be exploited in the design of observers and tailor the design to the source of observability. It has been suggested in [29] that for linear time-varying systems, components with different observability properties can be separated from each other by means of decomposition. In the context of output regulation and internal model design, [30] extends these ideas to hybrid systems with linear maps and periodic jumps with outputs during flows only. The goal of an internal model is to extract and capture the dynamics that generate the outputs. In that way, observability decompositions are relevant to extracting the dynamics "seen from the outputs". Indeed, the internal model approach reveals if a component of the dynamics is instantaneously observable during flows from the flow output and if the non-observable component becomes *visible* in the observable states at jumps. The rest of the dynamics, called the *invisible dynamics*, does not play a role in internal model design approach. This idea has been proposed in [30] for hybrid systems with periodic jumps and only a flow output.

[1] Meaning that all solutions with identically zero outputs converge asymptotically to zero when they are complete (resp., are identically zero).

Similar ideas have been exploited in switched systems, where observability can be gained by accumulating information from individual non-observable sub-systems under *persistent switching* [9,11]. Observability-based decompositions are then used to build observers combining the information of each modes. Similarly for hybrid systems with linear maps, [31,32] designs observer based on the decomposition of the state into two parts, one that is instantaneously observable from the flow output, and the other (at least) detectable from the jump output as well as the combination of flows and jumps. Actually, we proved in [33] that the observability brought by the combination of linear flows and jumps can be efficiently quantified by a *hybrid observability Gramian* over a sufficiently large hybrid time window, leading to a *systematic* Kalman-like observer without any need for state decomposition. Unfortunately, such a linear design hardly applies to the nonlinear setting, so we rely here on decomposition-based observers for hybrid systems with *nonlinear* maps, following a preliminary work in [23].

1.2 Contributions of this paper

Our main contributions are as follows. First, the state is assumed to be decomposed into two parts, where the first one is instantaneously observable during flows from the flow output. We explicitly construct and couple observers estimating each mentioned part of the state depending on their observability properties, namely an arbitrarily fast high-gain flow-based observer exploiting the flow output for the first part and a jump-based observer exploiting the (fictitious) extended output for the second one. Input-to-state properties of each observer are exploited to handle the interconnections at jumps. More precisely, for a wide class of hybrid systems with nonlinear maps following a structure that is affine in certain state components (see system (1)), we propose two designs for the jump-based part of the observer: an LMI-based design exploiting a *detectability* property, and a KKL-based one inspired by the Kazantzis-Kravaris/Luenberger (KKL) method [34–36] under an *observability* property. Thanks to general Lyapunov-based conditions to couple observers [23], a high-gain-like result is obtained, where the global exponential stability (GES) of the estimation error is achieved in the original coordinates if the high-gain flow-based observer is pushed sufficiently fast. The main differences with respect to the preliminary work in [23] are: i) We provide an improved version of the LMI-based observer design proposed in [23], followed by constructive methods to implement it; ii) We provide a novel KKL-based design with assumptions made along the real solutions and the capability to converge with an arbitrarily fast rate; and iii) The observers are applied to state and uncertainty estimation in a walking robot.

Notation: Let $\mathbb{R}$ (resp., $\mathbb{N}$) denote the set of real numbers (resp., natural numbers, i.e., $\{0,1,2,\ldots\}$). Let $\mathbb{R}_{\geq a}$ (resp., $\mathbb{R}_{>a}$) denote $[a,+\infty)$ (resp., $(a,+\infty)$); similarly for $\mathbb{N}_{\geq a}$. Denote $\mathbb{R}^{m\times n}$ (resp., $\mathcal{S}^n_{>0}$) as the set of real $m\times n$ (resp., symmetric positive definite real $n\times n$) matrices. For some set S, let $S^{\mathbb{N}}$ be the set of sequences whose elements are in S. Given a set S, $\mathrm{cl}(S)$ is its closure and $\mathrm{int}(S)$ denotes its interior. Denote $\mathbb{B}$ as the closed unit ball centered at the origin of $\mathbb{R}^n$ and $\mathcal{S}+\delta\mathbb{B}$ as the set of points within a distance $\delta>0$ from any point in $\mathcal{S}$. Let $\Re(z)$ and $\Im(z)$ be the real and imaginary parts of the complex variable z, respectively. Let $\sigma_{\min}(A) := \sqrt{\lambda_{\min}(A^\top A)}$ (resp., $\sigma_{\max}(A) := \sqrt{\lambda_{\max}(A^\top A)}$) be the smallest (resp., largest) singular value of matrix A, where $\lambda_{\min}$ (resp., $\lambda_{\max}$) denote the minimal (resp., maximal) eigenvalue. Let $|\cdot|$ be the Euclidean norm and $\|\cdot\|$ the induced matrix norm which coincides with $\sigma_{\max}(\cdot)$. Let $A^\perp$ be the orthogonal complement of matrix A satisfying $A^\perp A=0$ and $A^\perp(A^\perp)^\top>0$, and $A^\dagger$ be the Moore-Penrose inverse of A [37]; if A is square and invertible, then $A^\dagger=A^{-1}$. Let $\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ be the diagonal matrix with entries λ_i, $i=1,2,\ldots,n$. Denote Id_n (resp., $0_{m\times n}$) as the $n\times n$ identity matrix (resp., $m\times n$ zero matrix; the dimensions can be neglected depending on the context. The symbol $\star$ in the matrix inequalities denotes the symmetric block, while in some long derivations, $\star^\top Px$ denotes $x^\top Px$, and $\star^\top x$ denotes $x^\top x$. The notions of class-$\mathcal{K}$ and class-$\mathcal{KL}$ functions used are from [38, Definitions 4.2 and 4.3]. For a given input $t\mapsto u(t)$ to $\dot{x}=f(x,u)$, $\Psi_{f(\cdot,u)}(x_0,t,\tau)$ is the associated flow operator from initial value x_0 at initial time t evaluated *after* τ time units; if f is autonomous, then this notation reduces to $\Psi_f(x_0,\tau)$. The notions of hybrid time domains and hybrid arcs, as well as their properties, can be found in [39, Section 2.2]. For a solution $(t,j)\mapsto x(t,j)$ to a hybrid system (see Definition 1), we denote $\mathrm{dom}\, x$ its domain [39, Definition 2.6], $\mathrm{dom}_t\, x$ (resp., $\mathrm{dom}_j\, x$) the domain's projection on the ordinary time (resp., jump) component, for $j\in\mathbb{N}$, $t_j(x)$ the unique time such that $(t_j(x),j)\in\mathrm{dom}\, x$ and $(t_j(x),j-1)\in\mathrm{dom}\, x$, and $\mathcal{T}_j(x):=\{t\in\mathrm{dom}_t(x):(t,j)\in\mathrm{dom}\, x\}$ the j-th flow interval. The mention of x is omitted when no confusion is possible. A solution x to a hybrid system is *complete* if $\mathrm{dom}\, x$ is unbounded and *Zeno* if it is complete and $\sup \mathrm{dom}_t\, x<+\infty$. For a function $V:\mathbb{R}^{n_\eta}\to\mathbb{R}$ and a hybrid system with state $\eta\in\mathbb{R}^{n_\eta}$ and input u, flow dynamics $\dot{\eta}=f(\eta,u)$ and jump dynamics $\eta^+=g(\eta,u)$, we denote $\dot{V}(\eta,u):=\langle\nabla V(\eta),f(\eta,u)\rangle$ where ∇V denotes the gradient of V as the derivative of V along the flows and $V^+(\eta,u):=V(g(\eta,u))$ as the value of V after a jump.

2 Problem formulation

2.1 Class of hybrid systems

Consider a hybrid system of the form [39]

$$\begin{cases} \left.\begin{aligned} \dot{\xi}_o &= f_o(\xi_o, u_c) \\ \dot{\xi}_{no} &= F_{no}\xi_{no} + U_{cno} \end{aligned}\right\} (\xi, u_c) \in C \\ \left.\begin{aligned} \xi_o^+ &= J_o(\xi_o, u_d) + J_{ono}(\xi_o, u_d)\xi_{no} \\ \xi_{no}^+ &= J_{no}(\xi_o, u_d)\xi_{no} + J_{noo}(\xi_o, u_d) \end{aligned}\right\} (\xi, u_d) \in D \end{cases} \tag{1a}$$

with state $\xi = (\xi_o, \xi_{no}) \in \mathbb{R}^{n_\xi}$, where $n_\xi = n_o + n_{no}$, F_{no} and U_{cno} are constant matrices, C and D are the flow and jump sets, respectively, with the flow and jump outputs respectively

$$y_c = h_c(\xi_o, u_c), \tag{1b}$$
$$y_d = H_{do}(\xi_o, u_d) + H_{dno}(\xi_o, u_d)\xi_{no}. \tag{1c}$$

Remark 1. *Model* (1) *covers time-varying systems [40,41] (by including the times t, j in the inputs or the state), impulsive and switched systems with state jumps as in [42] (by treating the switching signal as a known input), and continuous-time systems with (possibly multi-rate) sampled outputs, with* fast *sampled outputs treated as y_c and other* sporadic *outputs treated as y_d (by considering the sampling events as jumps triggering the availability of y_d)—see for instance [33, Example 3] for this modeling.*

Model (1) covers a wide class of hybrid systems, including mechanical systems with impacts, in which case the state ξ_o, typically containing positions and velocities, is observable during flows and some unknown constant parameters, captured by ξ_{no}, affect ξ_o at jumps in an affine way. These parameters typically become detectable from y_c through the way they affect ξ_o at jumps, namely from the *fictitious* output $J_{ono}(\xi_o, u_d)\xi_{no}$.

The jump times of each solution to system (1) are assumed to be detected without delay. Note that the robustness with respect to delays in jump detection is discussed in Remark 3. The outputs y_c and y_d may either come from sensor measurements or other known information about the state, for instance, the flow and jump conditions, which encode the fact that $(\xi, u_c) \in C$ and $(\xi, u_d) \in D$, respectively.

Let us define solutions to a hybrid system with inputs, which are considered in this paper.

Definition 1. *The hybrid arc $(t, j) \mapsto \xi(t, j)$ is solution to system*

$$\begin{cases} \dot{\xi} = f(\xi, u_c) & (\xi, u_c) \in C \\ \xi^+ = g(\xi, u_d) & (\xi, u_d) \in D \end{cases} \tag{2}$$

with flow input $t \mapsto u_c(t)$ defined on $\mathbb{R}_{\geq 0}$ and jump input $(u_{d,j})_{j\in\mathbb{N}}$ if:

- $(\xi(0,0), u_c(0)) \in \mathrm{cl}(C)$ *or* $(\xi(0,0), u_{d,0}) \in D$*;*
- *For all $j \in \mathrm{dom}_j\, \xi$ such that $\mathrm{int}(\mathcal{T}_j(\xi)) \neq \emptyset$, we have $(\xi(t,j), u_c(t)) \in C$ for all $t \in \mathrm{int}(\mathcal{T}_j(\xi))$ and $\dot{\xi}(t,j) = f(\xi(t,j), u_c(t))$ for almost all $t \in \mathcal{T}_j(\xi)$;*
- *For all $(t,j) \in \mathrm{dom}\,\xi$ such that $(t, j+1) \in \mathrm{dom}\,\xi$, we have $(\xi(t,j), u_{d,j}) \in D$ and $\xi(t, j+1) = g(\xi(t,j), u_{d,j})$.*

Similar to [39, Definition 2.7], a maximal solution is a solution that cannot be further extended. Note that in Definition 1, the hybrid arc can stop being defined after some time (i.e., it is not complete), even when the inputs given are complete.

The goal of this paper is to design an asymptotic observer for system $\mathcal{H}$ in (1) (as defined below in (5)), assuming that its jump times are known, based on an observability decomposition (1) and the coupling of flow- and jump-based observers [23]. Since in practice, we may be interested in estimating only certain trajectories of "physical interest" from specific initial conditions, we denote as Ξ_0 the set containing the initial conditions of the trajectories to be estimated and $\mathfrak{U}_c$ (resp., $\mathfrak{U}_d$) a set of locally bounded continuous (resp., discrete) inputs of interest, defined on $\mathbb{R}_{\geq 0}$ (resp., $\mathbb{N}$). Then, we denote $\mathcal{S}_{\mathcal{H}}(\Xi_0, \mathfrak{U}_c \times \mathfrak{U}_d)$ as the set of maximal solutions to $\mathcal{H}$ initialized in Ξ_0 with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$, as defined in Definition 1. Let us denote $\mathcal{U}_c \subseteq \mathbb{R}^{n_{u,c}}$ (resp., $\mathcal{U}_d \subseteq \mathbb{R}^{n_{u,d}}$) as the set of points that the inputs $\mathfrak{u}_c \in \mathfrak{U}_c$ (resp., $\mathfrak{u}_d \in \mathfrak{U}_d$) take values in.

Following [20], our design relies on available information about the possible duration of flow intervals between successive jumps in the trajectories of interest, as defined next.

Definition 2. *For a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$ and some $j_m \in \mathbb{N}$, we say that a hybrid arc $(t,j) \mapsto x(t,j)$ has* flow lengths within $\mathcal{I}$ after jump $j_m \in \mathbb{N}$ *if:*

(1) $0 \leq t - t_j(x) \leq \sup \mathcal{I}$ *for all* $(t,j) \in \mathrm{dom}\, x$*;*
(2) $t_{j+1}(x) - t_j(x) \in \mathcal{I}$ *holds for all* $j \in \mathbb{N}_{\geq j_m}$ *if* $\sup \mathrm{dom}_j\, x = +\infty$*, and for all* $j \in \{j_m, j_m + 1, \ldots, \sup \mathrm{dom}_j\, x - 1\}$ *otherwise.*

In brief, $\mathcal{I}$ contains all the possible lengths of the flow intervals between successive jumps, at least after some time. In Definition 2, Item 1 is to bound the length of the flow intervals not covered by Item 2, namely possibly the first one, which is $[0, t_1]$, and the last one, which is $\mathrm{dom}_t\, x \cap [t_{J(x)}, +\infty)$, where $t_{J(x)}$ is the time when the last jump happens (when the number of jumps is finite). For some classes of systems where the jump times of solutions are determined by exogenous inputs and thus are independent of trajectories, such as those in Remark 1, j_m can be zero.

As noticed in [20], the observer type to use depends on $\mathcal{I}$. In particular, [20] studies how to design:

- A *flow-based* observer with an innovation term during flows only, exploiting the observability of the full state during flows obtained from measuring y_c, when $0 \notin \mathcal{I}$, i.e., $\min \mathcal{I} > 0$, and the solutions exhibit a dwell time after some time;
- A *jump-based* observer with an innovation term at jumps only, exploiting the detectability or observability/distinguishability of the full state via the combination of flows and jumps obtained from y_d available at the jumps only, when $\mathcal{I}$ is bounded, i.e., when the solutions exhibit persistent jumps.

As outlined in Section 1.2, in this paper, we consider the case where the component ξ_o of the state ξ in system (1) is *instantaneouly observable* from y_c during flows, while the rest of the state ξ_{no} draws observability from the coupling of flows and jumps. The following assumption formulates these properties. It follows that both flows and jumps need to be exploited to reconstruct the state and that neither (eventually) continuous nor discrete/ Zeno solutions are allowed: both flows and jumps need to be persistent, as assumed next. Moreover, because we are interested in *asymptotic* observers (in the sense that convergence holds asymptotically in hybrid time—see (5) below), we assume completeness of solutions.

Assumption 1. *Given system* (1)*, the following hold:*

(A1.1) There exist $j_m \in \mathbb{N}$ *and compact sets* $\Xi_o \subset \mathbb{R}^{n_o}$, $\Xi_{no} \subset \mathbb{R}^{n_{no}}$, $\Xi_0 \subset \Xi_o \times \Xi_{no} \subset \mathbb{R}^{n_\xi}$, *and* $\mathcal{I} \subset \mathbb{R}_{>0}$ *such that* $C \subseteq \Xi_o \times \Xi_{no} \times \mathcal{U}_c$, $D \subseteq \Xi_o \times \Xi_{no} \times \mathcal{U}_d$, *and each solution in* $\mathcal{S}_{\mathcal{H}}(\Xi_0, \mathfrak{U}_c \times \mathfrak{U}_d)$ *is complete, remains in* $\Xi := \Xi_o \times \Xi_{no}$, *and has flow lengths within* $\mathcal{I}$ *after jump* j_m *(see Definition 2);*

(A1.2) The flow pair (f_o, h_c) *is independent of* ξ_{no} *and is instantaneously observable, in the sense that, for any* $\mathfrak{u}_c \in \mathfrak{U}_c$, *the knowledge of* $t \mapsto h_c(\xi_o(t), u_c(t))$ *for an arbitrarily short amount of time uniquely determines the solution* $t \mapsto \xi_o(t)$ *to* $\dot{\xi}_o = f_o(\xi_o, u_c)$ *as long as* $(\xi_o(t), \mathfrak{u}_c(t)) \in \{(\xi_o, u_c) \in \mathbb{R}^{n_o} \times \mathcal{U}_c : \exists \xi_{no} \in \mathbb{R}^{n_{no}} : (\xi_o, \xi_{no}, u_c) \in C\}$*;*

(A1.3) The maps f_o, J_o, J_{ono}, J_{no}, J_{noo}, H_{do}, *and* H_{dno} *are locally Lipschitz with respect to* ξ_o, *uniformly in* $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$.[2] *The maps* J_{no}, J_{ono}, *and* H_{dno} *are locally bounded with respect to* ξ_o, *uniformly in* $u_d \in \mathcal{U}_d$.[3]

[2] That is, take f_o for instance. Then, for any compact set $\mathcal{C} \subset \mathbb{R}^{n_o}$, there exists $L > 0$ such that, for all $(\xi_{o,a}, \xi_{o,b}) \in \mathcal{C} \times \mathcal{C}$ and for all $u_c \in \mathcal{U}_c$, we have $|f_o(\xi_{o,a}, u_c) - f_o(\xi_{o,b}, u_c)| \leq L|\xi_{o,a} - \xi_{o,b}|$. Similarly for the other maps.

[3] That is, take J_{no} for instance. Then, for any compact set $\mathcal{C} \subset \mathbb{R}^{n_o}$, there exists $M > 0$ such that, for all $\xi_o \in \mathcal{C}$ and for all $u_d \in \mathcal{U}_d$, we have $|J_{no}(\xi_o, u_d)| \leq M$. Similarly for the other maps.

Remark 2. *Assuming* $\min \mathcal{I} > 0$ *is not only to ensure an infinite amount of flows in the maximal solutions. Indeed,* $\sup \mathrm{dom}_t\, x = +\infty$ *could happen even for solutions with flow lengths vanishing to zero. It is the ratio/ balance between flows and jumps that is important to ensure that the estimation error contracting during flows decreases sufficiently to compensate for its possible increase at jumps, and vice versa. In that sense, this persistence of flows could be relaxed into an average-dwell-time condition with* τ_m *replaced by the average dwell time by appropriately modifying the Lyapunov analysis as in [43, Proposition IV.1] or [20, Theorem 3.1].*

Example 1. *As an application, we estimate unknown uncertainties entering through the jumps in a walking robot. The considered model of a two-link bipedal robot has state* (θ, ω) *with angular position* $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$, *angular velocity* $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, *and torque input* $u_c \in \mathbb{R}^2$. *The flow dynamics satisfy (with matrices and parameters in [44]):*

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = (M(\theta))^{-1}(u_c - C(\theta, \omega)\omega - G(\theta)). \end{cases} \tag{3}$$

We choose, during flows only, the state-feedback energy tracking controller as in [45], with parameters in [44]. An impact happens if the swing leg hits the ground when both legs have suitable velocities, namely:

$$\theta_1 + \theta_2 = 0, \qquad \omega_1 \geq 0, \qquad \omega_2 \geq 0, \qquad \theta_1 > 0.$$

The role of the legs are swapped at the impact, so that $\theta^+ = J_s\theta$ *with* $J_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Then, we derive the post-impact velocities following the method explained in [46] (instead of [44]), but introducing unknown tangential and normal uncertainties* $\delta = (\delta_t, \delta_n) \in \mathbb{R}^2$ *at the impact of the swing leg, modeling possible rebound and slip. More precisely, we derive the relation:*

$$\begin{pmatrix} M_e(\theta) & -(E_J(\theta))^\top \\ E_J(\theta) & 0 \end{pmatrix} \begin{pmatrix} \mathrm{diag}(J_s, \mathrm{Id})\omega_e^+ \\ F \end{pmatrix} = \begin{pmatrix} M_e(\theta)\omega_e \\ \delta \end{pmatrix}, \tag{4}$$

where $(\omega_e^+, F) = (\omega^+, v^+, F)$ *stands for the post-impact extended velocity and contact force, with* $E_J(\theta)$ *and the extended mass matrix* $M_e(\theta)$ *given in [46]. The post-impact angular velocities* ω^+ *are obtained by solving* (4) *in* (ω^+, v^+, F) *as a function of* (θ, ω, δ) *by matrix inversion and extracting the first two components corresponding to* ω^+. *This means that* ω^+ *is affine in* δ, *taking the form* $\omega^+ = g_1(\theta)\omega + g_2(\theta)\delta$, *where* g_1 *and* g_2 *are nonlinear functions of* θ *taking values in* $\mathbb{R}^{2\times 2}$ *and deduced from solving* (4). *In terms of measurements,* $y_c = \theta$ *is obtained during flows through encoders placed at the ankles and the hip, assuming the foot is flat on the ground. On the other hand,* y_d *is unavailable to encode the fact that*

we avoid using sensor outputs at the impact due to a loss of reliability (apart from the required jump detection obtained by impact sensors). The resulting system fits in the form (1)*, with* $\xi_o = (\theta, \omega) \in \mathbb{R}^2 \times \mathbb{R}^2$*,* $\xi_{no} = \delta \in \mathbb{R}^2$*, and dynamics given by* $F_{no} = 0_{2\times 2}$*,* $U_{cno} = 0_{2\times 1}$*,* $J_o(\xi_o) = (J_s\theta, g_1(\theta)\omega) \in \mathbb{R}^2 \times \mathbb{R}^2$*,* $J_{ono}(\xi_o) = \begin{pmatrix} 0_{2\times 2} \\ g_2(\theta) \end{pmatrix} \in \mathbb{R}^{4\times 2}$*,* $J_{no}(\xi_o) = \mathrm{Id}_2$*, and* $J_{noo}(\xi_o) = 0_{2\times 1}$*. Solutions of interest are known to be bounded in physical compact sets, and since* $(\theta, \omega) = (y_c, \dot{y}_c)$*,* ξ_o *is instantaneously observable from* y_c*. Moreover, the maps are both locally Lipschitz and locally bounded in* ξ_o*. We can also show that the matrices to be inverted in* (3) *and* (4) *are uniformly invertible. Therefore, Assumption 1 is verified for this system.*

2.2 Ingredients for asymptotic observer design

In this paper, our goal is to design an asymptotic observer for system (1), as introduced below, assuming we know: i) the exogenous signals u_c and u_d, ii) the output(s) y_c during flows and/or y_d at jumps, iii) the jump times of the plant, and iv) some information about the possible flow lengths, as in Item (A1.1) of Assumption 1. Since the jump times of the plant are known, following [20], a *synchronized* asymptotic observer has dynamics of the form

$$\begin{cases} \dot{\hat{\zeta}} = \mathcal{F}(\hat{\zeta}, y_c, u_c) & \text{when (1) flows} \\ \hat{\zeta}^+ = \mathcal{G}(\hat{\zeta}, y_d, u_d) & \text{when (1) jumps} \end{cases} \tag{5a}$$

with the estimate $\hat{\xi}$ obtained from a solution $(t, j) \mapsto \hat{\zeta}(t, j)$ to dynamics (5a) as

$$\hat{\xi}(t, j) = \Upsilon(\hat{\zeta}(t, j), t, j), \tag{5b}$$

where $\hat{\zeta} \in \mathbb{R}^{n_\zeta}$ is the observer state; $\mathcal{F}$, $\mathcal{G}$, and Υ are the observer dynamics and output maps designed together with an initialization set $\mathcal{Z}_0 \subseteq \mathbb{R}^{n_\zeta}$ such that the dependence of Υ on time (t, j) is only through the inputs $(\mathfrak{u}_c, \mathfrak{u}_d)$ and each maximal solution $(\xi, \hat{\zeta})$ to the cascade (1)-(5) initialized in $\Xi_0 \times \mathcal{Z}_0$ and with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$ is complete and verifies

$$\lim_{\substack{t+j\to+\infty \\ (t,j)\in \mathrm{dom}\,\xi (=\mathrm{dom}\,\hat{\xi})}} |\xi(t, j) - \hat{\xi}(t, j)| = 0. \tag{6}$$

Remark 3. *As shown in [20, Section 6], the dwell time after jump* j_m*, given by Item (A1.1) of Assumption 1, typically allows us to ensure the robustness of the observer with respect to delays in the detection of the system jumps, namely when the jumps of observer* (5) *are not triggered simultaneously with the jumps of system* (1)*, but slightly later. More precisely, the semi-global practical stability of the estimation error could be obtained over the time intervals after such delays. Robustness against delays in jump detection but in the context of constant parameter estimation in hybrid systems is established in [47, Chapter 7]. Keeping these in mind, in the rest of this paper, we will assume that the jump times are perfectly known and observers are triggered without any delay, giving exact synchronization of the observer with the system, to focus on observer design for different forms of system* (1)*.*

Following our preliminary results in [23], as illustrated in Figure 1, we proceed to estimate ξ_o from y_c sufficiently fast during flows using a high-gain flow-based observer, then estimate ξ_{no} via a jump-based observer exploiting an observability property of an appropriately defined discrete-time system with extended jump output $(y_d, J_{ono}(\xi_o, u_d)\xi_{no})$ and known input ξ_o. Note that this extended output is *fictitious*, in the sense that it is not available at jumps. However, it can be used in the analysis by exploiting the interaction between ξ_{no} and ξ_o at jumps and the fact that ξ_o is estimated arbitrarily fast during flows from y_c.

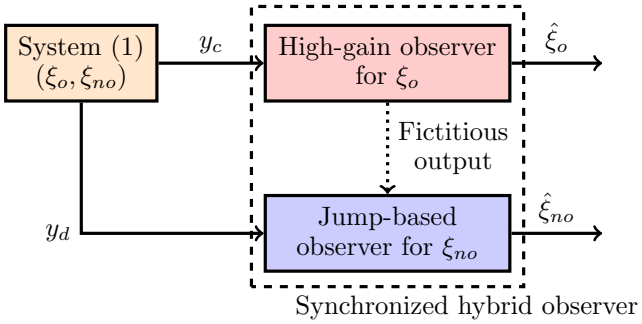

Fig. 1. Illustration of our decomposition-based observer. The known inputs (u_c, u_d) are neglected for clarity.

The observer designs in this paper thus combine two ingredients: a high-gain flow-based observer for ξ_o and jump-based observers designed for an equivalent discrete-time system for ξ_{no}. First, we describe the class of considered flow-based observers for ξ_o, which will be common to all the designs developed in this paper. More precisely, we assume available a high-gain flow-based observer of the form

$$\dot{\hat{\xi}}_o = \hat{f}_{o,\ell}(\hat{\xi}_o, \tau, y_c, u_c), \tag{7}$$

where τ is a timer that keeps track of the time elapsed since the previous jump and has dynamics $\dot{\tau} = 1$ and $\tau^+ = 0$, and where the jump dynamics assigning $\hat{\xi}_o^+$ are to be chosen later, satisfying the following assumption.

Assumption 2. *Suppose Assumption 1 holds and with* $\mathcal{I}$ *from its Item (A1.1), define* $\tau_M := \max \mathcal{I}$*. There exist a map* $\hat{f}_{o,\ell}$*, scalars* $\ell_1^\star > 0$*,* $\lambda_c > 0$*,* $\bar{b}_o > 0$*, and a rational function* $\underline{b}_o > 0$ *such that for all* $\ell > \ell_1^\star$*, there exists a function* $V_{o,\ell} : \mathbb{R}^{n_o} \times \mathbb{R}^{n_o} \times \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}$ *such that:*

(A2.1) For all $(u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$, $\xi = (\xi_o, \xi_{no}) \in \mathbb{R}^{n_\xi}$ such that $(\xi, u_c) \in C$ or $(\xi, u_d) \in D$, $\hat{\xi} = (\hat{\xi}_o, \hat{\xi}_{no}) \in \mathbb{R}^{n_\xi}$, and $\tau \in [0, \tau_M]$,

$$\underline{b}_o(\ell)|\xi_o - \hat{\xi}_o|^2 \leq V_{o,\ell}(\xi_o, \hat{\xi}_o, \tau) \leq \overline{b}_o|\xi_o - \hat{\xi}_o|^2;$$

(A2.2) For all $u_c \in \mathcal{U}_c$, $\xi \in \mathbb{R}^{n_\xi}$ such that $(\xi, u_c) \in C$, $\hat{\xi} \in \mathbb{R}^{n_\xi}$, and $\tau \in [0, \tau_M]$,

$$\dot{V}_{o,\ell}(\xi_o, \hat{\xi}_o, \tau, u_c) \leq -\ell\lambda_c V_{o,\ell}(\xi_o, \hat{\xi}_o, \tau),$$

where $\dot{V}_{o,\ell}$ is the Lie derivative of $V_{o,\ell}$ along the flow vector field $(f_o(\xi_o, u_c), \hat{f}_{o,\ell}(\hat{\xi}_o, \tau, y_c, u_c), 1)$.

Example 2. *We illustrate how high-gain observers meet the requirements in Assumption 2. Assume that system (1) is such that*

$$\dot{\xi}_o = A\xi_o + \Phi(\xi_o, u_c), \qquad y_c = \xi_{o,1} = H\xi_o, \tag{8}$$

with A, H, and Φ in triangular observable form [48], where Φ is locally Lipschitz with respect to ξ_o, uniformly with respect to u_c. A classical high-gain observer [48] is

$$\dot{\hat{\xi}}_o = A\hat{\xi}_o + \Phi(\mathrm{sat}_o(\hat{\xi}_o), u_c) + \ell\mathcal{L}(\ell)K(y_c - H\hat{\xi}_o), \tag{9}$$

where $\ell > 0$, $\mathcal{L}(\ell) = \mathrm{diag}(1, \ell, \ldots, \ell^{n_o-1})$, $K = (k_1, k_2, \ldots, k_{n_o})$ chosen independently of ℓ such that $A - KH$ is Hurwitz, and sat_o a bounded Lipschitz saturation map active outside of the compact set Ξ_o. It can be checked that the Lyapunov function

$$V_{o,\ell}(z_o, \hat{z}_o) = (z_o - \hat{z}_o)^\top(\mathcal{L}(\ell))^{-1}P(\mathcal{L}(\ell))^{-1}(z_o - \hat{z}_o),$$

where the constant matrix $P = P^\top > 0$ is a solution to $P(A - KH) + (A - KH)^\top P \leq -aP$, for some $a > 0$, verifies Assumption 2.

Example 3. *Consider the walking robot in Example 1. Because $\dot{\xi}_o$ takes the form (8), we build for ξ_o a high-gain flow-based observer of the form (9), in a block-wise way (namely with $\theta \in \mathbb{R}^2$ as the first state and $\omega \in \mathbb{R}^2$ as the second state). In simulations, we will pick $K = (5, 6)$ and $\ell = 10$, with sat_o defined based on some bounds obtained by simulating the system solutions of interest (we pick a saturation level of 10^3).*

For simplicity, in the following, we employ the projection of the flow/jump sets onto the state space Ξ, given by:

$$C_o := \{\xi_o \in \Xi_o : \exists(\xi_{no}, u_c) \in \Xi_{no} \times \mathcal{U}_c : (\xi_o, \xi_{no}, u_c) \in C\}, \tag{10a}$$
$$C_{no} := \{\xi_{no} \in \Xi_{no} : \exists(\xi_o, u_c) \in \Xi_o \times \mathcal{U}_c : (\xi_o, \xi_{no}, u_c) \in C\}, \tag{10b}$$
$$D_o := \{\xi_o \in \Xi_o : \exists(\xi_{no}, u_d) \in \Xi_{no} \times \mathcal{U}_d : (\xi_o, \xi_{no}, u_d) \in D\}, \tag{10c}$$
$$D_{no} := \{\xi_{no} \in \Xi_{no} : \exists(\xi_o, u_d) \in \Xi_o \times \mathcal{U}_d : (\xi_o, \xi_{no}, u_d) \in D\}. \tag{10d}$$

To estimate the full state $\xi = (\xi_o, \xi_{no})$ of system (1), we propose to combine the high-gain flow-based observer (7) for ξ_o with a jump-based observer for ξ_{no}. For this, following our preliminary work [23], we will rely on the observability/detectability properties of an equivalent (time-varying) discrete-time system, modeling the dynamics of $k \mapsto \xi_{no}(t_k, k)$ sampled after each jump, with output made of $H_{dno}(\xi_o, u_d)\xi_{no}$ appearing in y_d as well as $J_{ono}(\xi_o, u_d)\xi_{no}$ affecting ξ_o at jumps. More precisely, we consider

$$\xi_{no,k+1} = \mathcal{J}_{no}(\xi_{o,k}, u_{d,k}, \tau_k)\xi_{no,k}, \tag{11a}$$

with the extended output

$$y_k = \mathcal{H}(\xi_{o,k}, u_{d,k}, \tau_k)\xi_{no,k} \in \mathbb{R}^{n_{y,d,\mathrm{ext}}}, \tag{11b}$$

where $n_{y,d,\mathrm{ext}} := n_{y,d} + n_o$ and

$$\mathcal{H}(\xi_o, u_d, \tau) = \begin{pmatrix} \mathcal{H}_{dno}(\xi_o, u_d, \tau) \\ \mathcal{J}_{ono}(\xi_o, u_d, \tau) \end{pmatrix}, \tag{11c}$$

where

$$\mathcal{J}_{no}(\xi_o, u_d, \tau) := J_{no}(\xi_o, u_d)e^{F_{no}\tau}, \tag{11d}$$
$$\mathcal{H}_{dno}(\xi_o, u_d, \tau) := H_{dno}(\xi_o, u_d)e^{F_{no}\tau}, \tag{11e}$$
$$\mathcal{J}_{ono}(\xi_o, u_d, \tau) := J_{ono}(\xi_o, u_d)e^{F_{no}\tau}, \tag{11f}$$

with inputs $(\xi_{o,k}, u_{d,k}, \tau_k) \in (\Xi_o \cap D_o) \times \mathcal{U}_d \times \mathcal{I}$ for all $k \in \mathbb{N}$, where τ_k models the length of the k-th flow interval. System (11) can be interpreted as a time-varying discrete-time system, with dynamics and output of the form $\xi_{no,k+1} = \mathcal{A}(k)\xi_{no,k}$ and $y_k = \mathcal{C}(k)\xi_{no,k}$. From our preliminary work [23], we see that the possibility of designing a jump-based observer for system (1) relies on the detectability of the equivalent discrete-time system (11).

The next sections consist of observer designs for system (1), combining the flow-based observer (7) for ξ_o, with jump-based observers for ξ_{no}: first through an LMI-based design under quadratic detectability in Section 3, then through a KKL-based design under backward distinguishability in Section 4.

3 Observer design based on quadratic detectability

3.1 Observer construction

This section follows and extends the observer design developed in [23, Theorem 3], combining a high-gain flow-based observer with an LMI-based jump-based one. More precisely, the observer we propose for system (1)

takes the form

$$\begin{cases} \dot{\hat{\xi}}_o = \hat{f}_{o,\ell}(\hat{\xi}_o, \tau, y_c, u_c) \\ \dot{\hat{\xi}}_{no} = F_{no}\hat{\xi}_{no} + U_{cno} + e^{F_{no}\tau} K_d \frac{d}{dt} \Psi_{f_{o,\mathrm{sat}}(\cdot,u_c)}(\hat{\xi}_o, t, -\tau) \\ \dot{\tau} = 1 \\ \\ \hat{\xi}_o^+ = J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) + J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\, \mathrm{sat}_{no}(\hat{\xi}_{no}) \\ \hat{\xi}_{no}^+ = J_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\hat{\xi}_{no} + J_{noo}(\mathrm{sat}_o(\hat{\xi}_o), u_d) \\ \qquad + L_d(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)(y_d - H_{do}(\mathrm{sat}_o(\hat{\xi}_o), u_d) \\ \qquad - H_{dno}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\hat{\xi}_{no}) \\ \qquad - K_d J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)(\hat{\xi}_{no} - \mathrm{sat}_{no}(\hat{\xi}_{no})) \\ \tau^+ = 0 \end{cases} \tag{12}$$

with flows and jumps triggered at the same time as system (1), with $\hat{f}_{o,\ell}$ being a high-gain observer as described in (7), with gain $\ell > 0$ to be chosen, $f_{o,\mathrm{sat}}$ being a map that is globally Lipschitz with respect to ξ_o, uniformly in $u_c \in \mathcal{U}_c$, and equal to f_o on $\Xi_o \times \mathcal{U}_c$ (guaranteed to exist by Item (A1.3) of Assumption 1 and [49, Corollary 1]), $\Psi_{f_{o,\mathrm{sat}}}$ denoting the flow operator associated with $f_{o,\mathrm{sat}}$ as defined in the Notations above, sat_{no} being a bounded map such that $\mathrm{sat}_{no}(\xi_{no}) = \xi_{no}$ for all $\xi_{no} \in (\Xi_{no} \cap D_{no}) + \overline{c}_{no}\mathbb{B}$ for some $\overline{c}_{no} > 0$ and K_d and L_d being gains and sat_o being a bounded saturation function to be designed. Note that we propose an alternative expression as well as an approximation algorithm for $\frac{d}{dt}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\hat{\xi}_o, t, -\tau)$ in Section 3.2, along with a non-obvious interpretation of its role as a correction term in the observer.

The high-gain observer (7) for ξ_o, verifying Assumption 2, combined with i) the decoupling of ξ_o from ξ_{no} during flows, ii) the saturation of the impact of $\hat{\xi}_{no}$ on $\hat{\xi}_o$ at jumps through sat_{no}, and iii) the dwell time (after jump j_m), allows us to make the estimation error $\xi_o(t_{j+1}(\xi), j) - \hat{\xi}_o(t_{j+1}(\xi), j)$ before each jump arbitrarily small by choosing ℓ sufficiently large. We make the following assumption to design a jump-based observer for ξ_{no}, namely choose the gains L_d and K_d.

Assumption 3. *There exist $\overline{c}_o > 0$, $0 \leq a < 1$, a symmetric positive definite matrix $Q \in \mathbb{R}^{n_{no} \times n_{no}}$, and gains $K_d \in \mathbb{R}^{n_{no} \times n_o}$ and $(\xi_o, u_d, \tau) \mapsto L_d(\xi_o, u_d, \tau) \in \mathbb{R}^{n_{no} \times n_{y,d}}$ bounded on $((\Xi_o \cap D_o) + \overline{c}_o\mathbb{B}) \times \mathcal{U}_d \times \mathcal{I}$ such that for all $(\xi_o, u_d, \tau) \in ((\Xi_o \cap D_o) + \overline{c}_o\mathbb{B}) \times \mathcal{U}_d \times \mathcal{I}$,*

$$(\Phi(\xi_o, u_d, \tau))^\top Q \Phi(\xi_o, u_d, \tau) \leq aQ, \tag{13a}$$

where

$$\Phi(\xi_o, u_d, \tau) = \mathcal{J}_{no}(\xi_o, u_d, \tau) - \begin{pmatrix} L_d(\xi_o, u_d, \tau) & K_d \end{pmatrix} \begin{pmatrix} \mathcal{H}_{dno}(\xi_o, u_d, \tau) \\ \mathcal{J}_{ono}(\xi_o, u_d, \tau) \end{pmatrix}. \tag{13b}$$

Note that if Φ is continuous and the sets $\Xi_o \cap D_o$ and $\mathcal{U}_d$ are compact, then satisfying

$$(\Phi(\xi_o, u_d, \tau))^\top Q \Phi(\xi_o, u_d, \tau) < Q \tag{14}$$

on the compact set $(\xi_o, u_d, \tau) \in ((\Xi_o \cap D_o) + \overline{c}_o\mathbb{B}) \times \mathcal{U}_d \times \mathcal{I}$ for some $\overline{c}_o > 0$ implies the existence of $0 \leq a < 1$ such that (13) holds. Assumption 3 deals with the *detectability* of the discrete-time system (11) with inputs (ξ_o, u_d, τ). Contrary to L_d which may depend on (ξ_o, u_d, τ), K_d is required to be constant to perform the analysis (see in the proof of Theorem 1). This extra requirement is similar to the one we made in [31] for hybrid systems with linear maps. It is stronger than the notion of *quadratic detectability* [50] by the constant nature of K_d. Constructive methods to solve (13) are described later in Section 3.2.

Theorem 1. *Suppose Assumptions 1, 2, and 3 hold. Consider $\overline{c}_o$, K_d, and L_d defined in Assumption 3, and a bounded map sat_o such that $\mathrm{sat}_o(\hat{\xi}_o) = \hat{\xi}_o$ for all $\hat{\xi}_o \in (\Xi_o \cap D_o) + \overline{c}_o\mathbb{B}$. Then, there exists $\ell^\star > 0$ such that for any $\ell > \ell^\star$, there exist $\rho > 0$ and $\lambda > 0$ such that any maximal solution to the cascade (1)-(12) initialized in $\Xi_0 \times \mathbb{R}^{n_\xi} \times \{0\}$ with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$ is complete and verifies (with j_m from Item (A1.1) of Assumption 1)*

$$|\xi(t,j) - \hat{\xi}(t,j)| \leq \rho |\xi(0,0) - \hat{\xi}(0,0)| e^{-\lambda(t+j)}, \quad \forall (t,j) \in \mathrm{dom}\, \xi : j \geq j_m. \tag{15}$$

Proof. First, exploiting exponential decrease over rational growth, given $\overline{c}_o$ from Assumption 3 and using the compactness of Ξ_o, let $\ell_2^\star \geq \ell_1^\star$ (defined in Assumption 2) such that, for all $\ell > \ell_2^\star$,

$$\sqrt{\frac{\overline{b}_o}{\underline{b}_o(\ell)}} \left(\max_{\xi_o \in \Xi_o} |\xi_o| + M_o \right) e^{-\ell \frac{\lambda_c}{2} \tau_m} \leq \overline{c}_o, \tag{16}$$

where $\overline{b}_o$, $\underline{b}_o$, λ_c are defined in Assumption 2, $\tau_m := \min \mathcal{I} > 0$, and $M_o > 0$ is a bound of $J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) + J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\, \mathrm{sat}_{no}(\hat{\xi}_{no})$ (uniform in $u_d \in \mathcal{U}_d$) obtained from the definitions of sat_o, sat_{no}, and Item (A1.3) of Assumption 1. The proof then consists of three main parts: i) Define new coordinates $(z_{no}, \hat{z}_{no})$, replacing $(\xi_{no}, \hat{\xi}_{no})$ and obtain the state dynamics in those new coordinates, ii) Show that the Lyapunov conditions in [23, Theorem 1] hold after jump j_m and

obtain exponential stability of the estimation error in the new coordinates with initial time (t_{j_m}, j_m), where j_m comes from Item (A1.1) of Assumption 1, and iii) Recover the exponential stability in the ξ-coordinates with respect to the initial time.

Let us begin with the first part of this proof and consider the transformation $(\xi_o, \xi_{no}, \hat{\xi}_o, \hat{\xi}_{no}, \tau, t, j) \mapsto (\xi_o, z_{no}, \hat{\xi}_o, \hat{z}_{no}, \tau)$ with

$$z_{no} = \Psi_{f_{no}}(\xi_{no}, -\tau) - K_d \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau), \quad (17a)$$
$$\hat{z}_{no} = \Psi_{f_{no}}(\hat{\xi}_{no}, -\tau) - K_d \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\hat{\xi}_o, t, -\tau), \quad (17b)$$

where

$$\Psi_{f_{no}}(\xi_{no}, \tau) = e^{F_{no}\tau}\xi_{no} + \int_0^\tau e^{F_{no}(\tau - s)} U_{cno} ds. \quad (17c)$$

In the new coordinates, the dynamics of $(\xi_o, \hat{\xi}_o, \tau)$ are obtained by replacing ξ_{no} and $\hat{\xi}_{no}$ respectively with

$$\xi_{no} = \Psi_{f_{no}}(z_{no} + K_d \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau), \tau), \quad (18a)$$
$$\hat{\xi}_{no} = \Psi_{f_{no}}(\hat{z}_{no} + K_d \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\hat{\xi}_o, t, -\tau), \tau), \quad (18b)$$

and considering the extended inputs $\mathfrak{u}_{c,\mathrm{ext}}(t) = (\mathfrak{u}_c(t), t) \in \mathcal{U}_{c,\mathrm{ext}}$ and $\mathfrak{u}_{d,\mathrm{ext}}(j) = (\mathfrak{u}_d(j), t_{j+1}) \in \mathcal{U}_{d,\mathrm{ext}}$, with $\mathcal{U}_{c,\mathrm{ext}} = \mathcal{U}_c \times \mathbb{R}_{\geq 0}$ and $\mathcal{U}_{d,\mathrm{ext}} = \mathcal{U}_d \times \mathbb{R}_{\geq 0}$. Concerning the dynamics of z_{no} and $\hat{z}_{no}$, we start by showing that $\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau)$ is constant along solutions to the cascade (1)-(12) initialized in $\Xi_0 \times \mathbb{R}^{n_\xi} \times \{0\}$ with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$. To do that, pick $\ell > \ell_2^\star$ and pick a solution $(\xi, \hat{\xi}, \tau)$ to the cascade (1)-(12) initialized in $\Xi_0 \times \mathbb{R}^{n_\xi} \times \{0\}$ with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$ (which is complete thanks to Item (A1.1) of Assumption 1 and given the dynamics of observer (12) which do not allow finite-time escape thanks to saturation functions and the local boundedness of the maps of system (1) as in Item (A1.3) of Assumption 1). In the following, we refer to the jump times of this solution as t_j instead of $t_j(\xi)$ to ease the notations. Since the solution component ξ_o flows according to f_o with input $\mathfrak{u}_c \in \mathfrak{U}_c$, for each $j \in \mathrm{dom}_j\, \xi$ and for each $s \in [0, t - t_j]$, we have

$$\Psi_{f_o(\cdot,\mathfrak{u}_c)}(\xi_o(t,j), t, -s) = \xi_o(t - s, j).$$

Since the trajectory $t \mapsto \xi_o(t, j)$ remains in Ξ_o and τ is initialized as $\tau(0, 0) = 0$, we have for all $(t, j) \in \mathrm{dom}\, \xi_o$,

$$\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o(t,j), t, -\tau(t,j)) = \Psi_{f_o(\cdot,\mathfrak{u}_c)}(\xi_o(t,j), t, -\tau(t,j)).$$

In addition, by the definition of the dynamics, τ initialized as $\tau(0, 0) = 0$ is the time elapsed since the previous jump, namely $\tau(t, j) = t - t_j$ for all $j \in \mathrm{dom}_j\, \xi$. Therefore, exploiting again that ξ_o evolves according to f_o with input $\mathfrak{u}_c$, we have $\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o(t,j), t, -\tau(t,j)) = \Psi_{f_o(\cdot,\mathfrak{u}_c)}(\xi_o(t,j), t, -(t - t_j)) = \xi_o(t_j, j)$ for all $j \in \mathrm{dom}_j\, \xi$ and $t \in [t_j, t_{j+1}]$. Hence,

$$t \mapsto \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o(t,j), t, -\tau(t,j))$$

is constant during flow intervals. We also deduce that $\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau)$ remains in the compact set Ξ_o at all times. Similarly, since ξ_{no} evolves during flows along with the vector field $\xi_{no} \mapsto f_{no}(\xi_{no}) = F_{no}\xi_{no} + U_{cno}$, the quantity $\Psi_{f_{no}}(\xi_{no}, -\tau)$ is constant during flow intervals and remains in the compact set Ξ_{no} at all times. Then, the compact set

$$\mathcal{Z}_{no} := \{z_{no} \in \mathbb{R}^{n_{no}} : \exists (\xi_o, \xi_{no}) \in \Xi_o \times \Xi_{no}, \; z_{no} = \xi_{no} - K_d \xi_o\},$$

is such that along solutions to the cascade (1)-(12) initialized in $\Xi_0 \times \mathbb{R}^{n_\xi} \times \{0\}$ with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$, the image $(t, j) \mapsto z_{no}(t, j)$ defined in (17a) remains in $\mathcal{Z}_{no}$ at all times. Solutions to the cascade (1)-(12) that are initialized in $\Xi_0 \times \mathbb{R}^{n_\xi} \times \{0\}$ with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$ are such that the variable z_{no} takes the dynamics (using that $\tau^+ = 0$ and so $\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o^+, t^+, -\tau^+) = \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o^+, t, 0) = \xi_o^+$),

$$\begin{aligned}
\dot{z}_{no} &= -F_{no}e^{-F_{no}\tau}\xi_{no} + e^{-F_{no}\tau}\dot{\xi}_{no} - e^{-F_{no}\tau}U_{cno} \\
&= -F_{no}e^{-F_{no}\tau}\xi_{no} + e^{-F_{no}\tau}(F_{no}\xi_{no} + U_{cno}) \\
&\quad - e^{-F_{no}\tau}U_{cno} \\
&= 0, \qquad (19a) \\
z_{no}^+ &= \xi_{no}^+ - K_d \xi_o^+ \\
&= J_{no}(\xi_o, u_d)\xi_{no} + J_{noo}(\xi_o, u_d) - K_d(J_o(\xi_o, u_d) \\
&\quad + J_{ono}(\xi_o, u_d)\xi_{no}) \\
&= (J_{no}(\xi_o, u_d) - K_d J_{ono}(\xi_o, u_d))\xi_{no} + J_{noo}(\xi_o, u_d) \\
&\quad - K_d J_o(\xi_o, u_d) \\
&= \phi(\xi_o, u_d, \tau)\Big(z_{no} + K_d \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau) \\
&\quad + \int_0^\tau e^{-F_{no}s} U_{cno} ds\Big) + J_{noo}(\xi_o, u_d) \\
&\quad - K_d J_o(\xi_o, u_d), \qquad (19b)
\end{aligned}$$

where $\phi(\xi_o, u_d, \tau) = (J_{no}(\xi_o, u_d) - K_d J_{ono}(\xi_o, u_d))e^{F_{no}\tau}$,

and the variable $\hat{z}_{no}$ takes the dynamics

$$\begin{aligned}
\dot{\hat{z}}_{no} =& -F_{no}e^{-F_{no}\tau}\hat{\xi}_{no} + e^{-F_{no}\tau}\dot{\hat{\xi}}_{no} \\
&- K_d\frac{d}{dt}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\hat{\xi}_o,t,-\tau) - e^{-F_{no}\tau}U_{cno} \\
=& -F_{no}e^{-F_{no}\tau}\hat{\xi}_{no} + e^{-F_{no}\tau}(F_{no}\hat{\xi}_{no} + U_{cno} \\
&+ e^{F_{no}\tau}K_d\frac{d}{dt}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\hat{\xi}_o,t,-\tau)) \\
&- K_d\frac{d}{dt}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\hat{\xi}_o,t,-\tau) - e^{-F_{no}\tau}U_{cno} \\
=& \ 0, \qquad (20a) \\
\hat{z}_{no}^+ =& \ \hat{\xi}_{no}^+ - K_d\hat{\xi}_o^+ \\
=& \ J_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d)\hat{\xi}_{no} + J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d) \\
&+ L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)(y_d - H_{do}(\mathrm{sat}_o(\hat{\xi}_o),u_d) \\
&- H_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d)\hat{\xi}_{no}) \\
&- K_dJ_{ono}(\mathrm{sat}_o(\hat{\xi}_o),u_d)(\hat{\xi}_{no} - \mathrm{sat}_{no}(\hat{\xi}_{no})) \\
&- K_d(J_o(\mathrm{sat}_o(\hat{\xi}_o),u_d) \\
&+ J_{ono}(\mathrm{sat}_o(\hat{\xi}_o),u_d)\,\mathrm{sat}_{no}(\hat{\xi}_{no})) \\
=& \ (J_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d) - L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\times \\
&\times H_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d) - K_dJ_{ono}(\mathrm{sat}_o(\hat{\xi}_o),u_d))\hat{\xi}_{no} \\
&+ J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d) - K_dJ_o(\mathrm{sat}_o(\hat{\xi}_o),u_d) \\
&+ L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)(y_d - H_{do}(\mathrm{sat}_o(\hat{\xi}_o),u_d) \\
=& \ \Phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\bigg(\hat{z}_{no} + K_d\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\hat{\xi}_o,t,-\tau) \\
&+ \int_0^\tau e^{-F_{no}s}U_{cno}ds\bigg) \\
&+ J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d) - K_dJ_o(\mathrm{sat}_o(\hat{\xi}_o),u_d) \\
&+ L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)(y_d - H_{do}(\mathrm{sat}_o(\hat{\xi}_o),u_d)), \qquad (20b)
\end{aligned}$$

where Φ is defined in Assumption 3. The flow and jump sets are subsets of $\Xi_o \times \mathcal{Z}_{no} \times \mathcal{U}_c$ and $\Xi_o \times \mathcal{Z}_{no} \times \mathcal{U}_d$, respectively. Now, we deduce the estimation error dynamics. For brevity, let us denote

$$\begin{aligned}
\Upsilon &= z_{no} + K_d\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\xi_o,t,-\tau) + \int_0^\tau e^{-F_{no}s}U_{cno}ds, \\
\hat{\Upsilon} &= \hat{z}_{no} + K_d\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\hat{\xi}_o,t,-\tau) + \int_0^\tau e^{-F_{no}s}U_{cno}ds.
\end{aligned}$$

Then, we see that

$$\begin{aligned}
y_d &= H_{do}(\xi_o,u_d) + H_{dno}(\xi_o,u_d)e^{F_{no}\tau}\Upsilon, \\
z_{no}^+ &= \phi(\xi_o,u_d,\tau)\Upsilon + J_{noo}(\xi_o,u_d) - K_dJ_o(\xi_o,u_d), \\
\hat{z}_{no}^+ &= \Phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\hat{\Upsilon} + J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d) \\
&\quad - K_dJ_o(\mathrm{sat}_o(\hat{\xi}_o),u_d) \\
&\quad + L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)(H_{do}(\xi_o,u_d) \\
&\quad - H_{do}(\mathrm{sat}_o(\hat{\xi}_o),u_d) + H_{dno}(\xi_o,u_d)e^{F_{no}\tau}\Upsilon).
\end{aligned}$$

Define the estimation error $\tilde{z}_{no} := z_{no} - \hat{z}_{no}$. Then, we get

$$\begin{aligned}
\tilde{z}_{no}^+ &= \phi(\xi_o,u_d,\tau)\Upsilon - \Phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\hat{\Upsilon} \\
&- L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)H_{dno}(\xi_o,u_d)e^{F_{no}\tau}\Upsilon + (J_{noo}(\xi_o,u_d) \\
&- J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d)) - K_d(J_o(\xi_o,u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o),u_d)) \\
&- L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)(H_{do}(\xi_o,u_d) - H_{do}(\mathrm{sat}_o(\hat{\xi}_o),u_d)).
\end{aligned}$$

Now, add and subtract both the terms $\phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\Upsilon$ and $L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)H_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d)e^{F_{no}\tau}\Upsilon$ to get

$$\begin{aligned}
\tilde{z}_{no}^+ &= \Phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)(\Upsilon - \hat{\Upsilon}) \\
&\quad + (\phi(\xi_o,u_d,\tau) - \phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau) \\
&\quad - L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\times \\
&\quad \times (H_{dno}(\xi_o,u_d) - H_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d))e^{F_{no}\tau})\Upsilon \\
&\quad + (J_{noo}(\xi_o,u_d) - J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d)) \\
&\quad - K_d(J_o(\xi_o,u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o),u_d)) \\
&- L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)(H_{do}(\xi_o,u_d) - H_{do}(\mathrm{sat}_o(\hat{\xi}_o),u_d)).
\end{aligned}$$

Now, see that $\Upsilon - \hat{\Upsilon} = \tilde{z}_{no} + K_d(\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\xi_o,t,-\tau) - \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\hat{\xi}_o,t,-\tau))$ and use the expression of Υ. As a result, the estimation error $\tilde{z}_{no}$ takes the dynamics

$$\begin{aligned}
\dot{\tilde{z}}_{no} &= 0, \\
\tilde{z}_{no}^+ &= \Phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\tilde{z}_{no} + \Phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)K_d\times \\
&\quad \times (\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\xi_o,t,-\tau) - \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\hat{\xi}_o,t,-\tau)) \\
&\quad + (\phi(\xi_o,u_d,\tau) - \phi(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau) \\
&\quad - L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\times \\
&\quad \times (H_{dno}(\xi_o,u_d) - H_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d))e^{F_{no}\tau})\times \\
&\quad \times \bigg(z_{no} + K_d\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathtt{u}_c)}(\xi_o,t,-\tau) \\
&\quad + \int_0^\tau e^{-F_{no}s}U_{cno}ds\bigg) \\
&\quad + (J_{noo}(\xi_o,u_d) - J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d)) \\
&\quad - K_d(J_o(\xi_o,u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o),u_d)) \\
&\quad - L_d(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\times \\
&\quad \times (H_{do}(\xi_o,u_d) - H_{do}(\mathrm{sat}_o(\hat{\xi}_o),u_d)).
\end{aligned}$$

Let us move to the second part of this proof. First, from i) Assumption 2, ii) the fact that $t_{j+1} - t_j \geq \tau_m$ according to Item (A1.1) of Assumption 1, and iii) the choice of

$\ell > \ell_2^\star$ satisfying (16), we have for all $j \in \mathbb{N}_{\geq j_m}$,

$$\begin{aligned}
&|\xi_o(t_{j+1}, j) - \hat{\xi}_o(t_{j+1}, j)| \\
&\leq \sqrt{\frac{\overline{b}_o}{\underline{b}_o(\ell)}} |\xi_o(t_j, j) - \hat{\xi}_o(t_j, j)| e^{-\ell \frac{\lambda_c}{2} \tau_m} \\
&\leq \sqrt{\frac{\overline{b}_o}{\underline{b}_o(\ell)}} \left(\max_{\xi_o \in \Xi_o} |\xi_o| + M_o \right) e^{-\ell \frac{\lambda_c}{2} \tau_m} \leq \overline{c}_o.
\end{aligned}$$

Then, by the definition of sat_o and since $\xi_o(t_{j+1}, j) \in \Xi_o \cap D_o$, we have, for all $j \in \mathbb{N}_{\geq j_m}$,

$$\mathrm{sat}_o(\hat{\xi}_o(t_{j+1}, j)) = \hat{\xi}_o(t_{j+1}, j).$$

Hence, the condition in Assumption 3 is satisfied with $\mathrm{sat}_o(\hat{\xi}_o)$ replacing ξ_o, making $\Phi(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)$ Schur and so the dynamics of $\tilde{z}_{no}$ are contracting (since the input $u_d \in \mathfrak{U}_d$ takes values in $\mathcal{U}_d$). [4] Consider the Lyapunov function

$$V_{no}(z_{no}, \hat{z}_{no}) = (z_{no} - \hat{z}_{no})^\top Q (z_{no} - \hat{z}_{no}), \tag{21}$$

with Q in (13). Denote $\underline{c}_Q > 0$ and $\overline{c}_Q > 0$, respectively, as the minimum and maximum eigenvalues of Q. Then, we have for all $(z_{no}, \hat{z}_{no}) \in \mathcal{Z}_{no} \times \mathbb{R}^{n_{no}}$,

$$\underline{c}_Q |z_{no} - \hat{z}_{no}|^2 \leq V_{no}(z_{no}, \hat{z}_{no}) \leq \overline{c}_Q |z_{no} - \hat{z}_{no}|^2. \tag{22}$$

For all $(\xi_o, z_{no}) \in (\Xi_o \cap C_o) \times \mathcal{Z}_{no}$, $(\hat{\xi}_o, \hat{z}_{no}) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}}$ and for all $(\tau, t, u_c) \in [0, \tau_M] \times \mathbb{R}_{\geq 0} \times \mathcal{U}_c$, we have

$$\dot{V}_{no}(\xi_o, \hat{\xi}_o, z_{no}, \hat{z}_{no}, \tau, t, u_c) = 0,$$

along the respective flow dynamics. By the global Lipschitzness of $f_{o,\mathrm{sat}}$ with respect to ξ_o, uniformly with respect to u_c, Lemma 2 in Appendix A allows us to show that $\Psi_{f_{o,\mathrm{sat}}(\cdot, \mathrm{u}_c)}(\cdot, t, -\tau)$ is Lipschitz, uniformly with respect to $(t, \tau) \in \mathbb{R}_{\geq 0} \times [0, \tau_M]$. With sat_o defined as in Theorem 1 and sat_{no} defined in observer (12), there exist $L_o > 0$ and $L_{no} > 0$ such that

$$\begin{aligned}
&|\xi_o - \mathrm{sat}_o(\hat{\xi}_o)| \leq L_o |\xi_o - \hat{\xi}_o|, \\
&\quad \forall (\xi_o, \hat{\xi}_o) \in (\Xi_o \cap D_o) \times \mathbb{R}^{n_o}, && \text{(23a)} \\
&|\xi_{no} - \mathrm{sat}_{no}(\hat{\xi}_{no})| \leq L_{no} |\xi_{no} - \hat{\xi}_{no}|, \\
&\quad \forall (\xi_{no}, \hat{\xi}_{no}) \in (\Xi_{no} \cap D_{no}) \times \mathbb{R}^{n_{no}}. && \text{(23b)}
\end{aligned}$$

[4] Note that in our preliminary work [23], it is assumed that $\mathrm{sat}_o(\hat{\xi}_o)$ takes value in $\Xi_o \cap D_o$, where Assumption 3 holds. Here, one of our highlights is that the power of the high-gain observer to converge arbitrarily fast is exploited, in combination with a dwell time, to bring the condition in Assumption 3 from the sequence of real solutions $\xi_o(t_{j+1}, j)$ to the sequence of estimates $\mathrm{sat}_o(\hat{\xi}_o(t_{j+1}, j))$, along the common $\mathrm{dom}\, \xi$.

Indeed, if $\hat{\xi}_o \in (\Xi_o \cap D_o) + \overline{c}_o \mathbb{B}$ then $\mathrm{sat}_o(\hat{\xi}_o) = \hat{\xi}_o$ and the property holds with $L_o = 1$; on the other hand, if $\hat{\xi}_o \notin (\Xi_o \cap D_o) + \overline{c}_o \mathbb{B}$ then $|\xi_o - \hat{\xi}_o| \geq \overline{c}_o$ and thus $|\xi_o - \mathrm{sat}_o(\hat{\xi}_o)| = |\,\mathrm{sat}_o(\xi_o) - \mathrm{sat}_o(\hat{\xi}_o)| \leq 2M_o' \leq \frac{2M_o'}{\overline{c}_o} |\xi_o - \hat{\xi}_o|$, where $M_o' > 0$ is the bound of sat_o; therefore, take $L_o = \max\left\{1, \frac{2M_o'}{\overline{c}_o}\right\}$. The proof of (23b) follows similarly. Thanks to Item (A1.3) of Assumption 1, Assumption 3, and Young's inequality on the cross terms, there exist $c_1 \in [0, 1)$, $c_2 > 0$, $c_3 > 0$, and $c_4 > 0$ such that for any $\kappa > 0$, for all $(\xi_o, z_{no}) \in (\Xi_o \cap D_o) \times \mathcal{Z}_{no}$, for all $(\hat{\xi}_o, \hat{z}_{no}) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}}$, and for all $(\tau, u_c, u_d) \in \mathcal{I} \times \mathcal{U}_c \times \mathcal{U}_d$, we have

$$\begin{aligned}
&V_{no}^+(\xi_o, \hat{\xi}_o, z_{no}, \hat{z}_{no}, \tau, t, u_c, u_d) \\
&\leq \left(c_1 + \frac{c_2}{\kappa}\right) V_{no}(z_{no}, \hat{z}_{no}) + (c_3 \kappa + c_4) |\xi_o - \hat{\xi}_o|^2 \\
&\leq \left(c_1 + \frac{c_2}{\kappa}\right) V_{no}(z_{no}, \hat{z}_{no}) + \frac{c_3 \kappa + c_4}{\underline{b}_o(\ell)} V_{o,\ell}(\xi_o, \hat{\xi}_o, \tau),
\end{aligned}$$

along the jump dynamics of $\tilde{z}_{no}$, where the latter inequality is obtained from Assumption 2. Pick κ large enough so that $c_1 + \frac{c_2}{\kappa} \in [0, 1)$ and so V_{no} satisfies the second item of each condition of [23, Theorem 1] (note that $\frac{c_3 \kappa + c_4}{\underline{b}_o(\ell)}$ is rational in ℓ because so is $\underline{b}_o(\ell)$). Using

$$\begin{aligned}
&\xi_{no} - \hat{\xi}_{no} = e^{F_{no} \tau} (z_{no} - \hat{z}_{no}) \\
&+ K_d (\Psi_{f_{o,\mathrm{sat}}(\cdot, \mathrm{u}_c)}(\xi_o, t, -\tau) - \Psi_{f_{o,\mathrm{sat}}(\cdot, \mathrm{u}_c)}(\hat{\xi}_o, t, -\tau)),
\end{aligned}$$

and since τ remains in $[0, \tau_M]$ and $\Psi_{f_{o,\mathrm{sat}}(\cdot, \mathrm{u}_c)}(\cdot, t, -\tau)$ is Lipschitz, uniformly with respect to $(t, \tau) \in \mathbb{R}_{\geq 0} \times [0, \tau_M]$, using Young's inequality, we deduce that there exist $c_5 > 0$, $c_6 > 0$, $c_7 > 0$, and $c_8 > 0$ such that for all $(\xi_o, z_{no}, \hat{\xi}_o, \hat{z}_{no}, \tau, u_c, t) \in \Xi_o \times \mathcal{Z}_{no} \times \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times [0, \tau_M] \times \mathcal{U}_c \times \mathbb{R}_{\geq 0}$, $(\xi_{no}, \hat{\xi}_{no})$ defined in (18) verifies

$$\begin{aligned}
|\xi_{no} - \hat{\xi}_{no}| &\leq c_5 |\xi_o - \hat{\xi}_o| + c_6 |z_{no} - \hat{z}_{no}|, \\
|z_{no} - \hat{z}_{no}| &\leq c_7 |\xi_o - \hat{\xi}_o| + c_8 |\xi_{no} - \hat{\xi}_{no}|.
\end{aligned}$$

Then, there exist $c_9 > 0$ and $c_{10} > 0$ such that

$$|\xi - \hat{\xi}| \leq c_9 |(\xi_o, z_{no}) - (\hat{\xi}_o, \hat{z}_{no})|, \tag{24a}$$

$$|(\xi_o, z_{no}) - (\hat{\xi}_o, \hat{z}_{no})| \leq c_{10} |\xi - \hat{\xi}|. \tag{24b}$$

On the other hand, we have

$$\begin{aligned}
\xi_o^+ - \hat{\xi}_o^+ &= J_o(\xi_o, u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) + J_{ono}(\xi_o, u_d) \xi_{no} \\
&\quad - J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\, \mathrm{sat}_{no}(\hat{\xi}_{no}) \\
&= J_o(\xi_o, u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) \\
&\quad + (J_{ono}(\xi_o, u_d) - J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)) \xi_{no} \\
&\quad + J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)(\xi_{no} - \mathrm{sat}_{no}(\hat{\xi}_{no})).
\end{aligned} \tag{25}$$

Because ξ_{no} is bounded and thanks to (23b), there exist $c_{11} > 0$ and $c_{12} > 0$ and from these, $d_o(\ell) := \frac{\overline{b}_o c_{11}}{\underline{b}_o(\ell)} > 0$ (rational in ℓ because so is $\underline{b}_o$) and $d_{ono} := \overline{b}_o c_{12} > 0$, such that $V_{o,\ell}$ in Assumption 2 satisfies

$$\begin{aligned} V^+_{o,\ell}(\xi,\hat{\xi},\tau,u_d) &\le \overline{b}_o|\xi_o^+ - \hat{\xi}_o^+|^2 \\ &\le \overline{b}_o c_{11}|\xi_o - \hat{\xi}_o|^2 + \overline{b}_o c_{12}|\xi_{no} - \hat{\xi}_{no}|^2 \\ &\le \frac{\overline{b}_o c_{11}}{\underline{b}_o(\ell)} V_{o,\ell}(\xi_o,\hat{\xi}_o,\tau) + \overline{b}_o c_{12}|\xi_{no} - \hat{\xi}_{no}|^2 \\ &\le d_o(\ell) V_{o,\ell}(\xi_o,\hat{\xi}_o,\tau) + d_{ono}|\xi_{no} - \hat{\xi}_{no}|^2, \end{aligned}$$

where $V^+_{o,\ell}$ denotes the jump of $V_{o,\ell}$ along $(J_o(\xi_o,u_d) + J_{ono}(\xi_o,u_d)\xi_{no}, J_o(\mathrm{sat}_o(\hat{\xi}_o),u_d) + J_{ono}(\mathrm{sat}_o(\hat{\xi}_o),u_d)\,\mathrm{sat}_{no}(\hat{\xi}_{no})))$, according to Assumption 2. This inequality, along with Assumption 2, shows that $V_{o,\ell}$ satisfies the inequalities involving it in [23, Theorem 1] in the new coordinates. Applying [23, Theorem 1] starting from hybrid time (t_{j_m}, j_m) with $z_o = \xi_o$ and with z_{no} defined above, we deduce that there exists $\ell_3^\star \ge \ell_2^\star$ such that for any $\ell > \ell_3^\star$, there exist $\rho_1(\ell) > 0$ and $\lambda(\ell) > 0$ such that

$$\begin{aligned} &|(\xi_o, z_{no})(t,j) - (\hat{\xi}_o, \hat{z}_{no})(t,j)| \\ &\le \rho_1(\ell)|(\xi_o, z_{no})(t_{j_m}, j_m) - (\hat{\xi}_o, \hat{z}_{no})(t_{j_m}, j_m)|e^{-\lambda(\ell)(t+j)}, \\ &\qquad \forall (t,j) \in \mathrm{dom}\,\xi, j \ge j_m. \end{aligned} \quad (26)$$

Now, we move to the last part of this proof, providing an exponential decreasing bound of the estimation error $(\xi_o, z_{no})(t_{j_m}, j_m) - (\hat{\xi}_o, \hat{z}_{no})(t_{j_m}, j_m)$ compared to its value at time $(0,0)$. Consider the Lyapunov function

$$W(\xi_o,\hat{\xi}_o,z_{no},\hat{z}_{no}) = V_{o,\ell}(\xi_o,\hat{\xi}_o,\tau) + V_{no}(z_{no},\hat{z}_{no}),$$

which verifies for all $(u_c,u_d) \in \mathcal{U}_c \times \mathcal{U}_d$, for all $\xi = (\xi_o,\xi_{no}) \in \mathbb{R}^{n_\xi}$ such that $(\xi,u_c) \in C$ or $(\xi,u_d) \in D$, for all $\hat{\xi} = (\hat{\xi}_o,\hat{\xi}_{no}) \in \mathbb{R}^{n_\xi}$, for all $\tau \in [0,\tau_M]$, and for all $(z_{no},\hat{z}_{no}) \in \mathcal{Z}_{no} \times \mathbb{R}^{n_{no}}$,

$$\begin{aligned} \underline{b}_o(\ell)|\xi_o - \hat{\xi}_o|^2 + \underline{c}_Q|z_{no} - \hat{z}_{no}|^2 &\le W(\xi_o,\hat{\xi}_o,z_{no},\hat{z}_{no}) \\ &\le \overline{b}_o|\xi_o - \hat{\xi}_o|^2 + \overline{c}_Q|z_{no} - \hat{z}_{no}|^2. \end{aligned}$$

For any $\ell > \ell_1^\star$, for all $(\xi_o, z_{no}) \in (\Xi_o \cap C_o) \times \mathcal{Z}_{no}$, for all $(\hat{\xi}_o,\hat{z}_{no}) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}}$, and for all $(\tau,t,u_c) \in [0,\tau_M] \times \mathbb{R}_{\ge 0} \times \mathcal{U}_c$, we have

$$\dot{W}(\xi_o,\hat{\xi}_o,z_{no},\hat{z}_{no},\tau,t,u_c) = -\ell\lambda_c V_{o,\ell}(\xi_o,\hat{\xi}_o,\tau) \le 0,$$

along the respective flow dynamics. Thanks to boundedness in solutions and the observer jump map brought by Item (A1.3) of Assumption 1 and the saturation functions, there exists $c_{13}(\ell) > 0$ such that before jump j_m, for all $(\xi_o, z_{no}) \in (\Xi_o \cap D_o) \times \mathcal{Z}_{no}$, for all $(\hat{\xi}_o,\hat{z}_{no}) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}}$, and for all $(\tau,u_d) \in [0,\tau_M] \times \mathcal{U}_d$,

$$W^+(\xi_o,\hat{\xi}_o,z_{no},\hat{z}_{no},\tau,t,u_d) \le c_{13}(\ell) W(\xi_o,\hat{\xi}_o,z_{no},\hat{z}_{no}),$$

along the respective jump dynamics. So there exists $c_{14}(\ell) > 0$ such that

$$\begin{aligned} &W(\xi_o(t_{j_m},j_m),\hat{\xi}_o(t_{j_m},j_m),z_{no}(t_{j_m},j_m),\hat{z}_{no}(t_{j_m},j_m)) \\ &\quad \le c_{14}(\ell) W(\xi_o(0,0),\hat{\xi}_o(0,0),z_{no}(0,0),\hat{z}_{no}(0,0)). \end{aligned}$$

Consequently, there exists $c_{15}(\ell) > 0$ such that

$$\begin{aligned} &|(\xi_o, z_{no})(t_{j_m}, j_m) - (\hat{\xi}_o, \hat{z}_{no})(t_{j_m}, j_m)| \\ &\qquad \le c_{15}(\ell)|(\xi_o, z_{no})(0,0) - (\hat{\xi}_o, \hat{z}_{no})(0,0)|. \end{aligned} \quad (27)$$

From (24), (26), and (27), we deduce that for all $(t,j) \in \mathrm{dom}\,\xi$ such that $j \ge j_m$,

$$\begin{aligned} &|\xi(t,j) - \hat{\xi}(t,j)| \\ &\le c_9|(\xi_o, z_{no})(t,j) - (\hat{\xi}_o, \hat{z}_{no})(t,j)| \\ &\le c_9\rho_1(\ell)|(\xi_o, z_{no})(t_{j_m}, j_m) - (\hat{\xi}_o, \hat{z}_{no})(t_{j_m}, j_m)|e^{-\lambda(\ell)(t+j)} \\ &\le c_9\rho_1(\ell)c_{15}(\ell)|(\xi_o, z_{no})(0,0) - (\hat{\xi}_o, \hat{z}_{no})(0,0)|e^{-\lambda(\ell)(t+j)} \\ &\le c_9\rho_1(\ell)c_{15}(\ell)c_{10}|\xi(0,0) - \hat{\xi}(0,0)|e^{-\lambda(\ell)(t+j)}, \end{aligned}$$

and so Theorem 1 follows. □

3.2 Constructive methods to implement observer (12)

Observer (12) requires computing the correction term $\frac{d}{dt}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathrm{u}_c)}(\hat{\xi}_o, t, -\tau)$ in $\dot{\hat{\xi}}_{no}$. A first solution to avoid computing this term is to implement the observer in the new coordinates defined in the proof of Theorem 1, where $\hat{z}_{no}$ replaces the observer state $\hat{\xi}_{no}$ with dynamics (19) and $\hat{\xi}_{no}$ recovered from $\hat{z}_{no}$ at all times by inverting (17b), namely

$$\hat{\xi}_{no} = \Psi_{f_{no}}(\hat{z}_{no} + K_d\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathrm{u}_c)}(\hat{\xi}_o, t, -\tau), \tau). \quad (28)$$

In this case, a numerical scheme must be used to integrate backward $f_{o,\mathrm{sat}}$ and compute the term $\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathrm{u}_c)}(\hat{\xi}_o, t, -\tau)$ in (18b). Note though that, as shown in the proof of Theorem 1, this term estimates the value of ξ_o at the beginning of the flow interval, i.e., $\xi_o(t_j, j)$, which is constant during flows: it is thus sufficient to update its estimate only "from time to time," as allowed by the available computational power, at a frequency independent of the integration steps of the observer, and at the jump times for observer implementation. This is rendered possible by the fact that $\hat{\xi}_{no}$ is used in the dynamics of observer (12) only at the jump times to compute $\hat{\xi}_o^+$.

When $(\xi_o, t) \mapsto f_{o,\mathrm{sat}}(\xi_o, \mathfrak{u}_c(t))$ is continuous and C^1 with respect to ξ_o, a second option to implement (12) is to notice by Lemma 3 in Appendix A, that observer (12) can equivalently be written and implemented with

$$\dot{\hat{\xi}}_{no} = F_{no}\hat{\xi}_{no} + U_{cno} - e^{F_{no}\tau} K_d \frac{\partial \Psi_{f_{o,\mathrm{sat}}}}{\partial \xi_o}(\hat{\xi}_o, t, -\tau) \times \\ \times \left(f_{o,\mathrm{sat}}(\hat{\xi}_o, u_c) - \hat{f}_{o,\ell}(\hat{\xi}_o, \tau, y_c, u_c) \right). \quad (29)$$

Intuitively, the update of $\hat{\xi}_{no}$ during flows comes from the mismatch between $f_{o,\mathrm{sat}}$ and $\hat{f}_{o,\ell}$ (given in observer (12)) at $\hat{\xi}_o$, which typically remains zero only when $\hat{\xi}_o = \xi_o$: if an estimation error in ξ_{no} triggers an estimation error in ξ_o^+, it becomes visible through y_c and corrected during the subsequent flow interval.

Then, we exploit Lemma 4 in Appendix A to propose an approximation strategy of the gain $\frac{\partial \Psi_{f_{o,\mathrm{sat}}}}{\partial \xi_o}(\hat{\xi}_o, t, -\tau)$ in (29), when it cannot be exactly computed. Indeed, Lemma 4 shows that, when $(\xi_o, t) \mapsto f_{o,\mathrm{sat}}(\xi_o, \mathfrak{u}_c(t))$ is continuous and C^1 with respect to ξ_o, for any given $(\hat{\xi}_o, t) \in \mathbb{R}^{n_o} \times \mathbb{R}_{\geq 0}$, the pair

$$\left(\Psi_{f_{o,\mathrm{sat}}}(\hat{\xi}_o, t, -\tau), \frac{\partial \Psi_{f_{o,\mathrm{sat}}}}{\partial \xi_o}(\hat{\xi}_o, t, -\tau) \right)$$

is obtained by integrating backward, during τ unit(s) of time, the system

$$\frac{d}{ds}\zeta(s) = f_{o,\mathrm{sat}}(\zeta(s), \mathfrak{u}_c(s)), \quad (30a)$$

$$\frac{d}{ds}\Phi_o(s) = \frac{\partial f_{o,\mathrm{sat}}}{\partial \xi_o}(\zeta(s), \mathfrak{u}_c(s))\Phi_o(s), \quad (30b)$$

from final condition $(\zeta(t), \Phi_o(t)) = (\hat{\xi}_o, \mathrm{Id})$ at time t. In other words, the gain $\frac{\partial \Psi_{f_{o,\mathrm{sat}}}}{\partial \xi_o}(\hat{\xi}_o, t, -\tau)$ in (29) is approximated by executing Algorithm 1 at each time step, based on, for example, a forward Euler discretization. While we have found in simulations that this scheme works with a small enough step size—see in Sections 3.3 and 4.3, any other time discretization scheme of (30) (with possibly varying steps) can be used, depending on the properties of the differential equation (30) (e.g., stiffness, invariance), and the available computational power. Note that the gain $\frac{\partial \Psi_{f_{o,\mathrm{sat}}}}{\partial \xi_o}(\hat{\xi}_o, t, -\tau)$ in (29) appears only in front of a term that vanishes when $\hat{\xi}_o$ converges to ξ_o (when $\hat{f}_{o,\ell}$ is properly designed), thus preserving the possibility of asymptotic stability if the approximation is precise enough. In many cases, the correct sign of the gain may suffice.

Now, concerning the choice of the gains $(L_d(\cdot), K_d)$ in Assumption 3, according to Lemma 6 in Assumption A, we see that $Q = Q^\top > 0$ and $a > 0$ satisfying a strict

Algorithm 1 Approximating $\frac{\partial \Psi_{f_{o,\mathrm{sat}}}}{\partial \xi_o}(\hat{\xi}_o, t, -\tau)$ in (29)

Require: the map $(\xi_o, t) \mapsto f_{o,\mathrm{sat}}(\xi_o, \mathfrak{u}_c(t))$, the point $(\hat{\xi}_o, t, \tau)$, and a step size $\Delta > 0$
Initialize $\zeta = \hat{\xi}_o$ and $\Phi_o = \mathrm{Id}$
for $k = 1 : \left\lfloor \frac{\tau}{\Delta} \right\rfloor$ **do**
$\zeta = \zeta - \Delta f_{o,\mathrm{sat}}(\zeta, \mathfrak{u}_c(t - (k-1)\Delta))$
$\Phi_o = \Phi_o - \Delta \frac{\partial f_{o,\mathrm{sat}}}{\partial \xi_o}(\zeta, \mathfrak{u}_c(t - (k-1)\Delta))\Phi_o$
end for
Output Φ_o

version of (13) for some $(L_d(\cdot), K_d)$ exist only if they are solution to, for all $(\xi_o, u_d, \tau) \in (\Xi_o \cap D_o) \times \mathcal{U}_d \times \mathcal{I}$,

$$\begin{pmatrix} ((\mathcal{H}(\xi_o, u_d, \tau))^\perp)^\top Q (\mathcal{H}(\xi_o, u_d, \tau))^\perp & \star \\ Q\mathcal{J}_{no}(\xi_o, u_d, \tau)(\mathcal{H}(\xi_o, u_d, \tau))^\perp & aQ \end{pmatrix} > 0. \quad (31)$$

This is an LMI in (Q, aQ), treating aQ as a new variable. If such (a, Q) are obtained, the gains $L_d(\cdot)$ and K_d can be found by using (13) with Q and a known.

If $((\Xi_o \cap D_o) + \bar{c}_o\mathbb{B}) \times \mathcal{U}_d \times \mathcal{I}$ has infinitely many points, then there is an infinite number of LMIs to solve. Actually, it is worth noting that the exponential term $e^{F_{no}\tau}$ contained in all the τ-dependent matrices in (31) can be expanded using residue matrices termed in [17], as

$$e^{F_{no}\tau} = \sum_{i=1}^{\sigma_r} \sum_{j=1}^{m_i^r} R_{ij} e^{\lambda_i \tau} \frac{\tau^{j-1}}{(j-1)!} + \sum_{i=1}^{\sigma_c} \sum_{j=1}^{m_i^c} 2e^{\Re(\lambda_i)\tau} \\ \times (\Re(R_{ij}) \cos(\Im(\lambda_i)\tau) - \Im(R_{ij}) \sin(\Im(\lambda_i)\tau)) \frac{\tau^{j-1}}{(j-1)!},$$

where σ_r and σ_c are the numbers of distinct real eigenvalues and complex conjugate eigenvalue pairs; m_i^r and m_i^c are the multiplicity of the real eigenvalue λ_i and of the complex conjugate eigenvalue pair λ_i, λ_i^* in the minimal polynomial of F_{no}; $R_{ij} \in \mathbb{R}^{n_{no} \times n_{no}}$ are matrices corresponding to the residues associated to the partial fraction expansion of $(s\,\mathrm{Id} - F_{no})^{-1}$. This in turn allows $e^{F_{no}\tau}$ to be written as a finite sum of matrices affine in a finite number of scalar functions $\beta_{ij} = e^{\lambda_i \tau}\tau^{j-1}$, $\gamma_{ij} = e^{\Re(\lambda_i)\tau}\cos(\Im(\lambda_i)\tau)\tau^{j-1}$, and $\gamma_{ij}^* = e^{\Re(\lambda_i)\tau}\sin(\Im(\lambda_i)\tau)\tau^{j-1}$. Then, it implies that if $\Xi_o \cap D_o$ and $\mathcal{U}_d$ are compact, (13) or (31) can be solved using a *polytopic* approach, i.e., the LMIs are satisfied for all $(\xi_o, u_d, \tau) \in ((\Xi_o \cap D_o) + \bar{c}_o\mathbb{B}) \times \mathcal{U}_d \times \mathcal{I}$ compact if they are satisfied at the finite number of vertices of the polytope formed by these scalar functions when (ξ_o, u_d, τ) takes values in $((\Xi_o \cap D_o) + \bar{c}_o\mathbb{B}) \times \mathcal{U}_d \times \mathcal{I}$. Alternatively, the matrix inequalities can be solved in a *grid-based* approach assuming a particular structure of L_d followed by post-analysis of the solution's stability as in [50], possibly with a theoretical proof as in [18].

3.3 *LMI-based observer design for a walking robot*

Let us now design observer (12) for the walking robot in Example 1. To do this, we need to solve (13) for this system. Because there is no jump output y_d, we arbitrarily choose $L_d = 0$. Then, the gain $K_d \in \mathbb{R}^{2\times 4}$ is found by solving

$$\left(\mathrm{Id}_2 - K_d \begin{pmatrix} 0 \\ g_2(\theta) \end{pmatrix}\right)^\top Q \left(\mathrm{Id}_2 - K_d \begin{pmatrix} 0 \\ g_2(\theta) \end{pmatrix}\right) < Q, \tag{32}$$

for $\theta \in \mathbb{R}^2$ in a subset of the jump set D, namely with $\theta_1 = -\theta_2$ and $\theta_1 > 0$. In fact, we solve (32) in a simple and practical way as follows. By examining the solutions of interest starting from a range of initial conditions, we see that θ_1 is around 0.2 (rad) at the impacts and thus design $K_d = \begin{pmatrix} 0_{2\times 2} & K_d' \end{pmatrix}$ satisfying (32) for $\theta_1 = 0.2$ (rad) by pole placement at zero. This design is made possible by the fact that the matrix $g_2(\theta)$ is invertible at $\theta = (0.2, -0.2)$ (rad), so that indeed the pair $(\mathrm{Id}_2, g_2(\theta))$ is observable at that point and K_d' can be designed. Then, thanks to the stability margin in the unit circle, we should have the eigenvalues of $\mathrm{Id}_2 - K_d J_{ono}(\theta)$, for θ in some small interval around $\theta_1 = -\theta_2 = 0.2$ (rad), remain within the unit circle. To illustrate this, we show in Figure 2 the largest magnitude of the eigenvalues of $\mathrm{Id}_2 - K_d J_{ono}(\theta)$ obtained by taking 200 random samples in $\theta_1 = -\theta_2 \in [0.17, 0.23]$ (rad). These magnitudes remain under 1, so the eigenvalues lie safely inside the unit circle. Thus, (32) holds at and sample points, and by continuity, it should hold for all $\theta_1 = -\theta_2 \in [0.17, 0.23]$ (rad) (but it is not guaranteed with the same Q for all θ).

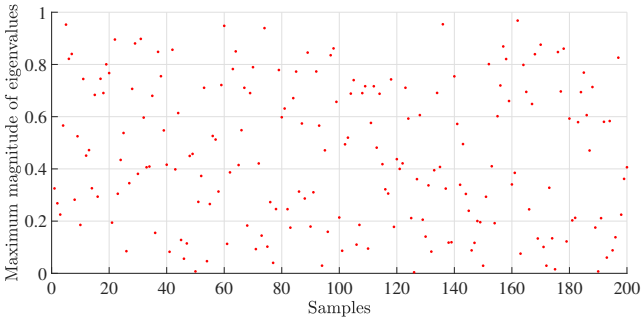

Fig. 2. Maximum magnitude of eigenvalues of $\mathrm{Id}_2 - K_d J_{ono}(\theta)$ with 200 values of θ randomly sampled around $\theta_1 = -\theta_2 \in [0.17, 0.23]$ (rad).

The high-gain observer for ξ_o is the same as in Example 3. With K_d and the high-gain observer at hand, we are now ready to implement the hybrid observer (12). However, following the recommendations in Section 3.2, in order to avoid the computation of the Jacobian of the flow operator, we choose here to implement the observer dynamics in the z-coordinates, as in the proof of Theorem 1, where $\hat{z}_{no}$ replaces $\hat{\xi}_{no}$ with dynamics (19) and $\hat{\xi}_{no}$ is recovered from $\hat{z}_{no}$ with (18b). Actually, exploiting the fact that ξ_{no} is constant in this application and that the estimate $\hat{\xi}_{no}$ is only needed at the jumps in (19) for the observer, we propose to obtain $\hat{\xi}_{no}$ using (18b) not at all times, but at some (possibly varying) rate, to allow for a sporadic but more precise computation. In our case, we use (18b) to get $\hat{\xi}_{no}$ after each 0.0001 (s) and hold its value in between the samples.

Simulation results are shown in Figure 3. Before the first impact, we estimate ξ_o thanks to its instantaneous observability, while ξ_{no} is not available yet. The convergence starts right after the first impact, when ξ_o, which now contains information about ξ_{no}, becomes observable from y_c during the next flow interval. This illustrates the idea of the fictitious output, an interesting phenomenon brought by the hybrid nature of the system.

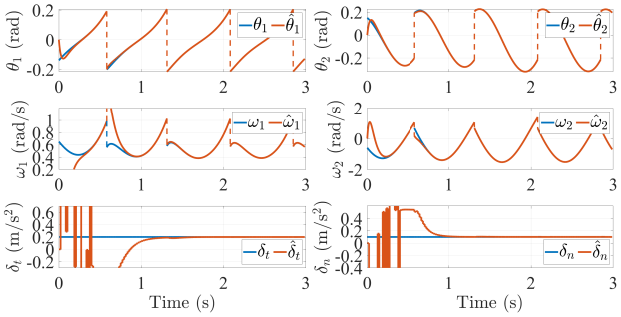

Fig. 3. State and impact uncertainty estimation in a bipedal robot based on quadratic detectability.

4 Observer design based on uniform backward distinguishability

In this section, we replace the LMI-based design of the observer for ξ_{no} in system (1) with a KKL-based one, updating the gains K_d and L_d dynamically along the solution to this system, instead of solving (13). To do that, we rely on the KKL observer design proposed in [36] applied to the equivalent discrete-time system (11).

4.1 *Discrete-time KKL observer design for system* (11)

Consider the discrete-time system (11) for a given sequence $(\xi_{o,k}, u_{d,k}, \tau_k)_{k\in\mathbb{N}}$. Following the spirit of the KKL observer [36], we search for a transformation $(T_k)_{k\in\mathbb{N}}$ such that, in the new coordinates $\zeta_k := T_k \xi_{no,k} \in \mathbb{R}^{n_\zeta}$ (with $n_\zeta \in \mathbb{N}$ to be defined later), system (11) follows the dynamics

$$\zeta_{k+1} = \gamma A \zeta_k + B y_k, \tag{33}$$

where $A \in \mathbb{R}^{n_\zeta \times n_\zeta}$ is Schur, $B \in \mathbb{R}^{n_\zeta \times n_{y,d,\mathrm{ext}}}$ (recall that $n_{y,d,\mathrm{ext}} = n_{y,d} + n_o$) is such that the pair (A, B) is controllable, and $\gamma \in (0, 1]$ is a design parameter. Then,

it follows that the transformation $(T_k)_{k\in\mathbb{N}}$ must be such that for the given sequence $(\xi_{o,k}, u_{d,k}, \tau_k)_{k\in\mathbb{N}}$,

$$T_{k+1}\mathcal{J}_k = \gamma A T_k + B\mathcal{H}_k, \tag{34}$$

where we use the abbreviations $\mathcal{J}_k = \mathcal{J}_{no}(\xi_{o,k}, u_{d,k}, \tau_k)$ and $\mathcal{H}_k = \mathcal{H}(\xi_{o,k}, u_{d,k}, \tau_k)$. The interest of this form is that the observer for system (33) in the ζ-coordinates is a simple filter of the output, namely,

$$\hat{\zeta}_{k+1} = \gamma A\hat{\zeta}_k + By_k, \tag{35}$$

making the estimation error $\tilde{\zeta}_k = \zeta_k - \hat{\zeta}_k$ verify

$$\tilde{\zeta}_{k+1} = \gamma A\tilde{\zeta}_k, \tag{36}$$

and thus exponentially stable because γA is Schur. Then, if $(T_k)_{k\in\mathbb{N}}$ is uniformly left-invertible after some discrete time $k^\star \in \mathbb{N}$, the estimate defined by $\hat{\xi}_{no,k} = T_k^*\hat{\zeta}_k$, where $(T_k^*)_{k\in\mathbb{N}}$ is a bounded sequence of left inverses of $(T_k)_{k\in\mathbb{N}}$ verifying $T_k^*T_k = \mathrm{Id}$ for all $k \in \mathbb{N}_{\geq k^\star}$, is such that the estimation error $\xi_{no,k} - \hat{\xi}_{no,k}$ is exponentially stable and converges to zero after $k^\star$. From [36, Corollary 1], we know that this is possible when the following property holds.

Definition 3. *System* (11)*, fed with* $(\xi_{o,k}, u_{d,k}, \tau_k)_{k\in\mathbb{N}}$ *such that the matrix* $\mathcal{J}_k$ *is invertible for all* $k \in \mathbb{N}$*, is* uniformly backward distinguishable *if, there exists* $\alpha > 0$ *and for each* $i \in \{1, 2, \ldots, n_{y,d,\mathrm{ext}}\}$*, there exists* $m_i \in \mathbb{N}_{>0}$ *such that, for all* $k \in \mathbb{N}_{\geq\overline{m}}$ *where* $\overline{m} := \max_{i\in\{1,2,\ldots,n_{d,\mathrm{ext}}\}} m_i$*, the backward distinguishability matrix sequence* $(\mathcal{O}_k^{bw})_{k\in\mathbb{N}}$ *defined as*

$$\mathcal{O}_k^{bw} = (\mathcal{O}_{1,k}^{bw}, \mathcal{O}_{2,k}^{bw}, \ldots, \mathcal{O}_{n_{y,d,\mathrm{ext}},k}^{bw}) \in \mathbb{R}^{(\sum_{i=1}^{n_{y,d,\mathrm{ext}}} m_i)\times n_{no}}, \tag{37a}$$

where

$$\mathcal{O}_{i,k}^{bw} := \begin{pmatrix} \mathcal{H}_{i,k-1}\mathcal{J}_{k-1}^{-1} \\ \mathcal{H}_{i,k-2}\mathcal{J}_{k-2}^{-1}\mathcal{J}_{k-1}^{-1} \\ \cdots \\ \mathcal{H}_{i,k-m_i}\mathcal{J}_{k-m_i}^{-1}\mathcal{J}_{k-(m_i-1)}^{-1}\cdots\mathcal{J}_{k-1}^{-1} \end{pmatrix}, \tag{37b}$$

where $\mathcal{H}_{i,k}$ *denotes the* i^{th} *row of* $\mathcal{H}_k$*, has full rank and satisfies* $(\mathcal{O}_k^{bw})^\top\mathcal{O}_k^{bw} \geq \alpha\,\mathrm{Id} > 0$.

The property in Definition 3 is equivalent to the fact that

$$\sum_{i=1}^{n_{y,d,\mathrm{ext}}}\sum_{j=k-m_i}^{k-1} \star^\top\mathcal{H}_{i,j}\mathcal{J}_j^{-1}\ldots\mathcal{J}_{k-2}^{-1}\mathcal{J}_{k-1}^{-1} \geq \alpha\,\mathrm{Id} > 0,$$

so that, when all observability indexes m_i are equal, it coincides with Kalman's *uniform complete observability* (see [51, Condition (13)], [52, Assumption 2-3], and [53, Definition 3]). This property can also be checked using the *forward* observability matrix as in [36, Remark 4], which is much easier to compute, when all the m_i are the same and there exists $c_{\mathcal{J}} > 0$ such that $(\mathcal{J}_k^{-1})^\top\mathcal{J}_k^{-1} \geq c_{\mathcal{J}}\,\mathrm{Id}$ for all $k \in \mathbb{N}$. Note that a discrete-time Kalman-like observer [52,53] could seem like a possible alternative to the KKL one since it requires the same observability condition as we have just shown and exhibits a strict Lyapunov function. However, the gain that is multiplied with the fictitious output in the observer must be *constant* during flows for us to perform the analysis (similar to K_d in (13)), which is not the case in a Kalman-like observer (unless the pair $(\mathcal{J}_{no}(\xi_o, u_d, \tau), \mathcal{H}_{dno}(\xi_o, u_d, \tau))$ at the jump times along the solution is UCO, without the need for the fictitious output). This constancy of the gain is ensured in the KKL-based design since it relies on a transformation into a time-invariant form with a chosen constant gain B_{ono} (see below in the proof of Theorem 2). But indeed, if UCO is guaranteed from y_d and without the fictitious output, one could use the Kalman-like observer [52,53] in place of the KKL-based one for a more systematic design.

Lemma 1, which is a particular case of [36, Theorems 2 and 3], then states the existence of $(T_k)_{k\in\mathbb{N}}$ that is uniformly bounded and uniformly left-invertible after some time, assuming i) uniform invertibility of the matrix sequence $(\mathcal{J}_{no}(\xi_{o,k}, u_{d,k}, \tau_k))_{k\in\mathbb{N}}$, ii) uniform boundedness of $(\mathcal{H}_{dno}(\xi_{o,k}, u_{d,k}, \tau_k))_{k\in\mathbb{N}}$ and $(\mathcal{J}_{ono}(\xi_{o,k}, u_{d,k}, \tau_k))_{k\in\mathbb{N}}$, and iii) uniformly backward distinguishability of the discrete-time system (11). Note that the injectivity of each T_k without uniformity in time can be obtained from (non-uniform) backward distinguishability conditions [36, Section V], which may suffice in some cases to ensure convergence of the KKL observer as seen in [36, Example 1], but is not sufficient in general.

Lemma 1. *Consider* $c_{\mathcal{J}_{no}^{-1}} > 0$*,* $c_{\mathcal{H}_{dno}} > 0$*,* $c_{\mathcal{J}_{ono}} > 0$*,* $\alpha > 0$*,* $c_{T,0} > 0$*, and for each* $i \in \{1, 2, \ldots, n_{y,d,\mathrm{ext}}\}$*, a positive integer* m_i *and a controllable pair* $(\tilde{A}_i, \tilde{B}_i) \in \mathbb{R}^{m_i\times m_i} \times \mathbb{R}^{m_i}$ *with* $\tilde{A}_i$ *Schur. Define* $n_\zeta = \sum_{i=1}^{n_{y,d,\mathrm{ext}}} m_i$*. For any* $k_m^\star \in \mathbb{N}_{\geq\overline{m}+1}$ *where* $\overline{m} = \max_{i\in\{1,2,\ldots,n_{d,\mathrm{ext}}\}} m_i$*, there exists* $0 < \gamma^\star \leq 1$ *such that for any* $0 < \gamma < \gamma^\star$*, there exist* $\underline{c}_T(\gamma) > 0$*,* $\overline{c}_T(\gamma) > 0$*, and an integer* $k^\star \leq k_m^\star$ *such that for any* $T_0 \in \mathbb{R}^{n_\zeta\times n_{no}}$ *verifying* $\|T_0\| \leq c_{T,0}$ *and for any sequence* $(\xi_{o,k}, u_{d,k}, \tau_k)_{k\in\mathbb{N}}$ *such that:*

- *(i) The matrix* $\mathcal{J}_{no}(\xi_{o,k}, u_{d,k}, \tau_k)$ *is invertible and* $\|(\mathcal{J}_{no}(\xi_{o,k}, u_{d,k}, \tau_k))^{-1}\| \leq c_{\mathcal{J}_{no}^{-1}}$ *for all* $k \in \mathbb{N}$*;*
- *(ii)* $\|\mathcal{H}_{dno}(\xi_{o,k}, u_{d,k}, \tau_k)\| \leq c_{\mathcal{H}_{dno}}$ *and* $\|J_{ono}(\xi_{o,k}, u_{d,k}, \tau_k)\| \leq c_{\mathcal{J}_{ono}}$ *for all* $k \in \mathbb{N}$*;*
- *(iii) System* (11) *fed with* $(\xi_{o,k}, u_{d,k}, \tau_k)_{k\in\mathbb{N}}$ *is uniformly backward distinguishable with parameters* α *and* m_i *(as in Definition 3);*

we have that the (unique) sequence $(T_k)_{k\in\mathbb{N}}$ *initialized as*

T_0 and satisfying (34) *with*

$$A = \mathrm{diag}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_{n_{y,d,\mathrm{ext}}}) \in \mathbb{R}^{n_\zeta \times n_\zeta}, \tag{38a}$$
$$B = \mathrm{diag}(\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_{n_{y,d,\mathrm{ext}}}) \in \mathbb{R}^{n_\zeta \times n_{y,d,\mathrm{ext}}}, \tag{38b}$$

verifies $\|T_k\| \leq \bar{c}_T(\gamma)$ *for all* $k \in \mathbb{N}$ *and* $T_k^\top T_k \geq (\underline{c}_T(\gamma))^2 \,\mathrm{Id}$ *for all* $k \in \mathbb{N}_{\geq k^\star}$.

Proof. The sequence $(T_k)_{k\in\mathbb{N}}$ is defined uniquely from T_0 by (34) and the assumed invertibility of $\mathcal{J}_{no}(\xi_{o,k}, u_{d,k}, \tau_k)$. The rest is a particular case of [36, Theorems 2 and 3]. □

Next, in Section 4.2, we exploit the results from this section for system (1).

4.2 KKL-based observer design for system (1)

Inspired by the developments in the previous section, we make the following assumption.

Assumption 4. *Given j_m defined in Item (A1.1) of Assumption 1, assume that there exist $c_{\mathcal{J}_{no}^{-1}} > 0$, $\alpha > 0$, and $m_i \in \mathbb{N}_{>0}$ for each $i \in \{1, 2, \dots, n_{y,d,\mathrm{ext}}\}$, such that along every complete solution $\xi \in \mathcal{S}_{\mathcal{H}}(\Xi_0, \mathfrak{U}_c \times \mathfrak{U}_d)$, the sequences $(\xi_{o,j}, u_{d,j}, \tau_j)_{j\in\mathbb{N}_{\geq j_m}}$ defined as $\xi_{o,j} = \xi_o(t_{j+1}, j)$, $u_{d,j} = \mathfrak{u}_d(j)$, and $\tau_j = t_{j+1} - t_j$ for each $j \in \mathbb{N}_{\geq j_m}$, are such that:*

- *(A4.1) The matrix $\mathcal{J}_{no}(\xi_{o,j}, u_{d,j}, \tau_j)$ is invertible and $\|(\mathcal{J}_{no}(\xi_{o,j}, u_{d,j}, \tau_j))^{-1}\| \leq c_{\mathcal{J}_{no}^{-1}}$ for all $j \in \mathbb{N}_{\geq j_m}$;*
- *(A4.2) System* (11) *fed with $(\xi_{o,j}, u_{d,j}, \tau_j)_{j\in\mathbb{N}_{\geq j_m}}$ is uniformly backward distinguishable with parameters α and m_i for each $i \in \{1, 2, \dots, n_{y,d,\mathrm{ext}}\}$.*

Remark 4. *Contrary to what is typically the case in discrete-time systems obtained from discretizing a physical system, the jump map of a hybrid system may not be invertible since it is not a discretization of some continuous-time dynamics. However, here $\mathcal{J}_{no}$ combines flow and jump dynamics, i.e., models the dynamics of the hybrid system sampled at jumps. Thus, it is reasonable to expect its invertibility along solutions. Besides, in applications of jump parameter estimation $\mathcal{J}_{no}(\xi_o, u_d, \tau) = \mathrm{Id}$ at all times. Note though that, while the invertibility of the dynamics is sufficient but typically not necessary for linear Kalman(-like) designs [54,33] (this assumption is generally used for analysis only), it is here required to implement the observer (see the dynamics $\hat{T}^+$ in observer* (39) *below), so it plays a much more crucial role.*

A difficulty in exploiting the discrete-time KKL observer (35) is that the discrete-time output y_k is not fully available at jumps since it contains fictitious outputs. The KKL-based observer we propose for system (1) has the form

$$\begin{cases} \dot{\hat{\xi}}_o = \hat{f}_{o,\ell}(\hat{\xi}_o, \tau, y_c, u_c) \qquad \dot{\hat{\eta}} = 0 \qquad \dot{\hat{T}} = 0 \qquad \dot{\tau} = 1 \\ \\ \hat{\xi}_o^+ = J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) + J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\,\mathrm{sat}_{no}(\hat{\xi}_{no}) \\ \hat{\eta}^+ = \gamma A\hat{\eta} + \gamma A B_{ono} \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\hat{\xi}_o, t, -\tau) \\ \qquad\quad + \gamma A \hat{T} \displaystyle\int_0^\tau e^{-F_{no}s} U_{cno} ds \\ \qquad\quad + B_{dno}\Big(y_d - H_{do}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\Big) \\ \qquad\quad + \hat{T}^+ J_{noo}(\mathrm{sat}_o(\hat{\xi}_o), u_d) - B_{ono} J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) \\ \hat{T}^+ = \big(\gamma A \hat{T} + B_{dno}\mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau) \\ \qquad\quad + B_{ono}\mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)\big) \times \\ \qquad\quad \times \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)) \\ \tau^+ = 0 \end{cases} \tag{39a}$$

with

$$\hat{\xi}_{no} = e^{F_{no}\tau}\,\mathrm{inv}_{\underline{c}_T(\gamma)}(\hat{T})(\hat{\eta} + B_{ono}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\hat{\xi}_o, t, -\tau)) + \int_0^\tau e^{F_{no}(\tau-s)} U_{cno} ds, \tag{39b}$$

with flows and jumps triggered at the same time as system (1), with $\hat{f}_{o,\ell}$ being a high-gain observer (7), $(\mathcal{J}_{no}, \mathcal{H}_{dno}, \mathcal{J}_{ono})$ defined in system (11), sat_o being a bounded map such that $\mathrm{sat}_o(\hat{\xi}_o) = \hat{\xi}_o$ for all $\hat{\xi}_o \in (\Xi_o \cap D_o) + \bar{c}_o\mathbb{B}$ for some $\bar{c}_o > 0$ to be designed, sat_{no} and $f_{o,\mathrm{sat}}$ fixed as in observer (12), $\ell > 0$, $\gamma \in (0, 1]$ and $(A, B_{dno}, B_{ono}) \in \mathbb{R}^{n_\eta \times n_\eta} \times \mathbb{R}^{n_\eta \times n_{y,d}} \times \mathbb{R}^{n_\eta \times n_o}$ (for some dimension $n_\eta \in \mathbb{N}$ to be defined later) being design parameters to be chosen, $\mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}$ and $\mathrm{inv}_{\underline{c}_T(\gamma)}$ being extended left inverse maps defined as

$$M \mapsto \mathrm{inv}_{\underline{c}_M}(M) = \begin{cases} M^\dagger & \text{if } \sigma_{\min}(M) \geq \underline{c}_M \\ \frac{\sigma_{\min}(M)}{\underline{c}_M} M^\dagger & \text{if } 0 < \sigma_{\min}(M) \leq \underline{c}_M \\ 0 & \text{if } \sigma_{\min}(M) = 0, \end{cases} \tag{39c}$$

for some fixed saturation level $\underline{c}_{\hat{\mathcal{J}}_{no}} < \frac{1}{c_{\mathcal{J}_{no}^{-1}}}$ with $c_{\mathcal{J}_{no}^{-1}}$ given in Item (A4.1) of Assumption 4 and some saturation level $\underline{c}_T(\gamma)$ to be designed. Those dynamics are picked so that, modulo some estimation errors on ξ_o, $\hat{T}$ coincides at jumps with $(T_k)_{k\in\mathbb{N}}$ studied in Section 4.1 and the corresponding discrete-time KKL error dynamics (36) appear in some way after a certain change of coordinates.

Theorem 2. *Suppose Assumptions 1, 2, and 4 hold. Define $n_\eta = n_\zeta = \sum_{i=1}^{n_{y,d,\mathrm{ext}}} m_i$ (with m_i coming from Item (A4.2) of Assumption 4) and consider, for each $i \in \{1, 2, \dots, n_{y,d,\mathrm{ext}}\}$, a controllable pair $(\tilde{A}_i, \tilde{B}_i) \in$*

$\mathbb{R}^{m_i\times m_i}\times\mathbb{R}^{m_i}$ *with* $\tilde{A}_i$ *Schur. Let*

$$A=\operatorname{diag}(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_{n_{y,d,\text{ext}}})\in\mathbb{R}^{n_\eta\times n_\eta},\tag{40a}$$
$$B_{dno}=\operatorname{diag}(\tilde{B}_1,\tilde{B}_2,\dots,\tilde{B}_{n_{y,d}})\in\mathbb{R}^{n_\eta\times n_{y,d}},\tag{40b}$$
$$B_{ono}=\operatorname{diag}(\tilde{B}_{n_{y,d}+1},\tilde{B}_{n_{y,d}+2},\dots,\tilde{B}_{n_{y,d,\text{ext}}})\in\mathbb{R}^{n_\eta\times n_o}.\tag{40c}$$

There exists $\overline{c}_o>0$ *in the definition of* sat_o *such that, given any* $T_0\in\mathbb{R}^{n_\eta\times n_{no}}$*, there exist* $0<\gamma^\star\le 1$ *and* $\ell^\star>0$ *such that for any* $0<\gamma<\gamma^\star$ *and for any* $\ell>\ell^\star$*, there exists* $\underline{c}_T(\gamma)>0$ *such that any maximal solution to the cascade* (1)-(39) *initialized in* $\Xi_0\times\mathbb{R}^{n_o}\times\mathbb{R}^{n_\eta}\times\{T_0\}\times\{0\}$ *with inputs* $(\mathfrak{u}_c,\mathfrak{u}_d)\in\mathfrak{U}_c\times\mathfrak{U}_d$*, is complete and verifies* (6).

Remark 5. *Note that in this KKL-based design, as a difference to the LMI-based one in Theorem 1, the stability of the estimation error with respect to its initial conditions cannot be formulated. This is because the KKL estimate* $\hat{\xi}_{no}$ *is fed back to* $\hat{\xi}_o^+$ *which will then enter* $\hat{T}^+$ *and eventually* $\hat{\eta}^+$*: even if the estimates are initialized exactly at the right values of the states, the non-injectivity of* $\hat{T}$ *before a certain time could still trigger an error in* $\hat{\xi}_{no}$*, which would propagate in the whole estimate. However, the asymptotic convergence in* (6) *still guarantees the estimation of* ξ*. Actually, exponential convergence of* (ξ,T)*, with any arbitrarily fast rate, can be achieved by selecting* ℓ *large and* γ *small, but only after a certain number of jumps and with respect to the estimation error at that later time.*

Proof. This proof consists of three main parts: i) Choose $\overline{c}_o$ (for the definition of sat_o), $\gamma^\star$, $\ell^\star$, and $\underline{c}_T(\gamma)$ allowing us to guarantee some preliminary bounds and injectivity properties on the maps and variables, ii) Define a change of coordinates for the ξ_{no} state into some target η-coordinates, and obtain the state dynamics in those new coordinates (along solutions), and iii) Define a Lyapunov function and apply [23, Theorem 2] to obtain exponential stability in the new coordinates after a certain time and retrieve asymptotic convergence in the ξ-coordinates.

Let us begin with the first preliminary part of this proof. We start by choosing $\overline{c}_o$. Along every complete solution $\xi\in\mathcal{S}_{\mathcal{H}}(\Xi_0,\mathfrak{U}_c\times\mathfrak{U}_d)$, according to Item (A1.1) of Assumption 1 and Item (A4.1) of Assumption 4, denoting $\xi_{o,j}=\xi_o(t_{j+1},j)$, $u_{d,j}=\mathfrak{u}_d(j)$, and $\tau_j=t_{j+1}-t_j$, we have $\xi_{o,j}\in\Xi_o$ for all $j\in\mathbb{N}$ and

$$\|(\mathcal{J}_{no}(\xi_{o,j},u_{d,j},\tau_j))^{-1}\|=\sqrt{\lambda_{\max}((\star^\top(\mathcal{J}_{no}(\xi_{o,j},u_{d,j},\tau_j))^{-1})}\le c_{\mathcal{J}_{no}^{-1}},$$

for all $j\in\mathbb{N}_{\ge j_m}$. It follows that $(\star^\top(\mathcal{J}_{no}(\xi_{o,j},u_{d,j},\tau_j))^{-1}\le c^2_{\mathcal{J}_{no}^{-1}}\,\text{Id}$ and thus we have $\star^\top\mathcal{J}_{no}(\xi_{o,j},u_{d,j},\tau_j)\ge\frac{1}{c^2_{\mathcal{J}_{no}^{-1}}}\,\text{Id}$ for all $j\in\mathbb{N}_{\ge j_m}$. Applying Lemma 7 in Appendix A after jump j_m with M therein being $\mathcal{J}_{no}$, the sequences $(\xi_j)_{j\in\mathbb{N}}$ and $(u_j)_{j\in\mathbb{N}}$ therein being $(\xi_{o,j})_{j\in\mathbb{N}_{\ge j_m}}$ and $(u_{d,j},\tau_j)_{j\in\mathbb{N}_{\ge j_m}}$ respectively, the scalar c therein being $\frac{1}{c^2_{\mathcal{J}_{no}^{-1}}}$, and the scalar c' therein being $\underline{c}^2_{\hat{\mathcal{J}}_{no}}$ for the chosen $\underline{c}_{\hat{\mathcal{J}}_{no}}<\frac{1}{c_{\mathcal{J}_{no}^{-1}}}$ in the definition of $\text{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}$, we deduce that there exists $\overline{c}_{o,1}>0$ such that if $|\xi_{o,j}-\hat{\xi}_{o,j}|\le\overline{c}_{o,1}$ for all $j\in\mathbb{N}_{\ge j_m}$, then $\star^\top\mathcal{J}_{no}(\hat{\xi}_{o,j},u_{d,j},\tau_j)\ge\underline{c}^2_{\hat{\mathcal{J}}_{no}}\,\text{Id}$ for all $j\in\mathbb{N}_{\ge j_m}$, which implies that $(\mathcal{J}_{no}(\hat{\xi}_{o,j},u_{d,j},\tau_j))_{j\in\mathbb{N}_{\ge j_m}}$ is uniformly invertible and is such that $\text{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\hat{\xi}_{o,j},u_{d,j},\tau_j))=(\mathcal{J}_{no}(\hat{\xi}_{o,j},u_{d,j},\tau_j))^{-1}$ for all $j\in\mathbb{N}_{\ge j_m}$.

Next, from Item (A1.3) of Assumption 1 and the compactness of $\mathcal{I}$, the maps $\mathcal{J}_{no}$, $\mathcal{H}_{dno}$, and $\mathcal{J}_{ono}$ are locally Lipschitz with respect to ξ_o, uniformly in $(u_d,\tau)\in\mathcal{U}_d\times\mathcal{I}$, and so are the maps $\mathcal{O}_j^{bw}$ defined in (37) for all $j\in\mathbb{N}_{\ge j_m+\overline{m}}$. Applying Lemma 7 in Appendix A after jump $j_m+\overline{m}$ with M therein being $\mathcal{O}_j^{bw}$, the sequence $(\xi_j)_{j\in\mathbb{N}}$ therein being $(\xi_{o,j-1},\xi_{o,j-2},\dots,\xi_{o,j-\overline{m}})_{j\in\mathbb{N}_{\ge j_m+\overline{m}}}$, the sequence $(u_j)_{j\in\mathbb{N}}$ therein being

$$(u_{d,j-1},u_{d,j-2},\dots,u_{d,j-\overline{m}},\tau_{j-1},\tau_{j-2},\dots,\tau_{j-\overline{m}})_{j\in\mathbb{N}_{\ge j_m+\overline{m}}},$$

the scalar c therein being α, and the scalar c' therein being any $0<\alpha'<\alpha$, we deduce that there exists $\overline{c}_{o,2}>0$ such that if $|\xi_{o,j}-\hat{\xi}_{o,j}|\le\overline{c}_{o,2}$ for all $j\in\mathbb{N}_{\ge j_m+\overline{m}}$, then $(\hat{\mathcal{O}}_j^{bw})_{j\in\mathbb{N}_{\ge j_m+\overline{m}}}$ defined as in (37) but fed with $(\hat{\xi}_{o,j},u_{d,j},\tau_j)_{j\in\mathbb{N}_{\ge j_m+\overline{m}}}$ is such that each $\hat{\mathcal{O}}_j^{bw}$ has full rank and satisfies $(\hat{\mathcal{O}}_j^{bw})^\top\hat{\mathcal{O}}_j^{bw}\ge\alpha'\,\text{Id}>0$ for all $j\in\mathbb{N}_{\ge j_m+\overline{m}}$.

Pick $\overline{c}_o=\min\{\overline{c}_{o,1},\overline{c}_{o,2}\}$. It follows that if $|\xi_{o,j}-\hat{\xi}_{o,j}|\le\overline{c}_o$ for all $j\in\mathbb{N}_{\ge j_m}$, then Item (A4.1) of Assumption 4 holds after jump j_m for $(\mathcal{J}_{no}(\hat{\xi}_{o,j},u_{d,j},\tau_j))_{j\in\mathbb{N}_{\ge j_m}}$ in place of $(\mathcal{J}_{no}(\xi_{o,j},u_{d,j},\tau_j))_{j\in\mathbb{N}_{\ge j_m}}$ with $\frac{1}{\underline{c}_{\hat{\mathcal{J}}_{no}}}$ replacing $c_{\mathcal{J}_{no}^{-1}}$, and Item (A4.2) of Assumption 4 holds after jump $j_m+\overline{m}$ for system (11) fed with $(\hat{\xi}_{o,j},u_{d,j},\tau_j)_{j\in\mathbb{N}_{\ge j_m+\overline{m}}}$ in place of $(\xi_{o,j},u_{d,j},\tau_j)_{j\in\mathbb{N}_{\ge j_m+\overline{m}}}$ with the same m_i, $i\in\{1,2,\dots,n_{d,\text{ext}}\}$, and with α' replacing α.

Now fix $T_0\in\mathbb{R}^{n_\xi\times n_{no}}$. We next choose a first upper bound $\gamma_1^\star$ for γ and $\underline{c}_T(\gamma)$. Thanks to Item (A1.1) of Assumption 1 and Item (A1.3) of Assumption 1, there exists $\overline{c}_{T,m}>0$ such that along any maximal solution $(\xi,\hat{\xi}_o,\hat{\eta},\hat{T},\tau)$ to the cascade (1)-(39) initialized in $\Xi_0\times\mathbb{R}^{n_o}\times\mathbb{R}^{n_\eta}\times\{T_0\}\times\{0\}$ with inputs $(\mathfrak{u}_c,\mathfrak{u}_d)\in\mathfrak{U}_c\times\mathfrak{U}_d$, with $\overline{c}_o$ in sat_o fixed in the previous step, with any $\gamma\in(0,1]$, any $\ell>\ell_1^\star$, and any $\underline{c}_T(\gamma)$, any

solutions $(T_j)_{j\in\mathbb{N}}$ and $(\hat{T}_j)_{j\in\mathbb{N}}$ to

$$\begin{aligned} T_{j+1} &= (\gamma A T_j + B_{dno}\mathcal{H}_{dno}(\xi_{o,j}, u_{d,j}, \tau_j) \\ &\quad + B_{ono}\mathcal{J}_{ono}(\xi_{o,j}, u_{d,j}, \tau_j))\times \\ &\quad \times \mathrm{inv}_{\underline{c}_{\mathcal{J}_{no}}}(\mathcal{J}_{no}(\xi_{o,j}, u_{d,j}, \tau_j)), \qquad (41a) \\ \hat{T}_{j+1} &= (\gamma A \hat{T}_j + B_{dno}\mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_{o,j}), u_{d,j}, \tau_j) \\ &\quad + B_{ono}\mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_{o,j}), u_{d,j}, \tau_j))\times \\ &\quad \times \mathrm{inv}_{\underline{c}_{\mathcal{J}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_{o,j}), u_{d,j}, \tau_j)), \qquad (41b) \end{aligned}$$

both initialized as T_0, where $\xi_{o,j} = \xi_o(t_{j+1}, j)$, $u_{d,j} = \mathfrak{u}_d(j)$, $\tau_j = \tau(t_{j+1}, j)$, and $\hat{\xi}_{o,j} = \hat{\xi}_o(t_{j+1}, j)$, are such that

$$\|T_j\| \le \overline{c}_{T,m}, \qquad \|\hat{T}_j\| \le \overline{c}_{T,m}, \qquad \forall j \in \mathbb{N}, j \le j_m. \tag{42}$$

Now, let us study what happens after jump j_m. First, observe that, if $|\xi_o(t_{j+1}, j) - \hat{\xi}_o(t_{j+1}, j)| \le \overline{c}_o$ for all $j \in \mathbb{N}_{\ge j_m}$, $(T_j)_{j\in\mathbb{N}}$ and $(\hat{T}_j)_{j\in\mathbb{N}}$ are solution to (34) after jump j_m, fed respectively with $(\xi_{o,j}, u_{d,j}, \tau_j)_{j\in\mathbb{N}}$ and $(\mathrm{sat}_o(\hat{\xi}_{o,j}), u_{d,j}, \tau_j)_{j\in\mathbb{N}}$. From Item (A1.3) of Assumption 1, since $\tau_j \in \mathcal{I}$ after jump j_m, there exist $c_{\mathcal{H}_{dno}} > 0$ and $c_{\mathcal{J}_{ono}} > 0$ such that all these matrix sequences are uniformly bounded for all $j \in \mathbb{N}_{\ge j_m}$:

$$\begin{aligned} \|\mathcal{H}_{dno}(\xi_{o,j}, u_{d,j}, \tau_j)\| &\le c_{\mathcal{H}_{dno}}, \\ \|\mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_{o,j}), u_{d,j}, \tau_j)\| &\le c_{\mathcal{H}_{dno}}, \\ \|J_{ono}(\xi_{o,j}, u_{d,j}, \tau_j)\| &\le c_{\mathcal{J}_{ono}}, \\ \|J_{ono}(\mathrm{sat}_o(\hat{\xi}_{o,j}), u_{d,j}, \tau_j)\| &\le c_{\mathcal{J}_{ono}}. \end{aligned}$$

Then, according to Lemma 1 starting from j_m (i.e., with $\overline{c}_{T,m}$ playing the role of $\overline{c}_{T,0}$), for any $j_m^\star \in \mathbb{N}_{\ge j_m+\overline{m}+1}$, there exists $0 < \gamma_1^\star \le 1$ such that for all $0 < \gamma < \gamma_1^\star$, there exist $\underline{c}_T(\gamma) > 0$ and $\overline{c}_T(\gamma) > 0$ such that for any maximal solution $(\xi, \hat{\xi}_o, \hat{\eta}, \hat{T}, \tau)$ to the cascade (1)-(39) initialized in $\Xi_0 \times \mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \{T_0\} \times \{0\}$ with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$, and verifying $|\xi_o(t_{j+1}, j) - \hat{\xi}_o(t_{j+1}, j)| \le \overline{c}_o$ for all $j \in \mathbb{N}_{\ge j_m}$, the solutions $(T_j)_{j\in\mathbb{N}}$ and $(\hat{T}_j)_{j\in\mathbb{N}}$ are both uniformly left-invertible for all $j \in \mathbb{N}_{\ge j_m^\star}$ and uniformly bounded for all $j \in \mathbb{N}_{\ge j_m}$, i.e.,

$$\begin{aligned} T_j^\top T_j &\ge (\underline{c}_T(\gamma))^2 \,\mathrm{Id}, && \forall j \in \mathbb{N}_{\ge j_m^\star}, \\ \|T_j\| &\le \overline{c}_T(\gamma), && \forall j \in \mathbb{N}_{\ge j_m}, \\ \hat{T}_j^\top \hat{T}_j &\ge (\underline{c}_T(\gamma))^2 \,\mathrm{Id}, && \forall j \in \mathbb{N}_{\ge j_m^\star}, \\ \|\hat{T}_j\| &\le \overline{c}_T(\gamma), && \forall j \in \mathbb{N}_{\ge j_m}. \end{aligned} \tag{43}$$

Finally, we pick a lower bound $\ell_2^\star$ for ℓ. Exploiting exponential decrease over rational growth, given $\overline{c}_o$ picked above and using the compactness of Ξ_o, let $\ell_2^\star \ge \ell_1^\star$ (defined in Assumption 2) such that, for all $\ell > \ell_2^\star$,

$$\sqrt{\frac{\overline{b}_o}{\underline{b}_o(\ell)}}\left(\max_{\xi_o\in\Xi_o}|\xi_o| + M_o\right)e^{-\ell\frac{\lambda_c}{2}\tau_m} \le \overline{c}_o, \tag{44}$$

where $\overline{b}_o$, $\underline{b}_o$, λ_c are defined in Assumption 2, $\tau_m := \min\mathcal{I} > 0$, and $M_o > 0$ is a bound of $J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) + J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\,\mathrm{sat}_{no}(\hat{\xi}_{no})$ (uniform in $u_d \in \mathcal{U}_d$) obtained from the definitions of sat_o, sat_{no}, and Item (A1.3) of Assumption 1.

Now, pick $0 < \gamma < \gamma_1^\star$ and $\ell > \ell_2^\star$. Consider a solution $(\xi, \hat{\xi}_o, \hat{\eta}, \hat{T}, \tau)$ to the cascade (1)-(39) initialized in $\Xi_0 \times \mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \{T_0\} \times \{0\}$ with inputs $(\mathfrak{u}_c, \mathfrak{u}_d) \in \mathfrak{U}_c \times \mathfrak{U}_d$ (which is complete thanks to Item (A1.1) of Assumption 1 and given the dynamics of observer (39) which do not allow finite-time escape thanks to saturation functions and the local boundedness of the maps of system (1) as in Item (A1.3) of Assumption 1), with the chosen (A, B_{dno}, B_{ono}), the saturation maps sat_o, sat_{no}, and the inverse maps $\mathrm{inv}_{\underline{c}_{\mathcal{J}_{no}}}$, and $\mathrm{inv}_{\underline{c}_T(\gamma)}$. In the following, we refer to the jump times of this solution as t_j instead of $t_j(\xi)$ to ease the notations. First, from i) Assumption 2, ii) the fact that $t_{j+1} - t_j \ge \tau_m$ according to Item (A1.1) of Assumption 1, and iii) the choice of $\ell > \ell_2^\star$ satisfying (44), we have for all $j \in \mathbb{N}_{\ge j_m}$,

$$\begin{aligned} &|\xi_o(t_{j+1}, j) - \hat{\xi}_o(t_{j+1}, j)| \\ &\le \sqrt{\frac{\overline{b}_o}{\underline{b}_o(\ell)}}|\xi_o(t_j, j) - \hat{\xi}_o(t_j, j)|e^{-\ell\frac{\lambda_c}{2}\tau_m} \\ &\le \sqrt{\frac{\overline{b}_o}{\underline{b}_o(\ell)}}\left(\max_{\xi_o\in\Xi_o}|\xi_o| + M_o\right)e^{-\ell\frac{\lambda_c}{2}\tau_m} \le \overline{c}_o. \end{aligned}$$

Then, by the definition of sat_o and since $\xi_o(t_{j+1}, j) \in \Xi_o \cap D_o$, we have, for all $j \in \mathbb{N}_{\ge j_m}$,

$$\mathrm{sat}_o(\hat{\xi}_o(t_{j+1}, j)) = \hat{\xi}_o(t_{j+1}, j).$$

Over the solution's time domain, we finally introduce the hybrid arc T with the same dimension as $\hat{T}$ and initialized as $T(0, 0) = T_0$, with dynamics $\dot{T} = 0$ during flows and at jumps given by

$$\begin{aligned} T^+ &= (\gamma A T + B_{dno}\mathcal{H}_{dno}(\xi_o, u_d, \tau) \\ &\quad + B_{ono}\mathcal{J}_{ono}(\xi_o, u_d, \tau))\,\mathrm{inv}_{\underline{c}_{\mathcal{J}_{no}}}(\mathcal{J}_{no}(\xi_o, u_d, \tau)). \end{aligned} \tag{45}$$

Since $\dot{T} = 0$ and $\dot{\hat{T}} = 0$ during flows, $(T(t_j, j))_{j\in\mathbb{N}}$ and $(\hat{T}(t_j, j))_{j\in\mathbb{N}}$ coincide for all $j \in \mathbb{N}$ respectively with $(T_j)_{j\in\mathbb{N}}$ and $(\hat{T}_j)_{j\in\mathbb{N}}$ studied above, with $\xi_{o,j} = \xi_o(t_{j+1}, j)$, $\hat{\xi}_{o,j} = \hat{\xi}_o(t_{j+1}, j)$, $u_{d,j} = \mathfrak{u}_d(j)$, and $\tau_j = t_{j+1} - t_j = \tau(t_{j+1}, j)$ for all $j \in \mathbb{N}_{\ge j_m}$. Therefore, since $0 < \gamma < \gamma_1^\star$ and $|\xi_o(t_{j+1}, j) - \hat{\xi}_o(t_{j+1}, j)| \le \overline{c}_o$ for all $j \in \mathbb{N}_{\ge j_m}$, we deduce from above that

$$(T(t,j))^\top T(t,j) \ge (\underline{c}_T(\gamma))^2\,\mathrm{Id}, \forall (t,j) \in \mathrm{dom}\,\xi_o, j \ge j_m^\star, \tag{46a}$$

$$(\hat{T}(t,j))^\top \hat{T}(t,j) \ge (\underline{c}_T(\gamma))^2\,\mathrm{Id}, \forall (t,j) \in \mathrm{dom}\,\xi_o, j \ge j_m^\star. \tag{46b}$$

It follows that for all $(t,j) \in \operatorname{dom}\xi_o$ with $j \geq j_m^\star$, the map $\operatorname{inv}_{\underline{c}_T(\gamma)}$ defined as in (39c) is such that $\operatorname{inv}_{\underline{c}_T(\gamma)}(T(t,j)) = (T(t,j))^\dagger$ and $\operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T}(t,j)) = (\hat{T}(t,j))^\dagger$, which are left inverses of $T(t,j)$ and $\hat{T}(t,j)$, respectively. Note that we also recover from above that for all $(t,j) \in \operatorname{dom}\xi_o$, $\|T(t,j)\|$ and $\|\hat{T}(t,j)\|$ are upper-bounded by $\bar{c}_{T,m}$ if $j \leq j_m$ and by $\bar{c}_T(\gamma)$ if $j \geq j_m$.
Now, we go to the second part of this proof. To exploit the results in Section 4.1, we define, over the time domain of the considered solution, the hybrid arc $\eta : \operatorname{dom}\xi \to \mathbb{R}^{n_\eta}$ as

$$\begin{aligned}\eta(t,j) = T(t,j)\Psi_{f_{no}}(\xi_{no}(t,j), -\tau(t,j)) \\ - B_{ono}\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o(t,j), t, -\tau(t,j)),\end{aligned} \quad (47)$$

where $\Psi_{f_{no}}(\cdot,\tau)$ is defined as in (17c). Arguing in a similar way as in the proof of Theorem 1 above, we notice that

$$\begin{aligned}&\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o(t,j), t, -\tau(t,j)) \\ &= \Psi_{f_o(\cdot,\mathtt{u}_c)}(\xi_o(t,j), t, -(t-t_j)) = \xi_o(t_j,j) \in \Xi_o, \\ &\Psi_{f_{no}}(\xi_{no}(t,j), -\tau(t,j)) = \xi_{no}(t_j,j) \in \Xi_{no},\end{aligned}$$

for all $j \in \operatorname{dom}_j \xi$ and $t \in [t_j, t_{j+1}]$. We deduce that η remains at all times in the compact set

$$\begin{aligned}\Xi_\eta(\gamma) := \{\eta \in \mathbb{R}^{n_\eta} : \exists (\xi_o, T, \xi_{no}) \in \Xi_o \times \mathcal{P}_T(\gamma) \times \Xi_{no}, \\ \eta = T\xi_{no} - B_{ono}\xi_o\},\end{aligned} \quad (48)$$

with $\mathcal{P}_T(\gamma) := \{T \in \mathbb{R}^{n_\eta \times n_{no}} : \|T\| \leq \max\{\bar{c}_{T,m}, \bar{c}_T(\gamma)\}\}$ where $\bar{c}_{T,m}$ is in (42) and $\bar{c}_T(\gamma)$ is in (43). We deduce also that η verifies $\dot{\eta} = 0$ during flows. From Item (A1.1) of Assumption 1, Item (A4.1) of Assumption 4, and the choice of $\underline{c}_{\hat{\mathcal{J}}_{no}}$, we have after time (t_{j_m}, j_m), $\operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\xi_{o,j}, u_{d,j}, \tau_j)) = (\mathcal{J}_{no}(\xi_o, u_d, \tau))^{-1}$ so that at jumps (using that $\tau^+ = 0$ and so $\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o^+, t^+, -\tau^+) = \Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o^+, t, 0) = \xi_o^+$),

$$\begin{aligned}\eta^+ &= T^+\xi_{no}^+ - B_{ono}\xi_o^+ \\ &= T^+\left(\mathcal{J}_{no}(\xi_o, u_d, \tau)e^{-F_{no}\tau}\xi_{no} + J_{noo}(\xi_o, u_d)\right) \\ &\quad - B_{ono}\left(J_o(\xi_o, u_d) + J_{ono}(\xi_o, u_d)\xi_{no}\right) \\ &= (\gamma AT + B_{dno}\mathcal{H}_{dno}(\xi_o, u_d, \tau) + B_{ono}\mathcal{J}_{ono}(\xi_o, u_d, \tau)) \\ &\quad \times e^{-F_{no}\tau}\xi_{no} + T^+J_{noo}(\xi_o, u_d) \\ &\quad - B_{ono}\left(J_o(\xi_o, u_d) + J_{ono}(\xi_o, u_d)\xi_{no}\right) \\ &= \gamma A\eta + \gamma AB_{ono}\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o, t, -\tau) \\ &\quad + \gamma AT\int_0^\tau e^{-F_{no}s}U_{cno}ds + B_{dno}H_{dno}(\xi_o, u_d)\xi_{no} \\ &\quad + T^+J_{noo}(\xi_o, u_d) - B_{ono}J_o(\xi_o, u_d) \\ &= \gamma A\eta + \gamma AB_{ono}\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o, t, -\tau) \\ &\quad + \gamma AT\int_0^\tau e^{-F_{no}s}U_{cno}ds + B_{dno}(y_d - H_{do}(\xi_o, u_d)) \\ &\quad + T^+J_{noo}(\xi_o, u_d) - B_{ono}J_o(\xi_o, u_d).\end{aligned}$$

On the other hand, from (47), (46) and the expression of $\Psi_{f_{no}}$, we can express ξ_{no} as a function of $(\xi_o, \eta, T, \tau, t)$ and input $\mathtt{u}_c$ after time $(t_{j_m^\star}, j_m^\star)$ as

$$\begin{aligned}\xi_{no} &= \Psi_{f_{no}}\left(T^\dagger(\eta + B_{ono}\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o, t, -\tau)), \tau\right) \\ &= e^{F_{no}\tau}T^\dagger(\eta + B_{ono}\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o, t, -\tau)) \\ &\quad + \int_0^\tau e^{F_{no}(\tau-s)}U_{cno}ds,\end{aligned} \quad (49)$$

which is known to be in Ξ_{no}. With $\hat{\eta}$ and $\hat{T}$ of dynamics as in the observer (39), the estimation errors $\tilde{\eta} := \eta - \hat{\eta}$ and $\tilde{T} := T - \hat{T}$ verify $\dot{\tilde{\eta}} = 0$ and $\dot{\tilde{T}} = 0$ during flows and at jumps, after time (t_{j_m}, j_m),

$$\begin{aligned}&\tilde{T}^+ \\ &= \gamma A(T(\mathcal{J}_{no}(\xi_o, u_d, \tau))^\dagger - \hat{T}\operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau))) \\ &\quad + B_{dno}(\mathcal{H}_{dno}(\xi_o, u_d, \tau)(\mathcal{J}_{no}(\xi_o, u_d, \tau))^\dagger \\ &\quad - \mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)\operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau))) \\ &\quad + B_{ono}(\mathcal{J}_{ono}(\xi_o, u_d, \tau)(\mathcal{J}_{no}(\xi_o, u_d, \tau))^\dagger \\ &\quad - \mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)\operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau))) \\ &= \gamma A\tilde{T}\operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)) + \gamma AT\times \\ &\quad \times ((\mathcal{J}_{no}(\xi_o, u_d, \tau))^\dagger - \operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau))) \\ &\quad + B_{dno}((\mathcal{H}_{dno}(\xi_o, u_d, \tau) - \mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau))\times \\ &\quad \times (\mathcal{J}_{no}(\xi_o, u_d, \tau))^\dagger + \mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)\times \\ &\quad \times ((\mathcal{J}_{no}(\xi_o, u_d, \tau))^\dagger - \operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau))) \\ &\quad + B_{ono}((\mathcal{J}_{ono}(\xi_o, u_d, \tau) - \mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau))\times \\ &\quad \times (\mathcal{J}_{no}(\xi_o, u_d, \tau))^\dagger + \mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau)\times \\ &\quad \times ((\mathcal{J}_{no}(\xi_o, u_d, \tau))^\dagger - \operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o), u_d, \tau))),\end{aligned}$$

and

$$\begin{aligned}\tilde{\eta}^+ &= \gamma A\tilde{\eta} + \gamma A\tilde{T}\int_0^\tau e^{-F_{no}s}U_{cno}ds + \gamma AB_{ono}\times \\ &\quad \times (\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o, t, -\tau) - \Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\hat{\xi}_o, t, -\tau)) \\ &\quad - B_{dno}(H_{do}(\xi_o, u_d) - H_{do}(\mathrm{sat}_o(\hat{\xi}_o), u_d)) \\ &\quad + T^+J_{noo}(\xi_o, u_d) - \hat{T}^+J_{noo}(\mathrm{sat}_o(\hat{\xi}_o), u_d) \\ &\quad - B_{ono}(J_o(\xi_o, u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d)) \\ &= \gamma A\tilde{\eta} + \gamma A\tilde{T}\int_0^\tau e^{-F_{no}s}U_{cno}ds + \gamma AB_{ono}\times \\ &\quad \times (\Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\xi_o, t, -\tau) - \Psi_{f_{o,\mathrm{sat}(\cdot,\mathtt{u}_c)}}(\hat{\xi}_o, t, -\tau)) \\ &\quad - B_{dno}(H_{do}(\xi_o, u_d) - H_{do}(\mathrm{sat}_o(\hat{\xi}_o), u_d)) \\ &\quad + \tilde{T}^+J_{noo}(\mathrm{sat}_o(\hat{\xi}_o), u_d) \\ &\quad + T^+(J_{noo}(\xi_o, u_d) - J_{noo}(\mathrm{sat}_o(\hat{\xi}_o), u_d)) \\ &\quad - B_{ono}(J_o(\xi_o, u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d)).\end{aligned}$$

Plugging the expression of $\tilde{T}^+$ above into the one in $\tilde{\eta}^+$,

we obtain

$$\tilde{\eta}^+ = \gamma A\tilde{\eta} + \gamma A\tilde{T}u_1 + \gamma v_1 + w_1, \tag{50a}$$
$$\tilde{T}^+ = \gamma A\tilde{T}U_2 + \gamma V_2 + W_2, \tag{50b}$$

where

$$\begin{aligned}
u_1 &= \int_0^\tau e^{-F_{no}s}U_{cno}ds \\
&\quad + \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau))J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d),\\
v_1 &= AB_{ono}(\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o,t,-\tau) - \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\hat{\xi}_o,t,-\tau))\\
&\quad + AT((\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger\\
&\quad - \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)))J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d),\\
w_1 &= B_{dno}(((\mathcal{H}_{dno}(\xi_o,u_d,\tau) - \mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau))\\
&\quad \times (\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger + \mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\\
&\quad \times ((\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger - \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)))\\
&\quad \times J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d)\\
&\quad - (H_{do}(\xi_o,u_d) - H_{do}(\mathrm{sat}_o(\hat{\xi}_o),u_d)))\\
&\quad + B_{ono}(((\mathcal{J}_{ono}(\xi_o,u_d,\tau) - \mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau))\\
&\quad \times (\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger + \mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\\
&\quad \times ((\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger - \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)))\\
&\quad \times J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d) - (J_o(\xi_o,u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o),u_d)))\\
&\quad + T^+(J_{noo}(\xi_o,u_d) - J_{noo}(\mathrm{sat}_o(\hat{\xi}_o),u_d)),\\
U_2 &= \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)),\\
V_2 &= AT((\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger - \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau))),\\
W_2 &= B_{dno}((\mathcal{H}_{dno}(\xi_o,u_d,\tau) - \mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau))\\
&\quad \times (\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger + \mathcal{H}_{dno}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\\
&\quad \times ((\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger - \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)))\\
&\quad + B_{ono}((\mathcal{J}_{ono}(\xi_o,u_d,\tau) - \mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau))\\
&\quad \times (\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger + \mathcal{J}_{ono}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau)\\
&\quad \times ((\mathcal{J}_{no}(\xi_o,u_d,\tau))^\dagger - \mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\mathcal{J}_{no}(\mathrm{sat}_o(\hat{\xi}_o),u_d,\tau))),
\end{aligned}$$

with the expression of T^+ from (45).

This is the third part of this proof. We are ready to start a Lyapunov analysis in the new coordinates, with η replacing ξ_{no} in view of applying [23, Theorem 2] after time $(t_{j_m^\star}, j_m^\star)$. To place ourselves in the framework of [23], we define $z = (z_o, z_{no})$ where $z_o = \xi_o$ and $z_{no} = (\eta, T)$, as well as $\hat{z} = (\hat{z}_o, \hat{z}_{no})$ where $\hat{z}_o = \hat{\xi}_o$ and $\hat{z}_{no} = (\hat{\eta}, \hat{T})$. We also consider the extended inputs $(\mathfrak{u}_{c,\mathrm{ext}}, \mathfrak{u}_{d,\mathrm{ext}})$ defined by $\mathfrak{u}_{c,\mathrm{ext}}(t) = (\mathfrak{u}_c(t), t) \in \mathcal{U}_{c,\mathrm{ext}}$ and $\mathfrak{u}_{d,\mathrm{ext}}(j) = (\mathfrak{u}_d(j), t_{j+1}) \in \mathcal{U}_{d,\mathrm{ext}}$, with $\mathcal{U}_{c,\mathrm{ext}} = \mathcal{U}_c \times \mathbb{R}_{\geq 0}$ and $\mathcal{U}_{d,\mathrm{ext}} = \mathcal{U}_d \times \mathbb{R}_{\geq 0}$. We have seen that z remains at all times in the compact set $\Xi_o \times \Xi_\eta(\gamma) \times \mathcal{P}_T(\gamma)$. Moreover, after time $(t_{j_m^\star}, j_m^\star)$, we know that, given an input trajectory $\mathfrak{u}_c \in \mathfrak{U}_c$, $(z, u_{c,\mathrm{ext}}, \tau) \in C_z^{\mathfrak{u}_c}(\gamma)$ during flows and $(z, u_{d,\mathrm{ext}}, \tau) \in D_z^{\mathfrak{u}_c}(\gamma)$ at jumps where

$$\begin{aligned}
C_z^{\mathfrak{u}_c}(\gamma) &= \Big\{(\xi_o,\eta,T,u_c,t,\tau) \in \\
&\quad (\Xi_o \cap C_o) \times \Xi_\eta(\gamma) \times \mathcal{C}_T(\gamma) \times \mathcal{U}_c \times \mathbb{R}_{\geq 0} \times [0,\tau_M] : \\
&\quad \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o,t,-\tau) \in \Xi_o, \exists \xi_{no} \in \Xi_{no} : \\
&\quad \eta = T\Psi_{f_{no}}(\xi_{no},-\tau) - B_{ono}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o,t,-\tau)\Big\},\\
D_z^{\mathfrak{u}_c}(\gamma) &= \Big\{(\xi_o,\eta,T,u_d,t,\tau) \in \\
&\quad (\Xi_o \cap D_o) \times \Xi_\eta(\gamma) \times \mathcal{C}_T(\gamma) \times \mathcal{U}_d \times \mathbb{R}_{\geq 0} \times \mathcal{I} : \\
&\quad \mathcal{J}_{no}(\xi_o,u_d,\tau) \in \mathcal{C}_{\mathcal{J}_{no}}, \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o,t,-\tau) \in \Xi_o, \\
&\quad \exists \xi_{no} \in \Xi_{no} : \\
&\quad \eta = T\Psi_{f_{no}}(\xi_{no},-\tau) - B_{ono}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o,t,-\tau)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}_{\mathcal{J}_{no}} &= \left\{\mathcal{J}_{no} \in \mathbb{R}^{n_{no}\times n_{no}} : \frac{1}{c^2_{\mathcal{J}_{no}^{-1}}}\,\mathrm{Id} \leq \mathcal{J}_{no}^\top\mathcal{J}_{no} \leq \bar{c}^2_{\mathcal{J}_{no}}\,\mathrm{Id}\right\},\\
\mathcal{C}_T(\gamma) &= \left\{T \in \mathbb{R}^{n_\eta\times n_{no}} : (\underline{c}_T(\gamma))^2\,\mathrm{Id} \leq T^\top T \leq (\bar{c}_T(\gamma))^2\,\mathrm{Id}\right\},
\end{aligned}$$

for some appropriate $\bar{c}_{\mathcal{J}_{no}} > 0$ guaranteed to exist by Item (A1.3) of Assumption 1 and the compactness of $[0,\tau_M]$ (depending only on Ξ_o, $\mathcal{U}_d$, and τ_M). Consider the Lyapunov function

$$V_{no}(z_{no},\hat{z}_{no}) = (\eta-\hat{\eta})^\top Q(\eta-\hat{\eta}) + \|T-\hat{T}\|^2, \tag{51}$$

where $Q = Q^\top > 0$ is a solution to $A^\top QA < Q$, which exists because A is Schur. Denote $\underline{c}_Q > 0$ and $\bar{c}_Q > 0$, respectively, as the minimum and maximum eigenvalues of Q. Then, for all $z_{no} = (\eta, T)$ and for all $\hat{z}_{no} = (\hat{\eta}, \hat{T})$ in $\mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\eta \times n_{no}}$, we have

$$\underline{c}_Q|\eta-\hat{\eta}|^2 + \|T-\hat{T}\|^2 \leq V_{no}(z_{no},\hat{z}_{no}) \leq \bar{c}_Q|\eta-\hat{\eta}|^2 + \|T-\hat{T}\|^2. \tag{52}$$

Besides, whatever input trajectory $\mathfrak{u}_c \in \mathfrak{U}_c$, for all $(\xi_o,\eta,T,u_c,t,\tau) \in C_z^{\mathfrak{u}_c}(\gamma)$ and for all $(\hat{\xi}_o,\hat{\eta},\hat{T}) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\eta \times n_{no}}$, we have

$$\dot{V}_{no}(\xi_o,\hat{\xi}_o,\eta,\hat{\eta},T,\hat{T},\tau,t,u_c) = 0, \tag{53a}$$

along the respective flow dynamics. Now, we upper bound V_{no}^+ at jumps. By the global Lipschitzness of $f_{o,\mathrm{sat}}$ with respect to ξ_o, uniformly with respect to u_c, Lemma 2 in Appendix A allows us to show that $\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\cdot,t,-\tau)$ is Lipschitz, uniformly with respect to $(t,\tau) \in \mathbb{R}_{\geq 0} \times [0,\tau_M]$ and $\mathfrak{u}_c \in \mathfrak{U}_c$. We also show that sat_o and sat_{no} satisfy (23) as in the proof of Theorem 1. Besides, applying Lemma 9 in Appendix A to $\mathrm{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}$ and then $\mathrm{inv}_{\underline{c}_T(\gamma)}$, we deduce that there exists $L_{\hat{\mathcal{J}}_{no}} > 0$

such that

$$\|\operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\hat{\mathcal{J}}_{no})\| \leq \frac{1}{\underline{c}_{\hat{\mathcal{J}}_{no}}}, \forall \hat{\mathcal{J}}_{no} \in \mathbb{R}^{n_{no}\times n_{no}},$$
$$\|\mathcal{J}_{no}^{\dagger} - \operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}(\hat{\mathcal{J}}_{no})\| \leq L_{\hat{\mathcal{J}}_{no}}\|\mathcal{J}_{no} - \hat{\mathcal{J}}_{no}\|,$$
$$\forall (\mathcal{J}_{no}, \hat{\mathcal{J}}_{no}) \in \mathcal{C}_{\mathcal{J}_{no}} \times \mathbb{R}^{n_{no}\times n_{no}},$$

and there exists $L_{\hat{T}}(\gamma) > 0$ such that

$$\|\operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T})\| \leq \frac{1}{\underline{c}_T(\gamma)}, \forall \hat{T} \in \mathbb{R}^{n_\eta \times n_{no}},$$
$$\|T^{\dagger} - \operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T})\| \leq L_{\hat{T}}(\gamma)\|T - \hat{T}\|,$$
$$\forall (T, \hat{T}) \in \mathcal{C}_T \times \mathbb{R}^{n_\eta \times n_{no}}.$$

Exploiting the boundedness of u_1 and U_2 (independently of γ), we apply Lemma 8 in Appendix A to the Lyapunov function (51) and estimation error $(\tilde{\eta}, \tilde{T})$ with jump dynamics (50), to compute V_{no}^{+}. From the global Lipschitzness of $f_{o,\mathrm{sat}}(\cdot, u_c)$ uniformly in $u_c \in \mathcal{U}_c$, the property of sat_o that we have just proven, and the properties of $\operatorname{inv}_{\underline{c}_{\hat{\mathcal{J}}_{no}}}$ and $\operatorname{inv}_{\underline{c}_T(\gamma)}$ proven above, we get that (v_1, w_1, V_2, W_2) can be upper-bounded by $|\xi_o - \hat{\xi}_o|$, with gains depending on γ. As a result, picking $c_1 \in [0, 1)$, we deduce that there exists $0 < \gamma_2^{\star} \leq \gamma_1^{\star}$ such that for any $0 < \gamma < \gamma_2^{\star}$, there exists $c_2(\gamma) > 0$ such that, whatever input trajectory $\mathfrak{u}_c \in \mathfrak{U}_c$, for all $(\xi_o, \eta, T, u_d, t, \tau) \in D_z^{\mathfrak{u}_c}(\gamma)$ and for all $(\hat{\xi}_o, \hat{\eta}, \hat{T}) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\eta \times n_{no}}$,

$$\begin{aligned} &V_{no}^{+}(\xi_o, \hat{\xi}_o, \eta, \hat{\eta}, T, \hat{T}, \tau, t, u_d) \\ &\leq \gamma c_1 V_{no}(z_{no}, \hat{z}_{no}) + c_2(\gamma)|\xi_o - \hat{\xi}_o|^2 \\ &\leq \gamma c_1 V_{no}(z_{no}, \hat{z}_{no}) + \frac{c_2(\gamma)}{\underline{b}_o(\ell)} V_{o,\ell}(\xi_o, \hat{\xi}_o, \tau), \end{aligned} \tag{53b}$$

along the respective jump dynamics, where the latter inequality is obtained from Assumption 2. From (52) and (53), we see that V_{no} satisfies the second item of all conditions of [23, Theorem 2] (note that $\frac{c_2(\gamma)}{\underline{b}_o(\ell)}$ is rational in ℓ because so is $\underline{b}_o(\ell)$). Now, we check if $V_{o,\ell}$ also satisfies all the first items in those conditions. For that, we need to upper bound $\xi_o^{+} - \hat{\xi}_o^{+}$ given in (25), and thus $\xi_{no} - \hat{\xi}_{no}$. Combining (49) with (39b), we obtain

$$\begin{aligned} &\xi_{no} - \hat{\xi}_{no} \\ &= e^{F_{no}\tau}\left(T^{\dagger}\eta - \operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T})\hat{\eta}\right) \\ &\quad + e^{F_{no}\tau}\Big(T^{\dagger}B_{ono}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau) \\ &\quad - \operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T})B_{ono}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\hat{\xi}_o, t, -\tau)\Big) \\ &= e^{F_{no}\tau}\big((T^{\dagger} - \operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T}))\eta + \operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T})(\eta - \hat{\eta})\big) \\ &\quad + e^{F_{no}\tau}\Big(\big(T^{\dagger} - \operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T})\big)B_{ono}\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau) \\ &\quad + \operatorname{inv}_{\underline{c}_T(\gamma)}(\hat{T})B_{ono}\times \\ &\quad \times \big(\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau) - \Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\hat{\xi}_o, t, -\tau)\big)\Big). \end{aligned}$$

By the boundedness of Ξ_o, $\Xi_\eta(\gamma)$, and $[0, \tau_M]$, by the uniform Lipschitzness of $\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\cdot, t, -\tau)$ for $(t, \tau) \in \mathbb{R}_{\geq 0} \times [0, \tau_M]$, and the properties of $\operatorname{inv}_{\underline{c}_T(\gamma)}$ shown above with a bound depending only on $\underline{c}_T(\gamma)$, there exist $c_3(\gamma) > 0$, $c_4(\gamma) > 0$, and $c_5(\gamma) > 0$ such that whatever input trajectory $\mathfrak{u}_c \in \mathfrak{U}_c$, for all $(\xi_o, \eta, T, t, \tau) \in \Xi_o \times \Xi_\eta(\gamma) \times \mathcal{C}_T(\gamma) \times \mathbb{R}_{\geq 0} \times [0, \tau_M]$ such that $\Psi_{f_{o,\mathrm{sat}}(\cdot,\mathfrak{u}_c)}(\xi_o, t, -\tau) \in \Xi_o$ and for all $(\hat{\xi}_o, \hat{\eta}, \hat{T}) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\eta \times n_{no}}$, ξ_{no} and $\hat{\xi}_{no}$ defined in (49) and (39b) verify

$$|\xi_{no} - \hat{\xi}_{no}| \leq c_3(\gamma)|\xi_o - \hat{\xi}_o| + c_4(\gamma)|\eta - \hat{\eta}| + c_5(\gamma)\|T - \hat{T}\|. \tag{54}$$

Thanks to Item (A1.3) of Assumption 1 and (23b), there exist $c_6 > 0$, $c_7 > 0$, $c_8(\gamma) > 0$, and $c_9(\gamma) > 0$ and from these, $d_o(\ell, \gamma) := \frac{\overline{b}_o c_8(\gamma)}{\underline{b}_o(\ell)} > 0$ (rational in ℓ because so is $\underline{b}_o$) and $d_{ono}(\gamma) := \overline{b}_o c_9(\gamma) > 0$ such that $V_{o,\ell}$ in Assumption 2 satisfies for all $(\xi_o, \eta, T, u_d, t, \tau) \in D_z^{\mathfrak{u}_c}(\gamma)$ and for all $(\hat{\xi}_o, \hat{\eta}, \hat{T}) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\eta \times n_{no}}$ (with ξ_{no} and $\hat{\xi}_{no}$ still given by (49) and (39b)),

$$\begin{aligned} &V_{o,\ell}^{+}(z, \hat{z}, \tau, u_d) \\ &\leq \overline{b}_o|\xi_o^{+} - \hat{\xi}_o^{+}|^2 \\ &\leq \overline{b}_o|J_o(\xi_o, u_d) + J_{ono}(\xi_o, u_d)\xi_{no} \\ &\quad - (J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) + J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)\,\mathrm{sat}_{no}(\hat{\xi}_{no}))|^2 \\ &\leq \overline{b}_o|J_o(\xi_o, u_d) - J_o(\mathrm{sat}_o(\hat{\xi}_o), u_d) \\ &\quad + (J_{ono}(\xi_o, u_d) - J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d))\xi_{no} \\ &\quad + J_{ono}(\mathrm{sat}_o(\hat{\xi}_o), u_d)(\xi_{no} - \mathrm{sat}_{no}(\hat{\xi}_{no}))|^2 \\ &\leq \overline{b}_o c_6|\xi_o - \hat{\xi}_o|^2 + \overline{b}_o c_7|\xi_{no} - \hat{\xi}_{no}|^2 \\ &\leq \overline{b}_o c_8(\gamma)|\xi_o - \hat{\xi}_o|^2 + \overline{b}_o c_9(\gamma)|(\eta, T) - (\hat{\eta}, \hat{T})|^2 \\ &\leq \frac{\overline{b}_o c_8(\gamma)}{\underline{b}_o(\ell)} V_{o,\ell}(\xi_o, \hat{\xi}_o, \tau) + \overline{b}_o c_9(\gamma)|(\eta, T) - (\hat{\eta}, \hat{T})|^2 \\ &\leq d_o(\ell, \gamma)V_{o,\ell}(z_o, \hat{z}_o, \tau) + d_{ono}(\gamma)|z_{no} - \hat{z}_{no}|^2, \end{aligned}$$

along the respective jump dynamics, because $\xi_{no} \in \Xi_{no}$ by the definition of $D_z^{\mathfrak{u}_c}(\gamma)$, where we use the norm $|\cdot|$ for the vector-matrix element-wise concatenation like (η, T) and Assumption 2. This inequality, along with Assumption 2, shows that $V_{o,\ell}$ satisfies the inequalities involving it in [23, Theorem 2] in the new coordinates, uniformly in the chosen input trajectory $\mathfrak{u}_c \in \mathfrak{U}_c$. Applying [23, Theorem 2] after jump $j_m^\star$, we deduce that given any $\lambda > 0$, there exist $0 < \gamma_3^\star \leq \gamma_2^\star$ and $\ell_3^\star \geq \ell_2^\star$ such that for any $0 < \gamma < \gamma_3^\star$ and for any $\ell > \ell_3^\star$, there exists $\rho_1(\ell, \gamma) > 0$ such that for any of those solutions,

$$|(\xi_o, \eta, T)(t,j) - (\hat{\xi}_o, \hat{\eta}, \hat{T})(t,j)| \leq \rho_1(\ell,\gamma)|(\xi_o, \eta, T)(t_{j_m^\star}, j_m^\star) - (\hat{\xi}_o, \hat{\eta}, \hat{T})(t_{j_m^\star}, j_m^\star)|e^{-\lambda(t+j)}, \quad \forall (t,j) \in \operatorname{dom}\xi, j \geq j_m^\star. \quad (55)$$

Combining this with (54), we obtain Theorem 2. □

4.3 KKL-based observer design for a walking robot

Let us now design observer (39) for the walking robot in Example 1. A first difference with the quadratic detectability-based design in Section 3.3 is that, because ξ_{no} is constant, i.e., its dynamics matrix J_{no} is identity, solving (32) requires, in particular, the observability of the pair $(\mathrm{Id}_2, g_2(\theta))$, i.e., invertibility of the matrix $g_2(\theta)$, at each fixed θ. In this section, uniform backward distinguishability instead draws observability from a certain number m of past outputs. Thus, it only requires the concatenation of all $g_2(\theta)$ encountered in the considered window to be left-invertible (instead of each $g_2(\theta)$). However, this advantage disappears in the case of periodic solutions and we simply pick $m_1 = m_2 = 0$ (no information provided by the first two components of $J_{ono}(\theta)$), and $m_3 = m_4 = 1$ ($g_2(\theta)$ directly invertible), which gives us $n_\eta = \sum_{i=1}^4 m_i = 2$. A second important advantage is that the KKL-based design (39) of this section is systematic and does not require offline computation of the gains as in (32). The price to pay lies in its higher complexity.

The high-gain observer for ξ_o is the same as in Example 3. For the jump-based observer estimating ξ_{no}, since $n_\eta = 2$ and given the choice of the m_i, we choose $A = \mathrm{diag}(0.1, 0.2)$, $B_{ono} = \begin{pmatrix} 0_{2\times 2} & \mathrm{Id}_2 \end{pmatrix}$, an empty B_{dno}, and $\gamma = 0.4$. This means we use only the last two components of $J_{ono}(\theta)$, which is $g_2(\theta)$, as the new output matrix for observer implementation. We still recover $\hat{\xi}_{no}$ using (18b) at the rate of 0.0001 (s).

Simulation results are shown in Figure 4. Like in the previous section, the convergence of $\hat{\xi}_{no}$ can only start after the first impact after ξ_{no} has become visible from the fictitious output.

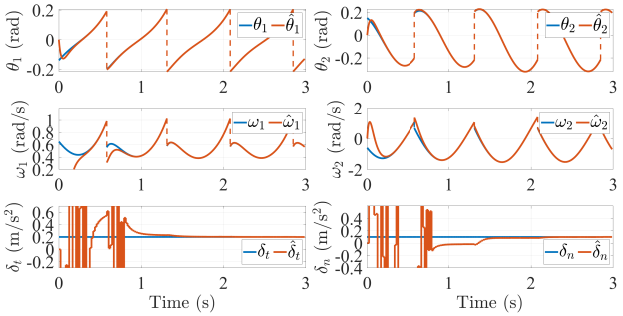

Fig. 4. State and impact uncertainty estimation in a bipedal robot based on uniform backward distinguishability.

5 Conclusion

Table 1 then summarizes and compares the two designs proposed in this section. Compared to the LMI-based observer, the KKL-based one seems to require stronger conditions, but it is more systematic, namely, we can implement it without checking observability, and it provides arbitrarily fast convergence (after a certain time).

We propose novel observer designs for hybrid systems with nonlinear affine structures and known jump times by combining observers estimating different parts of the state. The proposed observers are applied to estimate state and impact uncertainties in a walking robot.

Future work includes designing observers for hybrid systems with fully nonlinear maps in the case of known jump times, as well as treating the more challenging case of unknown jump times.

References

[1] P. Bernard and R. G. Sanfelice, "On Notions of Detectability and Observers for Hybrid Systems," *59th IEEE Conference on Decision and Control*, pp. 5767–5772, 2020.

[2] J. Kim, H. Shim, and J. H. Seo, "State Estimation and Tracking Control for Hybrid Systems by Gluing the Domains," *IEEE Transactions on Automatic Control*, vol. 64, no. 7, pp. 3026–3033, 2018.

[3] G. Q. B. Tran, S. Garcia, P. Bernard, F. Di Meglio, and R. G. Sanfelice, "Towards Gluing KKL Observer for Hybrid Systems with Unknown Jump Times," in *63rd IEEE Conference on Decision and Control*, Dec. 2024.

[4] P. Bernard and R. G. Sanfelice, "Semiglobal High-Gain Hybrid Observer for a Class of Hybrid Dynamical Systems with Unknown Jump Times," *IEEE Transactions on Automatic Control*, Jan. 2024.

[5] Z.-H. Guan, T.-H. Qian, and X. Yu, "On Controllability and Observability for a Class of Impulsive Systems," *Systems & Control Letters*, vol. 47, pp. 247–257, 2002.

[6] G. Xie and L. Wang, "Controllability and Observability of a Class of Linear Impulsive Systems," *Journal of Mathematical Analysis and Applications*, vol. 304, pp. 336–355, 2005.

	LMI-based observer (12) (Section 3)	KKL-based observer (39) (Section 4)
Time domain	Persistence of both flows and jumps (Item (A1.1) of Assumption 1)	
Hybrid model	No assumption	Invertibility of the matrices $(\mathcal{J}_{no}(\xi_{o,j}, u_{d,j}, \tau_j))_{j\in\mathbb{N}_{\geq j_m}}$ along solutions (Item (A4.1) of Assumption 4)
Observability	Instantaneous observability of ξ_o and a form of quadratic detectability of system (11) (Assumption 3)	Instantaneous observability of ξ_o and uniform backward distinguishability of system (11) along solutions (Item (A4.2) of Assumption 4)
State dimension	$n_\xi + n_p + 1$	$n_o + n_p + \left(\sum_{i=1}^{n_{d,\text{ext}}} m_i\right) + \left(\sum_{i=1}^{n_{d,\text{ext}}} m_i\right) n_{no} + 1$
Convergence rate	Determined by the solution to a matrix inequality	Arbitrarily fast, achieved after a certain time

Table 1
Observer designs for hybrid systems (1) with affine structures and known jump times. All the conditions here are *sufficient*.

[7] E. A. Medina and D. A. Lawrence, "Reachability and Observability of Linear Impulsive Systems," *Automatica*, vol. 44, no. 5, pp. 1304–1309, 2008.

[8] S. Zhao and J. Sun, "Controllability and Observability for a Class of Time-varying Impulsive Systems," *Nonlinear Analysis: Real World Applications*, vol. 10, pp. 1370–1380, 2009.

[9] A. Tanwani, H. Shim, and D. Liberzon, "Observability for Switched Linear Systems: Characterization and Observer Design," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 891–904, 2013.

[10] A. Alessandri and P. Coletta, "Switching Observers for Continuous-time and Discrete-time Linear Systems," *2001 American Control Conference*, vol. 3, pp. 2516–2521, 2001.

[11] A. Tanwani, H. Shim, and D. Liberzon, "Comments on "Observability of Switched Linear Systems: Characterization and Observer Design"," *IEEE Transactions on Automatic Control*, vol. 60, no. 12, pp. 3396–3400, 2015.

[12] F. Deza, E. Busvelle, J. Gauthier, and D. Rakotopara, "High Gain Estimation for Nonlinear Systems," *Systems & Control Letters*, vol. 18, no. 4, pp. 295 – 299, 1992.

[13] J. Sur and B. Paden, "Observers for Linear Systems with Quantized Output," *1997 American Control Conference*, pp. 3012–3016, 1997.

[14] T. Raff and F. Allgöwer, "Observers with Impulsive Dynamical Behavior for Linear and Nonlinear Continuous-Time Systems," *46th IEEE Conference on Decision and Control*, pp. 4287–4292, 2007.

[15] T. Raff, M. Kogel, and F. Allgower, "Observer with Sample-and-hold Updating for Lipschitz Nonlinear Systems with Nonuniformly Sampled Measurements," in *2008 American Control Conference*, pp. 5254–5257, 2008.

[16] T. Ahmed-Ali, L. Burlion, F. Lamnabhi-Lagarrigue, and C. Hann, "A Sampled-data Observer with Time-varying Gain for a Class of Nonlinear Systems with Sampled-measurements," in *53rd IEEE Conference on Decision and Control*, pp. 316–321, 2014.

[17] F. Ferrante, F. Gouaisbaut, R. Sanfelice, and S. Tarbouriech, "State Estimation of Linear Systems in the Presence of Sporadic Measurements," *Automatica*, vol. 73, pp. 101–109, 2016.

[18] A. Sferlazza, S. Tarbouriech, and L. Zaccarian, "Time-Varying Sampled-Data Observer With Asynchronous Measurements," *IEEE Transactions on Automatic Control*, vol. 64, no. 2, pp. 869–876, 2019.

[19] H. Ríos, J. Dávila, and A. R. Teel, "State Estimation for Linear Hybrid Systems with Periodic Jumps and Unknown Inputs," *International Journal of Robust and Nonlinear Control*, vol. 30, no. 15, pp. 5966–5988, 2020.

[20] P. Bernard and R. G. Sanfelice, "Observer Design for Hybrid Dynamical Systems with Approximately Known Jump Times," *Automatica*, vol. 141, p. 110225, 2022.

[21] E. A. Medina and D. A. Lawrence, "State Estimation for Linear Impulsive Systems," *2009 American Control Conference*, pp. 1183–1188, 2009.

[22] P. Bernard, V. Andrieu, and D. Astolfi, "Observer Design for Continuous-time Dynamical Systems," *Annual Reviews in Control*, vol. 53, pp. 224–248, 2022.

[23] G. Q. B. Tran, P. Bernard, and R. G. Sanfelice, "Coupling Flow and Jump Observers for Hybrid Systems with Nonlinear Maps and Known Jump Times," *22nd IFAC World Congress*, 2023.

[24] A. Tanwani and S. Trenn, "Detectability and Observer Design for Switched Differential–algebraic Equations," *Automatica*, vol. 99, pp. 289–300, 2019.

[25] H. Ríos, J. Davila, and A. R. Teel, "Linear Hybrid Systems With Periodic Jumps: A Notion of Strong Observability and Strong Detectability," *IEEE Transactions on Automatic Control*, vol. 65, no. 6, pp. 2640–2646, 2020.

[26] A. Balluchi, L. Benvenutia, M. D. D. Benedetto, and A. Sangiovanni-Vincentelli, "The Design of Dynamical Observers for Hybrid Systems: Theory and Application to an Automotive Control Problem," *Automatica*, vol. 49, no. 4, pp. 915–925, 2013.

[27] F. Lin, L. Y. Wang, W. Chen, and M. P. Polis, "On Observability of Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 11, pp. 6074–6081, 2022.

[28] F. Lin, L. Y. Wang, G. Yin, M. Polis, and W. Chen, "On Detectability of a Class of Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 68, no. 9, pp. 5509–5521, 2023.

[29] P. D'Alessandro, A. Isidori, and A. Ruberti, "A New Approach to the Theory of Canonical Decomposition of Linear Dynamical Systems," *SIAM Journal on Control*, vol. 11, no. 1, pp. 148–158, 1973.

[30] N. Cox, L. Marconi, and A. R. Teel, "Isolating Invisible Dynamics in the Design of Robust Hybrid Internal Models," *Automatica*, vol. 68, pp. 56–68, 2016.

[31] G. Q. B. Tran, P. Bernard, F. Di Meglio, and L. Marconi, "Observer Design based on Observability Decomposition for Hybrid Systems with Linear Maps and Known Jump Times," *61st IEEE Conference on Decision and Control*, 2022.

[32] G. Q. B. Tran, P. Bernard, and L. Marconi, "Observer Design for Hybrid Systems with Linear Maps and Known Jump Times," in *Hybrid and Networked Dynamical Systems* (R. Postoyan, P. Frasca, E. Panteley, and L. Zaccarian, eds.), Springer, 2024.

[33] G. Q. B. Tran and P. Bernard, "Kalman-like Observer for Hybrid Systems with Linear Maps and Known Jump Times," in *62nd IEEE Conference on Decision and Control*, 2023.

[34] V. Andrieu and L. Praly, "On the Existence of a Kazantzis–Kravaris/Luenberger Observer," *SIAM Journal on Control and Optimization*, vol. 45, no. 2, pp. 432–456, 2006.

[35] P. Bernard and V. Andrieu, "Luenberger Observers for Nonautonomous Nonlinear Systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 1, pp. 270–281, 2019.

[36] G. Q. B. Tran and P. Bernard, "Arbitrarily Fast Robust KKL Observer for Nonlinear Time-Varying Discrete Systems," *IEEE Transactions on Automatic Control*, vol. 69, no. 3, pp. 1520–1535, 2024.

[37] R. Penrose, "A Generalized Inverse for Matrices," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 51, no. 3, p. 406–413, 1955.

[38] H. K. Khalil, "Nonlinear Systems, Prentice-Hall," *Upper Saddle River, NJ*, vol. 3, 1996.

[39] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid Dynamical Systems: Modeling, Stability, and Robustness," 2012.

[40] R. G. Sanfelice and A. R. Teel, "Asymptotic Stability in Hybrid Systems via Nested Matrosov Functions," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1569–1574, 2009.

[41] R. G. Sanfelice, J. B. Biemond, N. van de Wouw, and W. M. H. Heemels, "An Embedding Approach for the Design of State-feedback Tracking Controllers for References with Jumps," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 11, pp. 1585–1608, 2014.

[42] H. Shim and A. Tanwani, "Hybrid-type Observer Design based on a Sufficient Condition for Observability in Switched Nonlinear Systems," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 6, pp. 1064–1089, 2014.

[43] D. Liberzon, D. Nešić, and A. R. Teel, "Lyapunov-based Small-gain Theorems for Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1395–1410, 2014.

[44] F. Asano, M. Yamakita, N. Kamamichi, and Z. Luo, "A Novel Gait Generation for Biped Walking Robots based on Mechanical Energy Constraint," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 3, pp. 565–573, 2004.

[45] A. Goswami, B. Espiau, and A. Keramane, "Limit Cycles and their Stability in a Passive Bipedal Gait," in *Proceedings of IEEE International Conference on Robotics and Automation*, vol. 1, pp. 246–251 vol.1, 1996.

[46] T. Chen and B. Goodwine, "Robust Gait Design for a Compass Gait Biped on Slippery Surfaces," *Robotics and Autonomous Systems*, vol. 140, p. 103762, 2021.

[47] R. S. Johnson, *Parameter Estimation for Hybrid Dynamical Systems*. University of California, Santa Cruz, 2023.

[48] J. Gauthier, H. Hammouri, and S. Othman, "A Simple Observer for Nonlinear Systems Applications to Bioreactors," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 875–880, 1992.

[49] E. J. McShane, "Extension of Range of Functions," *Bulletin of the American Mathematical Society*, vol. 40, no. 12, pp. 837 – 842, 1934.

[50] F. Wu, *Control of Linear Parameter Varying Systems*. PhD thesis, University of California, Berkeley, 1995.

[51] J. J. Deyst and C. Price, "Conditions for Asymptotic Stability of the Discrete Minimum-variance Linear Estimator," *IEEE Transactions on Automatic Control*, vol. 13, pp. 702–705, 1968.

[52] A. Ţiclea and G. Besançon, "Exponential Forgetting Factor Observer in Discrete Time," *Systems & Control Letters*, vol. 62, no. 9, pp. 756–763, 2013.

[53] Q. Zhang, "On Stability of the Kalman Filter for Discrete Time Output Error Systems," *Systems & Control Letters*, vol. 107, pp. 84–91, July 2017.

[54] J. B. Moore and B. D. O. Anderson, "Coping with Singular Transition Matrices in Estimation and Control Stability Theory," *International Journal of Control*, vol. 31, pp. 571–586, 1980.

[55] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, vol. 6. Springer Science & Business Media, 1998.

[56] J. K. Hale, *Ordinary Differential Equations*. Courier Corporation, 2009.

[57] R. Skelton, T. Iwasaki, and K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis, 01 2017.

[58] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*. SIAM, 2009.

A Technical lemmas

Lemma 2. *Consider $\mathfrak{U}$, a subspace of locally bounded functions $\mathfrak{u} : \mathbb{R}_{\geq 0} \to \mathcal{U} \subset \mathbb{R}^{n_u}$, a continuous map $f : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_\xi}$, and the continuous-time dynamics*

$$\dot{\xi} = f(\xi, u), \tag{A.1}$$

with state $\xi \in \mathbb{R}^{n_\xi}$ and input $\mathfrak{u} \in \mathfrak{U}$. Let $\tau_M > 0$. Assume that f is globally Lipschitz with respect to ξ, uniformly in $u \in \mathcal{U}$. Then, for any $\mathfrak{u} \in \mathfrak{U}$ and for any $(t, \tau) \in \mathbb{R}_{\geq 0} \times [0, \tau_M]$ with $\tau \leq t$, the map[5] $\Psi_{f(\cdot,\mathfrak{u})}(\cdot, t, -\tau)$ is defined on $\mathbb{R}^{n_\xi}$ and is globally Lipschitz, uniformly in $(t, \tau) \in \mathbb{R}_{\geq 0} \times [0, \tau_M]$ and in $\mathfrak{u} \in \mathfrak{U}$.

Proof. By assumption, there exists $L > 0$ such that $|f(\xi_a, u) - f(\xi_b, u)| \leq L|\xi_a - \xi_b|$ for all $(\xi_a, \xi_b) \in \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\xi}$ and for all $u \in \mathcal{U}$. From [55, Proposition C.3.8], we deduce that the map $\Psi_{f(\cdot,\mathfrak{u})}(\cdot, t, -\tau)$ is defined on $\mathbb{R}^{n_\xi}$, for any $\mathfrak{u} \in \mathfrak{U}$ and for any $(t, \tau) \in \mathbb{R}_{\geq 0} \times [0, \tau_M]$ with $\tau \leq t$. Let any $\mathfrak{u} \in \mathfrak{U}$. For any $(\xi_a, \xi_b, t, \tau) \in \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\xi} \times \mathbb{R}_{\geq 0} \times [0, \tau_M]$ with $\tau \leq t$, we have

$$\Psi_{f(\cdot,\mathfrak{u})}(\xi_a, t, -\tau) = \xi_a + \int_t^{t-\tau} f(\Psi_{f(\cdot,\mathfrak{u})}(\xi_a, t, s-t), \mathfrak{u}(s))ds,$$
$$\Psi_{f(\cdot,\mathfrak{u})}(\xi_b, t, -\tau) = \xi_b + \int_t^{t-\tau} f(\Psi_{f(\cdot,\mathfrak{u})}(\xi_b, t, s-t), \mathfrak{u}(s))ds.$$

[5] See Notation in Section 1.

and by subtracting both sides and using the triangle inequality,

$$|\Psi_{f(\cdot,\mathfrak{u})}(\xi_a,t,-\tau)-\Psi_{f(\cdot,\mathfrak{u})}(\xi_b,t,-\tau)|\leq|\xi_a-\xi_b| + \int_{t-\tau}^{t}|f(\Psi_{f(\cdot,\mathfrak{u})}(\xi_a,t,s-t),\mathfrak{u}(s)) - f(\Psi_{f(\cdot,\mathfrak{u})}(\xi_b,t,s-t),\mathfrak{u}(s))|ds,$$

so that, since $\mathfrak{u}(s)\in\mathcal{U}$ for all $s\in[t-\tau,t]\subset\mathbb{R}_{\geq 0}$,

$$|\Psi_{f(\cdot,\mathfrak{u})}(\xi_a,t,-\tau)-\Psi_{f(\cdot,\mathfrak{u})}(\xi_b,t,-\tau)|\leq|\xi_a-\xi_b| + \int_{t-\tau}^{t}L|\Psi_{f(\cdot,\mathfrak{u})}(\xi_a,t,s-t)-\Psi_{f(\cdot,\mathfrak{u})}(\xi_b,t,s-t)|ds.$$

Using Grönwall's inequality, we get

$$|\Psi_{f(\cdot,\mathfrak{u})}(\xi_a,t,-\tau)-\Psi_{f(\cdot,\mathfrak{u})}(\xi_b,t,-\tau)|\leq|\xi_a-\xi_b|e^{L\tau} \leq|\xi_a-\xi_b|e^{L\tau_M},$$

because $\tau\in[0,\tau_M]$, which concludes the proof. □

Lemma 3. *Consider a time-varying continuous-time system $\dot{\xi}=f(\xi,t)$ with f being continuous and of class C^1 with respect to ξ. For any differentiable time function $t\mapsto\hat{\xi}(t)$ and a modified time $t\mapsto\tau(t)$ such that $\dot{\tau}(t)=1$, we have, at any time t where $\Psi_f(\hat{\xi}(t),t,-\tau(t))$ is defined,*

$$\frac{d\Psi_f}{dt}(\hat{\xi}(t),t,-\tau(t)) = -\frac{\partial\Psi_f}{\partial\xi}(\hat{\xi}(t),t,-\tau(t))(f(\hat{\xi}(t),t)-\dot{\hat{\xi}}(t)). \quad (A.2)$$

Proof. First, from [56, Chapter 1, Theorem 3.3], since f is continuous and C^1 with respect to ξ, we get for any (ξ,t) and any τ such that $\Psi_f(\xi,t,\tau)$ is defined $f(\Psi_f(\xi,t,\tau),t+\tau)=\frac{\partial\Psi_f}{\partial\xi}(\xi,t,\tau)f(\xi,t)+\frac{\partial\Psi_f}{\partial t}(\xi,t,\tau)$. Using this expression with (ξ,t,τ) replaced by $(\hat{\xi}(t),t,-\tau(t))$, we then have

$$\begin{aligned}&\frac{d\Psi_f}{dt}(\hat{\xi}(t),t,-\tau(t))\\ &=\frac{\partial\Psi_f}{\partial\xi}(\hat{\xi}(t),t,-\tau(t))\dot{\hat{\xi}}(t)+\frac{\partial\Psi_f}{\partial t}(\hat{\xi}(t),t,-\tau(t))\\ &\quad-f(\Psi_f(\hat{\xi}(t),t,-\tau(t)),t-\tau(t))\\ &=-\frac{\partial\Psi_f}{\partial\xi}(\hat{\xi}(t),t,-\tau(t))\left(f(\hat{\xi}(t),t)-\dot{\hat{\xi}}(t)\right).\end{aligned}$$

The proof is completed. □

Lemma 4. *[56, Chapter 1, Theorem 3.3] Consider the time-varying system*

$$\dot{\xi}=f(\xi,t), \quad (A.3)$$

where f is continuous and of class C^1 with respect to ξ and define $\Psi_f(\xi_0,t_0,\tau)$ as the solution to system (A.3) initialized as ξ_0 at time t_0 and flowing during τ time unit(s), and a modified time $t\mapsto\tau(t)$ such that $\dot{\tau}=1$. Let $\Phi_f(\xi_0,t_0,\tau)=\frac{\partial\Psi_f}{\partial\xi_0}(\xi_0,t_0,\tau)$. Then (ξ,Φ_f) is solution to the dynamics

$$\dot{\xi}=f(\xi,t), \qquad \dot{\Phi}_f=\frac{\partial f}{\partial\xi}(\xi,t)\Phi_f, \quad (A.4)$$

initialized as (ξ_0,Id).

Lemma 5. *[57, Theorem 2.3.12] There exist matrices B, C, and $P=P^\top$ such that*

$$\begin{cases}B^\perp P(B^\perp)^\top<0 & or \quad BB^\top>0\\ (C^\top)^\perp P((C^\top)^\perp)^\top<0 & or \quad C^\top C>0\end{cases} \quad (A.5)$$

if and only if there exists a matrix Y such that

$$BYC+(BYC)^\top+P<0. \quad (A.6)$$

Lemma 6. *Assume that there exist $a>0$, $Q=Q^\top>0$, and $\rho\mapsto L(\rho)$ such that, for some set $\mathcal{P}$ and some $\rho\mapsto(J(\rho),H(\rho))$, we have, for all $\rho\in\mathcal{P}$,*

$$(J(\rho)-L(\rho)H(\rho))^\top Q(J(\rho)-L(\rho)H(\rho))-aQ<0. \quad (A.7)$$

Then,[6] *the same Q and a are solutions to*

$$\begin{pmatrix}((H(\rho))^\perp)^\top Q(H(\rho))^\perp & \star\\ QJ(\rho)(H(\rho))^\perp & aQ\end{pmatrix}>0, \quad \forall\rho\in\mathcal{P}. \quad (A.8)$$

Proof. Using Schur's lemma at each $\rho\in\mathcal{P}$, we deduce that (A.7) is equivalent to

$$\begin{pmatrix}Q & \star\\ QJ(\rho)-QL(\rho)H(\rho) & aQ\end{pmatrix}>0, \quad \forall\rho\in\mathcal{P}. \quad (A.9)$$

We introduce the variable $Y(\rho)=-QL(\rho)$ for $\rho\in\mathcal{P}$ and rewrite (A.9) as

$$\begin{pmatrix}0\\ \mathrm{Id}\end{pmatrix}Y(\rho)\begin{pmatrix}H(\rho) & 0\end{pmatrix}+\begin{pmatrix}(H(\rho))^\top\\ 0\end{pmatrix}(Y(\rho))^\top\begin{pmatrix}0 & \mathrm{Id}\end{pmatrix} + \begin{pmatrix}Q & \star\\ QJ(\rho) & aQ\end{pmatrix}>0, \quad \forall\rho\in\mathcal{P}. \quad (A.10)$$

[6] In contrast to Lemma 5, the other implication cannot be stated due to the constant nature of Q and a.

Now apply Lemma 5 to (A.10) at each $\rho \in \mathcal{P}$ with $B = \begin{pmatrix} 0 \\ \mathrm{Id} \end{pmatrix}$, $C(\rho) = \begin{pmatrix} H(\rho) & 0 \end{pmatrix}$, and $P(\rho) = -\begin{pmatrix} Q & \star \\ QJ(\rho) & aQ \end{pmatrix}$. Since $BB^\top = \begin{pmatrix} 0 \\ \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$ and $(C(\rho))^\top C(\rho) = \begin{pmatrix} (H(\rho))^\top \\ 0 \end{pmatrix} \begin{pmatrix} H(\rho) & 0 \end{pmatrix} = \begin{pmatrix} (H(\rho))^\top H(\rho) & 0 \\ 0 & 0 \end{pmatrix}$ are not positive definite (for all $\rho \in \mathcal{P}$), we deduce that (A.10) is equivalent to

$$\begin{pmatrix} \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} Q & \star \\ QJ(\rho) & aQ \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix} > 0,$$

$$\begin{pmatrix} ((H(\rho))^\perp)^\top & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} Q & \star \\ QJ(\rho) & aQ \end{pmatrix} \begin{pmatrix} (H(\rho))^\perp & 0 \\ 0 & \mathrm{Id} \end{pmatrix} > 0,$$

for all $\rho \in \mathcal{P}$. While the first condition becomes $Q > 0$ which is trivial, the second one gives us (A.8). □

Lemma 7. *Let $\mathcal{U}$ be a set, $\mathcal{C} \subset \mathbb{R}^{n_\xi}$ be a compact set, and $\mathcal{C}_s$ be a subset of $\mathcal{C}^{\mathbb{N}}$. Consider $M : \mathbb{R}^{n_\xi} \times \mathcal{U} \to \mathbb{R}^{m \times m}$ a matrix-valued function that is both locally bounded and locally Lipschitz with respect to ξ, uniformly in $u \in \mathcal{U}$. Assume that there exists $c > 0$ such that for all $(\xi_k)_{k\in\mathbb{N}} \in \mathcal{C}_s$ and for all $(u_k)_{k\in\mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$, we have*

$$(M(\xi_k, u_k))^\top M(\xi_k, u_k) \geq c\,\mathrm{Id}, \qquad \forall k \in \mathbb{N}. \quad \text{(A.11)}$$

Then, for any $0 < c' < c$, there exists $\bar{c} > 0$ such that for any $(\hat{\xi}_k)_{k\in\mathbb{N}}$ for which

$$\exists (\xi_k)_{k\in\mathbb{N}} \in \mathcal{C}_s : |\xi_k - \hat{\xi}_k| \leq \bar{c}, \qquad \forall k \in \mathbb{N}, \quad \text{(A.12)}$$

and for any $(u_k)_{k\in\mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$, we have

$$(M(\hat{\xi}_k, u_k))^\top M(\hat{\xi}_k, u_k) \geq c'\,\mathrm{Id}, \qquad \forall k \in \mathbb{N}. \quad \text{(A.13)}$$

Proof. Pick $0 < c' < c$ and some $\delta > 0$. Since M is locally bounded with respect to ξ, uniformly in $u \in \mathcal{U}$, let $c_M > 0$ be the bound of $\|M(\xi, u)\|$ on $\mathcal{C} \times \mathcal{U}$. Moreover, since M is locally Lipschitz with respect to ξ, uniformly in $u \in \mathcal{U}$, let $L_M > 0$ be its Lipschitz constant on $(\mathcal{C} + \delta\mathbb{B}) \times \mathcal{U}$. Then, given any sequences $(\hat{\xi}_k)_{k\in\mathbb{N}}$, $(\hat{\xi}_k)_{k\in\mathbb{N}}$ of $\mathbb{R}^{n_\xi}$ and any $(u_k)_{k\in\mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$, we have for all vectors $x \in \mathbb{R}^m$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\star^\top M(\hat{\xi}_k, u_k)x \\
&= \star^\top (M(\xi_k, u_k) + \Delta M(\xi_k, \hat{\xi}_k, u_k))x \\
&= \star^\top M(\xi_k, u_k)x + 2x^\top (\Delta M(\xi_k, \hat{\xi}_k, u_k))^\top M(\xi_k, u_k)x \\
&\qquad + \star^\top \Delta M(\xi_k, \hat{\xi}_k, u_k)x \\
&\geq cx^\top x - |2x^\top (\Delta M(\xi_k, \hat{\xi}_k, u_k))^\top M(\xi_k, u_k)x| \\
&\qquad - |\star^\top \Delta M(\xi_k, \hat{\xi}_k, u_k)x| \\
&\geq (c - 2\|M(\xi_k, u_k)\|\|\Delta M(\xi_k, \hat{\xi}_k, u_k)\| \\
&\qquad - \|\Delta M(\xi_k, \hat{\xi}_k, u_k)\|^2)x^\top x,
\end{aligned}$$

where $\Delta M(\xi_k, \hat{\xi}_k, u_k) = M(\hat{\xi}_k, u_k) - M(\xi_k, u_k)$. Let $\bar{c} > 0$ be such that $\bar{c} \leq \delta$ and $c - 2c_M L_M \bar{c} - L_M^2 \bar{c}^2 > c'$. If (A.12) holds, then it follows that $(\xi_k)_{k\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ and $(\hat{\xi}_k)_{k\in\mathbb{N}} \in (\mathcal{C} + \delta\mathbb{B})^{\mathbb{N}}$. Thanks to the local boundedness and local Lipschitzness of M, we have for all $(\xi_k)_{k\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$, for all $(\hat{\xi}_k)_{k\in\mathbb{N}} \in (\mathcal{C} + \delta\mathbb{B})^{\mathbb{N}}$, and for all $(u_k)_{k\in\mathbb{N}} \in \mathcal{U}^{\mathbb{N}}$, that $\|M(\xi_k, u_k)\| \leq c_M$ and $\|\Delta M(\xi_k, \hat{\xi}_k, u_k)\| \leq L_M|\xi_k - \hat{\xi}_k|$ for all $k \in \mathbb{N}$, and (A.13) follows. □

Lemma 8. *Let A be a Schur non-zero matrix and $Q = Q^\top > 0$ be a solution to $A^\top QA < Q$. Consider three functions of variable $(\tilde{z}_1, \tilde{Z}_2)$:*

$$\begin{aligned}
g_1(\tilde{z}_1, \tilde{Z}_2) &= \gamma A\tilde{z}_1 + \gamma A\tilde{Z}_2 u_1 + \gamma v_1 + w_1, \\
g_2(\tilde{Z}_2) &= \gamma A\tilde{Z}_2 U_2 + \gamma V_2 + W_2, \\
V(\tilde{z}_1, \tilde{Z}_2) &= \tilde{z}_1^\top Q\tilde{z}_1 + \|\tilde{Z}_2\|^2,
\end{aligned}$$

with $\gamma > 0$ and inputs $(u_1, v_1, w_1, U_2, V_2, W_2)$, where $(\tilde{z}_1, u_1, v_1, w_1)$ are vectors and $(\tilde{Z}_2, U_2, V_2, W_2)$ are matrices of appropriate dimensions, and some constant $m > 0$. There exist $d_1, d_2, d_3, d_4, d_5 > 0$ such that for any $\gamma > 0$, for all $(\tilde{z}_1, \tilde{Z}_2)$ and for all $(u_1, v_1, w_1, U_2, V_2, W_2)$ such that $|u_1|^2 \leq m$ and $\|U_2\|^2 \leq m$, we have

$$\begin{aligned}
V(g_1(\tilde{z}_1, \tilde{Z}_2), g_2(\tilde{Z}_2)) &\leq \gamma^2 d_1 V(\tilde{z}_1, \tilde{Z}_2) + \gamma^2 d_2 |v_1|^2 \\
&\quad + d_3|w_1|^2 + \gamma^2 d_4 \|V_2\|^2 + d_5\|W_2\|^2. \quad \text{(A.14)}
\end{aligned}$$

Proof. Using the fact that $\|U\|^2 = \|U^\top U\|$ for the 2-norm and Young's inequality, for all $(\tilde{z}_1, \tilde{Z}_2)$ and for all $(u_1, v_1, w_1, U_2, V_2, W_2)$ such that $|u_1|^2 \leq m$ and $\|U_2\|^2 \leq$

m, we have

$$
\begin{aligned}
&V(g_1(\tilde{z}_1,\tilde{Z}_2),g_2(\tilde{Z}_2))\\
&=\gamma^2\tilde{z}_1^\top A^\top QA\tilde{z}_1+\gamma^2u_1^\top\tilde{Z}_2^\top A^\top QA\tilde{Z}_2u_1+\gamma^2v_1^\top Qv_1\\
&\quad+w_1^\top Qw_1+2\gamma^2\tilde{z}_1^\top A^\top QA\tilde{Z}_2u_1+2\gamma^2\tilde{z}_1^\top A^\top Qv_1\\
&\quad+2\gamma\tilde{z}_1^\top A^\top Qw_1+2\gamma^2u_1^\top\tilde{Z}_2^\top A^\top Qv_1\\
&\quad+2\gamma u_1^\top\tilde{Z}_2^\top A^\top Qw_1+2\gamma v_1^\top Qw_1\\
&\quad+\|\gamma^2U_2^\top\tilde{Z}_2^\top A^\top A\tilde{Z}_2U_2+\gamma^2V_2^\top V_2+W_2^\top W_2\\
&\quad+2\gamma^2U_2^\top\tilde{Z}_2^\top A^\top V_2+2\gamma U_2^\top\tilde{Z}_2^\top A^\top W_2+2\gamma V_2^\top W_2\|\\
&\le\gamma^2\left(c_1\tilde{z}_1^\top Q\tilde{z}_1+c_2\|\tilde{Z}_2\|^2\right)+\gamma^2c_3|v_1|^2+c_4|w_1|^2\\
&\quad+\gamma^2c_5\|V_2\|^2+c_6\|W_2\|^2,
\end{aligned}
$$

for some $c_i>0$, $i=1,2,\dots,6$, some of which depend on m. Thus, Lemma 8 follows. □

Lemma 9. *Consider the inverse function* $\mathrm{inv}_{\underline{c}_M}:\mathbb{R}^{m\times n}\to\mathbb{R}^{n\times m}$ *with* $m\ge n$ *defined in* (39c) *for some level* $\underline{c}_M>0$. *For some* $\bar{c}_M\ge\underline{c}_M$, *define* $\mathcal{C}_M=\{M\in\mathbb{R}^{m\times n}:\underline{c}_M^2\,\mathrm{Id}\le M^\top M\le\bar{c}_M^2\,\mathrm{Id}\}$. *The map* $\mathrm{inv}_{\underline{c}_M}$ *is such that:*

- *For all* $\hat{M}\in\mathbb{R}^{m\times n}$, *we have* $\|\mathrm{inv}_{\underline{c}_M}(\hat{M})\|\le\frac{1}{\underline{c}_M}$;
- *There exists* $L_{\mathrm{inv}}>0$ *such that for all* $(M,\hat{M})\in\mathcal{C}_M\times\mathbb{R}^{m\times n}$, *we have* $\|M^\dagger-\mathrm{inv}_{\underline{c}_M}(\hat{M})\|\le L_{\mathrm{inv}}\|M-\hat{M}\|$.

Proof. Let us start by proving two intermediary results. The first one is that for any full-rank matrix $M\in\mathbb{R}^{m\times n}$, if a matrix $\hat{M}\in\mathbb{R}^{m\times n}$ is not full-rank, then $\|M-\hat{M}\|\ge\sigma_{\min}(M)$. Indeed, since $\hat{M}$ is not full-rank, there exists $x\in\mathbb{R}^n\setminus\{0\}$ such that $\hat{M}x=0$. We then have $|(M-\hat{M})x|^2=|Mx|^2=x^\top\hat{M}^\top\hat{M}x\ge(\sigma_{\min}(M))^2|x|^2$ and it follows that $|(M-\hat{M})x|\ge\sigma_{\min}(M)|x|$ and thus $\|M-\hat{M}\|\ge\sigma_{\min}(M)$ (since $|x|\neq0$).
Now we prove the second intermediary result that given $\mathcal{C}_M$ being compact, there exists $c_M>0$ such that if $\|M-\hat{M}\|\le c_M$ for some $M\in\mathcal{C}_M$, then $\hat{M}$ is such that $\sigma_{\min}(\hat{M})\ge\frac{\underline{c}_M}{2}$. Indeed, assume the contrary and construct sequences $(M_k)_{k\in\mathbb{N}}\in\mathcal{C}_M^{\mathbb{N}}$ and $(\hat{M}_k)_{k\in\mathbb{N}}\in(\mathbb{R}^{m\times n})^{\mathbb{N}}$ such that $\|M_k-\hat{M}_k\|\le\frac{1}{k}$ and $\sigma_{\min}(\hat{M}_k)<\frac{\underline{c}_M}{2}$ for all $k\in\mathbb{N}$. By the compactness of $\mathcal{C}_M$, we can extract a subsequence from $(M_k)_{k\in\mathbb{N}}$ converging to M^* in $\mathcal{C}_M$. To alleviate the notations, we do not denote this extraction. Then, $(\hat{M}_k)_{k\in\mathbb{N}}$ also converges to $\hat{M}^*$. It follows that $\sigma_{\min}(\hat{M}^*)\ge\underline{c}_M$, which is a contradiction by continuity of $\sigma_{\min}$.
To prove the first item of Lemma 9, we now show that $\mathrm{inv}_{\underline{c}_M}$ is bounded in norm on $\mathbb{R}^{m\times n}$. The case of not being full-rank is trivial. For all $\hat{M}\in\mathbb{R}^{m\times n}$ such that $\sigma_{\min}(\hat{M})>0$, we get that either $\sigma_{\min}(\hat{M})\ge\underline{c}_M$ and then $\|\mathrm{inv}_{\underline{c}_M}(\hat{M})\|=\|\hat{M}^\dagger\|\le\frac{1}{\underline{c}_M}$, or $0<\sigma_{\min}(\hat{M})\le\underline{c}_M$ and then $\|\mathrm{inv}_{\underline{c}_M}(\hat{M})\|=\left\|\left(\frac{\sigma_{\min}(\hat{M})}{\underline{c}_M}\right)\hat{M}^\dagger\right\|\le\frac{\sigma_{\min}(\hat{M})}{\underline{c}_M\sigma_{\min}(\hat{M})}\le\frac{1}{\underline{c}_M}$. So, $\mathrm{inv}_{\underline{c}_M}$ is bounded in norm on $\mathbb{R}^{m\times n}$ by $\frac{1}{\underline{c}_M}$.
To prove the second item of Lemma 9, we pick $M\in\mathcal{C}_M$ and consider four cases of $\hat{M}\in\mathbb{R}^{m\times n}$. First, if $\hat{M}^\top\hat{M}\ge\underline{c}_M^2\,\mathrm{Id}$, i.e., $\sigma_{\min}(M)\ge\underline{c}_M$, then according to [58, Theorem 10.4.5], we have

$$
\begin{aligned}
\|M^\dagger-\mathrm{inv}_{\underline{c}_M}(\hat{M})\|&=\|M^\dagger-\hat{M}^\dagger\|\\
&\le3\|M^\dagger\|\|\hat{M}^\dagger\|\|M-\hat{M}\|\\
&\le\frac{3}{\underline{c}_M^2}\|M-\hat{M}\|.
\end{aligned}
$$

Second, if $\hat{M}$ is not full-rank, from the first intermediary result above, we deduce that since M is full-rank, $\|M-\hat{M}\|\ge\sigma_{\min}(M)$ and so

$$
\begin{aligned}
\|M^\dagger-\mathrm{inv}_{\underline{c}_M}(\hat{M})\|&=\|M^\dagger\|\\
&=\|M^\dagger\|^2\sigma_{\min}(M)\\
&\le\frac{1}{\underline{c}_M^2}\|M-\hat{M}\|.
\end{aligned}
$$

Third, if $0<\sigma_{\min}(\hat{M})\le\underline{c}_M$ and $\|M-\hat{M}\|\le c_M$, then we have that $\tilde{M}:=\|\hat{M}^\dagger\|\underline{c}_M\hat{M}$ is such that $\tilde{M}^\top\tilde{M}\ge\underline{c}_M^2\,\mathrm{Id}$ and by applying the first case to it, we get

$$
\begin{aligned}
\|M^\dagger-\mathrm{inv}_{\underline{c}_M}(\hat{M})\|&=\left\|M^\dagger-\frac{1}{\|\hat{M}^\dagger\|\underline{c}_M}\hat{M}^\dagger\right\|\\
&=\|M^\dagger-(\|\hat{M}^\dagger\|\underline{c}_M\hat{M})^\dagger\|\\
&\le\frac{3}{\underline{c}_M^2}\|M-\tilde{M}\|\\
&\le\frac{3}{\underline{c}_M^2}\|M-\|\hat{M}^\dagger\|\underline{c}_M\hat{M}\|\\
&\le\frac{3}{\underline{c}_M}\left\|\left(\frac{1}{\underline{c}_M}-\|\hat{M}^\dagger\|\right)M\right\|\\
&\quad+\frac{3}{\underline{c}_M}\|\hat{M}^\dagger\|\|M-\hat{M}\|.
\end{aligned}
$$

Since $\|M^\dagger\|\le1/\underline{c}_M\le\|\hat{M}^\dagger\|$, we get that $|1/\underline{c}_M-\|\hat{M}^\dagger\||\le|\|M^\dagger\|-\|\hat{M}^\dagger\||\le\|M^\dagger-\hat{M}^\dagger\|$ and so using [58,

Theorem 10.4.5] again, we get

$$\begin{aligned}\|M^\dagger - \mathrm{inv}_{\underline{c}_M}(\hat{M})\| &\leq \frac{3}{\underline{c}_M}\|M\|\|M^\dagger - \hat{M}^\dagger\| \\ &\quad + \frac{3}{\underline{c}_M^2}\|\hat{M}^\dagger\|\underline{c}_M\|M - \hat{M}\| \\ &\leq \frac{9}{\underline{c}_M}\|M\|\|M^\dagger\|\|\hat{M}^\dagger\|\|M - \hat{M}\| \\ &\quad + \frac{3}{\underline{c}_M}\|\hat{M}^\dagger\|\|M - \hat{M}\| \\ &\leq \frac{3}{\underline{c}_M}\left(3\frac{\overline{c}_M}{\underline{c}_M} + 1\right)\|\hat{M}^\dagger\|\|M - \hat{M}\|.\end{aligned}$$

Now from the second intermediary result, since $\|M - \hat{M}\| \leq c_M$, we have $\sigma_{\min}(\hat{M}) \geq \frac{\underline{c}_M}{2}$ and we get $\|M^\dagger - \mathrm{inv}_{\underline{c}_M}(\hat{M})\| \leq 6/\underline{c}_M^2(3\overline{c}_M/\underline{c}_M + 1)\|M - \hat{M}\|$. Fourth, if $0 < \sigma_{\min}(\hat{M}) \leq \underline{c}_M$ and $\|M - \hat{M}\| \geq c_M$, from the boundedness of $\mathrm{inv}_{\underline{c}_M}$ proven above, we get $\|M^\dagger - \mathrm{inv}_{\underline{c}_M}(\hat{M})\| \leq 2/\underline{c}_M \leq 2/(\underline{c}_M c_M)\|M - \hat{M}\|$. Combining all these four cases, we get the result by letting $L_{\mathrm{inv}} = \max\{3/\underline{c}_M^2, 6/\underline{c}_M^2(3\overline{c}_M/\underline{c}_M + 1), 2/(\underline{c}_M c_M)\}$. □