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Sébastien Biebler

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Robust complex heterodimensional cycles

Sébastien Biebler

January 13, 2025

Abstract

A diffeomorphism f has a heterodimensional cycle if it displays two (transitive) hyperbolic sets K and K' with different indices such that the unstable set of K intersects the stable one of K' and vice versa. We prove that it is possible to find robust heterodimensional cycles for families of polynomial automorphisms of \mathbb{C}^3 . The proof is based on Bonatti-Díaz blenders.

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1 Introduction

The dynamics of uniformly hyperbolic systems such as the horseshoe introduced by Smale is particularly well understood. Such systems were originally conjectured to be dense in the 1960's. However it was soon realized that this is not true. There are two main mechanisms that yield robustly non-hyperbolic behaviors: robust heterodimensional cycles (obtained by Abraham and Smale [AS]) and robust homoclinic tangencies (this is the celebrated Newhouse phenomenon [Ne1, Ne2]). Later Palis [Pa1, Pa2] conjectured that diffeomorphisms exhibiting either a heterodimensional cycle or a homoclinic tangency are dense in the complement of the closure of the hyperbolic ones.

In this paper, we are interested in extending the phenomenon of robust heterodimensional cycles to the setting of complex dynamics.

A compact invariant set K for a diffeomorphism f of a manifold is a **hyperbolic basic set** if K is hyperbolic, transitive and locally maximal. Its **index** denotes the dimension of the unstable

manifold of any of its points (all unstable manifolds of points of K have the same dimension since the hyperbolic set K is transitive).

Definition. Let f be a diffeomorphism of a manifold of dimension 3 or higher having two hyperbolic basic sets K and K' . We say that f has a **heterodimensional cycle** associated to K and K' if:

1. the indices of K and K' are not equal,
2. the stable sets $W^s(K)$ and $W^s(K')$ intersect respectively the unstable sets $W^u(K')$ and $W^u(K)$.

Bonatti and Díaz showed in [BD3] that heterodimensional cycles of codimension 1 can be C^1 -stabilized. In a recent breakthrough, Li and Turaev [LT1] solved this C^r -persistence problem in any regularity $r = 2, \dots, \infty, \omega$ (one can also refer to [LT2, LLST]).

Let $\text{Aut}(\mathbb{C}^3)$ and $\text{Aut}_d(\mathbb{C}^3)$ be the spaces of holomorphic automorphisms and polynomial automorphisms of degree at most d of \mathbb{C}^3 . A polynomial automorphism of degree at most d is an element of $\text{Aut}(\mathbb{C}^3)$ whose component functions are polynomials of degree at most d . We endow these spaces with the topology induced by local uniform convergence of the map and its inverse.

Our main result shows that we can find robust heterodimensional cycles for polynomial automorphisms of \mathbb{C}^3 :

Theorem. *There exists a polynomial automorphism f of \mathbb{C}^3 such that every $g \in \text{Aut}(\mathbb{C}^3)$ sufficiently close to f displays a heterodimensional cycle associated to the hyperbolic continuations of a hyperbolic basic set of index 2 and a saddle periodic point of index 1.*

Corollary. *There exists an integer $d \geq 2$ and a nonempty open set $\mathcal{U}_d \subset \text{Aut}_d(\mathbb{C}^3)$ such that every $g \in \mathcal{U}_d$ displays a heterodimensional cycle associated to the hyperbolic continuations of a hyperbolic basic set of index 2 and a saddle periodic point of index 1.*

In particular, our results can be seen as an analogous of Buzzard's work [Bu] on complex Newhouse phenomenon (see also [Du2, ALZ]) for heterodimensional cycles instead of tangencies.

The main tool to obtain persistent heterodimensional cycles is a type of hyperbolic sets with very special fractal properties called **blender**. Introduced by Bonatti and Díaz in [BD1], blenders admit several definitions. Here is one:

Definition. *A hyperbolic basic set K_f for a diffeomorphism f of a manifold of dimension 3 or higher is a **cs-blender** if there is a nonempty open set Δ of embedded d -dimensional disks, with d smaller than the index of K_f , such that for every C^1 -perturbation g of f , the stable set $W^s(K_g)$ of the hyperbolic continuation K_g of K_f for g intersects any disk in Δ .*

We will use the blender property to obtain in \mathbb{C}^3 robust intersections between the one-dimensional unstable manifold of a saddle periodic point of index 1, and the one-dimensional stable set of a hyperbolic basic set of index 2, which was not expected a priori. Notice that blenders have many

other powerful applications in dynamical systems, see for example [ACW, AST, Be1, Be2, Bi1, Bi2, BD2, Du1, Ta, GTV].

Remark that the degree of the maps we obtain is unknown since we used a Runge approximation argument. It is then natural to ask:

Question. Is it possible to give an explicit degree d in the main Corollary ?

This paper is organized as follows. We define successively a polynomial p of \mathbb{C} , then a Hénon map H of \mathbb{C}^2 and finally a polynomial automorphism F_1 of \mathbb{C}^3 which is a skew-product over H displaying both a horseshoe of index 2 and a saddle point of index 1 (Section 2). Then we show that this horseshoe satisfies an open covering property on the third coordinate and hence the blender property (Section 3). Finally, we create an initial heterodimensional cycle and use the blender property to make it robust (Section 4).

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2 A skew-product polynomial automorphism

2.1 One dimensional dynamics

In this subsection, we introduce a polynomial p displaying a repelling Cantor set and a sink.

Let us start with a few notation. For every $z \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(z, r) \subset \mathbb{C}$ be the closed disk of center z and radius r . We set:

$$\mathbb{D} := \mathbb{D}(0, 1).$$

We fix once and for all $\eta := 10^{-4}$ and also define the following disks:

$$\begin{aligned} \mathbb{D}_0 &:= \mathbb{D}(1/4, \eta), & \mathbb{D}_1 &:= \mathbb{D}(i/4, \eta), & \mathbb{D}_2 &:= \mathbb{D}(-1/4, \eta), & \mathbb{D}_3 &:= \mathbb{D}(-i/4, \eta), \\ \mathbb{D}_4 &:= \mathbb{D}(3, 1), & \text{and} & & \mathbb{D}_5 &:= \mathbb{D}(3, \eta). \end{aligned}$$

For $j = 0, 1, 2, 3$, let ℓ_j be the expanding affine map of linear coefficient $1/\eta$ sending \mathbb{D}_j bijectively onto \mathbb{D} . Let ℓ_4 be the affine contraction of linear coefficient η sending \mathbb{D}_4 bijectively onto \mathbb{D}_5 . In particular, ℓ_4 displays an attracting fixed point equal to 3 with multiplier equal to η .

Proposition 2.1. *For every $\varepsilon > 0$, there exists a polynomial map $p : \mathbb{C} \rightarrow \mathbb{C}$ such that:*

$$\forall 0 \leq j \leq 4, \forall z \in \mathbb{D}_j, \quad |p(z) - \ell_j(z)| < \varepsilon \quad \text{and} \quad |p'(z) - \ell_j'(z)| < \varepsilon.$$

Proof. This is an immediate consequence of the Runge's theorem. □

Remark 2.2. The polynomial p has an attracting fixed point close to 3 with multiplier close to η .

Corollary 2.3. *For every $\delta > 0$ small enough, $p^{-1}(\mathbb{D}(\delta, 1))$ contains (at least) four connected components $\Delta_{\delta, j}$, $j \in \{0, 1, 2, 3\}$, which are topological disks close to \mathbb{D}_j for the Hausdorff topology. Moreover each $\Delta_{\delta, j}$ depends continuously on δ .*

2.2 Two dimensional dynamics

Here we define a Hénon map H from the polynomial p defined in the latter subsection. We take $\varepsilon > 0$ small and a polynomial p as given by Proposition 2.1.

Let us take $b \in \mathbb{C}^*$ and let us consider the Hénon map:

$$H : (z, w) \in \mathbb{C}^2 \mapsto (p(z) + bw, z) \in \mathbb{C}^2.$$

Recall that for $b \neq 0$ the map H is a polynomial automorphism of \mathbb{C}^2 . When $|b| > 0$ is small enough, by Corollary 2.3, the set $H^{-1}(\mathbb{D}^2) \cap \mathbb{D}^2$ has (at least) four connected components U_j , with:

$$U_j := \bigcup_{w \in \mathbb{D}} \Delta_{-bw, j} \times \{w\}, \quad j \in \{0, 1, 2, 3\}.$$

Recall that each $\Delta_{-bw, j}$ is a topological disk close to \mathbb{D}_j for the Hausdorff topology. Therefore U_j is close to $\mathbb{D}_j \times \mathbb{D}$.

Proposition 2.4. *There exists $b_o > 0$ such that for any $b \in \mathbb{C}^*$ satisfying $|b| < b_o$, the Hénon map H displays a hyperbolic basic set of saddle type*

$$K_o := \bigcap_{n \in \mathbb{Z}} H^n \left(\bigsqcup_{0 \leq j \leq 3} U_j \right)$$

and an attracting fixed point S_o close to $(3, 3)$ whose basin contains $\mathbb{D}(3, 1)^2$.

Proof. **We first show that K_o is a hyperbolic basic set of saddle type** (see Theorem 3.1 in [FS] for the proof of a very similar result). The set K_o is clearly compact, invariant by H and locally maximal. We are going to show that K_o is a horseshoe in the sense of Definition 6.5.2 of [KH]. The image $V_j := H(U_j)$ is equal to:

$$V_j = H(U_j) = \{(p(z) + bw, z) \mid w \in \mathbb{D}, z \in \Delta_{-bw, j}\},$$

which is close to the curve $\{(p(z), z) \mid z \in \mathbb{D}_j\}$ when b is small. Notice that the second coordinate projection of U_j is equal to \mathbb{D} . Also for every $w \in \mathbb{D}$, the first coordinate projection of the restriction of H to $U_j \cap (\mathbb{D} \times \{w\}) = \Delta_{-bw, j} \times \{w\}$ is a bijection onto $p(\Delta_{-bw, j}) + bw = \mathbb{D}$. Therefore the connected component U_j is full in the sense of Definition 6.5.1 of [KH]. We also have:

$$U_j \subset \text{int } \mathbb{D} \times \mathbb{D} \quad \text{and} \quad V_j \subset \mathbb{D} \times \text{int } \mathbb{D}.$$

We now define the two following constant cone fields:

$$\chi^u = \{(v_1, v_2) \in \mathbb{C}^2 : |v_2| \leq 10^{-3} \cdot |v_1|\} \quad \text{and} \quad \chi^s = \{(v_1, v_2) \in \mathbb{C}^2 : |v_1| \leq 10^{-3} \cdot |v_2|\}.$$

Notice that at any point $(z, w) \in \bigsqcup_{0 \leq j \leq 3} U_j$ the differential DH is equal to:

$$\begin{pmatrix} p'(z) & b \\ 1 & 0 \end{pmatrix}$$

with $p'(z)$ close to $1/\eta = 10^4$ by Proposition 2.1. Therefore, if b is small enough, any non zero vector in χ^u is sent into $\text{int } \chi^u$ and expanded by a factor larger than 10^3 by DH . Also at any point $(z, w) \in \bigsqcup_{0 \leq j \leq 3} V_j$ the differential DH^{-1} is equal to:

$$\begin{pmatrix} 0 & 1 \\ 1/b & -p'(w)/b \end{pmatrix}$$

with $p'(w)$ again close to $1/\eta = 10^4$. Therefore, if b is small enough, any non zero vector in χ^s is sent into $\text{int } \chi^s$ and expanded by a factor larger than 10^3 by DH^{-1} . Then K_o is a horseshoe in the sense of Definition 6.5.2 of [KH]. According to the discussion following this definition, K_o is also hyperbolic (this is an immediate consequence of the cone field criterion (Corollary 6.4.8 of [KH]) and topologically conjugate to a shift. In particular, it is transitive and therefore a hyperbolic basic set (of saddle type).

Now we show that H displays a sink. Recall that p displays a sink of period 1 close to 3 by Remark 2.2. Hence the map $(z, w) \mapsto (p(z), z)$ has a fixed point close to $(3, 3)$. Therefore by the implicit function theorem, for b small, the map H displays the continuation of a fixed point S_o close to $(3, 3)$. The differential of H at any $(z, w) \in \mathbb{D}(3, 1)^2$ is equal to:

$$\begin{pmatrix} p'(z) & b \\ 1 & 0 \end{pmatrix}$$

where $p'(z)$ is close to $\eta = 10^{-4}$ and b is small. Therefore the two eigenvalues of the differential of H at S_o are smaller than 1 in modulus and thus S_o is a sink for H . Since $p(\mathbb{D}_4)$ is close to \mathbb{D}_5 and since b is small, $\mathbb{D}(3, 1)^2$ is sent into itself by H . Also the differential of H^2 at any $(z, w) \in \mathbb{D}(3, 1)^2$ is close to:

$$\begin{pmatrix} 10^{-8} & 0 \\ 10^{-4} & 0 \end{pmatrix}$$

Then, by the mean value inequality, the distance of $H^2(z, w)$ to S_o is smaller than half the distance of (z, w) to S_o , hence $H^n(z, w)$ tends to S_o when n is large. Thus the basin of S_o contains $\mathbb{D}(3, 1)^2$. \square

2.3 Three dimensional dynamics

In this subsection, we define a polynomial automorphism F_1 of \mathbb{C}^3 which is a skew-product over the Hénon map H . We start with the following one-dimensional intermediate result:

Lemma 2.5. *For every $\zeta > 0$, there exists a polynomial map $q : \mathbb{C} \rightarrow \mathbb{C}$ such that:*

$$\forall 0 \leq j \leq 3, \forall z \in \mathbb{D}_j, \quad |q(z) - \frac{1}{10}e^{i\frac{j\pi}{2}}| < \zeta \quad \text{and} \quad |q'(z)| < \zeta,$$

$$\forall z \in \mathbb{D}_4, \quad |q(z) + \frac{1}{3}| < \zeta \quad \text{and} \quad |q'(z)| < \zeta.$$

Proof. This is again an immediate application of the Runge's theorem. \square

Let us fix once and for all:

$$\lambda := \frac{10}{9}.$$

We take $\zeta > 0$ small and a polynomial q as given by Lemma 2.5. We consider the Hénon map H given by Proposition 2.4. The following map is a polynomial automorphism of \mathbb{C}^3 for $b \neq 0$:

$$F_1 : (z, w, t) \in \mathbb{C}^3 \mapsto (H(z, w), \lambda t + q(z)) = (p(z) + bw, z, \lambda t + q(z)).$$

We set:

$$J := \{0, 1, 2, 3\}.$$

Notice that $\mathbb{D}^3 \cap F_1^{-1}(\mathbb{D}^3)$ displays (at least) four connected components $R^j(F_1)$, $j \in J$, close to:

$$\mathbb{D}_j \times \mathbb{D} \times \frac{1}{\lambda} \cdot (\mathbb{D} - \frac{1}{10}e^{i\frac{\pi}{2}}).$$

For every F close to F_1 , for every $j \in J$, we denote by $R^j = R^j(F)$ the connected component of $\mathbb{D}^3 \cap F^{-1}(\mathbb{D}^3)$ which is the continuation of $R^j(F_1)$. Its image $F(R^j)$ is close to $\{(p(z), z) \mid z \in \mathbb{D}_j\} \times \mathbb{D}$.

Let us consider the two following constant cone fields:

$$C^u := \{(v_1, v_2, v_3) \in \mathbb{C}^3 : |v_2| \leq 10^{-3} \cdot \|(v_1, v_3)\|_2\}$$

$$\text{and } C^s := \{(v_1, v_2, v_3) \in \mathbb{C}^3 : \|(v_1, v_3)\|_2 \leq 10^{-3} \cdot |b| \cdot |v_2|\}.$$

Proposition 2.6. *For every automorphism F close to F_1 , any non zero vector in C^u is sent into $\text{int } C^u$ and expanded by a factor larger than $\frac{1+\lambda}{2}$ by $D_{(z,w,t)}F$ for any $(z, w, t) \in \bigsqcup_{j \in J} R^j$.*

Moreover any non zero vector C^s is sent into $\text{int } C^s$ and expanded by a factor larger than 10^3 by $D_{(z,w,t)}F^{-1}$ for any $(z, w, t) \in \bigsqcup_{j \in J} F(R^j)$.

Proof. Note that the differentials of F and its inverse are close on $\bigsqcup_{j \in J} R^j$ and $\bigsqcup_{j \in J} F(R^j)$ to:

$$\begin{pmatrix} p'(z) & b & 0 \\ 1 & 0 & 0 \\ q'(z) & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1/b & -p'(w)/b & 0 \\ 0 & -q'(w) & 1/\lambda \end{pmatrix}$$

where p' is close to $1/\eta = 10^4$ and both q' and b are small. The result follows. \square

In the following result, we introduce the two hyperbolic basic sets whose continuations will be later involved in the formation of heterodimensional cycles.

Proposition 2.7. *There exists a neighborhood \mathcal{U}_1 of F_1 in $\text{Aut}(\mathbb{C}^3)$ such that every $F \in \mathcal{U}_1$ displays a hyperbolic basic set*

$$K = \bigcap_{n \in \mathbb{Z}} F^n \left(\bigsqcup_{j \in J} R^j \right)$$

of index 2 included in $\mathbb{D}(0, 1/2)^2 \times \mathbb{D}$ and a saddle fixed point S of index 1 close to $(3, 3, 3)$.

Proof. We first show that $K_1 = K(F_1)$ is a hyperbolic basic set of index 2. The set K_1 is clearly compact, F_1 -invariant and locally maximal. By Proposition 2.6 and by the cone field criterion, the set K_1 is also hyperbolic with unstable (resp. stable) dimension equal to 2 (resp. 1). Finally, let us show that K_1 is transitive.

Fact 2.8. *For every $(z, w) \in K_o$, the set K_1 intersects the line $\{(z, w, t) \mid t \in \mathbb{C}\}$ at exactly one point $(z, w, \xi(z, w))$, and the map $(z, w) \in K_o \mapsto \xi(z, w) \in \mathbb{C}$ is continuous.*

Proof. We fix $(z, w) \in K_o$. Let us denote for every $n \in \mathbb{Z}$ by I_n the third coordinate projection of $F_1^n(\{H^{-n}(z, w)\} \times \mathbb{D})$. We notice that I_n is a disk of diameter λ^n . Also $F_1(\{H^{-n-1}(z, w)\} \times \mathbb{D})$ is close to $\{H^{-n}(z, w)\} \times (\lambda \cdot \mathbb{D} + \frac{1}{10}e^{i\frac{j\pi}{2}})$ for some $j \in J$, with $(\lambda \cdot \mathbb{D} + \frac{1}{10}e^{i\frac{j\pi}{2}}) \ni \mathbb{D}$. Therefore we have $I_n \Subset I_{n+1}$ for every $n \in \mathbb{Z}$. Thus $\bigcap_{n \in \mathbb{Z}} I_n$ is reduced to a point $\xi(z, w) \in \mathbb{D}$. But $\{(z, w)\} \times \bigcap_{n \in \mathbb{Z}} I_n$ is precisely equal to the intersection of K_1 with $\{(z, w, t) \mid t \in \mathbb{C}\}$.

Then the continuity of ξ follows from the continuity of F_1 . \square

We recall that K_o is transitive for the Hénon map H . We can therefore pick $(z, w) \in K_o$ such that $\{H^n(z, w) \mid n \geq 0\}$ is dense in K_o . Then the sequence $\{F_1^n(z, w, \xi(z, w)) \mid n \geq 0\}$ is dense in K_1 . Indeed, for every $(\alpha, \beta, \xi(\alpha, \beta)) \in K_1$, we can find $H^n(z, w)$ arbitrarily close to (α, β) . Since ξ is continuous, then $\xi(H^n(z, w))$ is close to $\xi(\alpha, \beta)$. Thus $F_1^n(z, w, \xi(z, w)) = (H^n(z, w), \xi(H^n(z, w)))$ is close to $(\alpha, \beta, \xi(\alpha, \beta))$. This shows that K_1 is transitive for F_1 .

This shows that K_1 is a hyperbolic basic set of index 2 for F_1 . By robustness of hyperbolic basic sets, the set K is still a hyperbolic basic set of index 2 for F for every F close to F_1 .

Now we show that F displays a saddle fixed point S of index 1. Recall that H displays an attracting fixed point $S_o = (S_o^1, S_o^2)$ close to $(3, 3)$ by Proposition 2.4. Since q is close to $-1/3$ on \mathbb{D}_4 , the number $t_1 := (1 - \lambda)^{-1} \cdot q(S_o^1)$ is close to $-9 \cdot (-1/3) = 3$ and $S_1 = (S_o, t_1)$ is a fixed point for F_1 close to $(3, 3, 3)$. Moreover the differential of F_1 at S_1 is close to:

$$\begin{pmatrix} p'(3) & b & 0 \\ 1 & 0 & 0 \\ q'(3) & 0 & \lambda \end{pmatrix}$$

where $p'(3)$ is close to $\eta = 10^{-4}$ and both $q'(3)$ and b are small. Thus $D_{S_1}F_1$ has two eigenvalues smaller than 1 in modulus and one larger than 1 in modulus. Thus F_1 displays a saddle fixed point S_1 of index 1 close to $(3, 3, 3)$. The result follows by robustness of hyperbolic basic sets. \square

Definition 2.9. *For every $k \geq 1$ and $\mathbf{w} = (j_0 \cdots j_k) \in J^k$, we define the following set:*

$$R^{\mathbf{w}} := \bigcap_{0 \leq n \leq k} F^{-n}(R^{j_n}).$$

We also denote:

$$\partial_z R^{\mathbf{w}} := R^{\mathbf{w}} \cap F^{-k-1}(\partial \mathbb{D} \times \mathbb{D}^2), \quad \partial_w R^{\mathbf{w}} := R^{\mathbf{w}} \cap (\mathbb{D} \times \partial \mathbb{D} \times \mathbb{D}) \quad \text{and} \quad \partial_t R^{\mathbf{w}} := R^{\mathbf{w}} \cap F^{-k-1}(\mathbb{D}^2 \times \partial \mathbb{D}).$$

Proposition 2.10. For every $\mathbf{s} = (j_k)_{k \geq 0} \in J^{\mathbb{N}}$, the set $W^{\mathbf{s}} := \bigcap_{n \geq 0} R^{(j_0, \dots, j_n)}$ is a local stable manifold of K which is a holomorphic graph over $w \in \text{int } \mathbb{D}$ with tangent spaces in C^s .

For every $\mathbf{u} = (j_k)_{k < 0} \in J^{\mathbb{Z}^*}$, the set $W^{\mathbf{u}} := \bigcap_{n > 0} F^n(R^{(j_{-n}, \dots, j_{-1})})$ is a local unstable manifold of K which is a holomorphic graph over $(z, t) \in (\text{int } \mathbb{D})^2$ with tangent spaces in C^u .

Proof. By Proposition 2.6, for every $\mathbf{s} \in J^{\mathbb{N}}$, the set $W^{\mathbf{s}}$ intersects each $\{w = w_o\}$, with $w_o \in \mathbb{D}$, at exactly one point depending continuously on w_o . Thus $W^{\mathbf{s}}$ is a continuous graph over $w \in \mathbb{D}$.

Let $M \in W^{\mathbf{s}}$. For every $n \geq 0$, notice that $F^n(M)$ belongs to the set $F^n(R^{(s_1, \dots, s_n)}) \cap R^{(s_{n+1}, \dots, s_{2n})}$. The latter is included in $\bigcap_{-n \leq k \leq n} F^k(\bigsqcup_{j \in J} R^j)$, which is a small neighborhood of K . Thus the distance of $F^n(M)$ to K is small when n is large and so $W^{\mathbf{s}}$ is a local stable manifold of K . In particular it is a holomorphic graph. Still by Proposition 2.6, its tangent spaces are in C^s .

The same proof is working for unstable manifolds. \square

We also define:

$$W_{loc}^s(K) = \bigcup_{\mathbf{s} \in J^{\mathbb{N}}} W^{\mathbf{s}} \quad \text{and} \quad W_{loc}^u(K) = \bigcup_{\mathbf{u} \in J^{\mathbb{Z}^*}} W^{\mathbf{u}}.$$

The saddle point S and its stable/unstable manifolds will be of particular interest to us:

Definition 2.11. The local stable (resp. unstable) manifold $W_{loc}^s(S)$ (resp. $W_{loc}^u(S)$) of S is the connected component of $W^s(S) \cap \mathbb{D}(3, 1)^3$ (resp. $W^u(S) \cap \mathbb{D}(3, 1)^3$) containing S .

Up to restricting the neighborhood \mathcal{U}_1 of F_1 defined by Proposition 2.7, we obtain:

Fact 2.12. The local stable manifold $W_{loc}^s(S)$ of the saddle point S is C^1 -close to:

$$\{(z, w, 3) \mid z, w \in \mathbb{D}(3, 1)\}.$$

Proof. This is an immediate consequence of the hyperbolic continuation of local stable manifolds (indeed, for $F = F_1$ with $q = -1/3$ on \mathbb{D}_4 , $W_{loc}^s(S_1)$ is simply equal to $\{(z, w, 3) \mid z, w \in \mathbb{D}(3, 1)\}$ since the basin of the sink S_o of the Hénon map H contains $\mathbb{D}(3, 1)^2$ by Proposition 2.4). \square

Fact 2.13. The local unstable manifold $W_{loc}^u(S)$ of the saddle point S is C^1 -close to:

$$\{(3, 3, 3 + t) \mid t \in \mathbb{D}\}.$$

Proof. This is an immediate consequence of the hyperbolic continuation of local unstable manifolds (indeed, for $F = F_1$ with $q = -1/3$ on \mathbb{D}_4 , $W_{loc}^u(S_1)$ is simply equal to $\{(3, 3, 3 + t) \mid t \in \mathbb{D}\}$). \square

Remark 2.14. The set \mathcal{U}_1 can be taken as the ϵ_1 - C^0 -neighborhood of F_1 on some small neighborhood of $W_{loc}^s(K_1) \cup F_1^{-1}(W_{loc}^u(K_1)) \cup W_{loc}^s(S_1) \cup F_1^{-1}(W_{loc}^u(S_1))$, for some small $\epsilon_1 > 0$.

3 Blender property

In this section, we show that the compact hyperbolic set K for the automorphism $F \in \mathcal{U}_1$ displays the blender property. We first define the following constant cone field:

$$C^{uu} = \{v = (v_1, v_2, v_3) \in \mathbb{C}^3 : \|(v_2, v_3)\|_2 \leq 10^{-3} \cdot |v_1|\}.$$

Lemma 3.1. *For every $F \in \mathcal{U}_1$, any non zero vector in C^{uu} is sent into $\text{int } C^{uu}$ and expanded by a factor larger than 10^3 by $D_{(z,w,t)}F$ for any $(z, w, t) \in \bigsqcup_{j \in J} R^j$.*

Proof. The proof is exactly the same as in the proof of Proposition 2.6. \square

Definition 3.2. *A uu -curve is a continuous graph $\mathcal{C} = \{(z, c_2(z), c_3(z)) \mid z \in \mathbb{D}\} \subset \mathbb{D} \times (\text{int } \mathbb{D})^2$ such that $z \mapsto (c_2(z), c_3(z))$ is holomorphic on $\text{int } \mathbb{D}$ and $(1, c'_2(z), c'_3(z)) \in C^{uu}$ for every $z \in \text{int } \mathbb{D}$.*

Here is the main result of this section. It is the aforementioned blender property.

Proposition 3.3. *Every uu -curve $\mathcal{C} \subset \mathbb{D}^2 \times \mathbb{D}(0, 1/2)$ intersects $W_{loc}^s(K)$ for every $F \in \mathcal{U}_1$.*

The key to prove Proposition 3.3 is the following one-dimensional open covering property stated in Sublemma 3.4. Let us first define the four following affine maps:

$$L_0(t) := \frac{9}{10}t - \frac{9}{100}, \quad L_1(t) := \frac{9}{10}t - \frac{9i}{100}, \quad L_2(t) := \frac{9}{10}t + \frac{9}{100} \quad \text{and} \quad L_3(t) := \frac{9}{10}t + \frac{9i}{100}.$$

We remark that the third coordinate of F^{-1} is close to L_j on $F(R^j)$.

Sublemma 3.4. *For any $t \in \mathbb{D}(0, 1/2)$, there exists $\tau \in \mathbb{D}(0, 0.497)$ and $j \in J$ such that $t = L_j(\tau)$.*

Proof. By symmetry, it suffices to show that $\{re^{i\theta} \mid 0 \leq r \leq 0.5, -\pi/4 \leq \theta \leq \pi/4\}$ is included in the image of $\mathbb{D}(0, 0.497)$ by L_2 . For every $0 \leq r \leq 0.5$ and $-\pi/4 \leq \theta \leq \pi/4$, we have $re^{i\theta} \in L_2(\mathbb{D}(0, 0.497))$ if and only if $|re^{i\theta} - 0.09| \leq 0.9 \cdot 0.497$, that is if and only if:

$$\sqrt{r^2 - 0.18 \cdot r \cos(\theta) + 0.09^2} \leq 0.9 \cdot 0.497.$$

Rapid computations show this is true if the inequality is satisfied for $r = 0.5$ and $\theta = \pi/4$, which is the case since $0.5^2 - 0.18 \cdot 0.5 \cdot \sqrt{2}/2 + 0.09^2 < 0.2 < 0.9^2 \cdot 0.497^2$. \square

We are now ready to prove Proposition 3.3.

Proof of Proposition 3.3. It is enough to show that for every automorphism $F \in \mathcal{U}_1$, there is $\mathbf{s} = (j_0, j_1, \dots) \in J^{\mathbb{N}}$ such that for every $n \geq 0$, the curve \mathcal{C} intersects $R^{(j_1, \dots, j_n)}$. Then \mathcal{C} has a nonempty intersection with the local stable manifold $W^s = \bigcap_{n \geq 0} R^{(j_0, \dots, j_n)}$. The result then follows immediately by induction from the following Lemma 3.5. \square

Lemma 3.5. *Let $\mathcal{C} \subset \mathbb{D}^2 \times \mathbb{D}(0, 1/2)$ be a uu -curve. Then for every $F \in \mathcal{U}_1$, there is $j \in J$ such that \mathcal{C} intersects R^j and $F(\mathcal{C} \cap R^j)$ is a uu -curve included in $\mathbb{D}^2 \times \mathbb{D}(0, 1/2)$.*

Proof of Lemma 3.5. We start with the following intermediate result:

Sublemma 3.6. *Let \mathcal{C} be a uu -curve included in $\mathbb{D}^2 \times \mathbb{D}(0, 1/2)$. Then for every $F \in \mathcal{U}_1$, for any $j \in J$, the curve \mathcal{C} intersects the interior of R^j but does not intersect $\partial_w R^j \cup \partial_t R^j$.*

Proof. Let us denote $z_j = 1/4$ if $j = 0$, $= i/4$ if $j = 1$, $= -1/4$ if $j = 2$ and $= -i/4$ if $j = 3$. We remark that R^j contains the point $(z_j, c_2(z_j), c_3(z_j)) \in \mathcal{C}$ in its interior since $c_2(z_j) \in \text{int } \mathbb{D}$ and $c_3(z_j) \in \mathbb{D}(0, 1/2)$ while $\{z_j\} \times \text{int } \mathbb{D} \times \mathbb{D}(0, 1/2) \subset \text{int } R^j$ (indeed R^j is close to $\mathbb{D}_j \times \mathbb{D} \times L_j(\mathbb{D})$ with $L_j(\mathbb{D}) \ni \mathbb{D}(0, 1/2)$). Therefore \mathcal{C} intersects $\text{int } R^j$. Also since $\mathcal{C} \subset \mathbb{D} \times \text{int } \mathbb{D} \times \mathbb{D}(0, 1/2)$ while $\partial_w R^j \subset \mathbb{D} \times \partial \mathbb{D} \times \mathbb{D}$ and $\partial_t R^j \cap (\mathbb{D}^2 \times \mathbb{D}(0, 1/2)) = \emptyset$, the result follows. \square

Sublemma 3.7. *Let \mathcal{C} be a uu -curve intersecting the interior of R^j , for $j \in J$, but not $\partial_w R^j \cup \partial_t R^j$. Then $F(\mathcal{C} \cap R^j)$ is a uu -curve.*

Proof. It is immediate from Lemma 3.1 that $F(\mathcal{C} \cap R^j)$ has its tangent spaces in C^{uu} . Moreover, if we denote by π_z the first coordinate projection, the map $\psi : M \in \mathcal{C} \cap R^j \mapsto \pi_z(F(M)) \in \mathbb{C}$ is open (since holomorphic and non constant) on the interior of $\mathcal{C} \cap R^j$. Thus $\partial \text{Im} \psi$ is included in $\psi(\partial(\mathcal{C} \cap R^j)) = \psi(\mathcal{C} \cap \partial_z R^j) \subset \partial \mathbb{D}$. Since $\text{Im} \psi \subset \mathbb{D}$, we conclude that $\partial \text{Im} \psi = \partial \mathbb{D}$. Thus $F(\mathcal{C} \cap R^j)$ is a graph over $z \in \mathbb{D}$ and so a uu -curve. \square

Let us now conclude the proof of Lemma 3.5. By Sublemma 3.4, there exist $\tau \in \mathbb{D}(0, 0.497)$ and $j \in J$ such that $c_3(0) = L_j(\tau)$. Also, by Sublemma 3.6, the curve \mathcal{C} intersects R^j but not $\partial_w R^j \cup \partial_t R^j$. Then, by Sublemma 3.7, we have that $F(\mathcal{C} \cap R^j)$ is a uu -curve. Since any point $(z, c_2(z), c_3(z)) \in \mathcal{C} \cap R^j$ has its third coordinate $c_3(z)$ which is 10^{-3} -close to $c_3(0)$, the preimage of $c_3(z)$ by L_j is in $\mathbb{D}(0, 0.497 + \lambda \cdot 10^{-3}) \subset \mathbb{D}(0, 0.499)$. Since the third coordinate of F^{-1} is close to $L_j(t)$ on $F(R^j)$, it follows that $F(\mathcal{C} \cap R^j)$ is a uu -curve included in $\mathbb{D}^2 \times \mathbb{D}(0, 1/2)$. \square

4 Creating robust cycles

We now create robust heterodimensional cycles and conclude the proof of the main Theorem. We will take the composition of F_1 with finitely many polynomial automorphisms of the form:

$$(z, w, t) \mapsto (z, w, t + P(z, w)) \text{ or } (z, w, t) \mapsto (z, w + P(z, t), t) \text{ or } (z, w, t) \mapsto (z + P(w, t), w, t)$$

where $P : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial. So the final map will also be a polynomial automorphism.

4.1 Robust intersections between $W^s(S)$ and $W^u(K)$

We first create a robust intersection between the stable manifold $W^s(S)$ of the continuation S of the saddle point S_1 and the unstable set $W^u(K)$ of the continuation K of the basic set K_1 , by perturbing F_1 . This is the easiest part since the intersection will be transverse and hence robust.

Proposition 4.1. *There exists $F_2 \in \mathcal{U}_1$ and a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of F_2 such that every $F \in \mathcal{U}_2$ displays a nonempty intersection between $W^s(S)$ and $W^u(K)$.*

Proof of Proposition 4.1. Let $A_1 \in K_1$ be a fixed point of F_1 . Let us take $\omega > 0$ small. We set $\gamma_1 := 1 + 2\omega$ and $\alpha'_1 := 3.9$. The following is an immediate consequence of Proposition 2.10 and Fact 2.12 applied to $F = F_1$:

Fact 4.2. *There exist $M_1 = (\gamma_1, \beta_1, \gamma_1) \in F_1(W_{loc}^u(A_1))$ and $N_1 = (\alpha'_1, \alpha'_1, \gamma'_1) \in W_{loc}^s(S_1)$.*

Let $t \in \mathbb{C} \mapsto P_1(t)$ be a polynomial close to 1 on $\mathbb{D}(\gamma_1, \omega)$ and to 0 on $\mathbb{D} \sqcup \mathbb{D}(3, 1)$. Let $z \in \mathbb{C} \mapsto P_2(z)$ be a polynomial close to 1 on $\mathbb{D}(\alpha'_1, \omega)$ and close to 0 on $\mathbb{D} \sqcup \mathbb{D}(3, 1/2)$. Such polynomials exist by the Runge theorem. We set:

$$\begin{aligned} g_1(z, w, t) &:= (z + P_1(t) \cdot (\alpha'_1 - \gamma_1), w, t), \\ g_2(z, w, t) &:= (z, w + P_2(z) \cdot (\alpha'_1 - \beta_1), t), \\ \text{and } g_3(z, w, t) &:= (z, w, t + P_2(z) \cdot (\gamma'_1 - \gamma_1)). \end{aligned}$$

Notice that $g_3 \circ g_2 \circ g_1$ sends M_1 close to N_1 . Now we set:

$$F_2 := g_3 \circ g_2 \circ g_1 \circ F_1.$$

Fact 4.3. *The map F_2 belongs to the neighborhood \mathcal{U}_1 defined in Proposition 2.7 and Remark 2.14.*

Proof. The first and third coordinate projections of $F_1(W_{loc}^s(K_1)) \cup W_{loc}^u(K_1) \cup F_1(W_{loc}^s(S_1)) \cup W_{loc}^u(S_1)$ are respectively included in $\mathbb{D} \sqcup \mathbb{D}(3, 1/2)$ and $\mathbb{D} \sqcup \mathbb{D}(3, 1)$. Therefore the maps g_1, g_2 and g_3 are all close to the identity on a neighborhood of the latter set and hence $F_2 := g_3 \circ g_2 \circ g_1 \circ F_1$ is close to F_1 on $W_{loc}^s(K_1) \cup F_1^{-1}(W_{loc}^u(K_1)) \cup W_{loc}^s(S_1) \cup F_1^{-1}(W_{loc}^u(S_1))$. \square

In particular F_2 displays the hyperbolic continuations K_2 of K_1 and S_2 of S_1 . Also by Fact 2.12, there is a bidisk Δ^s embedded in $W_{loc}^s(S_2)$ which is C^1 -close to

$$N_1 + \{(z, w, 0) \mid z, w \in \mathbb{D}(0, \omega)\}.$$

Let A_2 be the continuation of A_1 for F_2 . By Proposition 2.10 and since ω is small, there is a bidisk Δ^u embedded in $F_1(W_{loc}^u(A_2))$ which is a graph of the form:

$$\Delta^u = M_1 + \{(z, w(z, t), t) \mid z, t \in \mathbb{D}(0, \omega)\},$$

where $w(0, 0)$ is small and the tangent spaces of Δ^u are included in a cone slightly larger than C^u (in particular $w(z, t) \in \mathbb{D}(0, \omega)$ for $(z, t) \in \mathbb{D}(0, \omega)^2$). Since M_1 is sent close to N_1 by $g_3 \circ g_2 \circ g_1$ and the restriction of $g_3 \circ g_2 \circ g_1$ to $M_1 + \mathbb{D}(0, \omega)^3$ is close to a translation, the set $(g_3 \circ g_2 \circ g_1)(\Delta^u)$ (which is included in $F_2(W_{loc}^u(A_2)) \subset W^u(A_2)$) is C^1 -close to:

$$N_1 + \{(z, w(z, t), t) \mid z, t \in \mathbb{D}(0, \omega)\}.$$

The directions of the tangent spaces of Δ^s are close to $\mathbb{C} \cdot (1, 0) + \mathbb{C} \cdot (0, 1)$ while those of $(g_3 \circ g_2 \circ g_1)(\Delta^u)$ are included in a cone slightly larger than C^u . Therefore $\Delta^s \subset W_{loc}^s(S_2)$ and $(g_3 \circ g_2 \circ g_1)(\Delta^u) \subset W^u(A_2)$ have a transverse intersection close to $N_1 + (0, w(0, 0), 0) \approx N_1$. In particular, this intersection is robust, which ends the proof of Proposition 4.1. \square

Remark 4.4. The set \mathcal{U}_2 can be taken as the subset of \mathcal{U}_1 formed by maps which are ϵ_2 - C^0 -close to F_2 on the ω -neighborhood of $F_1^{-1}(M_1)$, for some small $\epsilon_2 > 0$.

4.2 Robust intersections between $W^u(S)$ and $W^s(K)$

We now create a robust intersection between the unstable manifold $W^u(S)$ of the continuation S of the saddle point S_2 and the stable set $W^s(K)$ of the continuation K of the basic set K_2 . This is the difficult part since both the unstable manifold of S and the stable manifolds of points of K are one-dimensional so we can not apply a transversality argument here. To overcome this difficulty, we will use here the blender property to obtain the robustness.

Proposition 4.5. *There exist $F_3 \in \mathcal{U}_2$ and a neighborhood $\mathcal{U} \subset \mathcal{U}_2$ of F_3 such that every $F \in \mathcal{U}$ displays a nonempty intersection between $W^u(S)$ and $W^s(K)$.*

4.2.1 Obtaining an initial intersection between $W^u(S)$ and $W^s(K)$.

We now perturb the automorphism F_2 into F_3 in order to get an intersection point between $W^u(S_3)$ and $W^s(K_3)$. We start by selecting a particular periodic point in K_2 :

Fact 4.6. *There is a periodic point $B_2 \in K_2$ of F_2 of third coordinate in $\mathbb{D}(0, 1/10)$.*

Proof. By Proposition 3.3, there is a point in $W_{loc}^s(K_2)$ of third coordinate 0. Since by Proposition 2.10 any local stable manifold has its tangent spaces in C^s , there is a point of K_2 of third coordinate close to 0. Then, by density of periodic points in K_2 , there is a periodic point $B_2 \in K_2$ of third coordinate in $\mathbb{D}(0, 1/10)$. \square

Let us take again $\omega > 0$ small. We set $\gamma_2 := 2 - 2\omega$ and $\beta'_2 := -0.9$. The following is an immediate consequence of Proposition 2.10 and Fact 2.13 together with Fact 4.3:

Fact 4.7. *There exist $M_2 = (\alpha_2, \beta_2, \gamma_2) \in F_2(W_{loc}^u(S_2))$ and $N_2 = (\alpha'_2, \beta'_2, \gamma'_2) \in W_{loc}^s(B_2)$.*

Let $t \in \mathbb{C} \mapsto Q_1(t)$ be a polynomial close to 1 on $\mathbb{D}(\gamma_2, \omega)$ and close to 0 on $\mathbb{D} \sqcup \mathbb{D}(3, 1)$. Similarly let $w \in \mathbb{C} \mapsto Q_2(w)$ be a polynomial close to 1 on $\mathbb{D}(\beta'_2, \omega)$ and close to 0 on $\mathbb{D}(0, 1/2) \sqcup \mathbb{D}(3, 1)$. Let us also take $\mu \in \mathbb{D}(0, 1/10)$. Then we define:

$$\begin{aligned} h_1(z, w, t) &:= (z, w + Q_1(t) \cdot (\beta'_2 - \beta_2), t), \\ h_2(z, w, t) &:= (z, w, t + Q_2(w) \cdot (\gamma'_2 - \gamma_2)), \\ \text{and } h_3(z, w, t) &:= (z + Q_2(w) \cdot (\alpha'_2 - \alpha_2 + \mu + (t - \gamma'_2)), w, t). \end{aligned}$$

Notice that $h_3 \circ h_2 \circ h_1$ sends M_2 close to N_2 . Now we set:

$$F_3 := h_3 \circ h_2 \circ h_1 \circ F_2.$$

Fact 4.8. *The map F_3 belongs to the neighborhood \mathcal{U}_2 defined in Proposition 4.1 and Remark 4.4.*

Proof. The same proof as for Fact 4.3 shows that $F_3 \in \mathcal{U}_1$. Also recall that $F_1^{-1}(M_1)$ is sent close to N_1 by F_2 . Since $N_1 = (3.9, 3.9, \gamma'_1)$ with $\gamma'_1 \in \mathbb{D}(0, 1/2)$, the maps h_1 , h_2 and h_3 are all close to the identity on a ball around N_1 of radius bounded from below independently of ω . Since ω is small, this implies that F_3 is close to F_2 on the ω -neighborhood of $F_1^{-1}(M_1)$ and so $F_3 \in \mathcal{U}_2$. \square

By Proposition 2.10, there is a disk U_μ embedded in the local stable manifold of the continuation $B_3 \in K_3$ of $B_2 \in K_2$ for F_3 which is close to:

$$\{(\alpha'_2, \beta'_2 + w, \gamma'_2) \mid w \in \mathbb{D}(0, \omega)\}. \quad (4.1)$$

Let S_3 be the continuation of S_2 for F_3 . By Fact 2.13 and since ω is small, there is a disk \tilde{V}_μ embedded in $F_2(W_{loc}^u(S_3))$ which is close to $\{(\alpha_2, \beta_2, \gamma_2 + t) \mid t \in \mathbb{D}(0, \omega)\}$. Then its image is $V_\mu := (h_3 \circ h_2 \circ h_1)(\tilde{V}_\mu) \subset F_3(W_{loc}^u(S_3)) \subset W^u(S_3)$ and is close to:

$$\{(\alpha'_2 + \mu + t, \beta'_2, \gamma'_2 + t) \mid t \in \mathbb{D}(0, \omega)\}. \quad (4.2)$$

We will use the following independent technical result:

Lemma 4.9. *Let $(U_\mu)_{\mu \in \mathbb{D}}$ be a holomorphic family of graphs $w \in \text{int } \mathbb{D} \mapsto (U_\mu^1(w), w, U_\mu^3(w)) \in \mathbb{C}^3$, where U_μ^1 and U_μ^3 are small. Let $(V_\mu)_{\mu \in \mathbb{D}}$ be a holomorphic family of graphs $t \in \text{int } \mathbb{D} \mapsto (V_\mu^1(t), V_\mu^2(t), t) \in \mathbb{C}^3$. Suppose that $\mu \mapsto V_\mu^1(0)$ is close to μ and that $V_\mu^2(t) \in \text{int } \mathbb{D}$. Then there exists $\mu_o \in \mathbb{D}$ such that U_{μ_o} intersects V_{μ_o} .*

Proof. Up to the change of coordinates $(z, w, t) \mapsto (z - U_\mu^1(w), w, t - U_\mu^3(w))$ which is close to the identity, we can suppose that $U_\mu = \{(0, w, 0) \mid w \in \mathbb{D}\}$. Notice that V_μ intersects the plane $\{t = 0\}$ at a unique point equal to $(V_\mu^1(0), V_\mu^2(0), 0)$. Also the map $\mu \in \text{int } \mathbb{D} \mapsto V_\mu^1(0)$ is holomorphic and close to $\lambda \mapsto \lambda$. Then by the Rouché theorem there is $\mu_o \in \mathbb{D}(0, 1/2)$ such that $V_{\mu_o}^1(0) = 0$. Then the point $(0, V_{\mu_o}^2(0), 0)$ belongs to $U_{\mu_o} \cap V_{\mu_o}$. \square

As a consequence of Lemma 4.9 together with Eq. (4.1) and Eq. (4.2), we obtain:

Proposition 4.10. *The unstable manifold $W^u(S_3)$ intersects the local stable manifold $W_{loc}^s(B_3)$ at a point N such that $T_N W^u(S_3)$ is close to $\mathbb{C} \cdot (1, 0, 1)$.*

4.2.2 Obtaining robust intersections between $W^u(S_3)$ and $W^s(K_3)$ using the blender property.

We now conclude the proofs of Proposition 4.5 and of the main Theorem. We use the blender property to make robust the initial intersection between $W^u(S_3)$ and $W^s(K_3)$ we obtained above.

Proposition 4.11. *Let Δ_o be an embedded open disk intersecting $W_{loc}^s(B_3)$ at a point N such that $T_N \Delta_o$ is close to $\mathbb{C} \cdot (1, 0, 1)$. Then there exists $n \geq 1$ such that for every map F close to F_3 and every embedded disk Δ close to Δ_o , the intersection $F^n(\Delta) \cap (\mathbb{D}^2 \times \mathbb{D}(0, 1/2))$ contains a uu -curve.*

Proof. Let us consider the following matrix:

$$\mathcal{M}_o := \begin{pmatrix} 10^4 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

For every matrix \mathcal{M} close to \mathcal{M}_o and every vector v close to $(1, 0, 1)$, we have $\mathcal{M} \cdot v \in \text{int } C^{uu}$. Observe that the differential of F_3 at any point in $W_{loc}^s(B_3)$ is close to \mathcal{M}_o . Thus, if Δ_o intersects

$W_{loc}^s(B_3)$ at N with a direction close to $\mathbb{C} \cdot (1, 0, 1)$, the direction of $F_3(\Delta_o)$ at $F_3(N) \in W_{loc}^s(B_3)$ is in $\text{int } C^{uu}$. Then, by invariance of the cone C^{uu} (see Lemma 3.1) and by expansion in the strong unstable direction, for n large enough, $F_3^n(\Delta_o)$ contains a holomorphic graph Δ_n over z varying in a neighborhood of \mathbb{D} with tangent spaces in $\text{int } C^{uu}$ (in particular it contains a uu -curve). If moreover n is a multiple of the period of B_3 , then Δ_n intersects $W_{loc}^s(B_3)$ at a point $F_3^n(N)$ close to B_3 . Since the third coordinate of B_3 is in $\mathbb{D}(0, 1/10)$ by Fact 4.6, the curve Δ_n is included in $\mathbb{C}^2 \times \mathbb{D}(0, 1/3)$. Then, by continuity, for every map F close to F_3 and every Δ close to Δ_o , we have that $F^n(\Delta) \cap (\mathbb{D}^2 \times \mathbb{D}(0, 1/2))$ contains a uu -curve. \square

Proof of Proposition 4.5. By Proposition 4.10 and then Proposition 4.11, for every F in a neighborhood $\mathcal{U} \subset \mathcal{U}_2$ of F_3 , the unstable manifold $W^u(S)$ contains a uu -curve $\mathcal{C} \subset \mathbb{D}^2 \times \mathbb{D}(0, 1/2)$. Therefore by Proposition 3.3, there is a nonempty intersection between $\mathcal{C} \subset W^u(S)$ and $W_{loc}^s(K)$. This ends the proof of Proposition 4.5. \square

Proof of the Main Theorem. It follows immediately from Proposition 4.1 and Proposition 4.5. \square

Proof of the main Corollary. We just take $\mathcal{U}_d := \mathcal{U} \cap \text{Aut}_d(\mathbb{C}^3)$ with $d := \deg F_3$ to conclude. \square

sebastien.biebler@imj-prg.fr

Sorbonne Université, Université Paris-Cité, CNRS

Institut de Mathématiques de Jussieu-Paris Rive Gauche

75005 Paris, France

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