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# NORM ATTAINING OPERATORS INTO LOCALLY ASYMPTOTICALLY MIDPOINT UNIFORMLY CONVEX BANACH SPACES

A. FOVELLE

ABSTRACT. We prove that if  $Y$  is a locally asymptotically midpoint uniformly convex Banach space which has either a normalized, symmetric basic sequence that is not equivalent to the unit vector basis in  $\ell_1$ , or a normalized sequence with upper  $p$ -estimates for some  $p > 1$ , then  $Y$  does not satisfy Lindenstrauss' property B.

## 1. INTRODUCTION

The study of the denseness of norm-attaining operators started with the seminal paper by Bishop and Phelps [5], in which they prove that every functional can be approximated by norm-attaining ones. In this same paper, they ask the following question: given  $X$  and  $Y$  two Banach spaces, does the set of norm-attaining operators from  $X$  to  $Y$ , denoted by  $\text{NA}(X, Y)$  (that is  $T \in \text{NA}(X, Y)$  if  $\|Tx\| = \|T\|$  for some  $x \in B_X$ , the unit ball of  $X$ ) is dense in  $\mathcal{L}(X, Y)$ , the space of all (linear continuous) bounded operators? This question was answered by the negative in 1963 by Lindenstrauss [15], who also gave some positive examples. Following [15], we say that a Banach space  $Y$  has property B if  $\text{NA}(X, Y)$  is dense in  $\mathcal{L}(X, Y)$  for every Banach space  $X$ . The negative example of Lindenstrauss was the following: any strictly convex space containing an isomorphic copy of  $c_0$  fails property B. Thoroughly studied, important examples of Banach spaces having or failing property B have been given since (see for exemple [15] [17] [12] [18] [13] [3]).

Let us mention that the first example of a reflexive space failing property B is due to Gowers in 1990 [10], who proved that  $\ell_p$  does not have property B when  $p \in (1, \infty)$ .

In [4], Aguirre proved that strictly convex spaces satisfying the extra condition of having either a normalized, symmetric basic sequence which is not equivalent to the unit vector basis in  $\ell_1$ , or a normalized sequence with upper  $p$ -estimates for some  $p > 1$  (see Section 2 for the definition), do not satisfy property B. This was extended to all strictly convex Banach spaces by Acosta [1], who also proved that the same results holds for infinite-dimensional  $L_1(\mu)$  spaces [2].

The result of Aguirre was enough to deduce what he describes as the main result of his paper, namely: every infinite-dimensional uniformly convex Banach space fails property B. This is the result we will generalize in the asymptotic setting. To be more specific, we will prove the following theorem.

**Theorem A.** *Let  $Y$  be a locally AMUC space which has either a normalized, symmetric basic sequence which is not equivalent to the unit vector basis in  $\ell_1$ , or a normalized sequence with upper  $p$ -estimates for some  $1 < p < \infty$ . Then  $Y$  fails property B.*

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## 2. DEFINITIONS AND NOTATION

All Banach spaces in these notes are assumed to be real. We denote the closed unit ball of a Banach space  $X$  by  $B_X$ , and its unit sphere by  $S_X$ . Given a Banach space  $X$  with norm  $\|\cdot\|_X$ , we simply write  $\|\cdot\|$  as long as it is clear from the context on which space it is defined.

First, let us define locally asymptotically midpoint uniformly convex spaces.

**2.1. Locally AMUC Banach spaces.** Let  $Y$  be a Banach space. Let us denote by  $\text{cof}(Y)$  the set of all closed subspaces of  $Y$  of finite codimension. For  $y \in S_Y$  and  $t \in \mathbb{R}^+$ , let

$$\begin{aligned}\tilde{\delta}_Y(y, t) &= \sup_{E \in \text{cof}(Y)} \inf_{z \in S_E} \max\{\|y + tz\|, \|y - tz\|\} - 1 \\ &= \sup_{E \in \text{cof}(Y)} \inf_{\substack{z \in E \\ \|z\| \geq 1}} \max\{\|y + tz\|, \|y - tz\|\} - 1\end{aligned}$$

where the second equality follows from the fact that, for every  $z \in Z$ , the map  $s \in (0, \infty) \mapsto \max\{\|y + sz\|, \|y - sz\|\}$  is non-decreasing.

We say that  $Y$  is *locally asymptotically midpoint uniformly convex* (locally AMUC) if  $\tilde{\delta}_Y(y, t) > 0$  for every  $y \in S_Y$  and every  $t > 0$ . This is a local version of property AMUC, defined in [6]:  $Y$  is *asymptotically midpoint uniformly convex* (AMUC) if  $\tilde{\delta}_Y(t) := \inf_{y \in S_Y} \tilde{\delta}_Y(y, t) > 0$  for every  $t > 0$ .

**Proposition 1** (Corollary 2.3 [6]). *If  $Y$  is locally AMUC, then for every  $y \in S_Y$  and every  $t > 0$ , there exists  $\delta > 0$  such that*

$$\limsup \max\{\|y + tz_n\|, \|y - tz_n\|\} \geq 1 + \delta$$

for every weakly null sequence  $(z_n)_{n \in \mathbb{N}} \subset Y$  such that  $\|z_n\| \geq 1$  for every  $n \in \mathbb{N}$ . Moreover, the converse holds if  $Y$  does not contain  $\ell_1$ .

We now introduce the family of spaces we will use as domain spaces for the counterexamples.

**2.2. A family of Banach spaces.** We say that a sequence  $w = (w_n)_{n \in \mathbb{N}}$  of positive numbers is *admissible* if it is decreasing,  $w_1 = 1$  and  $w \in c_0 \setminus \ell_1$ . If  $w$  is an admissible sequence, we can define an associated Banach space  $d_*(w)$  as follows: let

$$d_*(w) = \left\{ x = (x_n)_{n \in \mathbb{N}} \in c_0; \lim_{n \rightarrow +\infty} \frac{\sum_{k=1}^n \tilde{x}_k}{\sum_{k=1}^n w_k} = 0 \right\}$$

where  $(\tilde{x}_n)_{n \in \mathbb{N}}$  is the decreasing rearrangement of  $(|x_n|)_{n \in \mathbb{N}}$ , endowed with the norm

$$\forall x \in d_*(w), \|x\| = \sup_{n \in \mathbb{N}} \frac{\sum_{k=1}^n \tilde{x}_k}{\sum_{k=1}^n w_k}.$$

The space  $d_*(w)$  is known to be a predual of a Lorentz sequence space (see [8], [16]) and it has a symmetric basis  $(e_n)_{n \in \mathbb{N}}$  that shares properties with the one of  $c_0$ . We will in particular use the following one, which proof can be found in [11], or [10] in the special case  $w = (1/n)_{n \in \mathbb{N}}$ .

**Lemma 2.** *For every  $x \in S_{d_*(w)}$ , we can find  $m \in \mathbb{N}$  and  $\delta \in (0, 1)$  so that*

$$\|x + \lambda e_n\| \leq 1$$

for every  $n \geq m$  and every  $\lambda \in \mathbb{K}$  so that  $|\lambda| \leq \delta$ .

We finish this subsection with definitions about sequences

**2.3. Symmetric basic sequences and upper  $p$ -estimates.** First of all, let us recall that a *basic* sequence is an infinite sequence that is a basis of its closed linear span. If  $(x_n)_{n \in \mathbb{N}}$  is a basis of a Banach space  $X$ , it is said to be *symmetric* if every permutation  $(x_{\sigma(n)})_{n \in \mathbb{N}}$  if  $(x_n)_{n \in \mathbb{N}}$  is a basis of  $X$ , equivalent to the basis  $(x_n)_{n \in \mathbb{N}}$ .

Let us now recall the definition of having upper  $p$ -estimates,  $1 < p < \infty$ , for a sequence. Let  $X$  be a Banach space and  $p \in (1, \infty)$ . We say that a normalized sequence  $(x_n)_{n \in \mathbb{N}}$  of elements of  $X$  has *upper  $p$ -estimates* if there exists a constant  $C > 0$  such that

$$\left\| \sum_{k=1}^n a_k x_k \right\| \leq C \left( \sum_{k=1}^n |a_k|^p \right)^{1/p}$$

for every  $n \in \mathbb{N}$  and every sequence of scalars  $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ .

If we denote by  $(f_n)_{n \in \mathbb{N}}$  the canonical vector basis of  $\ell_p$ , one can note that the existence of a normalized sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  with upper  $p$ -estimates is equivalent to the existence of an operator  $\alpha \in \mathcal{L}(\ell_p, X)$  satisfying  $\|\alpha(f_n)\| = 1$  for every  $n \in \mathbb{N}$ .

For a non-exhaustive list of Banach spaces with upper  $p$ -estimates, one can refer to [9], [7].

### 3. RESULTS

Theorem A will be deduced from the following key proposition, which links norm-attaining operators and locally AMUC Banach spaces.

**Proposition 3.** *If  $Y$  is a locally AMUC Banach space and  $w$  is an admissible sequence, then every norm-attaining operator  $T \in \text{NA}(d_*(w), Y)$  from  $d_*(w)$  into  $Y$  satisfies  $\lim \|Te_n\| = 0$ .*

*Proof.* Let  $T \in \text{NA}(d_*(w), Y)$ . Without loss of generality, we can assume  $\|T\| = 1$ . Then, there exists  $x \in B_{d_*(w)}$  so that  $\|Tx\| = \|T\| = 1$  and by assumption on  $Y$  and Proposition 1, for every  $t > 0$  we can find  $\delta(t) > 0$  such that

$$\limsup \max\{\|Tx + tz_n\|, \|Tx - tz_n\|\} \geq 1 + \delta(t)$$

for every weakly null sequence  $(z_n)_{n \in \mathbb{N}} \subset Y$  such that  $\|z_n\| \geq 1$  for every  $n \in \mathbb{N}$ .

Moreover, by Lemma 2, we can find  $\delta \in (0, 1)$  and  $m \in \mathbb{N}$  so that  $x \pm \delta e_n \in B_{d_*(w)}$  for every  $n \geq m$ .

Assume now by contradiction that  $\eta = \lim \|Te_n\| > 0$  and let  $\epsilon = \delta(\frac{\delta\eta}{2}) > 0$ . Up to extraction, we can assume without loss of generality that  $\|Te_n\| \geq \frac{\eta}{2}$  for every  $n \in \mathbb{N}$ . Since the sequence  $(Te_n)_{n \in \mathbb{N}}$  is weakly null, we get

$$1 + \epsilon \leq \limsup \max\{\|Tx + \delta Te_n\|, \|Tx - \delta Te_n\|\}.$$

As  $\|x \pm \delta e_n\| \leq 1$  for every  $n \geq m$ , the contradiction follows from

$$1 + \epsilon \leq \limsup \max\{\|T(x + \delta e_n)\|, \|T(x - \delta e_n)\|\} \leq \|T\| \leq 1. \quad \square$$

Let us now describe how to deduce Theorem A from the previous proposition. We will do it in two steps, starting with locally AMUC spaces that contain a normalized, symmetric basic sequence which is not equivalent to the unit vector basis in  $\ell_1$ . In [4], Aguirre proved the following result.

**Proposition 4** (Proposition 4 [4]). *Let  $Y$  be a Banach space containing a normalized, symmetric basic sequence  $(y_n)_{n \in \mathbb{N}}$  which is not equivalent to the unit vector basis in  $\ell_1$ . Then there is an admissible sequence  $w$  and an operator  $T \in \mathcal{L}(d_*(w), Y)$  such that*

$$\forall n \in \mathbb{N}, Te_n = y_n.$$

As a consequence, we immediately get

**Theorem 5.** *Let  $Y$  be a locally AMUC Banach space which has a normalized, symmetric basic sequence  $(y_n)_{n \in \mathbb{N}}$  which is not equivalent to the unit vector basis in  $\ell_1$ . Then  $Y$  fails property B.*

From now on, let  $w = (\frac{1}{n})_{n \in \mathbb{N}}$ , and  $G = d_*(w)$ , the space used by Gowers in [10] to prove that  $\ell_p$  does not have property B for  $1 < p < \infty$ .

Proposition 7 from [4] asserts that if  $Y$  is a Banach space in which we can find a normalized sequence with upper  $p$ -estimates, then there exists a noncompact operator  $T$  from  $G$  into  $Y$ . By looking at the proof, one even gets the following proposition. For sake of completeness, we introduce a proof.

**Proposition 6.** *Let  $Y$  be a Banach space in which we can find a normalized sequence  $(y_n) \subset Y$  with upper  $p$ -estimates. Then there exists a bounded linear operator  $T$  from  $G$  into  $Y$  such that  $\|Te_n\| = 1$  for every  $n \in \mathbb{N}$ .*

*Proof.* From the hypothesis on the sequence  $(y_n)_{n \in \mathbb{N}}$ , we get a bounded linear operator  $U \in \mathcal{L}(\ell_p, Y)$  sending the elements of the canonical basis of  $\ell_p$  on the sequence  $(y_n)_{n \in \mathbb{N}}$ . Moreover, the space  $G$  is known to be contained in  $\ell_p$  and the formal identity defines a bounded linear operator  $S \in \mathcal{L}(G, \ell_p)$  (see [10], p.149). To conclude, just note that  $T = US \in \mathcal{L}(G, Y)$  satisfies  $\|Te_n\| = \|y_n\| = 1$  for every  $n \in \mathbb{N}$ .  $\square$

The following result follows immediately.

**Theorem 7.** *Let  $Y$  be a locally AMUC space which has a normalized sequence with upper  $p$ -estimates,  $1 < p < \infty$ . Then  $Y$  fails property B.*

Theorem A gives us new examples of spaces without property B. Before mentioning such a class of new examples, let us remind the definition of  $\ell_p$ -sums. Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of Banach spaces and  $p \in [1, \infty)$ . We define the sum  $(\sum_{n \in \mathbb{N}} X_n)_{\ell_p}$  to be the space of sequences  $(x_n)_{n \in \mathbb{N}}$ , where  $x_n \in X_n$  for all  $n \in \mathbb{N}$ , such that  $\sum_{n \in \mathbb{N}} \|x_n\|_{X_n}^p$  is finite, and we set

$$\|(x_n)_{n \in \mathbb{N}}\| = \left( \sum_{n \in \mathbb{N}} \|x_n\|_{X_n}^p \right)^{\frac{1}{p}}.$$

One can check that  $(\sum_{n \in \mathbb{N}} X_n)_{\ell_p}$ , endowed with the norm  $\|\cdot\|$  defined above, is a Banach space. In addition, it is known that if  $(F_n)_{n \in \mathbb{N}}$  is a sequence of finite dimensional spaces, then  $(\sum_{n \in \mathbb{N}} F_n)_{\ell_p}$  is locally AMUC. More precisely, they are asymptotically uniformly convex (see [14] for example), therefore AMUC and thus locally AMUC. Now, we can state our last result.

**Corollary 8.** *Let  $(F_n)_{n \in \mathbb{N}}$  be a sequence of non-trivial finite dimensional spaces,  $p \in (1, \infty)$ . Then any infinite dimensional subspace of  $(\sum_{n \in \mathbb{N}} F_n)_{\ell_p}$  fails property B.*

#### 4. OPEN QUESTIONS

**Problem 1.** Does Theorem A hold for all locally AMUC Banach spaces?

One could start by the answering the following intermediate question.

**Problem 2.** If  $(F_n)$  is a sequence of non-trivial finite-dimensional spaces, does  $(\sum_{n \in \mathbb{N}} F_n)_{\ell_1}$  fail property B?

We will finish this paper by recalling the following longstanding open problem, even for the 2-dimensional Euclidean space.

**Problem 3.** Do finite dimensional spaces have property B?

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