SEMICLASSICAL RESOLVENT ESTIMATES FOR THE MAGNETIC SCHR ÖDINGER OPERATOR

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Abstract. We obtain semiclassical resolvent estimates for the Schrödinger operator $(ih\nabla + b)^2 + V$ in $\mathbb{R}^d$, $d \geq 3$, where $0 < h \ll 1$ is a semiclassical parameter, V and b are real-valued electric and magnetic potentials independent of h. If $V \in L^\infty(\mathbb{R}^d)$, $b \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$, $\operatorname{div} b \in L^\infty(\mathbb{R}^d)$ satisfy $V(x) = \mathcal{O}\left(|x|^{-1-\epsilon}\right)$, $b(x) = \mathcal{O}\left(|x|^{-1-\epsilon}\right)$, $\operatorname{div} b(x) = \mathcal{O}\left(|x|^{-1-\epsilon}\right)$, $\epsilon > 0$, for $|x| \gg 1$, we prove that the norm of the weighted resolvent is bounded by $\exp\left(Ch^{-2}\log(h^{-1})\right)$, $C > 0$. We get better resolvent bounds for electric potentials which are Hölder with respect to the radial variable and magnetic potentials which are Hölder with respect to the space variable. For long-range electric potentials which are Lipschitz with respect to the radial variable and long-range magnetic potentials which are Lipschitz with respect to the space variable we obtain a resolvent bound of the form $\exp\left(Ch^{-1}\right)$, $C > 0$.

Key words: Schrödinger operator, magnetic potentials, resolvent estimates.

1. Introduction and statement of results

In this paper we are going to study the resolvent of the Schrödinger operator

$$P(h) = (ih\nabla + b(x))^2 + V(x) \quad \text{in} \quad \mathbb{R}^d, d \geq 3,$$

where $0 < h \ll 1$ is a semiclassical parameter, ∇ is the gradient, $V \in L^\infty(\mathbb{R}^d;\mathbb{R})$ and $b = (b_1, ..., b_d) \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$ are electric and magnetic potentials, respectively. We are interested in bounding the quantity

$$g_s^\pm(h,\varepsilon) := \log \left\|(|x|+1)^{-s}(P(h) - E \pm i\varepsilon)^{-1}(|x|+1)^{-s}\right\|_{L^2\to L^2}$$

from above by an explicit function of h, independent of ε. Here $L^2 := L^2(\mathbb{R}^d)$, $0 < \varepsilon < 1$, $s > 1/2$ is independent of h and $E > 0$ is a fixed energy level independent of h. Our goal is to obtain the best possible upper bounds for $g_s^\pm(h,\varepsilon)$ for potentials which are Lipschitz, Hölder or just L^∞. There have been recently many papers studying this problem when $b \equiv 0$. To our best knoweledge, no such results exist for non-trivial magnetic potentials when $d \geq 2$ and it seems that this paper is the first one where upper bounds for $g_s^\pm$ are proved in this case. When $d = 1$ and b not identically zero, semiclassical resolvent bounds have been recently proved in [9] for a very large class of electric and magnetic potentials. Note also that sharp high-frequency resolvent bounds for the operator $P(1)$ with L^∞ potentials are proved in [11] when $d \geq 2$ and in [16] when $d \geq 3$. In [7] exponential high-frequency resolvent bounds for the operator $P(1)$ with smooth potentials on non-compact Riemannian manifolds have been recently proved, extending the results in [1].

We will be looking for bounding $g_s^\pm$ for the largest possible class of electric and magnetic potentials that our method allows to cover. To describe it we introduce the polar coordinates $r = |x|$, $w = x/|x|$. We suppose that $V = V_L + V_S$, $b = b^L + b^S$, where V_L, b^L (resp. V_S, b^S) are long-range (resp. short-range) electric and magnetic potentials satisfying the following conditions.

We suppose that V_L satisfies

$$V_L(rw) \le p(r), \tag{1.1}$$

where $p > 0$ is a decreasing function such that $p(r) \to 0$ as $r \to \infty$. We also suppose that $V_L(rw)$ is Lipschitz with respect to the radial variable r and its first derivative satisfies the bound

$$\partial_r V_L(rw) \le C(r+1)^{-1-\rho}. \tag{1.2}$$

Hereafter $C > 0$ and $0 < \rho \ll 1$ will denote positive constants that may change from line to line. The short-range part $V_S \in L^\infty(\mathbb{R}^d)$ is supposed to satisfy

$$|V_S(rw)| \le C(r+1)^{-1-\rho}. \tag{1.3}$$

The long-range part $b^L \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$ is supposed to be Lipschitz with respect to r and satisfies

$$|\partial_r^k b^L(rw)| \le C(r+1)^{-k-\rho}, \quad k = 0, 1. \tag{1.4}$$

The short-range part $b^S \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$ satisfies

$$|b^S(rw)| \le C(r+1)^{-1-\rho}. \tag{1.5}$$

Finally, we suppose that $\operatorname{div} b^S \in L^\infty(\mathbb{R}^d)$ exists and satisfies

$$|\operatorname{div} b^S(rw)| \le C(r+1)^{-1-\rho}. \tag{1.6}$$

This condition is fulfilled if each b_j^S is Lipschitz with respect to the variable x_j and $\partial_{x_j} b_j^S(x) = \mathcal{O}\left(|x|^{-1-\rho}\right)$ for $|x| \gg 1$.

Throughout this paper, given two vectors $a, b \in \mathbb{R}^d$, $a \cdot b$ will denote the scalar product. Our first result is the following

Theorem 1.1. *In addition to the above conditions we assume either the condition*

$$|w \cdot b^L(rw)| \le C(r+1)^{-1-\rho}, \tag{1.7}$$

or that the function $w \cdot b^L(rw)$ is Lipschitz in w and satisfies

$$|\partial_w^\beta (w \cdot b^L(rw))| \le C, \quad |\beta| = 1. \tag{1.8}$$

Then, there exist constants $C > 0$, $0 < h_0 \ll 1$, independent of h and ε, such that the bound

$$g_s^\pm(h, \varepsilon) \le Ch^{-2} \log(h^{-1}) \tag{1.9}$$

holds for all $0 < h \le h_0$. If $V_S \equiv 0$ and $b^S \equiv 0$, under the condition (1.8), we have the better bound

$$g_s^\pm(h, \varepsilon) \le Ch^{-1}. \tag{1.10}$$

The condition (1.8) is fulfilled if the function $x \cdot b^L(x)$ is Lipschitz in x. Note that the bound (1.10) is proved in [9] when $d = 1$, while for $d \ge 3$ it seems to be new. Proving (1.10) when $d = 2$ in the presence of a magnetic potential remains an open problem. When $b \equiv 0$, however, the bound (1.10) is well-known. Indeed, it was proved in [2] when $d \ge 3$ for potentials satisfying (1.1) with $p(r) = C(r+1)^{-\rho}$, and extended to the general case in [5] and [18]. When $d = 2$ the bound (1.10) is proved in [13] for potentials $V \in C^1(\mathbb{R}^2)$ satisfying (1.1) with $p(r) = C(r+1)^{-\rho}$ as well as the condition

$$|\nabla V(x)| \le C(|x|+1)^{-1-\rho}. \tag{1.11}$$

The bound (1.10) has been recently proved in [12] for long-range Lipschitz potentials having singularities at the origin in all dimensions $d \ge 2$. When $d = 1$ the bound (1.10) is proved in [4] for potentials $V \in L^1(\mathbb{R})$ and in [9] for more general measure potentials. It turns out that (1.10) also holds for $V \in L^\infty(\mathbb{R}^d)$, $d \ge 2$, provided V is compactly supported and depends only on the

radial variable r. Indeed, this is proved in [3] when $d \geq 2$. A simpler proof of this result is given in [21] when $d \geq 3$. Without a radial symmetry or some regularity of the potential, however, the bound (1.10) seems to fail even for compactly supported potentials. Indeed, for potentials $V \in L^\infty$ satisfying

$$|V(x)| \leq C(|x|+1)^{-\delta} \tag{1.12}$$

where $\delta > 1$, the following bounds are known to hold:

$$g_s^{\pm}(h,\varepsilon) \leq Ch^{-4/3}\log(h^{-1}) \quad \text{if} \quad \delta > 2, \tag{1.13}$$

$$g_s^{\pm}(h,\varepsilon) \leq C_\epsilon h^{-\frac{2\delta}{2\delta-1}-\epsilon} \quad \text{if} \quad 1 < \delta \leq 2, \tag{1.14}$$

for every $0 < \epsilon \ll 1$. When $d \geq 3$ the bound (1.13) is proved in [6] and in [17] if $\delta > 3$. Previously, (1.13) was proved for compactly supported L^∞ potentials in [8] and [14]. The bound (1.13) is extended in [19] to more general Riemannian manifolds. The bounds (1.13) and (1.14) have been recently proved in [15] when $d \geq 2$ for potentials $V = V_L + V_S$ with V_L satisfying (1.1) and (1.2), and V_S satisfying (1.12). In fact, the conditions on the behavior of V_L and V_S at infinity in this paper are a little bit more general than this. Moreover, the potential in [15] is allowed to have singularities at the origin. It turns out that (1.13) and (1.14) can be improved for potentials $V \in L^\infty$ depending only on r. Indeed, for radial potentials satisfying (1.12) it is shown in [20] and [21] that when $d \geq 3$ we have the bounds

$$g_s^{\pm}(h,\varepsilon) \leq Ch^{-\frac{\delta}{\delta-1}} \quad \text{if} \quad \delta > 4, \tag{1.15}$$

$$g_s^{\pm}(h,\varepsilon) \leq Ch^{-4/3} \quad \text{if} \quad 2 < \delta \leq 4, \tag{1.16}$$

$$g_s^{\pm}(h,\varepsilon) \leq Ch^{-\frac{2\delta}{2\delta-1}}\left(\log(h^{-1})\right)^{\frac{\delta+1}{2\delta-1}} \quad \text{if} \quad 1 < \delta \leq 2. \tag{1.17}$$

It turns out that the above bounds can be improved if some small regularity of the potential is assumed. To desribe these results, given $0 < \alpha \leq 1$ and $k > 0$, we introduce the space $C_k^\alpha(\overline{\mathbb{R}^+})$ of all Hölder functions a such that

$$\sup_{r' \geq 0:\, 0<|r-r'|\leq 1} \frac{|a(r)-a(r')|}{|r-r'|^\alpha} \leq C(r+1)^{-k}, \quad \forall r \in \overline{\mathbb{R}^+}.$$

We suppose that the function $V(r,w) := V(rw)$ satisfies the condition

$$V(\cdot,w) \in C_{3+\rho}^\alpha(\overline{\mathbb{R}^+}), \quad 0 < \alpha < 1, \tag{1.18}$$

uniformly in $w \in \mathbb{S}^{d-1}$. For potentials V satisfying (1.1) and (1.18) and $d \geq 3$ the following bound is known to hold:

$$g_s^{\pm}(h,\varepsilon) \leq Ch^{-4/(\alpha+3)}\log(h^{-1}). \tag{1.19}$$

The bound (1.19) is proved in [6] and in [18] if $\rho = 1$. It has been extended in [18] to the case $d = 2$ as well as to exterior domains for potentials which are Hölder with respect to the space variable x. The bound (1.19) is improved in [20] for radial potentials satisfying (1.1) and (1.18) with $\rho = 0$ to

$$g_s^{\pm}(h,\varepsilon) \leq Ch^{-4/(\alpha+3)}. \tag{1.20}$$

The proof of the above bounds is based on global Carleman estimates with suitable phase and weight functions. In order that the Carleman estimates work the phase and weight functions must satisfy some conditions. Therefore, the task consists of constructing explicitly these functions in such a way that we get the best possible bound for $g_s^{\pm}$. In fact, there are many ways to

do so and the class the potential belongs to is determined by the way we do it. This explains why in the papers mentioned above the same bound is proved for different classes of potentials.

To prove Theorem 1.1 we follow the same strategy as in the case $b \equiv 0$. The Carleman estimates for the magnetic Schrödinger operator, however, are much more difficult to handle with. The main difference is that conjugating this operator with $e^{\varphi/h}$, $\varphi > 0$ being a phase function depending only on r, makes appear a complex-valued effective potential of the form (see Section 3)

$$-2i\varphi'(r)w \cdot b(rw),$$

where $\varphi' > 0$ denotes the first derivative of φ. This term is not easy to handle with even in the case when $b^S \equiv 0$. One way to treat it is to consider it as a short-range potential and to add it to V_S. To do so, however, we need the condition (1.7). It allows us to make the Carleman estimates work. However, this approach does not lead to better bounds than what we have in (1.9). Therefore, we propose another way that requires the condition (1.8). This approach allows us to improve the bound (1.9) if $b_S \equiv 0$ or b_S is Hölder. To be more precise, given $0 < \alpha \leq 1$ and $k > 0$, we introduce the space $C^\alpha_k(\mathbb{R}^d)$ of all Hölder functions a such that

$$\sup_{x' \in \mathbb{R}^d:\, 0<|x-x'|\leq 1} \frac{|a(x) - a(x')|}{|x - x'|^\alpha} \leq C(|x| + 1)^{-k}, \quad \forall x \in \mathbb{R}^d.$$

Note that the case $\alpha = 1$ corresponds to the Lipschitz functions. We suppose that

$$V_S(\cdot, w) \in C^\alpha_{3+\rho}(\overline{\mathbb{R}^+}), \quad 0 < \alpha \leq 1, \quad \rho > 0, \tag{1.21}$$

uniformly in w, and

$$b^S \in C^{\alpha'}_{3/2+\rho_1}(\mathbb{R}^d; \mathbb{R}^d), \quad 0 < \alpha' \leq 1, \quad \rho_1 > 0. \tag{1.22}$$

We have the following

Theorem 1.2. *Assume that V_L satisfies (1.1), (1.2), b^L satisfies (1.4), (1.8), V_S satisfies (1.21) and (1.1) with possibly a new decreasing function p tending to zero, b^S satisfies (1.6), (1.22). We also suppose that b^L and b^S satisfy*

$$|b^L(rw)| + |b^S(rw)| \leq C(r + 1)^{-\rho_2}, \tag{1.23}$$

with some $\rho_2 > 0$ such that $\rho_1 + \rho_2 > 3/2$. Then we have the bound

$$g^\pm_s(h, \varepsilon) \leq Ch^{-n} \log(h^{-1}), \tag{1.24}$$

where

$$n = \max\left\{\frac{4}{\alpha + 3}, \frac{2}{\alpha' + 1}\right\}.$$

When $\alpha' \leq \frac{1}{2}$ the bound (1.24) also holds for L^∞ potentials V_S. More precisely, we have the following

Theorem 1.3. *Assume that V_L satisfies (1.1), (1.2), b^L satisfies (1.4), (1.8), b^S satisfies (1.6), (1.22) and (1.23), V_S satisfies*

$$|V_S(rw)| \leq C(r + 1)^{-\delta}, \tag{1.25}$$

with $\delta > 1$. Then we have the bounds

$$g^\pm_s(h, \varepsilon) \leq Ch^{-2/(\alpha'+1)} \log(h^{-1}) \quad \text{if} \quad \delta > \frac{1}{1 - \alpha'},\ \alpha' \leq \frac{1}{2}, \tag{1.26}$$

$$g^\pm_s(h, \varepsilon) \leq C_\epsilon h^{-\frac{2\delta}{2\delta-1}-\epsilon} \quad \text{if} \quad \alpha' \geq \frac{\delta - 1}{\delta},\ 1 < \delta \leq 2, \tag{1.27}$$

for every $0 < \epsilon \ll 1$ *independent of* h.

Most probably, the conditions in Theorems 1.1, 1.2 and 1.3 are not optimal, but they allow us to make a simple choice of the phase and weight functions, and hence to keep the proof of the Carleman estimates relatively short and simple. However, the bounds (1.9), (1.24) and (1.26) seem to be sharp even for compactly supported potentials, unless one assumes a radial symmetry of the potentials. Indeed, when $b \equiv 0$ and V radial, it is shown in [20] that the logarithmic term in (1.9), (1.24), (1.26) and the $h^{-\epsilon}$ term in (1.27) can be removed. Probably this is still the case for non-trivial radial magnetic and electric potentials.

The paper is organized as follows. In Section 2 we construct the phase and weight functions needed for the proof of the Carleman estimates. In Section 3 we prove Carleman estimates for the magnetic Schrödinger operator under the conditions of Theorem 1.1 needed for the proof of the bound (1.9). In Section 4 we improve the Carleman estimates when $V_S \equiv 0$ and $b^S \equiv 0$, which leads to the bound (1.10). In Section 5 we adapt the Carleman estimates to Hölder potentials V_S and b^S, which leads to the bound (1.24). In Section 6 we adapt the Carleman estimates to L^∞ potentials V_S and Hölder potentials b^S, which leads to the bounds (1.26) and (1.27). Finally, in Section 7 we show that the Carleman estimates imply the resolvent ones.

2. Construction of the phase and weight functions

We first construct the weight function μ as follows:

$$\mu(r) = \begin{cases} r^2 + r & \text{for} \quad 0 \le r \le 1, \\ r^2 + r^{2-\ell} & \text{for} \quad 1 \le r \le a, \\ (a^2 + a^{2-\ell})(1 + (a+1)^{-2s+1} - (r+1)^{-2s+1}) & \text{for} \quad r \ge a, \end{cases}$$

where $a \gg 1$ is either independent of h or it is of the form $a = h^{-m}$, $m > 0$. The parameters ℓ and s are independent of h such that $0 < \ell \ll 1$, $1/2 < s < 1$. Clearly, the first derivative of μ is given by

$$\mu'(r) = \begin{cases} 2r + 1 & \text{for} \quad 0 \le r < 1, \\ 2r + (2-\ell) r^{1-\ell} & \text{for} \quad 1 < r < a, \\ (2s-1)(a^2 + a^{2-\ell})(r+1)^{-2s} & \text{for} \quad r > a. \end{cases}$$

We have the following

Lemma 2.1. *We have the identity*

$$2r^{-1}\mu(r) - \mu'(r) = \begin{cases} 1 & \text{for} \quad 0 < r < 1, \\ \ell r^{1-\ell} & \text{for} \quad 1 < r < a. \end{cases} \tag{2.1}$$

For all $r > a$, *we have the lower bound*

$$2r^{-1}\mu(r) - \mu'(r) \ge a^2 (r+1)^{-1}. \tag{2.2}$$

For all $r \neq 1$, $r \neq a$, *we have*

$$\mu'(r) \ge (r+1)^{-2s}, \tag{2.3}$$

$$\frac{\mu(r)^j}{\mu'(r)} \lesssim a^{2j-2}(r+1)^{2s}, \tag{2.4}$$

for every $j \ge 1$.

Proof. The identity (2.1) is obvious, while (2.2) follows from the inequalities

$$2r^{-1}\mu(r)\geq 2(a^2+a^{2-\ell})(r+1)^{-1},\quad \mu'(r)\leq (2s-1)(a^2+a^{2-\ell})(r+1)^{-1}.$$

It is also clear that the inequality (2.3) holds as long as $a^2\geq (2s-1)^{-1}$. The bound (2.4) with $j=1$ is obvious, while for $j>1$ it follows from the fact that $\mu(r)\lesssim a^2$. □

We will now construct a phase function $\psi\in C^1([0,+\infty))$ such that $\psi(0)=0$ and $\psi(r)>0$ for $r>0$. We define the first derivative of ψ by

$$\psi'(r)=\begin{cases} c_\nu & \text{for}\quad 0\leq r\leq 1,\\ r^{-1}(1+r^{-\ell})^{-1/2}(1-(r+1)^{-\nu}) & \text{for}\quad 1\leq r\leq a,\\ K_a(r+1)^{-1-\kappa} & \text{for}\quad r\geq a,\end{cases}$$

where $0<\nu,\kappa\ll 1$ are independent of h to be fixed later on, and

$$c_\nu=2^{-1/2}(1-2^{-\nu}),\quad K_a=a^{-1}(a+1)^{1+\kappa}(1+a^{-\ell})^{-1/2}(1-(a+1)^{-\nu})=\mathcal{O}\left(a^\kappa\right),$$

are chosen in such a way that the function ψ' is Lipschitz. It is easy to check that the first derivative of ψ' satisfies

$$\psi''(r)=\begin{cases} 0 & \text{for}\quad 0\leq r<1,\\ \mathcal{O}(r^{-2}) & \text{for}\quad 1<r<a,\\ \mathcal{O}(a^\kappa r^{-2-\kappa}) & \text{for}\quad r>a.\end{cases}$$

Lemma 2.2. *For all $r\geq 0$ we have the bound*

$$\psi(r)\lesssim \log a. \tag{2.5}$$

Proof. We have

$$\max\psi=\int_0^1\psi'(r)dr+\int_1^a\psi'(r)dr+\int_a^\infty\psi'(r)dr$$
$$\leq c_\nu\int_0^1 dr+\int_1^a r^{-1}dr+K_a\int_a^\infty(r+1)^{-1-\kappa}dr=c_\nu+\log a+\kappa^{-1}(a+1)^{-\kappa}K_a.$$

□

The next lemma will play a crucial role in the proof of the Carleman estimates.

Lemma 2.3. *For $0<r<a$, $r\neq 1$, we have the inequality*

$$\left(\mu\psi'^2\right)'(r)\geq C_\nu(r+1)^{-1-\nu}, \tag{2.6}$$

with some constant $C_\nu>0$. For all $r>a$ we have the inequality

$$\left(\mu\psi'^2\right)'(r)\geq -\widetilde{C}_\kappa a^{-3+2s}\mu'(r), \tag{2.7}$$

with some constant $\widetilde{C}_\kappa>0$.

Proof. For $0<r<1$ we have

$$\left(\mu\psi'^2\right)'(r)=c_\nu^2(2r+1),$$

which clearly implies (2.6) in this case. For $1<r<a$ we have

$$\left(\mu\psi'^2\right)'(r)=((1-(r+1)^{-\nu})^2)'=2\nu(1-(r+1)^{-\nu})(r+1)^{-1-\nu}\geq 2\nu(1-2^{-\nu})(r+1)^{-1-\nu}.$$

For $r>a$, in view of (2.4) with $j=1$, we have

$$\left(\mu\psi'^2\right)'(r)=\mu'\psi'^2+2\mu\psi'\psi''\geq -2(r+1)^{2s}\mu'\psi'|\psi''|$$

$$\geq -Ca^{2\kappa}(r+1)^{-3-2\kappa+2s} \geq -Ca^{-3+2s},$$

as desired. □

3. Carleman estimates in the general case

Let μ and ψ be the functions introduced in the previous section and set $\varphi = \tau\psi$, where $\tau = \tau_0 h^{-1}$, $\tau_0 \gg 1$ being a parameter independent of h to be chosen later on. In what follows we set $\mathcal{D}_r = ih\partial_r$. In this section we will prove the following

Theorem 3.1. *Under the conditions of Theorem 1.1, there are constants $C > 0$ and $0 < h_0 \ll 1$ such that for all $0 < h \leq h_0$, $0 < \varepsilon \leq 1$, $s > 1/2$, and for all functions $f \in H^2(\mathbb{R}^d)$ satisfying*

$$(|x|+1)^s(P(h) - E \pm i\varepsilon)f \in L^2(\mathbb{R}^d)$$

we have the estimate

$$\|(|x|+1)^{-s}e^{\varphi/h}f\|_{L^2} \leq Cah^{-1}\|(|x|+1)^s e^{\varphi/h}(P(h) - E \pm i\varepsilon)f\|_{L^2}$$

$$+Ca\left(\tau h^{-1}\varepsilon\right)^{1/2}\|e^{\varphi/h}f\|_{L^2} + Ca\left(\tau^{-1}h^{-1}\varepsilon\right)^{1/2}\|e^{\varphi/h}\mathcal{D}_r f\|_{L^2}. \tag{3.1}$$

Proof. Clearly, it suffices to prove (3.1) for $0 < s - 1/2 \ll 1$ as this would imply (3.1) for all $s > 1/2$. So, in what follows s will be as in Section 2 with additional restrictions made later on. We write the operator $P(h)$ in the form

$$P(h) = -h^2\Delta + ihb^L \cdot \nabla + ih\nabla \cdot b^L + 2ihb^S \cdot \nabla + \widetilde{V}_L + \widetilde{V}_S,$$

where

$$\widetilde{V}_L = |b^L|^2 + V_L,$$
$$\widetilde{V}_S = ih\mathrm{div}\, b^S + 2b^S \cdot b^L + |b^S|^2 + V_S.$$

Note that our conditions guarantee that

$$|\widetilde{V}_S(x)| \lesssim (|x|+1)^{-1-\rho}, \quad \partial_r \widetilde{V}_L(x) \lesssim (|x|+1)^{-1-\rho}.$$

Moreover, $\widetilde{V}_L$ satisfies (1.1) with a new decreasing function p.

We will write $P(h)$ in the polar coordinates $(r, w) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}$, $r = |x|$, $w = x/|x|$. Recall that $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1}, r^{d-1}drdw)$. In what follows in this section we denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the scalar product in $L^2(\mathbb{S}^{d-1})$. We will make use of the identity

$$r^{(d-1)/2}\Delta r^{-(d-1)/2} = \partial_r^2 + r^{-2}\Delta_w - (d-1)(d-3)(2r)^{-2} \tag{3.2}$$

where Δ_w denotes the negative Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. We also have

$$r^{(d-1)/2}\partial_{x_j} r^{-(d-1)/2} = w_j\partial_r + r^{-1}q_j(w, \partial_w) \tag{3.3}$$

where q_j is a first-order differential operator on $\mathbb{S}^{d-1}$ which is antisymmetric with respect to the scalar product in $L^2(\mathbb{S}^{d-1})$. Clearly, we have the estimates

$$\|q_j(w, \partial_w)v\| \lesssim \|(-\Delta_w)^{1/2}v\| + \|v\| \quad \text{for all} \quad v \in H^1(\mathbb{S}^{d-1}). \tag{3.4}$$

Set $\Lambda = -h^2\Delta_w$, $\mathcal{D}_r = ih\partial_r$ and $Q_j = ihq_j(w, \partial_w)$. Then the operator Q_j is symmetric with respect to the scalar product in $L^2(\mathbb{S}^{d-1})$. Moreover, (3.4) implies the estimates

$$\|Q_j v\| \lesssim \|\Lambda^{1/2}v\| + h\|v\| \quad \text{for all} \quad v \in H^1(\mathbb{S}^{d-1}). \tag{3.5}$$

Set $u(r, w) = r^{(d-1)/2}e^{\varphi/h}f(rw)$ and

$$\mathcal{P}(h) = r^{(d-1)/2}(P(h) - E)r^{-(d-1)/2},$$
$$\mathcal{P}_\varphi(h) = e^{\varphi/h}\mathcal{P}(h)e^{-\varphi/h}.$$

We now define the function σ as follows. We put $\sigma=0$ if (1.7) holds and we put

$$\sigma(r,w)=\sum_{j=1}^d w_jb_j^L(rw)$$

if (1.8) holds. Using (3.2) and (3.3) we can write the operator $\mathcal{P}(h)$ in the coordinates (r,w) as follows

$$\mathcal{P}(h)=\mathcal{D}_r^2+r^{-2}\Lambda-E+h^2(d-1)(d-3)(2r)^{-2}+\widetilde{V}_L+\widetilde{V}_S$$
$$+r^{-1}\sum_{j=1}^d\left(b_j^LQ_j+Q_jb_j^L\right)+2\sum_{j=1}^d w_jb_j\mathcal{D}_r+2r^{-1}\sum_{j=1}^d b_j^SQ_j+ih\sum_{j=1}^d w_j\partial_r(b_j^L),$$

if (1.7) holds, and

$$\mathcal{P}(h)=(\mathcal{D}_r+\sigma)^2+r^{-2}\Lambda-E+h^2(d-1)(d-3)(2r)^{-2}+\widetilde{V}_L+\widetilde{V}_S-\sigma^2-2\sigma\sum_{j=1}^d w_jb_j^S$$
$$+r^{-1}\sum_{j=1}^d\left(b_j^LQ_j+Q_jb_j^L\right)+2\sum_{j=1}^d w_jb_j^S(\mathcal{D}_r+\sigma)+2r^{-1}\sum_{j=1}^d b_j^SQ_j,$$

if (1.8) holds. Since the function φ depends only on the variable r, in the first case we get

$$\mathcal{P}_\varphi(h)=\mathcal{D}_r^2+r^{-2}\Lambda-E+h^2(d-1)(d-3)(2r)^{-2}-2i\varphi'\mathcal{D}_r+W_L+W_S$$
$$+2\sum_{j=1}^d w_jb_j\mathcal{D}_r+r^{-1}\sum_{j=1}^d\left(b_j^LQ_j+Q_jb_j^L\right)+2r^{-1}\sum_{j=1}^d b_j^SQ_j,$$

where

$$W_L=\widetilde{V}_L-\varphi'^2,$$
$$W_S=\widetilde{V}_S+h\varphi''-2i\varphi'\sum_{j=1}^d w_jb_j+ih\sum_{j=1}^d w_j\partial_r(b_j^L).$$

In the second case we get

$$\mathcal{P}_\varphi(h)=(\mathcal{D}_r+\sigma)^2+r^{-2}\Lambda-E+h^2(d-1)(d-3)(2r)^{-2}-2i\varphi'(\mathcal{D}_r+\sigma)+W_L+W_S$$
$$+2\sum_{j=1}^d w_jb_j^S(\mathcal{D}_r+\sigma)+r^{-1}\sum_{j=1}^d\left(b_j^LQ_j+Q_jb_j^L\right)+2r^{-1}\sum_{j=1}^d b_j^SQ_j,$$

where

$$W_L=\widetilde{V}_L-\sigma^2-\varphi'^2,$$
$$W_S=\widetilde{V}_S+h\varphi''-2(i\varphi'+\sigma)\sum_{j=1}^d w_jb_j^S.$$

For $r>0$, $r\neq1$, $r\neq a$, introduce the function

$$F(r)=-\left\langle(r^{-2}\Lambda+h^2(d-1)(d-3)(2r)^{-2}-E+W_L)u(r,\cdot),u(r,\cdot)\right\rangle$$
$$+\|(\mathcal{D}_r+\sigma)u(r,\cdot)\|^2-2r^{-1}\sum_{j=1}^d\mathrm{Re}\left\langle b_j^LQ_ju(r,\cdot),u(r,\cdot)\right\rangle.$$

Observe now that in the first case we have

$$\frac{h}{2}\frac{d}{dr}\|\mathcal{D}_ru\|^2=\mathrm{Im}\left\langle\mathcal{D}_r^2u,\mathcal{D}_ru\right\rangle=\mathrm{Im}\left\langle\mathcal{P}_\varphi(h)u,\mathcal{D}_ru\right\rangle$$

$$-\mathrm{Im}\left\langle (r^{-2}\Lambda + h^2(d-1)(d-3)(2r)^{-2} - E + W_L + W_S)u, \mathcal{D}_r u\right\rangle$$
$$+2\varphi'\|\mathcal{D}_r u\|^2 - 2\sum_{j=1}^{d}\mathrm{Im}\left\langle w_j b_j \mathcal{D}_r u, \mathcal{D}_r u\right\rangle$$
$$-\sum_{j=1}^{d}\mathrm{Im}\left\langle r^{-1}(b_j^L Q_j + Q_j b_j^L)u, \mathcal{D}_r u\right\rangle - 2\sum_{j=1}^{d}\mathrm{Im}\left\langle r^{-1} b_j^S Q_j u, \mathcal{D}_r u\right\rangle,$$

while in the second case we have

$$\frac{h}{2}\frac{d}{dr}\|(\mathcal{D}_r+\sigma)u\|^2 = \mathrm{Im}\left\langle (\mathcal{D}_r+\sigma)^2 u, (\mathcal{D}_r+\sigma)u\right\rangle = \mathrm{Im}\left\langle \mathcal{P}_\varphi(h)u, (\mathcal{D}_r+\sigma)u\right\rangle$$
$$-\mathrm{Im}\left\langle (r^{-2}\Lambda + h^2(d-1)(d-3)(2r)^{-2} - E + W_L + W_S)u, (\mathcal{D}_r+\sigma)u\right\rangle$$
$$+2\varphi'\|(\mathcal{D}_r+\sigma)u\|^2 - 2\sum_{j=1}^{d}\mathrm{Im}\left\langle w_j b_j^S (\mathcal{D}_r+\sigma)u, (\mathcal{D}_r+\sigma)u\right\rangle$$
$$-\sum_{j=1}^{d}\mathrm{Im}\left\langle r^{-1}(b_j^L Q_j + Q_j b_j^L)u, (\mathcal{D}_r+\sigma)u\right\rangle$$
$$-2\sum_{j=1}^{d}\mathrm{Im}\left\langle r^{-1} b_j^S Q_j u, (\mathcal{D}_r+\sigma)u\right\rangle.$$

On the other hand, we have the identities

$$\mathrm{Im}\left\langle w_j b_j \mathcal{D}_r u, \mathcal{D}_r u\right\rangle = 0,$$
$$\mathrm{Re}\frac{d}{dr}\left\langle b_j^L Q_j u, u\right\rangle = \mathrm{Re}\left\langle \partial_r(b_j^L) Q_j u, u\right\rangle$$
$$-h^{-1}\mathrm{Im}\left\langle (b_j^L Q_j + Q_j b_j^L)u, \mathcal{D}_r u\right\rangle$$

in the first case, and

$$\mathrm{Im}\left\langle w_j b_j^S (\mathcal{D}_r+\sigma)u, (\mathcal{D}_r+\sigma)u\right\rangle = 0,$$
$$\mathrm{Re}\frac{d}{dr}\left\langle b_j^L Q_j u, u\right\rangle = \mathrm{Re}\left\langle \partial_r(b_j^L) Q_j u, u\right\rangle$$
$$-h^{-1}\mathrm{Im}\left\langle (b_j^L Q_j + Q_j b_j^L)u, (\mathcal{D}_r+\sigma)u\right\rangle - \mathrm{Re}\left\langle b_j^L u, [q_j,\sigma]u\right\rangle$$

in the second case. Using the above identities we find that in both cases the first derivative of F is given by

$$F'(r) = \frac{2}{r}\left\langle r^{-2}(\Lambda + (h/2)^2(d-1)(d-3))u, u\right\rangle - W_L'\|u\|^2$$
$$+2r^{-2}\sum_{j=1}^{d}\mathrm{Re}\left\langle b_j^L Q_j u, u\right\rangle - 2r^{-1}\sum_{j=1}^{d}\mathrm{Re}\left\langle \partial_r(b_j^L) Q_j u, u\right\rangle + 2r^{-1}\sum_{j=1}^{d}\mathrm{Re}\left\langle b_j^L u, [q_j,\sigma]u\right\rangle$$
$$+2h^{-1}\mathrm{Im}\left\langle \mathcal{P}_\varphi(h)u, (\mathcal{D}_r+\sigma)u\right\rangle + 4h^{-1}\varphi'\|(\mathcal{D}_r+\sigma)u\|^2 - 2h^{-1}\mathrm{Im}\left\langle \widetilde{W}_S u, (\mathcal{D}_r+\sigma)u\right\rangle,$$

where

$$\widetilde{W}_S = W_S + 2r^{-1}\sum_{j=1}^{d} b_j^S Q_j.$$

Thus we obtain the identity

$$(\mu F)' = \mu' F + \mu F'$$
$$= (2r^{-1}\mu - \mu')\left\langle r^{-2}(\Lambda + (h/2)^2(d-1)(d-3))u, u\right\rangle$$

$$+(E\mu'-(\mu W_L)')\|u\|^2+(\mu'+4h^{-1}\varphi'\mu)\|(\mathcal{D}_r+\sigma)u\|^2$$
$$-2(r^{-1}\mu)'\sum_{j=1}^d \mathrm{Re}\left\langle b_j^L Q_j u,u\right\rangle-2r^{-1}\mu\sum_{j=1}^d \mathrm{Re}\left\langle \partial_r(b_j^L)Q_j u,u\right\rangle$$
$$+2r^{-1}\mu\sum_{j=1}^d \mathrm{Re}\left\langle b_j^L u,[q_j,\sigma]u\right\rangle-2h^{-1}\mu\mathrm{Im}\left\langle \widetilde{W}_S u,(\mathcal{D}_r+\sigma)u\right\rangle$$
$$+2h^{-1}\mu\mathrm{Im}\left\langle (\mathcal{P}_\varphi(h)\pm i\varepsilon)u,(\mathcal{D}_r+\sigma)u\right\rangle\mp 2\varepsilon h^{-1}\mu\mathrm{Re}\left\langle u,(\mathcal{D}_r+\sigma)u\right\rangle.$$

In view of (3.5), we have the estimates

$$\left\|\widetilde{W}_S u\right\|^2\lesssim B_1\|u\|^2+B_2\left\|r^{-1}\Lambda^{1/2}u\right\|^2, \tag{3.6}$$

where

$$B_1(r)=(r+1)^{-2-\rho}+(r+1)^{-2-\rho}\varphi'^2+h^2\varphi''^2+h^2r^{-2}(r+1)^{-2-\rho},$$
$$B_2(r)=(r+1)^{-2-\rho},$$

and

$$\left|\left\langle b_j^L Q_j u,u\right\rangle\right|\lesssim B_3\|u\|\left\|\Lambda^{1/2}u\right\|+hB_3\|u\|^2$$
$$\lesssim\left(\gamma_1^{-1}(rB_3)^2+hB_3\right)\|u\|^2+\gamma_1\left\|r^{-1}\Lambda^{1/2}u\right\|^2, \tag{3.7}$$

$$\left|\left\langle \partial_r(b_j^L)Q_j u,u\right\rangle\right|\lesssim\left(\gamma_2^{-1}(rB_4)^2+hB_4\right)\|u\|^2+\gamma_2\left\|r^{-1}\Lambda^{1/2}u\right\|^2, \tag{3.8}$$

where γ_1 and γ_2 are arbitrary positive functions of r to be fixed below and

$$B_3(r)=(r+1)^{-\rho},\quad B_4(r)=(r+1)^{-1}B_3(r).$$

In view of (1.8), we also have

$$\left|\left\langle b_j^L u,[q_j,\sigma]u\right\rangle\right|\lesssim B_5\|u\|^2, \tag{3.9}$$

where $B_5(r)=B_3(r)$. Since $2r^{-1}\mu-\mu'>0$ and $d\geq 3$, by (3.6), (3), (3.8) and (3.9), we get the inequality

$$(\mu F)'\geq(2r^{-1}\mu-\mu')\left\langle r^{-2}\Lambda u,u\right\rangle$$
$$+(E\mu'-(\mu W_L)')\|u\|^2+(\mu'+4h^{-1}\varphi'\mu)\|(\mathcal{D}_r+\sigma)u\|^2$$
$$-2|(r^{-1}\mu)'|\sum_{j=1}^d\left|\left\langle b_j^L Q_j u,u\right\rangle\right|-2r^{-1}\mu\sum_{j=1}^d\left|\left\langle \partial_r(b_j^L)Q_j u,u\right\rangle\right|-2r^{-1}\mu\sum_{j=1}^d\left|\left\langle b_j^L u,[q_j,\sigma]u\right\rangle\right|$$
$$-3h^{-2}\mu^2(\mu'+4h^{-1}\varphi'\mu)^{-1}\left\|\widetilde{W}_S u\right\|^2-\frac{1}{3}(\mu'+4h^{-1}\varphi'\mu)\|(\mathcal{D}_r+\sigma)u\|^2$$
$$-\frac{3h^{-2}\mu^2}{\mu'}\|(\mathcal{P}_\varphi(h)\pm i\varepsilon)u\|^2-\frac{\mu'}{3}\|(\mathcal{D}_r+\sigma)u\|^2-\varepsilon h^{-1}\mu\|u\|\|(\mathcal{D}_r+\sigma)u\|$$
$$\geq A_1\|u\|^2+A_2\|r^{-1}\Lambda^{1/2}u\|^2-\frac{3h^{-2}\mu^2}{\mu'}\|(\mathcal{P}_\varphi(h)\pm i\varepsilon)u\|^2-\varepsilon h^{-1}\mu\|u\|\|(\mathcal{D}_r+\sigma)u\|,$$

where

$$A_1(r)=(\varphi'^2\mu)'+(E-\widetilde{V}_L)\mu'-C(r+1)^{-1-\rho}\mu-Ch^{-2}B_1\mu^2(\mu'+h^{-1}\varphi'\mu)^{-1}$$
$$-C|(r^{-1}\mu)'|\left(\gamma_1^{-1}(rB_3)^2+hB_3\right)-Cr^{-1}\mu\left(\gamma_2^{-1}(rB_4)^2+hB_4+B_5\right),$$
$$A_2(r)=2r^{-1}\mu-\mu'-Ch^{-2}B_2\mu^2(\mu'+h^{-1}\varphi'\mu)^{-1}$$
$$-C\gamma_1|(r^{-1}\mu)'|-C\gamma_2 r^{-1}\mu.$$

We now choose the functions γ_1 and γ_2 so that

$$C\gamma_1|(r^{-1}\mu)'| = C\gamma_2 r^{-1}\mu = \frac{1}{4}(2r^{-1}\mu - \mu').$$

Thus we conclude that the above inequality holds with

$$A_1(r) = (\varphi'^2\mu)' + (E - \widetilde{V}_L)\mu' - Cr^{-1}(r+1)^{-\rho}\mu - Ch^{-2}B_1\mu^2(\mu' + h^{-1}\varphi'\mu)^{-1}$$
$$-C(rB_3)^2M_1 - ChB_3M_2,$$
$$2A_2(r) = 2r^{-1}\mu - \mu' - Ch^{-2}B_2\mu^2(\mu' + h^{-1}\varphi'\mu)^{-1},$$

with a new constant $C > 0$, where

$$M_1(r) = (2r^{-1}\mu - \mu')^{-1}\left(|(r^{-1}\mu)'|^2 + (r(r+1))^{-2}\mu^2\right),$$
$$M_2(r) = |(r^{-1}\mu)'| + (r(r+1))^{-1}\mu.$$

Now we will use Lemmas 2.1 and 2.3 to bound the functions A_1 and A_2 from below. Observe first that $M_1(r) = M_2(r) = 1$ for $r < 1$. For $1 < r < a$ we have the bounds

$$M_1(r) \lesssim (r+1)^{-1+\ell}, \quad M_2(r) \lesssim 1,$$

while for $r > a$ we have the bounds

$$M_1(r) \lesssim (r+1)^{-3+2s}\mu'(r), \quad M_2(r) \lesssim (r+1)^{-2+2s}\mu'(r).$$

Therefore, for $0 < r < 1$ we have

$$A_1(r) \geq E\mu'(r) + C_\nu^\flat\tau^2 - Ch^{-2} - Ch^{-1}\tau, \quad C_\nu^\flat > 0,$$
$$A_2(r) \geq 2^{-1} - Ch^{-1}\tau^{-1}.$$

We take $\tau = \tau_0 h^{-1}$, where $\tau_0 \gg 1$ is independent of h. It is clear that if τ_0 is big enough we can arrange that

$$A_1(r) \geq E\mu'(r), \quad A_2(r) \geq 0, \quad \text{for} \quad 0 < r < 1.$$

For $1 < r < a$ we have $r^{-1}\mu(r) < \mu'(r)$ and

$$(rB_3)^2M_1(r) \lesssim (r+1)^{\ell-2\rho}\mu'(r) \lesssim (r+1)^{-\rho}\mu'(r) \tag{3.10}$$

if $\ell < \rho$. Hence

$$A_1(r) \geq \tau^2 C_\nu(r+1)^{-1-\nu} + (E - \widetilde{V}_L)\mu' - C(r+1)^{-\rho}\mu'$$
$$-Ch^{-1}\tau^{-1}(r+1)^{-\rho}\mu' - Ch\tau(r+1)^{-2}\mu' - Ch^{-1}\tau(r+1)^{-1-\rho},$$
$$A_2(r) \geq 2^{-1}\ell r^{1-\ell} - Ch^{-1}\tau^{-1}r^{1-\rho}.$$

Since $\widetilde{V}_L$ satisfies (1.1), there exists $\lambda_0 \gg 1$, independent of h and τ_0, such that

$$\widetilde{V}_L(rw) + C(r+1)^{-\rho} \leq E/4 \quad \text{for} \quad r \geq \lambda_0. \tag{3.11}$$

On the other hand, for $r < \lambda_0$, we have

$$\widetilde{V}_L\mu'(r) + C(r+1)^{-\rho}\mu'(r) \lesssim (r+1)^{-1-\nu}. \tag{3.12}$$

Similarly, there exists $\lambda \gg 1$, independent of h but depending on τ_0, such that

$$C\tau_0(r+1)^{-2} \leq E/4 \quad \text{for} \quad r \geq \lambda,$$

and

$$C\tau_0(r+1)^{-2}\mu'(r) \lesssim (r+1)^{-1-\nu} \quad \text{for} \quad r < \lambda.$$

Therefore, if we choose $\nu < \rho$, $\ell < \rho$, taking τ_0 big enough and h small enough we can arrange that

$$A_1(r) \geq 3^{-1}E\mu'(r), \quad A_2(r) \geq 0, \quad \text{for} \quad 1 < r < a.$$

We now take $1/2 < s < (1+\rho)/2$, $0 < \kappa < 1+\rho-2s$. In view of (1.1) with $\widetilde{V}_L$, for $r > a$ we have

$$A_1(r) \geq 2^{-1}E\mu'(r) - C\mu'(r)\left(\tau^2 a^{-3+2s} + a^{-\rho+2s-1} + \tau h^{-1}a^{-4+2s}\right),$$
$$A_2(r) \geq 2^{-1}a^2(r+1)^{-1}\left(1 - Ch^{-1}\tau^{-1}a^{-1}\right).$$

Taking $a = h^{-m}$ with $m > 0$ big enough and h small enough we can arrange that

$$A_1(r) \geq 3^{-1}E\mu'(r), \quad A_2(r) \geq 0, \quad \text{for} \quad r > a.$$

By the above inequalities we obtain

$$(\mu F)' \geq \frac{E}{3}\mu'\|u\|^2 - \frac{3h^{-2}\mu^2}{\mu'}\|(\mathcal{P}_\varphi(h) \pm i\varepsilon)u\|^2 - \varepsilon h^{-1}\mu\|u\|\|(\mathcal{D}_r + \sigma)u\|. \tag{3.13}$$

Integrating (3.13) with respect to r and using that

$$\int_0^\infty (\mu F)' dr = 0,$$

we obtain the estimate

$$\frac{E}{3}\int_0^\infty \mu'\|u\|^2 dr \leq 3h^{-2}\int_0^\infty \frac{\mu^2}{\mu'}\|(\mathcal{P}_\varphi(h) \pm i\varepsilon)u\|^2 dr + \varepsilon h^{-1}\int_0^\infty \mu\|u\|\|(\mathcal{D}_r + \sigma)u\|dr. \tag{3.14}$$

Using that $\mu(r) \lesssim \zeta(r)a^2$, where $\zeta(r) = r(r+1)^{-1}$, together with (2.3) and (2.4) we get from (3.14),

$$\int_0^\infty (r+1)^{-2s}\|u\|^2 dr \leq Ca^2h^{-2}\int_0^\infty (r+1)^{2s}\|(\mathcal{P}_\varphi(h) \pm i\varepsilon)u\|^2 dr$$

$$+C\varepsilon h^{-1}a^2\int_0^\infty \left(\tau\|u\|^2 + \tau^{-1}\zeta^2\|\mathcal{D}_r u\|^2\right) dr \tag{3.15}$$

with some constant $C > 0$ independent of h and ε. Since

$$\mathcal{D}_r u = r^{(d-1)/2}e^{\varphi/h}\mathcal{D}_r f + (i\varphi' + (d-1)(2r)^{-1})u,$$

we have

$$\zeta^2\|\mathcal{D}_r u\|^2 \lesssim \left\|r^{(d-1)/2}e^{\varphi/h}\mathcal{D}_r f\right\|^2 + \tau^2\|u\|^2. \tag{3.16}$$

Clearly, the estimate (3.1) follows from (3) and (3.16). □

4. Carleman estimates when $V_S \equiv 0$ and $b^S \equiv 0$

In this section we will prove the following

Theorem 4.1. *Under the conditions of Theorem 1.1 with (1.8), if $V_S \equiv 0$ and $b^S \equiv 0$, the estimate (3.1) holds true with new parameters $a, \tau \gg 1$ independent of h.*

Proof. We will make use of the Carleman estimates obtained in the previous section under the condition (1.8), which in our case take a much simpler form. Indeed, $V_S \equiv 0$, $b^S \equiv 0$ imply that $\widetilde{V}_S \equiv 0$ and $\widetilde{W}_S = h\varphi''$. Hence $B_1 = h^2\varphi''^2$ and $B_2 = 0$. The functions B_3, B_4 and B_5 remain unchanged. Therefore the functions A_1 and A_2 take the form

$$A_1(r) = (\varphi'^2\mu)' + (E - \widetilde{V}_L)\mu' - Cr^{-1}(r+1)^{-\rho}\mu$$
$$-Ch\tau\mu(\psi'')^2(\psi')^{-1} - C(rB_3)^2M_1 - ChB_3M_2,$$
$$2A_2(r) = 2r^{-1}\mu - \mu' > 0.$$

For $0 < r < a$, $r \neq 1$, in view of (3.10), (3.11) and (3.12), we have

$$A_1(r) \geq \tau^2 C_\nu (r+1)^{-1-\nu} + (E - \widetilde{V}_L)\mu' - C(r+1)^{-\rho}\mu' - Ch\tau(r+1)^{-2}\mu' \geq E\mu'/3,$$

if τ is taken big enough, independent of h, and h is taken small enough. For $r > a$ we have

$$A_1(r) \geq 2^{-1}E\mu' - C\mu'\left(\tau^2 a^{-3+2s} + a^{-\rho+2s-1}\right) \geq E\mu'/3,$$

if a is taken big enough, independent of h. Thus we conclude that the estimate (3) still holds in this case with parameters $a, \tau \gg 1$ independent of h. Clearly, this implies the theorem. □

5. Carleman estimates for Hölder potentials

In this section we will prove the following

Theorem 5.1. *Under the conditions of Theorem 1.2, the estimate (3.1) holds true with new parameters $a = h^{-m}$ and $\tau = \tau_0 h^{-n+1}$ with some positive constants m and τ_0.*

Proof. Let $\phi_0 \in C_0^\infty([0,1])$, $\phi_0 \geq 0$, be a real-valued function independent of h such that $\int_0^\infty \phi_0(t)dt = 1$. If V_S satisfies (1.21), we approximate it by the function

$$V_{\theta_1}(r,w) = \theta_1^{-1}\int_0^\infty \phi_0((r'-r)/\theta_1)V_S(r',w)dr' = \int_0^\infty \phi_0(t)V_S(r+\theta_1 t, w)dt$$

where $0 < \theta_1 < 1$ is a parameter depending on h to be fixed later on. We have

$$|V_S(r,w) - V_{\theta_1}(r,w)| \leq \int_0^\infty \phi_0(t)|V_S(r+\theta_1 t,w) - V_S(r,w)|dt$$

$$\lesssim \theta_1^\alpha (r+1)^{-3-\rho}\int_0^\infty t^\alpha \phi_0(t)dt \lesssim \theta_1^\alpha (r+1)^{-3-\rho}. \tag{5.1}$$

Since V_S satisfies (1.1), the above bound implies

$$V_{\theta_1}(r,w) \leq p(r) + \mathcal{O}((r+1)^{-3-\rho}). \tag{5.2}$$

Clearly, V_{θ_1} is C^1 with respect to the variable r and its first derivative is given by

$$\partial_r V_{\theta_1}(r,w) = -\theta_1^{-2}\int_0^\infty \phi_0'((r'-r)/\theta_1)V_S(r',w)dr'$$

$$= -\theta_1^{-1}\int_0^\infty \phi_0'(t)V_S(r+\theta_1 t,w)dt = -\theta_1^{-1}\int_0^\infty \phi_0'(t)(V_S(r+\theta_1 t,w) - V_S(r,w))dt$$

where we have used that $\int_0^\infty \phi_0'(t)dt = 0$. Hence

$$|\partial_r V_{\theta_1}(r,w)| \lesssim \theta_1^{-1+\alpha}(r+1)^{-3-\rho}\int_0^\infty t^\alpha |\phi_0'(t)|dt \lesssim \theta_1^{-1+\alpha}(r+1)^{-3-\rho}. \tag{5.3}$$

Let $\phi \in C_0^\infty(\mathbb{R}^d)$, $\phi \geq 0$, be a real-valued function independent of h such that $\int_{\mathbb{R}^d}\phi(x)dx = 1$. If b^S satisfies (1.22), we approximate it by the function

$$b_{\theta_2}(x) = \theta_2^{-d}\int_{\mathbb{R}^d}\phi((x'-x)/\theta_2)b^S(x')dx' = \int_{\mathbb{R}^d}\phi(y)b^S(x+\theta_2 y)dy$$

where $0 < \theta_2 < 1$ is a parameter depending on h to be fixed later on. We have

$$|b^S(x) - b_{\theta_2}(x)| \leq \int_{\mathbb{R}^d}\phi(y)|b^S(x+\theta_2 y) - b^S(x)|dy$$

$$\lesssim \theta_2^{\alpha'}(r+1)^{-3/2-\rho_1}\int_{\mathbb{R}^d}|y|^{\alpha'}\phi(y)dy \lesssim \theta_2^{\alpha'}(r+1)^{-3/2-\rho_1}. \tag{5.4}$$

By (1.23) and (5),

$$|b_{\theta_2}(rw)| \lesssim (r+1)^{-\rho_2} + (r+1)^{-3/2-\rho_1}. \tag{5.5}$$

If β is a multi-index such that $|\beta| = 1$, we have

$$\partial_x^\beta b_{\theta_2}(x) = -\theta_2^{-d-1} \int_{\mathbb{R}^d} (\partial_x^\beta \phi)((x'-x)/\theta_2) b^S(x') dx' = -\theta_2^{-1} \int_{\mathbb{R}^d} (\partial_y^\beta \phi)(y) b^S(x+\theta_2 y) dy$$
$$= -\theta_2^{-1} \int_{\mathbb{R}^d} (\partial_y^\beta \phi)(y)(b^S(x+\theta_2 y) - b^S(x)) dy.$$

Hence

$$|\partial_x^\beta b_{\theta_2}(x)| \le \theta_2^{-1} \int_{\mathbb{R}^d} |(\partial_y^\beta \phi)(y)||b^S(x+\theta_2 y) - b^S(x)| dy$$
$$\lesssim \theta_2^{-1+\alpha'}(r+1)^{-3/2-\rho_1} \int_{\mathbb{R}^d} |y|^{\alpha'} |(\partial_y^\beta \phi)(y)| dy \lesssim \theta_2^{-1+\alpha'}(r+1)^{-3/2-\rho_1}. \tag{5.6}$$

By (5) we deduce that

$$|\partial_r b_{\theta_2}(rw)| \lesssim \theta_2^{-1+\alpha'}(r+1)^{-3/2-\rho_1}, \tag{5.7}$$

$$|\partial_w^\beta (w \cdot b_{\theta_2}(rw))| \lesssim \theta_2^{-1+\alpha'}(r+1)^{-1/2-\rho_1}. \tag{5.8}$$

It is also easy to see that (1.6) implies

$$|\mathrm{div}\,(b^S(x) - b_{\theta_2}(x))| \lesssim (r+1)^{-1-\rho}. \tag{5.9}$$

We will make use of the Carleman estimates obtained in Section 3 under the condition (1.8) with V_S, V_L, b^S and b^L replaced by $V_S - V_{\theta_1}$, $V_L + V_{\theta_1}$, $b^S - b_{\theta_2}$ and $b^L + b_{\theta_2}$, respectively. Using the above estimates one can easily check that our assumptions guarantee that the new functions $\widetilde{V}_S$, $\widetilde{V}_L$ and σ satisfy the bounds

$$|\widetilde{V}_S| \lesssim \left(\theta_1^\alpha + \theta_2^{\alpha'}\right)(r+1)^{-3-\rho} + h(r+1)^{-1-\rho},$$
$$\partial_r(\widetilde{V}_L - \sigma^2) \lesssim (r+1)^{-1-\rho} + \left(\theta_1^{-1+\alpha} + \theta_2^{-1+\alpha'}\right)(r+1)^{-3-\rho},$$

with some $\rho > 0$. Moreover, $\widetilde{V}_L$ satisfies (1.1) with a new decreasing function p independent of the parameters θ_1 and θ_2. Then the functions B_j in this case become

$$B_1(r) = \left(\theta_1^{2\alpha} + \theta_2^{2\alpha'}\right)(r+1)^{-6-\rho} + h^2(r+1)^{-2-\rho} + h^2\theta_2^{2\alpha'} r^{-2}(r+1)^{-3-\rho}$$
$$+\theta_2^{2\alpha'}(r+1)^{-3-\rho}\varphi'^2 + h^2\varphi''^2,$$
$$B_2(r) = \theta_2^{2\alpha'}(r+1)^{-3-\rho}, \quad B_3(r) = (r+1)^{-\rho},$$
$$B_4(r) = (r+1)^{-1}B_3(r) + \theta_2^{-1+\alpha'}(r+1)^{-3/2-\rho},$$
$$B_5(r) = B_3(r) + \theta_2^{-1+\alpha'}(r+1)^{-2-\rho}.$$

For the new functions A_1 and A_2 we obtain

$$A_1(r) = \tau^2(\psi'^2\mu)' + (E - \widetilde{V}_L)\mu' - C\mu r^{-1}(r+1)^{-\rho}$$
$$-C\left(M_2\theta_1^{-1+\alpha} + M_2\theta_2^{-1+\alpha'} + M_1\theta_2^{-2+2\alpha'}\right)(r+1)^{-1-\rho}$$
$$-Ch^{-1}\tau^{-1}\left(\theta_1^{2\alpha} + \theta_2^{2\alpha'}\right)(r+1)^{-6-\rho}\mu(\psi')^{-1} - C\tau h^{-1}\theta_2^{2\alpha'}(r+1)^{-3-\rho}\mu\psi'$$
$$-Ch\tau^{-1}(r+1)^{-2-\rho}\mu(\psi')^{-1} - Ch\tau\mu(\psi'')^2(\psi')^{-1}$$
$$-C\theta_2^{2\alpha'}\mu^2(r\mu')^{-2}(r+1)^{-3-\rho}\mu' - C(rB_3)^2M_1 - ChB_3M_2,$$

$$2A_2(r) = 2r^{-1}\mu - \mu' - Ch^{-1}\tau^{-1}\theta_2^{2\alpha'}(r+1)^{-3-\rho}\mu(\psi')^{-1},$$

where the functions M_1 and M_2 are the same as in Section 3. Note that for $1 < r < a$ we have

$$\mu(\psi')^{-1} = \mathcal{O}(r^3), \quad \mu\psi' = \mathcal{O}(r), \quad \mu(\psi'')^2(\psi')^{-1} = \mathcal{O}(r^{-1}), \quad \mu^2(r\mu')^{-2} = \mathcal{O}(1).$$

Set

$$\chi_1 = \tau^{-2}\theta_1^{-1+\alpha}, \quad \eta_1 = h^{-1}\tau^{-3}\theta_1^{2\alpha}, \quad \chi_2 = \tau^{-2}\theta_2^{-2+2\alpha'}, \quad \eta_2 = h^{-1}\tau^{-1}\theta_2^{2\alpha'}.$$

We take $\theta_1 = h^{2/(3+\alpha)}$, $\theta_2 = h^{1/(1+\alpha')}$, $\tau = h^{-n+1}\tau_0$, where $\tau_0 \gg 1$ is independent of h. It is easy to check that with this choice we have the inequalities

$$\chi_1 \leq \tau_0^{-2}, \quad \eta_1 \leq \tau_0^{-3}, \quad \chi_2 \leq \tau_0^{-2}, \quad \eta_2 \leq \tau_0^{-1}.$$

Therefore, we can make all these parameters small enough by taking τ_0 big enough. On the other hand, if we choose $\nu < \rho$, $\ell < \rho$ and the above parameters are small enough, it is easy to see that we have $A_1 \geq E\mu'/3$ and $A_2 \geq 0$. Indeed, in view of (3.10), for $r < a$, $r \neq 1$, we have

$$\begin{aligned} A_1(r) &\geq \tau^2(C_\nu - C(\chi_1 + \eta_1 + \chi_2 + \eta_2))(r+1)^{-1-\nu} \\ &+(E - \widetilde{V}_L)\mu' - C(r+1)^{-\rho}\mu' - C(h\tau + \theta_2^{2\alpha'})\mu' \\ &\geq 2^{-1}\tau^2 C_\nu(r+1)^{-1-\nu} + (E - \widetilde{V}_L)\mu' - C(r+1)^{-\rho}\mu' - C(h\tau + \theta_2^{2\alpha'})\mu', \\ A_2(r) &\geq C_\ell - C\eta_2 > 0. \end{aligned}$$

Since $n < 2$, we can arrange $C(h\tau + \theta_2^{2\alpha'}) \leq E/4$ by taking h small enough. Therefore, by (3.11) and (3.12) we obtain the desired lower bound for A_1 in this case. For $r > a$ the lower bounds for A_1 and A_2 are obtained in the same way as in Section 3 taking $a = h^{-m}$ with $m > 0$ big enough. Thus we conclude that the estimate (3) still holds in this case with the new parameters a, τ, as desired. □

6. Carleman estimates for L^∞ potentials V_S and Hölder potentials b^S

In this section we will prove the following

Theorem 6.1. *Under the conditions of Theorem 1.3, the estimate (3.1) holds true with new parameters $a = h^{-m}$, $\tau = \tau_0 h^{-\frac{1-\alpha'}{1+\alpha'}}$ if $\delta > (1-\alpha')^{-1}$, $\alpha' \leq \frac{1}{2}$, and $\tau = h^{-\frac{1}{2\delta-1}-\epsilon}\tau_0$, $0 < \epsilon \ll 1$ being arbitrary, independent of h, if $\alpha' \geq 1 - \delta^{-1}$, $1 < \delta \leq 2$, with some positive constants m and τ_0.*

Proof. In this case we will make use of the Carleman estimates obtained in Section 3 under the condition (1.8) with b^S and b^L replaced by $b^S - b_{\theta_2}$ and $b^L + b_{\theta_2}$, respectively. The functions V_S and V_L remain unchanged. Then the new functions $\widetilde{V}_S$, $\widetilde{V}_L$ and σ satisfy the bounds

$$\begin{aligned} |\widetilde{V}_S| &\lesssim (r+1)^{-\delta} + \theta_2^{\alpha'}(r+1)^{-3-\rho} + h(r+1)^{-1-\rho}, \\ \partial_r(\widetilde{V}_L - \sigma^2) &\lesssim (r+1)^{-1-\rho} + \theta_2^{-1+\alpha'}(r+1)^{-3-\rho}. \end{aligned}$$

Moreover, $\widetilde{V}_L$ satisfies (1.1) with a new decreasing function p independent of the parameter θ_2. Then the functions B_j in this case become

$$\begin{aligned} B_1(r) &= (r+1)^{-2\delta} + \theta_2^{2\alpha'}(r+1)^{-6-\rho} + h^2(r+1)^{-2-\rho} + h^2\theta_2^{2\alpha'}r^{-2}(r+1)^{-3-\rho} \\ &+\theta_2^{2\alpha'}(r+1)^{-3-\rho}\varphi'^2 + h^2\varphi''^2, \\ B_2(r) &= \theta_2^{2\alpha'}(r+1)^{-3-\rho}, \quad B_3(r) = (r+1)^{-\rho}, \\ B_4(r) &= (r+1)^{-1}B_3(r) + \theta_2^{-1+\alpha'}(r+1)^{-3/2-\rho}, \\ B_5(r) &= B_3(r) + \theta_2^{-1+\alpha'}(r+1)^{-2-\rho}. \end{aligned}$$

For the new functions A_1 and A_2 we obtain

$$A_1(r) = \tau^2(\psi'^2\mu)' + (E - \widetilde{V}_L)\mu' - C\mu r^{-1}(r+1)^{-\rho}$$
$$-C\left(M_2\theta_2^{-1+\alpha'} + M_1\theta_2^{-2+2\alpha'}\right)(r+1)^{-1-\rho}$$
$$-Ch^{-1}\tau^{-1}(r+1)^{-2\delta}\mu(\psi')^{-1} - Ch^{-1}\tau^{-1}\theta_2^{2\alpha'}(r+1)^{-6-\rho}\mu(\psi')^{-1}$$
$$-C\tau h^{-1}\theta_2^{2\alpha'}(r+1)^{-3-\rho}\mu\psi' - Ch\tau^{-1}(r+1)^{-2-\rho}\mu(\psi')^{-1} - Ch\tau\mu(\psi'')^2(\psi')^{-1}$$
$$-C\theta_2^{2\alpha'}\mu^2(r\mu')^{-2}(r+1)^{-3-\rho}\mu' - C(rB_3)^2M_1 - ChB_3M_2,$$
$$2A_2(r) = 2r^{-1}\mu - \mu' - Ch^{-1}\tau^{-1}\theta_2^{2\alpha'}(r+1)^{-3-\rho}\mu(\psi')^{-1}.$$

We have to show that the functions A_1 and A_2 satisfy the same lower bounds as in the previous section. It suffices to do so for $r < a$ since for $r > a$ the lower bounds for A_1 and A_2 can be obtained in the same way as in Section 3 taking $a = h^{-m}$ with $m > 0$ big enough. Set

$$\chi = h^{-1}\tau^{-3}, \quad \chi_2 = \tau^{-2}\theta_2^{-2+2\alpha'}, \quad \eta_2 = h^{-1}\tau^{-1}\theta_2^{2\alpha'}.$$

We take $\theta_2 = h^{\frac{1}{1+\alpha'}}$, $\tau = h^{-\frac{1-\alpha'}{1+\alpha'}}\tau_0$, where $\tau_0 \gg 1$ is independent of h. Then we have

$$\chi_2 = \tau_0^{-2}, \quad \eta_2 = \tau_0^{-1}.$$

Consider first the case $\alpha' = \frac{1}{2}$ and $\delta > 2$. Then

$$\chi = \tau_0^{-3}.$$

Choose $\nu < \min\{\rho, 2(\delta-2)\}$. For $r < a$, $r \neq 1$, we have

$$A_1(r) \geq \tau^2(C_\nu - C(\chi + \chi_2 + \eta_2))(r+1)^{-1-\nu}$$
$$+(E - \widetilde{V}_L)\mu' - C(r+1)^{-\rho}\mu' - C(h\tau + \theta_2^{2\alpha'})\mu'$$
$$\geq 2^{-1}\tau^2C_\nu(r+1)^{-1-\nu} + (E - \widetilde{V}_L)\mu' - C(r+1)^{-\rho}\mu' - C(h\tau + \theta_2^{2\alpha'})\mu',$$
$$A_2(r) \geq C_\ell - C\eta_2 > 0,$$

if τ_0 is taken big enough. In the same way as in the previous section one can conclude that $A_1 \geq E\mu'/3$, which leads to the desired estimate in this case.

Let now $\alpha' < \frac{1}{2}$ and $(1-\alpha')^{-1} < \delta \leq 2$. Let $1 < T < a$ be a parameter of the form

$$T = T_0(h\tau)^{-\frac{1}{2(\delta-1)}},$$

where $T_0 \gg 1$ is independent of h and τ_0. For $T \leq r < a$ we have

$$Ch^{-1}\tau^{-1}(r+1)^{-2\delta}\mu(\psi')^{-1} \lesssim h^{-1}\tau^{-1}(r+1)^{-2\delta+2}\mu'(r) \lesssim T_0^{-2\delta+2}\mu'(r) \leq 4^{-1}E\mu'(r),$$

provided T_0 is taken big enough. On the other hand, for $r < T$, we have

$$Ch^{-1}\tau^{-1}(r+1)^{-2\delta}\mu(\psi')^{-1} \lesssim h^{-1}\tau^{-1}(r+1)^{-2\delta+3}$$
$$\lesssim \chi\tau^2(r+1)^{-1-\nu}T^{4-2\delta+\nu} \lesssim \widetilde{\chi}\tau^2(r+1)^{-1-\nu},$$

where

$$\widetilde{\chi} = \chi(h\tau)^{-\frac{2-\delta+\nu/2}{\delta-1}} = h^{-\frac{1+\nu/2}{\delta-1}}\tau^{-\frac{2\delta-1+\nu/2}{\delta-1}} \leq \tau_0^{-\frac{2\delta-1+\nu/2}{\delta-1}},$$

provided ν is chosen small enough. Hence we can make $\widetilde{\chi}$ small enough by taking τ_0 big enough, depending on T_0. Thus we conclude that the above lower bound for A_1 holds with χ replaced by $\widetilde{\chi}$ and E replaced by $3E/4$, which again leads to the inequality $A_1 \geq E\mu'/3$ in this case.

Let now $\alpha' \geq 1-\delta^{-1}$, $1<\delta\leq 2$. In this case we take $\tau = h^{-\frac{1}{2\delta-1}-\epsilon}\tau_0$, where $0<\epsilon\ll 1$ is arbitrary, independent of h, and $\tau_0 \gg 1$ is independent of h and ϵ. Let θ_2 be as above. Clearly, we have $\tau \geq h^{-\frac{1-\alpha'}{1+\alpha'}}\tau_0$, which implies the inequalities

$$\chi_2 \leq \tau_0^{-2}, \quad \eta_2 \leq \tau_0^{-1}.$$

Moreover, it is easy to see that the above bound of $\widetilde{\chi}$ still holds, provided ν is chosen small enough depending on ϵ. Therefore, the inequality $A_1 \geq E\mu'/3$ holds in this case, too.

□

7. Resolvent estimates

In this section we will show that the Carleman estimates (3.1), proved under various conditions, imply the resolvent bounds in Theorems 1.1, 1.2 and 1.3. These arguments are well-known when $b\equiv 0$, but the presence of a non-trivial magnetic potential does not change anything. The key points are the symmetry of the operator $P(h)$ on the Hilbert space $L^2(\mathbb{R}^d)$ and the fact that the potentials are L^∞. In what follows we will sketch the proof for the sake of completeness. It follows from (3.1) and Lemma 2.2 that we have the estimate

$$\|(|x|+1)^{-s}f\|_{L^2} \leq N_1\|(|x|+1)^s(P(h)-E\pm i\varepsilon)f\|_{L^2} + N_2\varepsilon^{1/2}\left(\|f\|_{L^2}+\tau^{-1}\|h\nabla f\|_{L^2}\right) \tag{7.1}$$

where

$$N_1 = ah^{-1}\exp\left(Ch^{-1}\tau\log a\right), \quad N_2 = (h\tau)^{1/2}N_1,$$

with a constant $C>0$ independent of h and ε. On the other hand, since the operator $P(h)$ is symmetric, we have

$$\varepsilon\|f\|_{L^2}^2 = \pm\mathrm{Im}\,\langle (P(h)-E\pm i\varepsilon)f, f\rangle_{L^2}$$

$$\leq \gamma^2 N_2^{-2}\|(|x|+1)^{-s}f\|_{L^2}^2 + \gamma^{-2}N_2^2\|(|x|+1)^s(P(h)-E\pm i\varepsilon)f\|_{L^2}^2$$

for every $\gamma>0$, which yields

$$N_2\varepsilon^{1/2}\|f\|_{L^2} \leq \gamma\|(|x|+1)^{-s}f\|_{L^2} + \gamma^{-1}N_2^2\|(|x|+1)^s(P(h)-E\pm i\varepsilon)f\|_{L^2}. \tag{7.2}$$

We also have

$$\mathrm{Re}\,\langle (P(h)-E\pm i\varepsilon)f, f\rangle_{L^2} = \|(ih\nabla+b)f\|_{L^2}^2 + \mathrm{Re}\,\langle (V-E)f, f\rangle_{L^2}$$

$$\geq \|h\nabla f\|_{L^2}^2 - C\|f\|_{L^2}^2.$$

Thus we get

$$\|h\nabla f\|_{L^2} \lesssim \|f\|_{L^2} + \|(P(h)-E\pm i\varepsilon)f\|_{L^2}. \tag{7.3}$$

Taking γ small enough, independent of N_2 and ε, by (7.1), (7.2) and (7.3) we obtain the estimate

$$\|(|x|+1)^{-s}f\|_{L^2} \lesssim (N_1+N_2^2)\|(|x|+1)^s(P(h)-E\pm i\varepsilon)f\|_{L^2}. \tag{7.4}$$

It follows from (7.4) that the resolvent estimate

$$\left\|(|x|+1)^{-s}(P(h)-E\pm i\varepsilon)^{-1}(|x|+1)^{-s}\right\|_{L^2\to L^2} \lesssim N_1 + h\tau N_1^2 \lesssim N_1^2 \tag{7.5}$$

holds for all $0<h\ll 1$. Clearly, (7.5) implies the desired bounds for $g_s^\pm$.

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