Dimension reduction and error estimates

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Abstract

The aim of this paper is twofold: to provide asymptotic behavior and error estimates results for an elliptic dimension reduction problem. The problem is posed in a bounded and thin domain $\Omega_\delta \doteq \mathcal{O} \times \omega_\delta \subset \mathbb{R}^m \times \mathbb{R}^n$, m, $n \geq 1$ where $\omega_\delta = \delta\omega$ when $\delta \to 0$ (the thickness of Ω_δ is of order δ while its diameter is of order 1). The limit problem is elliptic and is posed in $\mathcal{O}$. Under the assumption that $\mathcal{O}$ has a smooth boundary $(C^{1,1})$ and homogeneous Dirichlet boundary condition, we give two important results: an estimate of the global error, whose order is $\delta^{1/2}$, and an estimate of the L^2 global and H^1 interior errors, whose order is δ.

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1 Introduction

In this paper, we consider tools similar to those used to study elasticity problems for beams, plates or shells. We first transform the thin, variable domain $\Omega_\delta = \mathcal{O}\times\omega_\delta$ into a fixed domain $\Omega = \mathcal{O}\times\omega$ where $\mathcal{O}$, ω are respectively open sets in $\mathbb{R}^m\,(m \geq 1)$ and $\mathbb{R}^n\,(n \geq 1)$ with Lipschitz boundary, $\omega_\delta = \delta\omega$. This geometrical transformation is a dilation of ratio $1/\delta$, it takes $\{x'\}\times\omega_\delta$ to $\{x'\}\times\omega$ for all $x' \in \mathcal{O}$. Then, we define the first operator Π_δ used in this paper; the dimension reduction operator. This operator maps $L^p(\Omega_\delta)$ onto $L^p(\Omega)$, $p \in [1, +\infty]$ and is also one to one. The mean value operator $\mathcal{M}_{\omega_\delta}$, the second important tool used here, transforms any function ϕ belonging to $L^p(\Omega_\delta)$ into the function $\mathcal{M}_{\omega_\delta}(\phi)$ belonging to $L^p(\mathcal{O})$. $\mathcal{M}_{\omega_\delta}(\phi)(x')$ is the mean value of ϕ over the fiber $\{x'\}\times\omega$, $x' \in \mathcal{O}$. This operator captures the main part of the function ϕ. We therefore have the decomposition of a function $\phi \in L^p(\Omega_\delta)$ as follows:

$$\phi = \Phi + \overline{\phi}, \qquad \Phi = \mathcal{M}_{\omega_\delta}(\phi).$$

The function $\overline{\phi}$ is the residual part of ϕ.

We apply the above tools to study the sequence formed by the solutions, indexed by $\delta > 0$, of the following problems:

$$\begin{cases} \text{Find } u_\delta \in H^1_{\Gamma_\delta}(\Omega_\delta) \doteq \left\{v \in H^1(\Omega_\delta) \mid v = 0 \text{ on } \Gamma_\delta = \gamma\times\omega_\delta\right\} \text{ such that,} \\ \displaystyle\int_{\Omega_\delta} A_\delta \nabla u_\delta \cdot \nabla v dx = \int_{\Omega_\delta} f_\delta v dx, \qquad \forall v \in H^1_{\Gamma_\delta}(\Omega_\delta), \end{cases} \tag{1.1}$$

where $f_\delta \in L^2(\Omega_\delta)$, $\gamma \subset \partial\mathcal{O}$ (see Subsection 4.1 for more details).
We decompose $u_\delta = \mathcal{U}_\delta + \overline{u}_\delta$ where $\mathcal{U}_\delta = \mathcal{M}_{\omega_\delta}(u_\delta)$ and we use the operator Π_δ to obtain the asymptotic behavior of the residual term $\overline{u}_\delta$. We show that

$$\mathcal{U}_\delta \rightharpoonup \mathcal{U} \qquad \text{weakly in } H^1_\gamma(\mathcal{O}) \doteq \left\{\psi \in H^1(\mathcal{O}) \mid \psi = 0 \text{ a.e. on } \gamma\right\} \tag{1.2}$$

where $\mathcal{U}$ is the unique solution of the following problem (see Subsection 4.1 and Theorem 5.3 for more details):

$$\begin{cases} \text{Find } \mathcal{U} \in H^1_\gamma(\mathcal{O}) \text{ such that} \\ \displaystyle\int_{\mathcal{O}} A^* \nabla \mathcal{U} \cdot \nabla \phi \, dx' = \int_{\mathcal{O}} f \phi \, dx', \\ \forall \phi \in H^1_\gamma(\mathcal{O}), \end{cases} \tag{1.3}$$

where the matrix A^* depends on the limit of the sequence $\{\Pi_\delta(A_\delta)\}_\delta$ and also of the asymptotic behavior of the sequence $\{\Pi_\delta(\overline{u}_\delta)\}_\delta$.

Then, proceeding in a similar way to that developed in [7, 8], we prove errors estimates. Under the assumptions: $f_\delta = f \in L^2(\mathcal{O})$, $\partial\mathcal{O}$ of class $\mathcal{C}^{1,1}$ and the homogeneous lateral Dirichlet (or Neumann) boundary conditions on $\partial\mathcal{O}\times\omega_\delta$, we obtain the global H^1 error estimate (of order $\delta^{1/2}$), then the global L^2 error (of order δ) and the H^1 interior error which is of order δ.

Other mathematical problems involve with small geometrical parameters. This is the case, for example, in periodic homogenization. In [6], the authors have introduced the unfolding operator $\mathcal{T}_\varepsilon$, which makes it possible to deal in a fixed domain with both macroscopic and microscopic variables. Also in linear and nonlinear elasticity for thin domains, we can use a dimension reduction operator which allows to work in a fixed domain (see [2, 4]). In [6, Chapter 11] the authors deal with an elasticity problem that simultaneously combines homogenization and dimension reduction. In the case of periodic homogenization and elliptic problems, numerous papers and books have studied error estimates. The global error estimates were first obtained in Bensoussan and Papanicolaou's book [1] (see also [5] a book by D. Cioranecu and P. Donato and [13] a book by O. A. Oleinik and al.). In all these books, the result was obtained under the additional assumption that the correctors belong to $W^{1,\infty}(Y)$ ($Y = (0,1)^n$ being the reference cell). In [7], G. Griso obtained global error estimates without any additional assumptions about the correctors. All these authors assume that the homogenization problem is posed in a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary ($\mathcal{C}^{1,1}$) and homogeneous Dirichlet or Neumann boundary conditions. G.Griso in [8, Theorem 3.2] completed the first result of error estimates by giving interior H^1 and global L^2 error estimates. Interestingly, if ε is the size of the small cells, the global error estimate is of order $\varepsilon^{1/2}$ while the H^1 interior and L^2 error estimates are of order ε. More recent work has produced error estimates with a weaker assumption about the boundary (see [14, 15]). In elasticity also some error estimates have been obtained (see e.g. [9, 10, 12]).

The paper is organized as follows:

- In Section 2 we give the notations.
- Section 3 is divided into two parts. We first define the operators Π_δ (the dimension reduction operator) and $\mathcal{M}_{\omega_\delta}$ (the mean value operator) and then give their properties. We also introduce the decomposition of any function ϕ defined on Ω_δ, $\phi = \Phi + \overline{\phi}$ where $\Phi = \mathcal{M}_{\omega_\delta}(\phi)$.
- Section 4 is devoted to presenting the elliptic problem. Then, we give the estimates and the convergences of the terms of the decomposition of u_δ.

- Section 5 is devoted to limit problems. A limit problem posed in the reference domain $\Omega \cdot \mathcal{O} \times \omega$ and a second one, posed in $\mathcal{O}$, satisfied by the limit of the principal term of the decomposition of u_δ.
- In Section 6, under the assumptions that $\partial\mathcal{O}$ is of class $C^{1,1}$, $f_\delta = f \in L^2(\mathcal{O})$ and the homogeneous Dirichlet and Neumann conditions, we give the H^1 error estimate.
- In Section 7, under the same assumptions as Section 6, we give respectively the L^2 global error and H^1 interior error estimates.
- Appendix contains the proof of some technical lemmas.

In this paper, the constants appearing in the estimates will always be independent from δ.

2 Notations

- $\left(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{e}_{m+1}, \ldots, \mathbf{e}_{m+n}\right)$ is the usual basis of $\mathbb{R}^N$, $N \doteq m+n$, $m \geq 1,\ n \geq 1$,
- $\mathcal{O}$ a bounded domain in $\mathbb{R}^m$ with Lipschitz boundary,
- ω a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, the origin of $\mathbb{R}^n$ belongs to ω,
- γ a measurable subset of $\partial\mathcal{O}$ (possibly empty),
- $\Omega_\delta \doteq \mathcal{O} \times \omega_\delta$, $\omega_\delta \doteq \delta\omega$,
- $\Gamma_\delta \doteq \gamma \times \omega_\delta$,
- $\Omega \doteq \mathcal{O} \times \omega$,
- $\Gamma \doteq \gamma \times \omega$.

We denote $x' = (x_1, \ldots, x_m)$ the curent point of $\mathcal{O}$ and $x'' = (x_{m+1}, \ldots, x_{m+n})$ (resp. $X'' = (X_{m+1}, \ldots, X_{m+n})$) the curent point of ω_δ (resp. ω).
The measure of the set S is denoted $|S|$.

3 The operators Π_δ, $\mathcal{M}_{\omega_\delta}$ and the decomposition

3.1 The operators

Definition 3.1. *(Dimension reduction operator.) For ϕ Lebesgue-measurable on Ω_δ, the dimension reduction operator Π_δ is defined as follows:*

$$\Pi_\delta(\phi)(x', X'') = \phi(x', \delta X'') \qquad \textit{for a.e.} \qquad (x', X'') \in \Omega = \mathcal{O} \times \omega$$

$\Pi_\delta(\phi)$ is a Lebesgue-measurable function on Ω.

Proposition 3.2. *Properties of Π_δ ($p \in [1, +\infty]$)*

1. *Π_δ is a continuous linear operator from $L^p\Omega_\delta)$ into $L^p(\Omega)$,*

2. *For any $u, v \in L^p(\Omega_\delta)$, we have $\Pi_\delta(uv) = \Pi_\delta(u)\Pi_\delta(v)$.*

3. *For any $u \in L^p(\Omega_\delta)$,*

$$\delta^n \int_\Omega \Pi_\delta(u)(x', X'')dx'dX'' = \int_{\Omega_\delta} u(x', x'')dx'dx''.$$

4. *For any $v \in L^p(\Omega_\delta)$ the function $\Pi_\delta(v)$ belongs to $L^p(\Omega)$ and we have*

$$\|\Pi_\delta(v)\|_{L^p(\Omega)} = \delta^{-\frac{n}{p}} \|v\|_{L^p(\Omega_\delta)}. \tag{3.1}$$

5. *For any $v \in W^{1,p}(\Omega_\delta)$ the function $\Pi_\delta(v)$ belongs to $W^{1,p}(\Omega)$and we have*

$$\frac{\partial \Pi_\delta(v)}{\partial x'_i} = \Pi_\delta\Big(\frac{\partial v}{\partial x'_i}\Big), \qquad \frac{\partial \Pi_\delta(v)}{\partial X''_j} = \delta \Pi_\delta\Big(\frac{\partial v}{\partial x''_j}\Big),$$
$$(i,j) \in \{1, \dots, m\} \times \{m+1, \dots, m+n\}$$

and

$$\begin{aligned} \big\|\nabla_{x'} \Pi_\delta(v)\big\|_{L^p(\Omega)} &= \delta^{-\frac{n}{p}} \big\|\nabla_{x'} v\big\|_{L^p(\Omega_\delta)}, \\ \big\|\nabla_{X''} \Pi_\delta(v)\big\|_{L^p(\Omega)} &= \delta^{1-\frac{n}{p}} \big\|\nabla_{x''} v\big\|_{L^p(\Omega_\delta)}. \end{aligned} \tag{3.2}$$

Definition 3.3. *(Mean value operator.) For any function ϕ belonging to $L^1(\Omega_\delta)$ we define the mean value operator $\mathcal{M}_{\omega_\delta}$ as*

$$\mathcal{M}_{\omega_\delta}(\phi)(x') = \frac{1}{\delta^n |\omega|} \int_{\omega_\delta} \phi(x', x'')dx'' \quad \textit{for a.e. } x' \in \mathcal{O}.$$

Lemma 3.4. *For any $\phi \in L^p(\Omega_\delta)$, $p \in [1, +\infty]$, we have*

$$\|\mathcal{M}_{\omega_\delta}(\phi)\|_{L^p(\mathcal{O})} \leq \frac{1}{|\omega|^{\frac{1}{p}} \delta^{\frac{n}{p}}} \|\phi\|_{L^p(\Omega_\delta)}. \tag{3.3}$$

Moreover, if $\phi \in W^{1,p}(\Omega_\delta)$ then $\mathcal{M}_{\omega_\delta}(\phi)$ belongs to $W^{1,p}(\mathcal{O})$ and we have

$$\big\|\nabla_{x'} \mathcal{M}_{\omega_\delta}(\phi)\big\|_{L^p(\mathcal{O})} \leq \frac{1}{|\omega|^{\frac{1}{p}} \delta^{\frac{n}{p}}} \|\nabla_{x'} \phi\|_{L^p(\Omega_\delta)}. \tag{3.4}$$

Proof. Step 1. We prove (3.3).

Let p be in $(1, +\infty)$. Denote p' the conjugate of p, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $p' \in (1, +\infty)$.

Using the Hölder inequality we obtain

$$\begin{aligned} \|\mathcal{M}_{\omega_\delta}(\phi)\|^p_{L^p(\mathcal{O})} &= \int_{\mathcal{O}} |\mathcal{M}_{\omega_\delta}(\phi)(x')|^p dx' = \int_{\mathcal{O}} \Big| \frac{1}{\delta^n |\omega|} \int_{\omega_\delta} \phi(x', x'')dx'' \Big|^p dx' \\ &= \frac{1}{\delta^{np} |\omega|^p} \int_{\mathcal{O}} \Big| \int_{\omega_\delta} \phi(x', x'')dx'' \Big|^p dx' \\ &\leq \frac{1}{\delta^{np} |\omega|^p} \int_{\mathcal{O}} \Big[\Big(\int_{\omega_\delta} |\phi(x', x'')|^p dx'' \Big)^{\frac{1}{p}} \Big(\int_{\omega_\delta} dx'' \Big)^{\frac{1}{p'}} \Big]^p dx' \\ &= \frac{1}{\delta^{np} |\omega|^p} \|\phi\|^p_{L^p(\Omega_\delta)} |\omega_\delta|^{\frac{p}{p'}} = \frac{1}{\delta^{np} |\omega|^p} \|\phi\|^p_{L^p(\Omega_\delta)} \delta^{\frac{np}{p'}} |\omega|^{\frac{p}{p'}} \\ &= \delta^{n(\frac{p}{p'} - p)} |\omega|^{(\frac{p}{p'} - p)} \|\phi\|^p_{L^p(\Omega_\delta)}. \end{aligned}$$

Since $\dfrac{1}{p}+\dfrac{1}{p'}=1$, we have $\dfrac{p}{p'}-p=-1$. Then, the above inequality gives (3.3).
Step 2. We prove (3.4).
Observe that

$$\nabla_{x'}\mathcal{M}_{\omega_\delta}(\phi)(x')=\frac{1}{\delta^n|\omega|}\int_{\omega_\delta}\nabla_{x'}\phi(x',x'')dx''=\mathcal{M}_{\omega_\delta}\big(\nabla_{x'}\phi\big)(x')\qquad\text{for a.e. } x'\in\mathcal{O}.$$

Then, (3.3) yields (3.4) for $p\in(1,+\infty)$.
In the case $p=1$, we obtain the estimates (3.3)-(3.4) by direct calculation. In the case $p=+\infty$, we obtain the estimates (3.3)-(3.4) by passing to the limit ($p\to+\infty$). □

3.2 The decomposition

Any function ϕ belonging to $L^1(\Omega_\delta)$ is decomposed as

$$\phi=\Phi+\overline{\phi}\qquad\text{where }\ \Phi=\mathcal{M}_{\omega_\delta}(\phi)\ \text{ and }\ \overline{\phi}=\phi-\Phi. \tag{3.5}$$

Lemma 3.5. *For any $\phi\in L^p(\Omega_\delta)$, $p\in[1,+\infty]$, decomposed as* (3.5)*, we have $\Phi\in L^p(\mathcal{O})$, $\overline{\phi}\in L^p(\Omega_\delta)$ and the following estimates:*

$$\|\Phi\|_{L^p(\mathcal{O})}\le\frac{1}{|\omega|^{\frac{1}{p}}\delta^{\frac{n}{p}}}\|\phi\|_{L^p(\Omega_\delta)}\qquad\qquad\|\overline{\phi}\|_{L^p(\Omega_\delta)}\le 2\|\phi\|_{L^p(\Omega_\delta)}. \tag{3.6}$$

Moreover, if ϕ belongs to $W^{1,p}(\Omega_\delta)$ then $\Phi\in W^{1,p}(\mathcal{O})$, $\overline{\phi}\in W^{1,p}(\Omega_\delta)$ and

$$\begin{cases}\big\|\nabla_{x'}\Phi\big\|_{L^p(\mathcal{O})}\le\dfrac{1}{|\omega|^{\frac{1}{p}}\delta^{\frac{n}{p}}}\|\nabla_{x'}\phi\|_{L^p(\Omega_\delta)},\\ \|\overline{\phi}\|_{L^p(\Omega_\delta)}\le C\delta\|\nabla_{x''}\phi\|_{L^p(\Omega_\delta)},\\ \|\nabla_{x'}\overline{\phi}\|_{L^p(\Omega_\delta)}\le 2\|\nabla_{x'}\phi\|_{L^p(\Omega_\delta)},\qquad\|\nabla_{x''}\overline{\phi}\|_{L^p(\Omega_\delta)}=\|\nabla_{x''}\phi\|_{L^p(\Omega_\delta)}.\end{cases} \tag{3.7}$$

The constant does not depend on δ.

Proof. First, the estimate $(3.6)_1$ (resp. $(3.7)_1$) is (3.3) (resp. (3.4)).
Then, $(3.6)_1$ yields

$$\begin{aligned}\|\overline{\phi}\|_{L^p(\Omega_\delta)}&=\|\phi-\Phi\|_{L^p(\Omega_\delta)}\\&\le\|\phi\|_{L^p(\Omega_\delta)}+\|\Phi\|_{L^p(\Omega_\delta)}\\&\le\|\phi\|_{L^p(\Omega_\delta)}+\delta^{\frac{n}{p}}|\omega|^{\frac{1}{p}}\|\Phi\|_{L^p(\mathcal{O})}\le 2\|\phi\|_{L^p(\Omega_\delta)}.\end{aligned}$$

Thus, $(3.6)_2$ is proved.
Now, we prove $(3.7)_2$.
The Poincaré-Wirtinger inequality yields (see [3, Chapter 9])

$$\|\phi(x',\cdot)-\mathcal{M}_{\omega_\delta}(\phi)(x')\|_{L^p(\omega_\delta)}\le C\delta\|\nabla_{x''}\phi(x',\cdot)\|_{L^p(\omega_\delta)}\qquad\text{for a.e. } x'\in\mathcal{O}.$$

The constant only depends on ω. Then, the above inequality leads to $(3.7)_2$.
It remains to prove $(3.7)_{3,4}$.

First observe that $\nabla_{x''}\overline{\phi} = \nabla_{x''}\phi$, thus $(3.7)_3$ is proved.
Thus, we get

$$\begin{aligned}\|\nabla_{x'}\overline{\phi}\|_{L^p(\Omega_\delta)} &= \|\nabla_{x'}\phi - \nabla_{x'}\Phi\|_{L^p(\Omega_\delta)}\\ &\leq \|\nabla_{x'}\phi\|_{L^p(\Omega_\delta)} + \|\nabla_{x'}\Phi\|_{L^p(\Omega_\delta)}.\end{aligned}$$

We conclude using $(3.7)_1$. □

4 The elliptic problem, estimates, convergences

We recall the following result which is a consequence of Fubini's theorem:

Lemma 4.1. *For any $(\phi_1, \phi_2) \in L^1(\mathcal{O})\times L^1(\omega_\delta)$ (resp. $(\phi_1, \Phi_2) \in L^1(\mathcal{O})\times L^1(\omega)$) the function ϕ (resp. Φ) defined by*

$$\begin{gathered}\phi(x', x'') = \phi_1(x')\phi_2(x'') \quad \textit{for a.e. } (x', x'') \in \Omega_\delta,\\ \textit{(resp. } \Phi(x', X'') = \phi_1(x')\Phi_2(X'') \quad \textit{for a.e. } (x', X'') \in \Omega\textit{)}\end{gathered}$$

belongs to $L^1(\Omega_\delta)$ (resp. $L^1(\Omega)$) and we have

$$\|\phi\|_{L^1(\Omega_\delta)} = \|\phi_1\|_{L^1(\mathcal{O})}\|\phi_2\|_{L^1(\omega_\delta)}, \qquad \textit{(resp. } \|\Phi\|_{L^1(\Omega)} = \|\phi_1\|_{L^1(\mathcal{O})}\|\Phi_2\|_{L^1(\omega)} \textit{).}$$

4.1 The elliptic problem

For any $\Psi \in L^1(\mathcal{O})$ we denote $\mathcal{M}_{\mathcal{O}}(\Psi)$ the mean value of Ψ over $\mathcal{O}$

$$\mathcal{M}_{\mathcal{O}}(\Psi) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \Psi\, dx'.$$

From now on, any function v in $H^1_{\Gamma_\delta}(\Omega_\delta)$ will be decomposed as (3.5)

$$v = \mathcal{V} + \overline{v}, \qquad \mathcal{V} = \mathcal{M}_{\omega_\delta}(v), \qquad \mathcal{V} \in H^1_\gamma(\mathcal{O}), \quad \overline{v} \in H^1_{\Gamma_\delta}(\Omega_\delta). \tag{4.1}$$

Denote

$$H^1_{\Gamma_\delta}(\Omega_\delta) \doteq \begin{cases}\{v \in H^1(\Omega_\delta) \mid v = 0 \text{ on } \Gamma_\delta\ \} & \text{if } |\gamma| > 0,\\ \{v \in H^1(\Omega_\delta) \mid \mathcal{M}_{\mathcal{O}}(\mathcal{V}) = 0\ \} & \text{if } \gamma = \phi\end{cases}$$

and

$$H^1_\gamma(\mathcal{O}) \doteq \begin{cases}\{\Phi \in H^1(\mathcal{O}) \mid \Phi = 0 \text{ on } \gamma\ \} & \text{if } |\gamma| > 0,\\ \{\Phi \in H^1(\mathcal{O}) \mid \mathcal{M}_{\mathcal{O}}(\Phi) = 0\ \} & \text{if } \gamma = \phi.\end{cases}$$

Using the decomposition (4.1) of $v \in H^1_{\Gamma_\delta}(\Omega_\delta)$ and due to $(3.7)_2$ we have

$$\|\overline{v}\|_{L^2(\Omega_\delta)} \leq C\delta\|\nabla v\|_{L^2(\Omega_\delta)}. \tag{4.2}$$

Then,

- if $|\gamma| > 0$ the Poincaré inequality and $(3.7)_1$-$(3.2)_1$ lead to

  $$\|\mathcal{V}\|_{L^2(\mathcal{O})} \leq C\|\nabla_{x'}\mathcal{V}\|_{L^2(\mathcal{O})} \leq C\delta^{-\frac{n}{2}}\|\nabla v\|_{L^2(\Omega_\delta)}$$

  which in turn yields

  $$\|\mathcal{V}\|_{L^2(\Omega_\delta)} \leq C\|\nabla v\|_{L^2(\Omega_\delta)}, \tag{4.3}$$

- if $\gamma = \phi$ then, the Poincaré-Wirtinger inequality and $(3.7)_1$ lead to

$$\|\mathcal{V}\|_{L^2(\mathcal{O})} = \|\mathcal{V} - \mathcal{M}_{\mathcal{O}}(\mathcal{V})\|_{L^2(\mathcal{O})} \leq C\|\nabla_{x'}\mathcal{V}\|_{L^2(\mathcal{O})} \leq C\delta^{-\frac{n}{2}}\|\nabla v\|_{L^2(\Omega_\delta)}$$

which in turn yields

$$\|\mathcal{V}\|_{L^2(\Omega_\delta)} \leq C\|\nabla v\|_{L^2(\Omega_\delta)}. \tag{4.4}$$

As a consequence of the above estimates (4.2)-(4.3)-(4.4), in both cases we get

$$\|v\|_{L^2(\Omega_\delta)} \leq C\|\nabla v\|_{L^2(\Omega_\delta)}. \tag{4.5}$$

In this paper, we study the following elliptic problem:

$$\begin{cases} \text{Find } u_\delta \in H^1_{\Gamma_\delta}(\Omega_\delta), \\ \displaystyle\int_{\Omega_\delta} A_\delta \nabla u_\delta \cdot \nabla v dx = \int_{\Omega_\delta} f_\delta v dx, \qquad \forall v \in H^1_{\Gamma_\delta}(\Omega_\delta), \end{cases} \tag{4.6}$$

where A_δ is a $N\times N$ matrix. The entries $a_{ij,\delta}$ of the matrix A_δ satisfy

$$a_{ij,\delta}(x) = a_{ij}\Big(\frac{x''}{\delta}\Big), \qquad x = (x', x'') \in \Omega_\delta, \qquad a_{ij} \in L^\infty(\omega)$$

and $f_\delta \in L^2(\Omega_\delta)$. The functions a_{ij} also satisfy: there exists $c > 0$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(X'')\xi_i\xi_j \geq c|\xi|^2, \qquad \text{for a.e. } X'' \in \omega \quad \text{and} \quad \forall \xi \in \mathbb{R}^N. \tag{4.7}$$

We decompose u_δ as (3.5)

$$u_\delta = \mathcal{U}_\delta + \overline{u}_\delta, \qquad \mathcal{U}_\delta = \mathcal{M}_{\omega_\delta}(u_\delta), \qquad \mathcal{U}_\delta \in H^1_\gamma(\mathcal{O}), \quad \overline{u}_\delta \in H^1_{\Gamma_\delta}(\Omega_\delta). \tag{4.8}$$

Note that since $u_\delta \in H^1_{\Gamma_\delta}(\Omega_\delta)$, we have

$$\mathcal{U}_\delta \in H^1_\gamma(\mathcal{O}) \quad \text{and} \quad \overline{u}_\delta \in H^1_{\Gamma_d}(\Omega_\delta).$$

Lemma 4.2. *Problem* (4.6) *admits a unique solution satisfying*

$$\|u_\delta\|_{H^1(\Omega_\delta)} \leq C\|f_\delta\|_{L^2(\Omega_\delta)}. \tag{4.9}$$

Proof. The coercivity assumption and the Lax-Milgram theorem give the existence and uniqueness to the solution of the problem (4.6).
The Cauchy-Schwarz inequality and (4.5) yield

$$\Big|\int_{\Omega_\delta} f_\delta u_\delta dx\Big| \leq \|f_\delta\|_{L^2(\Omega_\delta)}\|u_\delta\|_{L^2(\Omega_\delta)} \leq C\|f_\delta\|_{L^2(\Omega_\delta)}\|\nabla u_\delta\|_{L^2(\Omega_\delta)}. \tag{4.10}$$

Now, we choose $v = u_\delta$ as test function in (4.6). Then, the coercivity assumption and (4.10) yield

$$c\|\nabla u_\delta\|^2_{L^2(\Omega_\delta)} \leq C\|f_\delta\|_{L^2(\Omega_\delta)}\|\nabla u_\delta\|_{L^2(\Omega_\delta)}.$$

The above together with (4.5) ends the proof of (4.9). □

4.2 Estimates of the decomposition terms

In this subsection, we consider the sequence $\{u_\delta\}_\delta$ of solutions to problem (4.6). We give estimates of the terms appearing in the decomposition of u_δ and then, the asymptotic behavior of these terms in the following subsection.

We recall that u_δ is decomposed as (4.8). We set

$$\overline{U}_\delta = \Pi_\delta(\overline{u}_\delta).$$

Lemma 4.3. *We have*

$$\|\mathcal{U}_\delta\|_{H^1_\gamma(\mathcal{O})} \le C\delta^{-\frac{n}{2}}\|f_\delta\|_{L^2(\Omega_\delta)},$$
$$\|\overline{U}_\delta\|_{L^2(\Omega)} + \|\nabla_{X''}\overline{U}_\delta\|_{L^2(\Omega)} \le C\delta^{1-\frac{n}{2}}\|f_\delta\|_{L^2(\Omega_\delta)}, \quad \|\nabla_{x'}\overline{U}_\delta\|_{L^2(\Omega)} \le C\delta^{-\frac{n}{2}}\|f_\delta\|_{L^2(\Omega_\delta)}.$$

Proof. The estimates of this lemma are the immediate consequences of (3.6)-(3.7), (4.9) and the properties of the operator Π_δ (see Proposition 3.2). □

Now, we assume that

$$\Pi_\delta(f_\delta) \rightharpoonup F \quad \text{weakly in } L^2(\Omega). \tag{4.11}$$

So, the sequence $\{\Pi_\delta(f_\delta)\}_\delta$ is uniformly bounded in $L^2(\Omega)$. Hence, f_δ satisfies

$$\|f_\delta\|_{L^2(\Omega_\delta)} = \delta^{\frac{n}{2}}\|\Pi_\delta(f_\delta)\|_{L^2(\Omega)} \le C\delta^{\frac{n}{2}}. \tag{4.12}$$

The above estimate (4.12) and those of Lemma 4.3 yield

$$\begin{aligned}
&\|\mathcal{U}_\delta\|_{H^1_\gamma(\mathcal{O})} \le C,\\
&\|\overline{U}_\delta\|_{L^2(\Omega)} + \|\nabla_{X''}\overline{U}_\delta\|_{L^2(\Omega)} \le C\delta,\\
&\|\nabla_{x'}\overline{U}_\delta\|_{L^2(\Omega)} \le C.
\end{aligned} \tag{4.13}$$

4.3 First convergences

For any $\overline{V} \in L^1(\Omega)$ we denote $\mathcal{M}_\omega\big(\overline{V}\big)(x')$ the mean value of $\overline{V}$ over the fiber $\{x'\}\times\omega$

$$\mathcal{M}_\omega\big(\overline{V}\big)(x') = \frac{1}{|\omega|}\int_\omega \overline{V}(x', X'')dX'', \qquad \text{for a.e. } x' \in \mathcal{O}.$$

$\mathcal{M}_\omega$ is a continuous linear operator from $L^1(\Omega)$ into $L^1(\mathcal{O})$.

Set

$$H^1_\#(\omega) \doteq \Big\{\overline{V} \in H^1(\omega) \;\Big|\; \mathcal{M}_\omega(\overline{V}) = 0\Big\},$$
$$L^2(\mathcal{O}; H^1_\#(\omega)) \doteq \Big\{\overline{V} \in L^2(\mathcal{O}; H^1(\omega)) \;\Big|\; \overline{V}(x', \cdot) \in H^1_\#(\omega) \quad \text{for a.e. } x' \text{ in } \mathcal{O}\Big\}.$$

We endow the space $L^2(\mathcal{O}; H^1_\#(\omega))$ with the semi-norm

$$\|\overline{V}\|_{LH_\#} = \big\|\nabla_{X''}\overline{V}\|_{L^2(\Omega)}.$$

Lemma 4.4. *The semi-norm $\|\cdot\|_{LH_\#}$ is a norm equivalent to the usual norm of $L^2(\mathcal{O}; H^1(\omega))$.*

Proof. We first have

$$\|\overline{V}\|^2_{LH_\#} = \|\nabla_{X''}\overline{V}\|^2_{L^2(\Omega)} \leq \|\overline{V}\|^2_{L^2(\Omega)} + \|\nabla_{X''}\overline{V}\|^2_{L^2(\Omega)} = \|\overline{V}\|^2_{L^2(\mathcal{O};H^1(\omega))}.$$

Now, observe that

$$\begin{aligned}\|\overline{V}\|^2_{L^2(\Omega)} &= \int_\Omega \left|\overline{V}(x', X'')\right|^2 dx' dX'' \\ &= \int_\Omega \left|\overline{V}(x', X'') - \mathcal{M}_\omega(\overline{V})(x')\right|^2 dx' dX''.\end{aligned}$$

Then, the Poincaré-Wirtinger inequality applied in the cross-sections leads to

$$\int_\omega \left|\overline{V}(x', X'') - \mathcal{M}_\omega(\overline{V})(x')\right|^2 dX'' \leq C \int_\omega \left|\nabla_{X''}\overline{V}(x', X'')\right|^2 dX'' \quad \text{for a.e. } x' \in \mathcal{O}.$$

Finally integrating with respect to x' yields

$$\|\overline{V}\|_{L^2(\Omega)} \leq C\|\nabla_{X''}\overline{V}\|_{L^2(\Omega)} \tag{4.14}$$

and thus

$$\|\overline{V}\|_{L^2(\mathcal{O};H^1(\omega))} \leq C\|\overline{V}\|_{LH_\#}.$$

This ends the proof of the lemma. □

Lemma 4.5. *There exist a subsequence of $\{\delta\}$, still denoted $\{\delta\}$, $\mathcal{U} \in H^1_\gamma(\mathcal{O})$ and $\overline{U} \in L^2(\mathcal{O}; H^1_\#(\omega))$ such that*

$$\begin{aligned}&\mathcal{U}_\delta \rightharpoonup \mathcal{U} && \text{weakly in } H^1_\gamma(\mathcal{O}), \\ &\frac{1}{\delta}\overline{U}_\delta \rightharpoonup \overline{U} && \text{weakly in } L^2(\mathcal{O}; H^1_\#(\omega)), \\ &\nabla_{x'}\overline{U}_\delta \rightharpoonup 0 && \text{weakly in } L^2(\Omega)^m.\end{aligned} \tag{4.15}$$

Proof. The convergences (4.15) are the consequences of the estimates (4.13). □

As a consequence of the above lemma we obtain the following corollary:

Corollary 4.6. *For the same subsequence of $\{\delta\}$, we have*

$$\Pi_\delta(u_\delta) \rightharpoonup \mathcal{U} \qquad \text{weakly in } H^1(\Omega).$$

5 Limit problems

Theorem 5.1. *Let u_δ be the solution to problem* (4.6) *decomposed as* (4.8) *($\mathcal{U}_\delta \in H^1_\gamma(\mathcal{O})$, $\overline{u}_\delta \in H^1_{\Gamma_\delta}(\Omega_\delta)$). For the whole sequence $\{\delta\}$ we have*

$$\begin{aligned}&\mathcal{U}_\delta \to \mathcal{U} && \text{strongly in } H^1_\gamma(\mathcal{O}), \\ &\frac{1}{\delta}\overline{U}_\delta \to \overline{U} && \text{strongly in } L^2(\mathcal{O}; H^1_\#(\omega)), \\ &\nabla_{x'}\overline{U}_\delta \to 0 && \text{strongly in } L^2(\Omega)^m\end{aligned} \tag{5.1}$$

where $\overline{U}_\delta = \Pi_\delta(\overline{u}_\delta)$.

The couple $(\mathcal{U}, \overline{U}) \in H^1_\gamma(\mathcal{O}) \times L^2(\mathcal{O}; H^1_\#(\omega))$ *is the unique solution to the following problem:*

$$\begin{cases} \displaystyle\int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \nabla_{X''}\overline{U} \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}\phi \\ \nabla_{X''}\overline{\Psi} \end{pmatrix} dx'dX'' = \int_\Omega F\phi\, dx'dX'' = |\omega| \int_{\mathcal{O}} \mathcal{M}_\omega(F)\phi\, dx', \\ \forall (\phi, \overline{\Psi}) \in H^1_\gamma(\mathcal{O}) \times L^2(\mathcal{O}; H^1_\#(\omega)). \end{cases} \tag{5.2}$$

Proof. Step 1. We show (5.2).

Let ϕ be in $H^1_\gamma(\mathcal{O})$ and $\overline{\Psi}$ in $\mathcal{C}^1(\overline{\Omega})$ such that $\overline{\Psi}$ vanishes in a neighborhood of $\partial\mathcal{O}\times\omega$.

We take $\phi_\delta = \phi + \overline{\Psi}_\delta$, where $\overline{\Psi}_\delta(x', x'') = \delta\overline{\Psi}\Big(x', \dfrac{x''}{\delta}\Big)$, as a test function in problem (4.6).

We have

$$\begin{aligned} \Pi_\delta(\phi_\delta) &\longrightarrow \phi \qquad \text{strongly in } H^1(\Omega), \\ \frac{1}{\delta}\Pi_\delta(\overline{\Psi}_\delta) = \overline{\Psi}, &\qquad \nabla_{x'}\Pi_\delta(\overline{\Psi}_\delta) = \delta\nabla_{x'}\overline{\Psi}. \end{aligned} \tag{5.3}$$

We transform (4.6) using Π_δ. This gives

$$\int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta + \nabla_{x'}\overline{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}\phi + \nabla_{x'}\Pi_\delta(\overline{\Psi}_\delta) \\ \dfrac{1}{\delta}\nabla_{X''}\Pi_\delta(\overline{\Psi}_\delta) \end{pmatrix} dx'dX'' = \int_\Omega \Pi_\delta(f_\delta)\Pi_\delta(\phi_\delta) dx'dX''.$$

We pass to the limit ($\delta \to 0$) using the convergences (4.11)-(4.15) and (5.3). We obtain (5.2) with $(\phi, \overline{\Psi})$ as a test function. By density, in $L^2(\mathcal{O}; H^1_\#(\omega))$, of the test functions belonging to $\mathcal{C}^1(\overline{\Omega})$ and vanishing in a neighborhood of $\partial\mathcal{O}\times\omega$, we obtain (5.2) for any function $(\phi, \overline{\Psi}) \in H^1_\gamma(\mathcal{O}) \times L^2(\mathcal{O}; H^1_\#(\omega))$. The coercivity assumption (4.7) and the Lax-Milgram theorem give the existence and uniqueness of the solution to problem (5.2). The uniqueness of the solution implies that the whole sequences converge to their limits.

Step 2. We show the strong convergences (5.1).

We chose $v = u_\delta$ as the test function in (4.6) then transform equality using Π_δ. This gives

$$\int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta + \nabla_{x'}\overline{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}\mathcal{U} + \nabla_{x'}\overline{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix} dx'dX'' = \int_\Omega \Pi_\delta(f_\delta)\mathcal{U}_\delta dx'dX''.$$

Then, the weak convergences (4.15)-(4.11) and the weak lower semi-continuity of the LHS we obtain

$$\begin{aligned} &\int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \nabla_{X''}\overline{U} \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \nabla_{X''}\overline{U} \end{pmatrix} dx'dX'' \\ &\le \liminf \Big(\int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta + \nabla_{x'}\overline{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}\mathcal{U} + \nabla_{x'}\overline{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix} dx'dX'' \Big) \\ &\le \limsup \Big(\int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta + \nabla_{x'}\overline{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}\mathcal{U} + \nabla_{x'}\overline{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix} dx'dX'' \Big) \\ &= \limsup \Big(\int_\Omega \Pi_\delta(f_\delta)\mathcal{U}_\delta dx'dX'' \Big) = \lim \Big(\int_\Omega \Pi_\delta(f_\delta)\mathcal{U}_\delta dx'dX'' \Big) = \int_\Omega F\,\mathcal{U} dx'dX''. \end{aligned}$$

Thus, the above inequalities are equalities and therefore we have

$$\begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta + \nabla_{x'}\overline{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix} \to \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \nabla_{X''}\overline{U} \end{pmatrix} \quad \text{strongly in } L^2(\Omega)^N.$$

Hence

$$\nabla_{x'}\mathcal{U}_\delta = \mathcal{M}_\omega\big(\nabla_{x'}\mathcal{U}_\delta + \nabla_{x'}\overline{U}_\delta\big) \to \nabla_{x'}\mathcal{U} \quad \text{strongly in } L^2(\Omega)^m,$$
$$\nabla_{x'}\overline{U}_\delta \to 0 \quad \text{strongly in } L^2(\Omega)^m,$$
$$\frac{1}{\delta}\nabla_{X''}\overline{U}_\delta \to \nabla_{X''}\overline{U} \quad \text{strongly in } L^2(\Omega)^m.$$

The strong convergences above as well as (4.14)-(4.15) give those in (5.1). □

Corollary 5.2. *We have*

$$\Pi_\delta(u_\delta) \to \mathcal{U} \quad \textit{strongly in } H^1(\Omega).$$

In order to obtain the problem satisfied by $\mathcal{U}$, we introduce m correctors $\overline{\chi}_i$, $1 \le i \le m$. They are are the solutions to

$$\begin{cases} \overline{\chi}_i \in H^1_\#(\omega), \quad i \in \{1,\dots,m\}, \\ \displaystyle\int_\omega A\begin{pmatrix} \mathbf{e}_i \\ \nabla_{X''}\overline{\chi}_i \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X''}\psi \end{pmatrix} dX'' = 0, \qquad \forall \psi \in H^1_\#(\omega) \end{cases} \tag{5.4}$$

where $(\mathbf{e}_1,\dots,\mathbf{e}_m)$ is the usual basis of $\mathbb{R}^m$.

Theorem 5.3. *We have*

$$\overline{U}(x',X'') = \sum_{i=1}^m \frac{\partial \mathcal{U}}{\partial x'_i}(x')\overline{\chi}_i(X'') \qquad \textit{for a.e. } (x',X'') \in \mathcal{O}\times\omega. \tag{5.5}$$

The function $\mathcal{U}$ is the unique solution to the following problem:

$$\begin{cases} \textit{Find } \mathcal{U} \in H^1_\gamma(\mathcal{O}) \textit{ such that} \\ \displaystyle\int_{\mathcal{O}} A^*\nabla\mathcal{U}\cdot\nabla\phi\,dx' = \int_{\mathcal{O}} \mathcal{M}_\omega(F)\,\phi\,dx', \\ \forall \phi \in H^1_\gamma(\mathcal{O}), \end{cases} \tag{5.6}$$

where the entries of the matrix $A^* = (a^*_{ij})_{1\le i,\,j\le m}$ are given by

$$a^*_{ij} = \frac{1}{|\omega|}\int_\omega A\begin{pmatrix} \mathbf{e}_i \\ \nabla_{X''}\overline{\chi}_i \end{pmatrix} \cdot \begin{pmatrix} \mathbf{e}_j \\ \nabla_{X''}\overline{\chi}_j \end{pmatrix} dX''. \tag{5.7}$$

Proof. Step 1. We express $\overline{U}$.

In (5.2) we take $\phi = 0$. Using (5.4), this leads to

$$\begin{aligned} &\int_\omega A(X'')\begin{pmatrix} 0 \\ \nabla_{X''}\overline{U}(x',X'') \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X''}\overline{\Psi}(X'') \end{pmatrix} dX'' \\ &= -\int_\omega A(X'')\begin{pmatrix} \nabla_{x'}\mathcal{U}(x') \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X''}\overline{\Psi}(X'') \end{pmatrix} dX'' \\ &= -\sum_{i=1}^m \frac{\partial \mathcal{U}}{\partial x'_i}\int_\omega A(X'')\begin{pmatrix} \mathbf{e}_i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X''}\overline{\Psi}(X'') \end{pmatrix} dX'' \\ &= \sum_{i=1}^m \frac{\partial \mathcal{U}}{\partial x'_i}\int_\omega A(X'')\begin{pmatrix} 0 \\ \nabla_{X''}\overline{\chi}_i(X'') \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X''}\overline{\Psi}(X'') \end{pmatrix} dX'' \\ &\qquad \text{for a.e. } x' \text{ in } \mathcal{O}, \quad \forall \overline{\Psi} \in H^1_\#(\omega). \end{aligned}$$

Thus

$$\int_\omega A\begin{pmatrix} 0 \\ \nabla_{X''}\Big(\overline{U}-\sum_{i=1}^m \dfrac{\partial \mathcal{U}}{\partial x_i'}\overline{\chi}_i\Big)\end{pmatrix}\cdot\begin{pmatrix}0\\ \nabla_{X''}\overline{\Psi}\end{pmatrix}dX''=0 \quad \text{a.e. in } \mathcal{O}, \qquad \forall \overline{\Psi}\in H^1_{\#}(\omega).$$

Hence

$$\overline{U}=\sum_{i=1}^m \frac{\partial \mathcal{U}}{\partial x_i'}\overline{\chi}_i.$$

Step 2. We give the entries of the matrix A^*.

In (5.2), we take $\overline{\Psi}=\sum_{i=1}^m \dfrac{\partial \phi}{\partial x_i}\overline{\chi}_i$. We obtain

$$\int_\Omega A\begin{pmatrix}\nabla_{x'}\mathcal{U}\\ \nabla_{X''}\overline{U}\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x'}\phi\\ \nabla_{X''}\overline{\Psi}\end{pmatrix}dx'dX''=\int_\Omega F\phi\,dx'dX''.$$

This gives

$$\begin{aligned}\int_\Omega \frac{1}{|\omega|}A\begin{pmatrix}\sum_{i=1}^m \dfrac{\partial \mathcal{U}}{\partial x_i}\mathbf{e}_i\\ \sum_{i=1}^m \dfrac{\partial \mathcal{U}}{\partial x_i}\nabla_{X''}\overline{\chi}_i\end{pmatrix}\cdot\begin{pmatrix}\sum_{j=1}^m \dfrac{\partial \phi}{\partial x_j}\mathbf{e}_j\\ \sum_{j=1}^m \dfrac{\partial \phi}{\partial x_j}\nabla_{X''}\overline{\chi}_j\end{pmatrix}dx'dX''&=\int_\Omega \frac{1}{|\omega|}F(x',X'')\phi(x')dx'dX''\\ &=\int_{\mathcal{O}}\mathcal{M}_\omega(F)\,\phi dx'.\end{aligned}$$

The above LHS is also equal to

$$\sum_{i=1}^m\sum_{j=1}^m\int_\Omega A\begin{pmatrix}\mathbf{e}_i\\ \nabla_{X''}\overline{\chi}_i\end{pmatrix}\cdot\begin{pmatrix}\mathbf{e}_j\\ \nabla_{X''}\overline{\chi}_j\end{pmatrix}\frac{\partial \mathcal{U}}{\partial x_i}\frac{\partial \phi}{\partial x_j}dx'dX''.$$

Hence, we obtain

$$\int_{\mathcal{O}}\sum_{i=1}^m\sum_{j=1}^m a_{ij}^*\frac{\partial \mathcal{U}}{\partial x_i}\frac{\partial \phi}{\partial x_j}dx'=\int_{\mathcal{O}}\mathcal{M}_\omega(F)\phi dx' \tag{5.8}$$

where the a_{ij}^* are given by (5.7).

Step 3. We prove the coercivity of the matrix A^*.

For every $\xi\in\mathbb{R}^m$, we have

$$\begin{aligned}\sum_{i=1}^m\sum_{j=1}^m a_{ij}^*\xi_i\xi_j&=\sum_{i=1}^m\sum_{j=1}^m\frac{1}{|\omega|}\int_\omega A\xi_i\begin{pmatrix}\mathbf{e}_i\\ \nabla_{X''}\overline{\chi}_i\end{pmatrix}\cdot\xi_j\begin{pmatrix}\mathbf{e}_j\\ \nabla_{X''}\overline{\chi}_j\end{pmatrix}dX'',\\ &=\frac{1}{|\omega|}\int_\omega A\begin{pmatrix}\xi\\ \sum_{i=1}^m \xi_i\nabla_{X''}\overline{\chi}_i\end{pmatrix}\cdot\begin{pmatrix}\xi\\ \sum_{j=1}^m \xi_j\nabla_{X''}\overline{\chi}_j\end{pmatrix}dX''.\end{aligned}$$

The coercivity of the matrix A (see (4.7)) yields

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}^{*}\xi_i\xi_j \geq \frac{1}{|\omega|}\int_{\omega} c\big(|\xi|^2 + \sum_{i=1}^{m}|\xi_i|^2|\nabla_{X''}\overline{\chi}_i|^2\big)dX'' \geq c|\xi|^2.$$

This proves the coercivity of the matrix A^*. □

From now on, we assume that

$$\begin{aligned} &\bullet \quad \partial\mathcal{O} \textit{ of class } C^{1,1},\\ &\bullet \quad f_\delta = f \in L^2(\mathcal{O}),\\ &\bullet \quad \gamma = \partial\mathcal{O} \quad \textit{homogeneous lateral Dirichlet condition},\\ &\quad \textit{or } \gamma = \phi \quad \textit{homogeneous Neumann condition} \end{aligned} \tag{5.9}$$

If $\gamma = \phi$ we further assume that

$$\mathcal{M}_{\mathcal{O}}(f) = 0. \tag{5.10}$$

6 Global error estimates

6.1 Preliminary lemmas

Now, problem 4.6 becomes

$$\begin{cases} \text{Find } u_\delta \in H^1_{\Gamma_\delta}(\Omega_\delta),\\ \displaystyle\int_{\Omega_\delta} A_\delta \nabla u_\delta \cdot \nabla v\, dx = \delta^n|\omega| \int_{\mathcal{O}} f\,\mathcal{V}\, dx,\\ \forall v \in H^1_{\Gamma_\delta}(\Omega_\delta), \quad v \text{ decomposed as (3.5)}, \;\; v = \mathcal{V} + \overline{v}. \end{cases} \tag{6.1}$$

and now, the estimate (4.9) is

$$\|u_\delta\|_{H^1(\Omega_\delta)} \leq C\|f_\delta\|_{L^2(\Omega_\delta)} = C\delta^{\frac{n}{2}}\|f\|_{L^2(\mathcal{O})}. \tag{6.2}$$

Lemma 6.1. *Under the assumptions* (5.9)-(5.10) *we have $\mathcal{U} \in H^2(\mathcal{O}) \cap H^1_\gamma(\mathcal{O})$ and*

$$\|\mathcal{U}\|_{H^2(\mathcal{O})} \leq C\|f\|_{L^2(\mathcal{O})}. \tag{6.3}$$

Proof. In the case of the lateral homogeneous Dirichlet condition, we apply [11, Theorem 9.15] to obtain $\mathcal{U} \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$and [11, Lemma 9.17] to obtain the estimate (6.3). In both cases we also refer to [3, Section 9.6]. □

Denote

- $\rho(x') = dist(x', \partial\mathcal{O})$ the distance between $x' \in \mathcal{O}$ and $\partial\mathcal{O}$.
- ρ_δ the function defined by

$$\rho_\delta(x') = \inf\Big\{\frac{\rho(x')}{\delta}, 1\Big\} \qquad \forall x' \in \mathcal{O}.$$

- $\mathcal{O}_\delta$ the open set

$$\mathcal{O}_\delta = \{x' \in \mathcal{O} \mid \rho(x') < \delta\}.$$

Note that we have

$$\begin{cases} 0 \le \rho_\delta(x') \le 1, & \text{for a.e. } x' \text{ in } \mathcal{O}, \\ \rho_\delta(x') = 1, & \text{for a.e. } x' \text{ in } \mathcal{O} \setminus \overline{\mathcal{O}_\delta}, \\ \|\nabla_{x'}\rho_\delta\|_{L^\infty(\mathcal{O})} = \dfrac{1}{\delta}. \end{cases}$$

Lemma 6.2. *For any $\psi \in H^1(\mathcal{O})$, we have*

$$\begin{aligned} &\|\psi\|_{L^2(\mathcal{O}_\delta)} \le C\delta^{1/2}\|\psi\|_{H^1(\mathcal{O})}, \\ &\|(1-\rho_\delta)\psi\|_{L^2(\mathcal{O})} \le C\delta^{\frac{1}{2}}\|\psi\|_{H^1(\mathcal{O})}, \\ &\|\rho_\delta\psi\|_{H^1(\mathcal{O})} \le C\delta^{-\frac{1}{2}}\|\psi\|_{H^1(\mathcal{O})}. \end{aligned} \tag{6.4}$$

Proof. Step 1. We prove $(6.4)_1$.
We first recall that since the boundary of $\mathcal{O}$ is smooth, the trace theorem gives (see [3, Chapter 9])

$$\|\psi\|_{L^2(\partial\mathcal{O})} \le C\|\psi\|_{H^1(\mathcal{O})}.$$

Then, from [6, Proposition12.2] we get

$$\|\psi\|_{L^2(\mathcal{O}_\delta)} \le C\big(\delta^{1/2}\|\psi\|_{L^2(\partial\mathcal{O})} + \delta\|\nabla\psi\|_{L^2(\mathcal{O})}\big) \le C\delta^{1/2}\|\psi\|_{H^1(\mathcal{O})},$$

from which $(6.4)_{1,2}$ follow since

$$\|(1-\rho_\delta)\psi\|_{L^2(\mathcal{O})} \le \|\psi\|_{L^2(\mathcal{O}_\delta)} \le C\delta^{\frac{1}{2}}\|\psi\|_{H^1(\mathcal{O})}.$$

Step 2. We prove $(6.4)_3$.
Due to $(6.4)_1$ we have

$$\begin{aligned} \|\nabla_{x'}(\rho_\delta\psi)\|_{L^2(\mathcal{O})} &\le \|(\nabla_{x'}\rho_\delta)\psi\|_{L^2(\mathcal{O})} + \|\rho_\delta\nabla_{x'}\psi\|_{L^2(\mathcal{O})} \\ &\le \|\nabla_{x'}\rho_\delta\|_{L^\infty(\mathcal{O}_\delta)}\|\psi\|_{L^2(\mathcal{O}_\delta)} + \|\nabla_{x'}\psi\|_{L^2(\mathcal{O})} \\ &\le \frac{1}{\delta}\|\psi\|_{L^2(\mathcal{O}_\delta)} + \|\nabla_{x'}\psi\|_{L^2(\mathcal{O})} \\ &\le C\delta^{-\frac{1}{2}}\|\psi\|_{H^1(\mathcal{O})} \end{aligned}$$

and

$$\|\rho_\delta\psi\|_{L^2(\mathcal{O})} \le \|\rho_\delta\|_{L^\infty(\mathcal{O}_\delta)}\|\psi\|_{L^2(\mathcal{O}_\delta)} \le \|\psi\|_{L^2(\mathcal{O}_\delta)} \le C\delta^{\frac{1}{2}}\|\psi\|_{H^1(\mathcal{O})}.$$

This ends the proof of $(6.4)_3$. □

We define, u_δ^{ap}, an approximated solution to problem (6.1) by

$$\begin{aligned} &u_\delta^{ap}(x) = \mathcal{U}(x') + \overline{u}_\delta^{ap}(x), \\ &\overline{u}_\delta^{ap}(x) = \delta\sum_{i=1}^{m}\rho_\delta(x')\frac{\partial\mathcal{U}}{\partial x_i}(x')\overline{\chi}_i\Big(\frac{x''}{\delta}\Big)^1, \qquad \text{for a.e. } x \in \Omega_\delta. \end{aligned} \tag{6.5}$$

Since $\mathcal{U} \in H^2(\Omega) \cap H^1_\gamma(\mathcal{O})$ the function u_δ^{ap} belongs to $H^1_{\Gamma_\delta}(\Omega_\delta)$.

[1] In the case of the homogeneous lateral Dirichlet boundary condition, we need to multiply the function

Theorem 6.3. *Under the assumptions* (5.9)-(5.10) *we have*

$$\|u_\delta - u_\delta^{ap}\|_{H^1(\Omega_\delta)} \le C\delta^{\frac{1}{2}+\frac{n}{2}}\|f\|_{L^2(\mathcal{O})}. \tag{6.6}$$

The constant depends on the matrix A and $\displaystyle\sum_{i=1}^{m}\|\overline{\chi}_i\|_{H^1(\omega)}$.

Proof. The essential part of the proof of estimate (6.6) consists in calculating and estimating the quantity

$$\int_{\Omega_\delta} A_\delta \nabla(u_\delta - u_\delta^{ap})\cdot\nabla\phi\, dx, \qquad \forall\phi\in H^1_{\Gamma_\delta}(\Omega_\delta).$$

We have for a.e. $x\in\Omega_\delta$

$$\nabla u_\delta^{ap}(x) = \underbrace{\begin{pmatrix} \nabla_{x'}\mathcal{U}(x') \\ \displaystyle\sum_{i=1}^{m}\frac{\partial\mathcal{U}}{\partial x_i}(x')\ \nabla_{X''}\overline{\chi}_i\big(\frac{x''}{\delta}\big)\end{pmatrix}}_{\mathbf{D}_I} + \underbrace{\begin{pmatrix} \displaystyle\sum_{i=1}^{m}\delta\nabla_{x'}\Big(\rho_\delta\frac{\partial\mathcal{U}}{\partial x_i}\Big)(x')\ \overline{\chi}_i\big(\frac{x''}{\delta}\big) \\ \displaystyle\sum_{i=1}^{m}\big(\rho_\delta(x')-1\big)\frac{\partial\mathcal{U}}{\partial x_i}(x')\ \nabla_{X''}\overline{\chi}_i\big(\frac{x''}{\delta}\big)\end{pmatrix}}_{\mathbf{D}_{II}}. \tag{6.7}$$

Step 1. In this step we show

$$\Big|\int_{\Omega_\delta} A_\delta\big(\nabla u_\delta^{ap} - \mathbf{D}_I\big)\cdot\nabla\phi\, dx\Big| \le C\delta^{\frac{1}{2}+\frac{n}{2}}\|\|f\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^2(\Omega_\delta)}, \qquad \forall\phi\in H^1_{\Gamma_\delta}(\Omega_\delta). \tag{6.8}$$

First, we estimate $\mathbf{D}_{II}$.

Due to Lemma 4.1 and (3.1) we have

$$\begin{aligned}
&\|\mathbf{D}_{II}\|_{L^2(\Omega_\delta)}\\
&\le \sum_{i=1}^{m}\Big(\delta\Big\|\nabla_{x'}\Big(\rho_\delta\frac{\partial\mathcal{U}}{\partial x_i}\Big)\Big\|_{L^2(\mathcal{O})}\|\overline{\chi}_i\Big(\frac{\cdot}{\delta}\Big)\|_{L^2(\omega_\delta)} + \Big\|(\rho_\delta-1)\frac{\partial\mathcal{U}}{\partial x_i}\Big\|_{L^2(\mathcal{O})}\|\nabla_{X''}\overline{\chi}_i\Big(\frac{\cdot}{\delta}\Big)\|_{L^2(\omega_\delta)}\Big)\\
&\le \sum_{i=1}^{m}\Big(\delta\Big\|\nabla_{x'}\Big(\rho_\delta\frac{\partial\mathcal{U}}{\partial x_i}\Big)\Big\|_{L^2(\mathcal{O})}\delta^{\frac{n}{2}}\|\overline{\chi}_i\|_{L^2(\omega)} + \Big\|(\rho_\delta-1)\frac{\partial\mathcal{U}}{\partial x_i}\Big\|_{L^2(\mathcal{O})}\delta^{\frac{n}{2}}\|\nabla_{X''}\overline{\chi}_i\|_{L^2(\omega)}\Big).
\end{aligned}$$

Then, the estimates $(6.4)_{2,3}$ and (6.3) lead to

$$\|\mathbf{D}_{II}\|_{L^2(\Omega_\delta)} \le C\delta^{\frac{1}{2}+\frac{n}{2}}\|\mathcal{U}\|_{H^2(\mathcal{O})}\Big(\sum_{i=1}^{m}\|\overline{\chi}_i\|_{H^1(\omega)}\Big) \le C\delta^{\frac{1}{2}+\frac{n}{2}}\|f\|_{L^2(\mathcal{O})}. \tag{6.9}$$

As a consequence of the above and (6.3), we obtain (6.8).

Step 2. In this step, we apply the dimension reduction operator Π_δ.

$\displaystyle\sum_{i=1}^{m}\frac{\partial\mathcal{U}}{\partial x_i}(x')\overline{\chi}_i\Big(\frac{x''}{\delta}\Big)$ by $\rho_\delta(x')$ to construct an approximated solution which belongs to $H^1_{\Gamma_\delta}(\Omega_\delta)$. In the Neumann case, there is no need to multiply by $\rho_\delta(x')$; the reader can reproduce the proofs of the theorems in this section and the next, and see that the results are the same. The proofs will be a little simpler in this case. Here, we have chosen to use the same approximated solution to avoid 2 proofs.

Let ϕ be in $H^1_{\Gamma_\delta}(\Omega_\delta)$, Lemma 3.5 gives $\Phi \in H^1_\gamma(\mathcal{O})$ and $\overline{\phi} \in H^1_{\Gamma_\delta}(\Omega_\delta)$ such that

$$\begin{aligned}
&\phi = \Phi + \delta\overline{\phi},\\
&\big\|\nabla_{x'}\Phi\big\|_{L^2(\mathcal{O})} \le \frac{1}{|\omega|^{\frac12}\delta^{\frac n2}}\|\nabla_{x'}\phi\|_{L^2(\Omega_\delta)},\\
&\|\overline{\phi}\|_{L^2(\Omega_\delta)} \le C\|\nabla_{x''}\phi\|_{L^2(\Omega_\delta)},\\
&\|\nabla_{x'}\overline{\phi}\|_{L^2(\Omega_\delta)} \le \frac{2}{\delta}\|\nabla_{x'}\phi\|_{L^2(\Omega_\delta)}, \qquad \|\nabla_{x''}\overline{\phi}\|_{L^2(\Omega_\delta)} = \frac{1}{\delta}\|\nabla_{x''}\phi\|_{L^2(\Omega_\delta)}.
\end{aligned} \tag{6.10}$$

Set $\overline{\Phi} = \Pi_\delta(\overline{\phi})$, we have $\overline{\Phi} \in H^1(\Omega)$ and from the above estimates (6.10) and Proposition 3.2 we have

$$\begin{aligned}
&\|\overline{\Phi}\|_{L^2(\Omega)} \le C\delta^{-n/2}\|\nabla_{x''}\phi\|_{L^2(\Omega_\delta)},\\
&\|\nabla_{x'}\overline{\Phi}\|_{L^2(\Omega)} \le 2\delta^{-1-n/2}\|\nabla_{x'}\phi\|_{L^2(\Omega_\delta)}, \qquad \|\nabla_{X''}\overline{\Phi}\|_{L^2(\Omega)} = \delta^{-n/2}\|\nabla_{x''}\phi\|_{L^2(\Omega_\delta)}.
\end{aligned} \tag{6.11}$$

Due to Proposition 3.2 and problem (5.2) we obtain

$$\int_{\Omega_\delta} A_\delta \mathbf{D}_I\cdot\nabla\phi\,dx = \delta^n\int_\Omega A\begin{pmatrix}\nabla_{x'}\mathcal{U}\\ \nabla_{X''}\overline{U}\end{pmatrix}\cdot\Pi_\delta(\nabla\phi)dx'dX'', \qquad \forall\phi\in H^1_{\Gamma_\delta}(\Omega_\delta).$$

Estimate $(8.1)_2$ in Lemma 8.1 (see Appendix), the fact that $\mathcal{U}$ belongs to $H^2(\mathcal{O})$ and $(3.7)_4$ imply that

$$\begin{aligned}
&\Big|\int_\Omega A\begin{pmatrix}\nabla_{x'}\mathcal{U}\\ \nabla_{X''}\overline{U}\end{pmatrix}\cdot\Big(\Pi_\delta(\nabla\phi) - \begin{pmatrix}\nabla_{x'}\Phi\\ \nabla_{X''}\overline{\Phi}\end{pmatrix}\Big)dx'dX''\Big|\\
&\qquad\le C\|A\|_{L^\infty(\omega)}\|\mathcal{U}\|_{H^2(\mathcal{O})}\Big(\sum_{i=1}^m\|\overline{\chi}_i\|_{H^1(\omega)}\Big)\big(\delta^{1-\frac n2}\|\nabla\phi\|_{L^2(\Omega_\delta)} + \delta^{\frac12-\frac n2}\|\nabla\phi\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)}\big)\\
&\qquad\le C\delta^{\frac12-\frac n2}\|f\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^2(\Omega_\delta)}.
\end{aligned}$$

Then, from the above we get

$$\begin{aligned}
&\Big|\int_{\Omega_\delta} A_\delta\mathbf{D}_I\cdot\nabla\phi\,dx - \delta^n\int_\Omega A\begin{pmatrix}\nabla_{x'}\mathcal{U}\\ \nabla_{X''}\overline{U}\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x'}\Phi\\ \nabla_{X''}\overline{\Phi}\end{pmatrix}dx'dX''\Big|\\
&\qquad\le C\delta^{\frac12+\frac n2}\|f\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^2(\Omega_\delta)}.
\end{aligned} \tag{6.12}$$

So, the above estimate and (5.2) give

$$\Big|\int_{\Omega_\delta} A_\delta\mathbf{D}_I\cdot\nabla\phi\,dx - \int_{\Omega_\delta} f\phi\,dx\Big| \le C\delta^{\frac12+\frac n2}\|f\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^2(\Omega_\delta)}.$$

Therefore, using (6.8)

$$\Big|\int_{\Omega_\delta} A_\delta(\nabla u_\delta - \nabla u_\delta^{ap})\cdot\nabla\phi\,dx\Big| \le C\delta^{\frac12+\frac n2}\|f\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^2(\Omega_\delta)}. \tag{6.13}$$

Step 3. We prove (6.6).

We take $\phi = u_\delta - u_\delta^{ap}$ as a test function in (6.13) and then, due to the coercivity of the matrix A_δ (see (4.7)), we deduce that

$$\|\nabla(u_\delta - u_\delta^{ap})\|_{L^2(\Omega_\delta)} \le C\delta^{\frac12+\frac n2}\|f\|_{L^2(\mathcal{O})}$$

which, using the fact that $u_\delta - u_\delta^{ap}$ belongs to $H^1_{\Gamma_\delta}(\Omega_\delta)$, leads to (6.6). □

Remark 6.4. *In the proof of Theorem 6.3, we only used the fact that $\mathcal{U} \in H^2(\mathcal{O})$, which means that if we have $\partial\mathcal{O}$ Lipschitz, $\mathcal{U} \in H^2(\mathcal{O})$ and γ a measurable subset of $\partial\mathcal{O}$ then, the estimate* (6.6) *still holds.*

7 L^2 global error and H^1 interior error estimates

7.1 L^2 global error estimate

First, we introduce a second family of correctors, the $\widetilde{\chi}_i$[2], $1 \leq i \leq m$, defined by

$$\begin{cases} \widetilde{\chi}_i \in H^1_{\#}(\omega), \quad i \in \{1, \dots, m\}, \\ \displaystyle\int_\omega A \begin{pmatrix} 0 \\ \nabla_{X''}\psi \end{pmatrix} \cdot \begin{pmatrix} \mathbf{e}_i \\ \nabla_{X''}\widetilde{\chi}_i \end{pmatrix} dX'' = 0, \qquad \forall \psi \in H^1_{\#}(\omega) \end{cases} \tag{7.1}$$

where $(\mathbf{e}_1, \dots, \mathbf{e}_m)$ is the usual basis of $\mathbb{R}^m$.

Theorem 7.1. *Under the assumptions* (5.9)-(5.10) *we have*

$$\begin{aligned} \|\mathcal{U}_\delta - \mathcal{U}\|_{L^2(\mathcal{O})} &\leq C\delta \|f\|_{L^2(\mathcal{O})}, \\ \|u_\delta - u_\delta^{ap}\|_{L^2(\Omega_\delta)} &\leq C\delta^{1+\frac{n}{2}} \|f\|_{L^2(\mathcal{O})}. \end{aligned} \tag{7.2}$$

The constants depend on the matrix A and of the H^1 norms of the two families of correctors.

Proof. Step 1. In this step we show that

$$\|\Pi_\delta(\nabla u_\delta)\|_{L^2(\mathcal{O}_\delta \times \omega)} \leq C\delta^{\frac{1}{2}} \|f\|_{L^2(\mathcal{O})}. \tag{7.3}$$

First, from (6.7)-(6.9) and the properties of Π_δ, we have

$$\left\| \Pi_\delta(\nabla u_\delta^{ap}) - \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \displaystyle\sum_{i=1}^m \frac{\partial \mathcal{U}}{\partial x_i} \nabla_{X''}\overline{\chi}_i \end{pmatrix} \right\|_{L^2(\Omega)} \leq C\delta^{\frac{1}{2}} \|f\|_{L^2(\mathcal{O})}.$$

Then, Lemmas 4.1-6.1-6.2 imply

$$\left\| \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \displaystyle\sum_{i=1}^m \frac{\partial \mathcal{U}}{\partial x_i} \nabla_{X''}\overline{\chi}_i \end{pmatrix} \right\|_{L^2(\mathcal{O}_\delta \times \omega)} \leq C \|\nabla_{x'}\mathcal{U}\|_{L^2(\mathcal{O}_\delta)} \Big(\sum_{i=1}^m \|\overline{\chi}_i\|_{H^1(\omega)} + 1 \Big) \leq C\delta^{\frac{1}{2}} \|f\|_{L^2(\mathcal{O})}.$$

As a consequence of Theorem 6.3 and the above, we get

$$\|\Pi_\delta(\nabla u_\delta)\|_{L^2(\mathcal{O}_\delta \times \omega)} \leq \|\Pi_\delta(\nabla u_\delta^{ap})\|_{L^2(\mathcal{O}_\delta \times \omega)} + \|\Pi_\delta(\nabla u_\delta - u_\delta^{ap})\|_{L^2(\Omega)} \leq C\delta^{\frac{1}{2}} \|f\|_{L^2(\mathcal{O})}.$$

Thus, (7.3) is proved.

We recall that the function u_δ is decomposed as (4.8).
Now, let Φ be in $H^2(\mathcal{O}) \cap H^1_\gamma(\mathcal{O})$. We set

$$\phi_\delta = \Phi + \widetilde{\phi}_\delta, \qquad \widetilde{\phi}_\delta(x) = \delta \sum_{i=1}^m \rho_\delta(x') \frac{\partial \Phi}{\partial x_i}(x') \widetilde{\chi}_i\Big(\frac{x''}{\delta}\Big), \quad \text{for a.e. } x \in \Omega_\delta. \tag{7.4}$$

[2] Observe that if A is a symmetric matrix then $\widetilde{\chi}_i = \overline{\chi}_i$.

Function ϕ_δ is an admissible test function (by construction, it belongs to $H^1_{\Gamma_\delta}(\Omega_\delta)$), we choose it in (4.6). This gives

$$\begin{aligned} &\int_{\Omega_\delta} A_\delta \nabla u_\delta \cdot \nabla \phi_\delta \, dx = \delta^n \big|\omega\big| \int_{\mathcal{O}} f\Phi \, dx' \\ \Longrightarrow \quad &\int_\Omega A\Pi_\delta(\nabla u_\delta) \cdot \Pi_\delta(\nabla \phi_\delta) \, dx' dX'' = \big|\omega\big| \int_{\mathcal{O}} f\Phi \, dx'. \end{aligned} \tag{7.5}$$

Step 2. In this step we show that

$$\Big| \int_\Omega A\Pi_\delta(\nabla u_\delta) \cdot \begin{pmatrix} \nabla_{x'}\Phi \\ \sum_{i=1}^m \dfrac{\partial \Phi}{\partial x_i} \nabla_{X''} \widetilde{\chi}_i \end{pmatrix} dx' dX'' - |\omega| \int_{\mathcal{O}} f\Phi \, dx' \Big| \le C\delta \|f\|_{L^2(\mathcal{O})} \|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}. \tag{7.6}$$

First, we show that

$$\Big| \int_\Omega A\Pi_\delta(\nabla u_\delta) \cdot \begin{pmatrix} \delta \sum_{i=1}^m \nabla_{x'} \Big(\rho_\delta \dfrac{\partial \Phi}{\partial x_i} \Big) \widetilde{\chi}_i \\ \sum_{i=1}^m (\rho_\delta - 1) \dfrac{\partial \Phi}{\partial x_i} \nabla_{X''} \widetilde{\chi}_i \end{pmatrix} dx' dX'' \Big| \le C\delta \|f\|_{L^2(\mathcal{O})} \|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}. \tag{7.7}$$

We have

$$\delta \nabla_{x'} \Big(\rho_\delta \frac{\partial \Phi}{\partial x_i} \Big) = \frac{\partial \Phi}{\partial x_i} \delta \nabla_{x'} \rho_\delta + \delta \rho_\delta \nabla_{x'} \Big(\frac{\partial \Phi}{\partial x_i} \Big).$$

Since the supports of $\dfrac{\partial \Phi}{\partial x_i} \delta \nabla_{x'} \rho_\delta$ and $(\rho_\delta - 1) \dfrac{\partial \Phi}{\partial x_i}$ are included in $\mathcal{O}_\delta$ and both functions $\delta \nabla_{x'} \rho_\delta$ and $1 - \rho_\delta$ are bounded by 1 in $L^\infty(\mathcal{O}_\delta)$, then proceeding as in Step 1 of the proof of Theorem 6.3 and due to the estimates $(6.4)_{1,2}$, (7.3) and (6.3), we obtain

$$\Big| \int_\Omega A\Pi_\delta(\nabla u_\delta) \cdot \begin{pmatrix} \Big(\sum_{i=1}^m \dfrac{\partial \Phi}{\partial x_i} \widetilde{\chi}_i \Big) \delta \nabla_{x'} \rho_\delta \\ \sum_{i=1}^m (\rho_\delta - 1) \dfrac{\partial \Phi}{\partial x_i} \nabla_{X''} \widetilde{\chi}_i \end{pmatrix} dx' dX'' \Big| \le C\delta \|f\|_{L^2(\mathcal{O})} \|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}. \tag{7.8}$$

We also have

$$\Big| \int_\Omega A\Pi_\delta(\nabla u_\delta) \cdot \begin{pmatrix} \delta \rho_\delta \sum_{i=1}^m \Big(\nabla_{x'} \dfrac{\partial \Phi}{\partial x_i} \Big) \widetilde{\chi}_i \Big) \\ 0 \end{pmatrix} dx' dX'' \Big| \le C\delta \|f\|_{L^2(\mathcal{O})} \|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}.$$

Hence, (7.7) is proved.

As a consequence of $(7.5)_2$ and (7.7) we get (7.6).

Step 3. In this step we show that

$$\begin{aligned} \Big| \int_\Omega A \begin{pmatrix} \nabla_{x'} \mathcal{U}_\delta \\ \dfrac{1}{\delta} \nabla_{X''} \Pi_\delta(\overline{u}_\delta) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}\Phi \\ \sum_{i=1}^m \dfrac{\partial \Phi}{\partial x_i} \nabla_{X''} \widetilde{\chi}_i \end{pmatrix} dx' dX'' - |\omega| \int_{\mathcal{O}} f\Phi \, dx' \Big| \\ \le C\delta \|f\|_{L^2(\mathcal{O})} \|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}. \end{aligned} \tag{7.9}$$

The estimate (8.4) (see Lemma 8.2 in Appendix) implies

$$\Big| \int_\Omega A\Big(\Pi_\delta(\nabla u_\delta) - \begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta \\ \frac{1}{\delta}\nabla_{X''}\Pi_\delta(\overline{u}_\delta)\end{pmatrix}\Big)\cdot\begin{pmatrix} \nabla_{x'}\Phi \\ \sum_{i=1}^m \frac{\partial\Phi}{\partial x_i}\nabla_{X''}\widetilde{\chi}_i\end{pmatrix} dx'dX''\Big| \le C\delta\|f\|_{L^2(\mathcal{O})}\|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}.$$

As a consequence of the above and (7.6) we obtain (7.9).

Step 4. In this step we show that

$$\begin{aligned}\Big| \int_\Omega A\begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta \\ \sum_{i=1}^m \frac{\partial\mathcal{U}_\delta}{\partial x_i}\nabla_{X''}\overline{\chi}_i\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x'}\Phi \\ 0\end{pmatrix} dx'dX'' - |\omega|\int_{\mathcal{O}} f\Phi\, dx'\Big| \\ \le C\delta\|f\|_{L^2(\mathcal{O})}\|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}.\end{aligned} \tag{7.10}$$

From the definition of the correctors $\widetilde{\chi}_i$ (see (7.1)), we get

$$\begin{aligned}&\int_\Omega A\begin{pmatrix}0\\ \frac{1}{\delta}\nabla_{X''}\Pi_\delta(\overline{u}_\delta)\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x'}\Phi\\ \sum_{i=1}^m \frac{\partial\Phi}{\partial x_i}\nabla_{X''}\widetilde{\chi}_i\end{pmatrix} dx'dX'' = 0\\ \text{and}\quad &\int_\Omega A\begin{pmatrix}0\\ \sum_{i=1}^m \frac{\partial\mathcal{U}_\delta}{\partial x_i}\nabla_{X''}\overline{\chi}_i\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x'}\Phi\\ \sum_{i=1}^m \frac{\partial\Phi}{\partial x_i}\nabla_{X''}\widetilde{\chi}_i\end{pmatrix} dx'dX'' = 0.\end{aligned} \tag{7.11}$$

The above equalities allow to replace $\frac{1}{\delta}\nabla_{X''}\Pi_\delta(\overline{u}_\delta)$ by $\sum_{i=1}^m \frac{\partial\mathcal{U}_\delta}{\partial x_i}\nabla_{X''}\overline{\chi}_i$ in (7.9). This leads to

$$\begin{aligned}\Big| \int_\Omega A\begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta \\ \sum_{i=1}^m \frac{\partial\mathcal{U}_\delta}{\partial x_i}\nabla_{X''}\overline{\chi}_i\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x'}\Phi \\ \sum_{i=1}^m \frac{\partial\Phi}{\partial x_i}\nabla_{X''}\widetilde{\chi}_i\end{pmatrix} dx'dX'' - |\omega|\int_{\mathcal{O}} f\Phi\, dx'\Big| \\ \le C\delta\|f\|_{L^2(\mathcal{O})}\|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}.\end{aligned} \tag{7.12}$$

As a consequence of the definition (5.4) of the correctors $\overline{\chi}_i$ and (7.12), the estimate (7.10) is proved.

Step 5. In this step we show $(7.2)_1$.

The definition of the entries of the matrix A^* and (7.10) yield

$$\Big| \int_{\mathcal{O}} A^*\nabla_{x'}\mathcal{U}_\delta\cdot\nabla_{x'}\Phi\, dx' - \int_{\mathcal{O}} f\Phi\, dx'\Big| \le C\delta\|f\|_{L^2(\mathcal{O})}\|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}. \tag{7.13}$$

Taking into account (5.6) and (7.13), we obtain

$$\Big| \int_{\mathcal{O}} A^*\nabla_{x'}\big(\mathcal{U}_\delta - \mathcal{U}\big)\cdot\nabla_{x'}\Phi\, dx'\Big| \le C\delta\|f\|_{L^2(\mathcal{O})}\|\nabla_{x'}\Phi\|_{H^1(\mathcal{O})}. \tag{7.14}$$

Now, let ψ_δ be in $H^1_\gamma(\mathcal{O})$ the solution of the variational problem

$$\int_{\mathcal{O}} A^*\nabla_{x'}\phi\cdot\nabla_{x'}\psi_\delta dx = \int_{\mathcal{O}}(\mathcal{U}_\delta - \mathcal{U})\phi dx,\quad \forall\phi\in H^1_\gamma(\mathcal{O}). \tag{7.15}$$

Due to assumptions (5.9)-(5.10), the function ψ_δ belongs to $H^2(\mathcal{O}) \cap H^1_\gamma(\mathcal{O})$ and satisfies

$$\|\psi_\delta\|_{H^2(\mathcal{O})} \leq C\big\|\mathcal{U}_\delta - \mathcal{U}\big\|_{L^2(\mathcal{O})}. \tag{7.16}$$

In (7.15), we take as test function $\phi = \mathcal{U}_\delta - \mathcal{U}$ and in (7.14) we choose $\Phi = \psi_\delta$. Then, thanks to the estimate (7.16) we obtain

$$\begin{aligned}\big\|\mathcal{U}_\delta - \mathcal{U}\big\|^2_{L^2(\mathcal{O})} &= \int_{\mathcal{O}} A^*\nabla_{x'}(\mathcal{U}_\delta - \mathcal{U}) \cdot \nabla_{x'}\psi_\delta dx \leq C\delta\|f\|_{L^2(\mathcal{O})}\|\nabla_{x'}\psi_\delta\|_{H^1(\mathcal{O})} \\ &\leq C\delta\|f\|_{L^2(\mathcal{O})}\big\|\mathcal{U}_\delta - \mathcal{U}\big\|_{L^2(\mathcal{O})}.\end{aligned}$$

This gives $(7.2)_1$.

Step 6. In this step we show $(7.2)_2$.

As a consequence of the estimates $(3.7)_2$ and (6.6) we obtain

$$\|\overline{u}_\delta - \overline{u}^{ap}_\delta\|_{L^2(\Omega_\delta)} \leq C\delta\|\nabla(u_\delta - u^{ap}_\delta)\|_{L^2(\Omega_\delta)} \leq C\delta^{\frac{3}{2}+\frac{n}{2}}\|f\|_{L^2(\mathcal{O})}.$$

The above estimate together with $(7.2)_1$ lead to $(7.2)_2$. □

7.2 H^1 interior error estimate

Theorem 7.2. *Under the assumptions* (5.9)-(5.10) *we have*

$$\|\rho\nabla(u_\delta - u^{ap}_\delta)\|_{L^2(\Omega_\delta)} \leq C\delta^{1+\frac{n}{2}}\|f\|_{L^2(\mathcal{O})}. \tag{7.17}$$

The constant depends on the matrix A and the H^1 norms of the two families of correctors.

Proof. Step 1. Preliminary results.

Let ϕ be in $H^1_{\Gamma_\delta}(\Omega_\delta)$. We decompose ϕ as (3.5)

$$\phi = \Phi + \overline{\phi} \qquad \text{where } \Phi = \mathcal{M}_{\omega_\delta}(\phi) \text{ and } \Phi \in H^1_\gamma(\mathcal{O}),\ \overline{\phi} \in H^1_{\Gamma_\delta}(\Omega_\delta).$$

First, from the estimate $(6.4)_1$, $(3.7)_{1,4}$ and the fact $\|\rho\|_{L^\infty(\mathcal{O}_\delta)} = \delta$, we have

$$\begin{aligned}\|\nabla_{x'}(\rho\Phi)\|_{L^2(\mathcal{O}_\delta)} &\leq \|\nabla_{x'}\rho\|_{L^\infty(\mathcal{O}_\delta)}\|\Phi\|_{L^2(\mathcal{O}_\delta)} + \|\rho\|_{L^\infty(\mathcal{O}_\delta)}\|\nabla_{x'}\Phi\|_{L^2(\mathcal{O}_\delta)} \\ &\leq C\delta^{\frac{1}{2}}\|\nabla_{x'}\Phi\|_{L^2(\mathcal{O})} + \delta\|\nabla_{x'}\Phi\|_{L^2(\mathcal{O}_\delta)} \\ &\leq C\delta^{\frac{1}{2}}\|\Phi\|_{H^1(\mathcal{O})} \leq C\delta^{\frac{1}{2}-\frac{n}{2}}\|\nabla_{x'}\phi\|_{L^2(\Omega_\delta)}\end{aligned}$$

and

$$\|\nabla_{x''}(\rho\overline{\phi})\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} = \|\rho\nabla_{x''}\overline{\phi}\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} \leq \|\rho\|_{L^\infty(\mathcal{O}_\delta)}\|\nabla_{x''}\overline{\phi}\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} \leq \delta\|\nabla\phi\|_{L^2(\Omega_\delta)}.$$

The above estimates yield

$$\|\nabla(\rho\phi)\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} \leq C\delta^{\frac{1}{2}}\|\nabla\phi\|_{L^p(\Omega_\delta)}. \tag{7.18}$$

Then, transforming by Π_δ and using (3.2) we obtain

$$\begin{aligned}\|\Pi_\delta\big(\nabla(\rho\phi)\big)\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} &\leq C\delta^{\frac{1}{2}-\frac{n}{2}}\|\nabla\phi\|_{L^2(\Omega_\delta)}, \\ \|\nabla_{X''}\Pi_\delta\big(\rho\phi\big)\big)\|_{L^2(\mathcal{O}_\delta\times\omega)} &\leq C\delta^{\frac{3}{2}-\frac{n}{2}}\|\nabla\phi\|_{L^2(\Omega_\delta)}.\end{aligned} \tag{7.19}$$

Besides, since $\phi \in H^1_{\Gamma_\delta}(\Omega_\delta)$, there exists a constant C which does not depend on δ such that (see (4.5))

$$\|\nabla(\rho\phi)\|_{L^2(\Omega_\delta)} \leq \|\phi\|_{L^2(\Omega_\delta)} + \|\nabla\phi\|_{L^2(\Omega_\delta)} \leq C\|\nabla\phi\|_{L^2(\Omega_\delta)}. \tag{7.20}$$

Step 2. In this step we show that

$$\begin{aligned} &\Big||\omega| \int_{\mathcal{O}} f\rho\,\Phi\,dx' - \int_\Omega A\,\Pi_\delta(\nabla u_\delta^{ap}) \cdot \Pi_\delta(\nabla\big(\rho\phi\big))dx'dX''\Big| \\ \leq & C\delta^{1-\frac{n}{2}}\|f\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^2(\Omega_\delta)}. \end{aligned} \tag{7.21}$$

In (5.2) we choose as a test function $\big(\rho\Phi, \frac{1}{\delta}\Pi_\delta(\rho\overline{\phi})\big)$. This gives

$$\int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \sum_{i=1}^m \dfrac{\partial \mathcal{U}}{\partial x_i}\nabla_{X''}\overline{\chi}_i \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x'}(\rho\Phi) \\ \dfrac{1}{\delta}\nabla_{X''}\Pi_\delta(\rho\overline{\phi}) \end{pmatrix} dx'dX'' = |\omega| \int_{\mathcal{O}} f\rho\Phi\,dx'. \tag{7.22}$$

Since all the components of $\begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \sum_{i=1}^m \dfrac{\partial \mathcal{U}}{\partial x_i}\nabla_{X''}\overline{\chi}_i \end{pmatrix}$ belong to $L^2(\omega, H^1(\mathcal{O}))$, using $(8.1)_2$, we obtain

$$\begin{aligned} &\Big|\int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \sum_{i=1}^m \dfrac{\partial \mathcal{U}}{\partial x_i}\nabla_{X''}\overline{\chi}_i \end{pmatrix} \cdot \left[\Pi_\delta(\nabla\big(\rho\phi\big)) - \begin{pmatrix} \nabla_{x'}(\rho\Phi) \\ \dfrac{1}{\delta}\nabla_{X''}\Pi_\delta\big(\rho\overline{\phi}\big) \end{pmatrix}\right] dx'dX''\Big| \\ \leq & C\|\nabla_{x'}\mathcal{U}\|_{H^1(\mathcal{O})}\Big(\sum_{i=1}^m \|\overline{\chi}_i\|_{H^1(\omega)}\Big)\big(\delta^{1-\frac{n}{2}}\|\nabla(\rho\phi)\|_{L^2(\Omega_\delta)} + \delta^{\frac{1}{2}-\frac{n}{2}}\|\nabla(\rho\phi)\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)}\big). \end{aligned}$$

Then, equality (7.22) together with the above estimate and (6.3)-(7.18)-(7.20) yield

$$\begin{aligned} &\Big||\omega| \int_{\mathcal{O}} f\rho\,\Phi\,dx' - \int_\Omega A \begin{pmatrix} \nabla_{x'}\mathcal{U} \\ \sum_{i=1}^m \dfrac{\partial \mathcal{U}}{\partial x_i}\nabla_{X''}\overline{\chi}_i \end{pmatrix} \cdot \Pi_\delta(\nabla\big(\rho\phi\big))dx'dX''\Big| \\ \leq & C\|f\|_{L^2(\mathcal{O})}\big(\delta^{1-\frac{n}{2}}\|\nabla\phi\|_{L^2(\Omega_\delta)} + \delta^{\frac{1}{2}-\frac{n}{2}}\|\nabla\phi)\|_{L^2(\mathcal{O}_\delta\times\omega)}\big) \leq C\delta^{1-\frac{n}{2}}\|f\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^2(\Omega_\delta)}. \end{aligned} \tag{7.23}$$

Now, we have

$$\begin{aligned} &\Big|\int_\Omega A \begin{pmatrix} \delta\sum_{i=1}^m \nabla_{x'}\Big(\rho_\delta\dfrac{\partial \mathcal{U}}{\partial x_i}\Big)\overline{\chi}_i \\ \sum_{i=1}^m (1-\rho_\delta)\dfrac{\partial \mathcal{U}}{\partial x_i}\nabla_{X''}\overline{\chi}_i \end{pmatrix} \cdot \Pi_\delta(\nabla\big(\rho\phi\big))dx'dX''\Big| \\ &\leq \Big|\int_\Omega A \begin{pmatrix} \Big(\sum_{i=1}^m \dfrac{\partial \mathcal{U}}{\partial x_i}\overline{\chi}_i\Big)\,\delta\nabla_{x'}\rho_\delta \\ \sum_{i=1}^m (1-\rho_\delta)\dfrac{\partial \mathcal{U}}{\partial x_i}\nabla_{X''}\overline{\chi}_i \end{pmatrix} \cdot \Pi_\delta(\nabla\big(\rho\phi\big))dx'dX''\Big| \\ &+ \Big|\int_\Omega A \begin{pmatrix} \delta\rho_\delta\sum_{i=1}^m \nabla_{x'}\Big(\dfrac{\partial \mathcal{U}}{\partial x_i}\Big)\overline{\chi}_i \\ 0 \end{pmatrix} \cdot \Pi_\delta(\nabla\big(\rho\phi\big))dx'dX''\Big|. \end{aligned} \tag{7.24}$$

Observe that the support of $\begin{pmatrix} \delta \sum_{i=1}^{m} \nabla_{x'} \big(\rho_\delta \frac{\partial \mathcal{U}}{\partial x_i}\big) \overline{\chi}_i \\ \sum_{i=1}^{m} (1-\rho_\delta) \frac{\partial \mathcal{U}}{\partial x_i} \nabla_{X''} \overline{\chi}_i \end{pmatrix}$ is included in $\mathcal{O}_\delta \times \omega$. Thus, as to prove (7.8) and thanks to $(7.19)_1$, we obtain

$$\Big| \int_\Omega A \begin{pmatrix} \big(\sum_{i=1}^{m} \frac{\partial \mathcal{U}}{\partial x_i} \overline{\chi}_i \big) \delta \nabla_{x'} \rho_\delta \\ \sum_{i=1}^{m} (1-\rho_\delta) \frac{\partial \mathcal{U}}{\partial x_i} \nabla_{X''} \overline{\chi}_i \end{pmatrix} \cdot \Pi_\delta(\nabla(\rho\phi)) dx' dX'' \Big| \le C \|\nabla_{x'} \mathcal{U}\|_{L^2(\mathcal{O}_\delta)} \|\Pi_\delta(\nabla\big(\rho\phi)\big)\|_{L^2(\mathcal{O}_\delta \times \omega)}$$
$$\le C\delta^{\frac{1}{2}} \|\nabla_{x'} \mathcal{U}\|_{H^1(\mathcal{O})} \delta^{\frac{1}{2}-\frac{n}{2}} \|\nabla\phi\|_{L^p(\Omega_\delta)} \le C\delta^{1-\frac{n}{2}} \|f\|_{L^2(\mathcal{O})} \|\nabla\phi\|_{L^2(\Omega_\delta)}.$$

The second term in the RHS of (7.24) is less than

$$C\delta \|\nabla_{x'} \mathcal{U}\|_{H^1(\mathcal{O})} \|\Pi_\delta(\nabla(\rho\phi)\|_{L^2(\Omega)} \le C\delta^{1-\frac{n}{2}} \|f\|_{L^2(\mathcal{O})} \|\nabla\phi\|_{L^2(\Omega_\delta)}.$$

Then, from the above estimates, (7.23) and (7.24) we derive (7.21).

Step 3. In this step we show (7.17).

Taking as test function $\rho\phi$ in (4.6) gives

$$\int_{\Omega_\delta} A_\delta \nabla u_\delta \cdot \nabla(\rho\phi)\, dx = \delta^n \big|\omega\big| \int_{\mathcal{O}} f\rho\Phi\, dx'.$$

The above equality and estimate (7.21) yield

$$\Big| \int_{\Omega_\delta} A_\delta \nabla(u_\delta - u_\delta^{ap}) \cdot \nabla(\rho\phi) dx \Big| \le C\delta^{1+\frac{n}{2}} \|f\|_{L^2(\mathcal{O})} \|\nabla\phi\|_{L^2(\Omega_\delta)}. \tag{7.25}$$

Now, as a test function in (7.25), we choose $\phi = \rho(u_\delta - u_\delta^{ap})$. So, (7.25) leads to

$$\begin{aligned} &\Big| \int_{\Omega_\delta} A_\delta \rho \nabla(u_\delta - u_\delta^{ap}) \cdot \rho \nabla(u_\delta - u_\delta^{ap})\, dx \Big| \\ &\qquad \le 2 \Big| \int_{\Omega_\delta} A_\delta \rho \nabla(u_\delta - u_\delta^{ap}) \cdot (u_\delta - u_\delta^{ap}) \nabla\rho\, dx \Big| \\ &\qquad + C\delta^{1+\frac{n}{2}} \|f\|_{L^2(\mathcal{O})} \big(\|\rho\nabla(u_\delta - u_\delta^{ap})\|_{L^2(\Omega_\delta)} + \|u_\delta - u_\delta^{ap}\|_{L^2(\Omega_\delta)} \big). \end{aligned} \tag{7.26}$$

The above and $(7.2)_2$ lead to

$$\Big| \int_{\Omega_\delta} A_\delta \rho \nabla(u_\delta - u_\delta^{ap}) \cdot (u_\delta - u_\delta^{ap}) \nabla\rho\, dx \Big| \le C \|\rho\nabla(u_\delta - u_\delta^{ap})\|_{L^2(\Omega_\delta)} \|u_\delta - u_\delta^{ap}\|_{L^2(\Omega_\delta)}$$
$$\le C\delta^{1+\frac{n}{2}} \|f\|_{L^2(\mathcal{O})} \|\rho\nabla(u_\delta - u_\delta^{ap})\|_{L^2(\Omega_\delta)}.$$

The above inequality, (7.26) and the coercivity assumption (4.7) yield

$$\begin{aligned} c\|\rho\nabla(u_\delta - u_\delta^{ap})\|^2_{L^2(\Omega_\delta)} &\le C\delta^{1+\frac{n}{2}} \|f\|_{L^2(\mathcal{O})} \|\rho(\nabla u_\delta - \nabla u_\delta^{ap})\|_{L^2(\Omega_\delta)} + C\delta^{2+n} \|f\|^2_{L^2(\mathcal{O})} \\ &\le \frac{c}{2} \|\rho(\nabla u_\delta - \nabla u_\delta^{ap})\|^2_{L^2(\Omega_\delta)} + \frac{C^2}{2c} \delta^{2+n} \|f\|^2_{L^2(\mathcal{O})} + C\delta^{2+n} \|f\|^2_{L^2(\mathcal{O})} \end{aligned}$$

which in turn gives (7.17). □

8 Appendix

Lemma 8.1. *For any $\phi \in H^1(\Omega_\delta)$ decomposed as* (3.5) *($\Phi \in H^1(\mathcal{O})$ and $\overline{\phi} \in H^1(\Omega_\delta)$). We have*

$$\begin{aligned}
&\Big\|\Pi_\delta(\nabla\phi) - \begin{pmatrix} \nabla_{x'}\Phi \\ \frac{1}{\delta}\nabla_{X''}\overline{\Phi} \end{pmatrix}\Big\|_{L^2(\omega;H^{-1}(\mathcal{O}))} \le C\delta^{1-\frac{n}{2}}\|\nabla\phi\|_{L^2(\Omega_\delta)}, \\
&\Big\|\Pi_\delta(\nabla\phi) - \begin{pmatrix} \nabla_{x'}\Phi \\ \frac{1}{\delta}\nabla_{X''}\overline{\Phi} \end{pmatrix}\Big\|_{L^2(\omega;(H^1(\mathcal{O}))')} \le C\delta^{1-\frac{n}{2}}\big(\|\nabla\phi\|_{L^2(\Omega_\delta)} + \delta^{-\frac{1}{2}}\|\nabla\phi\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)}\big)
\end{aligned} \tag{8.1}$$

where $\overline{\Phi} = \Pi_\delta(\overline{\phi})$, $(H^1(\mathcal{O}))'$ is the dual space of $H^1(\mathcal{O})$.
The constants do not depend on δ.

Proof. First, remind that $\nabla_{x'}\phi = \nabla_{x'}\Phi + \nabla_{x'}\overline{\phi}$, $\nabla_{x''}\phi = \nabla_{x''}\overline{\phi}$. Hence

$$\Pi_\delta(\nabla\phi) - \begin{pmatrix} \nabla_{x'}\Phi \\ \frac{1}{\delta}\nabla_{X''}\overline{\Phi} \end{pmatrix} = \begin{pmatrix} \nabla_{x'}\overline{\Phi} \\ 0 \end{pmatrix}.$$

Let Ψ be in $L^2(\omega; H^1_0(\mathcal{O}))^m$. We have

$$\int_\Omega \nabla_{x'}\overline{\Phi}\cdot\Psi dx'dX'' = -\int_\Omega \overline{\Phi}\,\mathrm{div}_{x'}(\Psi)dx'dX''.$$

From the estimates $(3.7)_{2,4}$ and (3.1) we obtain

$$\Big|\int_\Omega \overline{\Phi}\,\mathrm{div}_{x'}(\Psi)dx'dX''\Big| \le \|\overline{\Phi}\|_{L^2(\Omega)}\|\Psi\|_{L^2(\omega;H^1_0(\mathcal{O}))} \le C\delta^{1-\frac{n}{2}}\|\nabla\phi\|_{L^2(\Omega_\delta)}\|\Psi\|_{L^2(\omega;H^1_0(\mathcal{O}))}.$$

This yields $(8.1)_1$.
Now, let Ψ be in $L^2(\omega; H^1(\mathcal{O}))^m$. Estimate $(6.4)_1$ gives

$$\|\Psi\|_{L^2(\mathcal{O}_\delta\times\omega)} \le C\delta^{\frac{1}{2}}\|\Psi\|_{L^2(\omega;H^1(\mathcal{O}))}. \tag{8.2}$$

We have

$$\begin{aligned}
\int_\Omega \nabla_{x'}\overline{\Phi}\cdot\Psi dx'dX'' &= \int_\Omega \nabla_{x'}\overline{\Phi}\cdot(1-\rho_\delta)\Psi dx'dX'' + \int_\Omega \nabla_{x'}\overline{\Phi}\cdot\rho_\delta\Psi dx'dX'' \\
\int_\Omega \nabla_{x'}\overline{\Phi}\cdot\rho_\delta\Psi dx'dX'' &= -\int_\Omega \overline{\Phi}\,\rho_\delta\mathrm{div}_{x'}(\Psi)dx'dX'' - \int_\Omega \overline{\Phi}\,(\nabla_{x'}\rho_\delta)\cdot\Psi dx'dX''.
\end{aligned} \tag{8.3}$$

Then, from (8.2), (3.1), (3.2) and $(3.7)_{2,3}$ we get

$$\begin{aligned}
\Big|\int_\Omega \nabla_{x'}\overline{\Phi}\cdot(1-\rho_\delta)\Psi dx'dX''\Big| &\le \|\nabla_{x'}\overline{\Phi}\|_{L^2(\mathcal{O}_\delta\times\omega)}\|\Psi\|_{L^2(\mathcal{O}_\delta\times\omega)} \\
&\le C\delta^{-\frac{n}{2}}\|\nabla_{x'}\phi\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)}\delta^{\frac{1}{2}}\|\Psi\|_{L^2(\omega;H^1(\mathcal{O}))}, \\
\Big|\int_\Omega \nabla_{x'}\overline{\Phi}\cdot\Psi dx'dX''\Big| &\le C\|\overline{\Phi}\|_{L^2(\mathcal{O}\times\omega)}\|\Psi\|_{L^2(\omega;H^1(\mathcal{O}))} + \frac{C}{\delta}\|\overline{\Phi}\|_{L^2(\mathcal{O}_\delta\times\omega)}\|\Psi\|_{L^2(\mathcal{O}_\delta\times\omega)} \\
&\le C\big(\delta^{1-\frac{n}{2}}\|\nabla_{x''}\phi\|_{L^2(\Omega_\delta)} + \delta^{\frac{1}{2}-\frac{n}{2}}\|\nabla_{x''}\phi\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)}\big)\|\Psi\|_{L^2(\omega;H^1(\mathcal{O}))}.
\end{aligned}$$

This yields $(8.1)_2$. □

As a consequence of the previous lemma, we have

Lemma 8.2. *Let u_δ be the solution to problem* (4.6) *(decomposed as* (3.5) *with $\mathcal{U}_\delta \in H^1_\gamma(\mathcal{O})$ and $\overline{u}_\delta \in H^1_{\Gamma_\delta}(\Omega_\delta)$). We have*

$$\Big\|\Pi_\delta(\nabla u_\delta) - \begin{pmatrix} \nabla_{x'}\mathcal{U}_\delta \\ \dfrac{1}{\delta}\nabla_{X''}\overline{U}_\delta \end{pmatrix}\Big\|_{L^2(\omega;(H^1(\mathcal{O}))')} \leq C\delta\|f\|_{L^2(\mathcal{O})} \tag{8.4}$$

where $\overline{U}_\delta = \Pi_\delta(\overline{u}_\delta)$.

Proof. We have

$$u_\delta = (u_\delta - u_\delta^{ap}) + u_\delta^{ap}.$$

Hence

$$\nabla_{x''}u_\delta = \nabla_{x''}(u_\delta - u_\delta^{ap}) + \nabla_{x''}u_\delta^{ap}.$$

Using the estimate (6.6), we obtain

$$\delta^{1-\frac{n}{2}}\|\nabla_{x''}(u_\delta - u_\delta^{ap})\|_{L^2(\Omega_\delta)} \leq C\delta^{3/2}\|f\|_{L^2(\mathcal{O})}. \tag{8.5}$$

Besides, using $(6.4)_1$ we obtain (remind that $\nabla_{x''}u_\delta^{ap} = \nabla_{x''}\overline{u}_\delta^{ap}$)

$$\begin{aligned}\|\nabla_{X''}\Pi_\delta(\overline{u}_\delta^{ap})\|_{L^2(\mathcal{O}_\delta\times\omega)} &= \Big\|\sum_{i=1}^m \delta\rho_\delta\frac{\partial\mathcal{U}}{\partial x_i}\nabla_{X''}\overline{\chi}_i\Big\|_{L^2(\mathcal{O}_\delta\times\omega)} \\ &\leq C\|\delta\rho_\delta\|_{L^\infty(\mathcal{O}_\delta)}\|\nabla\mathcal{U}\|_{L^2(\mathcal{O}_\delta)}\Big(\sum_{i=1}^m \|\nabla_{X''}\overline{\chi}_i\|_{L^2(\omega)}\Big).\end{aligned}$$

Then, estimates $(3.2)_2$-(6.3)-$(6.4)_1$ and the fact that $\|\delta\rho_\delta\|_{L^\infty(\mathcal{O}_\delta)} = \delta$ yield

$$\delta^{1-\frac{n}{2}}\|\nabla_{x''}(\overline{u}_\delta^{ap})\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} = \|\nabla_{X''}\Pi_\delta(u_\delta^{ap})\|_{L^2(\mathcal{O}_\delta\times\omega)} \leq C\delta^{3/2}\|f\|_{L^2(\mathcal{O})}. \tag{8.6}$$

The above estimates (8.5)-(8.6) lead to

$$\delta^{1-\frac{n}{2}}\big\|\nabla_{x''}u_\delta\big\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} \leq C\delta^{3/2}\|f\|_{L^2(\mathcal{O})}. \tag{8.7}$$

Now, again from (6.6), we have

$$\delta^{1-\frac{n}{2}}\|\nabla_{x'}(u_\delta - u_\delta^{ap})\|_{L^2(\Omega_\delta)} \leq C\delta^{3/2}\|f\|_{L^2(\mathcal{O})}. \tag{8.8}$$

Furthermore, due to $(6.4)_1$ we get

$$\begin{aligned}\|\nabla_{x'}(u_\delta^{ap})\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} &\leq \|\nabla_{x'}\mathcal{U}\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} + \|\nabla_{x'}(\overline{u}_\delta^{ap})\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} \\ &\leq C\delta^{\frac{1}{2}+\frac{n}{2}}\|\mathcal{U}\|_{H^2(\mathcal{O})} + \|\nabla_{x'}(\overline{u}_\delta^{ap})\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)}.\end{aligned}$$

Then, Lemma 4.1 and again $(6.4)_1$ yield

$$\begin{aligned}\|\nabla_{x'}(\overline{u}_\delta^{ap})\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} &\leq \sum_{i=1}^m\Big(\Big\|\delta\rho_\delta\nabla_{x'}\Big(\frac{\partial\mathcal{U}}{\partial x_i}\Big)\overline{\chi}_i\Big(\frac{\cdot}{\delta}\Big)\Big\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} + \Big\|\nabla_{x'}\big(\delta\rho_\delta\big)\frac{\partial\mathcal{U}}{\partial x_i}\overline{\chi}_i\Big(\frac{\cdot}{\delta}\Big)\Big\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)}\Big) \\ &\leq \delta^{1+\frac{n}{2}}\|\mathcal{U}\|_{H^2(\mathcal{O})}\Big(\sum_{i=1}^m \|\overline{\chi}_i\|_{L^2(\omega)}\Big) + C\delta^{\frac{n}{2}}\|\nabla_{x'}\mathcal{U}\|_{L^2(\mathcal{O}_\delta)}\Big(\sum_{i=1}^m \|\overline{\chi}_i\|_{L^2(\omega)}\Big) \\ &\leq C\delta^{\frac{1}{2}+\frac{n}{2}}\|\mathcal{U}\|_{H^2(\mathcal{O})}.\end{aligned}$$

Then (6.3) and the above lead to

$$\delta^{1-\frac{n}{2}}\|\nabla_{x'}(\overline{u}_\delta^{ap})\|_{L^2(\mathcal{O}_\delta\times\omega_\delta)} \leq \delta^{3/2}\|f\|_{L^2(\mathcal{O})}.$$

Finally, the above, (8.8), (8.7) and $(8.1)_2$ give (8.4). □

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