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Asymptotic Analysis of Stokes Flow Through a Filter

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Abstract

This paper investigates Stokes flow through a filter composed of a rigid, thin, periodic, and closely packed array of parallel long rods with a non-circular, anisotropic cross-section, having the shape of a slot. The analysis is restricted to regions far from the top and bottom boundaries, where the flow becomes effectively two-dimensional due to the invariance of the structure in the third dimension. In 2D formulation, we will refer to the cross-section of the rods as obstacles. The obstacles are much thinner than the distance between them, so the filter has wide channels. The form of the obstacles is parametrized to enable detailed analysis. We explore the critical scaling of the filter permeability and various regimes based on the relationships between the obstacle distance, length and thickness. In addition, the paper presents three new tools: (1) estimates for small periodic obstacles with unilateral constraints; (2) a rescaled version of the Nečas inequality for thin domains, analogous to Korn's inequalities; and (3) a demonstration of the invariance of estimates under local diffeomorphisms (changes in obstacle shape), including the derivation of a sharp constant.

Keywords: Periodic unfolding method, Stokes flow, Neumann sieve, homogenization, Nečas inequality
Mathematics Subject Classification (2010): 35Q30, 35B27, 35J50, 47H05, 74K20, 76D07.

1 Introduction

This paper investigates Stokes flow through a filter composed of a rigid, thin, periodic, and closely packed array of parallel long rods with a non-circular, anisotropic cross-section, having the shape of a slot. The original three-dimensional region of the process is described as:

$$\Omega_{\varepsilon\delta} \doteq \Omega_{\varepsilon\delta} \times (-L_3, L_3)$$

where L_3 is a large positive real number defining the extent along the third coordinate x_3 . The analysis focuses on regions sufficiently far from the top and bottom boundaries of the structure ($x_3 = \pm L_3$), where boundary effects become negligible and the flow behaves effectively as two-dimensional. Moreover, due to the structural invariance with respect to translations along the third direction, the original three-dimensional (3D) problem can be simplified to a two-dimensional (2D) formulation in domain $\Omega_{\varepsilon\delta}$. This reduction significantly simplifies the analysis while retaining the essential physical characteristics of the problem. In further discussion, we will refer to the cross-sections of the rods, which act as flow barriers, as obstacles. This terminology helps to emphasize their role in modifying the fluid flow through the filter. Obstacles, parametrized by smoothed diamonds, are considered thin in the direction orthogonal to the flow, with their thickness much smaller than the distance between them, forming a plate structure

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with anisotropic double porosity and large channels. We study the critical scale for the permeability of the plate and different regimes with regard to the relation between distance, length and thickness of the obstacles.

The motivation to study the filter efficiency arose from the analysis of limiting behavior in fluid-solid interaction problems for Stokes flow through a porous thin plate. When pore size and plate thickness are of the same order, the limiting 2D interface becomes impermeable, blocking the flow in the asymptotic limit. As shown in [19, 27, 31], the velocity trace vanishes on the 2D interface separating the two fluid domains in such cases. This observation prompted the exploration of filters with larger pores, specifically a double-porosity structure, to ensure the permeability of the filter and derive a form of the interface Darcy law.

Our motivation for the obstacle shape stems from fundamental principles discussed in boundary layer theory (see [39]). In aero- and hydrodynamics, hole shapes such as Laval nozzles [18] and Venturi tubes [38] are known to accelerate or decelerate the flow by smoothly decreasing or increasing the cross-sectional area of the hole. These nozzles and tubes are widely used in industry. We expect the same effect when selecting the appropriate shape of the obstacle for microstructures¹.

There are numerous studies on Stokes flows or similar problems through Neumann-type sieves or porous plates [1, 2, 6, 7, 8, 12, 15, 16, 17, 24, 25, 26, 30, 34, 37], as well as in volume domains [1, 10, 11, 12, 14, 34]. These works differentiate between two configurations: one with volumetric periodicity of obstacles in all n directions (with n as the space dimension) and another involving an $n - 1$ periodic boundary layer with obstacles separating two bulk domains. In the n -periodic case, imposing Dirichlet conditions on small obstacles introduces an additional zero-order term in the second-order PDE governing the domain without obstacles. For the $n - 1$ periodic configuration (a sieve), this results in a Robin-type jump condition on the interface between the two bulk domains as the layer collapses to an $n - 1$ -dimensional hyperplane. The coefficient in front of the "strange term" differs by a factor of $\frac{1}{\sqrt{\varepsilon}}$ (see e.g. [12]).

There are new tools, useful for some classes of problems, developed in this paper:

The first new tool, presented in Section 4, extends the methods from [13, 28, 34] for estimating boundary layers under zero Dirichlet conditions on small periodic holes or obstacles to the case of unilateral non-penetration constraints, where only the normal component of the velocity and the tangential traction vanish. It also demonstrates that the channels satisfy Korn-type inequalities and can be classified as Korn-type domains.

One of the key auxiliary tools developed in this paper is the rescaling of the Nečas constant (see Appendix B), used for pressure estimates with respect to domain topology. For the Nečas inequality, also known as Lions' lemma, we refer to [3, 4, 5, 9, 32, 33, 35, 40].

Finally, the last useful result, presented in Appendix C, is the invariance of the estimates under local diffeomorphisms (obstacle shape changes) and the derivation of a sharp constant that can be used, e.g., in multi-scale shape optimization problems. Similar results, demonstrating that the diffeomorphism applied to the microstructure remains in the corrector and does not influence the estimates or limits, were also shown in [23, 29, 31], while it was demonstrated in [22] that the diffeomorphism applied to the macro-domain only affects the macro-problem in the limit.

The paper is organized as follows. Section 2 presents the geometric setting of the problem. Three small geometric parameters are introduced to parametrically describe the obstacles in the 2D setting: ε represents the distance between the filter wires, while δ_1 and δ_2 describe the cross-sectional shapes of the wires.

In Section 3, the strong and weak formulations for the 2D Stokes flow through two bulk domains separated by a rigid filter or porous plate are provided. At the left boundary of the channel, the initial inlet pressure is specified, while at the right boundary, a free outlet condition is applied. The no-penetration condition is imposed on the walls of the channel and the obstacles.

¹ It is also interesting to note that the pillars of bridges built a few centuries ago have an elongated, pointed shape to facilitate the flow of the river they span (you can see this when you visit Paris, for example).

Section 4 provides important tools for obtaining estimates and was described above.

Section 5 is dedicated to obtaining a priori estimates for the velocity and pressure. The estimates for pressure are derived using the tools developed in Section 4. To estimate the pressure, we apply the Nečas inequality, the properties of which were discussed in detail in Sections B, C, providing a rigorous foundation for the analysis in the context of the Stokes flow problem.

Section 6 is dedicated to the study of the asymptotic behavior of the solution and the derivation of limit models. In this section, based on the previously derived estimates and the critical ratio $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2}$, we classify the limit models into three distinct regimes: $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow 0$, $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow \theta$, $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow \infty$. The first two regimes are analyzed in the present section. The third regime falls outside the scope of this work, as the estimates suggest that fluid flow in this regime is not governed by the shape of the inclusions. This regime may be investigated within the framework of an evolutionary problem, which would provide insights over extended time scales.

The unfolding operator, discussed in Appendix A, addresses the boundary layer and the unfolding of boundaries or interfaces, as demonstrated, for example, in [12, 20, 24, 25, 36].

2 The domain

We consider a viscous fluid occupying a two-dimensional domain. This domain is separated in two parts by a thin porous periodic filter. It depends on three small parameters δ_1 , δ_2 and ε , the first two are related to the size of the obstacles and the last to the size of the periodicity cell (see Fig. 2). Our goal is to study the asymptotic behavior and to obtain a limiting model when $\varepsilon, \delta_1, \delta_2 \rightarrow 0$.

We denote

$$\begin{aligned} \Omega &\doteq (-L_1, L_1) \times (-L_2, L_2), & \mathcal{O} &\doteq \{0\} \times (-L_2, L_2), \\ \Omega^- &\doteq (-L_1, 0) \times (-L_2, L_2), & \Omega^+ &\doteq (0, L_1) \times (-L_2, L_2) \\ \Omega_r^- &\doteq (-L_1, -r) \times (-L_2, L_2), & \Omega_r^+ &\doteq (r, L_1) \times (-L_2, L_2), \\ \mathcal{O}_r^\pm &\doteq \{\pm r\} \times (-L_2, L_2), & r &\in [0, L_1], \\ Y &\doteq (-2, 2) \times (0, 1), & Y_r &\doteq rY, \quad r \in (0, \varepsilon]. \end{aligned}$$

In this paper, we consider obstacles defined by two strictly positive parameters δ_1, δ_2 satisfying

$$\delta_1 \leq \frac{\varepsilon}{2}, \quad \delta_2 \leq \frac{\delta_1}{5}.$$

We denote F_δ the reference obstacle. F_δ is the open set whose boundary is given by the two curves parameterized by

$$M(t) = \begin{cases} t\mathbf{e}_1 - \mathbf{b}^u(t)\mathbf{e}_2, \\ t\mathbf{e}_1 + \mathbf{b}^d(t)\mathbf{e}_2, \end{cases} \quad t \in [-\delta_1, \delta_1],$$

where $\mathbf{b}^u(t) = \delta_2 \psi^u\left(\frac{t}{\delta_1}\right)$, $\mathbf{b}^d(t) = \delta_2 \psi^d\left(\frac{t}{\delta_1}\right)$, $t \in [-\delta_1, \delta_1]$, where ψ^u and ψ^d belong to the following set:

$$\begin{aligned} \mathcal{C} &\doteq \left\{ \psi \in W^{2,\infty}(\mathbb{R}) \mid \psi = 0 \text{ in } \mathbb{R} \setminus (-1, 1), \psi(t) \geq 0, \text{ and } |\psi'(t)| \leq 2, \forall t \in [-1, 1], \|\psi\|_{L^\infty(\mathbb{R})} = 1, \right. \\ &\quad \left. \text{there exist an interval } I \subset [-1, 1] \text{ and } c_\psi \in [-2, 2] \text{ such that } \psi'(t) = c_\psi, \forall t \in I \right\}. \end{aligned}$$

Note that

$$\begin{aligned} \|\mathbf{b}^u\|_{L^\infty(\mathbb{R})} &= \delta_2, & \|(\mathbf{b}^u)'\|_{L^\infty(\mathbb{R})} &\leq 2\frac{\delta_2}{\delta_1}, & \|(\mathbf{b}^u)''\|_{L^\infty(\mathbb{R})} &\leq C\frac{\delta_2}{\delta_1^2}, \\ \|\mathbf{b}^d\|_{L^\infty(\mathbb{R})} &= \delta_2, & \|(\mathbf{b}^d)'\|_{L^\infty(\mathbb{R})} &\leq 2\frac{\delta_2}{\delta_1}, & \|(\mathbf{b}^d)''\|_{L^\infty(\mathbb{R})} &\leq C\frac{\delta_2}{\delta_1^2}. \end{aligned} \tag{2.1}$$

The constant C depends on ψ^u and ψ^d .

Furthermore, we assume that the sum of the constants associated with the ψ^u and ψ^d functions (c_{ψ^u} and c_{ψ^d}) is non-zero.

Let us denote by (see Fig. 1)

$$F_\delta^1 = F_\delta \cap Y_\varepsilon \quad \text{and} \quad F_\delta^2 = (\varepsilon \mathbf{e}_2 + F_\delta) \cap Y_\varepsilon$$

the two portions of obstacles included in Y_ε .

Observe that ν^u is the unit normal vector to F_δ^1 and ν^d is the unit normal vector to F_δ^2 :

$$\nu^u(t) = \frac{(\mathbf{b}^u)'(t)\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{1 + [(\mathbf{b}^u)'(t)]^2}}, \quad \nu^d(t) = \frac{-(\mathbf{b}^d)'(t)\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{1 + [(\mathbf{b}^d)'(t)]^2}}, \quad t \in (-\delta_1, \delta_1). \quad (2.2)$$

Then, we define

$$Y_{\varepsilon\delta} = Y_\varepsilon \setminus \overline{F_\delta^1 \cup F_\delta^2}$$

and the domain $\mathbf{Y}_{\varepsilon\delta}$

$$\mathbf{Y}_{\varepsilon\delta} \doteq \bigcup_{p \in \{-N_\varepsilon, \dots, N_\varepsilon - 1\}} Y_{\varepsilon\delta}^p, \quad Y_{\varepsilon\delta}^p \doteq p\varepsilon \mathbf{e}_2 + Y_{\varepsilon\delta}$$

where $N_\varepsilon = \left\lceil \frac{L_2}{\varepsilon} \right\rceil \in \mathbb{N}^2$ and $p_\varepsilon \doteq \{-N_\varepsilon, \dots, N_\varepsilon\}$. The fluid-domain is

$$\Omega_{\varepsilon\delta} \doteq \Omega \setminus \bigcup_{p \in p_\varepsilon} (p\varepsilon \mathbf{e}_2 + \overline{F_\delta}).$$

We denote parts of boundary $\partial\Omega_{\varepsilon\delta}$ (see Fig. 2)

$$\begin{aligned} \Gamma_{\varepsilon\delta} &\doteq \partial\mathbf{Y}_{\varepsilon\delta} \cap \partial\Omega_{\varepsilon\delta} = \partial\Omega_{\varepsilon\delta} \setminus \partial\Omega, \\ \Gamma_1 &= (-L_1, L_1) \times \{\pm L_2\}, \\ \Gamma_2^{in} &= \{-L_1\} \times (-L_2, L_2), \\ \Gamma_2^{out} &= \{L_1\} \times (-L_2, L_2). \end{aligned}$$

We introduce the space of admissible velocities as

$$\mathbf{V}_{ad}(\Omega_{\varepsilon\delta}) \doteq \{\phi \in H^1(\Omega_{\varepsilon\delta})^2 \mid \phi \cdot \nu = 0 \text{ a.e. on } \Gamma_{\varepsilon\delta} \cup \Gamma_1\},$$

the normal components of velocities of the fluid vanish on $\Gamma_{\varepsilon\delta} \cup \Gamma_1$, which corresponds to the non-penetration condition, and its subspace of solenoidal fields as

$$\mathbf{V}_{ad,div0} \doteq \{\phi \in \mathbf{V}_{ad}(\Omega_{\varepsilon\delta}) \mid \nabla \cdot \phi = 0 \text{ a.e. in } \Omega_{\varepsilon\delta}\}.$$

²For simplicity we assume that ε is such that $L_2 = N_\varepsilon \varepsilon$.

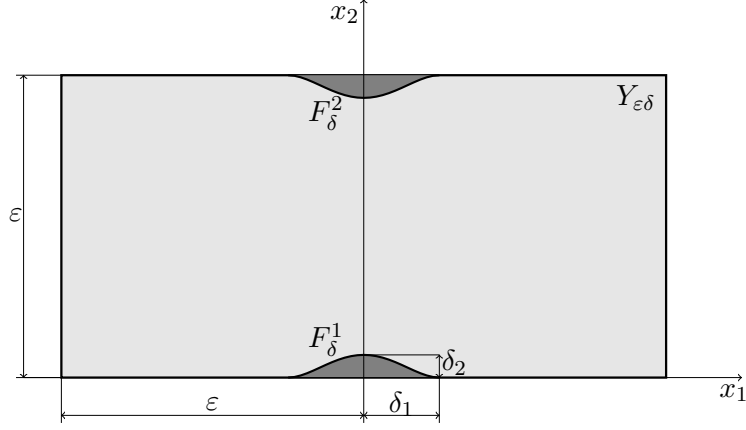


Figure 1: The open set $Y_{\epsilon\delta}$

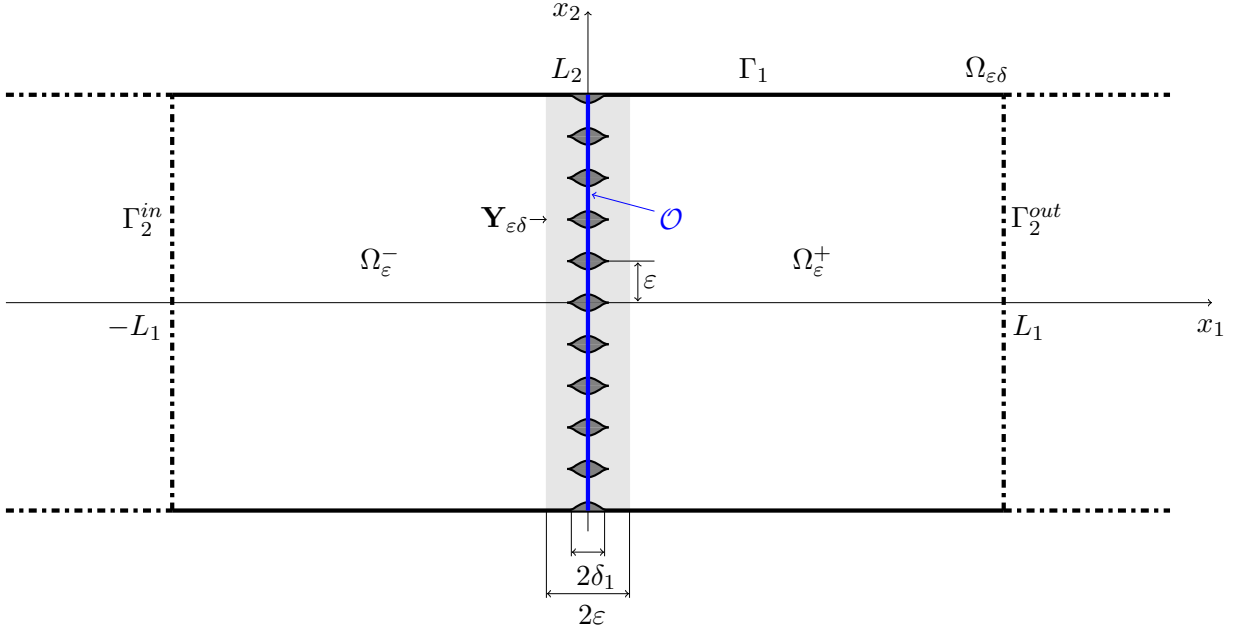


Figure 2: The fluid-domain $\Omega_{\epsilon\delta}$ with the set $\mathbf{Y}_{\epsilon\delta}$

3 The microscopic problem

The function $v_{\epsilon\delta} : \Omega_{\epsilon\delta} \rightarrow \mathbb{R}^2$ will denote the fluid velocity, while $p_{\epsilon\delta} : \Omega_{\epsilon\delta} \rightarrow \mathbb{R}$ the fluid pressure. In the fluid-domain $\Omega_{\epsilon\delta}$ we consider the following problem:

$$\left\{ \begin{array}{ll} -2\mu\nabla \cdot D(v_{\epsilon\delta}) + \nabla p_{\epsilon\delta} = 0 & \text{in } \Omega_{\epsilon\delta}, \\ \nabla \cdot v_{\epsilon\delta} = 0 & \text{in } \Omega_{\epsilon\delta}, \\ v_{\epsilon\delta} \cdot \nu = 0 & \text{on } \Gamma_{\epsilon\delta} \cup \Gamma_1, \\ D(v_{\epsilon\delta})\nu \cdot \tau = 0 & \text{on } \Gamma_{\epsilon\delta} \cup \Gamma_1, \\ (-2\mu D(v_{\epsilon\delta}) + p_{\epsilon\delta}\mathbf{I})\nu = \mathcal{P}\nu & \text{on } \Gamma_2^{in}, \\ (-2\mu D(v_{\epsilon\delta}) + p_{\epsilon\delta}\mathbf{I})\nu = 0 & \text{on } \Gamma_2^{out}, \end{array} \right. \quad (3.1)$$

where μ is a coefficient of viscosity of the fluid, $\mathcal{P} \in L^2(\Gamma_2^{in})$ specifies the boundary pressure; ν is a unit normal vector and τ is a unit tangential vector; $D(v)$ is the symmetric gradient of v .

Hence, the variational formulation to problem (3.1) is

$$\left\{ \begin{array}{l} \text{Find } (v_{\varepsilon\delta}, p_{\varepsilon\delta}) \in \mathbf{V}_{ad,div0} \times L^2(\Omega_{\varepsilon\delta}) \text{ such that} \\ 2\mu \int_{\Omega_{\varepsilon\delta}} D(v_{\varepsilon\delta}) : D(\phi) dx - \int_{\Omega_{\varepsilon\delta}} p_{\varepsilon\delta} \nabla \cdot \phi dx = \int_{\Gamma_2^{in}} \mathcal{P} \phi_1 dx_2, \quad \forall \phi \in \mathbf{V}_{ad}(\Omega_{\varepsilon\delta}), \end{array} \right. \quad (3.2)$$

Our aim is to study the asymptotic behavior of $(v_{\varepsilon\delta}, p_{\varepsilon\delta})$ as $(\varepsilon, \delta_1, \delta_2) \rightarrow (0, 0, 0)$.

4 Preliminary results

4.1 Some inequalities to obtain the estimates of the elements of $\mathbf{V}_{ad}(\Omega_{\varepsilon\delta})$

Let us denote $T_{d,r}$ the trapezoid whose two edges are (see Fig.3)

$$\{0\} \times (-d, d) \quad \text{and} \quad \{r\} \times (-r, r), \quad d \in (0, r/2).$$

We remind that for every $\Psi \in H^1(Y_r)$, we have

$$\left\{ \begin{array}{l} \|\Psi\|_{L^2(\{\pm r\} \times (0,r))}^2 \leq 2 \left(\frac{1}{r} \|\Psi\|_{L^2(Y_r)}^2 + r \|\partial_{x_1} \Psi\|_{L^2(Y_r)}^2 \right), \\ \|\Psi\|_{L^2(Y_r)}^2 \leq 4r \left(\|\Psi\|_{L^2(\{\pm r\} \times (0,r))}^2 + r \|\partial_{x_1} \Psi\|_{L^2(Y_r)}^2 \right). \end{array} \right. \quad (4.1)$$

Below, we recall a classical result (see [13], [28]).

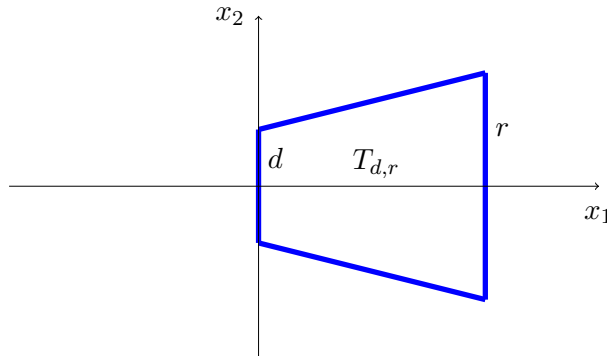


Figure 3: Illustration for Lemma 4.1

Lemma 4.1. *Let ϕ be in $H^1(T_{d,r})$, such that $\phi = 0$ a.e. on $\{0\} \times (-d, d)$. Then, we have*

$$\|\phi\|_{L^2(T_{d,r})} \leq Cr \sqrt{\ln\left(\frac{r}{d}\right)} \|\nabla \phi\|_{L^2(T_{d,r})}. \quad (4.2)$$

The constant does not depend on r and d .

4.2 Estimates of the elements of $\mathbf{V}_{ad}(\Omega_{\varepsilon\delta})$

Lemma 4.2. *For any $\phi \in \mathbf{V}_{ad}(\Omega_{\varepsilon\delta})$ there exists a_1 which depends on ϕ_1 such that*

$$\left\{ \begin{array}{l} \|\phi - a_1 \mathbf{e}_1\|_{H^1(\Omega_{\delta_1}^-)} \leq C \|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}, \\ \|\phi - a_1 \mathbf{e}_1\|_{H^1(\Omega_{\delta_1}^+)} \leq C \|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}. \end{array} \right. \quad (4.3)$$

The constants are independent of ε , δ_1 and δ_2 .

Proof. Step 1. We prove that there exists $a_1^-, a_1^+ \in \mathbb{R}$, such that

$$\begin{aligned}\|\phi - a_1^- \mathbf{e}_1\|_{H^1(\Omega_{\delta_1}^-)} &\leq C \|D(\phi)\|_{L^2(\Omega_{\delta_1}^-)}, \\ \|\phi - a_1^+ \mathbf{e}_1\|_{H^1(\Omega_{\delta_1}^+)} &\leq C \|D(\phi)\|_{L^2(\Omega_{\delta_1}^+)}.\end{aligned}\tag{4.4}$$

The constants are independent of δ_1 .

We only show (4.4)₁, the proof of (4.4)₂ is similar. In $\Omega_{\delta_1}^-$ the 2D-Korn inequality gives a rigid displacement $\mathbf{r}_{\delta_1}(x) = (a_1^- - b^- x_2)\mathbf{e}_1 + (a_2^- + b^- x_1)\mathbf{e}_2$, such that (see [21, Theorem 2.3])

$$\|\phi - \mathbf{r}_{\delta_1}\|_{H^1(\Omega_{\delta_1}^-)} \leq C \|D(\phi)\|_{L^2(\Omega_{\delta_1}^-)}.\tag{4.5}$$

The constant does not depend on δ_1 .

From the estimate (4.5) we derive the following trace inequality:

$$\|\phi - \mathbf{r}_{\delta_1}\|_{L^2(\Gamma_1 \cap \Omega_{\delta_1}^-)} \leq C \|\phi - \mathbf{r}_{\delta_1}\|_{H^1(\Omega_{\delta_1}^-)} \leq C \|D(\phi)\|_{L^2(\Omega_{\delta_1}^-)}.$$

Remind, that $\phi_2 = 0$ a.e. on $\Gamma_1 \cap \Omega_{\delta_1}^-$. Thus, we have

$$\|\mathbf{r}_{\delta_1,2}\|_{L^2(\Gamma_1 \cap \Omega_{\delta_1}^-)} \leq C \|D(\phi)\|_{L^2(\Omega_{\delta_1}^-)}$$

and then

$$|a_2^-| + |b^-| \leq C \|\mathbf{r}_{\delta_1,2}\|_{L^2(\Gamma_1 \cap \Omega_{\delta_1}^-)} \leq C \|D(\phi)\|_{L^2(\Omega_{\delta_1}^-)}.$$

Hence (4.4)₁ is proved.

Step 2. We prove the estimates (4.3).

First, we estimate $|a_1^- - a_1^+|$. From (4.1), we derive

$$\begin{aligned}\sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|\phi_1 - a_1^-\|_{L^2(\Omega_{\varepsilon\delta_2}^p)}^2 &\leq C \left(\sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|\phi_1 - a_1^-\|_{L^2(\Omega_{\varepsilon\delta_2}^p \cap \Omega_{\delta_1}^-)}^2 + \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \left\| \frac{\partial \phi_1}{\partial x_1} \right\|_{L^2(\Omega_{\varepsilon\delta_2}^p)}^2 \right) \leq C \|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}^2, \\ \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|\phi_1 - a_1^+\|_{L^2(\Omega_{\varepsilon\delta_2}^p)}^2 &\leq C \left(\sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|\phi_1 - a_1^+\|_{L^2(\Omega_{\varepsilon\delta_2}^p \cap \Omega_{\delta_1}^+)}^2 + \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \left\| \frac{\partial \phi_1}{\partial x_1} \right\|_{L^2(\Omega_{\varepsilon\delta_2}^p)}^2 \right) \leq C \|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}^2,\end{aligned}$$

where $\Omega_{\varepsilon\delta_2}^p = (-L_1, L_1) \times (p\varepsilon + \delta_2, (p+1)\varepsilon - \delta_2)$. The constants do not depend on ε , δ_1 and δ_2 .

Hence

$$\begin{aligned}2L_1(\varepsilon - 2\delta_2)N_\varepsilon |a_1^- - a_1^+|^2 &= \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|a_1^- - a_1^+\|_{L^2(\Omega_{\varepsilon\delta_2}^p)}^2 \leq 2 \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} (\|\phi_1 - a_1^-\|_{L^2(\Omega_{\varepsilon\delta_2}^p)}^2 + \|\phi_1 - a_1^+\|_{L^2(\Omega_{\varepsilon\delta_2}^p)}^2) \\ &\leq C \|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta}^+)}^2.\end{aligned}$$

Therefore $|a_1^- - a_1^+| \leq C \|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta}^+)}$. This and (4.4) gives (4.3), setting $a_1 = a_1^-$. \square

Lemma 4.3. *Assume $\delta_2 > 0$. For any $\phi \in \mathbf{V}_{ad}(\Omega_{\varepsilon\delta})$ we have*

$$\begin{aligned}\|\nabla \phi\|_{L^2(\Omega_{\varepsilon\delta})} + \|\phi_1 - a_1\|_{L^2(\Omega_{\varepsilon\delta})} + \|\phi_2\|_{L^2(\Omega_{\varepsilon\delta})} &\leq C \|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}, \\ \|\phi_1 - a_1\|_{L^2(\mathcal{O}_{\varepsilon}^\pm)} + \|\phi_2\|_{L^2(\mathcal{O}_{\varepsilon}^\pm)} &\leq C \|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}\end{aligned}\tag{4.6}$$

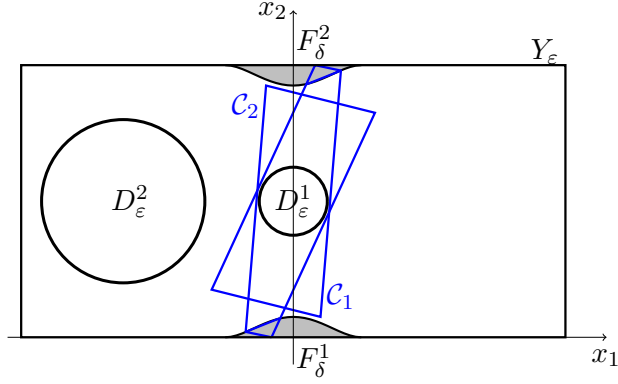


Figure 4: Illustration for Lemmas 4.3

and

$$\begin{aligned}
\|\phi_1\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} &\leq C\varepsilon\frac{\delta_1}{\delta_2}\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}, & \|\phi_2\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} &\leq C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}, \\
\|\phi_1\|_{L^2(\mathcal{O}_{\varepsilon}^{\pm})} &\leq C\sqrt{\varepsilon}\frac{\delta_1}{\delta_2}\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}, & \|\phi_2\|_{L^2(\mathcal{O}_{\varepsilon}^{\pm})} &\leq C\sqrt{\varepsilon}\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}.
\end{aligned} \tag{4.7}$$

Moreover, we have

$$\|\phi_1\|_{L^2(\Omega_{\varepsilon\delta})} \leq C\left(1 + \sqrt{\varepsilon}\frac{\delta_1}{\delta_2}\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\right)\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}. \tag{4.8}$$

The constants do not depend on ε , δ_1 and δ_2 .

Proof. Step 1. Preliminary constructions and definition.

We start with a few constructions (see Fig.4)

- D_{ε}^1 is the disc of radius $\frac{1}{16}\varepsilon$ centered in $(0, \varepsilon/2)$,
- D_{ε}^2 is the disc of radius $\frac{1}{3}\varepsilon$ centered in $(-\varepsilon, \varepsilon/2)$.

Step 2. We show that $Y_{\varepsilon\delta}$ is star-shaped with respect to the disc D_{ε}^1 .

Let $P(t) = (t, \mathbf{b}^u(t))$ ($t \in [-\delta_1, \delta_1]$) be an arbitrary point belonging to the boundary ∂F_{δ}^1 and $M = (x_1, x_2)$ be a point in the disc D_{ε}^1 . The interval $(P(t), M]$ is included in $Y_{\varepsilon\delta}$, otherwise the segment $[P(t), M]$ would contain another point $P(t') \neq P(t)$ of the boundary ∂F_{δ}^1 . Thus

$$\frac{x_2 - \mathbf{b}^u(t)}{x_1 - t} = \frac{\mathbf{b}^u(t) - \mathbf{b}^u(t')}{t - t'}.$$

We have

$$2\frac{\delta_2}{\delta_1} \geq \left| \frac{\mathbf{b}^u(t) - \mathbf{b}^u(t')}{t - t'} \right| = \left| \frac{x_2 - \mathbf{b}^u(t)}{x_1 - t} \right| \geq \frac{7\varepsilon/16 - \delta_2}{\varepsilon/16 + \delta_1} \geq \frac{7\varepsilon/16 - \varepsilon/10}{\varepsilon/16 + \varepsilon/2} = \frac{3}{5}.$$

This contradicts the fact that $\frac{\delta_2}{\delta_1} \leq \frac{1}{5}$ and proves that the interval $(P(t), M]$ is included in $Y_{\varepsilon\delta}$ for all $t \in [-\delta_1, \delta_1]$.

Similarly, we prove that the interval $(P(t), M]$, where $P(t) \in \partial F_{\delta}^2$ and $M \in D_{\varepsilon}^1$, is included in $Y_{\varepsilon\delta}$. Hence, the domain $Y_{\varepsilon\delta}$ is star-shaped with respect to the disc D_{ε}^1 .

Step 3. In this step we prove

$$\|\nabla\phi\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} + \|\phi_1 - a_1\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} + \|\phi_2\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} \leq C\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}. \quad (4.9)$$

Since the domain $Y_{\varepsilon\delta}^p$ is star-shaped with respect to the disc $p\varepsilon + D_\varepsilon^1$ and has a diameter less than 3ε , by the 2D Korn's inequality (see [21, Theorem 2.3]), there exists a rigid displacement

$$\mathbf{r}_p(x) = \begin{pmatrix} a_{p,1} + b_p(x_2 - p\varepsilon) \\ a_{p,2} - b_px_1 \end{pmatrix}, \quad x \in Y_{\varepsilon\delta}^p, \quad (4.10)$$

such that

$$\|\phi - \mathbf{r}_p\|_{L^2(Y_{\varepsilon\delta}^p)} \leq C\varepsilon\|D(\phi)\|_{L^2(Y_{\varepsilon\delta}^p)}, \quad \|\nabla(\phi - \mathbf{r}_p)\|_{L^2(Y_{\varepsilon\delta}^p)} \leq C\|D(\phi)\|_{L^2(Y_{\varepsilon\delta}^p)}, \quad (4.11)$$

where the constants do not depend on ε , δ_1 and δ_2 .

Now, we consider $D_\varepsilon^{2,p} \doteq p\varepsilon + D_\varepsilon^2 \in (\Omega_{\delta_1}^- \cup Y_{\varepsilon\delta}^p)$. We have

$$\begin{aligned} \varepsilon|b_p| &\leq C\|\nabla\mathbf{r}_p\|_{L^2(D_\varepsilon^{2,p})} \leq C(\|\nabla\phi\|_{L^2(D_\varepsilon^{2,p})} + \|\nabla(\phi - \mathbf{r}_p)\|_{L^2(Y_{\varepsilon\delta}^p)}), \\ \varepsilon(|a_{p,2}| + \varepsilon|b_p|) &\leq C\|\mathbf{r}_{p,2}\|_{L^2(D_\varepsilon^{2,p})} \leq C(\|\phi_2\|_{L^2(D_\varepsilon^{2,p})} + \|\phi_2 - \mathbf{r}_{p,2}\|_{L^2(Y_{\varepsilon\delta}^p)}), \\ \varepsilon(|a_{p,1} - a_1| + \varepsilon|b_p|) &\leq C\|\mathbf{r}_{p,1} - a_1\|_{L^2(D_\varepsilon^{2,p})} \leq C(\|\phi_1 - a_1\|_{L^2(D_\varepsilon^{2,p})} + \|\phi_1 - \mathbf{r}_{p,1}\|_{L^2(Y_{\varepsilon\delta}^p)}). \end{aligned}$$

Since the discs $D_\varepsilon^{2,p}$ ($p \in \Xi_\varepsilon$) are included in $\Omega_{\delta_1}^-$, the above estimates, those (4.11) and (4.3)₁ lead to

$$\begin{aligned} \sum_{p \in \Xi_\varepsilon} \varepsilon^2|b_p|^2 &\leq C\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}^2 & \sum_{p \in \Xi_\varepsilon} \varepsilon^2|a_{p,2}|^2 &\leq C\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}^2, \\ \sum_{p \in \Xi_\varepsilon} \varepsilon^2|a_{p,1} - a_1|^2 &\leq C\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}^2. \end{aligned} \quad (4.12)$$

As a consequence of (4.12) and again (4.11), we have

$$\begin{aligned} \|\nabla\phi\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 &= \sum_{p \in \Xi_\varepsilon} \|\nabla\phi\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \leq C \sum_{p \in \Xi_\varepsilon} \left(\varepsilon^2|b_p|^2 + \|\nabla(\phi - \mathbf{r}_p)\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \right) \leq C\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}^2, \\ \|\phi_2\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 &= \sum_{p \in \Xi_\varepsilon} \|\phi_2\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \\ &\leq C \sum_{p \in \Xi_\varepsilon} \left(\varepsilon^2|a_{p,2}|^2 + \varepsilon^4|b_p|^2 + \|\phi_2 - \mathbf{r}_{p,2}\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \right) \leq C\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}^2. \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} \|\phi_1 - a_1\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 &= \sum_{p \in \Xi_\varepsilon} \|\phi_1 - a_1\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \leq C \left(\sum_{p \in \Xi_\varepsilon} \varepsilon^2|a_{p,1} - a_1|^2 + \sum_{p \in \Xi_\varepsilon} \|\phi_1 - a_{p,1}\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \right) \\ &\leq C \sum_{p \in \Xi_\varepsilon} \left(\varepsilon^2|a_{p,1} - a_1|^2 + \varepsilon^4|b_p|^2 + \|\phi_1 - \mathbf{r}_{p,1}\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \right) \leq C\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}^2. \end{aligned} \quad (4.14)$$

The constants do not depend on ε and δ_1, δ_2 . This ends the proof of (4.9).

Then, (4.1)₁, (4.3) together with (4.9) yield the estimates (4.6).

Step 4. In this step we show that for every $\phi \in H^1(Y_{\varepsilon\delta})^2$ satisfying $\phi \cdot \nu^u = 0$ and $\phi \cdot \nu^d = 0$ a.e. on the boundary of the obstacles, we have

$$\|\phi_1\|_{L^2(Y_{\varepsilon\delta})} \leq C\varepsilon \frac{\delta_1}{\delta_2} \sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)} \|\nabla\phi\|_{L^2(Y_{\varepsilon\delta})}, \quad \|\phi_2\|_{L^2(Y_{\varepsilon\delta})} \leq C\varepsilon \sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)} \|\nabla\phi\|_{L^2(Y_{\varepsilon\delta})}. \quad (4.15)$$

By definition of the set \mathcal{C} there exist two intervals I_1 and I_2 included in $[-1, 1]$, such that

$$\begin{aligned}\nu^u(t) &= \nu_1 = \frac{c_1\delta_2\mathbf{e}_1 + \delta_1\mathbf{e}_2}{\sqrt{\delta_1^2 + c_1^2\delta_2^2}}, \quad t \in \delta_1 I_1, \\ \nu^d(t) &= \nu_2 = \frac{-c_2\delta_2\mathbf{e}_1 + \delta_1\mathbf{e}_2}{\sqrt{\delta_1^2 + c_2^2\delta_2^2}}, \quad t \in \delta_1 I_2.\end{aligned}\tag{4.16}$$

We remind that $c_1 + c_2 \neq 0$.

If $t \in \delta_1 I_1$ (resp. $\delta_1 I_2$) the part of the boundary of F_δ^1 (resp. F_δ^2) is flat and as a length of order δ_1 . Now, consider the cone \mathcal{C}_1 (resp. \mathcal{C}_2) whose edges pass through the ends of the flat segment of F_δ^1 (resp. F_δ^2) and which contains the disc D_ε^1 .

Since $\phi \cdot \nu_1$ (resp. $\phi \cdot \nu_2$) vanishes on the flat part of the boundary of F_δ^1 (resp. F_δ^2), we extend $\phi \cdot \nu_1$ (resp. $\phi \cdot \nu_2$) by 0 in the part of the cone \mathcal{C}_1 (resp. \mathcal{C}_2) containing a small part of F_δ^1 (resp. F_δ^2). Now, we apply the result of Lemma 4.1. Using simple geometric considerations, we can show that d is of order δ_1 and r of order ε . That gives

$$\|\phi \cdot \nu_1\|_{L^2(D_\varepsilon^1)} + \|\phi \cdot \nu_2\|_{L^2(D_\varepsilon^1)} \leq C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|\nabla\phi\|_{L^2(Y_{\varepsilon\delta})}.$$

As a consequence from above inequality and (4.16), we obtain

$$\|\phi_1\|_{L^2(D_\varepsilon^1)} \leq C\varepsilon\frac{\delta_1}{\delta_2}\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|\nabla\phi\|_{L^2(Y_{\varepsilon\delta})}, \quad \|\phi_2\|_{L^2(D_\varepsilon^1)} \leq C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|\nabla\phi\|_{L^2(Y_{\varepsilon\delta})}.$$

The constants do not depend on ε , δ_1 and δ_2 .

Since $Y_{\varepsilon\delta}$ is star-shaped with respect to the disc D_ε^1 , [21, Lemma 2.1] gives

$$\|\phi_i\|_{L^2(Y_{\varepsilon\delta})} \leq C(\|\phi_i\|_{L^2(D_\varepsilon^1)} + \varepsilon\|\nabla\phi_i\|_{L^2(Y_{\varepsilon\delta})}).$$

Then, the above inequalities yield (4.15).

Step 5. Due to the estimates (4.15), for $\mathbf{Y}_{\varepsilon\delta} = \bigcup_{p \in \Xi_\varepsilon} Y_{\varepsilon\delta}^p$ we have

$$\|\phi_1\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} \leq C\varepsilon\frac{\delta_1}{\delta_2}\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}, \quad \|\phi_2\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} \leq C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}.\tag{4.17}$$

Then, from (4.1)₁ and the above inequalities, we derive an estimate of the traces on $\mathcal{O}_\varepsilon^\pm$

$$\|\phi_1\|_{L^2(\mathcal{O}_\varepsilon^\pm)} \leq C\sqrt{\varepsilon}\frac{\delta_1}{\delta_2}\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}, \quad \|\phi_2\|_{L^2(\mathcal{O}_\varepsilon^\pm)} \leq C\sqrt{\varepsilon}\sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}\|D(\phi)\|_{L^2(\Omega_{\varepsilon\delta})}.\tag{4.18}$$

So, the estimates (4.7) are proved.

Finally, (4.7)₃ and (4.6)₁ lead to (4.8). □

5 A priori estimates

For every open set \mathcal{A} , we denote

$$\mathcal{M}_{\mathcal{A}}(\phi) = \frac{1}{|\mathcal{A}|} \int_{\mathcal{A}} \phi \, dx, \quad \forall \phi \in L^1(\mathcal{A}).$$

5.1 Estimates of $v_{\varepsilon\delta}$

Set

$$\mathfrak{c}(\varepsilon, \delta_1, \delta_2) \doteq 1 + \sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)}.$$

Lemma 5.1. *The following estimate holds:*

$$\|D(v_{\varepsilon\delta})\|_{L^2(\Omega_{\varepsilon\delta})} \leq C \mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \quad (5.1)$$

Moreover, we have

$$\begin{aligned} \|\nabla v_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta})} &\leq C \mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}, \\ \|v_{1,\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta})} &\leq C \mathfrak{c}(\varepsilon, \delta_1, \delta_2)^2 \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}, \\ \|v_{1,\varepsilon\delta}\|_{L^2(\mathcal{O}_{\varepsilon^\pm}^\pm)} &\leq C \sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)} \mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}, \\ \|v_{2,\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta})} &\leq C \mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}, \\ \|v_{2,\varepsilon\delta}\|_{L^2(\mathcal{O}_{\varepsilon^\pm}^\pm)} &\leq C \sqrt{\varepsilon} \sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)} \mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \end{aligned} \quad (5.2)$$

The constants C do not depend on ε , δ_1 and δ_2 .

Proof. We choose $\phi = v_{\varepsilon\delta}$ as test function in (3.2), this gives

$$2\mu \int_{\Omega_{\varepsilon\delta}} D(v_{\varepsilon\delta}) : D(v_{\varepsilon\delta}) dx = \int_{\Gamma_2^{in}} \mathcal{P} v_{\varepsilon\delta,1} dx_2$$

Then, by virtue of Cauchy-Schwarz inequality, we obtain

$$\|D(v_{\varepsilon\delta})\|_{L^2(\Omega_{\varepsilon\delta})}^2 \leq C \|\mathcal{P}\|_{L^2(\Gamma_2^{in})} \|v_{\varepsilon\delta,1}\|_{L^2(\Gamma_2^{in})}. \quad (5.3)$$

Constant C depends on L_1 , L_2 , μ and does not depend on ε , δ_1 , δ_2 .

We recall that for any $\psi \in H^1(\Omega_{\delta_1}^-)$ we have

$$\|\psi\|_{L^2(\Gamma_2^{in})} \leq C (\|\psi\|_{L^2(\Omega_{\delta_1}^-)} + \|\psi\|_{H^1(\Omega_{\delta_1}^-)}).$$

The constant does not depend on δ_1 . Then, due to (4.6)₁ and (4.8) we obtain

$$\|v_{\varepsilon\delta,1}\|_{L^2(\Gamma_2^{in})} \leq C \mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|D(v_{\varepsilon\delta})\|_{L^2(\Omega_{\varepsilon\delta})}. \quad (5.4)$$

From above inequality (5.4) and (5.3) we get (5.1). Then, the estimates of Lemma 4.3 lead us to the estimates (5.2). \square

5.2 Estimates of $p_{\varepsilon\delta}$

Lemma 5.2. *We have the following estimate of the pressure in $\Omega_{\varepsilon\delta}$:*

$$\|p_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta})} \leq C \mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \quad (5.5)$$

The constant is independent of ε , δ_1 and δ_2 .

Proof. Step 1. We prove

$$\|p_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon^-})} + \|p_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon^+})} \leq C \mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \quad (5.6)$$

First, the Nečas inequality (see the recalls in Section B) and (5.1) give

$$\|p_{\varepsilon\delta} - \mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta})\|_{L^2(\Omega_\varepsilon^\pm)} \leq C_{\Omega_\varepsilon^\pm} \|\nabla p_{\varepsilon\delta}\|_{H^{-1}(\Omega_\varepsilon^\pm)} \leq C \|D(v_{\varepsilon\delta})\|_{L^2(\Omega_\varepsilon^\pm)} \leq C\mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}.$$

The constants $C_{\Omega_\varepsilon^\pm}$ do not depend on $\varepsilon, \delta_1, \delta_2$ since from Proposition B.2, we have

$$C_{\Omega_\varepsilon^\pm} \leq \max \left\{ \frac{L_1 - \varepsilon/2}{2L_2}, \frac{2L_2}{L_1 - \varepsilon/2} \right\} C(Y) \leq \max \left\{ \frac{L_1}{2L_2}, \frac{2L_2}{L_1/2} \right\} C(Y).$$

Now, in (3.2) we choose a test function $\phi_0 \in H^1(\Omega)^2$ vanishing in $(-\varepsilon, \varepsilon) \times (-L_2, L_2)$ and such that

$$\int_{\Omega_\varepsilon^\pm} \nabla \cdot \phi_0 \, dx = - \int_{\Gamma_2^{in}} \phi_0 \cdot \mathbf{e}_1 \, dx_2 + \int_{\Gamma_2^{out}} \phi_0 \cdot \mathbf{e}_1 \, dx_2 = 1.$$

Choosing ϕ_0 as test function in (3.2) leads to

$$\int_{\Omega_\varepsilon^\pm} p_{\varepsilon\delta} \nabla \cdot \phi_0 \, dx = 2\mu \int_{\Omega_\varepsilon^\pm} D(v_{\varepsilon\delta}) : D(\phi_0) \, dx - \int_{\Gamma_2^{in}} \mathcal{P} \phi_0 \cdot \mathbf{e}_1 \, dx_2.$$

Thus

$$\begin{aligned} \left| \int_{\Omega_\varepsilon^\pm} p_{\varepsilon\delta} \nabla \cdot \phi_0 \, dx \right| &\leq 2\mu \int_{\Omega_\varepsilon^\pm} |D(v_{\varepsilon\delta}) : D(\phi_0)| \, dx + \int_{\Gamma_2^{in}} |\mathcal{P}| |\phi_0 \cdot \mathbf{e}_1| \, dx_2 \\ &\leq C (\|D(v_{\varepsilon\delta})\|_{L^2(\Omega_\varepsilon^\pm)} + \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}) \|D(\phi_0)\|_{L^2(\Omega_\varepsilon^\pm)} \leq C\mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \end{aligned}$$

Observe that since $\phi_0 = 0$ a.e. on $\mathcal{O}_\varepsilon^-$ and $\phi_0 \in H^1(\Omega)^2$ we have $\|\phi_0 \cdot \mathbf{e}_1\|_{L^2(\Gamma_2^{in})} \leq C \|D(\phi_0)\|_{L^2(\Omega_\varepsilon^-)}$ and $\|D(\phi_0)\|_{L^2(\Omega_\varepsilon^\pm)} \leq C$, the constant does not depend on ε . So, we get

$$\begin{aligned} \mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta}) &= \int_{\Omega_\varepsilon^\pm} \mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta}) \nabla \cdot \phi_0 \, dx = \int_{\Omega_\varepsilon^\pm} (\mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta}) - p_{\varepsilon\delta}) \nabla \cdot \phi_0 \, dx + \int_{\Omega_\varepsilon^\pm} p_{\varepsilon\delta} \nabla \cdot \phi_0 \, dx \\ &\leq \|\mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta}) - p_{\varepsilon\delta}\|_{L^2(\Omega_\varepsilon^\pm)} \|D(\phi_0)\|_{L^2(\Omega_\varepsilon^\pm)} + \left| \int_{\Omega_\varepsilon^\pm} p_{\varepsilon\delta} \nabla \cdot \phi_0 \, dx \right| \leq C\mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \end{aligned}$$

Similarly, we show that

$$-\mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta}) \leq C\mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}.$$

Hence $|\mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta})| \leq C\mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}$. This completes the proof of (5.6).

Step 2. We prove

$$\|p_{\varepsilon\delta}\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} \leq C\mathfrak{c}(\varepsilon, \delta_1, \delta_2) \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \quad (5.7)$$

Proposition B.2 (Property 2) and Lemma C.2 yield

$$\sum_{p \in \Xi_\varepsilon} \|p_{\varepsilon\delta} - \mathcal{M}_{Y_{\varepsilon\delta}^p}(p_{\varepsilon\delta})\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \leq C \|D(v_{\varepsilon\delta})\|_{L^2(\Omega_{\varepsilon\delta})}^2. \quad (5.8)$$

The constant does not depend on ε, δ_1 and δ_2 .

From (5.6) and (5.8) we obtain

$$\sum_{p \in \Xi_\varepsilon} \|\mathcal{M}_{Y_{\varepsilon\delta}^p}(p_{\varepsilon\delta})\|_{L^2(Y_{\varepsilon\delta}^p \cap \Omega_\varepsilon^-)}^2 \leq 2 \sum_{p \in \Xi_\varepsilon} \|p_{\varepsilon\delta} - \mathcal{M}_{Y_{\varepsilon\delta}^p}(p_{\varepsilon\delta})\|_{L^2(Y_{\varepsilon\delta}^p)}^2 + 2 \sum_{p \in \Xi_\varepsilon} \|p_{\varepsilon\delta}\|_{L^2(Y_{\varepsilon\delta}^p \cap \Omega_\varepsilon^-)}^2 \leq C \|D(v_{\varepsilon\delta})\|_{L^2(\Omega_{\varepsilon\delta})}^2.$$

Hence

$$\|p_{\varepsilon\delta}\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} \leq C \|D(v_{\varepsilon\delta})\|_{L^2(\Omega_{\varepsilon\delta})}.$$

This leads to (5.7) from which and (5.6) the estimate (5.5) follows. \square

6 Asymptotic behavior

Based on the a priori estimates in Lemma 4.3, we can distinguish three different classes.

- $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow 0$.
- $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow \theta \in (0, +\infty)$.
- $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow \infty$ and $\delta_2 = 0$.

This work will be devoted to examining the first two classes of obstacles, for which we will consider two distinct cases.

Case 1: $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow 0$.

Case 2: $\delta_1 = \kappa\varepsilon$, $\kappa \in (0, 1/2]$ and $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow \theta \in (0, +\infty)$.

Remark 6.1. *In Case 3, all estimates blow up.*

In each case, fluid movement within the $\Omega_{\varepsilon\delta}$ domain is influenced by the shape of the inclusions. In cases 1 and 2, fluid motion is strongly or partially slowed by the inclusions (inclusions control fluid motion). In the third case, fluid movement within the $\Omega_{\varepsilon\delta}$ domain is no longer controlled by the inclusions. The third case should be studied again in the context of an evolutionary problem, which would provide information over long time scales.

Lemmas 5.1-5.2 give us the following estimates for the cases 1 and 2:

Lemma 6.1. *Let parameters ε , δ_1 , δ_2 satisfy cases 1 or 2, we have*

$$\begin{aligned} \|D(v_{\varepsilon\delta})\|_{L^2(\Omega_{\varepsilon\delta})} &\leq C\|\mathcal{P}\|_{L^2(\Gamma_2^{in})} & \|v_{\varepsilon\delta}\|_{H^1(\Omega_{\varepsilon\delta})} &\leq C\|\mathcal{P}\|_{L^2(\Gamma_2^{in})}, \\ \|v_{1,\varepsilon\delta}\|_{L^2(\mathcal{O}_\varepsilon^\pm)} &\leq C\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)} \|\mathcal{P}\|_{L^2(\Gamma_2^{in})}, & \|v_{2,\varepsilon\delta}\|_{L^2(\mathcal{O}_\varepsilon^\pm)} &\leq C\sqrt{\varepsilon} \sqrt{\ln\left(\frac{\varepsilon}{\delta_1}\right)} \|\mathcal{P}\|_{L^2(\Gamma_2^{in})} \end{aligned} \quad (6.1)$$

and

$$\|p_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta})} \leq C\|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \quad (6.2)$$

The constants do not depend on ε , δ_1 and δ_2 .

From now on, since the pressure $p_{\varepsilon\delta}$ is uniformly bounded in $L^2(\Omega_{\varepsilon\delta})$ we extend it by 0 in $\Omega \setminus \overline{\Omega_{\varepsilon\delta}}$. The extension is still denoted $p_{\varepsilon\delta}$. So, we have

$$\|p_{\varepsilon\delta}\|_{L^2(\Omega)} \leq C\|\mathcal{P}\|_{L^2(\Gamma_2^{in})}. \quad (6.3)$$

The constant does not depend on ε , δ_1 and δ_2 .

6.1 Auxiliary tools

Before looking at the different cases, let us consider some auxiliary tools: an extension operator, which will facilitate the convergences in the domains Ω_ε^\pm , and the function Θ_ε , used to construct test functions. Let ϕ be in $H^1(\Omega_\varepsilon^-)$ (resp. $\psi \in H^+(\Omega_\varepsilon)$), we define the operators $\mathcal{E}_\varepsilon^-$ and $\mathcal{E}_\varepsilon^+$ by

$$\begin{aligned} \mathcal{E}_\varepsilon^-(\phi)(x) &= \begin{cases} \phi(x) & \text{for a.e. } x \in \Omega_\varepsilon^-, \\ \phi(-2\varepsilon - x_1, x_2) & \text{for a.e. } x \in \Omega^- \setminus \overline{\Omega_\varepsilon^-}, \end{cases} \\ \mathcal{E}_\varepsilon^+(\psi)(x) &= \begin{cases} \psi(x) & \text{for a.e. } x \in \Omega_\varepsilon^+, \\ \psi(2\varepsilon - x_1, x_2) & \text{for a.e. } x \in \Omega^+ \setminus \overline{\Omega_\varepsilon^+}, \end{cases} \end{aligned}$$

Lemma 6.2. *The extension operators $\mathcal{E}_\varepsilon^\pm$ from $H^1(\Omega_\varepsilon^\pm)$ into $H^1(\Omega^\pm)$ satisfy*

$$\begin{aligned} \forall \phi \in H^1(\Omega_\varepsilon^-), \quad & \|\mathcal{E}_\varepsilon^-(\phi)\|_{H^1(\Omega^-)} \leq C\|\phi\|_{H^1(\Omega_\varepsilon^-)}, \quad \|\mathcal{E}_\varepsilon^-(\phi)\|_{L^2(\mathcal{O})} \leq C(\|\phi\|_{L^2(\mathcal{O}_\varepsilon^-)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_\varepsilon^-)}), \\ \forall \phi \in H^1(\Omega_\varepsilon^+), \quad & \|\mathcal{E}_\varepsilon^+(\phi)\|_{H^1(\Omega^+)} \leq C\|\phi\|_{H^1(\Omega_\varepsilon^+)}, \quad \|\mathcal{E}_\varepsilon^+(\phi)\|_{L^2(\mathcal{O})} \leq C(\|\phi\|_{L^2(\mathcal{O}_\varepsilon^+)} + \varepsilon\|\nabla\phi\|_{L^2(\Omega_\varepsilon^+)}). \end{aligned}$$

Moreover, for every $\phi \in H^1(\Omega_{\varepsilon\delta})$ we have

$$\|\mathcal{E}_\varepsilon^+(\phi) - \mathcal{E}_\varepsilon^-(\phi)\|_{L^2(\mathcal{O})} \leq C\sqrt{\varepsilon}\|\nabla\phi\|_{L^2(\Omega_{\varepsilon\delta})}. \quad (6.4)$$

The constants do not depend on ε , δ_1 and δ_2 .

Proof. The estimates of $\mathcal{E}_\varepsilon^\pm(\phi)$, for $\phi \in H^1(\Omega_\varepsilon^\pm)$, are the immediate consequences of the definitions of these operators.

Now, let ϕ be in $H^1(\Omega_{\varepsilon\delta})$, to obtain (6.4), we have to estimate

$$\int_{-L_2}^{L_2} |\phi(2\varepsilon, x_2) - \phi(-2\varepsilon, x_2)|^2 dx_2.$$

Since the small domains $Y_{\varepsilon\delta}^p$, $p \in \Xi_\varepsilon$, are star-shaped with respect to a ball of radius $\varepsilon/16$ and have a diameter less than 3ε , the Poincaré-Wirtinger inequality gives

$$\|\phi - \mathcal{M}_\varepsilon(\phi)\|_{L^2(Y_{\varepsilon\delta}^p)} \leq C\varepsilon\|\nabla\phi\|_{L^2(Y_{\varepsilon\delta}^p)}.$$

The constant does not depend on ε and δ_2 . Then, we get the following estimates of the traces

$$\begin{aligned} \|\phi - \mathcal{M}_\varepsilon(\phi)\|_{L^2(\mathcal{O}_{2\varepsilon}^+)}^2 & \leq C\left(\frac{1}{\varepsilon}\|\phi - \mathcal{M}_\varepsilon(\phi)\|_{L^2(Y_{\varepsilon\delta}^p)}^2 + \varepsilon\left\|\frac{\partial\phi}{\partial x_1}\right\|_{L^2((\varepsilon, 2\varepsilon) \times (-L_2, L_2))}^2\right), \\ \|\phi - \mathcal{M}_\varepsilon(\phi)\|_{L^2(\mathcal{O}_{2\varepsilon}^-)}^2 & \leq C\left(\frac{1}{\varepsilon}\|\phi - \mathcal{M}_\varepsilon(\phi)\|_{L^2(Y_{\varepsilon\delta}^p)}^2 + \varepsilon\left\|\frac{\partial\phi}{\partial x_1}\right\|_{L^2((-2\varepsilon, -\varepsilon) \times (-L_2, L_2))}^2\right). \end{aligned}$$

As a consequence of the above estimates and a trace result (see e.g. (4.1)₁) we obtain

$$\int_{-L_2}^{L_2} |\phi(2\varepsilon, x_2) - \phi(-2\varepsilon, x_2)|^2 dx_2 \leq C\varepsilon\|\nabla\phi\|_{L^2(\Omega_{\varepsilon\delta})}^2.$$

The constant does not depend on ε . This leads to (6.4). \square

Now, we define the ε -periodic function $\psi_{\varepsilon\delta} \in W^{1,\infty}(0, 1)$ by

$$\psi_{\varepsilon\delta}(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{\delta_1}{\varepsilon}\right], \\ \frac{\varepsilon}{\delta_1}\left(2\frac{\delta_1}{\varepsilon} - t\right) & \text{if } t \in \left[\frac{\delta_1}{\varepsilon}, 2\frac{\delta_1}{\varepsilon}\right], \\ 0 & \text{if } t \in \left[2\frac{\delta_1}{\varepsilon}, 1\right] \end{cases}$$

and $\Theta_\varepsilon \in W^{1,\infty}(\Omega_{\varepsilon\delta})$ by

$$\Theta_\varepsilon(x) = \frac{\delta_2}{\delta_1} \left[\frac{d\psi^d}{dy_1} \left(\frac{x_1}{\delta_1} \right) \psi_{\varepsilon\delta} \left(\left\{ \frac{x_2}{\varepsilon} \right\} \right) - \frac{d\psi^u}{dy_1} \left(\frac{x_1}{\delta_1} \right) \psi_{\varepsilon\delta} \left(1 - \left\{ \frac{x_2}{\varepsilon} \right\} \right) \right] \quad \text{for a.e. } (x_1, x_2) \in \Omega_{\varepsilon\delta},$$

where for a.e. $t \in \mathbb{R}$, $\{t\}$ is the fractional part of t , it belongs to $[0, 1)$.

Observe that the support of Θ_ε is included in $\bigcup_{\xi \in \Xi_\varepsilon} (-\delta_1, \delta_1) \times (-2\delta_1 + \xi\varepsilon, 2\delta_1 + \xi\varepsilon)$.

Lemma 6.3. *We have*

$$\|\Theta_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta})} \leq C \frac{\delta_2}{\varepsilon^{1/2}}, \quad \|\nabla\Theta_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta})} \leq C \frac{\delta_2}{\delta_1 \varepsilon^{1/2}}.$$

The constants do not depend on ε , δ_1 and δ_2 .

Proof. We have

$$\|\Theta_\varepsilon\|_{L^\infty(\Omega_{\varepsilon\delta})} = 2 \frac{\delta_2}{\delta_1}, \quad \|\nabla\Theta_\varepsilon\|_{L^\infty(\Omega_{\varepsilon\delta})} \leq C \frac{\delta_2}{\delta_1^2}. \quad (6.5)$$

Since the measure of the support of Θ_ε is bounded by $C \frac{\delta_1^2}{\varepsilon}$, where the constants do not depend on ε , δ_1 and δ_2 . These lead to the estimates of the lemma. \square

6.2 Case 1: $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow 0$

We introduce the following spaces:

$$\begin{aligned} \mathbf{V}^{(1)}(\Omega^-) &\doteq \{\phi \in H^1(\Omega^-)^2 \mid \phi_2 = 0 \text{ a.e. on } \Gamma_1 \cap \Omega^-, \phi = 0 \text{ a.e. on } \mathcal{O}\}, \\ \mathbf{V}_{div0}^{(1)}(\Omega^-) &\doteq \{\phi \in \mathbf{V}^{(1)}(\Omega^-) \mid \nabla \cdot \phi = 0 \text{ a.e. on } \Omega^-\}, \\ \mathbf{V}^{(1)}(\Omega^+) &\doteq \{\phi \in H^1(\Omega^+)^2 \mid \phi_2 = 0 \text{ a.e. on } \Gamma_1 \cap \Omega^+, \phi = 0 \text{ a.e. on } \mathcal{O}\}, \\ \mathbf{V}_{div0}^{(1)}(\Omega^+) &\doteq \{\phi \in \mathbf{V}^{(1)}(\Omega^+) \mid \nabla \cdot \phi = 0 \text{ a.e. on } \Omega^+\}. \end{aligned}$$

Lemma 6.4. *Let $(v_{\varepsilon\delta}, p_{\varepsilon\delta})$ be the solution of the problem (3.2) and $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow 0$.*

Then there exist $v^- \in \mathbf{V}_{div0}^{(1)}(\Omega^-)$, $p^- \in L^2(\Omega^-)$ such that

$$\begin{aligned} v_{\varepsilon\delta} \mathbf{1}_{\Omega_\varepsilon^-} &\rightharpoonup v^- \quad \text{weakly in } L^2(\Omega^-)^2, \\ \nabla v_{\varepsilon\delta} \mathbf{1}_{\Omega_\varepsilon^-} &\rightharpoonup \nabla v^- \quad \text{weakly in } L^2(\Omega^-)^{2 \times 2}, \\ p_{\varepsilon\delta} \mathbf{1}_{\Omega^-} &\rightharpoonup p^- \quad \text{weakly in } L^2(\Omega^-)^2. \end{aligned} \quad (6.6)$$

The pairs $(v^-, p^-) \in \mathbf{V}_{div0}^{(1)}(\Omega^-) \times L^2(\Omega^-)$ is the solution to

$$2\mu \int_{\Omega^-} D(v^-) : D(\phi) dx - \int_{\Omega^-} p^- \nabla \cdot \phi dx = \int_{\Gamma_2^{in}} \mathcal{P} \phi_1 dx_2, \quad \forall \phi \in \mathbf{V}^{(1)}(\Omega^-). \quad (6.7)$$

In Ω_ε^+ we have

$$\begin{aligned} v_{\varepsilon\delta} \mathbf{1}_{\Omega_\varepsilon^+} &\rightharpoonup 0 \quad \text{weakly in } L^2(\Omega^+)^2, \\ \nabla v_{\varepsilon\delta} \mathbf{1}_{\Omega_\varepsilon^+} &\rightharpoonup 0 \quad \text{weakly in } L^2(\Omega^+)^{2 \times 2}, \\ p_{\varepsilon\delta} \mathbf{1}_{\Omega^+} &\rightharpoonup 0 \quad \text{weakly in } L^2(\Omega^+)^2. \end{aligned} \quad (6.8)$$

Moreover, we have

$$\begin{aligned} D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega^- \cap \Omega_{\varepsilon\delta}} &\rightarrow D(v^-) \quad \text{strongly in } L^2(\Omega^-)^{2 \times 2}, \\ D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega^+ \cap \Omega_{\varepsilon\delta}} &\rightarrow 0 \quad \text{strongly in } L^2(\Omega^+)^{2 \times 2}, \\ \text{and } v_{\varepsilon\delta} \mathbf{1}_{\Omega_\varepsilon^-} &\rightarrow v^- \quad \text{strongly in } L^2(\Omega^-)^2, \\ u_{\varepsilon\delta} \mathbf{1}_{\Omega_\varepsilon^+} &\rightarrow 0 \quad \text{strongly in } L^2(\Omega^+)^2. \end{aligned} \quad (6.9)$$

Proof. Step 1. From the estimates in (6.1)_{1,2}, we obtain, up to a subsequence of $\{\varepsilon, \delta_1, \delta_2\}$, that there exist $v^\pm \in H^1(\Omega^\pm)^2$, $p^\pm \in L^2(\Omega^\pm)$ such that

$$\begin{aligned} v_{\varepsilon\delta} \mathbf{1}_{\Omega^\pm} &\rightharpoonup v^\pm \quad \text{weakly in } L^2(\Omega^\pm)^2, \\ \nabla \cdot v_{\varepsilon\delta} \mathbf{1}_{\Omega^\pm} &\rightharpoonup 0 \quad \text{weakly in } L^2(\Omega^\pm), \\ \nabla v_{\varepsilon\delta} \mathbf{1}_{\Omega^\pm} &\rightharpoonup \nabla v^\pm \quad \text{weakly in } L^2(\Omega^\pm)^{2 \times 2}, \\ p_{\varepsilon\delta} \mathbf{1}_{\Omega^\pm} &\rightharpoonup p^\pm \quad \text{weakly in } L^2(\Omega^\pm)^2. \end{aligned} \tag{6.10}$$

The estimates (6.1)_{3,4} yield

$$v^\pm = 0 \quad \text{a.e. on } \mathcal{O} \cup (\Gamma_1 \cap \partial\Omega^\pm).$$

As a first consequence, v^- belongs to $\mathbf{V}_{div0}^{(1)}(\Omega^-)$ and $v^+ \in \mathbf{V}_{div0}^{(1)}(\Omega^+)$.

Step 2. We prove (6.6), (6.7).

Let ϕ be in $\mathbf{V}^{(1)}(\Omega^-)$ vanishing in a neighborhood of \mathcal{O} . We choose ϕ as test function in (3.2).

For ε sufficiently small, one has

$$2\mu \int_{\Omega_\varepsilon^-} D(v_{\varepsilon\delta}) : D(\phi) dx - \int_{\Omega_\varepsilon^-} p_{\varepsilon\delta} \nabla \cdot \phi dx = \int_{\Gamma_2^{in}} \mathcal{P}\phi_1 dx_2.$$

Passing to the limit and taking into account the above convergences yield (6.7). Then, a density argument gives (6.7) for any $\phi \in \mathbf{V}^{(1)}(\Omega^-)$. The uniqueness of the solution to problem (6.7) implies that the whole sequences go to their limits.

Step 3. We prove (6.8).

Now, in (3.2) we choose a test function $\phi \in H^1(\Omega)^2$ vanishing in $(-\varepsilon, \varepsilon) \times (-L_2, L_2)$. Choosing ϕ as test function in (3.2) leads to

$$2\mu \int_{\Omega_\varepsilon^-} D(v_{\varepsilon\delta}) : D(\phi) dx + 2\mu \int_{\Omega_\varepsilon^+} D(v_{\varepsilon\delta}) : D(\phi) dx - \int_{\Omega_\varepsilon^-} p_{\varepsilon\delta} \nabla \cdot \phi dx - \int_{\Omega_\varepsilon^+} p_{\varepsilon\delta} \nabla \cdot \phi dx = \int_{\Gamma_2^{in}} \mathcal{P}\phi_1 dx_2.$$

Passing to the limit and taking into account convergence (6.10), we get

$$2\mu \int_{\Omega^-} D(v^-) : D(\phi) dx + 2\mu \int_{\Omega^+} D(v^+) : D(\phi) dx - \int_{\Omega^-} p^- \nabla \cdot \phi dx - \int_{\Omega^+} p^+ \nabla \cdot \phi dx = \int_{\Gamma_2^{in}} \mathcal{P}\phi_1 dx_2.$$

So, the pair $(v^+, p^+) \in \mathbf{V}_{div0}^{(1)}(\Omega^+) \times L^2(\Omega^+)$ is the solution to

$$2\mu \int_{\Omega^+} D(v^+) : D(\phi) dx - \int_{\Omega^+} p^+ \nabla \cdot \phi dx = 0.$$

A density argument gives the above equality for any $\phi \in \mathbf{V}^{(1)}(\Omega^+)$. As a consequence, we obtain $(u^+, p^+) = (0, 0)$ and then the convergences (6.8) for the whole sequences.

Step 4. We prove (6.9).

First, observe that from the estimate (6.1)₂ and convergences (6.10) we have

$$\begin{aligned} D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega^- \cap \Omega_{\varepsilon\delta}} &\rightharpoonup D(v^-) \quad \text{weakly in } L^2(\Omega^-)^{2 \times 2}, \\ D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega^+ \cap \Omega_{\varepsilon\delta}} &\rightharpoonup 0 \quad \text{weakly in } L^2(\Omega^+)^{2 \times 2}. \end{aligned} \tag{6.11}$$

Now, in (3.2) we choose as a test function $v_{\varepsilon\delta}$

$$2\mu \int_{\Omega_{\varepsilon\delta}} D(v_{\varepsilon\delta}) : D(v_{\varepsilon\delta}) dx = \int_{\Gamma_2^{in}} \mathcal{P}v_{\varepsilon\delta,1} dx_2.$$

Hence

$$2\mu \int_{\Omega} (D(v_{\varepsilon\delta})\mathbf{1}_{\Omega_{\varepsilon\delta}}) : (D(v_{\varepsilon\delta})\mathbf{1}_{\Omega_{\varepsilon\delta}}) dx = \int_{\Gamma_2^{in}} \mathcal{P}v_{\varepsilon\delta,1} dx_2.$$

Then, by the convergences (6.11), (6.10) and the weak lower semicontinuity of the LHS we obtain

$$\begin{aligned} & 2\mu \int_{\Omega^-} D(v^-) : D(v^-) dx + 2\mu \int_{\Omega^+} D(v^+) : D(v^+) dx \leq \liminf \left(2\mu \int_{\Omega} (D(v_{\varepsilon\delta})\mathbf{1}_{\Omega_{\varepsilon\delta}}) : (D(v_{\varepsilon\delta})\mathbf{1}_{\Omega_{\varepsilon\delta}}) dx \right) \\ & \leq \limsup \left(2\mu \int_{\Omega} (D(v_{\varepsilon\delta})\mathbf{1}_{\Omega_{\varepsilon\delta}}) : (D(v_{\varepsilon\delta})\mathbf{1}_{\Omega_{\varepsilon\delta}}) dx \right) = \limsup \left(\int_{\Gamma_2^{in}} \mathcal{P}v_{\varepsilon\delta,1} dx_2 \right) \\ & = \lim \left(\int_{\Gamma_2^{in}} \mathcal{P}v_{\varepsilon\delta,1} dx_2 \right) = \int_{\Gamma_2^{in}} \mathcal{P}v_1^- dx_2 = 2\mu \int_{\Omega^-} D(v^-) : D(v^-) dx. \end{aligned}$$

So, the above inequalities are equalities and therefore the convergences in (6.9) are proved. \square

6.3 Case 2: $\delta_1 = \kappa\varepsilon$, $\kappa \in (0, 1/2]$ and $\sqrt{\varepsilon} \frac{\delta_1}{\delta_2} \rightarrow \theta \in (0, +\infty)$

Denote

$$Z = \mathbb{R} \times (0, 1), \quad Z_a = (-a, a) \times (0, 1), \quad a > 0, \quad Y = Z_1.$$

We introduce the following spaces:

$$\begin{aligned} \mathbf{V}(\Omega) &\doteq \{ \phi \in H^1(\Omega)^2 \mid \phi_2 = 0 \text{ a.e. on } \Gamma_1 \cup \mathcal{O} \}, \\ \mathbf{V}_{div0}(\Omega) &\doteq \{ \phi \in \mathbf{V}(\Omega) \mid \nabla \cdot \phi = 0 \text{ a.e. in } \Omega \}, \\ \mathbf{H}^1(Z) &\doteq \{ \Phi \in H_{loc}^1(Z) \mid \nabla_y \Phi \in L^2(Z)^2, \quad \Phi(0, \cdot) \in L^2(0, 1) \}, \\ \mathcal{H}^1(Z) &\doteq \{ \Phi \in \mathbf{H}^1(Z) \mid \Phi(y_1, 1) = \Phi(y_1, 0) \text{ for a.e. } y_1 \in \mathbb{R} \setminus [-\kappa, \kappa] \}, \\ \mathcal{H}_0^1(Z) &\doteq \{ \Phi \in \mathcal{H}^1(Z) \mid \Phi(y_1, 1) = \Phi(y_1, 0) = 0 \text{ for a.e. } y_1 \in (-\kappa, \kappa) \}, \\ \mathbf{W}(Z) &\doteq \{ (\Phi_1, \Phi_2) \in \mathcal{H}^1(Z) \times \mathcal{H}_0^1(Z) \mid \mathcal{M}_Y(\Phi_1) = 0 \}, \\ \mathbf{W}_{div0}(Z) &\doteq \{ (\Phi_1, \Phi_2) \in \mathbf{W}(Z) \mid \nabla_y \cdot \Phi = 0 \text{ a.e. in } Z \}. \end{aligned}$$

We endow $\mathbf{H}^1(Z)$ with the norm

$$\|\phi\|_{\mathbf{H}^1(Z)} = \sqrt{\|\nabla\phi\|_{L^2(Z)}^2 + \|\phi(0, \cdot)\|_{L^2(0,1)}^2}.$$

Lemma 6.5. *Let $(v_{\varepsilon\delta}, p_{\varepsilon\delta})$ be the solution to the problem (3.2).*

Then, there exists $(v, p) \in \mathbf{V}_{div0}(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} D(v_{\varepsilon\delta})\mathbf{1}_{\Omega_{\varepsilon\delta}} &\rightarrow D(v) && \text{strongly in } L^2(\Omega)^{2 \times 2}, \\ v_{\varepsilon\delta}\mathbf{1}_{\Omega_{\varepsilon\delta}} &\rightarrow v && \text{strongly in } L^2(\Omega)^2, \\ p_{\varepsilon\delta}\mathbf{1}_{\Omega_{\varepsilon\delta}^\pm} &\rightarrow p && \text{strongly in } L^2(\Omega). \end{aligned} \tag{6.12}$$

The pair $(v, p) \in \mathbf{V}_{div0}(\Omega) \times L^2(\Omega)$ is the unique solution to

$$2\mu \int_{\Omega} D(v) : D(\phi) dx - \int_{\Omega} p \nabla \cdot \phi dx = \int_{\Gamma_2^{in}} \mathcal{P}\phi_1 dx_2, \quad \forall \phi \in \mathbf{V}(\Omega). \tag{6.13}$$

Proof. Step 1. We show the convergences

$$\begin{aligned}\mathcal{E}_\varepsilon^\pm(v_{\varepsilon\delta}) &\rightharpoonup v|_{\Omega^\pm} && \text{weakly in } H^1(\Omega^\pm)^2, \\ p_{\varepsilon\delta} &\rightharpoonup p && \text{weakly in } L^2(\Omega).\end{aligned}\tag{6.14}$$

Using the operators $\mathcal{E}_\varepsilon^\pm$ (see Lemma 6.2) we extend the restrictions $v_{\varepsilon\delta}|_{\Omega_\varepsilon^\pm}$ to obtain functions belonging to $H^1(\Omega^\pm)^2$. Estimates of Lemmas 6.1 and 6.2 lead to

$$\begin{aligned}\|\mathcal{E}_\varepsilon^\pm(v_{\varepsilon\delta})\|_{H^1(\Omega^\pm)} &\leq C\|\mathcal{P}\|_{L^2(\Gamma_2^{in})}, \quad \|\mathcal{E}_\varepsilon^+(v_{\varepsilon\delta,1}) - \mathcal{E}_\varepsilon^-(v_{\varepsilon\delta,1})\|_{L^2(\mathcal{O})} \leq C\sqrt{\varepsilon}\|\mathcal{P}\|_{L^2(\Gamma_2^{in})}, \\ \|\mathcal{E}_\varepsilon^\pm(v_{\varepsilon\delta,2})\|_{L^2(\mathcal{O})} &\leq C\sqrt{\varepsilon}\|\mathcal{P}\|_{L^2(\Gamma_2^{in})}.\end{aligned}\tag{6.15}$$

The constants do not depend on ε , δ_1 and δ_2 .

Hence, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $(v, p) \in \mathbf{V}_{div0}(\Omega) \times L^2(\Omega)$, such that the convergences (6.14) hold.

Step 2. Let ϕ be in $\mathbf{V}(\Omega)$ such that $\phi = 0$ in Ω^+ and in a neighborhood of \mathcal{O} . We choose ϕ as test function in (3.2). Passing to the limit gives

$$2\mu \int_{\Omega^-} D(v) : D(\phi) dx - \int_{\Omega^-} p \nabla \cdot \phi dx = \int_{\Gamma_2^{in}} \mathcal{P} \phi_1 dx_2.\tag{6.16}$$

Then, we choose $\phi \in \mathbf{V}(\Omega)$ such that $\phi = 0$ in Ω^- and in a neighborhood of \mathcal{O} . Passing to the limit gives

$$2\mu \int_{\Omega^+} D(v) : D(\phi) dx - \int_{\Omega^+} p \nabla \cdot \phi dx = 0\tag{6.17}$$

A density argument gives (6.13) for all test functions in $\mathbf{V}(\Omega)$ such that $\phi = 0$ a.e. on \mathcal{O} .

Step 3. In this step we show that there exists $\hat{u} \in L^2(\mathcal{O}; \mathbf{W}_{div0}(Z))$ and $\hat{p} \in L^2(\mathcal{O} \times Z)$ such that

$$\begin{aligned}2\mu \int_{\Omega} D(v) : D(\phi) dx - \int_{\Omega} p \nabla \cdot \phi dx + \frac{2\mu}{\theta} \int_{\mathcal{O} \times Z} D_y(\hat{v}) : D_y(\Theta \mathbf{e}_2) \phi_1 dx_2 dy \\ - \frac{1}{\theta} \int_{\mathcal{O} \times Z} \hat{p} \partial_{y_2} \Theta \phi_1 dx_2 dy = \int_{\Gamma_2^{in}} \mathcal{P} \phi_1 dx_2, \quad \forall \phi \in \mathbf{V}(\Omega),\end{aligned}\tag{6.18}$$

Let ϕ_1 be in $\mathcal{C}^1(\bar{\Omega})$. We set

$$\phi_\varepsilon(x_1, x_2) = \phi_1(x_1, x_2)(\mathbf{e}_1 + \Theta_\varepsilon(x_1, x_2)\mathbf{e}_2), \quad (x_1, x_2) \in \Omega.\tag{6.19}$$

The function $\phi_\varepsilon \in \mathbf{V}(\Omega)$.

From the definition of ϕ_ε and estimates (6.5), we have

$$\|\phi_{\varepsilon,2}\|_{H^1(\Omega)} \leq C \left(\|\nabla \phi_1\|_{L^\infty(\Omega)} \frac{\delta_2}{\sqrt{\varepsilon}} + \|\phi_1\|_{L^\infty(\Omega)} \frac{\delta_2}{\delta_1 \sqrt{\varepsilon}} \right) \leq C \frac{\delta_2}{\delta_1 \sqrt{\varepsilon}} \|\phi\|_{W^{1,\infty}(\Omega)}\tag{6.20}$$

and then

$$\phi_\varepsilon \rightharpoonup \phi_1 \mathbf{e}_1 \quad \text{weakly in } H^1(\Omega)^2\tag{6.21}$$

since the support of Θ_ε is included in $\bigcup_{\xi \in \Xi_\varepsilon} (-\delta_1, \delta_1) \times (-2\delta_1 + \xi\varepsilon, 2\delta_1 + \xi\varepsilon)$ (its measure is of order ε).

We choose ϕ_ε as test function in (3.2). This gives

$$2\mu \int_{\Omega_{\varepsilon\delta}} D(v_{\varepsilon\delta}) : D(\phi_\varepsilon) dx - \int_{\Omega_{\varepsilon\delta}} p_{\varepsilon\delta} \nabla \cdot \phi_\varepsilon dx = \int_{\Gamma_2^{in}} \mathcal{P} \phi_{\varepsilon,1} dx_2.\tag{6.22}$$

First, we have

$$\begin{aligned} \int_{\Omega_\varepsilon^\pm} D(v_{\varepsilon\delta}) : D(\phi_\varepsilon) dx &= \int_{\Omega_\varepsilon^\pm} D(v_{\varepsilon\delta}) : D(\phi_1 \mathbf{e}_1) dx \rightarrow \int_{\Omega} D(v) : D(\phi_1 \mathbf{e}_1) dx, \\ \int_{\Gamma_2^{in}} \mathcal{P} \phi_{\varepsilon,1} dx_2 &= \int_{\Gamma_2^{in}} \mathcal{P} \phi_1 dx_2. \end{aligned} \quad (6.23)$$

It remains to obtain the limits of the quantities below in in $\mathbf{Y}_{\varepsilon\delta}$

$$\begin{aligned} \int_{\mathbf{Y}_{\varepsilon\delta}} D(v_{\varepsilon\delta}) : D(\phi_1 \Theta_\varepsilon \mathbf{e}_2) dx &= \int_{\mathbf{Y}_{\varepsilon\delta}} \Theta_\varepsilon D(v_{\varepsilon\delta}) : D(\phi_1 \mathbf{e}_2) dx + \int_{\mathbf{Y}_{\varepsilon\delta}} \phi_1 D(v_{\varepsilon\delta}) : D(\Theta_\varepsilon \mathbf{e}_2) dx, \\ \int_{\mathbf{Y}_{\varepsilon\delta}} p_{\varepsilon\delta} \partial_2(\phi_1 \Theta_\varepsilon) dx &= \int_{\mathbf{Y}_{\varepsilon\delta}} \Theta_\varepsilon p_{\varepsilon\delta} \partial_2 \phi_1 dx + \int_{\mathbf{Y}_{\varepsilon\delta}} \phi_1 p_{\varepsilon\delta} \partial_2 \Theta_\varepsilon dx. \end{aligned}$$

Due to the estimate (6.5)₁ the first terms in the right-hand sides tend to 0.

Now, we consider the second terms in the RHS.

Using the unfolding operator $\mathcal{T}_\varepsilon^*$, we obtain

$$\begin{aligned} \int_{\mathbf{Y}_{\varepsilon\delta}} \phi_1 p_{\varepsilon\delta} \partial_2 \Theta_\varepsilon dx &= \varepsilon \int_{\mathcal{O} \times Z} \mathcal{T}_\varepsilon^*(\phi_1) \mathcal{T}_\varepsilon^*(p_{\varepsilon\delta}) \mathcal{T}_\varepsilon^*(\partial_2 \Theta_\varepsilon) dx_2 dy \\ &= \int_{\mathcal{O} \times Z} \mathcal{T}_\varepsilon^*(\phi_1) \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(p_{\varepsilon\delta}) \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\partial_2 \Theta_\varepsilon) dx_2 dy \\ \text{and } \int_{\mathbf{Y}_{\varepsilon\delta}} \phi_1 D(v_{\varepsilon\delta}) : D(\Theta_\varepsilon \mathbf{e}_2) dx &= \varepsilon \int_{\mathcal{O} \times Z} \mathcal{T}_\varepsilon^*(\phi_1) \mathcal{T}_\varepsilon^*(D(v_{\varepsilon\delta})) : \mathcal{T}_\varepsilon^*(D(\Theta_\varepsilon \mathbf{e}_2)) dx_2 dy \\ &= \int_{\mathcal{O} \times Z} \mathcal{T}_\varepsilon^*(\phi_1) \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(D(v_{\varepsilon\delta})) : \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(D(\Theta_\varepsilon \mathbf{e}_2)) dx_2 dy. \end{aligned}$$

From Lemmas A.2, A.3, there exist a subsequence of ε , still denoted $\{\varepsilon\}$, $\hat{v} \in L^2(\mathcal{O}; \mathbf{W}_{div0}(Z))$ and $\hat{p} \in L^2(\mathcal{O} \times Z)$ such that

$$\begin{aligned} \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla v_{\varepsilon\delta}) &\rightharpoonup \nabla_y \hat{v} \quad \text{weakly in } L^2(\mathcal{O} \times Z)^2, \\ \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(p_{\varepsilon\delta}) &\rightharpoonup \hat{p} \quad \text{weakly in } L^2(\mathcal{O} \times Z), \\ \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla \Theta_\varepsilon) &\rightarrow \frac{1}{\theta} \nabla_y \Theta \quad \text{strongly in } L^2(\mathcal{O} \times Z)^2. \end{aligned} \quad (6.24)$$

The third convergence in (6.24) holds since

$$\sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla \Theta_\varepsilon) = \frac{1}{\sqrt{\varepsilon}} \nabla_y \mathcal{T}_\varepsilon^*(\Theta_\varepsilon) = \frac{\delta_2}{\delta_1 \sqrt{\varepsilon}} \nabla_y \Theta$$

and $\frac{\delta_2}{\delta_1 \sqrt{\varepsilon}} \rightarrow \frac{1}{\theta}$.

Passing to the limit gives

$$\int_{\mathbf{Y}_{\varepsilon\delta}} \phi_1 p_{\varepsilon\delta} \partial_2 \Theta_\varepsilon dx \rightarrow \frac{1}{\theta} \int_{\mathcal{O} \times Z} \phi_1(0, x_2) \hat{p}(x_2, y) \partial_{y_2} \Theta(y) dx_2 dy$$

and

$$\int_{\mathbf{Y}_{\varepsilon\delta}} \phi_1 D(v_{\varepsilon\delta}) : D(\Theta_\varepsilon \mathbf{e}_2) dx \rightarrow \frac{1}{\theta} \int_{\mathcal{O} \times Z} \phi_1(0, x_2) D_y(\hat{u}) : D_y(\Theta \mathbf{e}_2) dx_2 dy.$$

Hence, we obtain

$$\begin{aligned}
2\mu \int_{\Omega} D(v) : D(\phi_1 \mathbf{e}_1) dx - \int_{\Omega} p \nabla \cdot (\phi_1 \mathbf{e}_1) dx + \frac{2\mu}{\theta} \int_{\mathcal{O} \times Z} \phi_1 D_y(\hat{v}) : D_y(\Theta \mathbf{e}_2) dx_2 dy \\
- \frac{1}{\theta} \int_{\mathcal{O} \times Z} \phi_1 \hat{p} \partial_{y_2} \Theta dx_2 dy = \int_{\Gamma_2^{in}} \mathcal{P} \phi_1 dx_2.
\end{aligned} \tag{6.25}$$

A density of argument gives (6.24) for all functions $\phi = (\phi_1, 0) \in \mathbf{V}(\Omega)$. Equalities (6.16), (6.17) and (6.24) lead to (6.18) for every $\phi \in \mathbf{V}(\Omega)$.

Step 4. We show that

$$2\mu \int_{\mathcal{O} \times Z} D_y(\hat{v}) : D_y(\hat{\psi}) dx_2 dy - \int_{\mathcal{O} \times Z} \hat{p} \nabla_y \cdot \hat{\psi} dx_2 dy = 0, \quad \forall \hat{\psi} \in L^2(\mathcal{O}; \mathbf{W}(Z)). \tag{6.26}$$

Let Φ be in $\mathcal{C}_c^1(\mathcal{O})$ and $\hat{\phi}$ in $\mathbf{W}(Z) \cap \mathcal{C}^1(\bar{Z})^2$ such that $\hat{\phi}_2$ vanishes in the neighborhood of $[-\kappa, \kappa] \times \{0, 1\}$.

Now, consider the test function

$$\phi_\varepsilon(x) = \sqrt{\varepsilon} \Phi(x_2) \left(\hat{\phi}_1 \left(\frac{x}{\varepsilon} \right) (\mathbf{e}_1 + \Theta_\varepsilon(x) \mathbf{e}_2) + \hat{\phi}_2 \left(\frac{x}{\varepsilon} \right) \mathbf{e}_2 \right) \quad \text{for a.e. } x \in \Omega_{\varepsilon\delta}.$$

For ε small enough, ϕ_ε is an admissible test function belonging to $\mathbf{V}_{ad}(\Omega_{\varepsilon\delta})$. We take it as a test function in (3.2) and apply $\mathcal{T}_\varepsilon^*$ (see Proposition A.1 for the properties of this operator)

$$\begin{aligned}
2\mu \int_{\mathcal{O} \times Z} \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(D(v_{\varepsilon\delta})) : \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(D(\phi_\varepsilon)) dx_2 dy - \int_{\mathcal{O} \times Z} \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(p_{\varepsilon\delta}) \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla \cdot \phi_\varepsilon) dx_2 dy \\
= \int_{\Gamma_2^{in}} \mathcal{P}(x_2) \Phi(x_2) \sqrt{\varepsilon} \hat{\phi}_1 \left(-\frac{L_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) dx_2.
\end{aligned} \tag{6.27}$$

By definition of the function Θ_ε we have for a.e. $(x_2, y_1, y_2) \in \mathcal{O} \times Z$

$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon)(x_2, y_1, y_2) = \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\Phi)(x_2, y_2) \left(\hat{\phi}_1(y_1, y_2) \left(\mathbf{e}_1 + \frac{\delta_2}{\delta_1} \Theta(y_1, y_2) \mathbf{e}_2 \right) + \hat{\phi}_2(y_1, y_2) \mathbf{e}_2 \right)$$

and thus

$$\frac{1}{\sqrt{\varepsilon}} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) \rightarrow \Phi (\hat{\phi}_1 \mathbf{e}_1 + \hat{\phi}_2 \mathbf{e}_2) \quad \text{strongly in } L^2(\mathcal{O} \times Z)^2.$$

Consider now the gradient of the function ϕ_ε

$$\begin{aligned}
\sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla \phi_\varepsilon)(x_2, y_1, y_2) &= \frac{1}{\sqrt{\varepsilon}} \nabla_y \mathcal{T}_\varepsilon^*(\phi_\varepsilon)(x_2, y_1, y_2) \\
&= \nabla_y \left(\mathcal{T}_\varepsilon^*(\Phi)(x_2, y_2) \left(\hat{\phi}_1(y_1, y_2) \left(\mathbf{e}_1 + \frac{\delta_2}{\delta_1} \Theta(y_1, y_2) \mathbf{e}_2 \right) + \hat{\phi}_2(y_1, y_2) \mathbf{e}_2 \right) \right) \\
&= \varepsilon \Phi' \left(\varepsilon \left[\frac{x_2}{\varepsilon} \right] + \varepsilon y_2 \right) \left(\hat{\phi}_1(y_1, y_2) \left(\mathbf{e}_1 + \frac{\delta_2}{\delta_1} \Theta(y_1, y_2) \mathbf{e}_2 \right) + \hat{\phi}_2(y_1, y_2) \mathbf{e}_2 \right) \\
&+ \mathcal{T}_\varepsilon^*(\Phi)(x_2, y_2) \nabla_y \left(\hat{\phi}_1(y_1, y_2) \mathbf{e}_1 + \hat{\phi}_2(y_1, y_2) \mathbf{e}_2 \right) + \frac{\delta_2}{\delta_1} \mathcal{T}_\varepsilon^*(\Phi)(x_2, y_2) \nabla_y \left(\hat{\phi}_1(y_1, y_2) \Theta(y_1, y_2) \mathbf{e}_2 \right) \\
&\text{for a.e. } (x_2, y_1, y_2) \in \mathcal{O} \times Z.
\end{aligned}$$

Thus

$$\sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla \phi_\varepsilon) \rightarrow \Phi \nabla_y (\hat{\phi}_1 \mathbf{e}_1 + \hat{\phi}_2 \mathbf{e}_2) = \Phi \nabla_y \hat{\phi} \quad \text{strongly in } L^2(\mathcal{O} \times Z)^2. \tag{6.28}$$

Passing to the limit in the LHS of (6.27), we obtain

$$\begin{aligned}
2\mu \int_{\mathcal{O} \times Z} \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(D(v_{\varepsilon\delta})) : \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(D(\phi_\varepsilon)) dx_2 dy - \int_{\mathcal{O} \times Z} \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(p_{\varepsilon\delta}) \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla \cdot \phi_\varepsilon) dx_2 dy \\
\rightarrow 2\mu \int_{\mathcal{O} \times Z} \Phi D_y(\hat{u}) : D_y(\hat{\phi}) dx_2 dy - \int_{\mathcal{O} \times Z} \Phi \hat{p} \nabla_y \cdot \hat{\phi} dx_2 dy.
\end{aligned} \tag{6.29}$$

Now, we consider the RHS of (6.27).

Applying the 1D unfolding operator \mathcal{T}_ε with the macroscopic domain \mathcal{O} and the reference cell $(0, 1)$, we obtain

$$\begin{aligned} & \int_{\Gamma_2^{in}} \mathcal{P}(x_2) \Phi(x_2) \sqrt{\varepsilon} \widehat{\phi}_1 \left(-\frac{L_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) dx_2 \\ &= \int_{\mathcal{O}} \int_0^1 \mathcal{T}_\varepsilon(\mathcal{P}\Phi)(x_2, y_2) \sqrt{\varepsilon} \widehat{\phi}_1 \left(-\frac{L_1}{\varepsilon}, y_2 \right) dx_2 dy_2. \end{aligned}$$

We have

$$\mathcal{T}_\varepsilon(\mathcal{P}\Phi) \longrightarrow \mathcal{P}\Phi \quad \text{strongly in } L^2(\mathcal{O} \times (0, 1)).$$

Besides, Lemma 6.6 gives

$$\sqrt{\varepsilon} \widehat{\phi}_1 \left(-\frac{L_1}{\varepsilon}, \cdot \right) \rightharpoonup 0 \quad \text{weakly in } L^2(0, 1).$$

Thus, the RHS of (6.27) goes to 0.

As a result, passing to the limit in (6.27) give

$$\begin{aligned} & 2\mu \int_{\mathcal{O} \times Z} \Phi D_y(\widehat{u}) : D_y(\widehat{\phi}) dx_2 dy - \int_{\mathcal{O} \times Z} \Phi \widehat{p} \nabla_y \cdot \widehat{\phi} dx_2 dy = 0, \\ & \forall \Phi \in \mathcal{C}_c^1(\mathcal{O}) \quad \text{and} \quad \forall \widehat{\phi} \in \mathbf{W}(Z) \cap \mathcal{C}^1(\overline{Z}). \end{aligned} \tag{6.30}$$

Due to the density of the set of test functions in $L^2(\mathcal{O}; \mathbf{W}(Z))$, we get (6.26) for all $\widehat{\psi} \in L^2(\mathcal{O}; \mathbf{W}(Z))$.

Step 5. Existence and uniqueness of the solution.

Problem (6.26) admits the trivial solution $(\widehat{u}, \widehat{p}) = (0, 0)$. Therefore, (6.27) reduces to (6.13). Clearly, (6.13) admits a unique solution, so the whole sequences converge to their limits.

Step 6. We prove (6.12).

First, observe that from the estimate (6.1)₂ and convergence (6.14)₁ we have

$$\begin{aligned} & D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega^- \cap \Omega_{\varepsilon\delta}} \rightharpoonup D(v) \quad \text{weakly in } L^2(\Omega^-)^{2 \times 2}, \\ & D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega^+ \cap \Omega_{\varepsilon\delta}} \rightharpoonup D(v) \quad \text{weakly in } L^2(\Omega^+)^{2 \times 2}. \end{aligned} \tag{6.31}$$

In (3.2) and (6.13) choosing as tests function $v_{\varepsilon\delta}$ and v respectively, we obtain

$$2\mu \int_{\Omega} (D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega_{\varepsilon\delta}}) : (D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega_{\varepsilon\delta}}) dx = \int_{\Gamma_2^{in}} \mathcal{P}v_{\varepsilon\delta,1} dx_2.$$

and

$$2\mu \int_{\Omega} D(v) : D(v) dx = \int_{\Gamma_2^{in}} \mathcal{P}v_1 dx_2.$$

The weak lower semicontinuity of the LHS and the above convergences (6.31) give

$$\begin{aligned} & 2\mu \int_{\Omega} D(v) : D(v) dx \leq 2\mu \liminf \left(\int_{\Omega} (D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega_{\varepsilon\delta}}) : (D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega_{\varepsilon\delta}}) dx \right) \\ & \leq 2\mu \limsup \left(\int_{\Omega} (D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega_{\varepsilon\delta}}) : (D(v_{\varepsilon\delta}) \mathbf{1}_{\Omega_{\varepsilon\delta}}) dx \right) \\ & = \limsup \left(\int_{\Gamma_2^{in}} \mathcal{P}v_{\varepsilon\delta,1} dx_2 \right) = \lim \left(\int_{\Gamma_2^{in}} \mathcal{P}v_{\varepsilon\delta,1} dx_2 \right) = \int_{\Gamma_2^{in}} \mathcal{P}v_1 dx_2 = 2\mu \int_{\Omega} D(v) : D(v) dx, \end{aligned}$$

and the convergences in (6.12)_{1,2} are proved.

We prove the strong convergence (6.12)₃.

To do that, first in (3.2) and (6.13) we choose a test function $\phi \in H_0^1(\Omega)^2$ vanishing in $(-\varepsilon, \varepsilon) \times (-L_2, L_2)$. This leads to

$$\begin{aligned} \int_{\Omega_\varepsilon^\pm} p_{\varepsilon\delta} \nabla \cdot \phi \, dx &= 2\mu \int_{\Omega_\varepsilon^\pm} D(v_{\varepsilon\delta}) : D(\phi) \, dx, \\ \int_{\Omega_\varepsilon^\pm} p \nabla \cdot \phi \, dx &= 2\mu \int_{\Omega_\varepsilon^\pm} D(v) : D(\phi) \, dx. \end{aligned}$$

Subtracting the second from the first gives

$$\int_{\Omega_\varepsilon^\pm} (p_{\varepsilon\delta} - p) \nabla \cdot \phi \, dx = 2\mu \int_{\Omega_\varepsilon^\pm} (D(v_{\varepsilon\delta}) - D(v)) : D(\phi) \, dx. \quad (6.32)$$

Below, we show that $\|\nabla(p_{\varepsilon\delta} - p)\|_{H^{-1}(\Omega_\varepsilon^\pm)} \rightarrow 0$.

Indeed, using the definition (B.2), (6.32) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\nabla(p_{\varepsilon\delta} - p)\|_{H^{-1}(\Omega_\varepsilon^\pm)} &= \sup_{\phi \in H_0^1(\Omega_\varepsilon^\pm)^2, \|\nabla\phi\|_{L^2(\Omega_\varepsilon^\pm)} \neq 0} \frac{\int_{\Omega_\varepsilon^\pm} (p_{\varepsilon\delta} - p) \nabla \cdot \phi \, dx}{\|\nabla\phi\|_{L^2(\Omega_\varepsilon^\pm)}} \\ &= \sup_{\phi \in H_0^1(\Omega_\varepsilon^\pm)^2, \|\nabla\phi\|_{L^2(\Omega_\varepsilon^\pm)} \neq 0} \frac{2\mu \int_{\Omega_\varepsilon^\pm} D(v_{\varepsilon\delta} - v) : D(\phi) \, dx}{\|\nabla\phi\|_{L^2(\Omega_\varepsilon^\pm)}} \\ &\leq C \|D(v_{\varepsilon\delta} - v)\|_{L^2(\Omega_\varepsilon^\pm)} \end{aligned} \quad (6.33)$$

since

$$\|D(\phi)\|_{L^2(\Omega_\varepsilon^\pm)} \leq C \|\nabla\phi\|_{L^2(\Omega_\varepsilon^\pm)}, \quad \forall \phi \in H_0^1(\Omega_\varepsilon^\pm)^2.$$

The constant does not depend on ε . Now, we have

$$\|p_{\varepsilon\delta} - p\|_{L^2(\Omega_\varepsilon^\pm)} \leq \|(p_{\varepsilon\delta} - p) - \mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta} - p)\|_{L^2(\Omega_\varepsilon^\pm)} + 4L_1L_2 |\mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta} - p)|. \quad (6.34)$$

The Nečas inequality (see (B.1)) gives

$$\|(p_{\varepsilon\delta} - p) - \mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta} - p)\|_{L^2(\Omega_\varepsilon^\pm)} \leq C(\Omega_\varepsilon^\pm) \|\nabla(p_{\varepsilon\delta} - p)\|_{H^{-1}(\Omega_\varepsilon^\pm)}. \quad (6.35)$$

Let ω_1 and ω_2 be the following open subset of Ω_ε^- :

$$\omega_1 = \left(-L_1, -\frac{1}{3}L_1\right) \times (-L_2, L_2), \quad \omega_2 = \left(-\frac{2}{3}L_1 - \varepsilon, -\varepsilon\right) \times (-L_2, L_2).$$

Since the dimensions of these open sets do not depend on ε and these domains are isomorphic, their Nečas constants are equal and do not depend on ε . So, from (B.8) the Nečas constant $C(\Omega_\varepsilon^-)$ does not depend on ε , similarly we show that the Nečas constant $C(\Omega_\varepsilon^+)$ does not depend on ε .

Hence, from (6.33)-(6.35) we obtain

$$\|(p_{\varepsilon\delta} - p) - \mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta} - p)\|_{L^2(\Omega_\varepsilon^\pm)} \leq C \|\nabla(p_{\varepsilon\delta} - p)\|_{H^{-1}(\Omega_\varepsilon^\pm)} \leq C \|D(v_{\varepsilon\delta} - v)\|_{L^2(\Omega_\varepsilon^\pm)}.$$

Then, the above together with (6.34) lead to

$$\|p_{\varepsilon\delta} - p\|_{L^2(\Omega_\varepsilon^\pm)} \leq C \|D(v_{\varepsilon\delta} - v)\|_{L^2(\Omega_\varepsilon^\pm)} + 4L_1L_2 |\mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta} - p)|.$$

Besides, since

$$p_{\varepsilon\delta} \mathbf{1}_{\Omega_{\varepsilon\delta}} \rightharpoonup p \quad \text{weakly in } L^2(\Omega)$$

we have $|\mathcal{M}_{\Omega_\varepsilon^\pm}(p_{\varepsilon\delta} - p)| \rightarrow 0$. Finally, the strong convergence (6.12)₁ and the above imply (6.12)₃. \square

Lemma 6.6. For any function $\Psi \in \mathbf{H}^1(Z)$ we have

$$\forall a > 0, \quad \sqrt{\varepsilon}\Psi\left(\frac{a}{\varepsilon}, \cdot\right) \rightarrow 0 \quad \text{strongly in } L^2(0, 1). \quad (6.36)$$

Proof. First, for any $a > 0$, we have

$$\left\| \Psi\left(\frac{a}{\varepsilon}, \cdot\right) \right\|_{L^2(0,1)}^2 \leq 2\|\Psi(0, \cdot)\|_{L^2(0,1)}^2 + \frac{a}{\varepsilon} \left\| \frac{\partial \Psi}{\partial y_1} \right\|_{L^2(Z)}^2. \quad (6.37)$$

So, the sequence $\left\{ \sqrt{\varepsilon}\Psi\left(\frac{a}{\varepsilon}, \cdot\right) \right\}_\varepsilon$ is uniformly bounded in $L^2(0, 1)$.

Consider now, the following subspace of $\mathbf{H}^1(Z)$:

$$\mathbf{V}(Z) \doteq \left\{ \Psi \in \mathbf{H}^1(Z) \mid \exists b > 0 \text{ such that } \Psi \text{ is a constant function in } (b, +\infty) \times (0, 1) \right\}.$$

We know that $\mathbf{V}(Z)$ is dense in $\mathbf{H}^1(Z)$. Consequently, if Ψ belongs to this space, then we have

$$\sqrt{\varepsilon}\Psi\left(\frac{a}{\varepsilon}, \cdot\right) \rightarrow 0 \quad \text{strongly in } L^2(0, 1).$$

Now, let Ψ be in $\mathbf{H}^1(Z)$. For any $\eta > 0$ there exists $\Psi_b \in \mathbf{V}(Z)$, Ψ_b being a constant function in $(b, +\infty) \times (0, 1)$, such that

$$\|\Psi - \Psi_b\|_{\mathbf{H}^1(Z)} \leq \eta.$$

Then, from (6.37) we obtain

$$\sqrt{\varepsilon} \left\| \Psi\left(\frac{a}{\varepsilon}, \cdot\right) - \Psi_b\left(\frac{a}{\varepsilon}, \cdot\right) \right\|_{L^2(0,1)} \leq 2\sqrt{\varepsilon} \|(\Psi - \Psi_b)(0, \cdot)\|_{L^2(0,1)} + \sqrt{a} \|\nabla(\Psi - \Psi_b)\|_{L^2(Z)} \leq C\|\Psi - \Psi_b\|_{\mathbf{H}^1(Z)} \leq C\eta.$$

The constant does not depend on ε . Thus

$$\sqrt{\varepsilon} \left\| \Psi\left(\frac{a}{\varepsilon}, \cdot\right) \right\|_{L^2(0,1)} \leq \sqrt{\varepsilon} \left\| \Psi_b\left(\frac{a}{\varepsilon}, \cdot\right) \right\|_{L^2(0,1)} + C\eta.$$

For ε sufficiently small, this gives

$$\sqrt{\varepsilon} \left\| \Psi\left(\frac{a}{\varepsilon}, \cdot\right) \right\|_{L^2(0,1)} \leq (C + 1)\eta.$$

This ends the proof of (6.36). □

A The unfolding operators \mathcal{T}_ε and $\mathcal{T}_\varepsilon^*$

In this section we assume that $\delta_1 = \kappa\varepsilon$, $\kappa \in (0, 1/2]$, $\frac{\delta_2}{\delta_1\varepsilon^{1/2}} \rightarrow \theta \in [0, +\infty)$

For a.e. $t \in \mathbb{R}$, we have the following decomposition:

$$t = \varepsilon \left[\frac{t}{\varepsilon} \right] + \varepsilon \left\{ \frac{t}{\varepsilon} \right\},$$

where $[\cdot]$ is the integer part and $\{\cdot\}$ the fractional part.

Now, we introduce a specific unfolding operator which mixed the periodic unfolding operator in x_2 direction and a scaling in the direction x_1 .

Definition A.1 (The unfolding operator \mathcal{T}_ε). For every Lebesgue-measurable function ϕ on Ω , the unfolding operator $\mathcal{T}_\varepsilon : \Omega \rightarrow \mathcal{O} \times Z$ is defined as follows:

$$\mathcal{T}_\varepsilon(\phi)(x_2, y) \doteq \begin{cases} \phi\left(\varepsilon \left[\frac{x_2}{\varepsilon} \right] \mathbf{e}_2 + \varepsilon y\right) & \text{for a.e. } (x_2, y) = (x_2, y_1, y_2) \in \mathcal{O} \times Z_{L_1/\varepsilon}, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.1})$$

If a function ϕ is defined in $\Omega_{\varepsilon\delta}$ we extend it by 0 in the obstacles and denote with $\tilde{\phi}$ its extension. So, for every Lebesgue-measurable function ϕ on $\Omega_{\varepsilon\delta}$, the unfolding operators $\mathcal{T}_\varepsilon^* : \Omega_{\varepsilon\delta} \rightarrow \mathcal{O} \times Z$ is defined as follows:

$$\mathcal{T}_\varepsilon^*(\phi) = \mathcal{T}_\varepsilon(\tilde{\phi}). \quad (\text{A.2})$$

Denote

$$F_{\varepsilon\delta_2} \doteq \left\{ y \in Z \mid \frac{\delta_2}{\varepsilon} \psi^d(y_1) < y_2 < 1 - \frac{\delta_2}{\varepsilon} \psi^u(y_1) \right\}$$

and

$$\partial F_{\varepsilon\delta_2}^u \doteq \left\{ y_1 \in (-\kappa, \kappa) \mid y_2 = 1 - \frac{\delta_2}{\varepsilon} \psi^u(y_1) \right\}, \quad \partial F_{\varepsilon\delta_2}^d \doteq \left\{ y_1 \in (-\kappa, \kappa) \mid y_2 = \frac{\delta_2}{\varepsilon} \psi^d(y_1) \right\}$$

are parts of the boundary of the domain $F_{\varepsilon\delta_2}$.

Proposition A.1 (Properties of operators \mathcal{T}_ε and $\mathcal{T}_\varepsilon^*$). The linear operator \mathcal{T}_ε (resp. $\mathcal{T}_\varepsilon^*$) defined by (A.1) (resp. (A.2)) satisfies:

(i) for all functions $\psi_1, \psi_2 \in L^1(\Omega)$ (resp. $\phi_1, \phi_2 \in L^1(\Omega_{\varepsilon\delta})$)

$$\mathcal{T}_\varepsilon(\psi_1 \psi_2) = \mathcal{T}_\varepsilon(\psi_1) \mathcal{T}_\varepsilon(\psi_2), \quad (\text{resp. } \mathcal{T}_\varepsilon^*(\phi_1 \phi_2) = \mathcal{T}_\varepsilon^*(\phi_1) \mathcal{T}_\varepsilon^*(\phi_2)),$$

(ii) for every function $\psi \in L^1(\Omega)$ (resp. $\phi \in L^1(\Omega_{\varepsilon\delta})$)

$$\begin{aligned} \int_{\mathcal{O} \times Z} \mathcal{T}_\varepsilon(\psi)(x_2, y) dx_2 dy &= \frac{1}{\varepsilon} \int_{\Omega} \psi(x) dx, \\ (\text{resp. } \int_{\mathcal{O} \times Z} \mathcal{T}_\varepsilon^*(\phi)(x_2, y) dx_2 dX &= \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon\delta}} \phi(x) dx), \end{aligned}$$

(iii) for every function $\psi \in L^2(\Omega)$ (resp. $\phi \in L^2(\Omega_{\varepsilon\delta})$)

$$\begin{aligned} \|\mathcal{T}_\varepsilon(\psi)\|_{L^2(\mathcal{O} \times Z)} &= \frac{1}{\sqrt{\varepsilon}} \|\psi\|_{L^2(\Omega)}, \\ (\text{resp. } \|\mathcal{T}_\varepsilon^*(\phi)\|_{L^2(\mathcal{O} \times Z)} &= \frac{1}{\sqrt{\varepsilon}} \|\phi\|_{L^2(\Omega_{\varepsilon\delta})}, \end{aligned}$$

(iv) for every function $\psi \in H^1(\Omega)$ (resp. $\phi \in H^1(\Omega_{\varepsilon\delta})$)

$$\begin{aligned} \mathcal{T}_\varepsilon(\nabla \psi) &= \frac{1}{\varepsilon} \nabla_y \mathcal{T}_\varepsilon(\psi) & \text{a.e. in } \mathcal{O} \times Z_{L_1/\varepsilon}, \\ (\text{resp. } \mathcal{T}_\varepsilon^*(\nabla \phi) &= \frac{1}{\varepsilon} \nabla_y \mathcal{T}_\varepsilon(\psi) & \text{a.e. in } \mathcal{O} \times (F_{\varepsilon\delta_2} \cap Z_{L_1/\varepsilon}). \end{aligned}$$

Proof. The properties of the \mathcal{T}_ε are omitted here as they are similar to those obtained in [11][S.9.3]. The property (ii) for the operator $\mathcal{T}_\varepsilon^*$ follows from the fact that

$$\begin{aligned} \forall \phi \in L^1(\Omega_{\varepsilon\delta}) \quad \int_{\mathcal{O} \times Z} \mathcal{T}_\varepsilon^*(\phi)(x_2, y) dx_2 dy &= \int_{\mathcal{O} \times Z} \mathcal{T}_\varepsilon(\tilde{\phi})(x_2, y) dx_2 dy \\ &= \varepsilon \int_{\Omega} \tilde{\phi}(x) dx = \varepsilon \int_{\Omega_{\varepsilon\delta}} \phi(x) dx. \end{aligned}$$

The property (iii) for the operator $\mathcal{T}_\varepsilon^*$ follows from the fact that for any function $\phi \in H^1(\Omega_{\varepsilon\delta})$

$$\mathcal{T}_\varepsilon^*(\nabla\phi) = \frac{1}{\varepsilon}\nabla_y\mathcal{T}_\varepsilon^*(\phi) \quad \text{a.e. in } \mathcal{O} \times (F_{\varepsilon\delta_2} \cap Z_{L_1/\varepsilon}). \quad \square$$

Lemma A.1. *Let $\{w_\varepsilon\}_\varepsilon$ be a sequence of functions belonging to $H^1(\Omega_{\varepsilon\delta})$ satisfying*

$$\|w_\varepsilon\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} + \varepsilon\|w_\varepsilon\|_{H^1(\Omega_{\varepsilon\delta})} \leq C\varepsilon. \quad (\text{A.3})$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $W \in L^2(\mathcal{O}; \mathcal{H}^1(Z))$ such that

$$\begin{aligned} \frac{1}{\sqrt{\varepsilon}}\mathcal{T}_\varepsilon^*(w_\varepsilon) &\rightharpoonup W \quad \text{weakly in } L^2(\mathcal{O}; L^2_{loc}(Z)), \\ \frac{1}{\sqrt{\varepsilon}}\mathcal{T}_\varepsilon^*(\varepsilon\nabla w_\varepsilon) &\rightharpoonup \nabla_y W \quad \text{weakly in } L^2(\mathcal{O} \times Z)^2. \end{aligned} \quad (\text{A.4})$$

Proof. From the estimates (A.3), the definition of the operator $\mathcal{T}_\varepsilon^*$ and its properties (iii)-(iv), we have

$$\|\mathcal{T}_\varepsilon^*(w_\varepsilon)\|_{L^2(\mathcal{O} \times Z_1)} \leq C\sqrt{\varepsilon}, \quad \|\mathcal{T}_\varepsilon^*(w_\varepsilon)\|_{L^2(\mathcal{O} \times Z)} \leq \frac{C}{\sqrt{\varepsilon}}, \quad \|\mathcal{T}_\varepsilon^*(\varepsilon\nabla w_\varepsilon)\|_{L^2(\mathcal{O} \times Z)} \leq C\sqrt{\varepsilon}.$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, $W \in L^2(\mathcal{O}; L^2_{loc}(Z))$ and $F \in L^2(\mathcal{O} \times Z)^2$, such that

$$\begin{aligned} \frac{1}{\sqrt{\varepsilon}}\mathcal{T}_\varepsilon^*(w_\varepsilon) &\rightharpoonup W \quad \text{weakly in } L^2(\mathcal{O}; L^2_{loc}(Z))^3, \\ \frac{1}{\sqrt{\varepsilon}}\mathcal{T}_\varepsilon^*(\varepsilon\nabla w_\varepsilon) &\rightharpoonup F \quad \text{weakly in } L^2(\mathcal{O} \times Y)^2. \end{aligned}$$

Now, for any $a > 1$ and any test function $\phi \in \mathcal{C}_c^1(\mathcal{O} \times Z_a)^2$ (function with compact support in $\mathcal{O} \times Z_a$) we have for ε small enough

$$\int_{\mathcal{O} \times Z_a} \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \cdot \phi \, dx_2 \, dy = \frac{1}{\varepsilon} \int_{\mathcal{O} \times Z_a} \nabla_y \mathcal{T}_\varepsilon^*(w_\varepsilon) \cdot \phi \, dx_2 \, dy.$$

Then, from the above convergences

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\varepsilon}} \int_{\mathcal{O} \times Z_a} \mathcal{T}_\varepsilon^*(\varepsilon\nabla w_\varepsilon) \cdot \phi \, dx_2 \, dy = \int_{\mathcal{O} \times Z_a} F \cdot \phi \, dx_2 \, dy.$$

Besides

$$\begin{aligned} \sqrt{\varepsilon} \int_{\mathcal{O} \times Z_a} \nabla_y \mathcal{T}_\varepsilon^*(w_\varepsilon) \cdot \phi \, dx_2 \, dy &= -\sqrt{\varepsilon} \int_{\mathcal{O} \times Z_a} \mathcal{T}_\varepsilon^*(w_\varepsilon) \operatorname{div}_y(\phi) \, dx_2 \, dy, \\ \text{and } \lim_{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \int_{\mathcal{O} \times Z_a} \mathcal{T}_\varepsilon^*(w_\varepsilon) \operatorname{div}_y(\phi) \, dx_2 \, dy &= \int_{\mathcal{O} \times Z_a} W \operatorname{div}_y(\phi) \, dx_2 \, dy. \end{aligned}$$

Therefore, we have

$$\int_{\mathcal{O} \times Y} F \cdot \phi \, dx_2 \, dy = - \int_{\mathcal{O} \times Y} W \cdot \operatorname{div}_y(\phi) \, dx_2 \, dy.$$

Since $\mathcal{C}_c^1(\mathcal{O} \times Z_a)^2$ is dense in $L^2(\mathcal{O}, H_0^1(Z_a))^2$, the above equality is satisfied for any $\phi \in L^2(\mathcal{O}, H_0^1(Z_a))^2$. Hence, for any $a > 1$ we have $\nabla_y W = F$ a.e. in $\mathcal{O} \times Z_a$. Therefore $\nabla_y W = F$ a.e. in $\mathcal{O} \times Z$. So (A.4)₂ is proved.

To prove the periodicity, in the direction \mathbf{e}_2 , of the restriction of W to the domains $(-\infty, -\kappa) \times (0, 1)$ and $(\kappa, +\infty) \times (0, 1)$, we proceed as in [11, Theorem 1.36]. \square

³Here, for every $a \geq 1$ we have $\|\mathcal{T}_\varepsilon^*(w_\varepsilon)\|_{L^2(\mathcal{O} \times Z_a)} \leq C_a\sqrt{\varepsilon}$. The constant does not depend on ε , it depends on a .

We define the operator \mathcal{M}_ε from $L^1(\mathbf{Y}_{\varepsilon\delta})$ into $L^1(\mathcal{O})$ by

$$\begin{aligned}\mathcal{M}_\varepsilon(\Phi)(p\varepsilon) &= \frac{1}{|Y_{\varepsilon\delta}^p|} \int_{Y_{\varepsilon\delta}^p} \Phi(p\varepsilon\mathbf{e}_2 + x) dx, \quad \forall \Phi \in L^1(\mathbf{Y}_{\varepsilon\delta}), \\ \mathcal{M}_\varepsilon(\Phi)(x_2) &= \mathcal{M}_\varepsilon(\Phi)(p\varepsilon) \quad \forall x_2 \in [p\varepsilon, (p+1)\varepsilon), \quad p \in \{-N_\varepsilon, \dots, N_\varepsilon - 1\}.\end{aligned}$$

Lemma A.2. *Let $\{w_\varepsilon\}_\varepsilon$ be a sequence of functions belonging to $H^1(\Omega_{\varepsilon\delta})$ satisfying*

$$\|w_\varepsilon\|_{H^1(\Omega_{\varepsilon\delta})} \leq C. \quad (\text{A.5})$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $w \in H^1(\Omega)$ and $W \in L^2(\mathcal{O}; \mathcal{H}^1(Z))$ such that

$$\begin{aligned}\mathcal{E}_\varepsilon^\pm(w_\varepsilon) &\rightharpoonup w|_{\Omega^\pm} \quad \text{weakly in } H^1(\Omega^\pm)^2, \\ \mathcal{T}_\varepsilon^*(w_\varepsilon) &\rightharpoonup w|_{\mathcal{O}} \quad \text{weakly in } L^2(\mathcal{O}; L_{loc}^2(Z)), \\ \frac{1}{\sqrt{\varepsilon}} \mathcal{T}_\varepsilon^*(w_\varepsilon - \mathcal{W}_\varepsilon) &\rightharpoonup W \quad \text{weakly in } L^2(\mathcal{O}; L_{loc}^2(Z)), \\ \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) &\rightharpoonup \nabla_y W \quad \text{weakly in } L^2(\mathcal{O} \times Z)^2.\end{aligned} \quad (\text{A.6})$$

Moreover, we have

$$\int_Y W(x_2, y) dy = 0 \quad \text{for a.e. } x_2 \in (-L_2, L_2). \quad (\text{A.7})$$

Proof. Step 1. We prove (A.6)₁.

Using the operators $\mathcal{E}_\varepsilon^\pm$ and proceeding as in the proof of Lemma 6.2, we obtain that there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $w \in H^1(\Omega)$, such that the convergence (A.6)₁ holds.

Step 2. We introduce W .

We set

$$\mathcal{W}_\varepsilon = w_\varepsilon - \mathcal{M}_\varepsilon(w_\varepsilon).$$

The Poincaré-Wirtinger inequality and (A.5) give

$$\begin{aligned}\|\mathcal{W}_\varepsilon\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 &= \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|\mathcal{W}_\varepsilon\|_{L^2(Y_{\varepsilon\delta}^p)}^2 = \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|w_\varepsilon - \mathcal{M}_\varepsilon(w_\varepsilon)\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \leq C\varepsilon^2 \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|\nabla w_\varepsilon\|_{L^2(Y_{\varepsilon\delta}^p)}^2 \\ &= C\varepsilon^2 \|\nabla w_\varepsilon\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 \leq C\varepsilon^2.\end{aligned} \quad (\text{A.8})$$

Besides, we have

$$\|\nabla \mathcal{W}_\varepsilon\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 = \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|\nabla \mathcal{W}_\varepsilon\|_{L^2(Y_{\varepsilon\delta}^p)}^2 = \sum_{p=-N_\varepsilon}^{N_\varepsilon-1} \|\nabla w_\varepsilon\|_{L^2(Y_{\varepsilon\delta}^p)}^2 = \|\nabla w_\varepsilon\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 \leq C. \quad (\text{A.9})$$

Then, proceeding as in the proof of Lemma A.1, there exists $W \in L^2(\mathcal{O}; H^1(Z))$ such that

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}} \mathcal{T}_\varepsilon^*(\mathcal{W}_\varepsilon) &\rightharpoonup W \quad \text{weakly in } L^2(\mathcal{O}; L_{loc}^2(Z)), \\ \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) &= \frac{1}{\sqrt{\varepsilon}} \mathcal{T}_\varepsilon^*(\varepsilon \nabla w_\varepsilon) = \frac{1}{\sqrt{\varepsilon}} \mathcal{T}_\varepsilon^*(\varepsilon \nabla \mathcal{W}_\varepsilon) \rightharpoonup \nabla_y W \quad \text{weakly in } L^2(\mathcal{O} \times Z)^2\end{aligned} \quad (\text{A.10})$$

and (A.6)₃ holds.

Step 3. We prove (A.6)₂.

From estimates (4.1), (A.8), (A.9) we get the following estimates

$$\|w_\varepsilon - \mathcal{M}_\varepsilon(w_\varepsilon)\|_{L^2(\mathcal{O}_\varepsilon^-)}^2 = \|\mathcal{W}_\varepsilon\|_{L^2(\mathcal{O}_\varepsilon^-)}^2 \leq C \left(\frac{1}{\varepsilon} \|\mathcal{W}_\varepsilon\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 + \varepsilon \left\| \frac{\partial \mathcal{W}_\varepsilon}{\partial x_1} \right\|_{L^2(\mathbf{Y}_{\varepsilon\delta})}^2 \right) \leq C\varepsilon$$

and

$$\begin{aligned} \|w_{\varepsilon}|_{\mathcal{O}_{\varepsilon}^{-}} - \mathcal{E}_{\varepsilon}^{-}(w_{\varepsilon})|_{\mathcal{O}}\|_{L^2(-L_2, L_2)}^2 &= \|w_{\varepsilon}|_{\mathcal{O}_{\varepsilon}^{-}} - w_{\varepsilon}|_{\mathcal{O}_{2\varepsilon}^{-}}\|_{L^2(-L_2, L_2)}^2 = \int_{-L_2}^{L_2} |w_{\varepsilon}(-\varepsilon, x_2) - w_{\varepsilon}(-2\varepsilon, x_2)|^2 dx_2 \\ &= \int_{-L_2}^{L_2} \left(\int_{-2\varepsilon}^{-\varepsilon} \frac{\partial w_{\varepsilon}}{\partial x_1} dx_1 \right)^2 dx_2 \leq \varepsilon \int_{Y_{\varepsilon}^{-}} \left| \frac{\partial w_{\varepsilon}}{\partial x_1} \right|^2 dx \leq \varepsilon \|\nabla w_{\varepsilon}\|_{L^2(\Omega_{\varepsilon\delta})}^2 \leq C\varepsilon, \end{aligned}$$

where $Y_{\varepsilon}^{-} = (-2\varepsilon, -\varepsilon) \times (-L_2, L_2)$.

Then, we have

$$\|\mathcal{M}_{\varepsilon}(w_{\varepsilon}) - \mathcal{E}_{\varepsilon}^{-}(w_{\varepsilon})\|_{L^2(\mathcal{O})}^2 \leq C\varepsilon.$$

From the convergence (A.6) we derive that

$$\mathcal{E}_{\varepsilon}^{-}(w_{\varepsilon})|_{\mathcal{O}} \rightharpoonup w|_{\mathcal{O}} \quad \text{weakly in } L^2(\mathcal{O}).$$

Therefore

$$\mathcal{M}_{\varepsilon}(w_{\varepsilon}) \rightharpoonup w|_{\mathcal{O}} \quad \text{weakly in } L^2(\mathcal{O}),$$

which, together with the convergence (A.10)₁, leads

$$\mathcal{T}_{\varepsilon}^*(w_{\varepsilon}) = \mathcal{T}_{\varepsilon}^*(\mathcal{W}_{\varepsilon}) + \mathcal{T}_{\varepsilon}^*(\mathcal{M}_{\varepsilon}(w_{\varepsilon})) = \mathcal{T}_{\varepsilon}^*(\mathcal{W}_{\varepsilon}) + \mathcal{M}_{\varepsilon}(w_{\varepsilon}) \rightharpoonup 0 + w|_{\mathcal{O}} \quad \text{weakly in } L^2(\mathcal{O}; L_{loc}^2(Z)).$$

So convergence (A.6)₂ is proved.

Step 4. We show that W belongs to $L^2(\mathcal{O}; \mathcal{H}^1(Z))$.

Denote

$$\begin{aligned} Z_a'^- &\doteq (-a, -\kappa), & Z_a'^+ &\doteq (\kappa, a), & a &> 1. \\ \mathcal{O}_a'^{\pm} &\doteq (-L_2, L_2) \times Z_a'^{\pm} \end{aligned}$$

From (A.8)-(A.9) we obtain the estimates of the traces of $\mathcal{T}_{\varepsilon}^*(\mathcal{W}_{\varepsilon})$ on $\mathcal{O}_a'^{\pm}$ and $\mathcal{O}_a'^{\pm} + \mathbf{e}_2$

$$\|\mathcal{T}_{\varepsilon}^*(\mathcal{W}_{\varepsilon})\|_{L^2(\mathcal{O}_a'^{\pm})} \leq C\sqrt{\varepsilon}, \quad \|\mathcal{T}_{\varepsilon}^*(\mathcal{W}_{\varepsilon})\|_{L^2(\mathcal{O}_a'^{\pm} + \mathbf{e}_2)} \leq C\sqrt{\varepsilon}.$$

Now, let ψ be in $\mathcal{C}_c^1(\mathcal{O}_a'^{\pm})$. For ε small enough, we have

$$\begin{aligned} &\int_{\mathcal{O}_a'^{\pm}} \frac{1}{\sqrt{\varepsilon}} (\mathcal{T}_{\varepsilon}^*(\mathcal{W}_{\varepsilon})(x_2, y_1, 1) - \mathcal{T}_{\varepsilon}^*(\mathcal{W}_{\varepsilon})(x_2, y_1, 0)) \psi(x_2, y_1) dx_2 dy_1 \\ &= \int_{\mathcal{O}_a'^{\pm}} \frac{1}{\sqrt{\varepsilon}} \left(w_{\varepsilon} \left(\varepsilon \left[\frac{x_2}{\varepsilon} \right] \mathbf{e}_2 + \varepsilon y_1 \mathbf{e}_1 + \varepsilon \mathbf{e}_2 \right) - w_{\varepsilon} \left(\varepsilon \left[\frac{x_2}{\varepsilon} \right] \mathbf{e}_2 + \varepsilon y_1 \mathbf{e}_1 \right) \right) \psi(x_2, y_1) dx_2 dy_1 \\ &= \int_{\mathcal{O}_a'^{\pm}} \frac{1}{\sqrt{\varepsilon}} w_{\varepsilon} \left(\varepsilon \left[\frac{x_2}{\varepsilon} \right] \mathbf{e}_2 + \varepsilon y_1 \mathbf{e}_1 \right) (\psi(x_2 - \varepsilon, y_1) - \psi(x_2, y_1)) dx_2 dy_1 \\ &= \varepsilon \int_{\mathcal{O}_a'^{\pm}} \frac{1}{\sqrt{\varepsilon}} \mathcal{T}_{\varepsilon}^*(\mathcal{W}_{\varepsilon})(x_2, y_1, 0) \frac{\psi(x_2 - \varepsilon, y_1) - \psi(x_2, y_1)}{\varepsilon} dx_2 dy_1. \end{aligned}$$

Passing to the limit gives

$$\int_{\mathcal{O}_a'^{\pm}} (W(x_2, y_1, 1) - W(x_2, y_1, 0)) \psi(x_2, y_1) dx_2 dy_1 = 0.$$

As a consequence, W belongs to $L^2(\mathcal{O}; H_{per}^1(Z_a))$ for every $a > 1$, which proves the claim.

Step 5. We show (A.7).

By definition of \mathcal{W}_ε , we have

$$\int_{Y_{\varepsilon\delta}^p} \mathcal{W}_\varepsilon(x_2 + \varepsilon y) dy = 0 \quad \text{for a. e. } x_2 \in (p\varepsilon, p\varepsilon + \varepsilon), \quad p \in \{-N_\varepsilon, \dots, N_\varepsilon - 1\}.$$

The characteristic function $\mathbf{1}_{Y_{\varepsilon\delta}^p}$ strongly converges to $\mathbf{1}_{\mathcal{O} \times Y}$ in $L^2(\mathcal{O} \times Y)$. Then, transforming the above equality by unfolding, dividing by $\sqrt{\varepsilon}$ and passing to the limit, give (A.7). \square

As immediate consequence of Proposition A.1 we have

Lemma A.3. *Let $\{\rho_\varepsilon\}_\varepsilon$ be a sequence of functions belonging to $L^2(\Omega_{\varepsilon\delta})$ satisfying*

$$\|\rho_\varepsilon\|_{L^2(\Omega_{\varepsilon\delta})} \leq C. \quad (\text{A.11})$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and $\hat{\rho} \in L^2(\Omega \times Z)$, such that

$$\sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\rho_\varepsilon) \rightharpoonup \hat{\rho} \quad \text{weakly in } L^2(\mathcal{O} \times Z). \quad (\text{A.12})$$

Lemma A.4. *Let $\{v_\varepsilon\}_\varepsilon$ be a sequence of functions belonging to $\mathbf{V}_{ad}(\Omega_{\varepsilon\delta})(\Omega_{\varepsilon\delta})$ satisfying*

$$\|D(v_\varepsilon)\|_{L^2(\Omega_{\varepsilon\delta})} \leq C. \quad (\text{A.13})$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, $v_1 \in H^1(\Omega)$ and $\hat{v} = (\hat{v}_1, \hat{v}_2) \in L^2(\mathcal{O}; \mathbf{W}(Z))$ such that

$$\begin{aligned} \mathcal{E}_\varepsilon^\pm(v_{1,\varepsilon}) &\rightharpoonup v_1|_{\Omega^\pm} \quad \text{weakly in } H^1(\Omega^\pm)^2, \\ \mathcal{T}_\varepsilon^*(v_{1,\varepsilon}) &\rightharpoonup v_1|_{\mathcal{O}} \quad \text{weakly in } L^2(\mathcal{O}; L_{loc}^2(Z)), \\ \frac{1}{\sqrt{\varepsilon}} \mathcal{T}_\varepsilon^*(v_{2,\varepsilon}) &\rightharpoonup \hat{v}_2 \quad \text{weakly in } L^2(\mathcal{O}; L_{loc}^2(Z))^2, \\ \sqrt{\varepsilon} \mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) &\rightharpoonup \nabla_y \hat{v} \quad \text{weakly in } L^2(\mathcal{O} \times Z)^{2 \times 2}. \end{aligned} \quad (\text{A.14})$$

Moreover, if $\{v_\varepsilon\}_\varepsilon \in \mathbf{V}_{ad,div0}(\Omega_{\varepsilon\delta})$ then $\hat{v} \in L^2(\mathcal{O}; \mathbf{W}_{div0}(Z))$.

To prove the above lemma, we first need to prove the lemma bellow.

Let $\mathbf{Y}^d = (-\kappa, \kappa) \times (0, 1/2)$ and χ be a function belonging to $H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^d)$. We extend χ as a function defined in \mathbf{Y}^d by setting

$$\tilde{\chi}(y') = \begin{cases} \chi(y') & \text{for a.e. } y' = (y_1, y_2) \in F_{\varepsilon\delta_2} \cap \mathbf{Y}^d, \\ \chi\left(y_1, 2\frac{\delta_2}{\varepsilon}\psi^d\left(\frac{y_1}{\kappa}\right) - y_2\right) & \text{for a.e. } y' = (y_1, y_2) \in \mathbf{Y}^d \setminus \overline{F_{\varepsilon\delta_2} \cap \mathbf{Y}^d}. \end{cases}$$

Lemma A.5. $\tilde{\cdot}$ *is a continuous linear operator from $H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^d)$ into $H^1(\mathbf{Y}^d)$.*

Moreover, for any $\Psi \in H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^d)^2$ satisfying $\Psi \cdot \nu^d = 0$ on the boundary of the obstacles, we have

$$\|\tilde{\Psi}_2\|_{L^2((-\kappa, \kappa) \times \{0\})} \leq C \sqrt{\frac{\delta_2}{\varepsilon}} \|\Psi\|_{H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^d)}, \quad (\text{A.15})$$

where C is independent of ε and δ_2 .

Proof. Step 1. In this step we show that there exists a constant C independent of ε and δ_2 such that

$$\|\tilde{\chi}\|_{H^1(\mathbf{Y}^d)} \leq C \|\chi\|_{H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^d)}. \quad (\text{A.16})$$

⁴Periodicity in direction \mathbf{e}_2 with respect to the microscopic variable y_2 .

First, observe that a.e. in $F_{\varepsilon\delta_2}$ we have

$$\begin{aligned}\partial_{y_1}\tilde{\chi}(y_1, y_2) &= \partial_{y_1}\chi\left(y_1, 2\frac{\delta_2}{\varepsilon}\psi^d\left(\frac{y_1}{\kappa}\right) - y_2\right) + \frac{2}{\kappa}\frac{\delta_2}{\varepsilon}(\psi^d)'\left(\frac{y_1}{\kappa}\right)\partial_{y_2}\chi\left(y_1, 2\frac{\delta_2}{\varepsilon}\psi^d\left(\frac{y_1}{\kappa}\right) - y_2\right), \\ \partial_{y_2}\tilde{\chi}(y_1, y_2) &= -\partial_{y_2}\chi\left(y_1, 2\frac{\delta_2}{\varepsilon}\psi^d\left(\frac{y_1}{\kappa}\right) - y_2\right).\end{aligned}$$

Besides, the Jacobian determinant of the change of variables $(y_1, y_2) \mapsto \left(y_1, 2\frac{\delta_2}{\varepsilon}\psi^d\left(\frac{y_1}{\kappa}\right) - y_2\right)$ is equal to -1 . Then, a straightforward calculation yields (A.16). So, the operator $\tilde{\cdot}$ is a continuous linear map from $H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^d)$ into $H^1(\mathbf{Y}^d)$ (the linearity follows from its definition).

Step 2. We prove the estimate given in the lemma.

Let Ψ be in $H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^d)^2$ satisfying $\Psi \cdot \nu^d = 0$ on the boundary of the obstacles. First, observe that

$$-\frac{\delta_2}{\varepsilon}(\psi^d)'\left(\frac{y_1}{\kappa}\right)\Psi_1(y') + \Psi_2(y') = 0 \quad \text{a.e. on } \partial F_{\varepsilon\delta_2}^d.$$

Hence

$$\|\Psi_2\|_{L^2(\partial F_{\varepsilon\delta_2}^d)} \leq C\frac{\delta_2}{\varepsilon}\|\Psi_1\|_{L^2(\partial F_{\varepsilon\delta_2}^d)}.$$

A trace theorem (see [11]) gives

$$\|\Psi_1\|_{L^2(\partial F_{\varepsilon\delta_2}^d)} \leq C\|\Psi_1\|_{H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^d)}$$

and

$$\|\tilde{\Psi}_2\|_{L^2((-\kappa, \kappa) \times \{0\})}^2 \leq C(\|\Psi_2\|_{L^2(\partial F_{\varepsilon\delta_2}^d)}^2 + \frac{\delta_2}{\varepsilon}\|\partial_{y_2}\tilde{\Psi}_2\|_{L^2(F_{\varepsilon\delta_2}^d)}^2).$$

Then, (A.15) follows. \square

Remark A.1. Let $\mathbf{Y}^u = (-1, 1) \times (1/2, 1)$, the estimate

$$\|\tilde{\Psi}_2\|_{L^2((-\kappa, \kappa) \times \{1\})} \leq C\sqrt{\frac{\delta_2}{\varepsilon}}\|\Psi\|_{H^1(F_{\varepsilon\delta_2} \cap \mathbf{Y}^u)} \quad (\text{A.17})$$

is proved similarly.

Proof of Lemma A.4. Step 1. We prove (A.14).

Lemma 4.3 applied with the functions $v_{\varepsilon,1}$, $v_{\varepsilon,2}$ gives

$$\begin{aligned}\|v_{\varepsilon,1}\|_{H^1(\Omega_{\varepsilon\delta})} &\leq C, \\ \|v_{\varepsilon,2}\|_{L^2(\mathbf{Y}_{\varepsilon\delta})} + \varepsilon\|v_{\varepsilon,2}\|_{H^1(\Omega_{\varepsilon\delta})} &\leq C\varepsilon.\end{aligned}$$

Then, thanks to the Lemmas A.1, A.2 we obtain the convergences (A.14) with $\hat{v} \in L^2(\mathcal{O}; \mathcal{H}^1(Z))$.

Step 2. Let $\{v_\varepsilon\}_\varepsilon$ be in $\mathbf{V}_{ad}(\Omega_{\varepsilon\delta})(\Omega_{\varepsilon\delta})$. Since we also have

$$\frac{1}{\sqrt{\varepsilon}}\mathcal{T}_\varepsilon(\tilde{v}_{2,\varepsilon}) \rightharpoonup \hat{v}_2 \quad \text{weakly in } L^2(\mathcal{O}; L_{loc}^2(Z)).$$

Thanks to Lemma A.5 and Remark A.1, we obtain

$$\hat{v}_2 = 0 \quad \text{a.e. on } (-\kappa, \kappa) \times \{0, 1\}.$$

As a consequence $\hat{v} \in L^2(\mathcal{O}; \mathbf{W}(Z))$.

Step 3. Let $\{v_\varepsilon\}_\varepsilon \subset \mathbf{V}_{ad,div0}(\Omega_{\varepsilon\delta})$.

By virtue of the linearity of $\mathcal{T}_\varepsilon^*$ and $\text{div}_x v_\varepsilon = 0$, we have $\mathcal{T}_\varepsilon^*(\text{div}_x v_\varepsilon) = 0$ a.e. in $\mathcal{O} \times Z$.

On the other side, due to (A.14)₄, we obtain

$$\sqrt{\varepsilon}\mathcal{T}_\varepsilon^*(\text{div}_x v_\varepsilon) = \frac{1}{\sqrt{\varepsilon}}\text{div}_y \mathcal{T}_\varepsilon^*(v_\varepsilon) \rightharpoonup \text{div}_y \hat{v} \quad \text{weakly in } L^2(\mathcal{O} \times Z).$$

Hence, $\text{div}_y \hat{v} = 0$ a.e. in $\mathcal{O} \times Z$ and then $\hat{v} \in L^2(\mathcal{O}; \mathbf{W}_{div0}(Z))$. \square

B About the Nečas inequality

Below, we recall some definitions and a classical result.

Proposition B.1 (The Nečas inequality). *Let ω be a bounded domain with Lipschitz boundary. There exists a constant $C(\omega)$ depending on ω , such that*

$$\forall \phi \in L^2(\omega) \quad \|\phi - \mathcal{M}_\omega(\phi)\|_{L^2(\omega)} \leq C(\omega) \|\nabla \phi\|_{H^{-1}(\omega)}. \quad (\text{B.1})$$

1. The norm of $\|\nabla \phi\|_{H^{-1}(\omega)}$

$$\|\nabla \phi\|_{H^{-1}(\omega)} = \sup_{\Psi \in H_0^1(\omega)^2, \Psi \neq 0} \frac{\int_\omega \phi \operatorname{div}(\Psi) dx}{\|\nabla \Psi\|_{L^2(\omega)}}. \quad (\text{B.2})$$

Let $U \in H_0^1(\omega)^2$ be the solution to the following variational problem:

$$\int_\omega \nabla U : \nabla V dx = \int_\omega \phi \operatorname{div}(V) dx, \quad \forall V \in H_0^1(\omega)^2. \quad (\text{B.3})$$

Then, we have

$$\|\nabla \phi\|_{H^{-1}(\omega)} = \|\nabla U\|_{L^2(\omega)}. \quad (\text{B.4})$$

2. We define Nečas constant as

$$C(\omega) = \sup_{\phi \in L_{\#}^2(\omega) \setminus \{0\}} \frac{\|\phi\|_{L^2(\omega)}}{\|\nabla \phi\|_{H^{-1}(\omega)}}, \quad (\text{B.5})$$

where $L_{\#}^2(\omega) = \left\{ \phi \in L^2(\omega) \mid \int_\omega \phi dx = 0 \right\}$.

Proposition B.2. *The Nečas constant has the following property:*

1. $Y = (0, 1)^2$ and $Y_{\mathbf{a}} = (0, a_1) \times (0, a_2)$, $\mathbf{a} = (a_1, a_2)$, $a_1, a_2 > 0$

$$\min \left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1} \right\} C(Y) \leq C(Y_{\mathbf{a}}) \leq \max \left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1} \right\} C(Y). \quad (\text{B.6})$$

2. Let ω be a bounded domain in \mathbb{R}^2 with Lipschitz boundary. We have $C(\varepsilon\omega) = C(\omega)$ for all $\varepsilon > 0$.

3. Let $\mathbf{a} \in \mathbb{R}^2$ and $\mathbf{R} \in SO(2)$. We have $C(\mathbf{a} + \mathbf{R}\omega) = C(\omega)$.

4. Let ω and Ω in \mathbb{R}^2 with Lipschitz boundary such that $\omega \subset \Omega$. We have

$$\|\phi - \mathcal{M}_\omega(\phi)\|_{L^2(\omega)} \leq 2C(\Omega) \frac{\sqrt{|\Omega|}}{\sqrt{|\omega|}} \|\nabla \phi\|_{H^{-1}(\Omega)}, \quad \forall \phi \in L^2(\Omega). \quad (\text{B.7})$$

5. Let ω_1 and ω_2 be two bounded domains in \mathbb{R}^2 with Lipschitz boundary, such that $\omega_1 \cap \omega_2 \neq \emptyset$. We have

$$C(\omega_1 \cup \omega_2) \leq 2 \frac{\sqrt{|\omega_1|} + \sqrt{|\omega_2|}}{\sqrt{|\omega_1 \cap \omega_2|}} (C(\omega_1) + C(\omega_2)). \quad (\text{B.8})$$

Proof. Property 1. First, we introduce the change of variables

$$x_1 = \frac{y_1}{a_1}, \quad x_2 = \frac{y_2}{a_2} \quad (y_1, y_2) \in Y_{\mathbf{a}}, \quad (x_1, x_2) \in Y$$

and functions

$$\psi(x_1, x_2) = \psi_{\mathbf{a}}(a_1 x_1, a_2 x_2), \quad \psi_{\mathbf{a}} \in H_0^1(Y_{\mathbf{a}}), \quad \psi \in H_0^1(Y).$$

Then, for every $\Psi \in H_0^1(Y)^2$ we define $\Psi_{\mathbf{a}} \in H_0^1(Y_{\mathbf{a}})^2$ by

$$\Psi(x_1, x_2) = \frac{1}{a_1} \Psi_{\mathbf{a},1}(a_1 x_1, a_2 x_2) \mathbf{e}_1 + \frac{1}{a_2} \Psi_{\mathbf{a},2}(a_1 x_1, a_2 x_2) \mathbf{e}_2.$$

Now, let V be in $H_0^1(Y)^2$. We have

$$\begin{aligned} \nabla_x V &= \begin{pmatrix} \partial_{x_1} V_1 & \partial_{x_1} V_2 \\ \partial_{x_2} V_1 & \partial_{x_2} V_2 \end{pmatrix} = \begin{pmatrix} \partial_{y_1} V_{\mathbf{a},1} & \frac{a_1}{a_2} \partial_{y_1} V_{\mathbf{a},2} \\ \frac{a_2}{a_1} \partial_{y_2} V_{\mathbf{a},1} & \partial_{y_2} V_{\mathbf{a},2} \end{pmatrix}, \\ \operatorname{div}_x V &= \partial_{x_1} V_1 + \partial_{x_2} V_2 = \partial_{y_1} V_{\mathbf{a},1} + \partial_{y_2} V_{\mathbf{a},2} = \operatorname{div}_y V_{\mathbf{a}}. \end{aligned}$$

Thus

$$a_1 a_2 \|\nabla_x V\|_{L^2(Y)}^2 = \|\partial_{y_1} V_{\mathbf{a},1}\|_{L^2(Y_{\mathbf{a}})}^2 + \frac{a_2^2}{a_1^2} \|\partial_{y_2} V_{\mathbf{a},1}\|_{L^2(Y_{\mathbf{a}})}^2 + \frac{a_1^2}{a_2^2} \|\partial_{y_1} V_{\mathbf{a},2}\|_{L^2(Y_{\mathbf{a}})}^2 + \|\partial_{y_2} V_{\mathbf{a},2}\|_{L^2(Y_{\mathbf{a}})}^2$$

and

$$a_1 a_2 \int_Y \phi \operatorname{div}_x(V) dx = \int_{Y_{\mathbf{a}}} \phi_{\mathbf{a}} \operatorname{div}_y(V_{\mathbf{a}}) dy.$$

Now, let U be the solution to the variational problem (B.3) with ω and ϕ . Taking into account the above equalities, we have

$$\|\nabla_x \phi\|_{H^{-1}(Y)} = \|\nabla_x U\|_{L^2(Y)}$$

and

$$\min \left\{ \frac{a_1^2}{a_2^2}, \frac{a_2^2}{a_1^2} \right\} \|\nabla_z U_{\mathbf{a}}\|_{L^2(Y_{\mathbf{a}})}^2 \leq a_1 a_2 \|\nabla_x U\|_{L^2(Y)}^2 \leq \max \left\{ \frac{a_1^2}{a_2^2}, \frac{a_2^2}{a_1^2} \right\} \|\nabla_z U_{\mathbf{a}}\|_{L^2(Y_{\mathbf{a}})}^2.$$

Thus

$$\begin{aligned} \|\nabla_x \phi\|_{H^{-1}(Y)} &= \frac{\int_Y \phi \operatorname{div}_x(U) dx}{\|\nabla_x U\|_{L^2(Y)}} \leq \frac{\frac{1}{a_1 a_2} \int_{Y_{\mathbf{a}}} \phi_{\mathbf{a}} \operatorname{div}_y(U_{\mathbf{a}}) dy}{\min \left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1} \right\} \frac{1}{\sqrt{a_1 a_2}} \|\nabla_y U_{\mathbf{a}}\|_{L^2(Y_{\mathbf{a}})}}} \\ &\leq \frac{1}{\sqrt{a_1 a_2}} \max \left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1} \right\} \|\nabla_y \phi_{\mathbf{a}}\|_{H^{-1}(Y_{\mathbf{a}})}. \end{aligned}$$

Hence

$$\|\nabla_x \phi\|_{H^{-1}(Y)} \leq \frac{1}{\sqrt{a_1 a_2}} \max \left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1} \right\} \|\nabla_y \phi_{\mathbf{a}}\|_{H^{-1}(Y_{\mathbf{a}})}.$$

Then, due to the above inequality and the fact that $\|\phi_{\mathbf{a}}\|_{L^2(Y_{\mathbf{a}})} = \sqrt{a} \|\phi\|_{L^2(Y)}$, we obtain

$$\frac{\|\phi_{\mathbf{a}}\|_{L^2(Y_{\mathbf{a}})}}{\|\nabla_y \phi_{\mathbf{a}}\|_{H^{-1}(Y_{\mathbf{a}})}} \leq \max \left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1} \right\} \frac{\|\phi\|_{L^2(Y)}}{\|\nabla_x \phi\|_{H^{-1}(Y)}}.$$

This easily leads to $C(Y_{\mathbf{a}}) \leq \max \left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1} \right\} C(Y)$. This gives the right-hand side of (B.6). Then, we swap Y and $Y_{\mathbf{a}}$ to obtain the left-hand side of (B.6).

Property 2. Set $\omega_\varepsilon = \varepsilon\omega$. Consider the change of variables and functions

$$y_1 = \varepsilon x_1, \quad y_2 = \varepsilon x_2 \quad \text{and} \quad \tilde{V}(y_1, y_2) = V(x_1, x_2) \quad \text{for a.e. } (x_1, x_2) \in \omega, \quad V \in H_0^1(\omega)^2, \quad \tilde{V} \in H_0^1(\omega_\varepsilon)^2.$$

We have

$$\nabla_y \tilde{V} = \frac{1}{\varepsilon} \nabla_x V, \quad \operatorname{div}_y \tilde{V} = \frac{1}{\varepsilon} \operatorname{div}_x V.$$

Let U be the solution to the variational problem (B.3). We have

$$\int_{\omega_\varepsilon} \nabla_y \tilde{U} : \nabla_y \tilde{V} \, dy = \int_{\omega} \nabla_x U : \nabla_x V \, dx = \int_{\omega} \phi \operatorname{div}_x(V) \, dx = \frac{1}{\varepsilon} \int_{\omega_\varepsilon} \tilde{\phi} \operatorname{div}_y(\tilde{V}) \, dy, \quad \forall \tilde{V} \in H_0^1(\omega_\varepsilon)^2.$$

So, the function \tilde{U} is the solution to

$$\int_{\omega_\varepsilon} \nabla_y \tilde{U} : \nabla_y \tilde{V} \, dy = \frac{1}{\varepsilon} \int_{\omega_\varepsilon} \tilde{\phi} \operatorname{div}_y(\tilde{V}) \, dy, \quad \forall \tilde{V} \in H_0^1(\omega_\varepsilon)^2.$$

Hence

$$\|\nabla_x \phi\|_{H^{-1}(\omega)} = \|\nabla_x U\|_{L^2(\omega)} = \|\nabla_y \tilde{U}\|_{L^2(\omega_\varepsilon)} = \frac{1}{\varepsilon} \|\nabla_y \tilde{\phi}\|_{H^{-1}(\omega_\varepsilon)}. \quad (\text{B.9})$$

Now, (B.5) leads to

$$C(\omega_\varepsilon) = \sup_{\tilde{\phi} \in L_{\#}^2(\omega_\varepsilon) \setminus \{0\}} \frac{\|\tilde{\phi}\|_{L^2(\omega_\varepsilon)}}{\|\nabla_y \tilde{\phi}\|_{H^{-1}(\omega_\varepsilon)}} = \sup_{\phi \in L_{\#}^2(\omega) \setminus \{0\}} \frac{\|\phi\|_{L^2(\omega)}}{\|\nabla_x \phi\|_{H^{-1}(\omega)}} = C(\omega),$$

due to (B.9) and the fact that $\|\phi\|_{L^2(\omega)} = \varepsilon \|\tilde{\phi}\|_{L^2(\omega_\varepsilon)}$.

Property 3. Since the transformation $y = \mathbf{a} + \mathbf{R}x$ from ω into $\tilde{\omega} = \mathbf{a} + \mathbf{R}\omega$, $\mathbf{a} \in \mathbb{R}^2$ and $\mathbf{R} \in SO(2)$, does not change either the shape or size of the domain ω , then we obtain the same Nečas constant for both domains.

Property 4. Any function χ in $L^2(\omega)$ is extended by 0 in $\Omega \setminus \bar{\omega}$. The extension of χ is denoted $\tilde{\chi}$.

Let ϕ be in $L^2(\Omega)$ and

- $U \in H_0^1(\omega)^2$ the solution to the variational problem (B.3) with ω and ϕ replaced by its restriction to ω ,
- $\mathbf{U} \in H_0^1(\Omega)^2$ the solution to the variational problem (B.3) with ω replaced by Ω and the function ϕ .

We have

$$\begin{aligned} \int_{\omega} \nabla U : \nabla V \, dx &= \int_{\omega} \phi \operatorname{div}(V) \, dx, \quad \forall V \in H_0^1(\omega)^2 \quad \text{and} \quad \|\nabla \phi\|_{H^{-1}(\omega)} = \|\nabla U\|_{L^2(\omega)}, \\ \int_{\Omega} \nabla \mathbf{U} : \nabla \mathbf{V} \, dx &= \int_{\Omega} \phi \operatorname{div}(\mathbf{V}) \, dx, \quad \forall \mathbf{V} \in H_0^1(\Omega)^2 \quad \text{and} \quad \|\nabla \phi\|_{H^{-1}(\Omega)} = \|\nabla \mathbf{U}\|_{L^2(\Omega)}. \end{aligned}$$

We extend U by 0 outside of ω . The extension of U is \tilde{U} . So

$$\int_{\Omega} \nabla \tilde{U} : \nabla \tilde{U} \, dx = \int_{\omega} \nabla U : \nabla U \, dx = \int_{\omega} \phi \operatorname{div}(U) \, dx = \int_{\Omega} \phi \operatorname{div}(\tilde{U}) \, dx = \int_{\Omega} \nabla \mathbf{U} : \nabla \tilde{U} \, dx.$$

As a consequence

$$\|\nabla \mathbf{U}\|_{L^2(\Omega)}^2 = \|\nabla U\|_{L^2(\omega)}^2 + \|\nabla(\mathbf{U} - \tilde{U})\|_{L^2(\Omega)}^2.$$

Hence

$$\|\nabla\phi\|_{H^{-1}(\omega)} \leq \|\nabla\phi\|_{H^{-1}(\Omega)}. \quad (\text{B.10})$$

Now, we have

$$\begin{aligned} \|\phi - \mathcal{M}_\omega(\phi)\|_{L^2(\omega)} &\leq C(\omega)\|\nabla\phi\|_{H^{-1}(\omega)} \leq C(\omega)\|\nabla\phi\|_{H^{-1}(\Omega)} \\ \text{and } \|\phi - \mathcal{M}_\Omega(\phi)\|_{L^2(\Omega)} &\leq C(\Omega)\|\nabla\phi\|_{H^{-1}(\Omega)}. \end{aligned}$$

The second above inequality yields

$$\sqrt{|\omega|} |\mathcal{M}_\omega(\phi) - \mathcal{M}_\Omega(\phi)| \leq C(\Omega)\|\nabla\phi\|_{H^{-1}(\Omega)}.$$

Therefore

$$\begin{aligned} \|\phi - \mathcal{M}_\omega(\phi)\|_{L^2(\omega)} &\leq \|\phi - \mathcal{M}_\omega(\phi)\|_{L^2(\Omega)} \leq \|\phi - \mathcal{M}_\Omega(\phi)\|_{L^2(\Omega)} + \|\mathcal{M}_\omega(\phi) - \mathcal{M}_\Omega(\phi)\|_{L^2(\Omega)} \\ &\leq C(\Omega) \left(1 + \frac{\sqrt{|\Omega|}}{\sqrt{|\omega|}}\right) \|\nabla\phi\|_{H^{-1}(\Omega)}. \end{aligned}$$

This proves (B.7).

Property 5. Thanks to (B.10), for any $\phi \in L^2(\omega_1 \cup \omega_2)$, we have

$$\begin{aligned} \|\phi - \mathcal{M}_{\omega_1}(\phi)\|_{L^2(\omega_1)} &\leq C(\omega_1)\|\nabla\phi\|_{H^{-1}(\omega_1)} \leq C(\omega_1)\|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}, \\ \|\phi - \mathcal{M}_{\omega_2}(\phi)\|_{L^2(\omega_2)} &\leq C(\omega_2)\|\nabla\phi\|_{H^{-1}(\omega_2)} \leq C(\omega_2)\|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}, \\ \|\phi - \mathcal{M}_{\omega_1 \cup \omega_2}(\phi)\|_{L^2(\omega_1 \cup \omega_2)} &\leq C(\omega_1 \cup \omega_2)\|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}. \end{aligned}$$

So, we get

$$\begin{aligned} \sqrt{|\omega_1 \cap \omega_2|} |\mathcal{M}_{\omega_1 \cap \omega_2}(\phi) - \mathcal{M}_{\omega_1}(\phi)| &\leq C(\omega_1)\|\nabla\phi\|_{H^{-1}(\omega_1)} \leq C(\omega_1)\|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}, \\ \sqrt{|\omega_1 \cap \omega_2|} |\mathcal{M}_{\omega_1 \cap \omega_2}(\phi) - \mathcal{M}_{\omega_2}(\phi)| &\leq C(\omega_2)\|\nabla\phi\|_{H^{-1}(\omega_2)} \leq C(\omega_2)\|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}. \end{aligned}$$

Hence

$$\begin{aligned} \|\phi - \mathcal{M}_{\omega_1 \cap \omega_2}(\phi)\|_{L^2(\omega_1)} &\leq C(\omega_1) \left(1 + \frac{\sqrt{|\omega_1|}}{\sqrt{|\omega_1 \cap \omega_2|}}\right) \|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}, \\ \|\phi - \mathcal{M}_{\omega_1 \cap \omega_2}(\phi)\|_{L^2(\omega_2)} &\leq C(\omega_2) \left(1 + \frac{\sqrt{|\omega_2|}}{\sqrt{|\omega_1 \cap \omega_2|}}\right) \|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}. \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} \|\phi - \mathcal{M}_{\omega_1 \cap \omega_2}(\phi)\|_{L^2(\omega_1 \cup \omega_2)} &\leq \|\phi - \mathcal{M}_{\omega_1 \cap \omega_2}(\phi)\|_{L^2(\omega_1)} + \|\phi - \mathcal{M}_{\omega_1 \cap \omega_2}(\phi)\|_{L^2(\omega_2)} \\ &\leq (C(\omega_1) + C(\omega_2)) \frac{\sqrt{|\omega_1|} + \sqrt{|\omega_2|}}{\sqrt{|\omega_1 \cap \omega_2|}} \|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}, \end{aligned}$$

and then

$$\sqrt{|\omega_1 \cup \omega_2|} |\mathcal{M}_{\omega_1 \cup \omega_2}(\phi) - \mathcal{M}_{\omega_1 \cap \omega_2}(\phi)| \leq (C(\omega_1) + C(\omega_2)) \frac{\sqrt{|\omega_1|} + \sqrt{|\omega_2|}}{\sqrt{|\omega_1 \cap \omega_2|}} \|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}.$$

The above inequalities lead to

$$\|\phi - \mathcal{M}_{\omega_1 \cup \omega_2}(\phi)\|_{L^2(\omega_1 \cup \omega_2)} \leq 2(C(\omega_1) + C(\omega_2)) \frac{\sqrt{|\omega_1|} + \sqrt{|\omega_2|}}{\sqrt{|\omega_1 \cap \omega_2|}} \|\nabla\phi\|_{H^{-1}(\omega_1 \cup \omega_2)}.$$

Thus, estimate (B.8) is proved. \square

C Two lemmas

Denote

$$\mathcal{Y} = (-1, 1) \times (0, 3/4)$$

and

$$\mathcal{Y}_\psi = \left\{ (x_1, x_2) \in (-1, 1) \times \mathbb{R} \mid \psi(x_1) < x_2 < \frac{3}{4} + \psi(x_1) \right\}.$$

Note, that \mathcal{Y}_ψ is an open set with a Lipschitz boundary.

Lemma C.1. *Let ψ be an element of $\mathcal{C}^1([-1, 1])$ satisfying*

$$\psi(\pm 1) = 0, \quad \|\psi'\|_{L^\infty(-1,1)} \leq \frac{1}{8} \frac{1}{(1 + C(\mathcal{Y}))}, \quad (\text{C.1})$$

where

$$\mathcal{Y} = (-1, 1) \times (0, 3/4).$$

The Nečas constant $C(\mathcal{Y}_\psi)$ satisfies the following inequality:

$$C(\mathcal{Y}_\psi) \leq \frac{8}{7} C(\mathcal{Y}). \quad (\text{C.2})$$

Proof. The map \mathfrak{T} defined by

$$\mathfrak{T} : y = (y_1, y_2) \in \mathcal{Y} \mapsto x = (x_1, x_2) = \mathfrak{T}(y) = (y_1, y_2 + \psi(y_1)) \in \mathcal{Y}_\psi$$

is a \mathcal{C}^1 -diffeomorphism from \mathcal{Y} onto \mathcal{Y}_ψ .

Now, let p be in $L^2_{\#}(\mathcal{Y}_\psi) \setminus \{0\}$ and $\Psi \in H_0^1(\mathcal{Y}_\psi)^2$. Observe that the Jacobian of the change of variables is equal to 1. Hence, the function $p \circ \mathfrak{T}$ belongs to $L^2(\mathcal{Y}) \setminus \{0\}$. We have

$$\int_{\mathcal{Y}_\psi} p \operatorname{div}_x(\Psi) dx = \int_{\mathcal{Y}_\psi} p(x) \left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2} \right) (x) dx = \int_{\mathcal{Y}} p \circ \mathfrak{T}(y) \left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_2} \right) \circ \mathfrak{T}(y) dy$$

and

$$\begin{aligned} \frac{\partial \Psi_1}{\partial x_1}(x) &= \frac{\partial \Psi_1 \circ \mathfrak{T}}{\partial y_1}(y) + \frac{\partial \Psi_2 \circ \mathfrak{T}}{\partial y_2}(y) \psi'(y_1), \\ \frac{\partial \Psi_2}{\partial x_2}(x) &= \frac{\partial \Psi_2 \circ \mathfrak{T}}{\partial y_2}(y), \end{aligned} \quad \text{for a.e. } y \in \mathcal{Y}.$$

Hence

$$\begin{aligned} \left| \int_{\mathcal{Y}_\psi} p \operatorname{div}_x(\Psi) dx - \int_{\mathcal{Y}} p \circ \mathfrak{T}(y) \left(\frac{\partial \Psi_1 \circ \mathfrak{T}}{\partial y_1} + \frac{\partial \Psi_2 \circ \mathfrak{T}}{\partial y_2} \right) (y) dy \right| \\ \leq \|p\|_{L^2(\mathcal{Y}_\psi)} \|\nabla_y \Psi \circ \mathfrak{T}\|_{L^2(\mathcal{Y})} \|\psi'\|_{L^\infty(-1,1)}. \end{aligned} \quad (\text{C.3})$$

We have

$$\|p\|_{L^2(\mathcal{Y}_\psi)} = \|p \circ \mathfrak{T}\|_{L^2(\mathcal{Y})} \quad \text{and} \quad \begin{cases} \|\nabla_y \Psi \circ \mathfrak{T}\|_{L^2(\mathcal{Y})} \leq \|\nabla_x \Psi\|_{L^2(\mathcal{Y}_\psi)} (1 + \|\psi'\|_{L^\infty(-1,1)}), \\ \|\nabla_x \Psi\|_{L^2(\mathcal{Y}_\psi)} \leq \|\nabla_y \Psi \circ \mathfrak{T}\|_{L^2(\mathcal{Y})} (1 + \|\psi'\|_{L^\infty(-1,1)}). \end{cases}$$

As a consequence, if $\int_{\mathcal{Y}} p \circ \mathfrak{T} \operatorname{div}_y(\Psi \circ \mathfrak{T}) dy \geq 0$, then we have

$$\begin{aligned} & \frac{\int_{\mathcal{Y}} p \circ \mathfrak{T} \operatorname{div}_y(\Psi \circ \mathfrak{T}) dy}{\|\nabla_y \Psi \circ \mathfrak{T}\|_{L^2(\mathcal{Y})}} (1 - \|\psi'\|_{L^\infty(-1,1)}) - \|p\|_{L^2(\mathcal{Y}_\psi)} \|\psi'\|_{L^\infty(-1,1)} (1 + \|\psi'\|_{L^\infty(-1,1)}) \\ & \leq \frac{\int_{\mathcal{Y}_\psi} p \operatorname{div}_x(\Psi) dx}{\|\nabla_x \Psi\|_{L^2(\mathcal{Y}_\psi)}} \\ & \leq \frac{\int_{\mathcal{Y}} p \circ \mathfrak{T} \operatorname{div}_y(\Psi \circ \mathfrak{T}) dy}{\|\nabla_y \Psi \circ \mathfrak{T}\|_{L^2(\mathcal{Y})}} (1 + \|\psi'\|_{L^\infty(-1,1)}) + \|p\|_{L^2(\mathcal{Y}_\psi)} \|\psi'\|_{L^\infty(-1,1)} (1 + \|\psi'\|_{L^\infty(-1,1)}). \end{aligned}$$

So,

$$\begin{aligned} & \|\nabla_y(p \circ \mathfrak{T})\|_{H^{-1}(\mathcal{Y})} (1 - \|\psi'\|_{L^\infty(-1,1)}) - \|p\|_{L^2(\mathcal{Y}_\psi)} \|\psi'\|_{L^\infty(-1,1)} (1 + \|\psi'\|_{L^\infty(-1,1)}) \\ & \leq \|\nabla_x p\|_{H^{-1}(\mathcal{Y}_\psi)} \\ & \leq \|\nabla_y(p \circ \mathfrak{T})\|_{H^{-1}(\mathcal{Y})} (1 + \|\psi'\|_{L^\infty(-1,1)}) + \|p\|_{L^2(\mathcal{Y}_\psi)} \|\psi'\|_{L^\infty(-1,1)} (1 + \|\psi'\|_{L^\infty(-1,1)}). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} (1 - \|\psi'\|_{L^\infty(-1,1)}) \frac{1}{C(\mathcal{Y})} - \|\psi'\|_{L^\infty(-1,1)} (1 + \|\psi'\|_{L^\infty(-1,1)}) & \leq \frac{1}{C(\mathcal{Y}_\psi)} \\ & \leq \left(\frac{1}{C(\mathcal{Y})} + \|\psi'\|_{L^\infty(-1,1)} \right) (1 + \|\psi'\|_{L^\infty(-1,1)}). \end{aligned}$$

Therefore

$$\left| \frac{1}{C(\mathcal{Y}_\psi)} - \frac{1}{C(\mathcal{Y})} \right| \leq \|\psi'\|_{L^\infty(-1,1)} \left(\frac{1}{C(\mathcal{Y})} + \frac{9}{8} \right),$$

which implies

$$C(\mathcal{Y}_\psi) \leq \frac{C(\mathcal{Y})}{1 - \|\psi'\|_{L^\infty(-1,1)} \left(1 + \frac{9}{8} C(\mathcal{Y}) \right)}$$

and then the estimate given in the lemma. □

Lemma C.2. *Let ψ_1, ψ_2 be two elements of $\mathcal{C}^1([-1, 1])$ satisfying ($i \in \{1, 2\}$)*

$$\psi_i(\pm 1) = 0, \quad \psi_i(t) \geq 0, \quad \forall t \in [-1, 1], \quad \|\psi'_i\|_{L^\infty(-1,1)} \leq \frac{1}{8} \frac{1}{(1 + C(\mathcal{Y}))}.$$

Denote

$$\mathcal{Y}_{\psi_1, \psi_2} = \left\{ (x_1, x_2) \in (-1, 1) \times \mathbb{R} \mid \psi_1(x_1) < x_2 < 1 - \psi_2(x_1) \right\}.$$

$\mathcal{Y}_{\psi_1, \psi_2}$ is an open set with a Lipschitz boundary. The Nečas constant $C(\mathcal{Y}_{\psi_1, \psi_2})$ satisfies

$$C(\mathcal{Y}_{\psi_1, \psi_2}) \leq 16C(\mathcal{Y}).$$

Proof. The inequality on $C(\mathcal{Y}_{\psi_1, \psi_2})$ is an immediate consequence of Lemmas B.2 and C.1. □

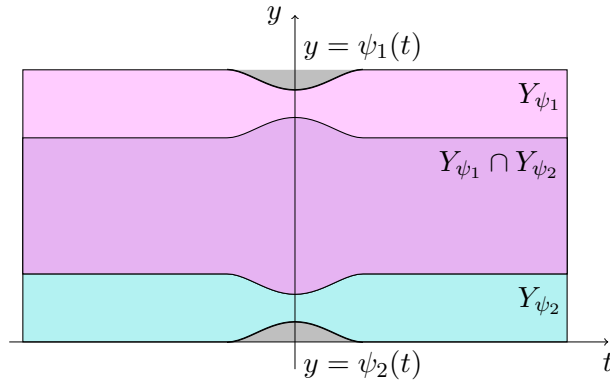


Figure 5: Illustration for Lemma C.2

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