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# WELL-POSEDNESS OF A TIME DISCRETIZATION SCHEME FOR A STOCHASTIC $p$ -LAPLACE EQUATION WITH NEUMANN BOUNDARY CONDITIONS

Caroline Bauzet, Kerstin Schmitz, Cédric Sultan  
and Aleksandra Zimmermann

**Abstract.** In this contribution, we are interested in the analysis of a semi-implicit time discretization scheme for the approximation of a parabolic equation driven by multiplicative colored noise involving a  $p$ -Laplace operator (with  $p \geq 2$ ), nonlinear source terms and subject to Neumann boundary conditions. Using the Minty-Browder theorem, we are able to prove the well-posedness of such a scheme.

*Keywords:* Stochastic PDE, Itô integral, multiplicative Lipschitz noise, Euler-Maruyama scheme,  $p$ -Laplace operator, Neumann boundary conditions.

*AMS classification:* 60H15, 35K55, 35K92.

## §1. Introduction

### 1.1. Statement of the problem and assumptions

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a stochastic basis with the usual assumptions on its filtration  $(\mathcal{F}_t)_{t \geq 0}^{\ddagger}$ ,  $T > 0$  and  $D$  be a bounded open domain of  $\mathbb{R}^d$  where  $d \geq 1$ . Having in mind applications in solid mechanics for the future (through the study of a  $p$ -Laplace perturbation of the stochastic nonlinear parabolic differential inclusion studied in [2]), we consider in this introductory work the Moreau-Yosida approximation of the maximal monotone operator  $\partial J_{[0,1]}$  (see, e.g., [1, 6]), denoted by  $(\psi_\varepsilon)_{\varepsilon > 0}$  in the sequel, where for any  $\varepsilon > 0$ ,  $\psi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  is defined for all  $v \in \mathbb{R}$  by

$$\psi_\varepsilon(v) = -\frac{(v)^-}{\varepsilon} + \frac{(v-1)^+}{\varepsilon} = \begin{cases} \frac{v}{\varepsilon}, & \text{if } v \leq 0, \\ 0, & \text{if } v \in [0, 1], \\ \frac{v-1}{\varepsilon}, & \text{otherwise.} \end{cases} \quad (1.1)$$

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<sup>‡</sup>i.e.,  $(\mathcal{F}_t)_{t \geq 0}$  is right continuous and  $\mathcal{F}_0$  contains all negligible sets of  $\mathcal{F}$ .

Then, for a fixed parameter  $\epsilon > 0$ , we are interested in the following problem:

$$\left\{ \begin{array}{ll} du_\epsilon + \left( \Delta_p^N u_\epsilon + \psi_\epsilon(u_\epsilon) \right) dt = G(u_\epsilon) d\mathcal{W}(t) + (\beta(u_\epsilon) + f) dt & \text{in } \Omega \times ]0, T[ \times D \\ u_\epsilon(0, \cdot) = u_0 & \text{in } \Omega \times D \\ \nabla u_\epsilon \cdot \mathbf{n} = 0 & \text{on } \Omega \times ]0, T[ \times \partial D \end{array} \right. \quad (1.2)$$

where  $\Delta_p^N u_\epsilon = -\operatorname{div}_x (|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon) + |u_\epsilon|^{p-2} u_\epsilon$  is the  $p$ -Laplace operator (with  $p \geq 2$ ) associated to homogeneous Neumann boundary conditions,  $\mathcal{W}$  is a  $(\mathcal{F}_t)_{t \geq 0}$ -adapted  $Q$ -Wiener process (for a given operator  $Q$  defined in the sequel),  $G, \beta, f$  and  $u_0$  are given data, and  $\mathbf{n}$  denotes the unit normal vector to  $\partial D$  outward to  $D$ .

Before stating the considered assumptions on these data, let us precise the stochastic framework. We will call  $\mathcal{P}_T$  the predictable  $\sigma$ -field on  $\Omega_T^\dagger$  defined by  $\Omega_T = \Omega \times ]0, T[$ . By a slight abuse of notation, for any separable Banach space  $X$  and  $1 \leq r < \infty$ , we will denote the space of predictable,  $r$ -integrable mappings  $u : \Omega_T \rightarrow X$  by  $L^r(\Omega_T; X)$ , *i.e.*, the measurability of  $u \in L^r(\Omega_T; X)$  is understood as the predictable measurability, and in particular  $u$  is a random variable with values in  $L^r(]0, T[; X)$ . Additionally, we fix a separable Hilbert space  $U$  such that  $L^2(D) \subset U$ , a non-negative symmetric trace class operator  $Q : U \rightarrow U$  satisfying  $Q^{1/2}(U) = L^2(D)$ , and an orthonormal basis  $(e_j)_j$  of  $U$  made of eigenvectors of  $Q$  with corresponding eigenvalues  $(\lambda_j)_j \subset [0, \infty[$ . At last, the separable Hilbert space of Hilbert-Schmidt operators from  $L^2(D)$  to  $L^2(D)$  is denoted by  $\operatorname{HS}(L^2(D))$ . Within this stochastic framework, we consider the following hypotheses on the data:

(H<sub>1</sub>):  $u_0 \in L^2(\Omega; L^2(D))$  is  $\mathcal{F}_0$ -measurable and verifies  $0 \leq u_0(\omega, x) \leq 1$ , for almost all  $(\omega, x) \in \Omega \times D$ .

(H<sub>2</sub>):  $G : L^2(D) \rightarrow \operatorname{HS}(L^2(D))$  is such that for any  $v \in L^2(D)$  and all  $j \in \mathbb{N}$ ,  $G(v)(Q^{1/2}(e_j)) = g_j(v)$  where  $g_j : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $\operatorname{supp} g_j \subset ]0, 1[$  for all  $j \in \mathbb{N}$ . Moreover, it is supposed that there exists  $L_g \geq 0$  such that for all  $r, s \in \mathbb{R}$ ,

$$\sum_{j=1}^{\infty} |g_j(r) - g_j(s)|^2 \leq L_g |r - s|^2.$$

In particular, for all  $v, w \in L^2(D)$ , we have

$$\|G(v) - G(w)\|_{\operatorname{HS}(L^2(D))}^2 \leq L_g \|v - w\|_{L^2(D)}^2.$$

(H<sub>3</sub>):  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  is a  $L_\beta$ -Lipschitz-continuous function (with  $L_\beta > 0$ ) such that, for convenience,  $\beta(0) = 0$ .

(H<sub>4</sub>):  $f \in L^2(\Omega_T; L^2(D))$ .

(H<sub>5</sub>): There exists  $(W_j)_j = \left( (W_j(t))_{t \geq 0} \right)_{j \in \mathbb{N}^*}$  a sequence of independent, real-valued Wiener processes with respect to  $(\mathcal{F}_t)_{t \geq 0}$  and such that

$$\forall t \geq 0, \mathcal{W}(t) := \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j W_j(t) = \sum_{j=1}^{\infty} Q^{1/2}(e_j) W_j(t).$$

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<sup>†</sup> $\mathcal{P}_T := \sigma(\{F_s \times (s, t] \mid 0 \leq s < t \leq T, F_s \in \mathcal{F}_s\} \cup \{F_0 \times \{0\} \mid F_0 \in \mathcal{F}_0\})$  (see [19, p. 33]). Then, a mapping defined on  $\Omega_T$  with values in a separable Banach space  $E$  is predictable if it is  $\mathcal{P}_T$ -measurable.

In the following, let us write  $V = W^{1,p}(D)$  endowed with the norm  $\|v\|_V^p = \int_D (|\nabla v|^p + |v|^p) dx$  and denote its dual space  $(W^{1,p}(D))^*$  by  $V^*$ . We are interested in a solution  $u_\epsilon$  for Problem (1.2) in the sense of the definition below:

**Definition 1.** A stochastic process  $u_\epsilon \in L^2(\Omega; \mathcal{C}([0, T[; L^2(D))) \cap L^p(\Omega_T; V)$  is a solution to (1.2) if it satisfies

$$u_\epsilon(t) - u_0 + \int_0^t \Delta_p^N u_\epsilon(s) + \psi_\epsilon(u_\epsilon(s)) ds = \int_0^t G(u_\epsilon(s)) d\mathcal{W}(s) + \int_0^t \beta(u_\epsilon(s)) + f(s) ds$$

in  $L^2(D)$  for all  $t \in [0, T]$ , and  $\mathbb{P}$ -a.s in  $\Omega$ , where the stochastic integral is defined as in [9, Section 4.2].

## 1.2. State of the art and aim of the study

We study the existence, uniqueness and measurability of solutions to a semi-implicit Euler-Maruyama time discretization scheme of a parabolic equation involving a  $p$ -Laplace operator (with  $p \geq 2$ ), nonlinear source terms and subject to homogeneous Neumann boundary conditions. On the right-hand side of the equation, we consider a multiplicative stochastic forcing given by a stochastic Itô integral with respect to a Hilbert space valued  $Q$ -Wiener process. The penalization term  $(\psi_\epsilon)_{\epsilon>0}$  as well as the homogeneous Neumann boundary condition are motivated by applications in solid mechanics, see, *e.g.*, [11] and [17] for more details. More precisely, our study can be seen as a first result towards the numerical analysis of a  $p$ -Laplace-Allen-Cahn type equation with constraint extending the existing contributions [2, 3] to the nonlinear setting. In the deterministic case, *i.e.*, without the presence of the stochastic force term,  $p$ -Laplace equations with Neumann boundary conditions are well-known. Non exhaustively, let us mention [10] and the references therein.

In particular, it is well-known that the Dirichlet  $p$ -Laplace operator is not coercive on the Sobolev space  $W^{1,p}$ , and instead, the Neumann  $p$ -Laplace operator may be considered. In the literature on stochastic PDEs, equations with  $p$ -Laplace type operators are usually addressed with Dirichlet boundary conditions for different values of  $1 < p < \infty$ . In the case of bounded domains, let us mention [4, 13, 12, 22] and the list is far from being complete. A semi-implicit Euler-Maruyama time discretization scheme for pseudomonotone SPDEs has been proposed in [23] in the case of multiplicative noise and in [24] for additive noise.

On unbounded domains, let us mention [18, 16, 21] concerning  $p$ -Laplace equations subject to additive or multiplicative noise with homogeneous Dirichlet boundary conditions.

Only a few results on stochastic  $p$ -Laplace equation with Neumann boundary conditions exist in the literature, see, *e.g.*, [14, 8].

To the best of our knowledge, a semi-implicit Euler-Maruyama time discretization scheme for Problem (1.2) with Neumann boundary conditions has not yet been proposed in the literature. The main result of this contribution is to prove the well-posedness as stated in Proposition 1. Future work will be devoted to the convergence of the scheme as the time discretization parameter and the regularization parameter  $\epsilon$  tend towards 0 simultaneously, in order to obtain existence of a solution to Problem (1.2) with the maximal monotone operator  $\partial\mathcal{I}_{[0,1]}$ .

## §2. Semi-implicit time discretization scheme

### 2.1. Presentation of the scheme and main result

For  $M \in \mathbb{N}^*$ , let  $0 = t_0 < t_1 < \dots < t_M = T$  be an equidistant subdivision of the interval  $[0, T]$  and let  $\tau := T/M = t_{n+1} - t_n$  for all  $n \in \llbracket 0, M-1 \rrbracket$ . For a fixed  $\epsilon > 0$ , we set  $u_{\epsilon,0} = u_0$ . Then our semi-implicit Euler-Maruyama scheme is constructed as follows: knowing for a fixed  $n \in \llbracket 0, M-1 \rrbracket$  a  $\mathcal{F}_{t_n}$ -measurable random variable with values in  $L^2(D)$ , denoted by  $u_{\epsilon,n}$ , we search for a  $\mathcal{F}_{t_{n+1}}$ -measurable random variable with values in  $V$ , labelled by  $u_{\epsilon,n+1}$ , such that  $\mathbb{P}$ -a.s in  $\Omega$  the following equality holds in  $L^2(D)$

$$u_{\epsilon,n+1} - u_{\epsilon,n} + \tau \left( \Delta_p^N u_{\epsilon,n+1} + \psi_\epsilon(u_{\epsilon,n+1}) \right) = G(u_{\epsilon,n}) \Delta_{n+1} \mathcal{W} + \tau (\beta(u_{\epsilon,n+1}) + f_n), \quad (2.1)$$

where  $\Delta_{n+1} \mathcal{W} := \mathcal{W}(t_{n+1}) - \mathcal{W}(t_n)$  and  $f_n = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} f(s) ds$ , for all  $n \in \llbracket 0, M-1 \rrbracket$ .

*Remark 1.* Note that as  $u_{\epsilon,n}$  is assumed to be  $\mathcal{F}_{t_n}$ -measurable, we have

$$\begin{aligned} G(u_{\epsilon,n}) \Delta_{n+1} \mathcal{W} &= \int_{t_n}^{t_{n+1}} G(u_{\epsilon,n}) d\mathcal{W} = \sum_{j=1}^{\infty} G(u_{\epsilon,n})(Q^{1/2} e_j) (W_j(t_{n+1}) - W_j(t_n)) \\ &= \sum_{j=1}^{\infty} g_j(u_{\epsilon,n}) (W_j(t_{n+1}) - W_j(t_n)). \end{aligned}$$

Since  $\Delta_{n+1} \mathcal{W}$  takes values in the Hilbert space  $U$  and  $G(u_{\epsilon,n}) \circ Q^{1/2}$  is a Hilbert-Schmidt operator from  $U$  to  $L^2(D)$ , the last infinite sum is a well-defined element of  $L^2(\Omega; L^2(D))$ .

The main result of this contribution is:

**Proposition 1.** *Let us assume that Hypotheses  $(H_1)$  to  $(H_5)$  are satisfied, consider a fixed parameter  $\epsilon > 0$ , a fixed  $M \in \mathbb{N}^*$ , define  $\tau = T/M$ ,  $t_n = n\tau$  ( $\forall n \in \llbracket 0, M-1 \rrbracket$ ) and  $u_{\epsilon,0} = u_0$ . Then, for any given  $\mathcal{F}_{t_n}$ -measurable random variable  $u_{\epsilon,n}$  taking values in  $L^2(D)$  (where  $n \in \llbracket 0, M-1 \rrbracket$ ), under the assumption  $\tau < \frac{1}{L_\beta}$ , there exists a unique  $\mathcal{F}_{t_{n+1}}$ -measurable random variable  $u_{\epsilon,n+1}$  with values in  $V$  solving (2.1).*

### 2.2. Proof of Proposition 1

The proof of Proposition 1 is based on Minty-Browder theorem, Pettis measurability theorem and calls on an algebraic inequality from [5, Lemma 2.1., p.107] stated below in Lemma 2. By denoting for any  $x, y$  in  $\mathbb{R}^d$ , the euclidean norm of  $x$  by  $|x|$ , and the associated scalar product of  $x$  and  $y$  by  $x \cdot y$ , such a lemma reads as follows:

**Lemma 2** (Algebraic inequality). *For any  $p \in [2, +\infty[$  and any  $d \in \mathbb{N}^*$ , there exists a constant  $C(p) > 0$  only depending on  $p$  such that*

$$\forall \zeta, \eta \in \mathbb{R}^d, \left( |\zeta|^{p-2} \zeta - |\eta|^{p-2} \eta \right) \cdot (\zeta - \eta) \geq C(p) |\zeta - \eta|^p.$$

Let us assume that Hypotheses  $(H_1)$  to  $(H_5)$  are fulfilled, consider a fixed parameter  $\epsilon > 0$ , a fixed  $M \in \mathbb{N}^*$  and define  $\tau = T/M$  satisfying the condition  $\tau < \frac{1}{L_\beta}$ . Denoting by  $(\cdot, \cdot)_{L^2(D)}$

the scalar product in  $L^2(D)$ , we define the operator  $A_{\varepsilon,\tau} : V \rightarrow V^*$  by

$$\langle A_{\varepsilon,\tau}(u), v \rangle_{V^*,V} := (u, v)_{L^2(D)} + \tau \int_D (|\nabla u|^{p-2} \nabla u \cdot \nabla v + |u|^{p-2} uv + \psi_\varepsilon(u)v - \beta(u)v) dx, \quad \forall u, v \in V.$$

Since our scheme (2.1) can be rewritten as

$$A_{\varepsilon,\tau}(u_{\varepsilon,n+1}) = u_{\varepsilon,n} + G(u_{\varepsilon,n})\Delta_{n+1}\mathcal{W} + \tau f_n,$$

our objective is to prove in a first step the invertibility of  $A_{\varepsilon,\tau}$  by using the theorem of Minty-Browder in order to get the existence of  $u_{\varepsilon,n+1}$  in  $V$ , and to show in a second step its  $\mathcal{F}_{t_{n+1}}$ -measurability thanks to Pettis measurability theorem.

**Step 1:** Existence of  $u_{\varepsilon,n+1}$

- Firstly, we show that  $A_{\varepsilon,\tau}$  is coercive. By the monotonicity of  $\psi_\varepsilon$ , the Lipschitz continuity of  $\beta$ , the fact that  $\psi_\varepsilon(0) = \beta(0) = 0$ , and the assumption  $\tau < \frac{1}{L_\beta}$ , one gets that for any  $u$  in  $V$ ,

$$\begin{aligned} \langle A_{\varepsilon,\tau}(u), u \rangle_{V^*,V} &= \|u\|_{L^2(D)}^2 + \tau \int_D (|\nabla u|^p + |u|^p + \psi_\varepsilon(u)u - \beta(u)u) dx \\ &\geq (1 - \tau L_\beta) \|u\|_{L^2(D)}^2 + \tau \|u\|_V^p \\ &\geq \tau \|u\|_V^p, \end{aligned}$$

and, since  $p \geq 2$ , one obtains that  $\frac{\langle A_{\varepsilon,\tau}(u), u \rangle_{V^*,V}}{\|u\|_V} \xrightarrow{\|u\|_V \rightarrow +\infty} +\infty$ , hence that  $A_{\varepsilon,\tau}$  is coercive.

- Secondly, we prove that  $A_{\varepsilon,\tau}$  is strictly monotone. For any  $u, v \in V$  such that  $u \neq v$ , owing again to the monotonicity of  $\psi_\varepsilon$ , the Lipschitz continuity of  $\beta$ , the assumption  $\tau < \frac{1}{L_\beta}$ , and the application of Lemma 2, we have:

$$\begin{aligned} &\langle A_{\varepsilon,\tau}(u) - A_{\varepsilon,\tau}(v), u - v \rangle_{V^*,V} \\ &= \langle A_{\varepsilon,\tau}(u), u - v \rangle_{V^*,V} - \langle A_{\varepsilon,\tau}(v), u - v \rangle_{V^*,V} \\ &= (u - v, u - v)_{L^2(D)} + \tau \int_D [(\psi_\varepsilon(u) - \psi_\varepsilon(v))(u - v) - (\beta(u) - \beta(v))(u - v)] dx \\ &\quad + \tau \int_D [ (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla(u - v) + (|u|^{p-2} u - |v|^{p-2} v)(u - v) ] dx \\ &\geq (1 - \tau L_\beta) \|u - v\|_{L^2(D)}^2 + \tau (C(p) \|\nabla(u - v)\|_{L^p(D)}^p + C(p) \|u - v\|_{L^p(D)}^p) \\ &\geq \tau C(p) \|u - v\|_V^p, \end{aligned}$$

which leads to the strict monotonicity of  $A_{\varepsilon,\tau}$ .

- Thirdly, we show that  $A_{\varepsilon,\tau}$  is hemicontinuous. To this end, we prove the stronger property of demicontinuity for  $A_{\varepsilon,\tau}$ , *i.e.*, that for all  $u \in V$ , for any  $(u_k)_{k \in \mathbb{N}} \subset V$  such that  $u_k \xrightarrow{k \rightarrow +\infty} u$  in  $V$ , then

$$\forall w \in V, \langle A_{\varepsilon,\tau}(u_k), w \rangle_{V^*,V} \xrightarrow{k \rightarrow +\infty} \langle A_{\varepsilon,\tau}(u), w \rangle_{V^*,V}.$$

We consider  $u$  in  $V$  and  $(u_k)_{k \in \mathbb{N}} \subset V$ , such that  $u_k \xrightarrow[k \rightarrow +\infty]{} u$  in  $V$ . Then we have in particular  $u_k \xrightarrow[k \rightarrow +\infty]{} u$  in  $L^p(D)$  and  $\nabla u_k \xrightarrow[k \rightarrow +\infty]{} \nabla u$  in  $(L^p(D))^d$ . Note that, since  $\psi_\epsilon$  and  $\beta$  are Lipschitz continuous, one gets that for any  $w \in V$ , as  $p \geq 2$ ,

$$(u_k, w)_{L^2(D)} + \tau \int_D (\psi_\epsilon(u_k) - \beta(u_k))w \, dx \xrightarrow[k \rightarrow +\infty]{} (u, w)_{L^2(D)} + \tau \int_D (\psi_\epsilon(u) - \beta(u))w \, dx. \quad (2.2)$$

Let us now focus on the analysis of the demicontinuity of the operator  $\Delta_p^N : V \rightarrow V^*$  defined for any  $u$  in  $V$  by  $\Delta_p^N u = -\operatorname{div}_x(|\nabla u|^{p-2} \nabla u) + |u|^{p-2}u$ . By applying the reverse Lebesgue dominated convergence theorem (see [7, Th. 4.9, p. 94]), there exist  $g_0, g_1 \in L^p(D)$  such that, up to a not relabeled subsequence of  $(u_k)_{k \in \mathbb{N}}$ ,

(i)  $u_k \xrightarrow[k \rightarrow +\infty]{} u$  a.e. on  $D$  and  $\nabla u_k \xrightarrow[k \rightarrow +\infty]{} \nabla u$  a.e. on  $D$ .

(ii) for all  $k \in \mathbb{N}$  and a.e. in  $D$ ,  $|u_k| \leq g_0$  and  $|\nabla u_k| \leq g_1$ .

Thus, by (i), a.e. in  $D$ ,  $|u_k|^{p-2}u_k \xrightarrow[k \rightarrow +\infty]{} |u|^{p-2}u$  and  $|\nabla u_k|^{p-2}\nabla u_k \xrightarrow[k \rightarrow +\infty]{} |\nabla u|^{p-2}\nabla u$ . Moreover, as  $p \geq 1$ , by (ii) one obtains that for all  $k \in \mathbb{N}$  and a.e. in  $D$ ,

$$|u_k|^{p-2}u_k \leq g_0^{p-1} \quad \text{and} \quad |\nabla u_k|^{p-2}\nabla u_k \leq g_1^{p-1}.$$

Since  $g_0, g_1 \in L^p(D)$ , then  $g_0^{p-1}, g_1^{p-1} \in L^{p'}(D)$  where  $p' = \frac{p}{p-1}$ , and therefore, by Lebesgue's dominated convergence theorem,

$$|u_k|^{p-2}u_k \xrightarrow[k \rightarrow +\infty]{} |u|^{p-2}u \text{ in } L^{p'}(D) \quad \text{and} \quad |\nabla u_k|^{p-2}\nabla u_k \xrightarrow[k \rightarrow +\infty]{} |\nabla u|^{p-2}\nabla u \text{ in } (L^{p'}(D))^d. \quad (2.3)$$

*A priori*, the convergence of  $(|u_k|^{p-2}u_k)_{k \in \mathbb{N}}$  and of  $(|\nabla u_k|^{p-2}\nabla u_k)_{k \in \mathbb{N}}$  may be only for a subsequence. But for any subsequences  $(|u_{k_l}|^{p-2}u_{k_l})_{l \in \mathbb{N}}$  and  $(|\nabla u_{k_l}|^{p-2}\nabla u_{k_l})_{l \in \mathbb{N}}$  respectively, we may extract a subsequence  $(k_{l_j})_{j \in \mathbb{N}}$  such that for any  $j \in \mathbb{N}$ , (i) and (ii) hold true, which is enough to obtain (2.3) for the whole sequences  $(|u_k|^{p-2}u_k)_{k \in \mathbb{N}}$  and  $(|\nabla u_k|^{p-2}\nabla u_k)_{k \in \mathbb{N}}$ .

Next, for any  $w$  in  $V$ , making use of the divergence theorem, we get

$$\begin{aligned} \langle \Delta_p^N(u_k), w \rangle_{V^*, V} &= \int_D -\operatorname{div}_x(|\nabla u_k|^{p-2}\nabla u_k)w + |u_k|^{p-2}u_k w \, dx \\ &= \int_D |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla w \, dx + \int_D |u_k|^{p-2}u_k w \, dx \\ &\xrightarrow[k \rightarrow +\infty]{} \int_D |\nabla u|^{p-2}\nabla u \cdot \nabla w \, dx + \int_D |u|^{p-2}u w \, dx \\ &= \langle \Delta_p^N(u), w \rangle_{V^*, V}. \end{aligned} \quad (2.4)$$

Finally, (2.2) combined with (2.4) give us the demicontinuity of  $A_{\epsilon, \tau}$ , and afterward its hemicontinuity.

- Fourthly, owing to the coercivity, hemicontinuity, and strict monotonicity of the operator  $A_{\epsilon, \tau}$ , one is able to apply the Minty-Browder theorem (see, e.g., [20, Theorem 2.14, p.38]), which allows us to affirm that  $A_{\epsilon, \tau}$  is invertible, of inverse operator  $A_{\epsilon, \tau}^{-1}$ .

Consequently, by setting  $u_{\epsilon,0} = u_0$  and  $t_n = n\tau$ ,  $\forall n \in \llbracket 0, M-1 \rrbracket$ , then for any given  $\mathcal{F}_{t_n}$ -measurable random variable  $u_{\epsilon,n}$  taking values in  $L^2(D)$  (with  $n \in \llbracket 0, M-1 \rrbracket$ ), there exists a unique element  $u_{\epsilon,n+1}$  in  $V$ , such that  $\mathbb{P}$ -a.s in  $\Omega$ ,

$$u_{\epsilon,n+1} = A_{\epsilon,\tau}^{-1}(u_{\epsilon,n} + G(u_{\epsilon,n})\Delta_{n+1}\mathcal{W} + \tau f_n).$$

and finally  $u_{\epsilon,n+1}$  solves (2.1), *a priori* in  $V^*$ , and *a posteriori* in  $L^2(D)$  thanks to the regularity of the data.

**Step 2:**  $\mathcal{F}_{t_{n+1}}$ -measurability of  $u_{\epsilon,n+1} : \Omega \rightarrow V$

In the following, let  $A_{\tau,\epsilon}^{-1}|_{L^2(D)}$  be the restriction of  $A_{\tau,\epsilon}^{-1}$  to  $L^2(D)$ . Our objective is to prove the measurability of  $A_{\tau,\epsilon}^{-1}|_{L^2(D)} : L^2(D) \rightarrow V$  which will lead us to the  $\mathcal{F}_{t_{n+1}}$ -measurability of  $u_{\epsilon,n+1} : \Omega \rightarrow V$ . Indeed, assuming that the above claim holds true, denoting the class of Borel sets of  $L^2(D)$  by  $\mathcal{B}(L^2(D))$  and the class of Borel sets of  $V$  by  $\mathcal{B}(V)$ , respectively, and setting for any  $n \in \llbracket 0, M-1 \rrbracket$ ,  $v_n = u_{\epsilon,n} + G(u_{\epsilon,n})\Delta_{n+1}\mathcal{W} + \tau f_n$ , one gets that for all  $\mathcal{A} \in \mathcal{B}(V)$ ,

$$(u_{\epsilon,n+1})^{-1}(\mathcal{A}) = (A_{\tau,\epsilon}^{-1}|_{L^2(D)} \circ v_n)^{-1}(\mathcal{A}) = v_n^{-1}\left(\left(A_{\tau,\epsilon}^{-1}|_{L^2(D)}\right)^{-1}(\mathcal{A})\right) \in \mathcal{F}_{t_{n+1}},$$

since  $(A_{\tau,\epsilon}^{-1}|_{L^2(D)})^{-1}(\mathcal{A}) \in \mathcal{B}(L^2(D))$  and  $v_n : \Omega \rightarrow L^2(D)$  is  $\mathcal{F}_{t_{n+1}}$ -measurable.

From Pettis measurability theorem (see [15, Th 1.1.6, p. 5]), since  $V$  is separable, in order to prove the measurability of  $A_{\tau,\epsilon}^{-1}|_{L^2(D)} : L^2(D) \rightarrow V$ , it is sufficient to show its weak measurability, *i.e.*, that for all  $v \in V^*$ , the application  $\phi_v : L^2(D) \rightarrow \mathbb{R}$  defined by

$$\phi_v(w) = \langle v, A_{\tau,\epsilon}^{-1}|_{L^2(D)}(w) \rangle_{V^*,V}, \quad \forall w \in L^2(D),$$

is measurable. To do so, we will prove the weak continuity of  $A_{\tau,\epsilon}^{-1}|_{L^2(D)}$  by showing that for every sequence  $(h_j)_{j \in \mathbb{N}} \subset L^2(D)$  converging strongly to some element  $h$  in  $L^2(D)$ , we have

$$\langle v, A_{\tau,\epsilon}^{-1}|_{L^2(D)}(h_j) \rangle_{V^*,V} \xrightarrow{j \rightarrow +\infty} \langle v, A_{\tau,\epsilon}^{-1}|_{L^2(D)}(h) \rangle_{V^*,V}$$

for any fixed  $v \in V^*$ . This will give us directly the continuity of  $\phi_v$  and subsequently its measurability.

Hence, we consider  $(h_j)_{j \in \mathbb{N}} \subset L^2(D)$  and  $h \in L^2(D)$  such that  $h_j \xrightarrow{j \rightarrow +\infty} h$  in  $L^2(D)$ . Then for any  $j \in \mathbb{N}$ , setting  $u_j = A_{\tau,\epsilon}^{-1}|_{L^2(D)}(h_j)$ , by application of  $A_{\tau,\epsilon}$  we arrive at the following equality in  $V^*$ :

$$u_j + \tau \left( \Delta_p^N u_j + \psi_\epsilon(u_j) - \beta(u_j) \right) = h_j. \quad (2.5)$$

Testing (2.5) with  $u_j \in V$  yields the following energy estimate for any  $\delta > 0$

$$\|u_j\|_{L^2(D)}^2 + \tau \|u_j\|_V^p \leq \frac{1}{2\delta} \|h_j\|_{L^2(D)}^2 + \frac{\delta}{2} \|u_j\|_{L^2(D)}^2 + \tau L_\beta \|u_j\|_{L^2(D)}^2.$$

By choosing  $\delta > 0$  such that  $1 - \frac{\delta}{2} - \tau L_\beta \geq 0$ , we get  $\|u_j\|_V^p \leq \frac{1}{2\delta\tau} \|h_j\|_{L^2(D)}^2$ , which proves the boundedness of  $(u_j)_{j \in \mathbb{N}}$  in  $V$ . Consequently, there exists  $u \in V$  such that, up to a not-relabelled subsequence,  $(u_j)_{j \in \mathbb{N}}$  converges weakly towards  $u$  in  $V$ , and as  $p \geq 2$ , by compact embedding of  $V$  in  $L^p(D)$  we also have, up to a not-relabelled subsequence, strong convergence of  $(u_j)_{j \in \mathbb{N}}$



towards  $u$  in  $L^2(D)$ . Moreover, by continuity of  $\beta$  and  $\psi_\epsilon$ , we then get the strong convergence in  $L^2(D)$  of  $(\beta(u_j))_{j \in \mathbb{N}}$  and  $(\psi_\epsilon(u_j))_{j \in \mathbb{N}}$  towards  $\beta(u)$  and  $\psi_\epsilon(u)$ , respectively. Additionally, denoting the conjugate exponent of  $p$  by  $p'$ , since

$$\| |u_j|^{p-2} u_j \|_{L^{p'}(D)} = \|u_j\|_{L^p(D)}^{p-1} \quad \text{and} \quad \| |\nabla u_j|^{p-2} \nabla u_j \|_{(L^{p'}(D))^d} = \| \nabla u_j \|_{(L^p(D))^d}^{p-1},$$

one gets that  $(|u_j|^{p-2} u_j)_{j \in \mathbb{N}}$  and  $(|\nabla u_j|^{p-2} \nabla u_j)_{j \in \mathbb{N}}$  are bounded in  $L^{p'}(D)$  and  $(L^{p'}(D))^d$ , respectively. Consequently, there exist  $y \in L^{p'}(D)$  and  $\mathbf{z} \in (L^{p'}(D))^d$  that are weak limits (up to not-relabelled subsequences) respectively of  $(|u_j|^{p-2} u_j)_{j \in \mathbb{N}}$  in  $L^{p'}(D)$  and of  $(|\nabla u_j|^{p-2} \nabla u_j)_{j \in \mathbb{N}}$  in  $(L^{p'}(D))^d$ . Using all these weak and strong convergences in (2.5), we obtain that for any  $v \in V$ ,

$$\int_D uv \, dx + \tau \int_D (-\operatorname{div}_x \mathbf{z} + y + \psi_\epsilon(u) - \beta(u))v \, dx = \int_D hv \, dx, \quad (2.6)$$

and taking  $v = u$  in (2.6) provides the following energy equality

$$\|u\|_{L^2(D)}^2 = -\tau \langle -\operatorname{div}_x \mathbf{z} + y, u \rangle_{V^*, V} - \tau \langle \psi_\epsilon(u), u \rangle_{L^2(D)} + \tau \langle \beta(u), u \rangle_{L^2(D)} + \langle h, u \rangle_{L^2(D)}. \quad (2.7)$$

In parallel, testing again (2.5) by  $u_j$  leads to

$$\|u_j\|_{L^2(D)}^2 + \tau \langle \Delta_p^N(u_j), u_j \rangle_{V^*, V} + \tau \langle \psi_\epsilon(u_j), u_j \rangle_{L^2(D)} - \tau \langle \beta(u_j), u_j \rangle_{L^2(D)} = \langle h_j, u_j \rangle_{L^2(D)},$$

and passing to the superior limit with respect to  $j$ , one arrives at

$$\|u\|_{L^2(D)}^2 = -\tau \limsup_{j \rightarrow +\infty} \langle \Delta_p^N(u_j), u_j \rangle_{V^*, V} - \tau \langle \psi_\epsilon(u), u \rangle_{L^2(D)} + \tau \langle \beta(u), u \rangle_{L^2(D)} + \langle h, u \rangle_{L^2(D)}. \quad (2.8)$$

From (2.7) and (2.8), one deduces

$$\limsup_{j \rightarrow +\infty} \langle \Delta_p^N(u_j), u_j \rangle_{V^*, V} = \langle -\operatorname{div}_x \mathbf{z} + y, u \rangle_{V^*, V}$$

Therefore, by Minty's monotonicity argument, we can affirm that  $\Delta_p^N(u) = -\operatorname{div}_x \mathbf{z} + y$ . Back into (2.6), we then obtain the following equality in  $V^*$

$$u + \tau \left( \Delta_p^N(u) + \psi_\epsilon(u) - \beta(u) \right) = h,$$

which leads to  $u = A_{\tau, \epsilon}^{-1} L^2(D)(h)$ .

At last, fixing  $v \in V^*$ , owing to the strong convergence of  $(u_j)_{j \in \mathbb{N}} = (A_{\tau, \epsilon}^{-1} L^2(D)(h_j))_{j \in \mathbb{N}}$  towards  $u$  in  $V$ , we may now conclude that

$$\lim_{j \rightarrow +\infty} \langle v, A_{\tau, \epsilon}^{-1} L^2(D)(h_j) \rangle_{V^*, V} = \lim_{j \rightarrow +\infty} \langle v, u_j \rangle_{V^*, V} = \langle v, u \rangle_{V^*, V} = \langle v, A_{\tau, \epsilon}^{-1} L^2(D)(h) \rangle_{V^*, V}$$

and we get back the weak continuity of the operator  $A_{\tau, \epsilon}^{-1} L^2(D) : L^2(D) \rightarrow V$ , which completes the proof of Proposition 1.

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