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Generators of $H^1(\Gamma, \partial\Gamma^c)$ with $\partial\Gamma^c \subset \partial\Gamma$ for Triangulated Surfaces Γ : Construction and Classification of Global Loops

Silvano Pitassi *

Abstract

Given a compact surface Γ embedded in \mathbb{R}^3 with boundary $\partial\Gamma$, our goal is to construct a set of representatives for a basis of the relative cohomology group $H^1(\Gamma, \partial\Gamma^c)$, where Γ^c is a specified subset of $\partial\Gamma$. To achieve this, we propose a novel graph-based algorithm with two key features: it is applicable to non-orientable surfaces, thereby generalizing previous approaches, and it has a worst-case time complexity that is linear in the number of edges of the mesh \mathcal{K} triangulating Γ . Importantly, this algorithm serves as a critical pre-processing step to address the low-frequency breakdown encountered in boundary element discretizations of integral equation formulations.

Keywords: relative cohomology; loop-star decomposition; global loops; electrical field integral equation.

1 Introduction

This paper bridges the fields of algebraic topology and computational electromagnetics. Specifically, it addresses a challenge arising from boundary element method computations for an electromagnetic scattering problem: computing a set of representatives for a basis of the *first relative cohomology group* $H^1(\Gamma, \partial\Gamma^c)$, where Γ is a triangulated surface embedded in \mathbb{R}^3 and $\partial\Gamma^c \subset \partial\Gamma$. To tackle this, we propose a worst-case linear-time algorithm.

Background and motivation Throughout this paper, we consider the following prototypical electromagnetic scattering problem, which serves as the primary motivation for the algorithm discussed here. Specifically, an incident harmonic electric field \mathbf{E}^i , propagating in a space with permittivity ϵ and permeability μ , impinges upon a triangulated surface Γ without boundary, which is a perfect conductor. This results in a scattered electric field \mathbf{E}^s . To determine \mathbf{E}^s , we solve the *Electric Field Integral Equation (EFIE)*:

$$\gamma_T \circ \mathfrak{U}_k(\mathbf{J}) = -\gamma_T \mathbf{E}^i,$$

which involves finding the induced surface current density \mathbf{J} on Γ , and subsequently, determining \mathbf{E}^s via the *EFIE integral operator* $\gamma_T \circ \mathfrak{U}_k$ [1]. Here, γ_T denotes the tangential trace operator on Γ .

Crucially, when the EFIE is discretized using the *Boundary Element Method (BEM)*, two numerical challenges arise as the frequency of the involved harmonic fields approaches zero—a well-documented phenomenon referred to as the *low-frequency breakdown*. The first challenge is the growth in the condition number of the resulting linear system after BEM discretization, while the second is the occurrence of round-off errors due to finite precision in the numerical integration of the EFIE's source term. These two effects can lead to inefficiencies in standard numerical solvers, requiring significantly higher computational resources (e.g., more memory or more solver iterations), or even causing the solution to fail to converge [2].

To address the low-frequency breakdown of the EFIE, the key idea is to decompose the current density \mathbf{J} as the sum $\mathbf{J} = \mathbf{J}_\Lambda + \mathbf{J}_\Sigma$, where \mathbf{J}_Λ is *solenoidal* (i.e., $\text{div}_\Gamma \mathbf{J}_\Lambda = 0$, with div_Γ denoting

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the surface divergence operator on Γ) and \mathbf{J}_Σ is non-solenoidal. This decomposition is essentially derived from the Hodge decomposition theorem applied to the L^2 -based *de Rham complex* on Γ , along with the definition of the first (absolute) cohomology group $H^1(\Gamma)$ as a quotient space, so that we can write:

$$\mathbf{J} = \underbrace{\text{curl}_\Gamma \boldsymbol{\psi}}_{\mathbf{J}_\Lambda} + \underbrace{\text{grad}_\Gamma \varphi}_{\mathbf{J}_\Sigma}, \quad (1)$$

where grad_Γ is the L^2 -adjoint of div_Γ , and \mathbf{g} represents a cohomology class in $H^1(\Gamma)$. The vector field $\boldsymbol{\psi}$ is referred to as the *stream function* or *local loop*, and \mathbf{g} is called a (*cohomology*) *generator* or *global loop*.

Two approaches for computing this decomposition have been proposed in the literature. Initially, the *loop-star* decomposition was introduced [3, 4]. This approach requires computing a set of generators for the image spaces Im curl_Γ and Im grad_Γ , as well as a set of generators for the cohomology space $H^1(\Gamma)$. More recently, *quasi-Helmholtz projectors* have been proposed as an alternative to avoid explicitly computing cohomology generators [5]. This method computes the non-solenoidal part \mathbf{J}_Σ via an orthogonal projection P_Σ onto Im grad_Γ , while the solenoidal part \mathbf{J}_Λ is found using the complementary orthogonal projection P_Λ .

However, we emphasize that the computation of generators of $H^1(\Gamma)$ is, in practice, strictly necessary to address the low-frequency breakdown of the EFIE, even when quasi-Helmholtz projectors are used. This point is supported by recent work [6], which introduces a technique for reducing round-off errors in the numerical evaluation of the EFIE's source term. Notably, the methodology outlined in Sections IV A and B of that paper clearly shows the need for a set of generators of $H^1(\Gamma)$, as they appear explicitly in the definition of the proposed quasi-Helmholtz projectors (cf. [6, Equations (24)–(28)]).

Review of cohomology computations in the numerical analysis community We begin by pointing out early works by Hiptmair and Ostrowski in [7] (which we refer to as the *HO-algorithm* hereafter), along with a non-exhaustive list of related works in [8–10]. These articles focus on a surface Γ without boundary and aim to compute generators of the first homology group $H_1(\Gamma)$. However, in numerical analysis and engineering applications, what is typically needed are generators of $H^1(\Gamma)$, rather than those of $H_1(\Gamma)$.

Since Γ is orientable (i.e., the boundary is a closed subset of \mathbb{R}^3), it is possible to recover generators of $H^1(\Gamma)$ from those of $H_1(\Gamma)$ using the Poincaré-Lefschetz Duality Theorem [11]. This approach is mentioned in the introduction of [7], though the actual implementation details are missing.

An algorithm that implements this construction was initially developed by Dłotko in [12] and later reorganized by Dłotko and Specogna in the series of works [13, 14] (referred to as the *DS-algorithm*).

Now, consider the case where Γ is a surface with a non-empty boundary $\partial\Gamma$. In this case, the DS-algorithm can still be applied with a simple modification of the construction. However, in the context of our electromagnetic application, the cohomology space $H^1(\Gamma)$ is no longer appropriate. We must instead account for the boundary condition $\mathbf{J} \cdot \mathbf{n}_{\partial\Gamma} = 0$ on $\partial\Gamma$, which arises from the EFIE model.

To satisfy this boundary condition, it can be shown that the vector field \mathbf{g} in (1) must belong to the first *relative cohomology space* $H^1(\Gamma, \partial\Gamma)$ [11]. Therefore, algorithms like the DS-algorithm or the HO-algorithm are unsuitable for computing generators of $H^1(\Gamma, \partial\Gamma)$, since $H^1(\Gamma)$ and $H^1(\Gamma, \partial\Gamma)$ are generally quite different spaces. As far as we are aware, no combinatorial algorithm has been devised to address this case.

In this paper, we build upon this observation and consider the more general problem of computing generators of $H^1(\Gamma, \partial\Gamma^c)$, where $\partial\Gamma^c$ is an arbitrary subset of the boundary $\partial\Gamma$ of Γ . This general version arises in the EFIE model, specifically for cases where the surface Γ is connected to external lumped-element circuits through suitable *contacts* or *ports*, such as in antennas and their connections with electronic circuits.

From a physical perspective, these connections result in net currents flowing through the body from one electrode to another, a phenomenon that would otherwise be absent. Crucially, this modifies the topology of the surface. For example, in the simplest case of a simply connected

surface Γ , these connections make the surface non-simply connected. We discuss the details of the EFIE model with contact regions in Section 3.

The main results and their novelty w.r.t. literature The primary objective of this paper is to provide an efficient algorithm for constructing and classifying the generators of the first relative cohomology group $H^1(\Gamma, \partial\Gamma^c)$, where Γ is an arbitrary triangulated surface embedded in \mathbb{R}^3 and $\partial\Gamma^c$ is a subset of its boundary. Specifically, we present two key contributions: one for the engineering community and the other for the mathematical community.

The first contribution is a rigorous definition of *global loops* [15] in the context of *cohomology theory*. In Definition 3.1 at the end of Section 3, we demonstrate that when global loops are defined as generators of the first relative cohomology group $H^1(\Gamma, \partial\Gamma^c)$ of a triangulated surface that excludes contact regions, the decomposition of the current density \mathbf{J} in (1) holds. This formal definition has several important implications. First, it covers cases where Γ has no contact regions and no boundary, in which the space $H^1(\Gamma, \Gamma^c)$ reduces to both $H^1(\Gamma, \partial\Gamma)$ and $H^1(\Gamma)$, since we have $\partial\Gamma^c = \partial\Gamma$ and $\partial\Gamma = \emptyset$, respectively. Thus, our definition naturally encompasses a variety of modeling contexts, correctly identifying the appropriate cohomology space for each case, whether relative or absolute.

Second, our definition is based directly on the *cochain complex* (see [16, Section 2.6]) supported on a triangulation \mathcal{K} of Γ , rather than the standard L^2 de Rham complex typically used in the engineering literature (see [2, Section IV]). In the latter framework, conforming basis functions are added to the set of standard *Raviart-Thomas basis functions* to span the de Rham first cohomology group. By contrast, our approach, grounded in the cochain complex, provides new insights into the construction of algorithms, which can be efficiently implemented using graph-based techniques. In particular, our algorithm has a worst-case time complexity that is linear in the number of edges of the triangulation, making it highly efficient for engineering applications, such as those involving the EFIE model. In fact, regarding the problem of computing global loops in engineering applications, it has even been suggested that “it is impossible to provide a complete description of this family of functions” [5].

Regarding the mathematical contribution, a significant novel aspect is that our algorithm is provably correct for non-orientable surfaces, such as the *Möbius strip*. Since we assume that Γ is embedded in \mathbb{R}^3 , it follows that Γ must have a non-empty boundary. This is because a non-orientable surface without boundary cannot be embedded in \mathbb{R}^3 (e.g., the Klein bottle), while every surface with boundary can be embedded in \mathbb{R}^3 . The key challenge for non-orientable surfaces is that a generator of the *torsion subgroup* appears when considering cohomology with integer coefficients (a fact evident from a canonical representation of the Möbius strip). The problem, then, is how to distinguish the torsion generator from other cohomology generators.

In the proof of correctness presented in Section 5, we show how to precisely identify the torsion generator. The proof relies on concepts from *discrete Morse theory*, which has several far-reaching consequences. First, it generalizes the construction used in the HO-algorithm, making our algorithm a natural extension of the HO-algorithm to the case of non-orientable surfaces. Second, it provides an alternative proof of the classical Classification Theorem for compact surfaces [17], which emerges as a corollary of our constructions.

Another important aspect is the treatment of contact regions. While the EFIE model in the presence of contacts has been studied in engineering works [18, 19], the focus has primarily been on the design of the coupling between the EFIE model and external circuit equations, as well as the quasi-static assumptions near contact regions. Mathematically, contributions have been made to establish links with relative cohomology, beginning with seminal ideas in [11] and revisited in works such as [20–22], among others. However, these contributions have largely focused on orientable domains, where the Poincaré-Lefschetz Duality Theorem can be applied.

In contrast, when torsion generators are present, additional complications arise, particularly the introduction of an extra generator associated with contact regions. This stands in stark contrast to the orientable case, where a strong physical interpretation is also available. Techniques that rely on duality theorems do not necessarily work in non-orientable cases, where the hypotheses underlying these theorems are not satisfied.

A glimpse on the impact of the proposed algorithm in practical application, as well as on possible extension

The proposed algorithm has several key features.

First, it provides an automatic procedure for computing global loops for surfaces, both with and without boundary, and for both orientable and non-orientable surfaces. Specifically, the main algorithm, presented in Section 4.2, is based on a three-step procedure that partitions the global loops into corresponding disjoint classes. These classes correspond, informally, to the presence of “handles”, “holes”, or contacts. The algorithm is designed to handle all of these cases automatically, adapting itself to provide the correct output for each situation.

Additionally, we emphasize that all the algorithms presented in Section 4 are entirely graph-based. This makes them straightforward to implement, as they do not require any external libraries or specialized procedures.

Second, the algorithm has a linear time worst-case complexity. Specifically, to run the algorithms described in Section 4, it is sufficient to have a list-based representation of a triangulation \mathcal{T} of the surface Γ , where each cell in \mathcal{T} has a corresponding data structure that stores the set of its neighboring cells. A general assumption, which holds in practical applications, is that the number of neighboring cells for each cell in \mathcal{T} is bounded by a constant, a property linked to the usual mesh regularity assumptions in boundary element methods. Considering this constant in the complexity analysis of standard spanning tree graph algorithms leads to algorithms with linear time worst-case complexity [23].

As another consideration, while the algorithm is very fast in practice, it does not aim to produce a minimal or shortest set of cohomology generators. In other words, the generated set of cohomology generators may not have the most compact support, meaning minimal length or the fewest number of edges.

For further insights, we refer to the works of Erickson et al. [24] for homology generators and Dlotko et al. [14] for cohomology generators, which are particularly relevant to our applications. While obtaining a truly minimal basis is often too slow for practical use, a trade-off between computational time and the quality of the basis obtained is usually necessary.

Finally, we emphasize that a crucial feature of our algorithm is its foundation in concepts from Discrete Morse Theory. As a result, the algorithms presented here can be applied whenever an optimal discrete Morse function can be computed. For instance, optimal discrete Morse functions can also be computed for pathological cases, such as surfaces connected by a single common vertex or by a single edge (see [25]). Moreover, these concepts can be extended to handle *non-manifold surfaces* (e.g., screens with junction points, as defined in [26, 27]). We hypothesize that simple modifications to our algorithms can also be applied to these cases, while maintaining the worst-case linear complexity. Indeed, identifying the manifold parts can be achieved in linear time by examining the coboundary elements of each edge. By fixing the orientation of a random polygon on each manifold part and iteratively orienting the others with a standard breadth-first search technique [23], we expect the algorithm to retain linear time complexity.

We believe that the combination of all these features opens up new possibilities for efficient numerical procedures to solve the EFIE.

2 Preliminaries from algebraic topology

In this preliminary section, we introduce concepts from algebraic topology and provide specific definitions tailored to our purposes. In particular, we define the concept of a cochain as a model for a discrete vector field.

In this paper, we consider two types of cochains. The first are integer-valued cochains, such as representatives of the generators of the first cohomology group over the integers. The second are real-valued cochains, which are used to model physical variables.

2.1 CW complex and triangulations

A *2-manifold Γ with boundary* is a Hausdorff topological space in which every point has a neighborhood homeomorphic to either \mathbb{R}^2 or the half-plane $\{(a, b) \in \mathbb{R}^2 \mid a \geq 0\}$. The *boundary* of Γ , denoted by $\partial\Gamma$, consists of all points with neighborhoods homeomorphic to the half-plane. We say that Γ is a *2-manifold without boundary*, or simply a 2-manifold, if $\partial\Gamma$ is empty.

Let us now introduce the concept of a CW complex \mathcal{K} defined on the 2-manifold Γ .

Definition (CW complex)

An (*open*) m -cell, for $m \in \{0, 1, 2\}$, is a topological space homeomorphic to the interior of an m -ball $\text{int } B^m$ (note that B^0 is a single point).

Definition 2.1 (CW complex). A (2-dimensional) *CW complex* \mathcal{K} (on Γ) is a partition of Γ into m -cells x such that the following holds: For each m -cell x , there exists a continuous surjection (the *gluing map*) $g_x : B^m \rightarrow x$ satisfying:

- (i) The restriction of g_x to the interior, $g_x : \text{int } B^m \rightarrow x$, is a homeomorphism.
- (ii) The image of the boundary, $g_x(\partial B^m)$, is covered by the closure of a finite number of k -cells with $0 \leq k < m$.

For $m \in \{0, 1, 2\}$, we denote by $\mathcal{K}^{(m)}$ the set of all m -cells of \mathcal{K} .

A CW complex \mathcal{K} is called *regular* if, for every m -cell x , the restriction of the gluing map $g_x : \partial B^m \rightarrow g_x(\partial B^m)$ is a homeomorphism.

A *regular* CW complex \mathcal{K} where each m -cell is an m -simplex is also referred to as a *triangulation*. In this case, we adopt a more familiar notation and terminology for the m -cells. Specifically: a *node* n is a 0-cell in $V := \mathcal{K}^{(0)}$, an *edge* e is a 1-cell in $E := \mathcal{K}^{(1)}$, and a *face* f is a 2-cell in $F := \mathcal{K}^{(2)}$. Accordingly, we denote a triangulation \mathcal{K} by $\mathcal{K} = (V, E, F)$.

A *CW subcomplex* \mathcal{S} of \mathcal{K} is a union of m -cells $x \in \mathcal{K}$ such that, if $x \subseteq \mathcal{S}$, then the closure of x is also a subset of \mathcal{S} .

If Γ has a non-empty boundary $\partial\Gamma$, we denote by $\partial\mathcal{K}$ the subcomplex of \mathcal{K} induced on $\partial\Gamma$. (Note that $\partial\Gamma$ is itself a manifold because Γ is.)

The *1-skeleton* of \mathcal{K} is the union $\mathcal{K}^{(0)} \cup \mathcal{K}^{(1)}$, i.e., the set of all 0- and 1-cells. It is straightforward to observe that the 1-skeleton of \mathcal{K} can be regarded as a graph whose vertices and edges are $\mathcal{K}^{(0)}$ and $\mathcal{K}^{(1)}$, respectively.

Using this identification, a *spanning tree* $T = (V_T, E_T)$ of \mathcal{K} refers to a spanning tree of the 1-skeleton of \mathcal{K} , treated as a graph.

2.2 (Co)chains and (co)homology

Let \mathcal{T} be a CW complex, and let G be either the module of integers \mathbb{Z} or reals \mathbb{R} . We start by fixing an orientation of all its m -cells (see, e.g., [28, Chapter 2]). The m th *chain group* $C_m(\mathcal{T}; G)$ is the group of all formal sums $c = \sum_{x \in \mathcal{T}^{(m)}} a_x x$ with $a_x \in G$. We call each element of $C_m(\mathcal{T}; G)$ a *m -chain*. For two m -chains $c = \sum_{x \in \mathcal{T}^{(m)}} a_x x$ and $c' = \sum_{x \in \mathcal{T}^{(m)}} b_x x$, we define their *scalar product* by

$$\langle c, c' \rangle := \sum_{x \in \mathcal{T}^{(m)}} a_x b_x.$$

The m th *cochain group* $C^m(\mathcal{T}; G)$ is the group of all (linear) maps $\mathbf{c} : C_m(\mathcal{T}; G) \rightarrow G$ such that $\mathbf{c}(c) = \mathbf{c}(\sum_{x \in \mathcal{T}^{(m)}} a_x x) = \sum_{x \in \mathcal{T}^{(m)}} a_x \mathbf{c}(x)$ for every $c \in C_m(\mathcal{T}; G)$. We call each element of $C^m(\mathcal{T}; G)$ a *m -cochain*. For a m -cochain $\mathbf{c} \in C^m(\mathcal{T}; G)$, its *support* is the set of all m -cells $x \in \mathcal{T}^{(m)}$ such that $\mathbf{c}(x) \neq 0$. Notice that to compute the value $\mathbf{c}(c)$ it is sufficient to know the values $\mathbf{c}(x)$ on every m -cell. Therefore, we can associate to the m -cochain \mathbf{c} the m -chain $\phi(\mathbf{c}) := \sum_{x \in \mathcal{T}^{(m)}} b_x x$ with $b_x = \mathbf{c}(x)$ for $x \in \mathcal{T}^{(m)}$ in such a way that, for each m -chain $c' \in C_m(\mathcal{T}; G)$, it holds

$$\mathbf{c}(c') = \langle \phi(\mathbf{c}), c' \rangle$$

For $m \in \{1, 2\}$, let $\iota : \mathcal{T}^{(m)} \times \mathcal{T}^{(m-1)} \rightarrow \mathbb{Z}$ be the *incidence number* between a m -cell and a $(m-1)$ -cell (see [29, Section 2.2], and, in particular, the cellular boundary formula on p. 140; equivalently, see [28]). If $\iota(x, x') \neq 0$, then we say that x is *incident* on x' , or simply that x and x' are incident. We use incidence numbers to define the *boundary map* $\partial_m : C_m(\mathcal{T}; G) \rightarrow C_{m-1}(\mathcal{T}; G)$ for $m \in \{1, 2\}$. For each m -cell $x \in \mathcal{T}^{(m)}$ we define $\partial_m x := \sum_{x' \in \mathcal{T}^{(m-1)}} \iota(x, x') x'$, and for a m -chain $c = \sum_{x \in \mathcal{T}^{(m)}} a_x x \in C_m(\mathcal{T}; G)$, by linearity, we define $\partial_m c := \sum_{x \in \mathcal{T}^{(m)}} a_x \partial_m x$. It can be checked

that $\partial_{m+1} \circ \partial_m = 0$ for $m \in \{0, 1\}$. The *coboundary map* $\delta^m : C^{m-1}(\mathcal{T}; G) \rightarrow C^m(\mathcal{T}; G)$, for $m \in \{0, 1\}$, is defined as follows. For a m -cochain $\mathbf{c} \in C^m(\mathcal{T}; G)$, $\delta^m \mathbf{c}$ is the (unique) $(m+1)$ -cochain such that $\langle \delta^m \mathbf{c}, c' \rangle = \langle \mathbf{c}, \partial_{m+1} c' \rangle$ for all $c' \in C_{m+1}(\mathcal{T}; G)$. Using $\partial_{m+1} \circ \partial_m = 0$, it can be checked that $\delta^m \circ \delta^{m+1} = 0$ for $m \in \{0, 1\}$, so that the (graded) family $(C^m(\mathcal{T}; G), \delta^m)$ with $m \in \{0, 1, 2\}$ is a (*cochain*) *complex* once we also set $\delta^2 := 0$.

The boundary operator gives rise to a classification of m -chains in \mathcal{T} . We define the subgroup $Z_m(\mathcal{T}; G) = \{c \in C_m(\mathcal{T}; G) \mid \partial_m c = 0\}$ and the group $B_m(\mathcal{T}; G) = \{\partial_{m+1} c \mid c \in C_{m+1}(\mathcal{T}; G)\}$. The m th *homology group* is then the quotient group $H_m(\mathcal{T}; G) := Z_m(\mathcal{T}; G)/B_m(\mathcal{T}; G)$.

In a similar way, we define the group $Z^m(\mathcal{T}; G) = \{c \in C^m(\mathcal{T}; G) \mid \delta^m c = \mathbf{0}\}$ and the group $B^m(\mathcal{T}; G) = \{\delta^{m-1} c \mid c \in C^{m-1}(\mathcal{T}; G)\}$. We call each element of $Z^m(\mathcal{T}; G)$ a *m-cocycle*. The m th *cohomology group* is then the quotient group $H^m(\mathcal{T}; G) := Z^m(\mathcal{T}; G)/B^m(\mathcal{T}; G)$. When the group G is the group of reals \mathbb{R} , in the following we will simply write $H^m(\mathcal{T})$ in place of $H^m(\mathcal{T}; \mathbb{R})$. A *cohomology basis* of $H^m(\mathcal{T}; G)$ is a set of equivalence classes of m -cocycles $\{[\mathbf{g}_1], \dots, [\mathbf{g}_N]\} \subset H^m(\mathcal{T}; G)$ such that every equivalence class in $H^m(\mathcal{T}; G)$ can be obtained in a unique way as a linear combination $\sum_{i=1}^N \alpha_i [\mathbf{g}_i]$ with $\alpha_i \in G$. In this case, we say that the set of m -cocycles $\{\mathbf{g}_1, \dots, \mathbf{g}_N\}$ are *cohomology generators* of $H^m(\mathcal{T}; G)$, and we call the number N the *rank* of $H^m(\mathcal{T}; G)$ and we denote it by $\text{rank}(H^m(\mathcal{T}; G))$.

In the following, we will also need the standard concept of *relative cohomology*. In relative cohomology, some parts of the complex may be considered irrelevant. In our case, we will focus on the CW subcomplex $\partial\mathcal{T}^c$ induced by \mathcal{T} on a closed subset of $\partial\Gamma^c \subset \partial\Gamma$. We define the m th *relative cochain group* by $C^m(\mathcal{T}, \partial\mathcal{T}^c; G) := C^m(\mathcal{T}; G)/C^m(\partial\mathcal{T}^c; G)$ (notice that $C_2(\partial\mathcal{T}^c; G)$ is the zero group since $\partial\mathcal{T}^c$ is 1-dimensional). Accordingly, the definition of the m th relative cohomology group $H^m(\mathcal{T}, \partial\mathcal{T}^c; G)$ remains unchanged with respect to the absolute version once relative 1-chains are used in the corresponding definitions. The following characterization of elements of $C^1(\mathcal{T}, \partial\mathcal{T}^c; G)$ will be useful later: a 1-cochain $\mathbf{c} \in C^1(\mathcal{T}; G)$ is in $C^1(\mathcal{T}, \partial\mathcal{T}^c; G)$ if and only if the support of \mathbf{c} does not contain any 1-cell in $\partial\mathcal{T}^c$.

The m th relative homology group $H^m(\mathcal{T}, \partial\mathcal{T}^c; \mathbb{Z})$ is an Abelian group by construction. Therefore, using the Fundamental Theorem for Finitely Generated Abelian Groups (see e.g. [11]), it is isomorphic to the direct sum of a free Abelian group made of $\beta_1 \in \mathbb{N}$ cohomology generators and a *torsion group* made of $\tau_1 \in \mathbb{N}$ *torsion generators*. In our target application, the relationship between $H^m(\mathcal{T}, \partial\mathcal{T}^c; \mathbb{Z})$ and $H^m(\mathcal{T}, \partial\mathcal{T}^c; \mathbb{R})$ will be important. In fact, for the numerical approximation of physical fields, we employ real coefficients, and hence generators of $H^m(\mathcal{T}, \partial\mathcal{T}^c; \mathbb{R})$, while computational topology algorithms employ integer coefficients and hence generators of $H^m(\mathcal{T}, \partial\mathcal{T}^c; \mathbb{Z})$. Despite this distinction, we notice that when $H^m(\mathcal{T}, \partial\mathcal{T}^c; \mathbb{Z})$ is found, the cohomology generators $H^m(\mathcal{T}, \partial\mathcal{T}^c; \mathbb{R})$ can be deduced from those of $H^m(\mathcal{T}, \partial\mathcal{T}^c; \mathbb{Z})$ by not taking into account the torsion ones; see Section 5.5 and, in particular, Corollary 1 for the formal statement of this fact.

3 Definition of global loops

Let Γ be a 2-manifold with boundary, embedded in \mathbb{R}^3 , and satisfying the conditions described in [1, Section 1.1]. As explained in the introduction, an induced current density \mathbf{J} on Γ arises either from an incident electric field or from an external circuit. In this section, we focus on the latter case and specifically review how the EFIE model on Γ can be coupled with lumped-element circuit models.

This coupling requires defining contact regions as the intersections between Γ and the physical connections of the circuits. To rigorously define these contact regions, we first assume that Γ is the boundary of a domain Ω in \mathbb{R}^3 . We then extend the discussion to the general case, where Γ represents a screen. This is modeled by assuming Ω to be a thin conductor, where the thickness is small compared to the wavelength and other characteristic lengths.

This distinction is necessary to justify the inclusion of contact regions in the definition of a suitable triangulation of Γ . It also lays the foundation for our general definition of global loops, presented in Definition 3.1.

3.1 EFIE modeling in presence of contacts

Assume that $\Gamma = \partial\Omega$. The electric field \mathbf{E} must satisfy the following condition:

$$\gamma_{\Gamma}\mathbf{E} = \mathbf{0} \quad \text{on } \Gamma. \quad (2)$$

Let \mathbf{E}^i and \mathbf{E}^s denote the incident and scattered electric fields from Γ , respectively, so that the total electric field \mathbf{E} is given by

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s. \quad (3)$$

In terms of electromagnetic potentials \mathbf{A} and φ , the scattered electric field can be written as

$$\mathbf{E}_s = -i\omega\mathbf{A} - \text{grad } \varphi. \quad (4)$$

The potentials \mathbf{A} and φ are expressed in terms of the equivalent current density \mathbf{J} and charge density σ on Γ as follows:

$$\mathbf{A}(\mathbf{x}) = \mu \int_{\Gamma} G_k(\mathbf{x}, \mathbf{x}') \mathbf{J}(\mathbf{x}') d\mathbf{x}', \quad \varphi(\mathbf{x}) = \frac{1}{\epsilon} \int_{\Gamma} G_k(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}') d\mathbf{x}',$$

where

$$G_k(\mathbf{x}, \mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

is the free-space Green's function, $k = \omega/c$, and $c = 1/\sqrt{\epsilon\mu}$.

We also need to consider the standard continuity equation, which relates the charge density σ to the current density \mathbf{J} :

$$i\omega\sigma = -\text{div}_{\Gamma} \mathbf{J} \quad \text{on } \Gamma. \quad (5)$$

Next, we introduce the concept of *contact*. Let $\Gamma^c \subset \Gamma$ be the subset where the physical connections of lumped circuits intersect Ω . Thus, Γ is partitioned as

$$\Gamma = \Gamma^c \cup \Gamma^{nc},$$

where $\Gamma^{nc} := \Gamma \setminus \Gamma^c$. We assume that Γ^c decomposes into disjoint open sets $\{\Gamma_i^c\}_{i=1}^{N^c}$, where each Γ_i^c is homeomorphic to an (open) disk. Each such Γ_i^c is referred to as a *contact*.

Each contact Γ_i^c represents an electrically small region, meaning that its diameter in \mathbb{R}^3 is small compared to the smallest characteristic wavelength. Consequently, for all $i \in \{1, \dots, N^c\}$, we have

$$\varphi = \text{const} = V_i \quad \text{and} \quad \gamma_{\Gamma}\mathbf{A} = \mathbf{0} \quad \text{on } \Gamma_i^c, \quad (6)$$

where V_i is the constant scalar potential enforced on Γ_i^c , which corresponds to the voltage of the corresponding circuit node. The potential difference is calculated with respect to the potential at infinity, assumed to be zero.

Now, consider the case of a conducting body Ω whose thickness is much smaller than the smallest characteristic wavelength. In this case, Ω can be modeled as a screen Γ with possibly a non-empty boundary $\partial\Gamma$, on which the following boundary condition holds:

$$\mathbf{J} \cdot \mathbf{n}_{\partial\Gamma} = 0 \quad \text{on } \partial\Gamma, \quad (7)$$

where $\mathbf{n}_{\partial\Gamma}$ is the unit outer normal to $\partial\Gamma$.

In the presence of contacts, the screen Γ is partitioned as $\Gamma = \Gamma^c \cup \Gamma^{nc}$, as described above. Furthermore, the boundary $\partial\Gamma$ can be partitioned as $\partial\Gamma = \partial\Gamma^c \cup \partial\Gamma^{nc}$, where $\partial\Gamma^c$ is the disjoint union of the contact regions.

This partitioning arises when a contact Γ_i^c intersects Ω in a subset of Γ , and we still want to include its effects in the model.

3.2 EFIE discretization and interpretation of discrete unknowns as real-valued cochains

To motivate the general definition of global loops, we first recall the initial steps of the *Galerkin method* for the EFIE, considering the case where Γ is a piecewise-smooth 2-manifold.

Let \mathcal{T} be a triangulation of Γ , and let W_h^1 denote the Sobolev space spanned by the standard Raviart-Thomas basis functions, supported on \mathcal{T} and satisfying the boundary condition on $\partial\Gamma$ as given in (7). The boundary condition (2) may be enforced on the Galerkin space W_h^1 by requiring that

$$\int_{\Gamma} \mathbf{E} \cdot \mathbf{j} = 0, \quad \forall \mathbf{j} \in W_h^1. \quad (8)$$

Substituting the expressions for \mathbf{E} from (3) and \mathbf{E}^s from (4) into (8), and using integration by parts with the two conditions (6) on Γ^c , we obtain the following equation:

$$i\omega\mu \int_{\Gamma^{nc}} \mathbf{A} \cdot \mathbf{j} dS + \int_{\Gamma^{nc}} \varphi \operatorname{div}_{\Gamma} \mathbf{j} dS = \sum_{i=1}^N \int_{\partial\Gamma_i^c} \varphi \mathbf{j} \cdot \mathbf{n}_{\partial\Gamma_i^c} dl + \int_{\Gamma^{nc}} \mathbf{E}^i \cdot \mathbf{j} dS, \quad (9)$$

where we have used the conditions (6) on each contact Γ_i^c . Notice that if the electric potential φ is assigned on $\partial\Gamma_i^c$, then the right-hand side of the equation is known. On the other hand, assigning the normal components of the current density $\mathbf{J} \cdot \mathbf{n}_{\partial\Gamma_i^c}$ is related to our formal definition of global loops. Also note that Γ^{nc} is generally the sub-manifold of Γ over which the field integrals are evaluated in the weak formulation (9). In the absence of contacts, we have $\Gamma^c = \emptyset$, so $\Gamma^{nc} = \Gamma$. Therefore, we can focus on a triangulation \mathcal{T} of Γ^{nc} .

Now, let $F_h^1 : W_h^1 \rightarrow C^1(\mathcal{T})$ be the *De Rham map* [16, 30], restricted to the space W_h^1 . The de Rham map is a *cochain isomorphism* between the cochain complex and the finite element complex supported on the triangulation \mathcal{T} . Specifically, by considering the boundary condition on $\partial\mathcal{T}$, the map F_h^1 is an isomorphism between W_h^1 and the set of *relative 1-cochains* $C^1(\mathcal{T}, \partial\mathcal{T})$ [31].

We are now ready to present the general definition of global loops.

Definition 3.1 (Definition of global loops). The set $GL = \{\mathbf{g}_i\}_{i=1}^{\beta_1(\Gamma^{nc})}$ of *global loops* on Γ is defined as a set of 1-cocycles, representing a basis of the first relative cohomology group $H^1(\mathcal{T}, \partial\mathcal{T}^c)$ of the triangulation \mathcal{T} of the closed sub-manifold Γ^{nc} , excluding the contact regions.

A crucial aspect of this definition is that the global loops are defined as 1-cochains supported on a triangulation \mathcal{T} , rather than as vector fields on Γ . As we will show in the subsequent sections, this (combinatorial) definition provides a more effective approach for handling topologically complex domains Ω , as it suggests efficient graph-based algorithms.

4 General algorithm to construct global loops

As formalized in Definition 3.1, global loops should be understood as generators of the first cohomology group $H^1(\mathcal{T}, \partial\mathcal{T}^c)$, where \mathcal{T} is a triangulation of the sub-manifold Γ^{nc} of Γ that excludes contact regions. The types of generators required depend on the topology of \mathcal{K} and the presence or absence of contact regions.

In this section, we present a general algorithm to compute these generators, which is detailed in Algorithm 5.

We assume that \mathcal{K} is connected for the purposes of this discussion. This assumption is not restrictive in general. Indeed, if \mathcal{T} consists of p connected components \mathcal{T}_i for $i \in \{1, \dots, p\}$, we have the following decomposition:

$$H^1(\mathcal{T}) \cong \bigoplus_{i=1}^p H^1(\mathcal{T}_i),$$

which means that it is sufficient to consider each connected component \mathcal{K}_i individually.

4.1 Interpreting cocycles as cycles on the dual graph \tilde{G}

In the constructions used in Algorithm 5, we require the definition of a special graph associated with the triangulation \mathcal{T} . This graph, known as the *dual graph*, is denoted by $\tilde{G} = (\tilde{V}, \tilde{E})$, where $\tilde{V} := F$ and $\tilde{E} := E$. It is defined using the (bijective) *duality map* $D : E \cup F \rightarrow \tilde{V} \cup \tilde{E}$ as follows:

- For each face $f \in F$, the corresponding *dual node* is $\tilde{n} = D(f) \in \tilde{V}$.
- For each edge $e \in E$, which is a common edge of two faces $f_e^{(1)}, f_e^{(2)} \in F$, the corresponding (non-oriented) *dual edge* is $\tilde{e} = \{D(f_e^{(1)}), D(f_e^{(2)})\} = D(e) \in \tilde{E}$.

Additionally, we need to describe how the duality map D acts on edges that lie on the boundary $\partial\mathcal{T}$ of \mathcal{T} :

- For each edge $e \in \partial\mathcal{T}$, the corresponding *dual node* is $\tilde{n} = D_\partial(e)$.
- For each edge $e \in \partial\mathcal{T}$, the corresponding *dual edge* is $\tilde{e} = \{D_\partial(e), D(f)\} = D(e)$, where f is the unique face in F incident to e .

A *path* \tilde{p} of \tilde{G} is a set of $l \in \mathbb{N}, l \geq 1$ distinct dual nodes $\tilde{n}_1, \dots, \tilde{n}_l$ and $l-1$ distinct dual edges $\tilde{e}_1, \dots, \tilde{e}_{l-1}$ such that $\tilde{e}_i = \{\tilde{n}_i, \tilde{n}_{i+1}\}$ for $i \in \{1, \dots, l-1\}$ ($l=1$ means the path reduces to the single dual node \tilde{n}_1). By duality via the map D , there is a corresponding set of distinct l -faces $f_1 = D^{-1}(\tilde{n}_1), \dots, f_l = D^{-1}(\tilde{n}_l)$ and $l-1$ distinct edges $e_1 = D^{-1}(\tilde{e}_1), \dots, e_{l-1} = D^{-1}(\tilde{e}_{l-1})$ of \mathcal{T} .

We use paths on the dual graph \tilde{G} to construct 1-cochains of \mathcal{T} according to the following procedure.

Algorithm 1 Construct a 1-cocycle \mathbf{g} on \mathcal{T} , taking as input a path $\tilde{p}_{e,e'}$ in \tilde{G} and a pair of edges (e, e') of \mathcal{T} , where $e \subset f_1$ and $e' \subset f_l$.

- 1: initialize $\mathbf{g} = \mathbf{0}$
- 2: set $\langle \mathbf{g}, e \rangle$ to 1 ▷ initialize the value of \mathbf{g} on the edge e to 1
- 3: **for** each dual edge \tilde{e}_i of $\tilde{p}_{e,e'}$ **do** ▷ each \tilde{e}_i is selected by walking \tilde{p} from \tilde{e}_1 to \tilde{e}_l
- 4: compute

$$\langle \mathbf{g}, e_i \rangle = -\frac{\iota(f_i, e_i)}{\iota(f_i, e_{i-1})} \langle \mathbf{g}, e_{i-1} \rangle$$

where $e_i = D^{-1}(\tilde{e}_i)$ and the conventions $e_0 := e$ and $e_l := e'$ are applied

- 5: **end for**
 - 6: **if** $\langle \mathbf{g}, e' \rangle = 0$ **then**
 - 7: **return** $\mathbf{0}$
 - 8: **else**
 - 9: **return** \mathbf{g}
 - 10: **end if**
-

4.2 Algorithm to construct global loops

The key idea behind the main procedure in Algorithm 5 is that the set GL of all global loops of \mathcal{T} can be partitioned into three subsets that are non-empty according to the presence of, informally, “handles”, “holes”, and contacts on \mathcal{T} , and that they can be computed independently using three distinct procedures.

4.3 Global loops corresponding to handles: the set GL^{ha}

The generators arising from the presence of “handles” are the more difficult to find, and the novel algorithm to compute them reads as follows.

Algorithm 2 Construct the set GL^{ha}

- 1: construct a spanning tree T on $\partial\mathcal{T}$
 - 2: extend T to a spanning tree on the whole \mathcal{T}
 - 3: construct a spanning tree \tilde{T} on the subgraph $\tilde{G}' = (\tilde{V}, \tilde{E} \setminus \bigcup_{e \in E_T} \{D(e)\})$ \triangleright i.e., on the subgraph of \tilde{G} made by the set of all dual edges in \tilde{G} that are not dual to an edge e in T
 - 4: **for** each connected component $\partial\mathcal{T}_k$ of $\partial\mathcal{T}$ **do**
 - 5: select an edge e of $\partial\mathcal{T}_k$
 - 6: add the dual edge $D(e)$ to $E_{\tilde{T}}$ and the dual node $D_\partial(e)$ to $V_{\tilde{T}}$
 - 7: **end for**
 - 8: let $E_c := \{e \in E \mid e \notin T, D(e) \notin \tilde{T}\}$ \triangleright i.e., the set of edges that is not in T and whose dual is not in \tilde{T}
 - 9: **for** each $e \in E_c$ **do**
 - 10: let $f_e^{(1)}, f_e^{(2)}$ be the unique two faces of \mathcal{T} incident on e
 - 11: construct the unique path \tilde{p}_e in \tilde{T} from $D(f_e^{(1)})$ to $D(f_e^{(2)})$
 - 12: apply Algorithm 1 with input \tilde{p}_e and the pair (e, e) to get \mathbf{g}_e^{ha} \triangleright the pair (e, e) is made of the same edge e
 - 13: **end for**
 - 14: let $E_c^{\text{II}} := \{e \in E_c \mid \mathbf{g}_e^{ha} = \mathbf{0}\}$ \triangleright i.e., the set of edges for which the 1-cochain \mathbf{g}_e^{ha} computed from Algorithm 1 is the zero 1-cochain
 - 15: **if** $|E_c^{\text{II}}| = 0$ **then**
 - 16: **return** $GL^{ha} = \{\mathbf{g}_1^{ha}, \dots, \mathbf{g}_{N^{ha}}^{ha}\} = \{\mathbf{g}_e^{ha}\}_{e \in E_c}$ $\triangleright N^{ha}$ is equal to $|E_c|$
 - 17: **else** \triangleright i.e., if $|E_c^{\text{II}}| > 0$
 - 18: select an edge $e^* \in E_c^{\text{II}}$
 - 19: **for** $e \in E_c^{\text{II}} \setminus \{e^*\}$ **do**
 - 20: let $f_e^{(1)}, f_e^{(2)}$ be the unique two faces of \mathcal{T} incident on e
 - 21: let $f_{e^*}^{(1)}, f_{e^*}^{(2)}$ be the unique two faces of \mathcal{T} incident on e^*
 - 22: construct the unique path $\tilde{p}_{e,e^*}^{(1)}$ in \tilde{T} from $D(f_e^{(1)})$ to $D(f_{e^*}^{(1)})$
 - 23: construct the unique path $\tilde{p}_{e,e^*}^{(2)}$ in \tilde{T} from $D(f_e^{(2)})$ to $D(f_{e^*}^{(2)})$
 - 24: apply Algorithm 1 with input $\tilde{p}_{e,e^*}^{(1)}$ and (e, e^*) to get $\mathbf{g}_e^{(1)}$
 - 25: apply Algorithm 1 with input $\tilde{p}_{e,e^*}^{(2)}$ and (e, e^*) to get $\mathbf{g}_e^{(2)}$
 - 26: define \mathbf{g}_e^{ha} by

$$\langle \mathbf{g}_e^{ha}, e' \rangle = \begin{cases} \langle \mathbf{g}_e^{(1)}, e' \rangle & \text{if } e' \in \{e, e^*\}, \\ \langle \mathbf{g}_e^{(1)}, e' \rangle + \langle \mathbf{g}_e^{(2)}, e' \rangle & \text{if } e' \notin \{e, e^*\} \end{cases}$$
 - 27: **end for**
 - 28: **return** $GL^{ha} = \{\mathbf{g}_1^{ha}, \dots, \mathbf{g}_{N^{ha}}^{ha}\} = \{\mathbf{g}_e^{ha}\}_{e \in E_c \setminus \{e^*\}}$ $\triangleright N^{ha}$ is equal to $|E_c| - 1$
 - 29: **end if**
-

4.4 Global loops corresponding to holes: the set GL^h

The global loops that arise from the presence of “holes” are straightforward to construct. Let $\partial\mathcal{T}_k$ for $k \in \{1, \dots, N^h\}$ represent the connected components of $\partial\mathcal{T}$, and fix one of them, say $\partial\mathcal{T}_{N^h}$.

For each connected component $\partial\mathcal{T}_k$ with $k \in \{1, \dots, N^h - 1\}$, there corresponds a global loop \mathbf{g}_k^h . The support of this global loop consists of all edges that have exactly one of their nodes in $\partial\mathcal{T}_k$.

Algorithm 3 Construct the set GL^h

- 1: **for** each connected component $\partial\mathcal{T}_k$ of $\partial\mathcal{T}$ except $\partial\mathcal{T}_{N^h}$ **do**
- 2: define \mathbf{c}_k^h by

$$\langle \mathbf{c}_k^h, n \rangle := \begin{cases} 1 & \text{if } n \in \partial\mathcal{T}_k \\ 0 & \text{if } n \notin \partial\mathcal{T}_k \end{cases}$$

- 3: compute $\mathbf{g}_k^h := \delta^1 \mathbf{c}_k^h$ \triangleright \mathbf{c}_k^h is equal to 1 on each boundary node n of $\partial\mathcal{T}_k$ and 0 elsewhere
 - 4: **end for** \triangleright apply the coboundary map δ^1 to \mathbf{c}_k^h
 - 5: **return** $\{\mathbf{g}_1^h, \dots, \mathbf{g}_{N^h-1}^h\}$
-

4.5 Global loops corresponding to contacts: the set GL^c

The global loops arising from the presence of contacts are also straightforward to construct. Let $\partial\mathcal{T}_j^c$ for $j \in \{1, \dots, N^c\}$ denote the contacts of $\partial\mathcal{T}^c$, and fix one of them, say $\partial\mathcal{T}_{N^c}^c$.

Then, for each contact $\partial\mathcal{T}_j^c$ with $j \in \{1, \dots, N^c\}$, there corresponds a global loop \mathbf{g}_j^c , whose support consists of all edges whose duals belong to a specific path in the dual graph \tilde{G} , associated with the pair of contacts $\partial\mathcal{T}_j^c$ and $\partial\mathcal{T}_{N^c}^c$ of $\partial\mathcal{T}^c$.

Algorithm 4 Construct the set GL^c

- 1: let \tilde{T} be a spanning tree of \tilde{G}
- 2: select an edge e of $\partial\mathcal{T}_{N^c}^c$ and let f_e the unique face of \mathcal{T} incident on it
- 3: **for** each connected component $\partial\mathcal{T}_j^c$ of $\partial\mathcal{T}^c$ except $\partial\mathcal{T}_{N^c}^c$ **do**
- 4: select an edge e_j of $\partial\mathcal{T}_j^c$
- 5: let f_{e_j} the unique face of \mathcal{T} incident on e_j
- 6: construct the unique path $\tilde{p}_{e_j, e}$ in \tilde{T} from $D(f_{e_j})$ to $D(f_e)$
- 7: apply Algorithm 1 with input $\tilde{p}_{e_j, e}$ and (e_j, e) to get \mathbf{g}_j^c
- 8: **end for**
- 9: let E_c^{II} be the set computed in Algorithm 2 at line 13 \triangleright the set E_c and its subset E_c^{II}
 have already been computed in Algorithm 2 (see the order in which the algorithms are called in Algorithm 5)
- 10: **if** $|E_c^{\text{II}}| = 0$ **then**
- 11: **return** $GL^c := \{\mathbf{g}_1^c, \dots, \mathbf{g}_{N^c-1}^c\}$
- 12: **else** \triangleright i.e., if $|E_c^{\text{II}}| > 0$
- 13: select an edge $e^* \in E_c^{\text{II}}$
- 14: let $f_{e^*}^{(1)}, f_{e^*}^{(2)}$ be the unique two faces of \mathcal{T} incident on e^*
- 15: construct the unique path $\tilde{p}_{e, e^*}^{(1)}$ in \tilde{T} from $D(f_e)$ to $D(f_{e^*}^{(1)})$
- 16: construct the unique path $\tilde{p}_{e, e^*}^{(2)}$ in \tilde{T} from $D(f_e)$ to $D(f_{e^*}^{(2)})$
- 17: apply Algorithm 1 with input $\tilde{p}_{e, e^*}^{(1)}$ and (e, e^*) to get $\mathbf{g}_e^{(1)}$
- 18: apply Algorithm 1 with input $\tilde{p}_{e, e^*}^{(2)}$ and (e, e^*) to get $\mathbf{g}_e^{(2)}$
- 19: define $\mathbf{g}_{N^c}^c$ by

$$\langle \mathbf{g}_{N^c}^c, e' \rangle = \begin{cases} \langle \mathbf{g}_e^{(1)}, e' \rangle & \text{if } e' = e^*, \\ \langle \mathbf{g}_e^{(1)}, e' \rangle + \langle \mathbf{g}_e^{(2)}, e' \rangle & \text{if } e' \neq e^* \end{cases}$$

- 20: **return** $GL^c := \{\mathbf{g}_1^c, \dots, \mathbf{g}_{N^c}^c\}$
 - 21: **end if**
-

4.6 The main algorithm to construct the set GL of global loops

The algorithm to compute global loops combines together Algorithms 2 to 4 and reads as a three step procedure.

Algorithm 5 Construct of the set GL

- 1: apply Algorithm 2 to get global loops GL^{ha} corresponding to handles
 - 2: **if** $\partial\mathcal{T} \neq \emptyset$ **then** ▷ i.e., if \mathcal{T} has a boundary
 - 3: apply Algorithm 3 to get global loops GL^h corresponding to holes
 - 4: **if** $\partial\mathcal{T}^c \neq \emptyset$ **then** ▷ i.e., if \mathcal{T} has some contacts
 - 5: apply Algorithm 4 to get global loops GL^c corresponding to contacts
 - 6: **end if**
 - 7: **end if**
 - 8: **return** $GL = GL^{ha} \cup GL^h \cup GL^c$ ▷ return the set of all global loops
-

In the next section we prove the correctness of Algorithm 5.

5 Proof of correctness of Algorithm 5

We divide the proof of in nine steps.

5.1 Tools from Discrete Morse Theory

To start with, we recall a specific version of discrete Morse theory introduced by Kozlov in his book [32]. Let \mathcal{T} be a CW complex, and for $m \in \{1, 2\}$, let x be a m -cell and x' be a $(m-1)$ -cell of \mathcal{T} . If $\iota(x, x') \neq 0$, we write $x' \prec x$.

Definition 5.1 (Matching, acyclic matching). A *matching* M of m -cells on \mathcal{T} is a set of pairs (x, x') made of an m -cell and a $(m-1)$ -cell of \mathcal{T} such that:

- (i) If $(x, x') \in M$, then $x' \prec x$.
- (ii) If $(x_1, x'), (x_2, x') \in M$, then $x_1 = x_2$.

When $(x, x') \in M$, we write $x' = d(x)$. A matching M is *acyclic* if there does not exist a cycle of matched elements x_i for $i \in \{1, \dots, l\}$ of the form

$$x_1 \succ d(x_1) \prec x_2 \succ d(x_2) \prec \dots \prec x_l \succ d(x_l) \prec x_1, \quad (10)$$

with $l \geq 2$.

We say that an m -cell or $(m-1)$ -cell of \mathcal{T} is *matched* in M if it appears as the first or second member of a pair in M . The set of *critical cells* of \mathcal{T} is set of all cells of \mathcal{T} that are not matched in M .

The next lemma says that to every spanning tree on \mathcal{T} we can associate an acyclic matching.

Lemma 1 (Acyclic matching from spanning tree). *Let T be a spanning tree of \mathcal{T} . Then, there exist an acyclic matching M_T such that every vertex of T except one, say n , is matched in M_T . In particular, n can be chosen arbitrarily. Similarly, let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the dual graph of \mathcal{T} and let \tilde{T} be a spanning tree on \tilde{G} . Then, there exist an acyclic matching $M_{\tilde{T}}$ such that every dual vertex of \tilde{T} except one, say \tilde{n} , is matched in $M_{\tilde{T}}$. In particular, \tilde{n} can be chosen arbitrarily.*

Proof. The proof of these two results follows directly from the reasoning in [33, Section 4.2] for the cases $k = 0$ and $k = 2$ (notice that the construction for the case $k = 2$ is stated for 3-dimensional domains but it can be easily adapted to the 2-dimensional ones). \square

The next theorem can be regarded as the main theorem of Discrete Morse Theory for CW complexes, since it sheds light on the topological role of acyclic matchings.

Theorem 5.1. [32, Theorem 11.13] *Let M be an acyclic matching on \mathcal{T} , and denote by d_m the number of critical m -cells of \mathcal{T} . The following two assertions hold:*

- (i) *If the set of critical cells form a subcomplex \mathcal{T}_c of \mathcal{T} , then \mathcal{T}_c is a CW complex with d_m m -cells that is homotopy equivalent to \mathcal{T} .*
- (ii) *For every critical m -cell x and $(m-1)$ -cell x' in \mathcal{T}_c , let $S(x, x')$ be the set of all sequences s of matched cells of the form $s := (x \succ d(x_1) \prec x_1 \succ d(x_2) \prec x_2 \succ \dots \succ d(x_l) \prec x_l \succ x')$. For each $s \in S(x, x')$, let $w(s)$ be the m -chain computed from the sequence s as follows: $w(s) := w_l$, where, for all $i \in \{0, \dots, l\}$, w_i is computed by recursion as*

$$\begin{aligned} w_0 &:= x, \\ w_i &:= w_{i-1} + \xi_i x_i, \end{aligned} \tag{11}$$

with

$$\xi_i := -\frac{\langle \partial_{m+1} w_{i-1}, d(x_i) \rangle}{\langle \partial_{m+1} x_i, d(x_i) \rangle}, \tag{12}$$

for all $i \in \{1, \dots, l\}$. Then, there is a natural identification of cells of \mathcal{T}_c with the critical cells of \mathcal{T} such that, for any x, x' as above, the incidence number $\iota^M(x, x')$ in \mathcal{T}_c is given by

$$\iota^M(x, x') := \sum_{s \in S(x, x')} \langle \partial_m w(s), x' \rangle. \tag{13}$$

5.2 A property of the 1-cochain computed by Algorithm 1

Let \tilde{p} be a path in \tilde{G} , and denote by f_1, \dots, f_l the corresponding set of 2-cells and by e_1, \dots, e_{l-1} the associated set of 1-cells in \mathcal{K} . Define $M_{\tilde{p}}$ as the set of all pairs (f_i, e_i) for $i \in \{1, \dots, l-1\}$. It is straightforward to verify that $M_{\tilde{p}}$ forms an acyclic matching on \mathcal{K} , and that f_l is a critical 2-cell in $M_{\tilde{p}}$.

Accordingly, for each critical 1-cell e such that $e \subset f_1$, we define the sequence $s_{\tilde{p}}$ associated with the path \tilde{p} in $M_{\tilde{p}}$ as

$$s_{\tilde{p}} := (f_l \succ e_{l-1} \prec f_{l-1} \succ e_{l-2} \prec f_{l-2} \succ \dots \succ e_1 \prec f_1 \succ e). \tag{14}$$

Lemma 2. *Let \tilde{p} be a path in \tilde{G} , and let (e, e') be a pair of 1-cells of \mathcal{T} such that $e \subset f_1$ and $e' \subset f_l$. Let \mathbf{g} be the 1-cochain of \mathcal{T} computed by Algorithm 1. If $\mathbf{g} \neq \mathbf{0}$, then $\langle \mathbf{g}, e \rangle = 1$, and the following holds:*

$$\langle \mathbf{g}, e' \rangle = -\frac{\iota(\xi_l f_l, e')}{\iota(\xi_1 f_1, e)}, \tag{15}$$

where the coefficients ξ_1, ξ_l are defined as in (12) for the sequence $s_{\tilde{p}}$.

Furthermore, the following holds for all $i \in \{1, \dots, l\}$:

$$\langle \mathbf{g}_e, \partial_2(\xi_i f_i) \rangle = 0. \tag{16}$$

Proof. By the definition of the \mathbf{g} in Algorithm 1 at line 1 and 4, we have that $\langle \mathbf{g}, e \rangle = 1$, and for each $i \in \{1, \dots, l\}$, the following holds:

$$\langle \mathbf{g}, e_i \rangle = -\frac{\iota(f_i, e_i)}{\iota(f_i, e_{i-1})} \langle \mathbf{g}, e_{i-1} \rangle \tag{17}$$

where we define $e_0 := e$, $e_l := e'$ as the initial and final edges, respectively. This formula expresses the transformation of the 1-cochain as we move through the sequence of edges.

Note that (17) is independent of the specific orientation of each face f_i . In fact, for any coefficient $\chi_i \in \{-1, +1\}$ corresponding to an oriented face $\chi_i f_i$, we have $\iota(\chi_i f_i, e_i) = \chi_i \iota(f_i, e_i)$. This shows that the expression in (17) remains unchanged under the orientation change of the faces.

Next, we multiply each face f_i by the coefficient ξ_i , which is computed according to (12) using the sequence $s_{\bar{p}}$. By applying (17) recursively, and using the fact that $\iota(\xi_i f_i, e_i) = -\iota(\xi_i f_i, e_{i-1})$ for all $i \in \{1, \dots, l\}$ (this follows from the specific choice of coefficients ξ_i in (12), as well as from the recursion formula (11)), we obtain the desired result in (15).

To prove (16), we rewrite (17) as the following equation:

$$\langle \mathbf{g}, e_i \rangle \iota(\xi_i f_i, e_i) + \langle \mathbf{g}, e_{i-1} \rangle \iota(\xi_i f_i, e_{i-1}) = 0,$$

for each $i \in \{1, \dots, l\}$. This is simply a restatement of the transformation rule for the 1-cochain.

Finally, recalling that $\mathbf{g} = \mathbf{0}$ as initialized in Algorithm 1 at line 1, we conclude that this leads to the desired result in (16). \square

5.3 A first decomposition of global loops on \mathcal{T}

We now discuss a preparatory result for the first decomposition of global loops corresponding to contact regions.

Lemma 3 (Dimension computation via long exact sequences). *Let \mathcal{T} a 2-dimensional CW complex with a non-empty boundary $\partial\mathcal{T}$. Recalling that $\partial\mathcal{T}^{nc} := \partial\mathcal{T} \setminus \bigcup_{j=1}^{N^c} \partial\mathcal{T}_j^c$, where each $\partial\mathcal{T}_j^c$ is a contact region, the following expression relates the ranks of $H^1(\mathcal{T}, \partial\mathcal{T}^{nc})$ and $H^1(\mathcal{T}, \partial\mathcal{T})$:*

$$\text{rank}(H^1(\mathcal{T}, \partial\mathcal{T}^{nc})) - \text{rank}(H^1(\mathcal{T}, \partial\mathcal{T})) = \begin{cases} N^c - 1 & \text{if } \mathcal{T} \text{ is orientable,} \\ N^c & \text{otherwise.} \end{cases} \quad (18)$$

Proof. The proof consists in expressing the ranks of $H^1(\mathcal{T}, \partial\mathcal{T}^{nc})$ and $H^1(\mathcal{T}, \partial\mathcal{T})$ in terms of that $H^1(\mathcal{T})$ and in computing the ranks of some elementary cohomology spaces. Let us first consider the long exact sequence in relative cohomology [34, Theorem 13] for the pair $(\partial\mathcal{T}, \mathcal{T})$:

$$\begin{aligned} 0 \longrightarrow \underbrace{H^0(\mathcal{T}, \partial\mathcal{T})}_0 &\longrightarrow \underbrace{H^0(\mathcal{T})}_1 \longrightarrow \underbrace{H^0(\partial\mathcal{T})}_{N^h} \longrightarrow \\ &\longrightarrow H^1(\mathcal{T}, \partial\mathcal{T}) \longrightarrow H^1(\mathcal{T}) \longrightarrow \underbrace{H^1(\partial\mathcal{T})}_{N^h} \longrightarrow \\ &\longrightarrow H^2(\mathcal{T}, \partial\mathcal{T}) \longrightarrow \underbrace{H^2(\mathcal{T})}_0 \longrightarrow \underbrace{H^2(\partial\mathcal{T})}_0 \longrightarrow 0, \end{aligned} \quad (19)$$

where the ranks of the cohomology groups are indicated under the braces. In particular, N^h denotes the number of connected components of $\partial\mathcal{T}$.

Similarly, applying [34, Theorem 12] to the pair $(\partial\mathcal{T}^{nc}, \mathcal{T})$, we obtain the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \underbrace{H^0(\mathcal{T}, \partial\mathcal{T}^{nc})}_0 &\longrightarrow \underbrace{H^0(\mathcal{T})}_1 \longrightarrow \underbrace{H^0(\partial\mathcal{T}^{nc})}_{N^{nc}} \longrightarrow \\ &\longrightarrow H^1(\mathcal{T}, \partial\mathcal{T}^{nc}) \longrightarrow H^1(\mathcal{T}) \longrightarrow H^1(\partial\mathcal{T}^{nc}) \longrightarrow \\ &\longrightarrow \underbrace{H^2(\mathcal{T}, \partial\mathcal{T}^{nc})}_0 \longrightarrow \underbrace{H^2(\mathcal{T})}_0 \longrightarrow \underbrace{H^2(\partial\mathcal{T}^{nc})}_0 \longrightarrow 0, \end{aligned} \quad (20)$$

where N^{nc} represents the number of connected components of $\partial\mathcal{T}^{nc}$.

Now, let us note the following two facts:

1. In the exact sequence for $(\partial\mathcal{T}, \mathcal{T})$ (19), the rank of $H^2(\mathcal{T}, \partial\mathcal{T})$ is 1 if \mathcal{T} is orientable, and 0 if \mathcal{T} is non-orientable;
2. In the exact sequence for $(\partial\mathcal{T}^{nc}, \mathcal{T})$ (20), the rank of $H^1(\partial\mathcal{T}^{nc})$ satisfies the formula:

$$\text{rank}(H^1(\partial\mathcal{T}^{nc})) = N^{nc} - N^c, \quad (21)$$

which will now we prove.

Consider the set of all connected components $\partial\mathcal{T}_k$ of $\partial\mathcal{T}$, where $k \in \{1, \dots, N^h\}$. After potentially rearranging the indices, we assume that there is a number $p \in \{1, \dots, N^h\}$ such that the first p connected components, $\{\partial\mathcal{T}_1, \dots, \partial\mathcal{T}_p\}$, contain some contact regions $\partial\mathcal{T}_j^c$ for $j \in \{1, \dots, N^c\}$, while the remaining components $\{\partial\mathcal{T}_{p+1}, \dots, \partial\mathcal{T}_{N^h}\}$ do not. If $p = N^h$, then the second set is empty.

For each $k \in \{1, \dots, N^h\}$, define J_k^c as the set of indices $j \in \{1, \dots, N^c\}$ such that $\partial\mathcal{T}_j^c \subset \partial\mathcal{T}_k$. Let $|J_k^c|$ denote the cardinality of J_k^c , and define N_k^{nc} as the number of connected components of the set $\partial\mathcal{T}_k \setminus \bigcup_{j \in J_k^c} \partial\mathcal{T}_j^c$.

It is easy to check that $|J_k^c| = N_k^{nc}$ for all $k \in \{1, \dots, p\}$, and that $|J_k^c| = 0$ and $N_k^{nc} = 1$ for all $k \in \{p+1, \dots, N^h\}$. Hence, we have the relation

$$N^h - p = \sum_{k=p+1}^{N^h} N_k^{nc}.$$

Also, since $\text{rank}(H^1(\partial\mathcal{T}^{nc})) = N^h - p$, we obtain the formula:

$$N^{nc} = \sum_{k=1}^{N^h} N_k^{nc} = \sum_{k=1}^p N_k^{nc} + \sum_{k=p+1}^{N^h} N_k^{nc} = N^c + \text{rank}(H^1(\partial\mathcal{T}^{nc})),$$

which is the desired result (21).

Finally, applying the Euler characteristic formula [29, Theorem 2.44] to the exact sequences (19) and (20) and using facts 1 and 2, we can derive the formula (18). \square

We are now ready to prove the first decomposition of global loops.

Lemma 4 (First decomposition of global loops). *Let $GL^c = \{\mathbf{g}_1^c, \dots, \mathbf{g}_{N^{nc}}^c\}$ be the set of 1-cochains computed by Algorithm 4. Define $[GL^c] := \{\sum_{j=1}^{N^{nc}} \alpha_j [\mathbf{g}_j^c] \mid \alpha_j \in \mathbb{R}\}$ as the linear span of the equivalence classes associated with each generator $\mathbf{g}_j^c \in GL^c$. Then, the following group isomorphism holds:*

$$H^1(\mathcal{T}, \partial\mathcal{T}^{nc}) \cong H^1(\mathcal{T}, \partial\mathcal{T}) \oplus [GL^c]. \quad (22)$$

Proof. First, note that in Algorithm 4 each 1-cochain $\mathbf{g}_j^c \in GL^c$ is computed using Algorithm 1 with input pairs (e, e') where $e, e' \in \partial\mathcal{T}$. As a result, by applying (15) and (16) from Lemma 2, we conclude that \mathbf{g}_j^c is 1-cocycle that is non-zero on exactly one 1-cell in the boundary $\partial\mathcal{T}_j^c \subset \partial\mathcal{T}$.

Hence, $\mathbf{g}_j^c \notin B^0(\mathcal{T}, \partial\mathcal{T}^{nc})$, because if \mathbf{g}_j^c were in $B^0(\mathcal{T}, \partial\mathcal{T}^{nc})$, it would be non-zero on more than 1-cell in $\partial\mathcal{T}_j^c$, which contradicts the earlier conclusion. Therefore, we have $[GL^c] \subset H^1(\mathcal{T}, \partial\mathcal{T}^{nc})$.

Furthermore, since \mathbf{g}_j^c is non-zero on exactly one 1-cell $\partial\mathcal{T}_j^c$, it follows that $H^1(\mathcal{T}, \partial\mathcal{T}) \cap [GL^c] = \{0\}$.

Next, let $Y \subset H^1(\mathcal{T}, \partial\mathcal{T}^{nc})$ be such that the following isomorphism holds:

$$H^1(\mathcal{T}, \partial\mathcal{T}^{nc}) \cong H^1(\mathcal{T}, \partial\mathcal{T}) \oplus Y.$$

By Lemma 3, we deduce that:

$$\text{rank}(Y) = \text{rank}(H^1(\mathcal{T}, \partial\mathcal{T}^{nc})) - \text{rank}(H^1(\mathcal{T}, \partial\mathcal{T})) = N^c - r, \quad (23)$$

where $r \in \{0, 1\}$ is the coefficient such that $r = 1$ if \mathcal{T} is orientable, and $r = 0$ otherwise.

We also observe that $[GL^c] \subset Y$, and that the set GL^c is linearly independent because each \mathbf{g}_j^c is non-zero on exactly one 1-cell of the j th connected component of $\partial\mathcal{T}^{nc}$. This implies that GL^c consists of $N^c - r$ independent 1-cocycles, making it a basis of Y . Therefore, we conclude that $[GL^c] = Y$, and we obtain the decomposition (22). \square

5.4 A second decomposition and a reduction of \mathcal{T} to a new CW complex $\bar{\mathcal{T}}$ with empty boundary

We now focus on $H^1(\mathcal{T}, \partial\mathcal{T})$, and we define the CW complex $\bar{\mathcal{T}}$ constructed from \mathcal{T} as described hereafter. For each connected component of the boundary $\partial\mathcal{T}_k$ with $k \in \{1, \dots, N^h\}$, we do the following operation: we consider a gluing map (in the CW sense as in Definition 2.1) attaching a closed disk D_k (i.e. a 2-ball) along the (polygonal) boundary $\partial\mathcal{T}_k$ and we contract D_k to a single 0-cell, say p_k . Accordingly, the CW complex structure of $\bar{\mathcal{T}}$ is obtained from that of \mathcal{T} by removing every 1-cell of $\partial\mathcal{T}$ and by identifying, for each $k \in \{1, \dots, N^h\}$, every 0-cell in $\partial\mathcal{T}_k$ with the 0-cell p_k in $\bar{\mathcal{T}}$. Notice that a 1-cocycle \mathbf{g} of \mathcal{T} is also a 1-cocycle of $\bar{\mathcal{T}}$ if and only if its support does not contain any 1-cell in $\partial\mathcal{T}$.

We now discuss a preparatory result similarly to the one in Lemma 3.

Lemma 5 (Dimension calculation via long exact sequences). *Let \mathcal{T} a 2-dimensional CW complex with a non-empty boundary $\partial\mathcal{T}$. The relationship between the ranks of $H^1(\mathcal{T}, \partial\mathcal{T})$ and $H^1(\bar{\mathcal{T}})$ is given by the following expression:*

$$\text{rank}(H^1(\mathcal{T}, \partial\mathcal{T})) - \text{rank}(H^1(\bar{\mathcal{T}})) = N^h - 1. \quad (24)$$

Proof. We proceed in a similar manner as in the proof of Lemma 3, by expressing the ranks of $H^1(\mathcal{T}, \partial\mathcal{T})$ and $H^1(\bar{\mathcal{T}})$ in terms of $H^1(\mathcal{T})$.

We start by considering the Mayer-Vietoris long exact sequence [34, Theorem 3, Chapter 11] for the triad $(\bar{\mathcal{T}}, \mathcal{T}, D)$, where D is defined as $D := \bigcup_{k=1}^{N^h} D_k$. The exact sequence is as follows:

$$\begin{aligned} 0 &\longrightarrow \underbrace{H^0(\bar{\mathcal{T}})}_1 \longrightarrow \underbrace{H^0(\mathcal{T})}_1 \oplus \underbrace{H^0(D)}_{N^h} \longrightarrow \underbrace{H^0(\mathcal{T} \cap D)}_{N^h} \longrightarrow \\ &\longrightarrow H^1(\bar{\mathcal{T}}) \longrightarrow H^1(\mathcal{T}) \oplus \underbrace{H^1(D)}_0 \longrightarrow \underbrace{H^1(\mathcal{T} \cap D)}_{N^h} \longrightarrow \\ &\longrightarrow H^2(\bar{\mathcal{T}}) \longrightarrow \underbrace{H^2(\mathcal{T})}_0 \oplus \underbrace{H^2(D)}_0 \longrightarrow \underbrace{H^2(\mathcal{T} \cap D)}_0 \longrightarrow 0, \end{aligned} \quad (25)$$

where the ranks of the corresponding cohomology groups are indicated under the braces.

Next, we apply the Euler characteristic formula [29, Theorem 2.44] to the exact sequences (25) and (19) from the proof of Lemma 3. Additionally, we observe that the rank of $H^2(\mathcal{T}, \partial\mathcal{T})$ is equal to the rank of $H^2(\bar{\mathcal{T}})$, regardless of whether \mathcal{T} is orientable or not. This leads us to the desired formula, (24). \square

We are now ready to state also the second decomposition of global loops.

Lemma 6 (Second decomposition of global loops). *Let $GL^h = \{\mathbf{g}_1^h, \dots, \mathbf{g}_{N^h-1}^h\}$ be the set of 1-cochains computed by Algorithm 3. Define $[GL^h] := \{\sum_{k=1}^{N^h-1} \beta_k [\mathbf{g}_k^h] \mid \beta_k \in \mathbb{R}\}$ as the linear span of the equivalence classes associated with each generator $\mathbf{g}_k^h \in GL^h$. Then, the following group isomorphism holds:*

$$H^1(\mathcal{T}, \partial\mathcal{T}) \cong H^1(\bar{\mathcal{T}}) \oplus [GL^h]. \quad (26)$$

Proof. We begin by recalling the definition of \mathbf{g}_k^h in Algorithm 3, specifically at lines 2 and 3. Each $\mathbf{g}_k^h \in GL^h$ is a 1-cocycle, and it follows that $\mathbf{g}_k^h \notin B^0(\mathcal{T}, \partial\mathcal{T})$. Therefore, we have that $[GL^h] \subset H^1(\mathcal{T}, \partial\mathcal{T})$.

Additionally, since $\mathbf{g}_k^h \in B^1(\bar{\mathcal{T}})$, we also deduce that $[GL^h] \cap H^1(\bar{\mathcal{T}}) = \{0\}$.

Now, let $Z \subset H^1(\mathcal{T}, \partial\mathcal{T})$ be a subspace such that $H^1(\mathcal{T}, \partial\mathcal{T}) \cong H^1(\bar{\mathcal{T}}) \oplus Z$. By Lemma 5, we conclude that the rank of Z is given by

$$\text{rank}(Z) = \text{rank}(H^1(\mathcal{T}, \partial\mathcal{T})) - \text{rank}(H^1(\bar{\mathcal{T}})) = N^h - 1. \quad (27)$$

Note that $[GL^h] \subset Z$. Since the set GL^h consists of 1-cocycles \mathbf{g}_k^h corresponding to the k th connected component $\partial\mathcal{T}_k$ of $\partial\mathcal{T}$, it is linearly independent. Therefore, the set GL^h contains $N^h - 1$ independent 1-cocycles, and thus it forms a basis for Z .

Consequently, we conclude that $[GL^h] = Z$, which gives us the decomposition (26). \square

Thanks to the two decompositions (22) and (26), we are reduced to showing that Algorithm 2 computes a set of generators of $H^1(\bar{\mathcal{T}})$.

5.5 Characterization of generators of $H^1(\bar{\mathcal{T}})$

Let us remind the Universal Coefficient Theorem for cohomology.

Theorem 5.2 ([29, Theorem 3.2]). *If a chain complex C of free Abelian groups has homology groups $H_n(C; G)$, then the cohomology groups $H^n(C; G)$ of the cochain complex $\text{Hom}(C; G)$ are determined by the exact sequence*

$$0 \rightarrow \text{Ext}(H_{n-1}(C); G) \rightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n(C); G) \rightarrow 0.$$

For our purposes, we do not need to delve into the definition of the Ext functor. It suffices to observe the following property: $\text{Ext}(Q; G) = 0$ when Q is a free group. For further details and a proof of this property, consult [29].

We now apply Theorem 5.2 to $\bar{\mathcal{T}}$ with $n = 1$ and $G = \mathbb{Z}$. In Theorem 5.2, the map $h : H^1(\bar{\mathcal{T}}) \rightarrow \text{Hom}(H_1(\bar{\mathcal{T}}), \mathbb{R})$ assigns to each generator $\mathbf{g} \in H^1(\bar{\mathcal{T}})$ the function $\langle \mathbf{g}, \cdot \rangle$, which maps an element $c \in H_1(\bar{\mathcal{T}})$ to the value $\langle \mathbf{g}, c \rangle$.

Note that if $\text{Ext}(H_0(\bar{\mathcal{T}}), \mathbb{R}) = 0$, then h is an isomorphism. Since we assume \mathcal{T} is connected, $\bar{\mathcal{T}}$ is also connected, and thus $H_0(\bar{\mathcal{T}}) = \mathbb{Z}$. Therefore, $\text{Ext}(H_0(\bar{\mathcal{T}}), \mathbb{R}) = 0$, and from the exactness of the sequence in Theorem 5.2, we conclude that $h : H^1(\bar{\mathcal{T}}) \rightarrow \text{Hom}(H_1(\bar{\mathcal{T}}), \mathbb{R})$ is an isomorphism.

As a result, we obtain the following characterization of generators of $H^1(\bar{\mathcal{T}})$ with real coefficients, in terms of a set of generators of $H_1(\bar{\mathcal{T}}; \mathbb{Z})$ (cf. the discussion at the end of Lemma 7).

Corollary 1. *Let $\{c_i\}_{i=1}^{\beta_1}$ be a set of homology generators of $H_1(\bar{\mathcal{T}}; \mathbb{Z})$, and let $\{\mathbf{g}_i\}_{i=1}^{\beta_1}$ be a set of 1-cocycles of $\bar{\mathcal{T}}$ such that*

$$\langle \mathbf{g}_i, c_j \rangle = \delta_{i,j}, \tag{28}$$

for all $i, j \in \{1, \dots, \beta_1\}$, where $\delta_{i,j}$ is the Kronecker delta. Then, the set $\{\mathbf{g}_i\}_{i=1}^{\beta_1}$ is basis of $H^1(\bar{\mathcal{T}})$.

5.6 Reduction of $\bar{\mathcal{T}}$ to a homotopy equivalent CW complex $\bar{\mathcal{T}}_c$

Let T and \tilde{T} be the spanning trees of \mathcal{T} and its dual graph \tilde{G} , as defined in Algorithm 1 between lines 1 and 5. These trees also define corresponding spanning trees of $\bar{\mathcal{T}}$ and its dual graph \tilde{G} , where we use the same symbols to denote them.

Let M_T and $M_{\tilde{T}}$ be the acyclic matchings constructed from the spanning trees T and \tilde{T} by applying Lemma 1 on $\bar{\mathcal{T}}$, respectively. Let $\bar{\mathcal{T}}_c$ denote the set of critical cells of $\bar{\mathcal{T}}$ with respect to the acyclic matching $M = M_{\tilde{T}} \cup M_T$.

Thus, $\bar{\mathcal{T}}_c$ forms a CW subcomplex of $\bar{\mathcal{T}}$, and by applying Theorem 5.1 to $\bar{\mathcal{T}}$ and M , we conclude that $\bar{\mathcal{T}}_c$ is homotopy equivalent to $\bar{\mathcal{T}}$. Since T is a spanning tree of $\bar{\mathcal{T}}$, each edge of T is matched in M_T to a vertex (a 0-cell) of $\bar{\mathcal{T}}$. As a result, M_T consists of $|\bar{\mathcal{T}}^{(0)}| - 1$ pairs.

Similarly, since \tilde{T} is a spanning tree of \tilde{G} , each edge of \tilde{T} is matched in $M_{\tilde{T}}$ to a vertex (a 2-cell, by duality) of \tilde{G} . Therefore, $M_{\tilde{T}}$ consists of $|\tilde{V}| - 1 = |\bar{\mathcal{T}}^{(2)}| - 1$ pairs.

Consequently, $\bar{\mathcal{T}}_c$ consists of:

- One 0-cell, which is the unique non-matched vertex of $\bar{\mathcal{T}}$ in M_T ,
- $|\bar{\mathcal{T}}^{(1)}| - (|\bar{\mathcal{T}}^{(0)}| - 1) - (|\bar{\mathcal{T}}^{(2)}| - 1) = 2 - \chi$ 1-cells, where χ is the Euler characteristic of $\bar{\mathcal{T}}$,
- One 2-cell, which is the unique non-matched vertex (again, by duality) of \tilde{G} in $M_{\tilde{T}}$.

It is important to note that the set of all 1-cells of $\bar{\mathcal{T}}_c$ is in one-to-one correspondence with the set E_c , as defined in Algorithm 2 at line 5. For the remainder of this paper, we will identify these two sets.

Now, let f denote the unique 2-cell of $\bar{\mathcal{T}}_c$, and consider the expression for the boundary map ∂_2^M on the CW complex $\bar{\mathcal{T}}_c$ with incidence numbers $\iota^M(f, e)$, as given by (13). Specifically, we have:

$$\partial_2^M f = \sum_{e \in E_c} \iota^M(f, e) e, \quad (29)$$

where $\iota^M(f, e)$ is the incidence number associated with the boundary of f along the edge e .

Since $\bar{\mathcal{T}}_c$ is homotopy equivalent to $\bar{\mathcal{T}}$ (by Theorem 5.1) and the boundary of $\bar{\mathcal{T}}$ is empty (as defined at the beginning of Section 5.4), we conclude that each coefficient $\iota^M(f, e)$ in the above sum can only take one of two possible values:

- $\iota^M(f, e) = 0$,
- $\iota^M(f, e) = 2\eta_e$, where $\eta_e \in \{-1, +1\}$.

Therefore, we can partition E_c into two subsets: $E_c = E_c^I \cup E_c^{II}$, where E_c^I and E_c^{II} consist of the edges $e \in E_c$ that satisfy conditions (1) and (2), respectively.

5.7 Differentiation between homology and torsion generators of $H_1(\bar{\mathcal{T}})$ passing through $\bar{\mathcal{T}}_c$

The characterization of the homology of $\bar{\mathcal{T}}$ is an immediate consequence of the Classification Theorem of Compact Surfaces [17]. In the next Lemma we provide an equivalent characterization which will be useful for our specific purposes.

Lemma 7 (Homology characterization of $\bar{\mathcal{T}}$). *For each $e \in E_c$, let c_e be the unique cycle in the graph $T \cup \{e\}$, oriented naturally according to the orientation induced by e . Let $E_c = E_c^I \sqcup E_c^{II}$, and denote by $g := |E_c| = 2 - \chi$ the genus of $\bar{\mathcal{T}}_c$. We distinguish between the following two cases:*

- (i) *If $|E_c^{II}| = 0$, then $H_1(\bar{\mathcal{T}}, \mathbb{Z}) = \mathbb{Z}^g$.*
- (ii) *If $|E_c^{II}| > 0$, then $H_1(\bar{\mathcal{T}}, \mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$. In this case, the torsion generator \bar{c}^* is given by:*

$$\bar{c}^* := \sum_{e \in E_c^{II}} \eta_e c_e, \quad (30)$$

where each $\eta_e \in \{-1, +1\}$ is the sign of the coefficient $\iota^M(f, e)$ in the expression of $\partial_2^M f$ as a linear combination of 1-cells $e \in E_c^{II}$ in (29).

Proof. Since $\bar{\mathcal{T}}_c$ has only one 0-cell, every 1-cell $e \in E_c$ forms a 1-cycle.

In the first case, where $|E_c^{II}| = 0$, we have $E_c = E_c^I$. Therefore, each 1-cell $e \in E_c$ is a homology generator of $H_1(\bar{\mathcal{T}}_c, \mathbb{Z})$, as $\partial_2^M f \neq e$. By applying Lemma 7, it follows that each cycle c_e , considered as a 1-cycle in $\bar{\mathcal{T}}$, is a homology generator of $H_1(\bar{\mathcal{T}}, \mathbb{Z})$.

In the second case, where $|E_c^{II}| > 0$, fix a 1-cell $e^* \in E_c^{II}$. The boundary operator $\partial_2^M f$ satisfies

$$\partial_2^M f = \sum_{e \in E_c^{II}} 2\eta_e e = 2\bar{c}^*,$$

where \bar{c}^* is the torsion generator of $\bar{\mathcal{T}}_c$, as defined in (30). For each $e \in E_c \setminus \{e^*\}$, the cell e is a homology generator of $H_1(\bar{\mathcal{T}}_c, \mathbb{Z})$, since $\partial_2^M f \neq e$. This means there are $g - 1$ independent homology generators.

Next, denote by c_e the unique 1-cycle of $\bar{\mathcal{T}}$ associated with each $e \in E_c$, as in the case where $|E_c^{II}| = 0$. Define the cycle c^* as

$$c^* := \sum_{e \in E_c^{II}} c_e.$$

By applying Lemma 7, and using the fact that $\bar{\mathcal{T}}$ is homotopy equivalent to $\bar{\mathcal{T}}_c$, it follows that each cycle c_e for $e \in E_c \setminus \{e^*\}$ is a homology generator of $H_1(\bar{\mathcal{T}}, \mathbb{Z})$. Furthermore, c^* is the torsion generator of $H_1(\bar{\mathcal{T}}, \mathbb{Z})$, since c^* is homotopy equivalent to the torsion 1-cycle \bar{c}^* of $\bar{\mathcal{T}}_c$ defined in (30). \square

5.8 Proof of correctness of Algorithm 2

Recall the definition of the spanning trees T and \tilde{T} in Algorithm 2 at lines 1 to 4. It follows that every path \tilde{p} in \tilde{G} does not contain any boundary edge of \mathcal{T} . Accordingly, the support of each 1-cochain \mathbf{g} of \mathcal{T} computed via Algorithm 1 does not contain any boundary edge of \mathcal{T} either (recall the formula in Algorithm 1 at line 4), and hence \mathbf{g} is also 1-cocycle of $\tilde{\mathcal{T}}$.

Lemma 8. *Let $e \in E_c$. The following two assertions hold:*

- (i) *Let \tilde{p}_e be the path defined in Algorithm 2 at lines 10 and 11, and let \mathbf{g}_e^{ha} be the corresponding 1-cochain of $\tilde{\mathcal{T}}$ computed at line 12 using Algorithm 1. Then, $e \in E_c^I$ if and only if \mathbf{g}_e^{ha} is a 1-cocycle of $\tilde{\mathcal{T}}$ whose restriction to $\tilde{\mathcal{T}}_c$ satisfies:*

$$\langle \mathbf{g}_e, e' \rangle = \delta_{e,e'}, \forall e' \in E_c. \quad (31)$$

- (ii) *Assume $|E_c^{II}| > 0$, and let e^* the 1-cell selected in Algorithm 2 at line 18. For each $e \in E_c^{II} \setminus \{e^*\}$, let $\tilde{p}_{e,e^*}^{(1)}, \tilde{p}_{e,e^*}^{(2)}$ be the paths defined in Algorithm 2 at lines 20 to 23. Let $\mathbf{g}_e^{(1)}$ and $\mathbf{g}_e^{(2)}$ denote the corresponding 1-cochains of $\tilde{\mathcal{T}}$ computed at lines 24 and 25 using Algorithm 1, and let \mathbf{g}_e^{ha} be the 1-cochain defined at line 26. Then, \mathbf{g}_e^{ha} is a 1-cocycle of $\tilde{\mathcal{T}}$ whose restriction to $\tilde{\mathcal{T}}_c$ satisfies:*

$$\langle \mathbf{g}_e, e' \rangle = \delta_{e,e'}, \forall e' \in E_c^{II} \setminus \{e^*\}. \quad (32)$$

Proof. (i) Let $e \in E_c$. Denote by f_1, \dots, f_l and e_1, \dots, e_{l-1} the set of 2-cells and 1-cells of \tilde{p}_e , respectively. Notice that we can identify the unique 2-cell f of $\tilde{\mathcal{T}}_c$ by setting $f := f_l$. This follows from the fact that every 2-cell of \mathcal{T} , except one, is matched in the acyclic matching M . The unique non-matched 2-cell in M can be chosen arbitrarily (as discussed in the property of $M_{\tilde{\mathcal{T}}}$ in Lemma 1). This matches with the natural identification of cells of $\tilde{\mathcal{T}}_c$ with the critical cells of $\tilde{\mathcal{T}}$ as stated in Theorem 5.1.

Let $s_{\tilde{p}_e}$ be the sequence associated with the path \tilde{p}_e in M . Applying Point (ii) in Theorem 5.1 to the pair f, e , we observe that $s_{\tilde{p}_e}$ is the unique sequence in the set $S(f, e)$. This is because f_1, f_l are the only 2-cells of $\tilde{\mathcal{T}}$ incident on e , with $f := f_l$.

Next, applying formula (13), we conclude that $e \in E_c^I$ if and only if

$$\iota^M(f, e) = \iota(\xi_1 f_1, e) + \iota(\xi_l f_l, e) = 0.$$

Thus, we obtain the following compatibility condition:

$$\iota(\xi_1 f_1, e) = -\iota(\xi_l f_l, e). \quad (33)$$

This is given by equation (33).

By Lemma 2, we know that the 1-cochain \mathbf{g}_e^{ha} satisfies $\langle \mathbf{g}_e^{ha}, e \rangle = 1$ and $\langle \mathbf{g}_e^{ha}, \xi_i f_i \rangle = 0$ for all $i \in \{1, \dots, l\}$. Since $e' = e$ in Algorithm 2 at line 12, it follows that \mathbf{g}_e^{ha} is a 1-cocycle if and only if the following condition

$$1 = \langle \mathbf{g}, e \rangle = \langle \mathbf{g}, e' \rangle$$

holds. Using (15) from Lemma 2, we conclude that this condition is exactly (33).

- (ii) Let $f_1^{(1)}, \dots, f_l^{(1)}$ and $e_1^{(1)}, \dots, e_{l-1}^{(1)}$ be the sets of 2-cells and 1-cells of $\tilde{p}_{e,e^*}^{(1)}$, and let $f_1^{(2)}, \dots, f_q^{(2)}$ and $e_1^{(2)}, \dots, e_{q-1}^{(2)}$ be the corresponding sets for $\tilde{p}_{e,e^*}^{(2)}$.

Let \tilde{p} be the unique path in \tilde{T} from $D(f_l^{(1)})$ to $D(f_1^{(2)})$ and let $s_{\tilde{p}}$ be the corresponding sequence in M . As in Point (i), we can identify $f := f_l^{(1)}$.

Apply Point (ii) of Theorem 5.1 to the pair f, e , and observe that $s_{\tilde{p}}$ and $s_{\tilde{p}_{e,e^*}^{(1)}}$ are the only two sequences in the set $S(f, e)$, since $f_1^{(1)}, f_1^{(2)}$ are the unique 2-cells of $\tilde{\mathcal{T}}$ incident on e . Then, applying formula (13) to the sequences $s_{\tilde{p}_{e,e^*}^{(1)}}$ and $s_{\tilde{p}}$, we have

$$\iota^M(f, e) = \iota(\xi_1^{(1)} f_1^{(1)}, e) + \iota(\xi_1^{(2)} f_1^{(2)}, e).$$

Since $e \in E_c^{\text{II}}$, by the definition of E_c^{II} in Section 5.7, we have $\iota^M(f, e) = 2\eta_e$ with $\eta_e \in \{-1, 1\}$. Moreover, since both 2-cells $f_1^{(1)}, f_1^{(2)}$ are also 2-cells of \mathcal{T} , we deduce that $\iota(\xi_1^{(1)} f_1^{(1)}, e), \iota(\xi_1^{(2)} f_1^{(2)}, e) \in \{-1, 1\}$. Therefore, we conclude that

$$\iota(\xi_1^{(1)} f_1^{(1)}, e) = \iota(\xi_1^{(2)} f_1^{(2)}, e) = \eta_e.$$

Similarly, let \tilde{p} be the unique path in \tilde{T} from $D(f_q^{(2)})$ to $D(f_l^{(1)})$. Apply again Point (ii) in Theorem 5.1 to the pair f, e^* , and observe that $s_{\tilde{p}}$ is the unique sequence of the set $S(f, e)$, since $f_l^{(1)}$ and $f_q^{(2)}$ are the only 2-cells of \tilde{T} incident on e .

Then, applying formula (13) to the sequence $s_{\tilde{p}}$, we have

$$\iota^M(f, e^*) = \iota(\xi_l^{(1)} f_l^{(1)}, e^*) + \iota(\xi_q^{(2)} f_q^{(2)}, e^*).$$

Since $e^* \in E_c^{\text{II}}$, we conclude, using similar reasoning as above, that

$$\iota(\xi_l^{(1)} f_l^{(1)}, e^*) = \iota(\xi_q^{(2)} f_q^{(2)}, e^*) = \eta_{e^*}$$

with $\eta_{e^*} \in \{-1, 1\}$.

By (16) in Lemma 2, we have that $\langle \mathbf{g}_e^{(1)}, \xi_i^{(1)} f_i^{(1)} \rangle = 0$ for all $i \in \{1, \dots, l\}$, and $\langle \mathbf{g}_e^{(2)}, \xi_i^{(2)} f_i^{(2)} \rangle = 0$ for all $i \in \{1, \dots, q\}$. Recalling the definition of \mathbf{g}_e^{ha} in Algorithm 2 at line 26, it follows that \mathbf{g}_e^{ha} is a 1-cocycle if and only if the condition

$$\langle \mathbf{g}_e^{(1)}, e^* \rangle = \langle \mathbf{g}_e^{(2)}, e^* \rangle$$

holds, along with $\langle \mathbf{g}_e^{(1)}, e \rangle = \langle \mathbf{g}_e^{(2)}, e \rangle$, which is true by the initialization in 1 at line 2.

Applying formula (15) in Lemma 2 to the sequences $s_{\tilde{p}_{e, e^*}^{(1)}}, s_{\tilde{p}_{e, e^*}^{(2)}}$, we deduce that

$$\langle \mathbf{g}_e^{(1)}, e^* \rangle = -\frac{\iota(\xi_l^{(1)} f_l^{(1)}, e^*)}{\iota(\xi_1^{(1)} f_1^{(1)}, e)} = -\frac{\eta_{e^*}}{\eta_e} = -\frac{\iota(\xi_q^{(2)} f_q^{(2)}, e^*)}{\iota(\xi_1^{(2)} f_1^{(2)}, e)} = \langle \mathbf{g}_e^{(2)}, e^* \rangle.$$

Thus, \mathbf{g}_e^{ha} is a 1-cocycle, and its restriction to $\tilde{\mathcal{T}}_c$ satisfies the desired condition, as given in (32). \square

Lemma 9. *Let $GL^{ha} = \{\mathbf{g}_1^{ha}, \dots, \mathbf{g}_{N^{ha}}\}$ be the set of 1-cochains computed by Algorithm 2, and let $[GL^{ha}] := \{\sum_{i=1}^{N^{ha}} \gamma_i [\mathbf{g}_i^{ha}] \mid \gamma_i \in \mathbb{R}\}$ be the linear span of the equivalence classes associated with each generator $\mathbf{g}_i \in GL^{ha}$. Then, the following isomorphism holds:*

$$H^1(\tilde{\mathcal{T}}) \cong [GL^{ha}]. \quad (34)$$

Proof. For each $e \in E_c$, let \mathbf{g}_e^{ha} be the 1-cocycle of $\tilde{\mathcal{T}}$ computed in Algorithm 2 at line 11 and 12. Applying Point (i) in Lemma 8, it follows that, if $\mathbf{g}_e^{ha} \neq \mathbf{0}$, then \mathbf{g}_e^{ha} is such that its restriction to $\tilde{\mathcal{T}}_c$ satisfies (31), and $e \in E_c^{\text{I}}$. On the other hand, if $\mathbf{g}_e^{ha} = \mathbf{0}$, then $e \in E_c^{\text{II}}$. This defines the partition $E_c = E_c^{\text{I}} \cup E_c^{\text{II}}$, and now examine the two possible cases for this partition.

In the case where $|E_c^{\text{II}}| = 0$, observe that by construction, the support of \mathbf{g}_e^{ha} does not include any edge of T . Consequently, for each $e' \in E_c$, the equality $\langle \mathbf{g}_e^{ha}, c_{e'} \rangle = \langle \mathbf{g}_e^{ha}, e' \rangle$ holds. Applying Corollary 1 in combination with Lemma 7, we deduce that $GL^{ha} = \{\mathbf{g}_e^{ha}\}_{e \in E_c}$ forms a set of generators of $H^1(\tilde{\mathcal{T}})$.

In the case where $|E_c^{\text{II}}| > 0$, let $e^* \in E_c^{\text{II}}$ denote the 1-cell fixed in Algorithm 2 at line 18. For each $e \in E_c^{\text{I}} \setminus \{e^*\}$, let \mathbf{g}_e^{ha} be the corresponding 1-cocycle of $\tilde{\mathcal{T}}$ computed in Algorithm 1 at line 26. By Point (ii) of Lemma 8, each \mathbf{g}_e^{ha} for $e \in E_c^{\text{I}} \setminus \{e^*\}$ is 1-cocycle of $\tilde{\mathcal{T}}$ whose restriction to $\tilde{\mathcal{T}}_c$ satisfies (32). Since the support of \mathbf{g}_e^{ha} does not contain any edge of T , we also have $\langle \mathbf{g}_e^{ha}, c_{e'} \rangle = \langle \mathbf{g}_e^{ha}, e' \rangle$ for all $e' \in E_c$, as well as $\langle \mathbf{g}_e^{ha}, c^* \rangle = \langle \mathbf{g}_e^{ha}, \bar{c}^* \rangle$.

Using Corollary 1 and Lemma 7, we conclude that $GL^{ha} = \{\mathbf{g}_e^{ha}\}_{e \in E_c \setminus \{e^*\}}$ is a set of generators of $H^1(\tilde{\mathcal{T}})$.

Thus, in both cases, the set GL^{ha} provides a valid set of generators of $H^1(\tilde{\mathcal{T}})$, determined by analyzing the contributions of 1-cocycles based on the partition $E_c = E_c^{\text{I}} \cup E_c^{\text{II}}$. \square

5.9 Conclusion

To conclude, it is sufficient to combine Lemmas 4, 6 and 9.

Lemma 10 (Correctness of Algorithm 5). *Let GL^{ha} , GL^h , GL^c denote the sets of global loops computed by Algorithm 2, Algorithm 3, and Algorithm 4, respectively. Then, we have the following isomorphism:*

$$H^1(\mathcal{T}, \partial\mathcal{T}^c) \cong [GL^{ha}] \oplus [GL^h] \oplus [GL^c].$$

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References

- [1] A. Buffa and S. H. Christiansen, “The electric field integral equation on lipschitz screens: definitions and numerical approximation,” *Numerische Mathematik*, vol. 94, pp. 229–267, 2003.
- [2] S. B. Adrian, A. Dely, D. Consoli, A. Merlini, and F. P. Andriulli, “Electromagnetic integral equations: Insights in conditioning and preconditioning,” *IEEE Open Journal of Antennas and Propagation*, vol. 2, pp. 1143–1174, 2021.
- [3] G. Vecchi, “Loop-star decomposition of basis functions in the discretization of the efie,” *IEEE Transactions on Antennas and Propagation*, vol. 47, no. 2, pp. 339–346, 1999.
- [4] F. P. Andriulli, “Loop-star and loop-tree decompositions: Analysis and efficient algorithms,” *IEEE Transactions on Antennas and Propagation*, vol. 60, no. 5, pp. 2347–2356, 2012.
- [5] F. P. Andriulli, K. Cools, I. Bogaert, and E. Michielssen, “On a well-conditioned electric field integral operator for multiply connected geometries,” *IEEE transactions on antennas and propagation*, vol. 61, no. 4, pp. 2077–2087, 2012.
- [6] B. Hofmann, T. F. Eibert, F. P. Andriulli, and S. B. Adrian, “A low-frequency stable, excitation agnostic discretization of the right-hand side for the electric field integral equation on multiply-connected geometries,” *IEEE Transactions on Antennas and Propagation*, vol. 71, no. 12, pp. 9277–9288, 2023.
- [7] R. Hiptmair and J. Ostrowski, “Generators of $H_1(\Gamma_h, \mathbb{Z})$ for triangulated surfaces: Construction and classification,” *SIAM Journal on Computing*, vol. 31, no. 5, pp. 1405–1423, 2002.
- [8] D. Eppstein, “Dynamic generators of topologically embedded graphs,” *ArXiv*, vol. cs.DS/0207082, 2002.
- [9] F. Lazarus, M. Pocchiola, G. Vegter, and A. Verroust, “Computing a canonical polygonal schema of an orientable triangulated surface,” in *Proceedings of the seventeenth annual symposium on Computational geometry*, pp. 80–89, 2001.
- [10] T. K. Dey, F. Fan, and Y. Wang, “An efficient computation of handle and tunnel loops via reeb graphs,” *ACM Transactions on Graphics (TOG)*, vol. 32, no. 4, pp. 1–10, 2013.
- [11] P. W. Gross and P. R. Kotiuga, *Electromagnetic theory and computation: a topological approach*, vol. 48. Cambridge University Press, 2004.
- [12] P. Dłotko, “A fast algorithm to compute cohomology group generators of orientable 2-manifolds,” *Pattern Recognition Letters*, vol. 33, no. 11, pp. 1468–1476, 2012.
- [13] P. Dłotko and R. Specogna, “Physics inspired algorithms for (co) homology computations of three-dimensional combinatorial manifolds with boundary,” *Computer Physics Communications*, vol. 184, no. 10, pp. 2257–2266, 2013.
- [14] P. Dłotko and R. Specogna, “Lazy cohomology generators: A breakthrough in (co) homology computations for cem,” *IEEE transactions on magnetics*, vol. 50, no. 2, pp. 577–580, 2014.
- [15] D. Wilton and A. Glisson, “On improving the stability of the electric field integral equation at low frequency,” in *Proc. IEEE Antennas and Propagation Soc. National Symp*, pp. 124–133, 1981.
- [16] D. N. Arnold, *Finite element exterior calculus*. SIAM, 2018.
- [17] J. H. Gallier, D. Xu, *et al.*, *A guide to the classification theorem for compact surfaces*. Springer, 2013.
- [18] G. Miano and F. Villone, “A surface integral formulation of maxwell equations for topologically complex conducting domains,” *IEEE transactions on antennas and propagation*, vol. 53, no. 12, pp. 4001–4014, 2005.
- [19] Y. Wang, D. Gope, V. Jandhyala, and C.-J. Shi, “Generalized kirchoff’s current and voltage law formulation for coupled circuit-electromagnetic simulation with surface integral equations,” *IEEE transactions on microwave theory and techniques*, vol. 52, no. 7, pp. 1673–1682, 2004.
- [20] S. Suuriniemi, J. Kangas, L. Kettunen, and T. Tarhasaari, “State variables for coupled circuit-field problems,” *IEEE transactions on magnetics*, vol. 40, no. 2, pp. 949–952, 2004.

- [21] S. Suuriniemi, J. Kangas, and L. Kettunen, “Driving a coupled field-circuit problem,” *COMPEL-The international journal for computation and mathematics in electrical and electronic engineering*, vol. 26, no. 3, pp. 899–909, 2007.
- [22] R. Hiptmair and J. Ostrowski, “Electromagnetic port boundary conditions: Topological and variational perspective,” *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, vol. 34, no. 3, p. e2839, 2021.
- [23] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to algorithms*. MIT press, 2022.
- [24] J. Erickson and K. Whittlesey, “Greedy optimal homotopy and homology generators,” in *SODA*, vol. 5, pp. 1038–1046, 2005.
- [25] T. Lewiner, H. Lopes, and G. Tavares, “Optimal discrete morse functions for 2-manifolds,” *Comput. Geom.*, vol. 26, pp. 221–233, 2003.
- [26] X. Claeys and R. Hiptmair, “Integral equations on multi-screens,” *Integral equations and operator theory*, vol. 77, no. 2, pp. 167–197, 2013.
- [27] X. Claeys and R. Hiptmair, “Integral equations for electromagnetic scattering at multi-screens,” *Integral Equations and Operator Theory*, vol. 84, pp. 33–68, 2016.
- [28] G. E. Cooke and R. L. Finney, *Homology of cell complexes*, vol. 2239. Princeton University Press, 2015.
- [29] A. Hatcher, “Algebraic topology,” 2001.
- [30] J. Dodziuk, “Finite-difference approach to the hodge theory of harmonic forms,” *American Journal of Mathematics*, vol. 98, no. 1, pp. 79–104, 1976.
- [31] G. F. Duff, “Differential forms in manifolds with boundary,” *Annals of Mathematics*, vol. 56, no. 1, pp. 115–127, 1952.
- [32] D. Kozlov, “Combinatorial algebraic topology,” in *Algorithms and computation in mathematics*, 2008.
- [33] S. Pitassi, R. Ghiloni, and R. Specogna, “Inverting the discrete curl operator: a novel graph algorithm to find a vector potential of a given vector field,” *Journal of Computational Physics*, vol. 466, p. 111404, 2022.
- [34] M. Spivak, “A comprehensive introduction to differential geometry, publish or perish,” *Inc., Berkeley*, vol. 2, 1979.