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A folk theorem for finitely repeated games with public monitoring

Johannes Hörner*, Jérôme Renault†

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Abstract

We provide the Folk theorem for finitely repeated games with public signals, with a small variation of the usual assumptions for finitely repeated games with perfect observation, and for discounted repeated games with public signals. Our proof uses using standard recursive methods and assumes the existence of a public correlation device. Three counterexamples show that our assumptions are tight.

Keywords: Finitely repeated games, public signals, public correlation device.

JEL codes: C72, C73

1 Introduction

The literature on finitely repeated games and discounted infinitely repeated games have proceeded somewhat independently in the last twenty years. Following Abreu, Pearce and Stacchetti (1990), tremendous progress has been accomplished in the analysis of infinitely repeated games with imperfect monitoring under discounting. Results in this literature have built on the fixed-point characterization that they give of the set of public perfect equilibrium payoffs, paving the way for a largely non-constructive characterization of the equilibrium payoff set, in particular, as discounting vanishes. See, among others, Fudenberg, Levine and Maskin (1994, hereafter FLM).

Clearly, no such fixed-point characterization exists in the case of finitely repeated games, as the (public perfect) equilibrium payoff set is not independent of the number of periods left.

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As a result, folk theorems for finitely repeated games have relied on explicit specifications of equilibrium strategies. This has led these authors to restrict attention to perfect monitoring. See Benoît and Krishna (1985), Gossner (1995) and Smith (1995), as well as Benoît and Krishna (1996) for a survey.

Yet the (Bellman-Shapley) operator involved in the definition of the fixed point applies just as well to the case of a finite horizon, giving us an immediate link between the equilibrium payoff sets that obtain as the horizon length varies. Similarly, the main idea behind the proof of FLM applies as well. Suppose that the (average) equilibrium payoff set converges to a strict subset of the feasible, and individually rational payoff set V . Then there exists a direction in which this limit set has a boundary that is locally smooth (in an appropriately defined sense) yet bounded away from the extreme point of V in that direction. Under the assumptions of FLM, then, a contradiction can be derived.

Some care must be taken in this argument, however. First, in the absence of discounting, the relative weights of current vs. continuation average payoffs in the definition of the operator are related to the number of periods left. Therefore, it is not exactly the same operator that is being applied repeatedly. As longer and longer horizons are considered, and as flow payoffs are assigned a vanishing weight, one must make sure that these weights do not decrease too fast in order for a contradiction to obtain.

More importantly, we show via examples that combining the assumptions that are made for the folk theorem with an infinite horizon under imperfect public monitoring (FLM), with those made by Benoît and Krishna (1985) for finitely repeated games with perfect monitoring is not enough. Their assumption of distinct Nash payoffs in the stage game must be strengthened. Indeed, we strengthen this assumption by assuming that the convex hull of this set of vectors has non-empty interior (Assumption B1). This assumption is added to the standard assumptions of pairwise full rank for some action profile (A2), and admissibility of the minmax action profiles (A3). We prove in the present manuscript that under B1, A2, A3 and the existence of a public correlation device, the finitely repeated Folk theorem holds, that is the sequence (E'_n) of public perfect equilibrium payoffs converges, as $n \rightarrow \infty$, to the set of feasible and individually rational payoffs. Furthermore, we show that our assumptions cannot be relaxed, in the sense that we exhibit games that satisfy any pair of our three assumptions B1, A2, A3, yet for which the Folk theorem does not hold, even when a public correlation device is allowed.

While there has been few systematic analyses of finitely repeated games since Benoît and

Krishna (1985) and Gossner (1995), very interesting examples have been produced by Mailath, Matthews and Sekiguchi (2002) to show how non-trivial equilibria can be constructed in finitely repeated with imperfect monitoring even as the stage game admits a unique Nash equilibrium. Such equilibria involve private strategies, and so are not covered by our analysis which is restricted to public equilibria, where the actions played only depend on the past public signals. However, all that matters for our argument is that the finitely repeated game admits a set of equilibrium payoffs whose convex hull has non-empty interior for some horizon length. This length can then be treated as an “end-game” possibly involving private strategies, and treated as a “blackbox” when defining public strategies.

Other related contributions involve Contou-Carrère and Tomala (2010) for the case of semi-standard monitoring, and González-Díaz (2006) for the case of Nash equilibria. Standard results or definitions follow Mailath and Samuelson (2006). Limits of sequences of compact subsets of \mathbb{R}^I are for the Hausdorff distance $d(A, B) = \max_{a \in A} \min_{b \in B} \|a - b\| + \max_{b \in B} \min_{a \in A} \|a - b\|$.

2 Model

We consider a finitely repeated game with public signals. We have a finite set of players I , and in the stage game each player i has a finite action set A_i and a payoff function $g_i : A \rightarrow \mathbb{R}$, where $A = \prod_{i \in I} A_i$ is the set of action profiles. As usual players are allowed to play randomly and consider expected payoffs, and the set of mixed action profiles is $\prod_{j \in I} \Delta(A_j)$. We denote by $g = (g_i)_{i \in I} : A \rightarrow \mathbb{R}^I$ the vector payoff function, which is extended by linearity to the set of correlated action profiles $\Delta(A)$. The n -stage repeated game is played at stages $t = 1, \dots, n$. Between the stages, action profiles are not observed, nor are realized payoffs. Given an action profile a in A there is a public signal y from a finite set Y that is publicly observed, the distribution of signals given a being denoted by $\pi(\cdot|a) \in \Delta(Y)$.

For every $n \geq 1$, we denote by E_n the set of *public perfect equilibrium payoffs* of the n -stage repeated game where each player i maximizes the expectation of the average payoff $\frac{1}{n} \sum_{t=1}^n g_i(a_t)$. And we denote by E'_n the set of *public perfect equilibrium payoffs* of the same game extended by a public correlation device, that is when in addition the players receive an independent public signal with uniform distribution in $[0, 1]$ at the beginning of each stage. It is known that E_n is a compact, typically not convex, subset of \mathbb{R}^I and converges when $n \rightarrow \infty$, for the Hausdorff distance, to the convex compact set $\lim_n E_n = \overline{\bigcup_{n \geq 1} E_n}$ (see Renault and Tomala, 2011). Similarly,

E'_n is compact and convex, and converges to the convex compact set $\lim_n E'_n = \overline{\bigcup_{n \geq 1} E'_n}$. The rates of convergence are unknown, and a main challenge is to characterize $\lim_n E_n$ and $\lim_n E'_n$ as functions of g and π .

We now introduce the recursive approach initiated by Abreu, Pearce and Stacchetti (1990). Let \mathcal{C} be the set of non empty compact subsets of \mathbb{R}^I , and for $C \in \mathcal{C}$ define $\Phi(C)$ as the set of mixed Nash equilibrium payoffs of the normal-form game with actions sets A_i for each player i , and payoffs given by the sum of g and continuation payoffs chosen from C :

$$\begin{aligned} \Phi: \mathcal{C} &\rightarrow \mathcal{C} \\ C &\mapsto \left\{ NEP \left(g(a) + \sum_{y \in Y} \pi(y | a) x(y) \right), \text{ some } x: Y \rightarrow C \right\}. \end{aligned}$$

$E_1 = \Phi(\{0\})$, and the recursive structure of public equilibria gives $(n+1)E_{n+1} = \Phi(nE_n)$ for each $n \geq 1$. Defining $\Psi(C) := \text{conv}(\Phi(C))$ for each C , we also get: $E'_1 = \Psi(\{0\})$, and $(n+1)E'_{n+1} = \Psi(nE'_n)$ for each $n \geq 1$. This recursive characterization of the sets E_n and E'_n is extensively used in the sequel.

The set of feasible payoffs is defined as $F = \text{conv}(\{g(a), a \in A\}) = g(\Delta(A))$, and the set of feasible and individually rational payoffs is $V = F \cap \{u \in \mathbb{R}^I, \forall i u_i \geq v_i\}$, where

$$v_i := \min_{\alpha_{-i} \in \prod_{j \neq i} \Delta(A_j)} \max_{\alpha_i \in \Delta(A_i)} g_i(\alpha_i, \alpha_{-i})$$

is the independent minmax of player i . Plainly, $E_n \subset E'_n \subset V$ for each n , so that $\lim_n E_n \subset \lim_n E'_n \subset V$.

3 Assumptions and results

We now recall a few standard assumptions and results from the literature. The first assumption requires the existence of distinct Nash equilibrium payoffs for each player in the stage game.

Assumption (A0): For each player i , there exist u and u' in E_1 s.t. $u_i \neq u'_i$.

Assumption (A1): V has non empty interior.

(A0) and (A1) were introduced by Benoît and Krishna (1985). In case of perfect monitoring (i.e., when the players perfectly observe the action profile played at the end of each stage), Gossner proved the following finitely repeated Folk theorem.

Theorem 1 (Gossner, 1995)

For repeated games with perfect monitoring, (A0) and (A1) imply $E_n \xrightarrow[n \rightarrow \infty]{} V$.

We now consider equilibrium payoffs of discounted games. For every $\delta \in [0, 1)$, we denote by E_δ the set of *public perfect equilibrium payoffs* of the infinitely repeated game where each player i maximizes the expectation of the discounted payoff $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} g_i(a_t)$. And we denote by E'_δ the set of *public perfect equilibrium payoffs* of the same game extended by a public correlation device.

Assumption (A2): (Pairwise Full Rank) For all $i \neq j$, there exists a mixed action profile α such that the family of vectors $\{\pi(a_i, \alpha_{-i}), a_i \in A_i\} \cup \{\pi(a_j, \alpha_{-j}), a_j \in A_j\}$ has rank $|A_i| + |A_j| - 1$.

Notice that for each α , $\pi(\alpha)$ belongs to both convex hulls $\text{conv}\{\pi(a_i, \alpha_{-i}), a_i \in A_i\}$ and $\text{conv}\{\pi(a_j, \alpha_{-j}), a_j \in A_j\}$ so the rank of $\{\pi(a_i, \alpha_{-i}), a_i \in A_i\} \cup \{\pi(a_j, \alpha_{-j}), a_j \in A_j\}$ is always at most $|A_i| + |A_j| - 1$. Assumption (A2) is useful to discriminate between a deviation of player i and a deviation of player j from a prescribed mixed action. As shown by Fudenberg, Levine and Maskin (1994, Lemma 6.2), this assumption implies that there exists an open and dense subset of action profiles that satisfy pairwise full rank for all pairs of players.

Assumption (A3): (Admissibility) For each i , there is a mixed action profile m^i minimaxing player i (i.e. $\max_{a_i \in A_i} g_i(a_i, m^i_{-i}) = v_i$) such that for all $j \neq i$ and $\alpha_j \in \Delta(A_j)$: if $\pi(m^i) = \pi(m^i_{-j}, \alpha_j)$ then $g_j(m^i) \geq g_j(m^i_{-j}, \alpha_j)$.

This assumption is due to Kandori and Matsushima (1998). If (A3) holds and m^i is prescribed, it punishes player i to his minmax level and no undetectable deviation is profitable. Fudenberg, Levine and Maskin proved the following discounted Folk theorem for repeated games with public signals.

Theorem 2 (FLM, 1994) (A1), (A2) and (A3) imply $E'_\delta \xrightarrow[\delta \rightarrow 1]{} V$.

In this manuscript we will give hypotheses implying that E'_n converges to V .

We recall the following elementary result, which establishes that any equilibrium payoff of the finitely repeated game, no matter how long the horizon, is necessarily an equilibrium payoff of the infinitely repeated game with low enough discounting (and the same monitoring structure).

Lemma 1 *For every n , it holds that*

$$E'_n \subseteq \lim_{\delta \rightarrow 1} E_\delta.$$

Proof. As is well known, $\lim_{\delta \rightarrow 1} E_\delta = \lim_{\delta \rightarrow 1} E'_\delta$ is the same whether or not a public randomization device is assumed (see Fudenberg, Levine and Takahashi, 2007), and for all $\delta' < 1$, $E_{\delta'} \subseteq \lim_{\delta \rightarrow 1} E_\delta$. The result then follows from the monotonicity of the Bellman-Shapley operator B_δ in δ (see Abreu, Pearce and Stacchetti, 1990, Theorem 6), using the convexity of E'_n . The proof is by induction. Clearly, $E_1 \subset \lim_{\delta \rightarrow 1} E_\delta$ and $E_1 \subseteq E_{\delta'}$ for some $\delta' > \frac{1}{2}$. Fix $\delta' > \frac{n}{n+1}$ such that $E_n \subseteq E_{\delta'}$. Then $E_{n+1} = B_{\frac{n}{n+1}}(E_n) \subseteq B_{\frac{n}{n+1}}(E_{\delta'}) \subseteq B_{\delta'}(E_{\delta'}) = E_{\delta'}$ and so also $E_{n+1} \subseteq \lim_{\delta \rightarrow 1} E_\delta$. ■

An immediate by-product (using Kandori, 1992) is that the convex hull of E_n is contained in the set of subgame-perfect Nash equilibria of the n -times repeated game under perfect monitoring with a correlation device. Hence, the conditions for the folk theorem (in PPE) to hold for finitely repeated games with imperfect monitoring are necessarily at least as strong as those for finitely repeated games with perfect monitoring, as well as those for the infinitely repeated game with imperfect public monitoring and vanishing discounting (by Lemma 1).

One might think that combining the usual assumptions (A0), (A1), (A2) and (A3) would be enough to obtain the Folk theorem for finitely repeated games with public signals, however the following example shows it is not the case.

4 An Example

We start with an example in which both Theorems 1 and 2 hold, yet with imperfect public monitoring, the finitely repeated Folk theorem fails. The game is illustrated in Figure 1.

	L	R		L	R
T	$(1, 1, 0)_{s_1}$	$(1, 1, 0)_{s_1}$	T	$(1, 1, 0)_{s_1}$	$(0, 0, 1)_{s_1}$
B	$(0, 0, 0)_{s_2}$	$(-1, 0, 0)_{s_3}$	B	$(0, 0, 1)_{s_2}$	$(0, 0, 1)_{\frac{1}{2}s_1 + \frac{1}{2}s_2}$
	W			E	

Figure 1: An Example in which the Folk Theorem Fails.

Signals are denoted by s_1, s_2, s_3 . Probabilities of signals are indicated in Figure 1. For example, under (B, R, E) , signals s_1 and s_2 have both probability $1/2$.

Minmax payoffs are all 0, as is readily checked. The set of feasible and individually rational payoffs is

$$V := \text{conv}\{(0, 0, 0), (0, 0, 1), (1, 1, 0), (0, 1/2, 0)\},$$

a set with non-empty interior. The Nash equilibria of the stage game are (B, R, E) , (T, L, W) , (T, L, E) and the convex combinations of (T, L, W) and (T, L, E) . Hence the stage game has two Nash payoff vectors: $E_1 = \{(1, 1, 0), (0, 0, 1)\}$. This implies that, for each player, there are two distinct Nash payoffs. It follows that the assumptions (A0) and (A1) are satisfied: under perfect monitoring, by Theorem 1 the equilibrium payoff set E_n would approach V as the number of repetitions goes to infinity.

Let us return to imperfect monitoring, but consider an infinite horizon, and assume that players maximize the average discounted sum of rewards. Note that there exists a mixed action profile α that has pairwise full rank for each pair of players: to see this, simply consider the mixed action profile where each player equally randomizes between his two actions. The induced law on signals is $s = \frac{9}{16}s_1 + \frac{5}{16}s_2 + \frac{2}{16}s_3$. With a unilateral deviation, player 1 induce laws over signals in the segment $[s_1, \frac{1}{8}s_1 + \frac{5}{8}s_2 + \frac{2}{8}s_3]$, player 2 in the segment $[\frac{1}{2}s_1 + \frac{1}{2}s_2, \frac{5}{8}s_1 + \frac{1}{8}s_2 + \frac{2}{8}s_3]$, and player 3 in the segment $[\frac{1}{2}s_1 + \frac{1}{4}s_2 + \frac{1}{4}s_3, \frac{5}{8}s_1 + \frac{3}{8}s_2]$. Since these segments only intersect at s , the mixed profile has pairwise full rank for any pair of players. Hence, Condition (A2) of FLM is satisfied. Furthermore note that, for each player i , there exists a Nash equilibrium of the stage game achieving the minmax payoff for i . Hence, this is an admissible action profile. So the standard assumptions (A2) and (A3) for the infinite-horizon folk theorem to hold are satisfied, so that Theorem 2 applies and $E_\delta \xrightarrow{\delta \rightarrow 1} V$.

Yet we claim that the Folk theorem fails if we consider imperfect monitoring and a finite

horizon. Denote by E the convex hull of E_1 , that is,

$$E = \{(\lambda, \lambda, 1 - \lambda), \lambda \in [0, 1]\}.$$

We have:

Lemma 2

$$\lim_n E_n = \lim_n E'_n = E.$$

Proof. We only need to show that $E'_n \subset E$ for each n . We proceed by induction.

Consider a perfect public equilibrium of the $n + 1$ -stage game, and denote by x (resp., y , z) the probabilities that at the first stage player 1 (P1) plays T (resp., player 2 (P2) plays L , player 3 (P3) plays W). Write $\alpha = (1 - x)(1 - y)z$, $\beta = (1 - x)y$ and $\gamma = (1 - x)(1 - y)(1 - z)/2$. The law of the public signal of stage 1 is $s = (x + \gamma)s_1 + (\beta + \gamma)s_2 + \alpha s_3$, and we have $x + \beta + 2\gamma + \alpha = 1$. By the induction hypothesis, the continuation payoffs are assumed to be in E and are denoted $(g(s_1), g(s_1), 1 - g(s_1))$, $(g(s_2), g(s_2), 1 - g(s_2))$ and $(g(s_3), g(s_3), 1 - g(s_3))$.

First, let us provide some intuition. If the profile (B, R, W) , which has individual full rank but not pairwise full rank, is played at stage 1, the deviation of P3 induces the distribution of signals $(u + v)/2$, the deviation of P1 induces u and the deviation of P2 induces v . Since $(u + v)/2$ lies in the convex hull of u and v , the continuation strategies would have to punish both P3, and a fictitious player with payoff the average of P1 and P2's payoffs, and this turns out to be impossible.

We now formalize this idea. Assume that $\alpha > 0$, that is, at the first stage (B, R, W) is played with positive probability. For the deviation by P1 playing T at stage 1 to be unprofitable, it is necessary that $g(u) < g(s) \stackrel{def}{=} (x + \gamma)g(u) + (\beta + \gamma)g(v) + \alpha g(w)$. This implies

$$g(u) < \frac{(\beta + \gamma)g(v) + \alpha g(w)}{\beta + \gamma + \alpha}.$$

Similarly, the deviation by P3 playing E at stage 1 should not be profitable, so

$$g(w) < (g(u) + g(v))/2.$$

And the deviation by P2 playing L at stage 1 should not be profitable either, that is, it holds

that $xg(u) + (1 - x)g(v) \leq g(s)$, which gives

$$(\gamma + \alpha)g(v) \leq \gamma g(u) + \alpha g(w).$$

The three displayed inequalities are not compatible, hence we must have $\alpha = 0$.

Assume now that $(1 - x)yz > 0$. If $y = 1$, P3 has a profitable (undetectable) deviation by playing E at stage 1. So $y < 1$, and $\alpha > 0$. But as we have shown, this is impossible.

As a consequence, we obtain $(1 - x)z = 0$, and the induced equilibrium payoff is in E . ■

5 The Folk theorem for finitely repeated games with public signals

It follows from the previous example that for the folk theorem to hold with imperfect public monitoring and a finite horizon, it is not enough to combine the usual assumptions invoked for the folk theorem under both perfect monitoring and a finite horizon, and imperfect monitoring and infinite horizon. We now strengthen both (A0) and (A1) into (B1).

Assumption (B1): The convex hull of E_1 has non empty interior.

As discussed in the introduction, this can be without loss of generality weakened to the assumption that the convex hull of the set E_n has non-empty interior, for some n (encompassing thereby games that might fail the assumption (A1), but satisfy Smith's (1995) assumption of recursively distinct Nash payoffs)¹.

Theorem 3 Folk Theorem for finitely repeated games and public monitoring:

$$(B1), (A2) \text{ and } (A3) \text{ imply } E'_n \xrightarrow[n \rightarrow \infty]{} V.$$

Notice that the convergence holds for the sequence (E'_n) , hence assuming a public correlation device. We do not know if under the same assumptions, $E_n \xrightarrow[n \rightarrow \infty]{} V$. Example 1 shows that (A2) and (A3) are not enough to imply the conclusion of Theorem 3. Examples 2 and 3 of section 7 will also show that (B1) and (A2), or (B1) and (A3), are not enough either.

¹(B1) can be even further weakened by assuming that the convex hull of the set of sequential equilibrium payoffs (in private strategies) of the n -times repeated game has non-empty interior. Of course, the resulting folk theorem would then be in private strategies.

6 Proof of the Folk theorem

Write $E := \lim_n E'_n$ and recall E is convex. Therefore, by a theorem due to Alexandrov, almost all its boundary points are normal, *i.e.*, the representing function is differentiable at these points. Therefore, if $E \neq V$, there exists a vector $\lambda \in S^1 := \{x \in \mathbb{R}^I : \|x\| = 1\}$ and a point $v \in \text{bd}(E)$ with (unique) normal vector λ , and such that $\max_{v' \in V} \lambda \cdot (v' - v) = 3\kappa > 0$.

Assume first that λ is not a coordinate direction, and let l be the line through v with direction λ . Let $\alpha \in \Delta(A)$ refers to an action profile in the open dense subset of action profiles satisfying pairwise full rank guaranteed by Assumption A2) such that $\lambda \cdot g(\alpha) \geq \lambda \cdot v + 2\kappa$ (fix one of them if multiple exist). Let $\bar{v} \in \mathbb{R}^I$ denote the unique point of l such that $\lambda \cdot \bar{v} = \lambda \cdot g(\alpha)$. Clearly, $\bar{v} \notin E$. For all $k \in \mathbb{R}$, let $H_\lambda(k) := \{x \in \mathbb{R}^I : \lambda \cdot x = k\}$, and $H_\lambda^+(k) := \{x \in \mathbb{R}^I : \lambda \cdot x \geq k\}$.

The pairwise full rank assumption at the action profile α ensures that there exists $x : Y \rightarrow \mathbb{R}^I$ such that α is a Nash equilibrium of the game with payoff function

$$g(\cdot) + \sum \pi(y|\cdot) x(y),$$

and $\lambda \cdot x(y) = 0$ for all $y \in Y$ (See Mailath and Samuelson, 2006, Lemma 8.1.1(4) and 9.2.2). Without loss of generality, we may assume that the payoff from this equilibrium is equal to \bar{v} (redefine each $x(y)$ as $x(y) + \bar{v} - g(\alpha) - \sum_{y \in Y} \pi(y|\alpha)x(y)$). Let $M = \max_y \|x(y)\|$, and set $\kappa_0 := \frac{\kappa}{2\sqrt{\kappa^2 + M^2}}$.

Given $z \in \mathbb{R}^I$, define the convex cone:

$$C_z := \{z' \in \mathbb{R}^I : \lambda \cdot (z - z') > \kappa_0 \|z - z'\|\}.$$

See Figure 2 (here, $\alpha = a$ is pure). Since A1 holds, E is the closure of its interior. Because E is smooth at v , there exists $k < \lambda \cdot v$ and a compact set $D \subset \mathbb{R}^I$ such that

$$H_\lambda(k) \cap C_v \subset D \subset \text{int}(E).$$

Because $(E_n)_n$ is a sequence of convex sets converging to E , we can assume that $D \subset E_n$ for n large. Let $v_n = \arg \max_{x \in l \cap E_n} \lambda \cdot x$ be the highest point of E_n on the line l , and set $k_n = \lambda \cdot v_n$. We restrict attention to $n \geq n_0$ such that $k_n - \kappa/n > k$. This implies that $C_{v_n} \cap H_\lambda^+(k) \subset E_n$, because $C_{v_n} \subset C_v$, v_n and D are in E_n , and E_n is convex.

Given v_n , let

$$w_n(y) := v_n + \frac{1}{n}x(y) - \frac{\kappa}{n}\lambda,$$

for all $y \in Y$. Note that

$$\frac{\lambda \cdot (v_n - w_n(y))}{\|v_n - w_n(y)\|} \geq \frac{\kappa}{\sqrt{\kappa^2 + M^2}} > \kappa_0,$$

so that $w_n(y) \in C_{v_n}$ and hence $w_n(y) \in E_n$ (note that $\lambda \cdot w_n(y) = k_n - \kappa/n > k$).

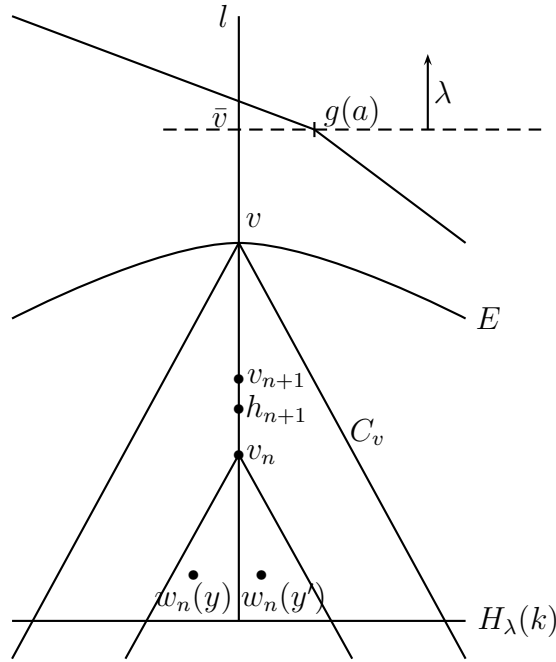


Figure 2: Proof of Theorem ??

Note that the action profile α is a Nash equilibrium of the game with payoff vector

$$\begin{aligned} & \frac{1}{n+1}g(\cdot) + \frac{n}{n+1} \sum \pi(y|\cdot) w_n(y) \\ &= \frac{1}{n+1} \left(g(\cdot) + \sum \pi(y|\cdot) x(y) \right) + \frac{n}{n+1}v_n - \frac{\kappa\lambda}{n+1}. \end{aligned}$$

Because $w_n \in E_n$, this implies that the resulting equilibrium payoff vector h_{n+1} is in E_{n+1} .

Observe that

$$\lambda \cdot (h_{n+1} - v_n) = \frac{1}{n+1} \lambda \cdot (g(\alpha) - v_n) - \frac{\kappa}{n+1} \geq \frac{\kappa}{n+1}.$$

Furthermore, observe that, by construction, h_{n+1} is on the line l (recall that $g(\alpha) + \sum \pi(y|\alpha)x(y) = \bar{v} \in l$). Hence,

$$k_{n+1} \geq \lambda \cdot h_{n+1} \geq \lambda \cdot v_n + \frac{\kappa}{n+1} = k_n + \frac{\kappa}{n+1}.$$

This is not possible, as it implies that $k_n \rightarrow \infty$.

Consider the case in which λ is a coordinate direction, *i.e.*, $\lambda = \pm e^i$, for some basis vector e^i . Then it is also the case, by Assumption A2 and A3 (see Mailath and Samuelson, 2006, Lemma 9.2.1(1) and (3)) that there exists $x(y)$ such that $\lambda \cdot x(y) = 0$ for all y , and such that the action profile a (resp., m^i) that maximizes (or is arbitrarily close to maximize, in case a fails pairwise full rank) $g_i(a)$ (resp., minmaxes player i) is an equilibrium of the game with payoffs $g(\cdot) + \sum_{y \in Y} \pi(y|\cdot)x(y)$. Whether $\lambda = e^i$ or $\lambda = -e^i$, the remainder of the proof is identical to the previous one.

7 Discussion and two examples

The example in Section 4 satisfies (A2) and (A3), but fails (A1), establishing its necessity for the folk theorem. The next example, Example 2, has some similarities with Example 3.1 in Renault and Tomala (2004). Here (A1) and (A3) are satisfied, but (A2) is not and the Folk theorem fails.

	L	R		L	R		L	R		L	R
T	(0, 0, 1)	(0, 3, 0)	T	(0, 4, 0)	(0, 4, 0)	T	(4, 0, 0)	(4, 0, 0)	T	(1, 1, 2)	(1, 1, 0)
B	(3, 0, 0)	(2, 2, 0)	B	(0, 4, 0)	(0, 4, 0)	B	(4, 0, 0)	(4, 0, 0)	B	(1, 1, 0)	(1, 1, 0)
	W			M_1			M_2			E	

Figure 3: Example 2, that satisfies (A1) and (A3), but not (A2).

This is a 3-player game where $A_1 = \{T, B\}$, $A_2 = \{L, R\}$ and $A_3 = \{W, M_1, M_2, E\}$. The public signal consists of the action and payoff of player 3. E_1 contains (0, 4, 0), (4, 0, 0), (1, 1, 0) and (1, 1, 2), hence the full dimensionality assumption A1 is satisfied. All minmax payoffs are 0, and can be obtained with Nash equilibria of the stage game, so admissibility (A3) also holds. The profile (T, L, W) has individual full rank, but there is no action profile in $\Delta(A)$ having pairwise

full rank for players 1 and 2, so A2 is not satisfied here. We claim that the Folk theorem does not hold.

Lemma. $E \subset \{x = (x_i)_i \in \mathbb{R}^3, x_1 + x_2 \geq 2\}$.

Proof. Let $\sigma = (\sigma_i)_i$ be an equilibrium of the n -stage game, with payoff $(\gamma_i(\sigma))_i$ in \mathbb{R}^3 . Consider the deviation τ_1 , resp. τ_2 , of player 1, resp. 2, which plays B , resp. R at every stage independently of the past. The point is that both profiles (τ_1, σ_{-1}) and (τ_2, σ_{-2}) induce the same distribution over sequences of player 3's actions. Let us denote by α_0 (resp. α_1 , resp. α_2 , resp. α_3), the expected frequencies across stages of player 3 playing W (resp. M_1 , resp. M_2 , resp. E_3) under this probability distribution. Since σ_1 is a best reply to σ_{-1} , we have $\gamma_1(\sigma) \geq 2\alpha_0 + 4\alpha_2 + \alpha_3$. Similarly σ_2 is a best reply to σ_{-2} , we have $\gamma_2(\sigma) \geq 2\alpha_0 + 4\alpha_1 + \alpha_3$. So $\gamma_1(\sigma) + \gamma_2(\sigma) \geq 4(1 - \alpha_3) + 2\alpha_3 \geq 2$.

Our last example below (see Figure 4) satisfies Assumptions (A1) (the Nash equilibria payoffs of the one-shot game include the vectors $(2, 0)$, $(2, 1)$ and $(3, 1)$) as well as Assumption (A2) (clearly, u uniquely discriminates among player 1's two actions, while v, w and x statistically distinguish player 2's actions, for any action profile such that player 1 does not assign probability 1 to T), yet fails Assumption (A3). (Player 2's action is not statistically identifiable, when player 1 plays the action T , which is part of the unique action profile that minmaxes player 1.)

	L	M	R
T	$(0, 0)_{u/4+v/4+w/4+x/4}$	$(1, 1)_{u/4+v/4+w/4+x/4}$	$(3, 1)_{u/4+v/4+w/4+x/4}$
B	$(-1, -1)_{u/2+v/4+w/8+x/8}$	$(2, 0)_{u/2+v/8+w/4+x/8}$	$(1, 0)_{u/2+v/8+w/8+x/4}$

Figure 4: Example 3, that satisfies A1 and A2, but not A3.

The minmax payoff of player 1 is 0 (achieved by the action profile (T, L)), yet the score $k(-e^1)$ of Fudenberg and Levine's (1994) algorithm is readily seen to be $1/2$, which is therefore a lower bound on the equilibrium payoff of player 1. The difficulty is that player 2's minmaxing action L is strictly dominated, yet deviations from L cannot be detected if player 1 plays T , as he should to minmax player 1. Hence, player 1 must assign positive probability to B , but given that it is costly to do so, continuation payoffs must be such that he gets rewarded for it. Yet, imperfect (full-support) monitoring ensures that this reward comes "too often," and so at an efficiency cost –the $1/2$ that separate the minmax payoff of player 1 from the score in direction $-e^1$.

8 Going further

The main limitation of Theorem 3 is that it considers the sequence (E'_n) , hence assumes a public randomization device. Dispensing with it raises significant issues that are already present in the case of perfect monitoring. Because the set of continuation payoff vectors need not be convex, it is not obvious that mixed action profiles can be enforced, as it requires continuation payoffs to be finely-tuned to the flow payoffs. Gossner's (1995) result for perfect monitoring does not readily adapt to the case of imperfect public monitoring, as approachability works best when signals (in his case, actions) can be attributed to specific players, as under perfect monitoring, or imperfect public but product monitoring. Whether his or other techniques can be extended to public monitoring with a finite horizon to dispense with public randomization (assuming it is possible at all) is an important open question.

We have chosen not to discount payoffs in the finitely repeated game, in line with the tradition of the literature. The result remains unchanged for vanishing discount factors. To the extent that the horizon length (T) can be viewed as the first occurrence of a discount factor equal to zero in some infinite sequence of discount factors, it is natural to wonder about a more general payoff criterion (*i.e.*, a characterization of discount factor sequences) for which the folk theorem would hold under Assumptions A1–A3. This is left for future research.

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