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Tilted inequalities and facets of the set covering polytope: a theoretical analysis

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Abstract

Given a ground-set of elements and a family of subsets, the set covering problem consists in choosing a minimum number of elements such that each subset contains at least one of the chosen elements. This research focuses on the set covering polytope, which is the convex hull of integer solutions to the set covering problem. We investigate the connection between the study of the facets of the set covering polytope and tilting theory. This theory studies how inequalities can be rotated around their contact points with a polyhedron in order to obtain inequalities inducing higher dimensional faces. To study this connection, we introduce the concept of *tilting vectors* which characterize the degrees of freedom of rotation of an inequality. These vectors characterize facet-defining inequalities and can be used to tilt inequalities with a similar procedure to the one used for arbitrary polyhedra. Additionally, we demonstrate that the computational effort needed to tilt an inequality can be reduced when the inequality has many null coefficients. Finally, we use the tilting vectors to extend several necessary and/or sufficient conditions for facets of the set covering polytope presented by several previous works of the literature.

Keywords: Set covering, Facets, Tilting, Tilting vectors, Valid inequalities

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1. Introduction

Given a ground-set \mathcal{E} of elements and a family \mathcal{S} of subsets of \mathcal{E} , the set covering problem consists in choosing a minimum number of elements of \mathcal{E} such that each subset in \mathcal{S} contains at least one of the chosen elements. A feasible set of chosen elements is referred to as a cover.

To model this problem, one can use a binary matrix B whose columns are indexed by the elements $e \in \mathcal{E}$ and rows by the subsets $s \in \mathcal{S}$. The coefficient B_{se} indicates whether the subset s contains the element e . If x_e takes the value 1 when element e is chosen, then the set covering problem can be modeled using the following mixed integer linear program:

$$\min_x \mathbf{1}x \tag{1a}$$

$$\text{subject to } Bx \geq \mathbf{1} \tag{1b}$$

$$x \in \{0, 1\}^{|\mathcal{E}|}. \tag{1c}$$

The set covering problem arises in many applications in a wide variety of fields such as logistics management [Mihelic and Robic, 2004, Cacchiani et al., 2014], crew scheduling [Caprara et al., 1999], manufacturing [Stanfel, 1989], data extraction and manipulation [Day, 1965], and medicine [Reggia et al., 1983]. Thus, understanding the structural properties of this problem and their impact on solution methods has very broad implications. Mathematically, we study the properties of the associated set covering polytope $\mathcal{Q}(B)$ which is the convex hull of the set $P = \{x \in \{0, 1\}^{|\mathcal{E}|} : Bx \geq \mathbf{1}\}$. A comprehensive polyhedral description of $\mathcal{Q}(B)$ via valid inequalities would allow solving the set covering problem using classical linear programming tools. However, even a partial description of the polytope would enable the design of fast and scalable enumeration-based methods.

Numerous families of valid inequalities have already been proposed in the literature (see the thesis of Borndörfer [1998] for an overview). A central question regarding these valid inequalities is whether they induce facets of the polytope $\mathcal{Q}(B)$, i.e., whether they are necessary to describe the polytope. There have been studies on proposing valid inequalities along with the condi-

tions under which they induce facets of the set covering polytope [Cornuéjols and Sassano, 1989, Nobili and Sassano, 1989, Balas and Ng, 1989].

Other polyhedral studies have approached the question by describing necessary and/or sufficient conditions for known valid inequalities to induce facets of $\mathcal{Q}(B)$. In this line of work, it is worth mentioning that Balas and Ng [1989] characterized all the facets having coefficients and right-hand sides in $\{0, 1, 2\}$. Similarly, Sánchez-García et al. [1998] and Saxena [2004] gave necessary and sufficient conditions for the inequalities with coefficients and right-hand sides in $\{0, 1, 2, 3\}$ to be facets. Another class of inequalities for which the characterization of facets has been studied is the rank inequalities, i.e., inequalities with binary coefficients but with arbitrary right-hand sides. Cornuéjols and Sassano [1989] proposed a necessary condition for these inequalities to be facets, while Sassano [1989] proposed a sufficient condition.

Instead of directly proposing facets of $\mathcal{Q}(B)$, a complementary approach to finding facets of $\mathcal{Q}(B)$ is to use dominance arguments to derive stronger inequalities from inequalities that are not facet defining. The approach referred to as *lifting* answers this question by posing the problem of finding the best set of coefficients for an inequality as a series of mixed-integer optimization problems [Padberg, 1973, Wolsey, 1976]. More recently, Chvátal et al. [2013] introduced a more general approach called *tilting*. The authors show that a non-facet defining valid inequality can be rotated around its contact points with the polyhedron and remain valid. They also describe a procedure to perform the largest such rotation and show that the resulting inequality induces a face of higher dimension. Thus, by applying this procedure at most as many times as the dimension of the ambient space, one ends up with a facet-defining inequality. The number of linearly independent axes on which a rotation can be performed corresponds to the degrees of freedom associated with the possible rotations.

In this work, we aim to deepen the understanding of the set covering polytope $\mathcal{Q}(B)$ for inequalities with arbitrary coefficients. To that end, we introduce a new mathematical object for the study of the facets of the set covering polytope: the *tilting vectors*. The importance of tilting vectors is twofold. First, they can be used to derive an alternative version of tilting

specialized for set covering. Second, tilting vectors characterize facets of the set covering polytope. Thus, they can be used as tools to provide new facet characterization results and proofs. We investigate this strong mathematical object leading to both methodological and topological contributions. While some of our contributions revisit known results, other contributions are entirely new. The revisiting contributions focus on the variation of tilting specialized to set covering by using tilting vectors:

- We present how tilting vectors characterize the facets of the set covering polytope.
- We show that under some conditions, facets can be obtained in one tilting operation.
- We design a simple procedure that obtains, from a valid inequality and a tilting vector, tilted inequalities that induce higher dimensional faces than the original inequality.

Our new contributions are the following:

- We show that the number of computations required to tilt a sparse inequality is significantly reduced compared to the dense case.
- We highlight that each null coefficient in a valid inequality may be associated with a specific tilting vector and that all the remaining tilting vectors have their support included in the support of the inequality. This implies that one can reduce the study of set covering facets from general inequalities to inequalities without null coefficients.
- We use tilting vectors to extend several results from the set covering literature. In particular, we extend from rank inequalities to arbitrary inequalities the necessary and sufficient conditions of Cornuéjols and Sassano [1989] and Sassano [1989].
- We provide an alternative proof for the characterization introduced by Balas and Ng [1989] for the facet-defining inequalities with coefficients and right-hand side in $\{0, 1, 2\}$. We also extend the results of Balas and Ng [1989] by characterizing the tilting vectors for these inequalities.

The remainder of this article is organized as follows. We begin by introducing notations and basic results in Section 2. Section 3 is dedicated to the introduction of the tilting vectors, their properties, and their specific use in the context of tilting for set covering. We then focus in Section 4 on how to exploit the null coefficients of set covering inequalities. In Section 5, we extend the necessary and sufficient conditions of Cornuéjols and Sassano [1989] and Sassano [1989] for rank inequalities to general inequalities. This last section also contains an extension of the work of Balas and Ng [1989] on inequalities with coefficients and right-hand sides in $\{0, 1, 2\}$. Finally, Section 6 concludes this article.

2. Notations and preliminaries

Following the notations in Sassano [1989], Cornuéjols and Sassano [1989], a set covering instance is associated with a bipartite graph $G = (\mathcal{E}, \mathcal{S}, A)$. The arc set A is composed of the pairs $(e, s) \in \mathcal{E} \times \mathcal{S}$ such that $e \in s$. In this case, a cover is a subset of nodes of \mathcal{E} such that each node in \mathcal{S} is connected to at least one node in the cover. An illustrative example can be found in Figure 1.

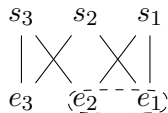


Figure 1: The bipartite graph associated with a set covering instance where $\mathcal{E} = \{1, 2, 3\}$ and $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. An example of cover (circled) is $\{1, 2\}$.

For a given subset $E \subseteq \mathcal{E}$, $N(E) = \{s \in \mathcal{S} \mid \exists e \in E, (e, s) \in A\}$ denotes the neighbors of the elements in E . In this context, a cover can equivalently be defined as a set of nodes in \mathcal{E} whose set of neighbors is \mathcal{S} . We will denote $\mathcal{Q}(G)$ the polytope of a set covering instance associated with a bipartite graph G .

Given a polyhedron \mathcal{Q} and a valid inequality $\alpha x \geq k$ for this polyhedron, the face associated with this inequality is $F = \{x \in \mathcal{Q} \mid \alpha x = k\}$. A facet of a polyhedron is a face whose dimension is the dimension of the polyhedron

minus one. To prevent our explanations from becoming too heavy and uneasy to read, we will not always differentiate the inequalities from the face they induce on the polyhedron $\mathcal{Q}(G)$. For the same reason, we will not always differentiate covers from their incidence vectors. This allows us, for example, to abuse the vocabulary and say that some covers are affinely independent or that an inequality is a facet. We will also denote x^E the incidence vector of a subset E of \mathcal{E} (typically x^c for a cover c).

For any vector $\alpha \in \mathbb{R}^{|\mathcal{E}|}$ and any sub-family $S \subset \mathcal{S}$, we will denote by $\gamma(\alpha, S)$ the minimum value of αx over the binary vectors $x \in \{0, 1\}^{|\mathcal{E}|}$ representing a set of elements covering all the subsets in S . For simplicity, we will use $\gamma(\alpha)$ in place of $\gamma(\alpha, \mathcal{S})$. In this work, when considering an inequality $\alpha x \geq k$, we assume that its right-hand side is minimum, *i.e.*, $k = \gamma(\alpha) = \min_{x \in \mathcal{Q}(G)} \alpha x$. To make this explicit, from now on we will denote $\alpha x \geq \gamma(\alpha)$ everywhere. Moreover, for a vector α and an integer p , we denote $E_p(\alpha) = \{e \in \mathcal{E} \mid \alpha_e = p\}$ the set of indices for which α has coefficient p .

A central object to the analysis of the strength of an inequality $\alpha x \geq \gamma(\alpha)$ is the set of covers that satisfy this inequality to equality. This set is denoted $C^=(\alpha)$ and contains most of the relevant information about the inequality. In particular, by definition, the inequality is a facet if and only if there is a number affinely independent covers in $C^=(\alpha)$ equal to the dimension of $\mathcal{Q}(G)$.

We now state a proposition taken from Nobili and Sassano [1989] that highlights some basic properties of the set covering polytope.

Proposition 1 (Nobili and Sassano [1989]).

- $\mathcal{Q}(G)$ is empty if and only if at least one subset in \mathcal{S} is empty;
- $\mathcal{Q}(G)$ is full-dimensional if and only if no subset in \mathcal{S} is a singleton.

If $\mathcal{Q}(G)$ is full-dimensional then, for each $e \in \mathcal{E}$:

- the inequality $x_e \geq 0$ defines a (trivial) facet of $\mathcal{Q}(G)$ unless for some subset $s \in \mathcal{S}$, $s \setminus \{e\}$ is a singleton;

- the inequality $x_e \leq 1$ defines a (trivial) facet of $\mathcal{Q}(G)$;
- every non-trivial facet of $\mathcal{Q}(G)$ is defined by an inequality of the form $\alpha x \geq \gamma(\alpha)$ where $\alpha \geq 0$ and $\gamma(\alpha) > 0$.

From now on, we will consider that the set covering polytope $\mathcal{Q}(G)$ is full-dimensional. This happens if and only if no subset in \mathcal{S} is a singleton and one can reduce any set covering instance to this case by setting to one the variable corresponding to elements present in singleton. Moreover, we will assume when considering an inequality $\alpha x \geq \gamma(\alpha)$ that $\alpha \geq 0, \gamma(\alpha) > 0$.

3. Tilting for set covering

In this section, we introduce the main concept of this article, the tilting vectors. After presenting how tilting vectors characterize the dimension of the face induced by an inequality, we discuss how tilting vectors can be used to tilt inequalities in the context of set covering. Finally, we discuss some computational aspects of the tilting procedure. For a presentation of tilting in the context of a general polyhedron see Appendix A or the work of Chvátal et al. [2013].

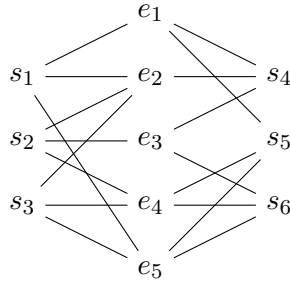
3.1. Tilting vectors

Let us now introduce the main concept of this article, the tilting vectors.

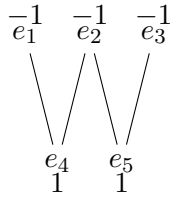
Definition 1. Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$ and let M_α be the matrix whose rows are the covers in $C^=(\alpha)$. A tilting vector β is a solution of the system $M_\alpha \beta = 0$.

In other words, a vector β is a tilting vector of the inequality $\alpha x \geq \gamma(\alpha)$ if it is orthogonal to all the covers in $C^=(\alpha)$. Note that, as the solution of a linear system, the set of tilting vectors of an inequality is a vector space. Moreover, although we should write “the tilting vectors of inequality $\alpha x \geq \gamma(\alpha)$ ”, for the sake of brevity, we will omit the reference to the studied inequality

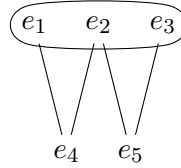
when this creates no confusion. A more graphical way to define tilting vectors, illustrated in Figure 2, is the following. Let $\mathcal{H}(\alpha) = (\mathcal{E}, C^=(\alpha))$ be the hypergraph with node set \mathcal{E} and with each hyper-edge representing a cover of $C^=(\alpha)$. A tilting vector is an assignment of weights β_e to the nodes of $\mathcal{H}(\alpha)$ such that the sum of the weights on the nodes of each hyper-edge is zero.



(a) An instance of set covering



(b) The hypergraph $\mathcal{H}(\alpha)$ for the inequality $x_1 + x_2 + x_3 + x_4 + x_5 \geq 2$ and the coefficients of a tilting vector



(c) The hypergraph $\mathcal{H}(\alpha)$ for the inequality $x_1 + x_2 + x_3 + 2x_4 + 2x_5 \geq 3$ for which the only tilting vector is null

Figure 2: A set covering instance: hypergraphs and tilting vectors corresponding to two inequalities

As mentioned earlier, tilting vectors can be used to tilt inequalities. A tilted inequality is defined as follows.

Definition 2. A tilted inequality corresponding to an inequality $\alpha x \geq \gamma(\alpha)$ and a tilting vector β is any valid inequality of the form $(\alpha + \epsilon\beta)x \geq \gamma(\alpha)$ where ϵ is a scalar.

A tilted inequality is a rotation of the original inequality around its set of contact points $C^=(\alpha)$ with the polytope $\mathcal{Q}(G)$. Indeed, each contact point of the original inequality is also a contact point of the tilted inequality. This is induced by $\beta x^c = 0$ for all $c \in C^=(\alpha)$ in the definition of the tilting vector β . This is illustrated in Figure 3.

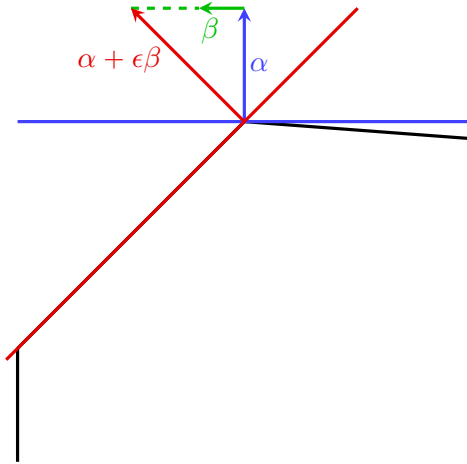


Figure 3: A valid inequality (blue) for a polyhedron with normal vector α , a tilting vector β (green), and the corresponding tilted inequality (red).

What makes the strength of the tilting vectors is that they can be used as tilting directions to obtain inequalities inducing higher dimensional faces and that they are a useful tool to study and characterize the facets of the set covering polytope. In the next sections, we introduce some of their basic relations to facets and their role in tilting.

3.2. Characterizing facets with tilting vectors

In this section, we highlight the relationship between the space of tilting vectors of an inequality and the dimension of the face it induces. More precisely, the following theorem links the dimension of the space of tilting vectors and the dimension of the face induced by an inequality.

Proposition 2. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$, let D_α be the dimension of its induced face, and let D_β be the dimension of the space of its tilting vectors. Then, $D_\alpha + D_\beta = |\mathcal{E}| - 1$.*

Proof is given in Appendix B.

With the above theorem we obtain as corollary that the existence of tilting vectors characterizes whether an inequality is a facet or not.

Corollary 1. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$. It is a facet of $\mathcal{Q}(G)$ if and only if its only tilting vector is the null vector.*

Corollary 1 is similar to Lemma 1 from Chvátal et al. [2013] for general tilting. The generalization from Corollary 1 to Proposition 2 can also be made for arbitrary polyhedra.

3.3. Tilting inequalities

Corollary 1 and Proposition 2 show that the existence of non-null tilting vectors certifies that the corresponding inequality is not a facet of $\mathcal{Q}(G)$. In such a case, it would be helpful to derive a facet or at least a higher dimensional face. This can be achieved by tilting the inequality. The next proposition shows that facets can be obtained in one tilting operation.

Proposition 3. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$. For each non-trivial facet containing $C^=(\alpha)$, there exists a tilting vector that can be used to tilt the valid inequality into the facet.*

Proof. Since we are considering non-trivial facets of the set covering polytope, the facet can be written $\alpha'x \geq \gamma(\alpha)$ through the right scaling. Take $\epsilon = 1$ and $\beta = \alpha' - \alpha$ as tilting vector. It is indeed a tilting vector since for each cover c in $C^=(\alpha)$ we have $\beta x^c = (\alpha' - \alpha)x^c = \gamma(\alpha) - \gamma(\alpha) = 0$. Note that if $\alpha x \geq \gamma(\alpha)$ was already a facet, then the only facet containing $C^=(\alpha)$ is $\alpha x \geq \gamma(\alpha)$ and the corresponding tilting vector β is equal to zero. \square

Although one can obtain any facet containing $C^=(\alpha)$ in one tilting operation, this requires the knowledge of the corresponding tilting vector which can be as difficult to obtain as the facet. However, any tilting vector can

be used to increase the dimension of the face of an inequality by at least one. Since tilting using tilting vectors does not change the right-hand side of the inequality, tilting operations cannot reach the trivial facets (which do not have a positive right-hand side). Thus, two special cases are required to handle the two types of trivial facets ($x \leq 1$ and $x \geq 0$). We first show in Proposition 4 that tilting with tilting vectors does lead to higher dimensional faces outside of the two special cases. The cases $x \leq 1$ and $x \geq 0$ are dealt with in Propositions 5 and 6, respectively. Note that for ease of reading, we have put the proofs of Propositions 4-6 in Appendix B.

Proposition 4. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$ such that, for each element $e \in \mathcal{E}$, there is at least one cover in $C^=(\alpha)$ that contains e , and one that does not. Let β be a non-null tilting vector for that inequality. Then, there exist positive numbers ϵ^+ , ϵ^- such that the tilted inequalities*

$$(\alpha + \epsilon^+ \beta)x \geq \gamma(\alpha) \tag{2}$$

$$(\alpha - \epsilon^- \beta)x \geq \gamma(\alpha) \tag{3}$$

are both valid inequalities inducing faces of strictly higher dimension.

Proposition 4 shows that under a unique condition, any non-facet defining inequality with positive coefficients can be decomposed as the sum of two non-trivial inequalities that induce faces of higher dimension. When this condition is not satisfied, it can be decomposed as the sum of one of the trivial facets ($x_e \geq 0$ or $x_e \leq 1$) and one other non-trivial inequality. These particular cases yield similar results and are treated in the next two theorems. These results are closely related to the theory of lifting inequalities that can be found in the literature [Padberg, 1973, Wolsey, 1976].

Proposition 5. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality and e an element of \mathcal{E} . Let η be such that $\eta + \gamma(\alpha)$ is the minimum of αx^c over the set of covers c not containing e . The following propositions are equivalent:*

- (i) e is contained in all covers in $C^=(\alpha)$;
- (ii) $\alpha - \gamma(\alpha)x^{\{e\}}$ is a tilting vector;

(iii) η is positive and $\alpha x \geq \gamma(\alpha)$ is the sum of $-\eta x_e \geq -\eta$ and $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$ with both inequalities being valid and the last one inducing a face of higher dimension.

Proposition 6. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality and e an element of \mathcal{E} . Let η be such that $\eta + \gamma(\alpha)$ is the minimum of αx^c over the set of covers c containing e . The following propositions are equivalent:*

- (i) e is not contained in any cover of $C^=(\alpha)$;
- (ii) $x^{\{e\}}$ is a tilting vector;
- (iii) η is positive and $\alpha x \geq \gamma(\alpha)$ is the sum of $\eta x_e \geq 0$ and $\alpha x - \eta x_e \geq \gamma(\alpha)$, with both inequalities being valid and the last one inducing a face of higher dimension.

In the context of the separation of a point from $Q(G)$, it is convenient to be able to decompose a separating inequality as the sum of two valid inequalities inducing faces of higher dimension. Indeed, since the inequalities of the decomposition sum to the original inequality, then at least one of them must also separate the point \hat{x} from $Q(G)$. Thus, given a separating inequality, one can always generate through tilting another separating inequality of higher dimension. In the case of Proposition 6, only the non-trivial inequality is guaranteed to be stronger than the original one (in Proposition 5, the trivial inequality is always a facet). This is, however, not an issue in practice. Indeed, in most cases, when a point is separated from $Q(G)$, it is the solution of a (linear) relaxation of the set covering model. In this case, the separated point satisfies the trivial inequalities. Thus, the non-trivial inequality of the decomposition, which is guaranteed to be stronger, must be separating the point from $Q(G)$.

3.4. Computation of the tilted inequalities

Let us now discuss how to compute the tilted inequalities that do induce higher dimensional faces. We start by discussing the two special cases treated in Propositions 5 and 6, where one of the tilted inequalities is one of the

trivial facets ($x_e \geq 0$ or $x_e \leq 1$). We then focus on the general case given by Proposition 4.

Special cases of Propositions 5 and 6: to be able to apply Proposition 4 and rule out the cases treated in Propositions 5 and 6, one needs to tilt the original inequality $\alpha x \geq \gamma(\alpha)$ so that each element $e \in \mathcal{E}$ is contained in a cover of $C^=(\alpha)$ and not contained in another cover of $C^=(\alpha)$. Assuming that at least one cover c in $C^=(\alpha)$ is known (it can be obtained by minimizing αx over $\mathcal{Q}(G)$), then for each element $e \in \mathcal{E}$, we know that either $e \in c \in C^=(\alpha)$ or $e \notin c \in C^=(\alpha)$. Thus, one needs to check only one of the two conditions of each element e . This can be done for each element by minimizing αx over $\mathcal{Q}(G)$ with the additional constraint $x_e = 1 - x_e^c$. Let us call η the minimal value and x^* the minimizer. If $x_e^c = 1$, then Proposition 5 tells us to replace the original inequality $\alpha x \geq \gamma(\alpha)$ by $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$; otherwise Proposition 6 tells us to replace the original inequality $\alpha x \geq \gamma(\alpha)$ by $\alpha x - \eta x_e \geq \gamma(\alpha)$. In both cases, the face induced by the new inequality contains x^c and the minimizer x^* which correspond to one cover containing e and one cover not containing e . Overall, the special cases can be treated in one initial call to the set covering oracle plus one call for each element.

General case of Proposition 4: the value of ϵ^+ , and similarly ϵ^- , can be determined with the following linear program:

$$\max \epsilon \tag{4}$$

$$\text{s. t. } (\alpha + \epsilon\beta)x^i \geq \gamma(\alpha) \quad \forall x^i \in \mathcal{Q}(G) \tag{5}$$

$$\epsilon \in \mathbb{R}. \tag{6}$$

This linear program can be solved with row generation and deciding whether there exists a row cutting off a value ϵ^* amounts to minimize $(\alpha + \epsilon^*\beta)x$ over $\mathcal{Q}(G)$. This is simply a call to a set covering oracle. At the end of the row generation, the only active row will correspond to a point that satisfies the tilted inequality to equality and is not orthogonal to β (otherwise ϵ would disappear from the constraint). This second fact implies that the point is affinely independent from the points in $C^=(\alpha)$. Note that since the above linear program has only one variable, no real linear programming machinery is required to solve it. Thus, the row generation algorithm can

be replaced by a simple algorithm iteratively calling a set covering oracle. This algorithm generates a sequence of candidates $\epsilon_1^*, \epsilon_2^*, \dots$ together with a sequence x_1^*, x_2^*, \dots where x_i^* minimizes $(\alpha + \epsilon_i^* \beta)x$ over $\mathcal{Q}(G)$. This iterative algorithm and sequences are illustrated in Figure 4.

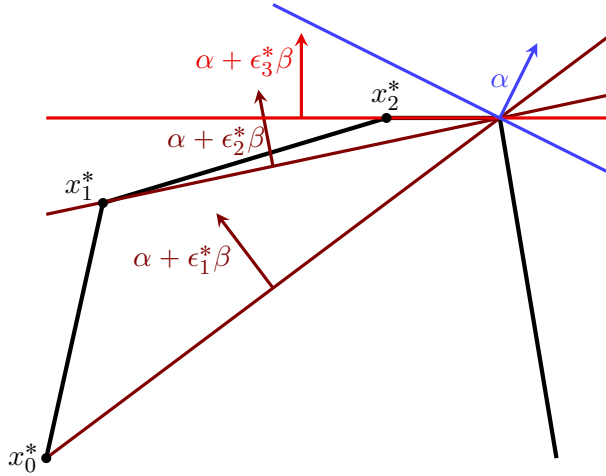


Figure 4: Illustration of the iterative algorithm solving the problem (4) - (6).

Another approach that finds ϵ^+ in only one call to a Branch & Cut algorithm is the following. For each point $y \in \mathcal{Q}(G)$ with $\beta y < 0$, one can define $\epsilon_y = \frac{\gamma(\alpha) - \alpha y}{\beta y}$ and y then satisfies the inequality $(\alpha + \epsilon_y \beta)x \geq \gamma(\alpha)$ to equality. This inequality is either invalid if there is a point of $\mathcal{Q}(G)$ violating it, or it is the optimal tilted inequality we are looking for. Thus, the point of $\mathcal{Q}(G)$ maximizing the smallest violation of this set of inequalities is on the optimal tilted inequality. Therefore, the following mixed integer linear

program computes the optimal tilted inequality:

$$\max z \tag{7}$$

subject to

$$z \leq \gamma(\alpha) - (\alpha - \epsilon_y \beta)x \quad \forall y \in \mathcal{Q}(G) \mid \beta y < 0 \tag{8}$$

$$x \in \mathcal{Q}(G) \tag{9}$$

$$z \in \mathbb{R}. \tag{10}$$

This program can be solved with a Branch & Cut algorithm. One can start with only one of the constraints (8) and every time an integer solution y^* with positive objective value is found, one can add to the problem the constraint associated with y^* .

4. Taking advantage of the null coefficients in set covering inequalities

In this section, we show how to take advantage of the null coefficients in an inequality. In particular, we highlight that each null coefficient in a valid inequality may be associated with a specific tilting vector and that all the remaining tilting vectors have their support included in the support of the inequality. The implications are twofold. First, we will show that one can reduce the study of facets from general inequalities to inequalities without null coefficients. Second, we show that the number of computations required to tilt a sparse inequality is significantly reduced compared to the dense case.

4.1. Tilting vectors associated with null coefficients of α

The next theorem highlights the special place that occupies the null coefficients of an inequality. In short, for an element $e_0 \in E_0(\alpha)$, *i.e.*, such that $\alpha_{e_0} = 0$, an important information is to know whether e_0 is contained in all the covers of $C^=(\alpha)$. If this is not the case, all tilting vectors have a null coefficient for e_0 . Otherwise, $(\alpha - \gamma(\alpha)x^{\{e_0\}})$ is a tilting vector.

Theorem 1. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$. Let E_0^* be the set of elements of $E_0(\alpha)$ contained in all the covers of $C^=(\alpha)$. A basis of the*

space of tilting vectors is $(\alpha - \gamma(\alpha)x^{\{e_0\}})_{e_0 \in E_0^*} \cup (\beta^1, \dots, \beta^p)$, where $(\beta^1, \dots, \beta^p)$ is a basis of the tilting vectors satisfying $\beta_{e_0} = 0$ for each $e_0 \in E_0(\alpha)$.

Proof. The vectors $(\alpha - \gamma(\alpha)x^{\{e_0\}})_{e_0 \in E_0^*}$ are tilting vectors since the elements of E_0^* are contained in all the covers of $C^=(\alpha)$.

A vector $\alpha - \gamma(\alpha)x^{\{e_0\}}$ is linearly independent from the other vector of this form and from $(\beta^1, \dots, \beta^p)$ since it is the only vector in the basis that has a non-zero coefficient for e_0 . Moreover, it is assumed that the vectors $(\beta^1, \dots, \beta^p)$ are independent from each other so that they can be part of a basis. Thus, we only need to show that all tilting vectors can be generated using the aforementioned vectors.

Let us now consider a tilting vector β . Note that for all e_0 in $E_0(\alpha) \subset E_0^*$, there is a cover c in $C^=(\alpha)$ not containing e_0 . Also note that $c \cup \{e_0\}$ belongs to $C^=(\alpha)$. Thus, $\beta_{e_0} = \beta(x^{c \cup \{e_0\}} - x^c) = 0 - 0 = 0$. Let us denote δ^{e_0} the vector $\frac{\beta_{e_0}}{\gamma(\alpha)}(\alpha - \gamma(\alpha)x^{\{e_0\}})$. It must be a tilting vector since $\alpha - \gamma(\alpha)x^{\{e_0\}}$ is one. Note that δ^{e_0} has only null coefficients for the elements in $E_0(\alpha)$ except for e_0 for which the coefficient is β_{e_0} . Thus, the vector $\beta - \sum_{e_0 \in E_0^*} \delta^{e_0}$ is a tilting vector with null coefficients for all elements in $E_0(\alpha)$. Thus, it can be decomposed as a linear combination of the $(\beta^1, \dots, \beta^p)$. Therefore, β can be decomposed as a linear combination of the $(\beta^1, \dots, \beta^p)$ and the $(\alpha - \gamma(\alpha)x^{\{e_0\}})_{e_0 \in E_0^*}$. \square

The above theorem shows that apart from the vectors $(\alpha - \gamma(\alpha)x^{\{e_0\}})_{e_0 \in E_0^*}$, the other tilting vectors have null coefficients for elements for which α does. As it turns out, these remaining tilting vectors correspond to the tilting vectors of $\alpha'x \geq \gamma(\alpha)$ for the polytope $\mathcal{Q}(G \setminus E_0(\alpha))$, where α' is the vector α without its zero coefficients. Recall that replacing $\mathcal{Q}(G)$ by $\mathcal{Q}(G \setminus \{e_0\})$ is the same as enforcing $x^{E_0} = 1$, i.e., enforcing e_0 to be in all covers.

Proposition 7. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$. The tilting vectors satisfying $\beta_{e_0} = 0$ for each $e_0 \in E_0(\alpha)$ are exactly the tilting vectors of the inequality $\alpha'x \geq \gamma(\alpha)$ for the polytope $\mathcal{Q}(G \setminus E_0(\alpha))$ with additional zero coefficients for $E_0(\alpha)$ where α' is the vector α without its zero coefficients.*

Proof. The tilting vectors for $\alpha x \geq \gamma(\alpha)$ are the vectors that satisfy $\beta x^c = 0$ for each cover c of $C^=(\alpha)$. When, the condition $\beta_{e_0} = 0$ for each $e_0 \in E_0(\alpha)$ is imposed, this is the same as having $\beta x^{c \setminus E_0(\alpha)} = 0$ for each cover c of $C^=(\alpha)$. The sets $(c \setminus E_0(\alpha))_{c \in C^=(\alpha)}$ are exactly the covers in $C^=(\alpha')$ for the polytope $\mathcal{Q}(G \setminus E_0(\alpha))$. Thus, the two sets of tilting vectors mentioned in the theorem are the solution set of the same system of equations. They are thus equal (up to the additional null coefficients). \square

By combining the above two results, we can show that the study of set covering facets can be reduced from arbitrary inequalities to inequalities without null coefficients. This result is crystallized in Proposition 8 which contains a necessary and sufficient condition for arbitrary inequalities to be facets. Although presented in a different manner, this result can be found in [Laurent, 1989, Cornuéjols and Sassano, 1989] for the special case of rank inequalities. It is also present in all facet characterizations for special cases we are aware of, such as those for facets with coefficients 0, 1, and 2 by Balas and Ng [1989] and 0, 1, 2, and 3 by Saxena [2004] and Sánchez-García et al. [1998].

Proposition 8. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$. It is a facet if and only if both of the following conditions are satisfied:*

- (i) $\alpha' x \geq \gamma(\alpha')$ is a facet of the polytope $\mathcal{Q}(G \setminus E_0(\alpha))$, where α' is the vector α without its zero coefficients;
- (ii) for all $e \in E_0(\alpha)$, there is a cover of $C^=(\alpha)$ not containing e .

With the above proposition, one can see that when searching for characterizations of facets, the study can be restricted to inequalities without null coefficients. Indeed, the conditions required in the general case will be the ones from the restricted case together with condition (ii) of Proposition 8.

4.2. Faster computation of tilting vectors for sparse inequalities

Most of the algorithms presented in Section 3.4 rely on calls to a set covering oracle for $\mathcal{Q}(G)$ to compute the tilted inequality (the arguments of

this section also work for the algorithm of the last paragraph of Section 3.4 relying on Branch & Bound). When the oracle is given an objective function with many null coefficients, one can expect the oracle to run much faster in practice. Indeed, setting the variables corresponding to these null coefficients to one does not increase the cost of the solution but covers several of the subsets to be covered. As a consequence, a smaller set covering problem can be solved without compromising the optimality of the solution. In particular, oracle calls for sparse objective functions may be much faster than finding a minimum cardinality cover. In the case of tilting, the objective function given to the oracle is always $\alpha + \epsilon\beta$ for $\epsilon \in [0, 1]$. Thus, if the tilting vector β has its support (set of non-zero coefficients) included in the one of α , the objective vector is as sparse as α . Note that in this case the tilted inequality, if non-trivial, will have a support included in the support of α , allowing one to re-apply tilting to the new equality with the same sparsity properties.

One can accelerate the computation of the tilted inequality when the tilting vector has its support included in the support of α . However, some tilting vectors have a larger support than α . This is the case of the tilting vector $\alpha - \gamma(\alpha)x^{\{e\}}$ appearing in Proposition 5 when e is outside of the support of α . However, one can greatly increase the dimension of original inequality by using only tilting vectors as sparse as α . In fact, once all sparse tilting vectors have been exploited, there only remain the aforementioned tilting vectors of Proposition 5. This fact is apparent in the following corollary of Theorem 1.

Corollary 2. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$. Let E_0^* be the set of elements of $E_0(\alpha) = \{e \in \mathcal{E} \mid \alpha_e = 0\}$ contained in all the covers of $C^=(\alpha)$. Assume that the only tilting vector satisfying $\beta_{e_0} = 0$ for all $e_0 \in E_0(\alpha)$ is $\beta = 0$. Then, a basis of the vector space of tilting vectors is $(\alpha - \gamma(\alpha)x^{\{e_0\}})_{e_0 \in E_0^*}$.*

Thus, given a sparse initial inequality, one can tilt it into a stronger inequality by using only tilting vectors whose support is included in the initial inequality. This will require only calls to a set covering oracle with

sparse objective functions which can be much quicker. Afterward, Corollary 2 shows that it only remains to check that the conditions in Proposition 5 are verified for the elements of $E_0(\alpha)$ which can be done with $|E_0(\alpha)|$ calls to the set covering oracle. Unfortunately, at each of these steps, the support of the inequality may increase its size which may slow down the last oracle calls.

5. Extending necessary and sufficient conditions for facets

In this section, we revisit several conditions from the literature that characterize facets of the set covering polytope. In particular, we extend the conditions of Cornuéjols and Sassano [1989] and Sassano [1989] from rank inequalities to arbitrary inequalities. We also give an alternative proof for the characterization of Balas and Ng [1989] on facets having coefficients and right-hand side in $\{0, 1, 2\}$. We finally complement their result by characterizing the tilting vectors for these inequalities.

5.1. Necessary conditions of Cornuéjols and Sassano [1989]

In their work, Cornuéjols and Sassano [1989] (Proposition 1) proposed a condition under which a rank inequality (*i.e.*, inequalities having only binary coefficients) is not a facet. We derive a new proof showing that under this condition one can pinpoint a tilting vector. The proof is not specific to rank inequalities which enables us to generalize their result to arbitrary inequalities. In this section, we consider inequalities without null coefficients but null coefficients can be taken into account by using Proposition 8.

To state the extended theorem and its proof, let us introduce a few notations and concepts. For any vector α and set of elements E , we will denote α_E the vector whose coefficient is α_e if e belongs to E and zero otherwise. Let us also introduce the concept of cutset as it is done in Cornuéjols and Sassano [1989]. This is based on the representation of set covering problems using graphs (see Section 2). Let E be a set of elements of \mathcal{E} and let $\bar{E} = \mathcal{E} \setminus E$. For each $E \subset \mathcal{E}$, the cutset S_E is the set of nodes adjacent to at least one node in E and one node in \bar{E} , *i.e.* $S_E = N(E) \cap N(\bar{E})$. Let us also recall that $\gamma(\alpha, S)$ denotes the minimum value of αx over the binary vectors

representing a cover of the subsets in S . Note that the result of Cornuéjols and Sassano [1989] corresponds to the special case $\alpha = \mathbf{1}$.

Proposition 9. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$ with $\alpha > 0$. If there is a non-critical cutset (i.e., for some $E \subset \mathcal{E}$, $\gamma(\alpha) = \gamma(\alpha, \mathcal{S} \setminus S_E)$), then the inequality $\alpha x \geq \gamma(\alpha)$ is not a facet.*

Proof. We will prove that if a cutset is not critical, one can derive a tilting vector.

Let us assume that for some $E \subset \mathcal{E}$, the cutset S_E is not critical, i.e., $\gamma(\alpha) = \gamma(\alpha, \mathcal{S} \setminus S_E)$. We show that in this case, $c \mapsto \alpha_E x^c$ is constant over all covers c in $C^=(\alpha)$. To that end, let us consider two covers c and c' of \mathcal{S} in $C^=(\alpha)$. Note that both $c \cap E$ and $c' \cap E$ cover $N(E) \setminus N(\bar{E})$ and also that both $c' \cap \bar{E}$ and $c \cap \bar{E}$ cover $N(\bar{E}) \setminus N(E)$. Thus, both $(c \cap E) \cup (c' \cap \bar{E})$ and $(c' \cap E) \cup (c \cap \bar{E})$ cover $\mathcal{S} \setminus S_E$. If one of these two covers of $\mathcal{S} \setminus S_E$ satisfied $\alpha x > \gamma(\alpha)$ then the other one would satisfy $\alpha x < \gamma(\alpha)$ which would contradict the criticality of S_E . Thus, they both satisfy $\alpha x = \gamma(\alpha)$. Thus, we have:

$$\begin{aligned} \alpha_E x^{c'} + \alpha_{\bar{E}} x^c &= \gamma(\alpha) \\ &= \alpha x^c \\ &= \alpha_E x^c + \alpha_{\bar{E}} x^c, \end{aligned}$$

which implies $\alpha_E x^{c'} = \alpha_E x^c$. Thus, $c \mapsto \alpha_E x^c$ is constant over all covers c in $C^=(\alpha)$. This means that $c \mapsto \alpha_{\bar{E}} x^c = \gamma(\alpha) - \alpha_E x^c$ is also constant. Thus, one can see that the vector $(\alpha_{\bar{E}} x^c)_{\alpha_E} - (\alpha_E x^c)_{\alpha_{\bar{E}}}$ is a tilting vector for $\alpha x \geq \gamma(\alpha)$. \square

In the proof above, we have shown that when a cutset S_E is not critical, then $\alpha_E x^c = \alpha_E x^{c'}$ for all $c, c' \in C^=(\alpha)$ and this enables us to highlight a tilting vector.

At this point, it is important to note that there are set covering instances where $\alpha_E x^c = \alpha_E x^{c'}$ for all $c, c' \in C^=(\alpha)$ but where for every $E' \subset \mathcal{E}$ the cutset $S_{E'}$ is critical.

Example. Let us consider a set covering instance with $\mathcal{E} = \{1, 2, 3, 4, 5, 6\}$, a set $E = \{1, 2, 3\}$ and $\alpha = \mathbf{1}$. The family of subsets \mathcal{S} of the set covering instance is $\mathcal{S} = \{s \subset \mathcal{E} \mid |s| = 3 \text{ and } |s \cap E| \neq 2\}$. There are two types of inclusion-wise minimal covers: the subset of \mathcal{E} of size 3 whose intersection with E has size 1; and the three covers formed of $\{1, 2, 3, e\}$ for $e \in \{4, 5, 6\}$. Among these inclusion-wise minimal covers, only the first type also has minimum size. Thus, all the minimal sized covers have an intersection with E of size 1 which means $\alpha_E x^c = \alpha_E x^{c'}$ for all $c, c' \in C^=(\alpha)$. However, every cutset is critical in this set covering instance. For instance, $\mathcal{S} \setminus S_E = \{\{1, 2, 3\}, \{4, 5, 6\}\}$, which can be covered with $\{1, 4\}$.

Thus, Proposition 9 can be extended as follows.

Proposition 10. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$ with $\alpha > 0$. If there exists a set $E \subset \mathcal{E}$ such that $\alpha_E x^c = \alpha_E x^{c'}$ for all $c, c' \in C^=(\alpha)$ then the inequality $\alpha x \geq \gamma(\alpha)$ is not a facet.*

5.2. Sufficient conditions of Sassano [1989]

In his work, Sassano [1989] presented a result (Lemma 3.1) giving a sufficient condition for the rank inequality $\mathbf{1}x \geq \gamma(\mathbf{1})$ to be a facet. We will discuss the underlying ideas of this lemma and generalize it to arbitrary inequalities. To that end, let us denote $N(e, e') = N(\{e\}) \cap N(\{e'\})$ the common neighbors to elements e and e' . The lemma of Sassano [1989] is built on the following notion of criticality.

Definition 3 (Critical graph, Sassano [1989]). *The critical graph G^* of the rank inequality $\mathbf{1}x \geq \gamma(\mathbf{1})$ has \mathcal{E} as node set and contains an edge $(e, e') \in \mathcal{E}^2$ if and only if $N(e, e')$ is critical i.e. $\gamma(\mathbf{1}) > \gamma(\mathbf{1}, \mathcal{S} \setminus N(e, e'))$.*

The lemma is then the following:

Lemma 1 (Sassano [1989]). *If the critical graph G^* is connected, then the inequality $\mathbf{1}x \geq \gamma(\mathbf{1})$ is a facet of $\mathcal{Q}(G)$.*

In order to generalize the lemma to arbitrary inequalities, let us introduce an adequate notion of criticality.

Definition 4 (Critical graph). *The critical graph G^* of the inequality $\alpha x \geq \gamma(\alpha)$ has \mathcal{E} as node set and contains an edge (e, e') between two elements for which $\alpha_e = \alpha_{e'}$ if and only the following equivalent conditions are verified :*

(i) $\gamma(\alpha) = \gamma(\alpha, \mathcal{S} \setminus N(e, e')) + \alpha_e$;

(ii) *there exists two covers $c \cup \{e\}$ and $c \cup \{e'\}$ in $C^=(\alpha)$.*

Equivalence. (i) \Rightarrow (ii) Let us call c a cover of $\mathcal{S} \setminus N(e, e')$ that achieves the value $\gamma(\alpha, \mathcal{S} \setminus N(e, e'))$. Then, $c \cup \{e\}$ and $c \cup \{e'\}$ both cover \mathcal{S} and satisfy $\alpha x^{c \cup \{e\}} = \alpha x^{c \cup \{e'\}} = \gamma(\alpha)$, which shows that condition (i) is satisfied.

(ii) \Rightarrow (i) Let $c \cup \{e\}$ and $c \cup \{e'\}$ be two covers of $C^=(\alpha)$. We know that c covers $\mathcal{S} \setminus N(\{e\})$ since $(c) \cup \{e\}$ is a cover. Similarly, it covers $\mathcal{S} \setminus N(\{e'\})$. Thus, c covers $\mathcal{S} \setminus N(e, e')$ and its incidence vector satisfies $\alpha x^c = \alpha x^{c \cup \{e\}} - \alpha_e = \gamma(\alpha) - \alpha_e$. Thus, we must have $\gamma(\alpha, \mathcal{S} \setminus N(e, e')) \leq \gamma(\alpha) - \alpha_e$. Since we also know that we always have $\gamma(\alpha, \mathcal{S} \setminus N(e, e')) \geq \gamma(\alpha) - \alpha_e$, we have proved that condition (i) is verified. \square

Note that the common neighbors to e and e' are critical when their removal induces a maximal decrease of the right-hand side from $\gamma(\alpha)$ to $\gamma(\alpha) - \alpha_e$. Indeed, we always have $\gamma(\alpha) \geq \gamma(\alpha, \mathcal{S} \setminus N(e, e')) \geq \gamma(\alpha) - \alpha_e$. Also, note that the critical graph can only contain edges between pairs of elements that share the same coefficients in the inequality. Indeed, the existence of two covers $c \cup \{e\}$ and $c \cup \{e'\}$ in $C^=(\alpha)$ implies that

$$\begin{aligned} \alpha_e - \alpha_{e'} &= \alpha x^{c \cup \{e\}} - \alpha x^{c \cup \{e'\}} \\ &= \gamma(\alpha) - \gamma(\alpha) \\ &= 0. \end{aligned}$$

We are now ready to generalize the lemma of Sassano [1989] as follows.

Lemma 2. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$ with $\alpha > 0$. If each connected component of its critical graph contains a cover of $C^=(\alpha)$ then it is a facet.*

Proof. The existence of two covers $c \cup \{e\}$ and $c \cup \{e'\}$ in $C^=(\alpha)$ means that for any tilting vector β , we have:

$$\begin{aligned}
\beta_e - \beta_{e'} &= \beta x^{c \cup \{e\}} - \beta x^{c \cup \{e'\}} \\
&= 0 - 0 \\
&= 0.
\end{aligned}$$

Hence, the presence of an edge in the critical graph implies that the two corresponding coefficients of any tilting vector are equal. Thus, the nodes in a connected component of the critical graph share the same coefficients in any tilting vector. If a connected component contains a cover c then $\sum_{e \in c} \beta_e = 0$ which means all the coefficients of the component are null since they are all equal. Thus, if each connected component contains a cover then the only tilting vector is the null vector and by Corollary 1, the inequality $\alpha x \geq \gamma(\alpha)$ induces a facet of $\mathcal{Q}(G)$. \square

Let us recall that an edge (e, e') in the critical graph can exist only when $\alpha_e = \alpha_{e'}$. Therefore, the connected components of the critical graph subdivide the sets of elements with equal coefficients. In some cases, one may not be able to prove that an inequality is a facet because not all connected components of the critical graph may contain a cover of $C^=(\alpha)$. However, even if only one component does, we can show that the coefficients of its elements in all tilting vectors are null. As shown in the following theorem, this has consequences on the facets obtainable from the original inequality $\alpha x \geq \gamma(\alpha)$. More precisely, the coefficients of the facet will be equal to the ones of α for these elements.

Theorem 2. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$ with $\alpha > 0$. If, for all tilting vectors β we have $\beta_e = 0$ for some element $e \in \mathcal{E}$, then all the non-trivial facets $\alpha' x \geq \gamma(\alpha)$ containing the face associated with $\alpha x \geq \gamma(\alpha)$ satisfy $\alpha'_e = \alpha_e$.*

Proof. Since we are considering non-trivial facets of the set covering polytope, the facet can be written $\alpha' x \geq \gamma(\alpha)$ through the right scaling. The vector $\alpha' - \alpha$ is a tilting vector since for each cover c in $C^=(\alpha)$ (which is included in $C^=(\alpha')$ by hypothesis) we have $(\alpha' - \alpha)x^c = \gamma(\alpha) - \gamma(\alpha) = 0$. Thus, we have $\alpha'_e - \alpha_e = (\alpha' - \alpha)_e = 0$ which means $\alpha'_e = \alpha_e$. \square

5.3. Inequalities with coefficients and right hand side in $\{0,1,2\}$

In their seminal paper, Balas and Ng [1989] studied all the inequalities for the set covering polytope with coefficients and right-hand side in $\{0,1,2\}$. In particular, they characterized the ones that induce facets of the set covering polytope. To understand their characterization, recall that $E_i(\alpha) = \{e \in \mathcal{E} | \alpha_e = i\}$ and let us introduce the 2-cover graph corresponding to an inequality $\alpha x \geq 2$. Its node set is $E_1(\alpha)$ and it contains an edge between e and e' when $E_0(\alpha) \cup \{e, e'\}$ is a cover (and thus belongs to $C^=(\alpha)$). Theorem 3 below states the characterization of Balas and Ng [1989] in a slightly different manner in order to fit the notations of this article. The three conditions are indexed with 0, 1, and 2 because they can be associated with the sets of all the coefficients 0, 1, and 2, respectively.

Theorem 3 (Balas and Ng [1989]). *Let $\alpha x \geq 2$ be a valid inequality for $\mathcal{Q}(G)$ with $\alpha \in \{0,1,2\}^{|\mathcal{E}|}$. It is a facet if and only if the following three conditions hold:*

0. *for each $e \in E_0(\alpha)$, there is a cover of $C^=(\alpha)$ not containing e ;*
1. *each connected component of the 2-cover graph contains an odd cycle (i.e. is not bipartite);*
2. *for each $e \in E_2(\alpha)$, $E_0(\alpha) \cup \{e\}$ is a cover.*

The above theorem characterizes facets corresponding to $\alpha x \geq 2$ with three conditions. We complement this result by characterizing the tilting vectors of these inequalities. In particular, each of the three conditions corresponds to a family of tilting vectors. In order to understand the tilting vector characterization, note that a classical result from graph theory is that a graph is bipartite if and only if it does not contain an odd cycle (see Theorem 1.2 in Bondy and Murty [1976]). Moreover, if a bipartite graph is connected, then it has a unique bipartition.

Theorem 4. *Let $\alpha x \geq 2$ be a valid inequality for $\mathcal{Q}(G)$ with $\alpha \in \{0, 1, 2\}^{\{E\}}$. A basis of the space of its tilting vectors is given by the juxtaposition of the following three families:*

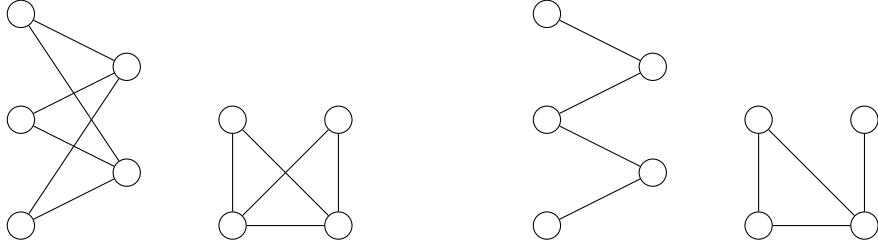
0. $\alpha - 2x^{\{e\}}$ for each $e \in E_0(\alpha)$ such that all covers of $C^=(\alpha)$ contain e ;
1. $x^U - x^V$ for each bipartite connected component of the 2-cover graph where U, V is the bipartition of the connected component;
2. $x^{\{e\}}$ for each $e \in E_2(\alpha)$ such that $E_0(\alpha) \cup \{e\}$ is a not cover.

We will now prove the characterization of tilting vectors, i.e., Theorem 4 which implies the theorem of Balas and Ng [1989] through Corollary 1. For this proof, we will use a lemma from the graph theory literature. For this lemma, we will define the notion of basis subgraph of an undirected graph $G = (U, E)$. An example of graph and basis subgraph is given in Figure 5. Note that the basis subgraph is not always uniquely defined. Let us denote G_1, \dots, G_n the subgraphs corresponding to the connected components of G . The basis subgraph has U as node set and contains a connected component G'_i for each G_i . If G_i is bipartite then the edges in G'_i are the edges of a spanning tree of G_i . If G_i is not bipartite then it contains an odd cycle c . Let us call e one of the edges of c . Then the edges of G'_i are composed of e and the edges of a spanning tree of G_i containing $c \setminus e$ (thus G'_i contains c). All of the components G'_i form the basis subgraph of G . In this lemma, we will not always differentiate an edge (u, v) from its incidence vector. This incidence vector has a null coefficient for each node of the graph except for u and v for which it has coefficient one.

Lemma 3. *Let $G = (U, E)$ be a graph. A basis of the linear span of the edges in E is given by the edges of a basis subgraph of G .*

For completion sake, we give a proof a Lemma 3 in Appendix B. We are now in a position to prove Theorem 4.

Proof. (Theorem 4) Let us denote N_β the number of tilting vectors mentioned in the theorem. We will prove that 1) these vectors are indeed tilting vectors of the inequality, 2) they are linearly independent and 3) they can be used to generate all tilting vectors.



(a) A graph with two connected components

(b) The corresponding basis subgraph

Figure 5: An undirected graph and its basis subgraph

1) If, for some $e \in E_2(\alpha)$, $E_0(\alpha) \cup \{e\}$ is a not cover, then, no cover of $C^=(\alpha)$ contains e . Thus, according to Proposition 6, $x^{\{e\}}$ is a tilting vector. Second, the vectors $\alpha - 2x^{\{e\}}$ are given by the Proposition 5. Third, note that the covers in $C^=(\alpha)$ take only two forms, $E_0 \cup \{e_2\}$ and $E_0 \cup \{e_1, e'_1\}$, where E_0 is a subset of $E_0(\alpha)$, e_2 belongs to $E_2(\alpha)$, and e_1, e'_1 belong to $E_1(\alpha)$. We want to show that for each of these covers c we have $(x^U - x^V)x^c = 0$. It is clear for the first type of covers since the support of $x^U - x^V$ and $x^{E_0 \cup \{e_2\}}$ are disjoint. For the second type, by definition of the 2-cover graph, (e_1, e'_1) is an edge of the 2-cover graph. If this edge is not contained in the connected component associated with $x^U - x^V$ then again the support of $x^U - x^V$ and $x^{E_0 \cup \{e_1, e'_1\}}$ are disjoint. Finally, if the edge (e_1, e'_1) is contained in the connected component associated with $x^U - x^V$ —since this connected component is bipartite—we either have $(e_1, e'_1) \in U \times V$ or $(e_1, e'_1) \in V \times U$. In both cases, we have $(x^U - x^V)x^c = -1 + 1 = 0$. Thus, $x^U - x^V$ is orthogonal to all covers of $C^=(\alpha)$ which makes it a tilting vector.

2) The tilting vectors in the second and third families are linearly independent as all their supports are disjoint. As for the first family of tilting vectors, they are also linearly independent of the others as each of them correspond to an element of $E_0(\alpha)$ for which they are the only one having a non-zero coefficient.

3) Let us now derive $|\mathcal{E}| - N_\beta$ linearly independent covers of $C^=(\alpha)$. Thanks to Proposition 2, this implies that the number of independent tilting vectors of $\alpha x \geq 2$ is less than N_β . This, in turn, means that the N_β tilting vectors of the theorem can be used to generate all the tilting vectors since they are independent. The covers will be separated into three families:

0. $c \cup (E_0(\alpha) \setminus e_0)$ for each $e \in E_0(\alpha)$ for which a cover c not containing e

exists;

1. $E_0(\alpha) \cup \{e_1, e'_1\}$ for each edge (e_1, e'_1) of the basis subgraph of the 2-cover graph;
2. $E_0(\alpha) \cup \{e_2\}$ for each $e_2 \in E_2(\alpha)$ for which $E_0(\alpha) \cup \{e_2\}$ is a cover.

Note that a basis subgraph has a number of edges equal to its number of nodes minus its number of bipartite connected components. Thus, one can see by pairing the above families of covers and the families of tilting vectors from the Theorem, that we have characterized $|E_0(\alpha)| + |E_1(\alpha)| + |E_2(\alpha)| = |\mathcal{E}|$ objects, covers and tilting vectors, in this proof. Hence, there is indeed a total of $|\mathcal{E}| - N_\beta$ covers of $C^=(\alpha)$ in the three previous families.

Let us now discuss the linear independence of the covers in the above three families. To that end, we will consider their incidence vectors as the columns of a matrix and show that this matrix is full column rank. If the first columns are the vectors of the family 0, then those of family 1 and then family 2, the matrix can be written as follows:

$$\begin{pmatrix} 1 - J_0 & 1 & 1 \\ Y_1 & X & 0 \\ Y_2 & 0 & J_2 \end{pmatrix}$$

In the previous matrix, J_0 is composed of an identity matrix of size η_0 — the size of the family 0 — on top of $|E_0(\alpha)| - \eta_0$ rows of zeros. J_2 is composed of an identity matrix of size η_2 — the size of the family 2 — on top of $|E_2(\alpha)| - \eta_2$ rows of zeros. Finally, X is the edge-node incidence matrix of the basis subgraph of the 2-cover graph. For each cover c associated with $e_0 \in E_0(\alpha)$ in family 0, the cover $c \cup \{e_0\}$ is either a cover of family 2 or it is equal to $E_0(\alpha) \cup \{e_1, e'_1\}$ for some edge (e_1, e'_1) of the 2-cover graph. In this second case, Lemma 3 tells us that the edge (e_1, e'_1) can be written as a linear combination of the edges in the basis subgraph. Note that the sum of the coefficients in the combination is 1. Thus, in both cases, the cover $c \cup \{e_0\}$ can be written as an affine combination of the covers in families 2 and 1. Therefore, by column manipulations, we can replace the previous matrix with the following matrix that has the same rank:

$$\begin{pmatrix} -J_0 & 1 & 1 \\ 0 & X & 0 \\ 0 & 0 & J_2 \end{pmatrix}$$

Since the matrices J_0 , J_2 , and X are full column rank, the complete matrix must also be. \square

6. Concluding remarks

In this work, we introduced a new mathematical object, the tilting vectors, which are derived from a variation of the tilting concepts introduced by Chvátal et al. [2013]. These vectors can be used to tilt set covering inequalities and provide tools to derive properties and proofs for facets of the set covering polytope. In particular, thanks to the tilting vectors, we were able to generalize some facet characterizations from rank inequalities to arbitrary inequalities. We also showed that the null coefficients in a set covering inequality can be treated separately. Indeed, one can study/tilt a set covering inequality by first ignoring its null coefficients. Special properties or tilting procedures can then be used to take them into account.

Although the study of the structure of the set covering problem has not received much recent attention, we believe that it remains an important topic of research. The set covering problem can be used to model any problem whose set of solutions X is monotonic ($x \in X$ and $x \leq y$ implies $y \in X$). This includes a wide variety of problems, such as covering problems (such as vertex covering, or feedback sets, for instance), packing problems (such as set packing, node packing, or independent sets), knapsack problems (single knapsack, multiple knapsack), and others. Thus, advances in understanding the structure of the set covering problem can be directly applied to multiple other optimization problems.

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Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Tilting for general polyhedra

In this section, we present the tilting procedure introduced by Chvátal et al. [2013]. Let $Q \subset \mathbb{R}^n$ be a polyhedron and let $\alpha x \geq \gamma(\alpha)$ be a valid inequality separating a point from Q and inducing a face F . The main idea of tilting is to rotate the inequality around its contact points Q in order to create a separating inequality inducing a face of dimension higher than the

one of F . This process can be repeated until a facet is obtained. Let us now formally introduce the concept of rotation of an inequality around a set of points.

Definition 5. Let $\pi x \geq \pi_0$ be an inequality satisfied at equality by a set of points X , i.e., $X \subseteq \{x \in \mathbb{R}^n : \pi x = \pi_0\}$. A rotation of $\pi x \geq \pi_0$ around X is another inequality $\mu x \geq \mu_0$ such that $X \subseteq \{x \in \mathbb{R}^n : \mu x = \mu_0\}$.

Now let \mathcal{Q}^\perp be the vector space defining the *implicit equations* of \mathcal{Q} , i.e., \mathcal{Q}^\perp is the set of the vectors q for which qx is constant over \mathcal{Q} . Also assume that we know a set $\{x^1, \dots, x^i\}$ of affinely independent vectors of the face F and a set $\{q_1, \dots, q_K\}$ of linearly independent vectors of \mathcal{Q}^\perp . They can both be empty initially and their size is denoted by I and K , respectively. By definition, a facet of \mathcal{Q} is a face of \mathcal{Q} that contains $\dim(\mathcal{Q})$ affinely independent points. Since $\dim(\mathcal{Q}) + \dim(\mathcal{Q}^\perp) = n$, if we can tilt the inequality $\alpha x \geq \gamma(\alpha)$ while extending the sets $\{x^1, \dots, x^i\}$ and $\{q_1, \dots, q_K\}$ until their sizes I and K sum to n , the resulting tilted inequality will be a facet of \mathcal{Q} .

Tilting the inequality $\alpha x \geq \gamma(\alpha)$ corresponds to rotating it around its contact points with \mathcal{Q} . Instead of considering all the possible degrees of freedom of rotation at once we will consider only one degree of freedom at a time and characterize it with a tilting direction. Formally, a tilting direction is a tuple $(\beta, \mu) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\beta x^i = \mu \quad \forall i \in \{1, \dots, I\}. \quad (\text{A.1a})$$

One can construct rotations of the inequality $\alpha x \geq \gamma(\alpha)$ by taking any linear combination of the inequalities $\alpha x \geq \gamma(\alpha)$ and $\beta x \geq \mu$. In particular, for a scalar λ , let us denote $I(\lambda, \alpha, \beta, \mu)$ the following rotation:

$$(\lambda\alpha + (1 - \lambda)\beta) x \geq \lambda\gamma(\alpha) + (1 - \lambda)\mu \quad (\text{A.2})$$

When no confusion is induced, we will simply use $I(\lambda)$ instead of $I(\lambda, \alpha, \beta, \mu)$. Using the above concepts and a procedure illustrated in Algorithm 1, one can tilt an inequality until it becomes a facet. In the remainder of this section, we detail each step of the procedure.

Step 0. Check whether $\alpha x \geq \gamma(\alpha)$ is an implicit equation of \mathcal{Q} : this can be done by maximizing αx over \mathcal{Q} . If the optimal value is $\gamma(\alpha)$,

Algorithm 1 Sketch of the tilting procedure

Step 0: Check whether $\alpha x \geq \gamma(\alpha)$ is an implicit equation of \mathcal{Q} ;
while $I + K < n$ **do** $\triangleright I = |\{x^1, \dots, x^i\}|$ and $K = |\{q_1, \dots, q_K\}|$
 Step 1: Obtain a tilting direction (β, μ) ;
 Step 2: Compute λ^* the smallest scalar such that $I(\lambda^*)$ is valid for \mathcal{Q} .
 At the same time a point $x^* \in \mathcal{Q}$ affinely independent from $\{x^1, \dots, x^i\}$ and satisfying $I(\lambda^*)$ to equality will be computed;
if $I(\lambda^*)$ is an implicit equation of \mathcal{Q} **then**
 Step 3a: add the coefficient vector $I(\lambda^*)$ to $\{q_1, \dots, q_K\}$;
else $\triangleright I(\lambda^*)$ is a proper face of \mathcal{Q}
 Step 3b: add x^* to $\{x^1, \dots, x^i\}$ and replace the inequality to be tilted by $I(\lambda^*)$

then $\alpha x \geq \gamma(\alpha)$ is an implicit equation. Otherwise, as a by-product, the maximization yields a point \bar{x} of \mathcal{Q} satisfying $\alpha \bar{x} > \gamma(\alpha)$. This makes \bar{x} affinely independent from the face F .

Step 1. Obtaining a tilting direction: to that end, let us consider a non-null solution (β, μ) of the following system:

$$\beta q_k = 0 \quad \forall k \in \{1, \dots, K\} \tag{A.3a}$$

$$\beta x^i = \mu \quad \forall i \in \{1, \dots, I\} \tag{A.3b}$$

$$\beta \bar{x} = \mu. \tag{A.3c}$$

Note that if the system above admits the null vector as a unique solution, then the inequality $\alpha x \geq \gamma(\alpha)$ is already a facet [Chvátal et al., 2013].

Step 2. Tilt the inequality: Upon finding a tilting direction (β, μ) , we find the smallest scalar λ^* such that the inequality $I(\lambda^*)$ remains a valid inequality for \mathcal{Q} . The resulting inequality is called the tilted inequality. Note that λ^* must be greater than 0 because for all $\lambda < 0$, $I(\lambda)$ is not valid for \bar{x} and thus not valid for \mathcal{Q} . An illustration of an original inequality, a tilting direction (β, μ) and a tilted inequality is provided in Figure A.6. This tilting step revolves around the following central result of the tilting routine.

Theorem 5. *The tilted inequality $I(\lambda^*)$ is satisfied to equality by a point x^* of \mathcal{Q} affinely independent from $\{x^1, \dots, x^i\}$.*

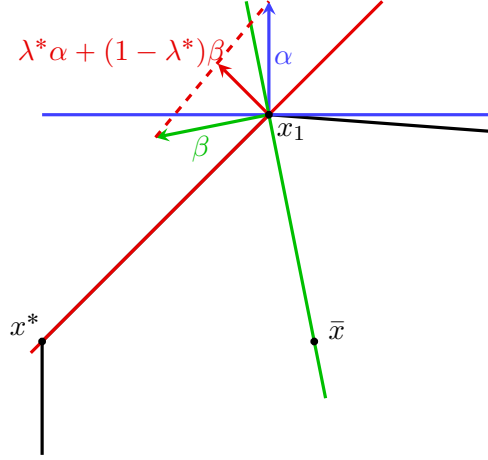


Figure A.6: A valid inequality (blue) for a polyhedron with normal vector α , a tilting direction β (green), and the corresponding tilted inequality (red).

Proof. Let us first exclude the case $\lambda^* = 0$ since in this case we can take $x^* = \bar{x}$.

Let $(\lambda_j)_{j \in \mathbb{N}}$ be a sequence converging towards λ^* with $\lambda^* > \lambda_j > 0$. By definition of λ^* , for every term in this sequence, the inequality $I(\lambda_j)$ is not valid for \mathcal{Q} . Thus, let us associate each λ_j with a point y_j of \mathcal{Q} not satisfying the inequality $I(\lambda_j)$. Let us note that since the inequality $I(\lambda_j)$ is not valid for \mathcal{Q} , either a vertex of \mathcal{Q} does not satisfy it, in which case, we set y_j equal to this vertex; or an extreme ray r of \mathcal{Q} exists such that $(\lambda_j \alpha + (1 - \lambda_j) \beta) r < 0$, in which case, we set y_j to be $x^i + r$ (which does not satisfy $I(\lambda_j)$ since x^i satisfies it to equality).

Now, since there is only a finite number of vertices and extreme rays of \mathcal{Q} , the sequence $(x_j)_{j \in \mathbb{N}}$ is a converging sequence over a finite set. Thus, it contains a constant sub-sequence. Let us define the candidate point x^* as the unique point in this constant sub-sequence and let us denote $(\lambda'_j)_{j \in \mathbb{N}}$ the associated sub-sequence of λ_j .

If x^* could be written as an affine combination of the points in $\{x^1, \dots, x^i\}$ then it would satisfy to equality all the inequalities $I(\lambda'_j)$. Since this is not true by construction of the sequence $(x^j)_{j \in \mathbb{N}}$ and thus by construction of x^* , the point x^* is affinely independent from $\{x^1, \dots, x^i\}$.

Let us now finish the proof by showing that x^* is satisfying the tilted inequality $I(\lambda^*)$ to equality. To that end let us denote $\delta(\lambda) = (\lambda \alpha + (1 -$

$\lambda)\beta)x^* - \lambda\gamma(\alpha) + (1 - \lambda)\mu$. Since the tilted inequality $I(\lambda^*)$ is valid for \mathcal{Q} we have $\delta(\lambda^*) \geq 0$. Moreover, since the sequence $(\lambda'_j)_{j \in \mathbb{N}}$ converges towards λ^* and x^* does not satisfy $I(\lambda'_j)$, the sequence $(\delta(\lambda'_j))_{j \in \mathbb{N}}$ is a sequence of negative numbers converging towards $\delta(\lambda^*)$. Thus, $\delta(\lambda^*) = 0$ which means that the point x^* satisfies the tilted inequality $I(\lambda^*)$ to equality. \square

The result of Theorem 5 relies on two properties of the point \bar{x} : $\bar{x} \in \mathcal{Q}$ and \bar{x} is affinely independent from $\{x^1, \dots, x^i\}$, which is implied by the fact that this point does not satisfy the inequality $\alpha x \geq \gamma(\alpha)$ to equality. To perform the computation of λ^* one can solve a non-linear program or make a sequence of calls to an oracle optimizing a linear function over \mathcal{Q} . These methods also provide the point x^* as a by-product. We refer to the works of Espinoza et al. [2010] and Chvátal et al. [2013] for a presentation on how to compute λ^* for general polyhedra. A tailored method for the set covering case is discussed in Section 3.4.

Step 3a. Case: $I(\lambda^*)$ is an implicit equation of \mathcal{Q} : this can be identified by maximizing $(\lambda^*\alpha + (1 - \lambda^*)\beta)x$ over \mathcal{Q} . If the result of this maximization equals $\lambda^*\gamma(\alpha) + (1 - \lambda^*)\mu$, then the tilted inequality is an implicit equation of \mathcal{Q} . Otherwise, it induces a proper face. Note that, when $I(\lambda^*)$ is an implicit equation, we have $\lambda^* = 0$ as otherwise \bar{x} can be shown to satisfy $\alpha\bar{x} = \gamma(\alpha)$ (which is by hypothesis false) using the following identities:

$$\begin{aligned} \alpha\bar{x} &= \frac{\lambda^*\alpha\bar{x}}{\lambda^*} \\ &= \frac{(\lambda^*\alpha + (1 - \lambda^*)\beta)\bar{x} - (1 - \lambda^*)\beta\bar{x}}{\lambda^*} \\ &= \frac{\lambda^*\gamma(\alpha) + (1 - \lambda^*)\mu - (1 - \lambda^*)\mu}{\lambda^*} \\ &= \gamma(\alpha). \end{aligned}$$

Thus, in this case the tilted inequality $I(\lambda^*)$ is in fact $\beta x \geq \mu$. Equations (A.3a) ensure that β is linearly independent from the set $\{q_1, \dots, q_K\}$. It can thus be added to it as a new independent implicit equation. The known dimensions of the space of implicit equations has been increased and one

can retry tilting the inequality $\alpha x \geq \gamma(\alpha)$ with a new solution of system (A.3a)-(A.3c) once it has been updated with the new implicit equation. The point \bar{x} can remain unchanged.

Step 3b. $I(\lambda^*)$ is a proper face of \mathcal{Q} : Since $I(\lambda^*)$ is a rotation of the original inequality, its induced face contains $\{x^1, \dots, x^i\}$. Moreover, Theorem 5 shows it also contains a new affinely independent point x^* . Thus, the point x^* can be added to $\{x^1, \dots, x^i\}$, effectively increasing the known dimensions of the induced face. One can continue the tilting procedure by tilting the inequality $I(\lambda^*)$. The only missing piece to continue the procedure is a point \bar{x} to define the system (A.3a)-(A.3c). If $\lambda^* \neq 0$, it can remain unchanged; otherwise \bar{x} can be taken as the maximizer of $(\lambda^*\alpha + (1 - \lambda^*)\beta)x$ over \mathcal{Q} which has already been computed to verify if $I(\lambda^*)$ is an implicit equation or a proper face of \mathcal{Q} .

Appendix B. Proofs of the unproven theorems propositions and lemmas

We now repeat Proposition 2 from Section 3.2 and present its proof.

Proposition 2. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$, let D_α be the dimension of its induced face, and let D_β be the dimension of the space of its tilting vectors. Then, $D_\alpha + D_\beta = |\mathcal{E}| - 1$.*

Proof. By definition, the dimension of the induced face is $N - 1$ where N is the number of affinely independent points of $\mathcal{Q}(G)$ that satisfy $\alpha x = \gamma(\alpha)$. Let M_α be the matrix whose rows are the covers in $C^=(\alpha)$, with $Im(M_\alpha)$ being its image, *i.e.*, $\{x \mid \exists y, x = M_\alpha y\}$, and $Ker(M_\alpha)$ being its kernel, *i.e.*, $\{x \mid M_\alpha x = 0\}$. We will show that $N = dim(Im(M_\alpha))$ and, since by definition the space of tilting vectors is $Ker(M_\alpha)$, we obtain from the rank theorem of matrices that $dim(Im(M_\alpha)) + D_\beta = |\mathcal{E}|$, which implies $D_\alpha + D_\beta = |\mathcal{E}| - 1$.

Thus, let us prove that $N = dim(Im(M_\alpha))$. First, the number of affinely independent points of $\mathcal{Q}(G)$ that satisfy $\alpha x = \gamma(\alpha)$ is also the number of affinely independent covers in $C^=(\alpha)$. Second, since the null vector does not satisfy $\alpha 0 = \gamma(\alpha)$, any affinely independent points of the hyperplane $\alpha x = \gamma(\alpha)$ are equivalently linearly independent. Thus, N is the number of linearly independent covers of $C^=(\alpha)$. This is exactly the dimension of $Im(M_\alpha^T)$ which is equal to $dim(Im(M_\alpha))$. \square

We now repeat the three theorems of Section 3.3 and present their proofs.

Proposition 4. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality for $\mathcal{Q}(G)$ such that, for each element $e \in \mathcal{E}$, there is at least one cover in $C^=(\alpha)$ that contains e , and one that does not. Let β be a tilting vector for that inequality. Then, there exist positive numbers ϵ^+ , ϵ^- such that the tilted inequalities*

$$(\alpha + \epsilon^+ \beta)x \geq \gamma(\alpha) \tag{B.1}$$

$$(\alpha - \epsilon^- \beta)x \geq \gamma(\alpha) \tag{B.2}$$

are both valid inequalities inducing faces of strictly higher dimension.

Proof. Sketch of the proof: we will show that, for $\epsilon \in [\epsilon_{min}, \epsilon_{max}]$, the inequality $(\alpha + \epsilon\beta)x \geq \gamma(\alpha)$ is valid for $\mathcal{Q}(G)$. Then we will show that with $\epsilon^- = -\epsilon_{min}$ and $\epsilon^+ = \epsilon_{max}$, we obtain faces of higher dimension.

We will make the proof only for ϵ^+ as the ϵ^- case is similar. Just note that the value of ϵ_{min} in the proof corresponding to ϵ^- would be the following:

$$\epsilon_{min} = \max_{x^c \in \mathcal{Q}(G), \beta x^c > 0} \frac{\gamma(\alpha) - \alpha x^c}{\beta x^c}$$

The inequality $(\alpha + \epsilon^+ \beta)x \geq \gamma(\alpha)$ is invalid for a cover c if and only if $\beta x^c < 0$ and $\epsilon^+ > \frac{\gamma(\alpha) - \alpha x^c}{\beta x^c}$. Thus, let us define ϵ_{max} as follows:

$$\epsilon_{max} = \min_{x^c \in \mathcal{Q}(G), \beta x^c < 0} \frac{\gamma(\alpha) - \alpha x^c}{\beta x^c}$$

ϵ_{max} positive: there are finitely many covers and for the covers for which $\gamma(\alpha) - \alpha x^c = 0$ we also have $\beta x^c = 0$. Thus, ϵ_{max} is the minimum over finitely many positive numbers which makes it positive.

ϵ_{max} finite: we now show that under the assumption that for each element $e \in \mathcal{E}$ there is a cover in $C^=(\alpha)$ that contains e and one that does not, ϵ_{max} is finite. Since $\beta \neq 0$, it has at least one non-zero coefficient. The corresponding element is contained in a cover of $C^=(\alpha)$ for which $\beta x^c = 0$. Thus, β has at least one other coefficient of the opposite sign and thus has at least one negative coefficient. Thus, for high enough values of ϵ^+ , the vector $\alpha + \epsilon^+ \beta$ has a negative coefficient for an element e' . There is at least one cover c in $C^=(\alpha)$ not containing e' , thus, $c \cap \{e'\}$ is a cover satisfying:

$$\begin{aligned} (\alpha + \epsilon^+ \beta)x^{c \cap \{e'\}} &= (\alpha + \epsilon^+ \beta)x^c + (\alpha + \epsilon^+ \beta)_{e'} \\ &= \gamma(\alpha) + \epsilon^+ 0 + (\alpha + \epsilon^+ \beta)_{e'} \\ &< \gamma(\alpha). \end{aligned}$$

Thus, for values of ϵ^+ where $\alpha + \epsilon^+\beta$ has a negative coefficient, the tilted inequality (2) is not valid. Thus, ϵ_{max} is finite as it is upper bounded by the largest value of ϵ such that $\alpha + \epsilon\beta \geq 0$.

Higher dimension face: let us take $\epsilon^+ = \epsilon_{max}$ and let c^* be a cover minimizing $\frac{\gamma(\alpha) - \alpha x^c}{\beta x^c}$ when $\beta x^c < 0$. We will show that the face of the tilted contains $C^=(\alpha) \cup \{c^*\}$. First, ϵ_{max} is defined so that every minimizer such as c^* will satisfy the tilted inequality to equality. On the other hand, the covers in $C^=(\alpha)$ also satisfy the tilted inequality (2) to equality as they are orthogonal to β (remember that the goal of tilting is to rotate the original inequality around its contact points with $\mathcal{Q}(G)$ in order to keep them in the face). Moreover, $\beta x^{c^*} \neq 0$ thus the cover c^* is affinely independent of the covers in $C^=(\alpha)$. Thus, the tilted inequality induces a face of strictly higher dimension. \square

Proposition 5. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality and e an element of \mathcal{E} . Let η be such that $\eta + \gamma(\alpha)$ is the minimum of αx^c over the set of covers c not containing e . The following propositions are equivalent:*

- (i) e is contained in all covers in $C^=(\alpha)$;
- (ii) $\alpha - \gamma(\alpha)x^{\{e\}}$ is a tilting vector;
- (iii) η is positive and $\alpha x \geq \gamma(\alpha)$ is the sum of $-\eta x_e \geq -\eta$ and $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$ with both inequalities being valid and the last one inducing a face of higher dimension.

Proof. (i) \Rightarrow (ii) If e is contained in all the covers in $C^=(\alpha)$ then for each cover in $C^=(\alpha)$, $(\alpha - \gamma(\alpha)x^{\{e\}})x = 0$. Thus, the vector $\alpha - \gamma(\alpha)x^{\{e\}}$ is a tilting vector.

(i) \Rightarrow (iii) Suppose that e is contained in all the covers in $C^=(\alpha)$. Then, all the covers c not containing e must satisfy $\alpha x^c > \gamma(\alpha)$. Thus, η is positive. Let us now show that the tilted inequality $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$ is valid for $\mathcal{Q}(G)$. For all $x \in \mathcal{Q}(G)$ we have $\alpha x \geq \gamma(\alpha)$. Thus, the incidence vector of each cover containing e satisfies $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$. Moreover, by definition of η , the incidence vector of each cover not containing e satisfies $\alpha x \geq \gamma(\alpha) + \eta$ and therefore also $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$. Thus, $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$ is valid for $\mathcal{Q}(G)$. Finally, let c^* be a cover that minimizes αx^c over the set of covers c not containing e . It must satisfy $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$ to equality. Note that all the covers c in $C^=(\alpha)$ also satisfy $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$ to equality as they contain e . Thus, the face associated with $\alpha x + \eta x_e \geq \gamma(\alpha) + \eta$ contains $C^=(\alpha) \cup \{c^*\}$. Since c^* contains e while the covers in $C^=(\alpha)$ do not, the

cover c^* is affinely independent of the covers in $C^=(\alpha)$. Thus, the tilted inequality induces a face of strictly higher dimension.

(iii) \Rightarrow (i) Suppose that $\alpha x \geq \gamma(\alpha)$ is the sum of $-\eta x_e \geq -\eta$ and another valid inequality with $\eta > 0$. Since both inequalities of the decomposition are valid for $\mathcal{Q}(G)$, then the incidence vector of $\mathcal{Q}(G)$ satisfying $\alpha x = \gamma(\alpha)$ (i.e., corresponding to covers of $C^=(\alpha)$) must satisfy both inequalities to equality. Thus, we have in particular $-\eta x_e = -\eta$ which implies $x_e = 1$ since $\eta > 0$. Thus, e is contained in all covers in $C^=(\alpha)$.

(ii) \Rightarrow (i) Suppose that $\alpha - \gamma(\alpha)x^{\{e\}}$ is a tilting vector. For each cover c in $C^=(\alpha)$, we have $\alpha x^c = \gamma(\alpha)$ and $(\alpha - \gamma(\alpha)x^{\{e\}})x^c = 0$. Thus, $\gamma(\alpha)x^{\{e\}}x^c = \gamma(\alpha)$ which means $x_e^c = 1$. \square

Proposition 6. *Let $\alpha x \geq \gamma(\alpha)$ be a valid inequality and e an element of \mathcal{E} . Let η be such that $\eta + \gamma(\alpha)$ is the minimum of αx^c over the set of covers c containing e . The following propositions are equivalent:*

- (i) e is not contained in any cover of $C^=(\alpha)$;
- (ii) $x^{\{e\}}$ is a tilting vector;
- (iii) η is positive and $\alpha x \geq \gamma(\alpha)$ is the sum of $\eta x_e \geq 0$ and $\alpha x - \eta x_e \geq \gamma(\alpha)$, with both inequalities being valid and the last one inducing a face of higher dimension.

Proof. (i) \Leftrightarrow (ii) e is not contained in any cover of $C^=(\alpha)$ if and only if for each cover in $C^=(\alpha)$, $x x^{\{e\}} = x_e = 0$ which is the definition of $x^{\{e\}}$ being a tilting vector.

(i) \Rightarrow (iii) Suppose that e is not contained in any cover of $C^=(\alpha)$. Then, all the covers c containing e must satisfy $\alpha x^c > \gamma(\alpha)$, thus η is positive. Let us now show that the tilted inequality $\alpha x - \eta x_e \geq \gamma(\alpha)$ is valid for $\mathcal{Q}(G)$. For all $x \in \mathcal{Q}(G)$ we have $\alpha x \geq \gamma(\alpha)$. Thus, the incidence vector of each cover not containing e satisfies $\alpha x - \eta x_e \geq \gamma(\alpha)$. Moreover, by definition of η , the incidence vector of each cover containing e satisfies $\alpha x \geq \gamma(\alpha) + \eta$ and thus also $\alpha x - \eta x_e \geq \gamma(\alpha)$. Thus, $\alpha x - \eta x_e \geq \gamma(\alpha)$ is valid for $\mathcal{Q}(G)$. Finally, let c^* be a cover that minimizes αx^c over the set of covers c containing e . It must satisfy $\alpha x - \eta x_e \geq \gamma(\alpha)$ to equality. Note that all the covers c in $C^=(\alpha)$ also satisfy $\alpha x - \eta x_e \geq \gamma(\alpha)$ to equality as they do not contain e . Thus, the face associated with $\alpha x - \eta x_e \geq \gamma(\alpha)$ contains $C^=(\alpha) \cup \{c^*\}$. Since c^* does not contain e while the covers in $C^=(\alpha)$ do, the cover c^* is affinely independent of the covers in $C^=(\alpha)$. Thus, the tilted inequality induces a face of strictly higher dimension.

(iii) \Rightarrow (i) Suppose that $\alpha x \geq \gamma(\alpha)$ is the sum of $\eta x_e \geq 0$ and another valid inequality with $\eta > 0$. Since both inequalities of the decomposition are valid for $\mathcal{Q}(G)$ then the incidence vector of $\mathcal{Q}(G)$ satisfying $\alpha x = \gamma(\alpha)$ (i.e. corresponding to covers of $C^=(\alpha)$) must satisfy both inequalities to equality. Thus, we have in particular $\eta x_e = 0$ which implies $x_e = 0$ since $\eta > 0$. Thus, e is not contained in any cover of $C^=(\alpha)$. \square

We now prove the Lemma 3 given in Section 5.3.

Lemma 3. *Let $G = (U, E)$ be a graph. A basis of the linear span of the edges in E is given by the edges of a basis subgraph of G .*

Proof. Recall that the incidence vector of an edge (u, v) has a null coefficient for each node of the graph except for u and v for which it has coefficient one.

We will separate the proof into two parts: 1) linear independence of the edges of the basis subgraph, 2) every edge of the main graph can be written as a linear combination of the edges of the basis subgraph.

1) We have linear independence of the edges of the different connected components as the edges in these groups have disjoint supports. We will thus analyze them separately.

Let us consider a connected graph. The edges of this graph are linearly independent if and only if the following linear system has only the null vector as solution. The system contains one variable for each edge and one equation for each node. The equation at a node asks for the sum of the variables of the incident edges to be equal to zero. Suppose one node u has exactly one incident edge e (u is a leaf of the graph). Then, the equation corresponding to u asks for the variable corresponding to e to be equal to zero. Setting this variable to zero equates to deleting it from the system and then the equation corresponding to u is trivially satisfied and can be removed too. The remaining system correspond to a graph similar to the initial one except u and e have been removed. Thus, when checking for the linear independence of the edge of a graph, the leaves of the graph can be removed.

Let us now consider a bipartite connected component of the main graph. The corresponding component in the basis subgraph is a spanning tree. Thus, the leaves can be removed iteratively until the whole tree is deleted thus showing that all the edges are linearly independent. In a similar fashion, for non-bipartite connected components, the leaves of the corresponding component in the basis subgraph can be removed until it is reduced to an odd cycle. The edges of an odd cycle are linearly independent (can be proved by induction on the size).

2) Let us show now that every edge of the main graph can be written as a linear combination of the edges of the basis subgraph. First, let us consider an edge e a bipartite component. The corresponding component in the basis subgraph is a spanning tree. Thus, either e is part of this tree or its addition creates a cycle. The created cycle has an even number of edges since the considered component of the main graph is bipartite. Thus, by assigning alternating $+1$ and -1 coefficients along the edges of the cycle, one can show that the edge e is linearly dependent of the edges of the spanning tree. Second, let us consider a non-bipartite connected components with k nodes. All the edges of the component have their endpoints inside the component, thus the linear span of these edges has dimension at most k . Since, the corresponding component in the basis subgraph contains k linearly independent edges, these edges generate all the edges of the component of the main graph. \square