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Nili Guttman-Beck, Avi Levin, Noam Malter, Michal Stern. Convert to Clique Method on Grid Hypergraphs. 2025. <hal-04873833>

HAL Id: hal-04873833

<https://hal.science/hal-04873833v1>

Preprint submitted on 8 Jan 2025

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Convert to Clique Method on Grid Hypergraphs

Nili Guttman-Beck * Avi Levin* Noam Malter* Michal Stern *†

January 5, 2025

Abstract

Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph, where V is a set of vertices and $\mathcal{S} = \{S_1, \dots, S_m\}$ is a set of clusters $S_i \subseteq V$ such that $\cup_{i=1}^m S_i = V$. The Clustered Spanning Tree by Trees problem determines whether a spanning tree exists on the complete graph on V , ensuring that each cluster induces a subtree. This paper focuses on grid hypergraphs, characterized by an intersection graph forming a grid, where no feasible solution exists.

We present the Convert to Clique method, where we choose a vertex that is contained in two clusters and insert it into all the other clusters. This method always generates a hypergraph with a feasible solution tree. In the case of grid hypergraphs, we prove that the Convert to Clique method guarantees a minimum number of insertions to gain feasibility.

1 Introduction

Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph, where V is a set of vertices and \mathcal{S} is a set of clusters (also known as hyperedges) S_1, \dots, S_m , with $\emptyset \neq S_i \subseteq V$ for $i \in \{1, \dots, m\}$, $\cup_{i=1}^m S_i = V$. The Clustered Spanning Tree by Trees problem, denoted by *CSTT*, is to decide whether there exists a tree-based tree support, which is a spanning tree in the complete graph on V , such that each cluster induces a subtree in the solution tree. When the clusters are pairwise disjoint, a feasible spanning tree always exists because we can span each cluster with a subtree and connect these subtrees using arbitrary edges. Therefore, we will focus on the problem when the clusters are not necessarily disjoint. See [1] for a detailed discussion on hypergraphs.

Given that most hypergraphs lack feasible solution trees, the issue of gaining feasibility holds significant importance. This paper concentrates on gaining feasible solution trees by inserting vertices into the clusters of given hypergraphs. Furthermore, we focus on a specific class of hypergraphs, where the intersection graph forms a grid.

An $n * m$ grid hypergraph, for $n \geq 2$ and $m \geq 2$, contains $n \cdot m$ clusters, such that its intersection graph forms a grid comprising n rows of nodes, with each row containing m nodes. The nodes in the intersection graph form a grid consisting of a $(n - 1) \cdot (m - 1)$ matrix of cycles, where each cycle comprises four nodes and shares two nodes with each neighboring cycle (see Example 2.3 and Figure 1 for an example of a grid hypergraph).

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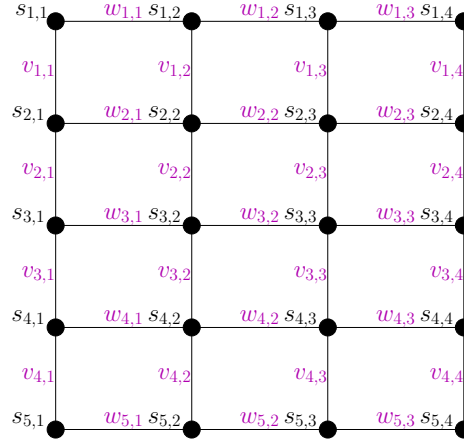


Figure 1: The intersection graph of a $5 * 4$ grid hypergraph, see Example 2.3. The nodes are labeled in black, while the vertices which constitute the intersections are marked in purple.

The primary concept presented in this paper is the introduction of the **Convert To Clique method**. This method involves selecting a vertex and inserting this vertex into all the clusters of the hypergraph that do not contain this vertex. It can be easily verified that this insertion creates a hypergraph with a feasible solution tree - a star with the inserted vertex as its center. The method name stems from the observation that, following the insertions, the intersection graph of the new hypergraph forms a clique.

The main result of this paper is that the minimum number of insertions required for a grid hypergraph is $nm - 2$. This outcome is surprising because, as will be shown later, for any hypergraph whose intersection graph forms a 4-node cycle, at least two insertions are necessary. However, in an $n * m$ grid intersection graph, there are $(n - 1) * (m - 1)$ such cycles, yet the required number of insertions, $nm - 2$, is smaller than the one obtained by applying two insertions for each 4-node cycle. For instance, in the hypergraph whose intersection graph is shown in Figure 1, there are 12 4-node cycles, but the required number of insertions is $nm - 2 = 18$ which is, of course, less than $2 * 12 = 24$.

This finding reveals the unexpected result that for a grid hypergraph, the Convert To Clique method generates a feasible insertion list with minimum cardinality, as long as the selected vertex is contained in two clusters of the given hypergraph.

Verifying whether a hypergraph has a feasible solution tree for *CSTT* can be performed in three manners. One approach to checking feasibility for *CSTT* is introduced independently by Duchet in [2], Flament in [4] and Slater in [12] and summarized by McKee and McMorris in [9]. This approach will be later presented in Theorem 2.9. Fagin in [3] and Tarjan and Yannakakis in [14] introduce a different approach of checking feasibility. A third approach, presented in [7], is based on finding a maximum spanning tree in a weighted version of the graph.

Considering *CSTT* with hypergraphs that do not have a feasible solution tree, Guttmann-Beck et al. in [7] give a characterization for inserting vertices to precisely one cluster in order to gain feasibility. Guttmann-Beck et al. in [6] present results considering gaining feasibility by removing vertices from clusters of a given hypergraph that lacks a feasible solution tree. Guttmann-Beck in [5] shows how to obtain a minimum cardinality feasible removal list or a minimum cardinality feasible insertion list of a given hypergraph by leveraging the smaller minimum cardinality feasible

removal or insertion lists from the subproblems created by decomposing the intersection graph into smaller instances.

We suggest three possible motivations for the *CSTT* problem. The first one comes from the area of communication networks and is presented in [13]. Given a complete graph, where vertices represent customers and edges represent links between customers, along with a collection of clusters of vertices, the objective is to construct a communication tree network. Each cluster induces a subtree in the solution tree, providing the network with group broadcast, group fault tolerance, and group privacy properties. When no feasible solution exists, customers may be inserted into more groups to achieve feasibility.

Another possible application of the *CSTT* problem arises from the area of databases with synchronous replications and is presented in [11]. Assume that we would like to construct a network comprising multiple databases, each employing synchronous replication. Each database copy resides in a network vertex, forming clusters, where a vertex may contain replicas of various databases, leading to non-disjoint clusters. The objective is to establish a spanning tree, ensuring that each database and its replicas form a connected subtree in the network. This facilitates seamless synchronization among different database copies. This task corresponds to the *CSTT* problem. When a feasible solution is unattainable, inserting certain database copies into vertices can be explored to achieve feasibility.

The third application arises from the area of key management for secure group communications and is presented in [10]. Suppose there are multiple non-disjoint user groups, and there is a need to create a communication network ensuring secure group communication for each. Periodically, distributing a new key among all group members is necessary. A solution for this, with minimal edge usage, is a feasible solution tree for the *CSTT* problem. This approach ensures that the subgraph induced by each group forms a connected subtree, ensuring privacy within the group. When a feasible solution is unattainable, inserting additional group members may be necessary to achieve feasibility.

The structure of this paper is as follows: Section 2 provides relevant definitions and previous results that serve as foundations for the discussion in this paper. Section 3 introduces hypergraphs with specific and basic intersection graph structures and presents results regarding the size of their feasible insertion lists. Section 4 formally presents the Convert to Clique method and establishes its minimality in gaining feasibility for grid hypergraphs.

2 Preliminaries

2.1 Definitions

This section includes some basic and important definitions that will be used throughout the paper. One of the definitions in this section is the definition of a feasible insertion list, one of the main concepts discussed in this paper. Using a feasible insertion list gains feasibility for a given hypergraph.

Definition 2.1 *Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph with vertex set V and $\mathcal{S} = \{S_1, \dots, S_m\}$, $S_i \subseteq V$ for $i \in \{1, \dots, m\}$, $\cup_{i=1}^m S_i = V$, a set of not necessarily disjoint clusters. The **intersection graph of H** , denoted by $\mathbf{G}_{int}(\mathcal{S})$, is defined to be a graph whose set of nodes is $\{s_1, \dots, s_m\}$, where s_i corresponds to $S_i \in \mathcal{S}$, and an edge (s_i, s_j) exists if and only if $S_i \cap S_j \neq \emptyset$.*

Note that in order to distinguish between the intersection graph before and after the insertion of vertices, we choose to represent an intersection graph as a function of \mathcal{S} .

Definition 2.2 An $n * m$ grid hypergraph, for $n \geq 2$ and $m \geq 2$, contains $n \cdot m$ clusters, such that its intersection graph forms a grid comprising n rows of nodes, with each row containing m nodes. The nodes in the intersection graph form a grid consisting of a $(n - 1) \cdot (m - 1)$ matrix of cycles, where each cycle comprises four nodes and shares two nodes with each neighboring cycle. Note that the intersection of every three clusters in a grid hypergraph is empty.

In an intersection graph of a grid hypergraph, we say that row i is **above** row j and row j is **below** row i , if $i < j$. Similarly, we say that cluster $S_{i,k}$ is **above** cluster $S_{j,k}$ and cluster $S_{j,k}$ is **below** cluster $S_{i,k}$ when $i < j$.

Example 2.3 Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph with vertex set $V = \{v_{i,j} : 1 \leq i \leq 4, 1 \leq j \leq 4\} \cup \{w_{i,j} : 1 \leq i \leq 5, 1 \leq j \leq 3\}$ and clusters set $\mathcal{S} = \{S_{i,j} : 1 \leq i \leq 5, 1 \leq j \leq 4\}$. The vertices and the clusters satisfy that $\{v_{i,j}\} = S_{i,j} \cap S_{i+1,j}$ and $\{w_{i,j}\} = S_{i,j} \cap S_{i,j+1}$. The intersection graph of H is demonstrated in Figure 1 where the black dots represent the nodes corresponding to the clusters, and the purple notations on the edges represent the vertices that constitute the intersection of the two clusters that correspond to the endpoints of the edge. For example $S_{1,1} = \{v_{1,1}, w_{1,1}\}$, $S_{1,2} = \{v_{1,2}, w_{1,1}, w_{1,2}\}$ and $S_{3,2} = \{v_{2,2}, v_{2,3}, w_{3,1}, w_{3,2}\}$. The intersection graph $G_{int}(\mathcal{S})$ is a $5 * 4$ grid, and therefore, the hypergraph is a $5 * 4$ grid hypergraph.

Definition 2.4 Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a family of sets. \mathcal{S} satisfies **the Helly Property** if the following holds: For every $\mathcal{S}' \subseteq \mathcal{S}$, if every pair of members of \mathcal{S}' has a common element, then all the members of \mathcal{S}' have a common element. In other words, if every $S_i, S_j \in \mathcal{S}'$ satisfy $S_i \cap S_j \neq \emptyset$, then $\bigcap_{S_i \in \mathcal{S}'} S_i \neq \emptyset$.

Remark 2.5 Since the Helly Property is trivially satisfied for any family of two subsets, we will only verify its validity for families with at least three subsets.

Definition 2.6 Given a graph G , a **chordless cycle** in G is a cycle with at least four nodes and no chords. A graph is **chordal** when it contains no chordless cycle.

Definition 2.7 Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph. **IL** is an **insertion list of H** if IL is a list of pairs: $IL = \{(v_1, S_{i_1}), \dots, (v_k, S_{i_k})\}$ with $v_j \notin S_{i_j}$, such that inserting vertex v_j to cluster S_{i_j} , for every pair $(v_j, S_{i_j}) \in IL$, creates a new hypergraph, denoted by $H + IL$. We also denote by $\mathcal{S} + IL$ the cluster set of the new hypergraph and by $G_{int}(\mathcal{S} + IL)$ the corresponding intersection graph.

If $H + IL$ has a feasible solution tree, we say that IL is a **feasible insertion list of H**. Refer to Example 2.8 for an example of a feasible insertion list.

Denote $mIL(H) = \min \{|IL| : IL \text{ is a feasible insertion list of } H\}$, the cardinality of such a list. Example 2.8 demonstrates an example of a minimum cardinality feasible insertion list.

Note that if a given hypergraph H has a feasible solution tree, it has an empty feasible insertion list and $mIL(H) = 0$.

Example 2.8 Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph with vertex set $V = \{1, 2, 3, 4\}$ and clusters set $\mathcal{S} = \{S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}\}$, where $S_{1,1} = \{1, 2\}$, $S_{1,2} = \{2, 3\}$, $S_{2,1} = \{1, 4\}$ and $S_{2,2} = \{3, 4\}$.

The intersection graph $G_{int}(\mathcal{S})$ is a chordless cycle of 4 nodes. As stated in Theorem 2.9, this hypergraph has no feasible solution tree.

Define $IL = \{(2, S_{2,1}), (3, S_{2,1}), (2, S_{2,2})\}$. In this case $\mathcal{S} + IL = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3, 4\}, \{2, 3, 4\}\}$. Path $1 - 2 - 3 - 4$ is a feasible solution for $H + IL$; therefore, IL is a feasible insertion list.

Define $IL2 = \{(2, S_{2,1}), (3, S_{2,1})\}$. In this case $\mathcal{S} + IL2 = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3, 4\}, \{3, 4\}\}$. As in the previous case, path $1 - 2 - 3 - 4$ is a feasible solution for $H + IL2$; therefore, $IL2$ is a feasible insertion list. As will be proven in Lemma 3.1, this is also a minimum cardinality feasible insertion list since any feasible insertion list must contain at least two insertions.

2.2 Previous results

This section contains definitions and previous results, which will be used throughout this paper. Theorem 2.9 gives sufficient and necessary conditions for the feasibility decision question, for a given hypergraph. Definitions 2.10 and 2.11 and Lemma 2.12 focus on induced hypergraphs and insertion lists. Definition 2.13, Notation 2.14, and Theorem 2.15 define and treat separating edges. Definition 2.16, Lemma 2.18 and Corollary 2.19 offer a way to create a new feasible insertion list with a smaller cardinality.

Theorem 2.9 ([2], [4], [12] and summarized in [9]) *A hypergraph $H = \langle V, \mathcal{S} \rangle$ has a feasible solution tree if and only if it satisfies the Helly Property and its intersection graph is chordal.*

Definition 2.10 *Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph and let $\mathcal{S}' \subseteq \mathcal{S}$ be a set of clusters. We define $H[\mathcal{S}']$ to be the induced hypergraph on \mathcal{S}' whose vertex set is $V(\mathcal{S}') = \bigcup_{S_i \in \mathcal{S}'} S_i$ and clusters set is \mathcal{S}' . Furthermore, the induced intersection graph on \mathcal{S}' is the intersection graph of $H[\mathcal{S}']$ and therefore can be denoted by $G_{int}(\mathcal{S}')$.*

Definition 2.11 *Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph. If $IL = \{(v_1, S_{i_1}), \dots, (v_k, S_{i_k})\}$ is an insertion list and $\mathcal{S}' \subseteq \mathcal{S}$ is a set of clusters, we define the induced insertion list $IL[\mathcal{S}']$ to be $\{(v, S_i) \mid (v, S_i) \in IL, v \in V, S_i \in \mathcal{S}'\}$.*

Denote by $IL[S_i] = \{v \mid (v, S_i) \in IL, v \in V\}$, the vertices inserted into S_i by IL .

The following lemma is stated and proven for the Clustered Spanning Tree by Paths, a restricted case of the Clustered Spanning Tree by Trees problem. Consequently, it can also be applied to the Clustered Spanning Tree by Trees problem discussed in our paper.

Lemma 2.12 ([8]) *Consider a hypergraph $H = \langle V, \mathcal{S} \rangle$. If IL is a feasible insertion list of H and $G_{int}(\mathcal{S}')$ is connected, for $\mathcal{S}' \subseteq \mathcal{S}$, then $IL[\mathcal{S}']$ is a feasible insertion list of $H[\mathcal{S}']$.*

Definition 2.13 (v_a, v_b) is a **separating edge** of a connected graph G if G contains edge (v_a, v_b) , such that deleting both nodes v_a and v_b from G results in a disconnected graph with ξ connected components, for $\xi \geq 2$. However, G remains connected if we delete only one of the nodes v_a or v_b .

Notation 2.14 *Consider a hypergraph $H = \langle V, \mathcal{S} \rangle$ with a connected intersection graph $G_{int}(\mathcal{S})$, which contains a separating edge (s_a, s_b) . Denote $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\xi$ the sets of clusters (sets of hyperedges) which correspond to the connected components of $G_{int}(\mathcal{S}) \setminus \{s_a, s_b\}$, and $H_j = H[\mathcal{S}_j \cup \{s_a, s_b\}]$, $j \in \{1, 2, \dots, \xi\}$.*

Theorem 2.15 ([5]) Consider a hypergraph $H = \langle V, \mathcal{S} \rangle$ whose intersection graph contains a separating edge (s_a, s_b) . For this hypergraph, $mIL(H) = \sum_{j=1}^{\xi} mIL(H_j)$.

Definition 2.16 Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph, IL an insertion list of H which includes a pair (v, S') with $S' \in \mathcal{S}$ and another vertex $u \in V$. We denote $IL_{v \rightarrow u}$ to be the new insertion list, created by replacing every appearance of v in IL by u . $IL_{v \rightarrow u} = (IL \setminus \{(v, S) \mid S \in SIL_v\}) \cup \{(u, S) \mid S \in SIL_v\}$ where $SIL_v = \{S \mid S \in \mathcal{S}, (v, S) \in IL\}$.

Note that, if there is a pair $(u, S) \in IL$ with $u \in S$, then inserting u into S according to $IL_{v \rightarrow u}$ does not change S .

Definition 2.17 Let $H = \langle V, \mathcal{S} \rangle$ be a hypergraph and $v \in V$. We denote $SC(v) = \{S \mid S \in \mathcal{S}, v \in S\}$, the set of clusters in \mathcal{S} which contain v .

Lemma 2.18 ([5]) Consider a hypergraph $H = \langle V, \mathcal{S} \rangle$ and IL a feasible insertion list of H . Consider a vertex $v \in V$ such that v is contained in a pair of IL . If there exists a vertex $u \in V$ with $SC(v) \subseteq SC(u)$ in H , then $IL_{v \rightarrow u}$ is a feasible insertion list of H .

Corollary 2.19 Consider a hypergraph $H = \langle V, \mathcal{S} \rangle$ and IL a minimum cardinality feasible insertion list of H . Consider a vertex $v \in V$ such that v is contained in a pair of IL . If there exists a vertex $u \in V$ with $SC(v) \subseteq SC(u)$ in H , then $IL_{v \rightarrow u}$ is a minimum cardinality feasible insertion list of H .

Proof: The inequality $|IL_{v \rightarrow u}| \leq |IL|$ holds because, for every pair (u, S) added to the list, a corresponding pair (v, S) is removed. The size of $IL_{v \rightarrow u}$ may be strictly smaller if there exists a cluster S such that both $(v, S) \in IL$ and either $(u, S) \in IL$ or $u \in S$. ■

3 Simple Structured Intersection Graphs

3.1 Chordless Cycles

In this section we prove the size of a minimum cardinality feasible insertion list when the intersection graph is a chordless cycle, or intersecting cycles which are arranged on a line, as shown in figure 2.

Lemma 3.1 Given a hypergraph $H = \langle V, \mathcal{S} \rangle$. If its intersection graph is a chordless cycle of $|\mathcal{S}|$ nodes, $|\mathcal{S}| \geq 4$, then $mIL(H) \geq |\mathcal{S}| - 2$ insertions.

Proof: According to the assumption of the lemma, the intersection graph G_{int} is a chordless cycle containing $|\mathcal{S}|$ nodes. The proof is by induction on $|\mathcal{S}|$.

Assume $|\mathcal{S}| = 4$. Consider C a cycle in the intersection graph, which contains four nodes, $C = s_1 - s_2 - s_3 - s_4 - s_1$.

According to Theorem 2.9, H has no feasible solution tree. Therefore, at least one chord has to be added to the intersection graph. Without loss of generality, suppose that this chord is edge $s_1 - s_3$. After the addition of this chord, the intersection graph contains two triangles, one containing

nodes s_1, s_2 and s_3 ; the other contains nodes s_1, s_3 and s_4 . Each set of clusters corresponding to a triangle must satisfy the Helly Property, and therefore requires an insertion. Note that it is possible that the insertion which added chord $s_1 - s_3$ also ensures that the Helly Property is satisfied for one of the triangles clusters sets. However, one insertion can not cause the Helly Property to be satisfied for both cluster sets. Therefore, at least two insertions to the cycle clusters are required. Note that we only count the insertions necessary to satisfy the Helly Property, not those required to add the chord. Therefore, the two required insertions indeed represent the minimum number needed. Furthermore, this proof also applies to cases where the cycle is part of the intersection graph, and the insertions use vertices outside the clusters which correspond to the cycle nodes.

Suppose the assumption of the lemma is correct for all hypergraphs (or induced hypergraphs) whose intersection graph contains a cycle of k nodes with $k \geq 4$ and $k < |\mathcal{S}|$. We prove it for $|\mathcal{S}|$. Consider C the chordless cycle, which is the intersection graph, containing $|\mathcal{S}| > 4$ nodes. According to Theorem 2.9, H has no feasible solution tree. Therefore, at least one chord $s_i - s_j$ has to be added to the intersection graph. After adding this chord, two cycles are created, C_1 containing k_1 nodes and C_2 containing k_2 nodes, with $|\mathcal{S}| = k_1 + k_2 - 2$. Without loss of generality, suppose that $k_1 \leq k_2$ which mean that $3 \leq k_1 < |\mathcal{S}|$ and $3 < k_2 < |\mathcal{S}|$. In $G_{int}(\mathcal{S}) \cup \{(s_i, s_j)\}$ chord (s_i, s_j) is a separating edge. According to Theorem 2.15, $mIL(H) = mIL(H[C_1]) + mIL(H[C_2])$. If $k_1 = 3$, then C_1 is a triangle. In this case, $k_1 - 2 = 1$ insertion is required to ensure that this triangle satisfies the Helly Property. Otherwise, $3 < k_1$ and according to the induction hypothesis, $mIL(H[C_1]) \geq k_1 - 2$. Similarly, $mIL(H[C_2]) \geq k_2 - 2$. Summing these two inequalities

$$mIL(H) = mIL(H[C_1]) + mIL(H[C_2]) \geq (k_1 - 2) + (k_2 - 2) = |\mathcal{S}| - 2.$$

Note that, as in the simple case of a cycle containing four nodes, it is possible that the insertion that added the first chord, may be an insertion that causes the Helly Property to be satisfied for a subset of clusters. For example, if the cycle contains nodes s_1, s_2, \dots, s_k and we add a vertex $v \in S_1 \cap S_2$ to S_3 , then a chord connecting s_1 and s_3 is added to the cycle. In this case $k_1 = 3$ and a triangle containing nodes s_1, s_2 and s_3 is created. The Helly Property is satisfied for this triangle, as $v \in S_1 \cap S_2 \cap (S_3 \cup IL[S_3])$. Inserting v into S_3 ensures that the Helly Property is satisfied and adds the chord $s_1 - s_3$. Since this insertion should only be counted once when determining the minimum number of required insertions, it is counted solely as the insertion needed to satisfy the Helly Property. ■

Definition 3.2 *A graph is a line of l intersecting cycles if it contains a spine path s_1, s_2, \dots, s_n and a set of l cycles, where each cycle is of the form $s_i, s_{i+1}, \dots, s_j, t, s_i$, containing exactly one node outside the spine path. Figure 2 shows a graph which is a line of 4 intersecting cycles.*

Lemma 3.3 *Given a hypergraph $H = \langle V, \mathcal{S} \rangle$ whose intersection graph is a line of l intersecting cycles, with $S \cap S' \cap S'' = \emptyset$ for every 3 different clusters S, S' and S'' , then $mIL(H) \geq |\mathcal{S}| - 2$ insertions.*

Proof: Consider Figure 2 for an intersection graph which is a specific case of a line of $l = 4$ intersecting cycles.

The proof is by induction on l . For $l = 1$, according to Lemma 3.1, the number of required insertions is at least $|\mathcal{S}| - 2$ insertions.

Suppose the assumption of the lemma is correct for $l - 1$. We prove it for l . Consider the hypergraph $H' \langle V', \mathcal{S}' \rangle$ induced on the $l - 1$ first cycles on the line, where the order of the cycles is

defined according to their last node on the spine path. If multiple cycles share the same final node along this path, select an arbitrary order among these cycles.

According to the induction hypothesis, the claim is true for H' and $mIL(H') \geq |\mathcal{S}'| - 2$. Let C' be the last cycle along the line, let k be the number of nodes in C' which are also contained in the spine path, and k_b the number of nodes shared by C' and $G_{int}(\mathcal{S}')$ in the spine path. In Figure 2 we have $k = 4$ and $k_b = 3$. Let t be the node of C' which is outside the spine line. There are two options:

1. t is also contained in one of the $l - 1$ first cycles. In this case $|\mathcal{S}| = |\mathcal{S}'| + k - k_b$. The last vertex of C' may be shared with a previous cycle, in this case $k = k_b$ and $|\mathcal{S}| = |\mathcal{S}'|$. Therefore, any feasible insertion list of H is also a feasible insertion list of H'' and $mIL(H) = mIL(H') \geq |\mathcal{S}'| - 2 = |\mathcal{S}| - 2$.

Otherwise, after implying the insertions to the clusters of $G_{int}(\mathcal{S}')$, the two ends of the shared path may be connected by an edge, thus gaining a chord in C' , reducing the size of C' to contain $k - k_b + 2$ nodes. If $k - k_b + 2 = 3$, to satisfy the Helly Property, which is necessary, and since the intersection of every three clusters of the hypergraph is empty, according to Theorem 2.9, at least $k - k_b = 1$ additional insertion is required. If $k - k_b + 2 \geq 4$, then according to Lemma 3.1, at least $(k - k_b + 2) - 2 = k - k_b$ more insertions are required to gain feasibility for the hypergraph induced on C' . In any case, $mIL(H) \geq mIL(H') + (k - k_b) \geq |\mathcal{S}'| - 2 + (k - k_b) = |\mathcal{S}| - 2$.

2. t is unique to C' . In this case $|\mathcal{S}| = |\mathcal{S}'| + k - k_b + 1$. It is possible that the last vertex of C' is shared with a previous cycle, in this case $k = k_b$ and $|\mathcal{S}| = |\mathcal{S}'| + 1$.

The proof for this case is very similar to the previous one and is omitted for the sake of readability.

■

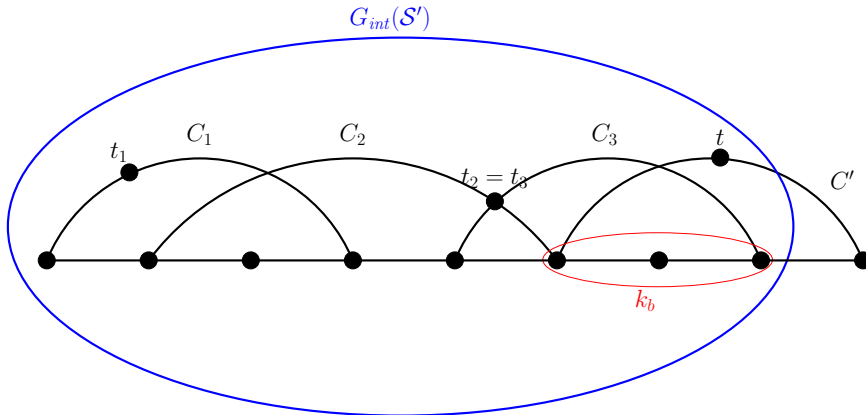


Figure 2: Figure for Definition 3.2 and Lemma 3.3.

3.2 $n * 2$ Grid Hypergraphs

In this section we prove that a minimum cardinality feasible insertion list of a $n * 2$ grid hypergraph contains $2n - 2$ insertions.

Theorem 3.4 Given $H = \langle V, \mathcal{S} \rangle$ a $n * 2$ grid hypergraph for $n \geq 2$, then $mIL(H) = 2n - 2$.

Proof: The proof is by induction of n .

When $n = 2$, the intersection graph is a cycle of 4 nodes $s_l^1 - s_r^1 - s_r^2 - s_l^2$, whose corresponding clusters are S_l^1, S_r^1, S_r^2 and S_l^2 , see Figure 3(a). According to Lemma 3.1, $mIL(H) \geq 2$.

Choose a vertex $v \in S_l^1 \cap S_l^2$ and let $IL_v = \{(v, S_r^1), (v, S_r^2)\}$. A star whose center is v is a feasible solution tree for $H + IL$. Thus, IL is a feasible insertion list with cardinality 2.

Combining the above two results yields $mIL(H) = 2$.

Suppose the claim is correct for $n - 1$ and prove it for n .

Suppose the nodes in $G_{int}(\mathcal{S})$ correspond to clusters $S_l^1, \dots, S_l^k, S_r^1, \dots, S_r^k$, where the left nodes correspond to S_l^1, \dots, S_l^k and the right nodes correspond to S_r^1, \dots, S_r^k , see Figure 3(b).

Let $\mathcal{S}_1 = \{S_l^1, \dots, S_l^{k-1}, S_r^1, \dots, S_r^{k-1}\}$ and let $H_1 = H[\mathcal{S}_1]$. Let $\mathcal{S}_2 = \{S_l^{k-1}, S_r^{k-1}, S_r^k, S_l^k\}$ and let $H_2 = H[\mathcal{S}_2]$.

H_1 is a $(n - 1) * 2$ grid hypergraph. According to the induction hypothesis $mIL(H_1) = 2(n - 1) - 2$. H_2 is a $2 * 2$ grid hypergraph. According to the basic step of the proof, $mIL(H_2) = 2$. According to Definition 2.13, (S_l^{k-1}, S_r^{k-1}) is a separating edge in $G_{int}(\mathcal{S})$ and according to Theorem 2.15, $mIL(H) = mIL(H_1) + mIL(H_2) = (2(n - 1) - 2) + 2 = 2n - 2$. ■

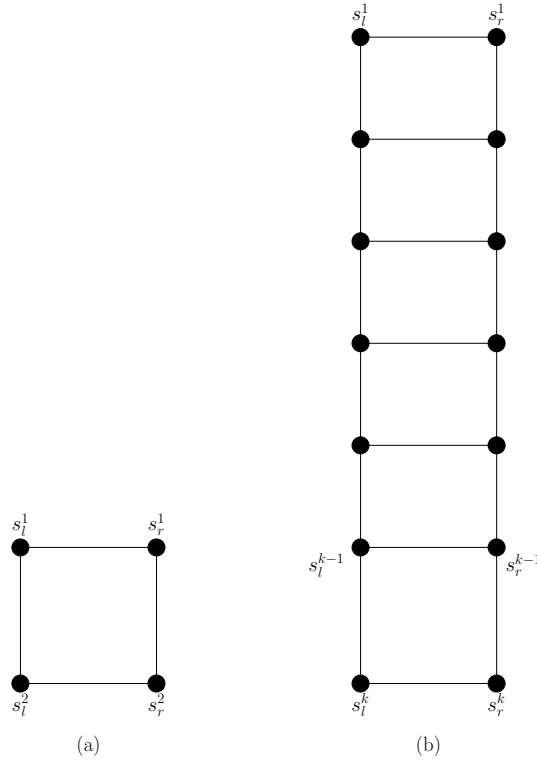


Figure 3: Figure for Theorem 3.4.

4 Convert To Clique Method

This section provides the main result of this paper. It defines and proves the minimality of the simple Convert to Clique method for grid hypergraphs.

To prove the minimality of the Convert to Clique method, we show that the minimum cardinality of a feasible insertion list for an $n * m$ grid hypergraph is $nm - 2$. The proof proceeds by induction on the number of columns m . Starting with a minimal insertion list, we compare it to the list induced on the leftmost $m - 1$ columns. The difference consists of insertions to the clusters in the rightmost column, that do not appear in the induced list, and additional insertions to the clusters in the $m - 1$ column that can be replaced by other insertions, reducing the cardinality of the induced list while preserving feasibility. This produces a feasible list, for the hypergraph induced on the leftmost $m - 1$ columns, that is smaller by at least n insertions compared to the original list. Combined with the induction hypothesis, this leads to the desired result.

We therefore focus on the $n * 2$ induced grid hypergraph created when considering only the rightmost two columns of the $n * m$ grid. We consider the $n * 2$ grid and divide its cluster set into two subsets: *LEFT* and *RIGHT*, according to their location on the grid. As will be shown later, we divide these two columns according to the number and type of vertices from clusters in *RIGHT* inserted into clusters in *LEFT*.

We start by defining the **Convert to Clique method**. Consider a hypergraph $H = \langle V, \mathcal{S} \rangle$, choose a vertex $v \in V$ and create the insertion list $CCIL_v = \{(v, S) | S \notin SC(v)\}$. The method derives its name from the observation that the intersection graph $G_{int}(\mathcal{S} + CCIL_v)$ is a clique.

Lemma 4.1 *Consider a hypergraph $H = \langle V, \mathcal{S} \rangle$. For every $v \in V$, the insertion list $CCIL_v$, created according to the Convert to Clique method, is a feasible insertion list.*

Proof: In $H + CCIL_v$, vertex v is contained in all the clusters. Therefore, the star whose center is v is a feasible solution tree for $H + CCIL_v$. ■

Corollary 4.2 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph. For every $v \in S \cap S'$, where S and S' are two distinct clusters, the feasible insertion list $CCIL_v$, created according to the Convert to Clique method, has cardinality $nm - 2$ and therefore $mIL(H) \leq nm - 2$.*

As stated before, we focus on the $n * 2$ grid hypergraph induced on the two rightmost columns of the grid. We consider a feasible insertion list and count the number of insertions on the $n * 2$ grid from one side to the other side.

Definition 4.3 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph.*

*Let **LEFT** represent the set of clusters corresponding to the nodes in the left column of the two rightmost columns in the grid intersection graph. Similarly, let **RIGHT** denote the set of clusters corresponding to the right column of the same two rightmost columns in the grid intersection graph. Furthermore, **LEFTGRID** denotes the set of clusters corresponding to the $m - 1$ leftmost columns of the grid. Note that all the clusters in **LEFT** are also included in **LEFTGRID**.*

*B is a **block** if it is an induced $k * 2$ grid hypergraph, with $k \leq n$, that forms part of the two rightmost columns of the grid. Let **LEFT(B)** be the cluster set on its left side, and let **RIGHT(B)** be the cluster set on its right side.*

For a cluster $S \in \text{RIGHT}$ we denote by $\text{LEFT}(S)$ the cluster in LEFT which is the neighbor of S . Similarly, for a cluster $S \in \text{LEFT}$ we denote by $\text{RIGHT}(S)$ the cluster in RIGHT which is the neighbor of S .

A vertex v is a **Right to Left** vertex with respect to an insertion list IL , denoted by **RTL**, if IL contains pair (v, S_l) , where $v \in S \cap S'$ for $S \in \text{RIGHT}$, $S' \in \text{RIGHT}$, and $S_l \in \text{LEFT}$.

A vertex v is a **Semi Right to Left** vertex with respect to an insertion list IL , denoted by **SRTL**, if IL contains pair (v, S_l) , where $v \in S \cap S'$ for $S \in \text{RIGHT}$, $S' \in \text{LEFT}$, and $S_l \in \text{LEFT}$.

Example 4.4 (cont. of Example 2.3) Consider the hypergraph presented in Example 2.3 and the corresponding intersection graph demonstrated in Figure 1. In this hypergraph, $\text{LEFT} = \{S_{i,3} : 1 \leq i \leq 5\}$, $\text{RIGHT} = \{\{S_{i,4} : 1 \leq i \leq 5\}$ and $\text{LEFTGRID} = \{\{S_{i,j} : 1 \leq i \leq 5, 1 \leq j \leq 3\}$.

Suppose an insertion list IL contains pair $(v_{1,4}, S_{1,3})$. Since $v_{1,4} \in S_{1,4} \cap S_{2,4}$ with $S_{1,4} \in \text{RIGHT}$, $S_{2,4} \in \text{RIGHT}$ and $S_{1,3} \in \text{LEFT}$, it follows that $v_{1,4}$ is an **RTL** vertex.

Similarly, suppose that IL contains pair $(w_{2,3}, S_{1,3})$. Since $w_{2,3} \in S_{2,4} \cap S_{2,3}$ with $S_{2,4} \in \text{RIGHT}$, $S_{2,3} \in \text{LEFT}$ and $S_{1,3} \in \text{LEFT}$, it follows that $w_{2,3}$ is an **SRTL** vertex.

4.1 Clean Blocks

This section defines a clean block. Given a feasible insertion list, a clean block has no insertion from RIGHT to LEFT . Furthermore, this section contains proofs regarding the number of insertions to RIGHT in clean blocks.

Definition 4.5 Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, let B be a block which is an induced $k * 2$ hypergraph in H , with $k \leq n$, that forms part of the two rightmost columns of the grid. Let IL be an insertion list of H . B is a **clean block with respect to IL** if $IL[B]$ contains no **SRTL** and no **RTL** vertices and IL contains no **RTL** or **SRTL** vertices contained in clusters of $\text{RIGHT}(B)$.

Lemma 4.6 Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph and let IL be a feasible insertion list of H . If B is a $1 * 2$ clean block with respect to IL , then there is at least one insertion to RIGHT associated with this block. Note that it is an insertion to RIGHT and not necessarily $\text{RIGHT}(B)$.

Proof: Consider S_r which is the only cluster at $\text{RIGHT}(B)$. Since the node corresponding to S_r belongs to at least one cycle of four nodes, according to Theorem 2.9, at least one chord must be added to this cycle, causing S_r to belong to a newly created triangle. According to Theorem 2.9, a vertex should be contained in all three clusters, after the insertion performed according to IL . Thus, either there is a vertex v inserted into S_r , or there is a vertex u inserted from S_r . If v is inserted into S_r , then this is an insertion to $\text{RIGHT}(B) \subseteq \text{RIGHT}$. If $u \in S_r$ is the vertex which is inserted from S_r , if cannot be inserted into a cluster in LEFT , as this would make it an **SRTL** or an **RTL** vertex, contradicting the assumption that S_r belongs to a clean Block. Hence, u is inserted into a cluster in RIGHT . ■

Lemma 4.7 Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be a feasible insertion list of H . If B is a $2 * 2$ clean block with respect to IL , then there are at least 2 insertions to $\text{RIGHT}(B)$.

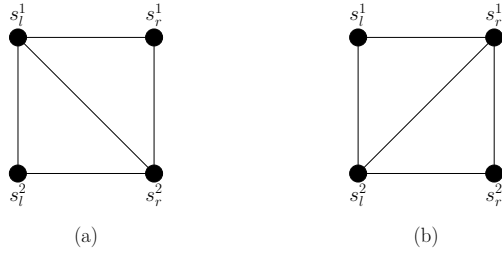


Figure 4: Figure for Lemma 4.7.

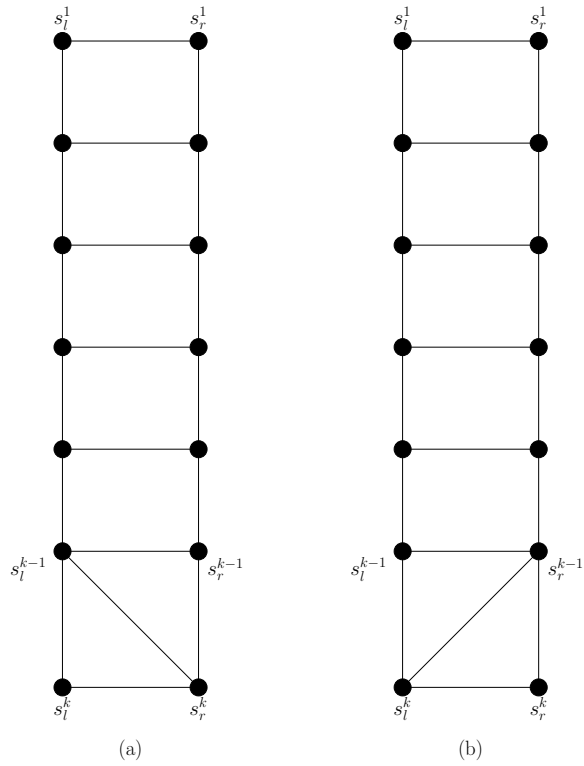


Figure 5: Figure for Lemma 4.8.

Proof: The intersection graph of B is a cycle with four nodes. Let S_l^1, S_l^2, S_r^1 and S_r^2 be the corresponding clusters. Since IL is a feasible insertion list of H , according to Theorem 2.9, at least one chord is added to the intersection graph, creating two triangles. Consider Figure 4(a) and (b) for the possible chords. Note that both chords may be added, but at least one of them must be added.

- Suppose a chord connecting the nodes that correspond to S_l^1 and S_r^2 is added, see Figure 4(a).

Consider triangle $s_l^1 - s_r^1 - s_r^2$, whose nodes correspond to clusters S_l^1, S_r^1 and S_r^2 . According to Theorem 2.9, in $B + IL$ there is a vertex v which is contained in $(S_l^1 \cup IL[S_l^1]) \cap (S_r^1 \cup IL[S_r^1]) \cap (S_r^2 \cup IL[S_r^2])$.

- If v is not contained in any of the clusters S_l^1, S_r^1 or S_r^2 , then it is inserted into S_r^1 and S_r^2 and creates 2 insertions to $RIGHT(B)$.
- If v is contained in exactly one of the clusters S_l^1, S_r^1 or S_r^2 , then it is inserted into at least one of the clusters S_r^1 or S_r^2 , and creates at least one insertion to $RIGHT(B)$.
- If v is contained in exactly two of the clusters S_l^1, S_r^1 or S_r^2 , It cannot be contained in $S_r^1 \cap S_r^2$, since in this case it would have been inserted into S_l^1 , contradicting the assumption that B is a clean block. Therefore, v is not contained in at least one of the clusters S_r^1 or S_r^2 and is inserted into this cluster, creating one insertion to $RIGHT(B)$.

To summarize, there is at least one insertion to $RIGHT(B)$ according to this triangle.

Consider triangle $s_l^1 - s_l^2 - s_r^2$, whose nodes correspond to clusters S_l^1, S_l^2 and S_r^2 . According to Theorem 2.9, in $B + IL[B]$ there is a vertex u which is contained in $(S_l^1 \cup IL[S_l^1]) \cap (S_l^2 \cup IL[S_l^2]) \cap (S_r^2 \cup IL[S_r^2])$. This vertex cannot be from S_r^2 , because, in this case, it is inserted into at least one of S_l^1 or S_l^2 , contradicting the assumption that B is a clean block. Therefore, u is inserted into S_r^2 , creating one insertion to $RIGHT(B)$ according to this triangle.

- Suppose a chord connecting the nodes that correspond to S_l^2 and S_r^1 is added, see Figure 4(b).

Consider triangle $s_l^2 - s_r^1 - s_r^2$, whose nodes correspond to clusters S_l^2, S_r^1 and S_r^2 .

Similar to the proof of the previous case considering triangle $s_l^1 - s_r^1 - s_r^2$, it can be shown that there is at least one insertion to $RIGHT(B)$ according to this triangle.

Consider triangle $s_l^1 - s_l^2 - s_r^1$, whose nodes correspond to clusters S_l^1, S_l^2 and S_r^1 . A proof similar to the one offered for triangle $s_l^1 - s_l^2 - s_r^2$ also applies for this triangle, confirming that there is one insertion to $RIGHT(B)$ according to this triangle. ■

Lemma 4.8 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be a feasible insertion list of H . If B is a $k * 2$ clean block with respect to IL , for $k \geq 2$, then IL contains at least k insertions to $RIGHT(B)$.*

Proof:

Suppose the nodes in B correspond to clusters $S_l^1, \dots, S_l^k, S_r^1, \dots, S_r^k$.

The proof is by induction on k . For $k = 2$, according to Lemma 4.7, there are at least two insertions to $RIGHT(B)$.

Suppose the claim is correct for $k - 1$, and we will prove it for k . Let B' be the block containing clusters $S_l^1, \dots, S_l^{k-1}, S_r^1, \dots, S_r^{k-1}$. Let $IL' = IL[B']$ be the induced insertion list on B' . According to Lemma 2.12, since IL is a feasible insertion list of H , IL' is feasible for B' . According to the induction hypothesis, IL' contains at least $k - 1$ insertions to $RIGHT(B')$.

In the cycle, whose nodes correspond to clusters $S_l^{k-1}, S_r^{k-1}, S_r^k$ and S_l^k there are 4 nodes. According to Theorem 2.9, at least one chord has to be added to this cycle. There are two options for the chord:

- The chord connecting the nodes that correspond to S_l^{k-1} and S_r^k is added, see Figure 5(a). Consider triangle $s_l^{k-1} - s_l^k - s_r^k$, whose nodes correspond to clusters S_l^{k-1}, S_l^k and S_r^k . According to Theorem 2.9, in $B + IL[B]$, the Helly Property should be satisfied in this triangle. Therefore, after the insertion, there is a vertex v which is contained in $(S_l^{k-1} \cup IL[S_l^{k-1}]) \cap (S_l^k \cup IL[S_l^k]) \cap (S_r^k \cup IL[S_r^k])$. Vertex v cannot be contained in S_r^k , since in this case, it is inserted into S_l^{k-1} , contradicting the assumption that B is a clean block with respect to IL . Therefore, vertex v is inserted into S_r^k . This insertion, combined with $k - 1$ insertions to $RIGHT(B')$ which exist according to the induction hypothesis, results in a total of at least k insertions to $RIGHT(B)$.
- The chord connecting the nodes that correspond to S_l^k and S_r^{k-1} is added, see Figure 5(b). Consider triangle $s_r^{k-1} - s_r^k - s_l^k$, whose nodes correspond to clusters S_r^{k-1}, S_r^k and S_l^k . According to Theorem 2.9, in $B + IL$ there is a vertex v which is contained in $(S_r^{k-1} \cup IL[S_r^{k-1}]) \cap (S_r^k \cup IL[S_r^k]) \cap (S_l^k \cup IL[S_l^k])$.
 - Suppose that vertex v is not contained in any of the clusters S_r^{k-1}, S_r^k or S_l^k . In this case, it is inserted into S_r^k . This insertion, combined with the insertions to $RIGHT(B')$ by IL' , results in a total of at least k insertions to $RIGHT(B)$.
 - Suppose that vertex v is contained in exactly one of the clusters S_r^{k-1}, S_r^k or S_l^k . However, v cannot be contained in either S_r^k or S_r^{k-1} . Otherwise, it would be inserted into S_l^k , which would contradict the assumption that B is a clean block with respect to IL . Therefore, v is contained in S_l^k and is inserted into S_r^k . This insertion and the insertions to $RIGHT(B')$ by IL' form at least k insertions to $RIGHT(B)$.
 - Suppose that vertex v is contained in exactly two of the clusters S_r^{k-1}, S_r^k or S_l^k . However, v cannot be contained in $S_r^{k-1} \cap S_r^k$, otherwise it would have been inserted into S_l^k , which would contradict the assumption that B is a clean block with respect to IL . Therefore, v is contained in $S_r^k \cap S_l^k$ and is inserted into S_r^{k-1} .

Let u be a vertex which satisfy $u \in S_r^{k-1}$. Consider $IL'_{v \rightarrow u}$. Since v is not contained in any of the clusters of B' , according to Lemma 2.18, $IL'_{v \rightarrow u}$ is a feasible insertion list of B' . Since $u \in S_r^{k-1}$, u is not inserted into S_r^{k-1} by $IL'_{v \rightarrow u}$, while IL contains pair (v, S_r^{k-1}) . Therefore, the number of insertions to $RIGHT(B')$ by IL' is bigger than the number of insertions to $RIGHT(B')$ by $IL'_{v \rightarrow u}$. Since B is a clean block with respect to IL , B' is also a clean block with respect to both IL' and to $IL'_{v \rightarrow u}$. According to the induction hypothesis, $IL'_{v \rightarrow u}$ contains at least $k - 1$ insertions to $RIGHT(B')$. And therefore IL' contains at least k insertions to $RIGHT(B') \subset RIGHT(B)$ and so does IL . ■

4.2 Ranges and Areas

This section presents a range, which describes the set of clusters in *LEFT*, which corresponds to *SRTL* and *RTL* vertices. In addition, this section describes and handles areas, which are sets of ranges that overlap.

Definition 4.9 Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be an insertion list of H . Consider v which is an *SRTL* vertex, let \mathbf{S}_v and $\mathbf{S}_l(v)$ be the clusters in *RIGHT* and *LEFT*, respectively, which contain v ($v \in S_v \cap S_l(v)$) in H .

If there is a cluster $S \in \text{LEFT}$ such that v is inserted into S ($v \notin S$ and $v \in S \cup IL(S)$) and S is above $S_l(v)$, then let $\mathbf{S}_u(v)$ be the highest such S , otherwise let $S_u(v) = S_l(v)$.

If there is a cluster $S \in \text{LEFT}$ such that v is inserted into S ($v \notin S$ and $v \in S \cup IL(S)$) and S is below $S_l(v)$, then let $\mathbf{S}_d(v)$ be the lowest such S , otherwise let $S_d(v) = S_l(v)$.

The **range** of v , denoted by $\mathbf{R}(v)$, is the block that contains all the clusters between $S_u(v)$ and $S_d(v)$, including both clusters and all their neighbor clusters in *RIGHT*.

In addition, let \mathbf{u}_v be the number of clusters from $\text{RIGHT}(S_u(v))$ to S_v , including $\text{RIGHT}(S_u(v))$ and excluding S_v . Similarly, define \mathbf{d}_v to be the number of clusters from $\text{RIGHT}(S_d(v))$ to S_v , including $\text{RIGHT}(S_d(v))$ and excluding S_v .

Remark 4.10 Note that it is possible that there is a cluster $S' \in \text{LEFT}$ with $v \notin S'$ and $v \in IL[S']$ (v is inserted into S') where $S' \neq S_u(V)$ and $S' \neq S_d(v)$. In this case, the row of S' lies below the row of $S_u(v)$ and above the row of $S_d(v)$.

Example 4.11 Consider the hypergraph presented in Example 2.3, with $\text{LEFT} = \{S_{i,3} : 1 \leq i \leq 5\}$ and $\text{RIGHT} = \{\{S_{i,4} : 1 \leq i \leq 5\}\}$.

Let $IL = \{(w_{3,3}, S_{4,3}), (w_{3,3}, S_{5,3})\}$. Note that IL is an insertion list, but not a feasible insertion list. In IL , $w_{3,3}$ is an *SRTL* vertex, with $S_{w_{3,3}} = S_{3,4}$, $S_l(w_{3,3}) = S_{3,3}$ and $S_u(w_{3,3}) = S_l(w_{3,3}) = S_{3,3}$ as $w_{3,3}$ is not inserted into a cluster above cluster $S_{3,3}$, and $S_d(w_{3,3}) = S_{5,3}$. In addition, $w_{3,3}$ is inserted into $S_{4,3}$, whose row is positioned below $S_{3,3}$ and above $S_{5,3}$. In this case, $R(w_{3,3})$ contains clusters $S_{3,3}$, $S_{4,3}$, $S_{5,3}$, $S_{3,4}$, $S_{4,4}$ and $S_{5,4}$, and its size is 3 as it contains three rows. Last, $u_{w_{3,3}} = 0$, and $d_{w_{3,3}} = 2$ counting clusters $S_{4,4}$ and $S_{5,4}$.

Definition 4.12 Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be an insertion list of H . Consider v which is an *RTL* vertex, let \mathbf{S}_v^1 and \mathbf{S}_v^2 be the two clusters in *RIGHT* which contain v , ($v \in S_v^1 \cap S_v^2$ in H) with S_v^1 above S_v^2 .

If there is a cluster $S \in \text{LEFT}$ such that v is inserted into S ($v \notin S$ and $v \in S \cup IL(S)$) and S is above $\text{LEFT}(S_v^1)$, then let $\mathbf{S}_u(v)$ be the highest such S , otherwise let $S_u(v) = \text{LEFT}(S_v^1)$.

If there is a cluster $S \in \text{LEFT}$ such that v is inserted into S ($v \notin S$ and $v \in S \cup IL(S)$) and S is below $\text{LEFT}(S_v^2)$, then let $\mathbf{S}_d(v)$ be the lowest such S , otherwise let $S_d(v) = \text{LEFT}(S_v^2)$.

The **range** of v , denoted by $\mathbf{R}(v)$, is the block that contains all the clusters between $S_u(v)$ and $S_d(v)$, including both clusters and their neighbor clusters in *RIGHT*.

In addition, let \mathbf{u}_v be the number of clusters from $\text{RIGHT}(S_u(v))$ to S_v^1 , including $\text{RIGHT}(S_u(v))$ and excluding S_v^1 . Similarly, define \mathbf{d}_v to be the number of clusters from $\text{RIGHT}(S_d(v))$ to S_v^2 , including $\text{RIGHT}(S_d(v))$ and excluding S_v^2 .

Example 4.13 Consider the hypergraph presented in Example 2.3, with $LEFT = \{S_{i,3} : 1 \leq i \leq 5\}$ and $RIGHT = \{\{S_{i,4} : 1 \leq i \leq 5\}$.

Let $IL = \{(v_{3,4}, S_{1,3}), (v_{3,4}, S_{2,3}), (v_{3,4}, S_{5,3})\}$, which is an insertion list of H (but not a feasible insertion list of H). In this case, $v_{3,4}$ is an RTL vertex, with $S_{v_{3,4}}^1 = S_{3,4}$ and $S_{v_{3,4}}^4 = S_{4,4}$, $S_u(v_{3,4}) = S_{1,3}$, $S_d(v_{3,4}) = S_{5,3}$. In addition, $v_{3,4}$ is inserted into $S_{2,3}$, whose row is positioned below $S_{1,3}$ and above $S_{5,3}$. In this case, $R(v_{3,4})$ contains all the clusters in $LEFT \cup RIGHT$ and its size is 5, as it contains five rows. Last, $u_{v_{3,4}} = 2$, counting clusters $S_{1,4}$ and $S_{2,4}$ and $d_{v_{3,4}} = 1$ counting only cluster $S_{5,4}$.

Definition 4.14 Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be a feasible insertion list of H . A vertex v forms a α -extra insertion with respect to IL , for $\alpha \in \{1, 2\}$, if

- $v \in S$ for $S \in RIGHT$.
- There is a feasible insertion list IL_g of $H[LEFTGRID]$ such that $|IL_g| \leq |IL[LEFTGRID]| - \alpha$ and IL_g does not contain any pair whose vertex is v .

Example 4.15 Let $H = \langle V, \mathcal{S} \rangle$ be a $2 * 3$ grid hypergraph with cluster set $\mathcal{S} = \{S_{1,1}, S_{1,2}, S_{1,3}, S_{2,1}, S_{2,2}, S_{2,3}\}$ and vertex set $V = \{v_{1,1}, v_{1,2}, v_{1,3}, w_{1,1}, w_{1,2}, w_{2,1}, w_{2,2}\}$, such that $S_{1,1} = \{v_{1,1}, w_{1,1}\}$, $S_{1,2} = \{w_{1,1}, w_{1,2}, v_{1,2}\}$, $S_{1,3} = \{w_{1,2}, v_{1,3}\}$, $S_{2,1} = \{v_{1,1}, w_{2,1}\}$, $S_{2,2} = \{w_{2,1}, w_{2,2}, v_{1,2}\}$ and $S_{2,3} = \{w_{2,2}, v_{1,3}\}$. Figure 6 presents the corresponding intersection graph. For this hypergraph $LEFTGRID = \{S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}\}$.

Consider $CCIL_{w_{2,2}} = \{(w_{2,2}, S_{1,1}), (w_{2,2}, S_{1,2}), (w_{2,2}, S_{1,3}), (w_{2,2}, S_{2,1})\}$, which is a feasible insertion list created by inserting $w_{2,2}$ into all the clusters which do not contain this vertex in H . Thus, $CCIL_{w_{2,2}}$ is a feasible insertion list generated by the Convert to Clique method, utilizing vertex $w_{2,2}$.

Define $IL_g = \{(v_{1,2}, S_{1,1}), (v_{1,2}, S_{2,1})\}$, which is a feasible insertion list of $H[LEFTGRID]$ with $|IL_g| = |CCIL_{w_{2,2}}[LEFTGRID]| - 1$. Hence $w_{2,2}$ forms a 1-extra insertion with respect to $CCIL_{w_{2,2}}$.

Consider $CCIL_{v_{1,3}} = \{(v_{1,3}, S_{1,1}), (v_{1,3}, S_{1,2}), (v_{1,3}, S_{2,1}), (v_{1,3}, S_{2,2})\}$, which is a feasible insertion list created by inserting $v_{1,3}$ into all the clusters which do not contain this vertex in H . Thus, $CCIL_{v_{1,3}}$ is a feasible insertion list generated by the Convert to Clique method, utilizing vertex $v_{1,3}$. In this case, $|IL_g| = |CCIL_{v_{1,3}}[LEFTGRID]| - 2$. Hence $v_{1,3}$ forms a 2-extra insertion with respect to $CCIL_{v_{1,3}}$.

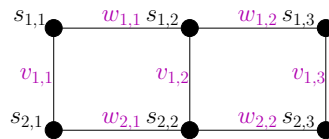


Figure 6: The intersection graph of a $2 * 3$ grid hypergraph presented in Example 4.15. The nodes are labeled in black, while the vertices which constitute the intersections are marked in purple.

Lemma 4.16 Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be a feasible insertion list. In a range of size k , with exactly one SRTL vertex, there are at least $k - 1$ insertions into clusters in $RIGHT$ of this range.

Proof: Let v be the *SRTL* vertex of the range.

Using the notation of a range in Definition 4.9, k represents the size of the range that includes all clusters between $S_u(v)$ and $S_d(v)$, as well as their neighboring clusters in *RIGHT*. Thus, $k = u_v + 1 + d_v$. Following the insertion of vertex v into $S_u(v)$ and $S_d(v)$, two chordless cycles are formed in the intersection graph: one cycle includes $s_u(v)$ and consists of $u_v + 2$ nodes, while the other includes $s_d(v)$ and consists of $d_v + 2$ nodes. Where $s_u(v)$ and $s_d(v)$ represent the nodes corresponding to the clusters $S_u(v)$ and $S_d(v)$, respectively. Refer to Figure 7 for examples of these blocks and the corresponding cycles. In this figure, the red lines indicate the chords formed in the intersection graph when v is inserted into $S_u(v)$ and $S_d(v)$. Moreover, in Figure 7(a) $u_v = 3$, $d_v = 2$ and $k = 6$, in Figure 7(b) $u_v = 1$, $d_v = 2$ and $k = 4$, and in Figure 7(c) $u_v = 0$, $d_v = 2$ and $k = 3$.

We claim that u_v insertions are required to resolve the cycle of size $u_v + 2$.

- Assume $u_v \geq 2$. In this case, a cycle with $u_v + 2$ nodes is formed, including the nodes corresponding to clusters S_v , $S_u(v)$ and *RIGHT*($S_u(v)$). Refer to Figure 7(a), where this cycle is demonstrated in green. According to Lemma 3.1, at least u_v insertions are required to achieve feasibility for this cycle. Since v is the only *SRTL* in this block, none of these insertions can be an insertion into $S_u(v)$. Therefore, *IL* requires at least u_v insertions into the clusters within this cycle that are also in *RIGHT* of this block.
- Assume $u_v = 1$. In this case, a triangle is formed by the nodes corresponding to clusters S_v , $S_u(v)$ and *RIGHT*($S_u(v)$). Refer to Figure 7(b), where this triangle is demonstrated in green. This triangle does not satisfy the Helly Property because $v \in S_v \cap S_l(v)$ is inserted into $S_u(v)$ but not into *RIGHT*($S_u(v)$). Consequently, one insertion ($u_v = 1$) is needed to satisfy the Helly Property. Since v is the only *SRTL* in this block, this insertion cannot insert a vertex into $S_u(v)$. Therefore, *IL* requires at least one insertion ($u_v = 1$) into the cluster within this triangle that is also in *RIGHT* of this block.
- Assume $u_v = 0$. Refer to Figure 7(c), where v is not inserted into clusters above $S_l(v)$ and therefore $S_u(v) = S_l(v)$. In this case $u_v = 0$ insertions are required.

Similarly, d_v insertions are required to resolve the second cycle, which contains $s_d(v)$. As proven above, these insertions must be made into the clusters in *RIGHT* of this block.

In total, *IL* contains at least $u_v + d_v = k - 1$ insertions to the clusters in *RIGHT* of this range.

■

Lemma 4.17 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let *IL* be a feasible insertion list. In a range with exactly one *SRTL* vertex there is at least one 1-extra insertion.*

Proof: Let v be the *SRTL* vertex of the range. According to Definition 4.3, $v \in S_v \cap S_l(v)$ is inserted into at least one cluster in *LEFT*. According to Lemma 2.12, *IL*[*LEFTGRID*] is a feasible insertion list of H [*LEFTGRID*].

Suppose that vertex v is inserted into S'_l , which is the neighbor cluster above $S_l(v)$, see Figure 8(a). Choose $u \in S_l(v) \cap S'_l$. In H [*LEFTGRID*], v is only contained in $S_l(v)$, whereas u is contained in both $S_l(v)$ and S'_l . Therefore, with respect to H [*LEFTGRID*] it holds that $SC(v) \subsetneq SC(u)$. According to Lemma 2.18, IL [*LEFTGRID*] $_{v \rightarrow u}$ is a feasible insertion list of H [*LEFTGRID*]. Since IL [*LEFTGRID*] contains insertion (v, S'_l) and in IL [*LEFTGRID*] $_{v \rightarrow u}$ this is a null insertion, as u

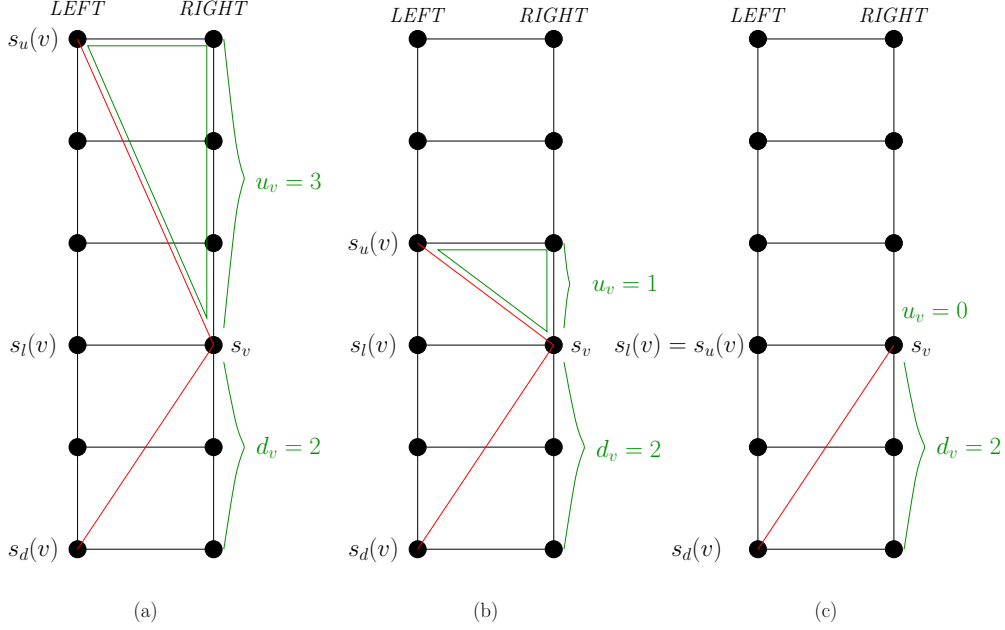


Figure 7: Figure for Lemma 4.16, where nodes $s_u(v)$, $s_l(v)$, $s_d(v)$ and s_v correspond to clusters $S_u(v)$, $S_l(v)$, $S_d(v)$ and S_v , respectively.

was originally contained in S'_l , we can conclude that $|IL[LEFTGRID]_{v \rightarrow u}| \leq |IL[LEFTGRID]| - 1$. Therefore, v is a 1-extra insertion with respect to IL .

A similar reasoning also applies if v is inserted into the neighbor cluster below $S_l(v)$.

Otherwise, vertex v is inserted into a cluster that is not adjacent to $S_l(v)$. Without loss of generality, assume that the nearest such cluster is S''_l , located above $S_l(v)$. In this case, a chordless cycle (or a triangle) is created between $S_l(v)$ and S''_l . This cycle (or triangle) requires at least one more insertion. Choose $u \in S_l(v) \cap S'_l$ where S'_l is the cluster above $S_l(v)$. In $H[LEFTGRID]$, it follows that $SC(v) \subsetneq SC(u)$ and according to Lemma 2.18, $IL[LEFTGRID]_{v \rightarrow u}$ is a feasible insertion list of $H[LEFTGRID]$.

Suppose that a triangle is formed by the clusters $S_l(v)$, S'_l and S''_l (consider Figure 8(b) where the triangle is depicted in green). In $H + IL[LEFTGRID]$, there is no vertex that satisfies the Helly Property for this set of clusters, so one more insertion is required. However, in $H[LEFTGRID] + IL[LEFTGRID]_{v \rightarrow u}$, vertex u is contained in all three clusters, since u was chosen to satisfy $u \in S_l(v) \cap S'_l$ and according to $IL[LEFTGRID]_{v \rightarrow u}$, vertex u is inserted into S''_l . Thus, u satisfies the Helly Property for this set of clusters. Hence, $|IL[LEFTGRID]_{v \rightarrow u}| \leq |IL[LEFTGRID]| - 1$ and v forms a 1-extra insertion with respect to IL .

Suppose that a cycle C containing $l \geq 4$ nodes is formed between $S_l(v)$ and S''_l (consider Figure 8(c) where the cycle is depicted in green)). According to Lemma 3.1, $IL[LEFTGRID]$ contains at least $l - 2$ insertions to achieve feasibility for this cycle. In $IL[LEFTGRID]_{v \rightarrow u}$ cycle C contains a chord between S'_l and S''_l . So C is divided into a triangle containing the nodes corresponding to $S_l(v)$, S'_l and S''_l , where u is the reason that the Helly Property is satisfied in this triangle, and a sub-cycle containing $l - 1$ nodes. Therefore, $IL[LEFTGRID]_{v \rightarrow u}$ requires only $l - 3$ more insertions, and $|IL[LEFTGRID]_{v \rightarrow u}| \leq |IL[LEFTGRID]| - 1$. Hence, v forms a 1-extra insertion with respect to IL . \blacksquare

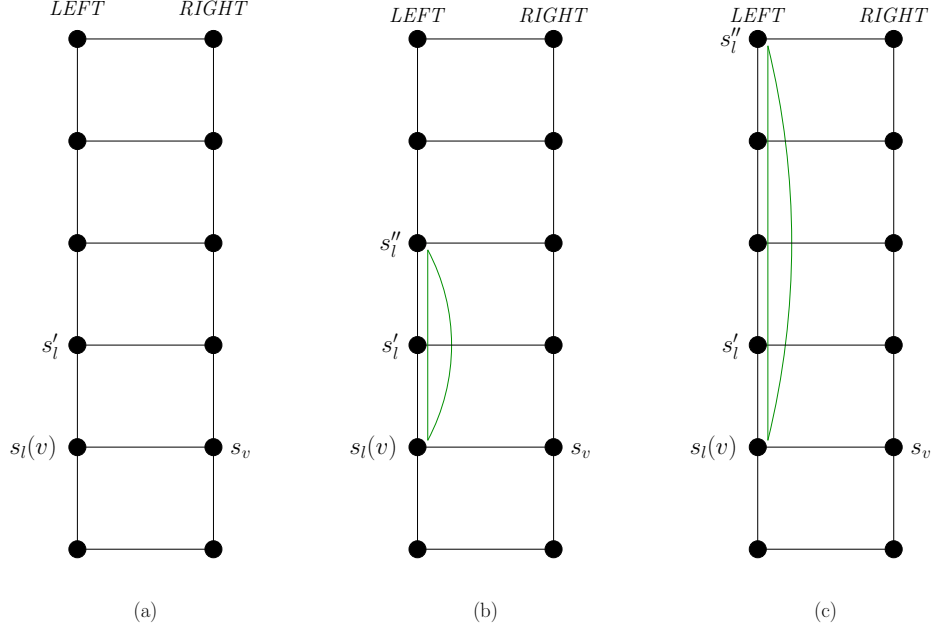


Figure 8: Figure for Lemma 4.17, where nodes s_l'' , s_l' , $s_l(v)$ and s_v correspond to clusters S_l'' , S_l' , $S_l(v)$ and S_v , respectively.

Corollary 4.18 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be a feasible insertion list. In a range of size k , with exactly one *SRTL* vertex, there are at least $k - 1$ insertions to the clusters in *RIGHT* of this range and at least one 1-extra insertion.*

Proof: Follows from Lemmas 4.16 and 4.17. ■

Lemma 4.19 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be a feasible insertion list of H . In a range of size k with exactly one *RTL* vertex, there are at least $k - 2$ insertions to the clusters in *RIGHT* of this range and at least one 2-extra insertion.*

Proof: Similar to the proof of Lemma 4.16, except that this is an *RTL* vertex, which is contained in two clusters in *RIGHT*: clusters S_v^1 and S_v^2 . In this case, $k = u_v + 2 + d_v$ and therefore the number of insertions to *RIGHT* is $k - 2$.

In addition, similar to the proof of Lemma 4.17, define $S_l^1 = \text{LEFT}(S_v^1)$ and $S_l^2 = \text{LEFT}(S_v^2)$ and choose $u \in S_l^1 \cap S_l^2$. With respect to $H[\text{LEFTGRID}]$, it holds that $SC(v) \subsetneq SC(u)$. According to Lemma 2.18, $IL[\text{LEFTGRID}]_{v \rightarrow u}$ is a feasible insertion list of $H[\text{LEFTGRID}]$, with $|IL[\text{LEFTGRID}]_{v \rightarrow u}| \leq |IL[\text{LEFTGRID}]| - 2$ and therefore v forms a 2-extra insertion with respect to IL . ■

Definition 4.20 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be an insertion list of H . Let U be a maximal set of *SRTL* and *RTL* vertices, which satisfies that for every $v \in U$, there exists $u \in U \setminus \{v\}$ with $R(v) \cap R(u) \neq \emptyset$. We denote an **area** by $\mathbf{A}(U) = \bigcup_{v \in U} \mathbf{R}(v)$. We define the size of an area as the number of rows in this area.*

*If an area contains only *SRTL* vertices, it is denoted as an **SRTL area**; otherwise, it contains at least one *RTL* vertex, and is denoted as an **RTL area**.*

Example 4.21 Let $H = \langle V, \mathcal{S} \rangle$ be a 6×2 grid hypergraph with cluster set $\mathcal{S} = \{S_{i,j} : 1 \leq i \leq 6, 1 \leq j \leq 2\}$ and vertex set $V = \{v_{i,j} : 1 \leq i \leq 5, 1 \leq j \leq 2\} \cup \{w_{i,1} : 1 \leq i \leq 6\}$. The intersection graph of H is described in Figure 9.

The vertices and clusters in H satisfy $\{v_{i,j}\} = S_{i,j} \cap S_{i+1,j}$ and $\{w_{i,j}\} = S_{i,j} \cap S_{i,j+1}$. Specifically, $\{w_{4,1}\} = S_{4,1} \cap S_{4,2}$ and $\{v_{2,2}\} = S_{2,2} \cap S_{3,2}$.

Suppose that $IL = \{(v_{2,2}, S_{1,1}), (v_{2,2}, S_{4,1}), (w_{4,1}, S_{3,1}), (w_{4,1}, S_{6,1})\}$. In Figure 9 the red lines indicate the chords formed when $v_{2,2}$ is inserted into $S_{1,1}$ and $S_{4,1}$, and the green lines indicate the chords formed when $w_{4,1}$ is inserted into $S_{3,1}$ and $S_{6,1}$.

In this case, $R(v_{2,2})$ contains rows 1 to 4 and $R(w_{4,1})$ contains rows 3 to 6. Since the ranges of $v_{2,2}$ and $w_{4,1}$ intersect, rows 1 to 6 form one area. Since $v_{2,2}$ is an *RTL* vertex and $w_{4,1}$ is an *SRTL* vertex, this area is an *RTL* area.

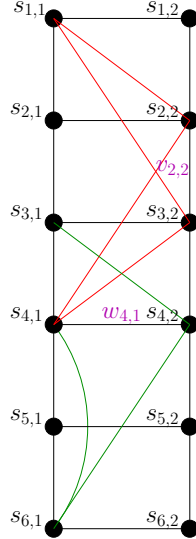


Figure 9: Figure for Example 4.21.

Lemma 4.22 Let $H = \langle V, \mathcal{S} \rangle$ be an $n \times m$ grid hypergraph, and let IL be a feasible insertion list of H . If an area $A(U)$ of size k is an *SRTL* area, then there are at least $k - 1$ insertions to the clusters in *RIGHT* of this area and at least one 1-extra insertion.

Proof: If $A(U)$ contains only one *SRTL* vertex, then according to Corollary 4.18, it contains at least $k - 1$ insertions to *RIGHT* and at least one 1-extra insertion to *LEFT*.

If the area contains more than one *SRTL* vertex, several cycles associated with the different *SRTL* vertices form a subgraph that consists a line of intersecting cycles. Where the spine path is composed of the nodes corresponding to the clusters in *RIGHT* of this area.

According to Lemma 3.3, the feasibility of a hypergraph induced on a set of cycles containing p clusters requires at least $p - 2$ insertions. This is the same number of insertions required for one cycle containing p nodes. Hence, a similar proof to the one given for Lemma 4.16, which assumed only one cycle, can determine that in an area with k rows, there are at least $k - 1$ insertions to *RIGHT*.

Furthermore, an *SRTL* vertex ensures at least one 1-extra insertion, as was proven in Lemma 4.17. \blacksquare

Lemma 4.23 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be a feasible insertion list of H . If an area $A(U)$ of size k is an *RTL* area, then there are at least $k - 2$ insertions to the clusters in *RIGHT* of this area and at least one 2-extra insertion.*

Proof: Similar to the proof of Lemma 4.22 and using Lemma 4.19. \blacksquare

Theorem 4.24 *Let $H = \langle V, \mathcal{S} \rangle$ be an $n * m$ grid hypergraph, and let IL be a feasible insertion list of H . Let l_1 be the number of *SRTL* areas and let l_2 be the number of *RTL* areas created by IL , then there are at least $n - l_1 - 2l_2$ insertions to *RIGHT*, and at least l_1 1-extra insertions and at least l_2 2-extra insertions.*

Proof: Let l_0 be the number of clean blocks created with respect to IL . Thus, IL creates l_0 clean blocks of sizes $k_1^c, \dots, k_{l_0}^c$, l_1 *SRTL* areas of sizes $k_1^s, \dots, k_{l_1}^s$ and l_2 *RTL* areas of sizes $k_1^r, \dots, k_{l_2}^r$. Since the grid intersection graph contains n rows $\sum_{i=1}^{l_0} k_i^c + \sum_{j=1}^{l_1} k_j^s + \sum_{q=1}^{l_2} k_q^r = n$.

According to Lemmas 4.6 and 4.8, in every clean block of size k_i^c , there are at least k_i^c insertions to *RIGHT*. According to Lemma 4.22, in every *SRTL* area of size k_j^s , there are at least $k_j^s - 1$ insertions to *RIGHT* and at least one 1-extra insertion. According to Lemma 4.23, in every *RTL* area of size k_q^r , there are at least $k_q^r - 2$ insertions to *RIGHT* and at least one 2-extra insertion.

Combining all the above, the number of insertions to *RIGHT* is at least $\sum_{i=1}^{l_0} k_i^c + \sum_{j=1}^{l_1} (k_j^s - 1) + \sum_{q=1}^{l_2} (k_q^r - 2) = n - l_1 - 2l_2$. In addition, there are at least l_1 1-extra insertions and at least l_2 2-extra insertions. \blacksquare

4.3 Main Result

This section contains the main result of the paper. We prove that any feasible insertion list of an $n * m$ grid hypergraph contains at least $nm - 2$ insertions. Furthermore, we prove that the Convert to Clique method creates a minimum cardinality feasible insertion list of an $n * m$ grid hypergraph.

Theorem 4.25 *If $H = \langle V, \mathcal{S} \rangle$ is an $n * m$ grid hypergraph for $n \geq 2$ and $m \geq 2$, then $mIL(H) \geq nm - 2$.*

Proof: The proof is by induction of m .

For $m = 2$ the claim is proved in Theorem 3.4. Suppose it is correct for $m - 1$, and we will prove it for m .

Suppose that $m > 2$ and consider $H[LEFTGRID]$. This is a $n * (m - 1)$ grid hypergraph. According to the induction hypothesis $mIL(H[LEFTGRID]) \geq n(m - 1) - 2$.

Let IL be a feasible insertion list of H , and let l_1 be the number of *SRTL* areas and l_2 the number of *RTL* areas, created by IL , respectively.

Consider $IL[LEFTGRID]$. According to Lemma 2.12, $IL[LEFTGRID]$ is a feasible insertion list of $H[LEFTGRID]$. According to Theorem 4.24, IL contains at least l_1 1-extra insertions and at least l_2 2-extra insertions.

According to Definition 4.14 and Theorem 4.24, there exists IL' a feasible insertion list of $H[LEFTGRID]$ such that $|IL'| \leq |IL[LEFTGRID]| - l_1 - 2l_2$, thus $|IL'| + l_1 + 2l_2 \leq |IL[LEFTGRID]|$. Since IL' is a feasible insertion list of $H[LEFTGRID]$, it holds that $n(m-1) - 2 \leq mIL(H[LEFTGRID]) \leq |IL'|$. Therefore, $|IL[LEFTGRID]| \geq |IL'| + l_1 + 2l_2 \geq (n(m-1) - 2) + l_1 + 2l_2$.

In addition, according to Theorem 4.24, IL contains at least $n - l_1 - 2l_2$ insertions into clusters in $RIGHT$.

Therefore,

$$\begin{aligned} |IL| &\geq |IL[LEFTGRID]| + n - l_1 - 2l_2 \\ &\geq (n(m-1) - 2) + l_1 + 2l_2 + (n - l_1 - 2l_2) = n(m-1) - 2 + n = nm - 2. \end{aligned}$$

Since this is true for every feasible insertion list IL , we can deduce that $mIL(H) \geq nm - 2$. ■

Corollary 4.26 *Let $H = \langle V, S \rangle$ be an $n * m$ grid hypergraph, $mIL(H) = nm - 2$ and the Convert to Clique method creates a minimum cardinally feasible insertion list $CCIL_v$, for every $v \in S \cap S'$, where S and S' are two distinct clusters.*

Proof: According to Corollary 4.2, the Convert to Clique method creates a feasible insertion list $CCIL_v$ with cardinality $nm - 2$, for every $v \in S \cap S'$. According to Theorem 4.25, $mIL(H) \geq nm - 2$. Hence, $CCIL_v$ is a minimum cardinality feasible insertion list of H with cardinality $nm - 2$. ■

5 Summary and Further Research

For the clustered spanning tree problem of a grid hypergraph, where the intersection graph is a grid, no feasible solution exists. In this paper we prove that the minimum cardinality of a feasible insertion list for an $n * m$ grid hypergraph is $nm - 2$. In addition, we present the Convert to Clique method, which involves selecting a vertex contained in two clusters and inserting it into all the other clusters. While this method always yields a hypergraph with a feasible solution tree, it provides a minimum cardinality feasible insertion list of grid hypergraphs.

We would like to investigate other intersection graph structures where we can provide minimum cardinality feasible insertion lists. For example, we might explore intersection graphs composed of larger cycles arranged in a grid pattern or those forming three-dimensional grids.

Another interesting question involves exploring hypergraphs with other conditions under which the Convert to Clique method provides a minimum cardinality feasible insertion list.

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