



HAL
open science

Dynamics of elastic bodies connected by a thin adhesive layer

Christian Licht, Frédéric Lebon, Alain Léger

► **To cite this version:**

Christian Licht, Frédéric Lebon, Alain Léger. Dynamics of elastic bodies connected by a thin adhesive layer. Alain Léger, Marc Deschamps. Ultrasonic wave propagation in non homogeneous media, 128, Springer, pp.99-110, 2009, Springer proceedings in physics 128, 10.1007/978-3-540-89105-5_9 . hal-04872416

HAL Id: hal-04872416

<https://hal.science/hal-04872416v1>

Submitted on 8 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Dynamics of elastic bodies connected by a thin adhesive layer

C. Licht, F. Lebon and A. Léger

Abstract We study a problem of wave propagation in a structure made of two tridimensional elastic bodies connected by a thin adhesive layer. We recall that, after several heuristic models, the bonding problem, which basically amounts to study how does the thin layer can be replaced by a simpler model without thickness, has been carried out in the case of equilibrium problems. The present work deals with the dynamical problem. We obtain that the problem of elastodynamics with a thin adhesive layer can be approximated, with a convergence result, by another problem in which the layer is changed into a mechanical constraint, which is precisely the same as the one of the equilibrium case.

1 Introduction

We study the effect of a thin adhesive layer in a problem of elastodynamics. We recall that the problem of the modelling of bonded solids goes back to ancient studies in the field of physics and acoustics (see e.g. [12, 6]) and has essentially given rheological models consisting in changing the layer of the adhesive into an areal distribution of springs. This kind of models was used in many fields of wave propagation analyses such as for instance seismology [5]. But it was observed that the range of applicability of these rheological models has not been clearly established and lots of studies tried to back up the models either by comparisons with experiments or, up to very recently, by comparison with finite elements calculations in which a mesh of a thin layer was used to take the behavior of the adhesive into ac-

C. Licht

CNRS, Laboratoire de Mécanique et Génie Civil, Place Eugene Bataillon, 34095 Montpellier, France e-mail: licht@lmgc.univ-montp2.fr

F. Lebon and A. Léger

CNRS, Laboratoire de Mécanique et d'Acoustique, 31, chemin Joseph Aiguier, 13402 Marseille Cedex 20, France e-mail: lebon, leger@lma.cnrs-mrs.fr

count [14].

The purpose of the present work is to make a contribution to the way of modelling a system made of two elastic bodies connected by an adhesive joint, and in particular to give a mathematical justification of rheological models. We shall get that the models given by the physicists years ago can be justified rigorously, which will settle the problem of the range of applicability of a given model, but we shall also get that these models may fail in some physical situations.

Let us now outline the main steps of the paper.

A first section recalls the problem of justifying the effect of a thin adhesive layer in equilibrium problems. This section only stands for a motivation since the equilibrium case has now widely been treated. The main idea of the justification deals with taking the thinness of the adhesive layer into account by an asymptotic analysis. We state the problem for a given thickness of the layer and we look for a limit, this means that we look at the changes of the problem or at the behavior of its solution, as the layer is thinner and thinner. Among different methods using tools of asymptotic analyses, we recall very briefly the basic notions of variational convergence.

The next sections of the paper concerns the dynamical problem. The classical statement of the elastodynamical problem is transformed into a first order problem for the pair (displacement, velocity) which gives the form of a so-called semi-group depending on the set of the mechanical parameters of the problem. Then the last parts show that, as the parameters tends to zero, this semi-group converges, in a sense which will be given, towards a limit which gives the convenient model.

2 The equilibrium problem

The mechanical analysis of soft thin adhesive bonded joints between two deformable bodies involves problems with several parameters. At least two of them are essential: the thickness of the joint, which is small with respect to those of the deformable bodies and the stiffness of the joint, which is usually lower than that of the bodies. In previous studies (see e.g. [2, 4, 7, 8, 11]), the bonding of two three-dimensional solids by an adhesive layer, within linear elasticity, small strains or finite strains, or viscoelasticity, has been handled by performing asymptotic analyses, i.e. by assuming that the thickness and the stiffness of the layer tend to zero.

Within this approach, the layer is replaced by a mechanical constraint. The layer no longer exists from the geometrical point of view, but is replaced by a jump condition taking the asymptotic behaviour of the parameters into account. When the adhesive is linearly elastic, the limit problem involves a transmission condition linking the stress vector to the jump of the displacement which occurs at the interface $\sigma \cdot n = \mathbb{K}[u]$. The structure of this interface law is similar in this case to that of the original constitutive equation, but the strain tensor is replaced by the symmetrized

tensorial product of the displacement jump by the unit normal vector. The coefficients of the tensor \mathbb{K} keeps the memory of the stiffness parameters of the original material. For example, for two-dimensional domains made of isotropic elastic material, the stiffness \mathbb{K} is a diagonal tensor which can be shared into normal and tangential parts:

$$\mathbb{K}_N = \lim_{s \rightarrow 0} \frac{\lambda + 2\mu}{\varepsilon}, \quad \mathbb{K}_T = \lim_{s \rightarrow 0} \frac{\mu}{\varepsilon}.$$

ε is the thickness of the adhesive, λ and μ are the Lamé coefficients of the adhesive, and parameter s stands for the triplet $(\lambda, \mu, \varepsilon)$. This result can be proved rigorously using variational convergence arguments [10]. Variational convergence is a notion of convergence of sequences of functions introduced during the 70's (see e.g. [3]). Let \mathcal{F}^s be an energy functional and let u_s be a minimum of this functional (i.e. an equilibrium state in equilibrium problems). The idea of the variational convergence is to define the "lowest" notion of convergence of \mathcal{F}^s to some functional \mathcal{F} when s tends to zero, which implies the convergence of both the minima and the minimizers of \mathcal{F}^s to those of \mathcal{F} .

In the case of the thin layer, the sequence $\{\mathcal{F}^s\}$ corresponds to the total energy in the adherents and in the adhesive. The variational limit is a sum of the total energy of the adherents together with an areal energy on a surface S which is the geometrical limit of the part of the domain made of the adhesive as its thickness tends to zero. The latter areal energy is given by:

$$\int_S \frac{1}{2} \mathbb{K}[u]^2 ds.$$

The rigorous proof of this result is obtained using a regularization operator R^ε which gives a good approximation of the solution u_s in the adhesive. For example, if the adhesive layer of thickness ε lies along a plane orthogonal to the third direction, the regularization can be taken as:

$$R^\varepsilon u(\cdot, x_3) = \frac{1}{2} \left\{ u(\cdot, \frac{\varepsilon}{2}) + u(\cdot, -\frac{\varepsilon}{2}) + \min(1, \frac{|x_3|}{\varepsilon}) (u(\cdot, \frac{\varepsilon}{2}) - u(\cdot, -\frac{\varepsilon}{2})) \right\}.$$

We now just recall the results in the form of the set of possible behaviors of the rheological model which can be used to replace the thin layer and to understand the equilibrium of the system adherent-adhesive under external loads. This is given in Table 1. It is worth seeing that some cases give exactly a rheological behavior with the explicit stiffness of the springs in the normal and tangential direction, but some other cases do not.

$\mu/\varepsilon \rightarrow 0$	$\sigma_N = 0,$ $\sigma_T = 0.$	$\sigma_N = \bar{\lambda} [u_N],$ $\sigma_T = 0.$	$[u_N] = 0,$ $\sigma_T = 0.$
$\mu/\varepsilon \rightarrow \bar{\mu}$	$\sigma_N = 2\bar{\mu} [u_N],$ $\sigma_T = \bar{\mu} [u_T].$	$\sigma_N = (\bar{\lambda} + 2\bar{\mu}) [u_N],$ $\sigma_T = \bar{\mu} [u_T].$	$[u_N] = 0,$ $\sigma_T = \bar{\mu} [u_T].$
$\mu/\varepsilon \rightarrow \infty$	$[u_N] = 0,$ $[u_T] = 0.$	$[u_N] = 0,$ $[u_T] = 0.$	$[u_N] = 0,$ $[u_T] = 0.$
	$\lambda/\varepsilon \rightarrow 0$	$\lambda/\varepsilon \rightarrow \bar{\lambda}$	$\lambda/\varepsilon \rightarrow \infty$

Table 1 Transmission conditions

3 Statement of the dynamical problem

Let Ω be a tridimensional domain with a Lipschitz continuous boundary having an intersection with the plane $\{x_3 = 0\}$ of non zero measure denote by S . Let this domain be divided into $\Omega^\varepsilon := \{x \in \Omega, |x_3| > \varepsilon\}$ and $B^\varepsilon := \{x \in \Omega, |x_3| < \varepsilon\}$. The parts Ω^ε will be referred to as the bodies, and the layer B^ε as the adhesive. The equations of the dynamics are the following:

$$(\mathcal{P}_s) \begin{cases} \gamma \frac{\partial^2 \mathbf{u}_s}{\partial t^2} - \operatorname{div} \sigma_s = \mathbf{f} & \text{in } \Omega \times [0, T] \\ \sigma_s = \begin{cases} a \mathbf{e}(\mathbf{u}_s) & \text{in } \Omega^\varepsilon \times [0, T] \\ \lambda \operatorname{tr}(\mathbf{e}(\mathbf{u}_s)) \mathbb{I}d + 2\mu \mathbf{e}(\mathbf{u}_s) & \text{in } B^\varepsilon \times [0, T] \end{cases} & (1) \\ \sigma_s n = g & \text{on } \Gamma_1, \quad \mathbf{u}_s = 0 & \text{on } \Gamma_0, \\ \mathbf{u}_s(x, 0) = \mathbf{u}_s^0(x), \quad v_s^0 = \frac{\partial \mathbf{u}_s(x, 0)}{\partial t} & \forall x \in \Omega. \end{cases}$$

(λ, μ) in B^ε and a in Ω^ε are the elasticity coefficients which are assumed to satisfy:
i) $a \in \mathbb{L}^\infty(\Omega; \mathcal{L}in(\mathbb{S}^3))$; $\exists \alpha > 0$ such that $a(x) \xi \cdot \xi \geq \alpha |\xi|_{\mathbb{S}^3}^2$ a. e. in Ω $\forall \xi \in \mathbb{S}^3$, $\mathcal{L}(\mathbb{S}^3)$ denotes the space of linear operators from the set \mathbb{S}^3 of symmetrical 3×3 matrixes onto itself;

ii) λ and μ are positive real numbers which will tend to zero.

The voluminal mass γ is such that:

i) $\exists \bar{\rho} \in \mathbb{L}^\infty(\Omega)$ such that $\bar{\rho}(x) \geq \rho_m > 0$ a. e. in Ω

ii) $\gamma(x) = \bar{\rho}(x)$ a. e. in Ω^ε , $\gamma(x) = \rho$ a. e. in B^ε , where ρ is a positive real number assumed to have a limit $\rho_L \geq 0$.

u_s is the displacement field, v_s the velocity field, σ_s the stress field and n the outer normal, (f, g) are the given external loads for which we assume that $\exists \varepsilon_0$ such that $\operatorname{supp}(g) \cap B^\varepsilon \neq \emptyset$

The index s means that this problem depends on a quadruplet of parameters $s := (\lambda, \mu, \varepsilon, \rho)$. It is well known that for smooth enough data f and g , this problem possesses a single solution u_s .

We now aim at studying the behavior of problem (\mathcal{P}_s) as the parameter s tends to zero. This is more easily carried out if we first put problem (\mathcal{P}_s) in the form of an evolution equation in a convenient function space which will be the set of admissible states of finite energy.

4 Changing the formulation of the problem

In order to study problem (\mathcal{P}_s) we first introduce the classical function space (see e.g. [1]) $\mathbb{H}_{\Gamma_0}^1(\Omega)^3 = \{u \in \mathbb{H}^1(\Omega)^3; u = 0 \text{ on } \Gamma_0\}$ and the following notations so that we shall go on with simple formula:

$$\begin{aligned} & \text{for } \phi, \psi \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3, \\ & a_s(\phi, \psi) = \int_{\Omega^\varepsilon} ae(\phi) \cdot e(\psi) dx + \int_{B^\varepsilon} \lambda \text{tr}(e(\phi)) \text{tr}(e(\psi)) + 2\mu e(\phi) e(\psi) dx. \end{aligned} \quad (2)$$

We then make the following regularity assumption on the surface loads:

Assumption \mathbb{H}_1
 $g \in \mathbf{C}^{2,1}([0, T]; \mathbb{L}^2(\Omega)^3)$.

Assumption \mathbb{H}_1 implies

$$\exists! u_s^e \in \mathbf{C}^{0,1}([0, T]; \mathbb{H}_{\Gamma_0}^1(\Omega)^3); \quad a_s(u_s^e, \phi) = \int_{\Gamma_1} g \phi ds, \quad \forall \phi \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3.$$

The phase space is then $\mathbb{H}_s = \{U = (u, v) \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3 \times \mathbb{L}^2(\Omega)^3\}$ endowed with the hilbertian norm and scalar product:

$$|U|_{\mathbb{H}_s}^2 = |(u, v)|_{\mathbb{H}_s}^2 := a_s(u, u) + \int_{\Omega} \gamma v v dx; \quad (U, U') = a_s(u, u') + \int_{\Omega} \gamma v v' dx.$$

Let us now define in \mathbb{H}_s an unbounded operator \mathbb{A}_s with a domain $\mathbb{D}(\mathbb{A}_s)$ as:

$$\left\{ \begin{array}{l} \mathbb{D}(\mathbb{A}_s) = \{U = (u, v) \in \mathbb{H}_s; v \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3 \text{ and } \exists! w \in \mathbb{L}^2(\Omega)^3 \text{ such that} \\ \int_{\Omega} \gamma w \cdot \phi dx + a_s(u, \phi) = 0, \forall \phi \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3\} \\ \mathbb{A}_s U = \mathbb{A}_s(u, v) = (v, w), \end{array} \right. \quad (3)$$

in such a way that, if

$$U_s^r = (u_s^r, v_s^r) := (u_s - u_s^e, v_s - v_s^e) \text{ and } F_s = \left(0, \frac{f}{\gamma} - \frac{dv_s^e}{dt}\right),$$

then problem (\mathcal{P}_s) is formally equivalent to the evolution equation in \mathbb{H}_s :

$$\frac{dU_s^r}{dt} = \mathbb{A}_s U_s^r + F_s, \quad U_s^r(0) = (u_s^0 - u_e(0), v_s^0 - v_e(0)). \quad (4)$$

Theorem 1. *Let assumption \mathbb{H}_1 be satisfied and assume $f \in \mathbf{C}^{0,1}([0, T]; \mathbf{L}^2(\Omega)^3)$ and $U^s \in \mathbb{D}(\mathbb{A}_s)$, then problem (\mathcal{P}_s) possesses a unique solution in $\mathbf{C}^1([0, T]; \mathbb{H}_s) \cap \mathbf{C}^0([0, T]; \mathbb{D}(\mathbb{A}_s))$.*

Due to Stone's theorem, the proof of theorem 1 just requires to show that operator \mathbb{A}_s is skew-adjoint. First, from the definition of operator \mathbb{A}_s we have $\forall U = (u, v) \in \mathbb{D}(\mathbb{A}_s)$, $(\mathbb{A}_s U, U)_{\mathbb{H}_s} = \int_{\Omega} \gamma w \cdot v dx + a_s(u, v) = 0$. Secondly, given an arbitrary $\Phi = (\Phi^1, \Phi^2)$ in \mathbb{H}_s , it is clear that for U such that $U - \mathbb{A}_s U = \Phi$ we have:

$$u \in \mathbf{H}_{\Gamma_0}^1(\Omega)^3, \quad \int_{\Omega} \gamma(u, \Phi^1 - \Phi^2) \phi dx + a_s(u, \phi) = 0 \quad \forall \phi \in \mathbf{H}_{\Gamma_0}^1(\Omega)^3. \quad (5)$$

The existence and uniqueness of u follows from Lax-Milgram lemma, and then:

$$U := (u, u - \Phi^1) \in \mathbb{D}(\mathbb{A}_s), \quad U - \mathbb{A}_s U = \Phi. \quad (6)$$

5 The asymptotic behavior as the parameters tend to zero

We shall now make use of Trotter's theory about the convergence of semi-groups of operators acting on variable Hilbert spaces. Let us first recall a brief definition of a semigroup. The notion of semi-groups follows from that of groups by removing the requirement of a symmetrical element. In the case of sets of operators this notion can be understood by the following example which can be seen as a basic introduction to the flows in the theory of ordinary differential equations. Let us assume that an ordinary differential equation of the following form is given:

$$\frac{du}{dt} = \mathbb{T}u,$$

the latter being associated with some initial data $u(0)$ and \mathbb{T} being some operator whose domain and properties will be given explicitly in the particular case we shall deal with. The solution is then formally $u^t := g^t u(0)$ and satisfies:

- i) for any positive real numbers t_1 and t_2 , $g^{t_1+t_2} = g^{t_1} \cdot g^{t_2}$,
- ii) $g(0) = \mathbb{I}$.

The set $\{g^t\}$ for $0 < t < \infty$ is called a one parameter semi-group generated by \mathbb{T} .

Let us now come back to problem (\mathcal{P}_s) and to its transformation into equa-

tion (4). Assume that the elasticity coefficients are such that the ratio λ/ε and μ/ε have finite limits as the thickness tends to zero: $\lambda/\varepsilon \rightarrow \bar{\lambda} \in (0, +\infty)$ and $\mu/\varepsilon \rightarrow \bar{\mu} \in (0, +\infty)$. This can be seen as a nondegeneracy assumption, but the cases $\bar{\lambda}, \bar{\mu} \in \{0, +\infty\}$ could be more or less studied in the same way.

Let $\mathbb{H}_{\Gamma_0}^1(\Omega \setminus S)^3 = \{u \in \mathbb{H}^1(\Omega \setminus S)^3; u = 0 \text{ on } \Gamma_0\}$. For any $u \in \mathbb{H}_{\Gamma_0}^1(\Omega \setminus S)^3$ let us now define two functions u^+ and u^- as elements of $\mathbb{H}^1(\Omega^\pm)$ which are the restrictions of functions u to $\Omega^\pm = \{x \in \Omega; \pm x_3 > 0\}$. In this way we can define a jump $[u] \in \mathbb{L}^2(S)$ as the difference between the traces on S of u^+ and u^- . Let n be the third axis of the frame of \mathbb{R}^3 that is the normal vector to S oriented from Ω^- to Ω^+ , and for any vector ξ of \mathbb{R}^3 let us denote $\xi_N = \xi \cdot n$ and $\xi \otimes_s \eta = \frac{1}{2}(\xi \otimes \eta + \eta \otimes \xi)$; $\xi, \eta \in \mathbb{R}^3$.

We can now define a continuous $\mathbb{H}_{\Gamma_0}^1(\Omega \setminus S)^3$ -elliptical bilinear form on $\mathbb{H}_{\Gamma_0}^1(\Omega \setminus S)^3$ as:

$$\begin{aligned} & \text{for } \phi, \psi \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3, \\ & a(\phi, \psi) = \int_{\Omega \setminus S} ae(\phi) \cdot e(\psi) dx + \int_S \bar{\lambda} [\phi]_N [\psi]_N + 2\bar{\mu} [\phi] \otimes_s n \cdot [\psi] \otimes_s n d\hat{x}, \end{aligned} \quad (7)$$

where \hat{x} denotes (x_1, x_2) if x is (x_1, x_2, x_3) .

The space \mathbb{H} of finite energy states in which the problem governing the asymptotic behavior of u_s will be formulated is then:

$$\mathbb{H} = \{U = (u, v) \in \mathbb{H}_{\Gamma_0}^1(\Omega \setminus S)^3 \times \mathbb{L}^2(\Omega)^3\} \quad (8)$$

endowed with the following norm and scalar product:

$$|U|_{\mathbb{H}}^2 = |(u, v)|_{\mathbb{H}}^2 := a(u, u) + \int_{\Omega} \bar{\rho} |v|^2 dx; \quad (U, U')_{\mathbb{H}} = a(u, u') + \int_{\Omega} \bar{\rho} v \cdot v' dx.$$

Since the space \mathbb{H} of states of finite energy is different from the natural phase space \mathbb{H}_s , we introduce a family of linear operators from \mathbb{H} to \mathbb{H}_s , $\mathbb{P}_s \in \mathcal{L}(\mathbb{H}, \mathbb{H}_s)$, which aim at "comparing" an element of \mathbb{H} with an element of \mathbb{H}_s :

$$U = (u, v) \in \mathbb{H} \longrightarrow \mathbb{P}_s U = (u_s, v_s) \in \mathbb{H}_s,$$

$$\begin{aligned} u_s(x) &= \mathbb{R}_\varepsilon u(x) := \frac{1}{2} \left\{ u(\hat{x}, x_3) + u(\hat{x}, -x_3) + \min\left(\frac{|x_3|}{\varepsilon}, 1\right) (u(\hat{x}, x_3) - u(\hat{x}, -x_3)) \right\}, \\ v_s(x) &= v(x) \text{ if } x \in \Omega^\varepsilon, \quad v_s(x) = v(x) \cdot \left(\frac{\bar{\rho}(x)}{\rho}\right)^{1/2} \text{ if } x \in B^\varepsilon. \end{aligned} \quad (9)$$

It is fundamental to observe that:

$$\begin{aligned} & i) \exists C > 0, |\mathbb{P}_s U|_{\mathbb{H}_s} \leq C |U|_{\mathbb{H}}, \quad \forall U \in \mathbb{H}, \quad \forall s \neq 0, \\ & ii) \lim_{s \rightarrow 0} |\mathbb{P}_s U|_{\mathbb{H}_s} = |U|_{\mathbb{H}}. \end{aligned} \quad (10)$$

In the same way, for $s \neq 0$, it is clear that:

$\exists! u^e \in \mathbf{C}^{0,1}([0, T]; \mathbf{H}_{\Gamma_0}^1(\Omega \setminus S)^3)$; such that $a_s(u^e, \phi) = \int_{\Gamma_1} g \phi ds, \forall \phi \in \mathbf{H}_{\Gamma_0}^1(\Omega \setminus S)^3$,

and we define the operator \mathbb{A} as:

$$\left\{ \begin{array}{l} \mathbb{D}(\mathbb{A}) = \{U = (u, v) \in \mathbf{H}; v \in \mathbf{H}_{\Gamma_0}^1(\Omega \setminus S)^3 \text{ and } \exists! w \in \mathbf{L}^2(\Omega)^3 \text{ such that} \\ \int_{\Omega} \bar{\rho} w \cdot \phi dx + a(u, \phi) = 0, \forall \phi \in \mathbf{H}_{\Gamma_0}^1(\Omega \setminus S)^3\} \\ \mathbb{A}U = \mathbb{A}(u, v) = (u, v), \end{array} \right. \quad (11)$$

Let us now introduce $F = (0, \frac{f}{\bar{\rho}} - \frac{du^e}{dt})$ and consider the following evolution equation in \mathbf{H} :

$$\frac{dU^r}{dt} = \mathbb{A}U^r + F, \quad U^r(0) = U_0^r. \quad (12)$$

In the same way as for operator \mathbb{A}_s , it is clear that operator \mathbb{A} is skew-adjoint in \mathbf{H} , so that we have:

Theorem 2. *Let assumption \mathbb{H}_1 be satisfied and assume $f \in \mathbf{C}^{0,1}([0, T]; \mathbf{L}^2(\Omega)^3)$ and $U_0^r \in \mathbb{D}(\mathbb{A})$, then equation (12) possesses a unique solution in $\mathbf{C}^1([0, T]; \mathbf{H}) \cap \mathbf{C}^0([0, T]; \mathbb{D}(\mathbb{A}))$.*

It is easily shown that $u := u^r + u^e$ satisfies formally the following problem (\mathcal{P})

$$(\mathcal{P}) \left\{ \begin{array}{l} \bar{\rho} \frac{\partial^2 u}{\partial t^2} - \operatorname{div} \sigma = f \quad \text{in } \Omega \setminus S \times [0, T] \\ \sigma_s = ae(u) \quad \text{in } \Omega \setminus S \times [0, T] \\ \sigma n = g \quad \text{on } \Gamma_1, \quad u_s = 0 \quad \text{on } \Gamma_0, \\ \left\{ \begin{array}{l} u(x, 0) = u_0^r(x) + u^e(x, 0) = u^0(x), \\ \partial_t u(x, 0) = v_0^r(x) + \partial_t u^e(x, 0) := v^0(x) \end{array} \right. \quad \forall x \in \Omega \\ \text{and} \\ [\sigma n] = 0, \quad \sigma n + \lambda[u]_n n + 2\mu[u] \otimes_s n = 0 \quad \text{on } S. \end{array} \right. \quad (13)$$

It remains to show that u_s converges to u when $s \rightarrow 0$. Let us recall the following result which has been established in the static case [10].

Proposition 1. *If $s \rightarrow 0$, then $u_s^e \rightarrow u^e$ in $\mathbf{L}^2(\Omega)^3$, $u_s \rightarrow u$ in $\mathbf{H}^1(\Omega^\eta)^3$, $\forall \eta > 0$, $a_s[u_s^e, u_s^e] \rightarrow a(u^e, u^e)$ and $u_s^e|_{x_3=\varepsilon} - u_s^e|_{x_3=-\varepsilon} \rightarrow [u^e]$ in $\mathbf{L}^2(S)$.*

This gives the convergence of U_s^e towards U^e . The convergence of U_s^r towards U^r will be given by Trotter's theory of approximation of semi-groups which roughly says that "if the stationary problems are converging, then the dynamical problems will also converge". More precisely (see [13])

Theorem 3. Assume operators \mathbb{A}_s and \mathbb{A} are anti-adjoint in the Hilbert spaces \mathbb{H}_s and \mathbb{H} and satisfy (9) and (10). Assume in addition:

i) $\lim_{s \rightarrow 0} |\mathbb{P}_s(\mathbb{I} - \mathbb{A})^{-1}f - (\mathbb{I} - \mathbb{A}_s)^{-1}\mathbb{P}_s f|_{\mathbb{H}_s} = 0 \quad \forall f \in \mathbb{H}$,

ii) $\lim_{s \rightarrow 0} |\mathbb{P}_s \mathbb{U}^0 - \mathbb{U}_s^0|_{\mathbb{H}_s} = 0$,

iii) $\lim_{s \rightarrow 0} \int_0^T |\mathbb{P}_s \mathcal{F}_s(t) - \mathcal{F}_s(t)|_{\mathbb{H}_s} dt = 0$,

If \mathbb{U}_s and \mathbb{U} are respectively the solutions to equations $\frac{d\mathbb{U}_s}{dt} = \mathbb{A}_s \mathbb{U}_s + \mathcal{F}_s$, $\mathbb{U}_s(0) =$

\mathbb{U}_s^0 and $\frac{d\mathbb{U}}{dt} = \mathbb{A} \mathbb{U} + \mathcal{F}$, $\mathbb{U}(0) = \mathbb{U}^0$,

then $\lim_{s \rightarrow 0} \text{Sup} \left\{ |\mathbb{P}_s \mathbb{U}(t) - \mathbb{U}_s(t)|_{\mathbb{H}_s}, t \in [0, T] \right\} = 0$.

This abstract result allows us to establish:

Theorem 4. Let assumption \mathbb{H}_1 be satisfied and assume $f \in \mathbf{C}^{0,1}([0, T]; \mathbb{L}^2(\Omega)^3)$.

Assume in addition that

$$\lim_{s \rightarrow 0} |\mathbb{P}_s \mathbb{U}^0 - \mathbb{U}_s^0|_{\mathbb{H}_s} = 0, \quad \mathbb{U}^0 - \mathbb{U}^e(0) \in \mathbb{D}(\mathbb{A}), \quad \mathbb{U}_s^0 - \mathbb{U}_s^e(0) \in \mathbb{D}(\mathbb{A}_s) \quad (14)$$

(compatibility/convergence assumption),

then the solutions \mathbb{U}_s^r and \mathbb{U}^r of problems (4) and (12) satisfy:

$$\lim_{s \rightarrow 0} \text{Sup} \left\{ |\mathbb{P}_s \mathbb{U}^r(t) - \mathbb{U}_s^r(t)|_{\mathbb{H}_s}, t \in [0, T] \right\} = 0.$$

Remark 1. This Trotter's convergence (that is $\lim_{s \rightarrow 0} |\mathbb{P}_s \mathbb{U}^0 - \mathbb{U}_s^0|_{\mathbb{H}_s} = 0$) is very natural and seems well suited from a mechanical point of view since it deals with a gap of energy, but it may be reassuring to compare this convergence with more usual points of view. This is the purpose of the next proposition.

Proposition 2. i) if $\lim_{s \rightarrow 0} |\mathbb{P}_s \mathbb{U} - \mathbb{U}_s|_{\mathbb{H}_s} = 0$, then $u_s \rightarrow u$ in $\mathbb{L}^2(\Omega)^3$ and $u_s \rightarrow u$ in $\mathbb{H}^1(\Omega^\eta)^3$, $\forall \eta > 0$,

ii) if $a_s(u_s, u_s) \rightarrow a(u, u)$ and $a_s(\mathbb{R}_\varepsilon u, u_s) \rightarrow a(u, u)$ then $\lim_{s \rightarrow 0} a_s(\mathbb{R}_\varepsilon u - u_s, \mathbb{R}_\varepsilon u - u_s) = 0$.

Proof:

The assumption of point i) implies that $a_s[u_s, u_s]$ is bounded, so that the result is established in [9] or [10].

Point ii) follows immediately from the already noticed fact that $a_s(\mathbb{R}_\varepsilon u, \mathbb{R}_\varepsilon u) \rightarrow a(u, u)$.

We can now prove theorem 4.

The proof simply consists in showing that assumptions i), ii) and iii) of Trotter's theorem are satisfied.

Points ii) and iii) are immediate consequences of assumption \mathbb{H}_1 together with propositions 1 and 2.

Remark 2. There exists many initial data satisfying (14):

$$\mathbb{U}_s^0 - \mathbb{U}_s^e(0) := (\mathbb{I} - \lambda \mathbb{A})^{-1} \mathbb{P}_s (\mathbb{I} - \lambda \mathbb{A}) (\mathbb{U}^0 - \mathbb{U}^e(0)) \quad \forall \lambda \in \mathbb{R} \setminus \{0\},$$

since one has Trotter's convergence of the resolvents of operators \mathbb{A} and \mathbb{A}_s .

Let's now focuss on point i). Let $U = (u, v)$ and $U_s = (u_s, v_s)$ satisfying respectively $U - \mathbb{A}U = \Phi$ and $U_s - \mathbb{A}_s U_s = \mathbb{P}_s \Phi$. From equations (5) and (6) we have:

$$\begin{cases} u_s \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3, \\ \int_{\Omega} \gamma u_s \cdot \phi \, dx + a_s(u_s, \phi) = \int_{\Omega} \gamma(\Phi_s^1 + \Phi_s^2) \phi \, dx \quad \forall \phi \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3, \\ v_s = u_s - \Phi_s^1, \end{cases} \quad (15)$$

$$\begin{cases} u \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3, \\ \int_{\Omega} \gamma u \cdot \phi \, dx + a(u, \phi) = \int_{\Omega} \gamma(\Phi^1 + \Phi^2) \phi \, dx \quad \forall \phi \in \mathbb{H}_{\Gamma_0}^1(\Omega)^3, \\ v = u - \Phi^1. \end{cases} \quad (16)$$

This means that u and u_s are solutions to a perturbation of equilibrium bonding problems, so that we only have to establish the convergence of the equilibrium problems, which has been carried out in [10]. Using the same tools we get that conditions ii) of proposition 2 are satisfied, which in turn establishes point i) of Trotter's theorem.

6 Concluding remarks

Starting from the physical problem (\mathcal{P}_s) of the wave propagation in two linearly elastic solids connected by a layer of thickness ε we have obtained a limit problem (\mathcal{P}) when the parameter s tends to zero, i.e. when the thickness ε of the layer tends to zero together with assumptions on the voluminal mass and on the elasticity coefficients. The mathematical meaning of this convergence result is that the dynamical behavior of two solids connected by a thin soft layer is asymptotically equivalent to the one of two solids connected by the following mechanical constraint:

$$\sigma n = \bar{\lambda} [u]_n \mathbb{1}d + 2\bar{\mu} [u] \otimes_s n = 0 \quad \text{with } \bar{\lambda}, \bar{\mu} = \lim(\lambda, \mu)/\varepsilon \quad (17)$$

The main worth seeing point is that condition (17) is the stationary condition which is obtained at the limit of the equilibrium bonding problem, which has been recalled in the first section. This has been explicitly carried out here in the case when the limits $\bar{\lambda}$ and $\bar{\mu}$ are finite and different from zero. The remaining cases $\bar{\lambda}, \bar{\mu} \in \{0, \infty\}$ could be handled with very slight changes so that it is not useful to write down this cases again here. Only the case where the voluminal mass ρ tends to infinity may be more difficult but this case is probably less interesting from a physical point of view.

Now it seems interesting to close this paper by some comments on the mechanical meaning of the result. According to the different cases of the limit behavior of the parameters, the set of results is given in Table 1. Let us describe the result in some particular cases.

- The case where $\lambda/\varepsilon \rightarrow \bar{\lambda}$ and $\mu/\varepsilon \rightarrow \bar{\mu}$ corresponds to the physical situation where the thickness of the layer is small with respect to the diameter of the whole domain and the stiffness of the layer is also small, "at the same order" with respect to the one of both other parts of domain. This could be seen as the generic case. The meaning of the result is that the thin layer will behave as a line of springs both in the normal and in the tangential directions. In a numerical model, say by a finite element method, the thin layer could be replaced by a set of linear elastic relations between the opposite nodes, with stiffnesses given by the limit problem.
- But when the thickness and the stiffness of the layer are both small but in such a way that the ratio of the thickness of the layer with respect to the diameter of the whole domain is much smaller than the ratio of the stiffnesses, the result is very different. This situation will correspond to the case where $\lambda/\varepsilon \rightarrow \infty$ and $\mu/\varepsilon \rightarrow \infty$ and the body will globally behave as two half bodies perfectly stuck.
- On the contrary if the physical problem is such that the thickness and the stiffness of the layer and both small but in such a way that the ratio of the thickness of the layer with respect to the diameter of the whole domain is much larger than the ratio of the stiffnesses then the body will behave as two completely separated parts.
- In intermediate cases which will for instance be modeled by the fact that $\lambda/\varepsilon \rightarrow (\infty \text{ or } \bar{\lambda})$ and $\mu/\varepsilon \rightarrow 0$, we may get a rigid or an elastic connection between the two parts in the normal direction but a free sliding in the tangential direction.

Now it remains to put the above results in correlation with previous analyses of bonded solids, and in particular with the occurrence of guided waves along the thin layer. As a first step, this guided waves analysis could be performed in the case of two half spaces connected by an infinite layer. Another point is that, from a physical point of view, bonding layers usually involve viscoelastic properties. Viscoelasticity in thin layers has already been taken into account in equilibrium problems, so that it can probably be introduced in the previous analysis in some cases. But the kind of viscoelastic behaviors encountered in physics seems to be precisely those for which important difficulties remain.

References

1. Adams R.A., Sobolev spaces, Academic Press, New York (1975)
2. Ait Moussa A., Modélisation et étude des singularités d'un joint collé, PhD thesis, Montpellier II University (1989)
3. Attouch H., Variational convergence for functions and operators, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984)
4. Geymonat G. and Krasucki F., Analyse asymptotique du comportement en flexion de deux plaques collées, C. R. Acad. Sci., I, 325, 307-314 ((1997)
5. Gupta R.N., Reflection of elastic waves from a linear transition layer, Bull. Seism. Soc. Am., 701-717 (1966)

6. Jones J.P. and Whittier J.S., Waves at flexibility bonded interface, *J. Appl. Mech.* 905-908 (1967)
7. Klarbring A., Derivation of the adhesively bonded joints by the asymptotic expansion method, *Int. J. Engineering Sci.*, 29, 493-512 (1991)
8. Lebon F., Ould-Kaoua A. and Licht C., Numerical study of soft adhesively bonded joints in finite elasticity, *Computational Mechanics*, 21, 134-140 (1997)
9. Licht C. and Michaille G., Une modélisation du comportement d'un joint collé, *C. R. Acad. Sci.*, I, 322, 295-300 (1996)
10. Licht C. and Michaille G., A Modeling of elastic adhesive bonded joints, *Advances in Mathematical Sciences and Applications*, 7, 711-740 (1997)
11. Suquet P., Discontinuities and Plasticity, in *Nonsmooth mechanics and applications*, CISM Courses and Lectures 302, Springer, Berlin (1998)
12. Thompson W.T., Transmission of elastic waves through a stratified solid medium, *J. Appl. Physics*, 21, 89-93 (1950)
13. Trotter H.F., Approximation of semi-groups of operators, *Pacific J. Maths.*, 28, 897-919 (1958)
14. Vlassie V. and Rousseau M., Acoustical validation of the rheological models for a structural bond, *Wave Motion*, 37, 333-349 (2003)