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ANALYTIC MICROLOCAL BOHR-SOMMERFELD EXPANSIONS

ANTIDE DURAFFOUR*

ABSTRACT. This article is devoted to Gevrey-analytic estimates in \hbar , of the Bohr-Sommerfeld expansion of the eigenvalues of self-adjoint pseudo-differential operators $P := p^w(x, \hbar D_x)$ acting on $L^2(\mathbb{R})$ in the regular case. We consider an interval of energies (E_1, E_2) in which the spectrum of P is discrete and such that the energy sets $\{p(x, \xi) = E\}$ are regular connected curves. Under some assumptions on the holomorphy of the symbol p , we will use the isometry between $L^2(\mathbb{R})$ and the Bargmann space to prove that the spectrum $\sigma_\hbar(P) \cap (E_1, E_2)$ of $p^w(x, \hbar D_x)$ is described by equations of the form

$$(0.1) \quad \int_{\text{orbit}} \xi dx + \pi\hbar + c_2(\lambda)\hbar^2 + \dots \in 2\pi\hbar\mathbb{Z}.$$

More precisely it is possible to build exponentially sharp quasimodes in the Bargmann space. This will allow us to describe the “Bohr-Sommerfeld” spectrum with precision $\mathcal{O}(e^{-\mathcal{C}/\hbar})$ when $\hbar \rightarrow 0^+$ for some small $\mathcal{C} > 0$. A precise examination of the principal symbols will provide an interpretation to the Maslov correction $\pi\hbar$ in the Bargmann space.

1. INTRODUCTION

The *Bohr-Sommerfeld* rules are one of the oldest “quantization” results for obtaining quantum levels from the Hamiltonian of classical mechanics. They appeared with Bohr’s model postulating that the orbits of the electrons in the hydrogen atoms must satisfy the quantization condition

$$(1.1) \quad \frac{1}{2\pi} \oint_{\text{orbit}} \xi dx \in \hbar\mathbb{Z}.$$

They were extended by Sommerfeld along elliptic orbits and made more precise by adding Maslov’s correction [Mas72] replacing $\hbar\mathbb{Z}$ by $\hbar(\mathbb{Z} + \frac{1}{2})$ in the previous equation. In mathematical terms, these rules are expected to give the eigenvalues of a (pseudo-)differential operator in terms of the geometry of its principal symbol. Except in very specific cases like the Harmonic oscillator, these “Bohr-Sommerfeld eigenvalues” are only approximations of the true eigenvalues, with an error term that tends to zero as the semiclassical parameter \hbar tends to zero. Historically, the Bohr-Sommerfeld rules were given for Schrödinger operators:

$$P = -\hbar^2 \frac{d^2}{dx^2} + V(x).$$

They describe regular eigenvalues, that is discrete eigenvalues in intervals with not critical value of V , with an error term of order $\mathcal{O}(\hbar^2)$, but it has been proved by Helffer and Robert [HR84] that these rules can be extended to 1D pseudo-differential operators $P = p^w(x, \hbar D_x)$

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(denoting $\hbar D_x := \frac{\hbar}{i} \frac{d}{dx}$) given by the Weyl quantization

$$(1.2) \quad \forall u \in \mathcal{S}(\mathbb{R}), \quad p^w(x, \hbar D_x)u := \frac{1}{2\pi\hbar} \iint_{\mathbb{R}^2} p\left(\frac{x+y}{2}, \xi\right) e^{i\xi(x-y)/\hbar} u(y) d\xi dy.$$

They give asymptotic expansions for eigenvalues that are exact to any order¹ $\mathcal{O}(\hbar^N)$, $N > 0$. In the region where the spectrum is discrete and the energy surfaces $E_\lambda := \{(x, \xi) \in \mathbb{R}^2 \mid p(x, \xi) = \lambda\}$ are regular connected curves, the eigenvalues admit a full asymptotic expansion starting as follows, for some $k \in \mathbb{Z}$

$$(1.3) \quad \lambda_k = A^{-1}(2\pi(k - 1/2)\hbar) + \mathcal{O}(\hbar^2) \text{ where } A(\lambda) := \int_{E_\lambda} \xi dx.$$

The Bohr-Sommerfeld rules are microlocal in essence as they are based upon quantities defined in the quantum phase-space. They do not necessarily describe the entire spectrum of the operator P , but only those eigenvalues that are associated with a regular, connected component of a level set of the classical Hamiltonian $p(x, \xi)$. The full spectral asymptotics, i.e. approximation with sharpness $\mathcal{O}(\hbar^\infty)$ of these eigenvalues, were tackled in many articles. For instance, they can be found in Rozenblum's article [Roz75] in the case of the circle or in Helffer-Robert [HR84] for a much broader presentation.

In scenarios of level sets containing several connected components, the approximate spectrum is obtained by taking the union of the Bohr-Sommerfeld eigenvalues of each component (although this statement is difficult to locate in the literature). From the viewpoint of quantum mechanics, separate components can actually interact, due to *tunnelling*. However, the impact of tunnelling on eigenvalues is expected to be exponentially small: $\mathcal{O}(e^{-\mathcal{E}/\hbar})$ and does not affect the asymptotic expansion. An interesting situation occurs when the Bohr-Sommerfeld expansions for two different components are identical; this occurs, for instance, for a Schrödinger operator with a symmetric double well. The usual Bohr-Sommerfeld expansion proves that the spectrum contains doublets of eigenvalues that are $\mathcal{O}(\hbar^\infty)$ -close to each other. However, it cannot prove the exponentially small splitting obtained by Helffer-Sjöstrand [HS84] in the case of the Schrödinger operator $-\Delta + V(x)$, and hence fails to confirm an aspect of the quantum tunnelling.

The goal of this paper is to reach an exponentially small $\mathcal{O}(e^{-\mathcal{E}/\hbar})$ but not optimal precision in the case of a single connected component in a broader setting than Schrödinger Operators.

Let us finally mention that the recent work of Hitrik-Zworski [HZ24] gives an analogue of Theorem 1.4 in the case of an elliptic singularity, in both self adjoint and non self adjoint settings.

1.1. Mathematical Formulation of the Problem. In order to be general enough, we will consider unbounded pseudo-differential operators defined on dense domains in $L^2(\mathbb{R})$. We will also need to introduce the holomorphic framework that will allow us to get exponentially accurate estimates.

¹We say that the precision is $\mathcal{O}(\hbar^\infty)$.

1.1.1. *Assumptions on the symbol.* We start with defining what a *tubular neighbourhood of a vector space* is, which will be useful for introducing the holomorphic setting.

Definition 1.1 (Tubular Neighbourhood). Letting $d \in \mathbb{N}^*$, a tubular neighbourhood² of a vector subspace $S \in \mathbb{C}^d$, is a small open neighbourhood of S in \mathbb{C}^d , satisfying $U + S = U$ and $U \subset S + \mathbb{B}_{\mathbb{C}^d}(0, r_0)$ for some $r_0 > 0$.

Note also that in Appendix A we have recalled the main results about *classical analytic symbols* developed in [Sjö82] necessary for understanding the formulation of the main Theorem 1.4.

We consider a pseudo-differential operator $P : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ given by the Weyl quantization

$$(1.4) \quad \forall u \in \mathcal{S}(\mathbb{R}), \quad Pu := \frac{1}{2\pi\hbar} \iint_{\mathbb{R}^2} e^{\frac{i}{\hbar}(x-y)\xi} p\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi,$$

We use order functions $\mathbf{m} \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}_+^*)$ defined in [Zwo12, Chapter 4.4] or [Mar02, Chapter 2] which among other satisfy, for some $N \in \mathbb{N}$,

$$(1.5) \quad \exists N \in \mathbb{N}, \exists C > 0, \forall (X, \widehat{X}) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad \mathbf{m}(X) \leq C \langle X - \widehat{X} \rangle^N \mathbf{m}(\widehat{X}),$$

where $\langle (x, \xi) \rangle = (1 + x^2 + \xi^2)^{\frac{1}{2}}$ is the usual Japanese bracket. We will see that the property (1.5) will be very useful when estimating norms of kernel operators with Schur's Lemma. In \mathbb{C}^2 , we extend this by requiring (1.5) with the notation

$$\forall (x, \xi) \in \mathbb{C}^2, \quad \langle (x, \xi) \rangle = \langle (\operatorname{Re}(x), \operatorname{Re}(\xi)) \rangle \text{ and } \mathbf{m}(x, \xi) := \mathbf{m}(\operatorname{Re}(x), \operatorname{Re}(\xi)).$$

We let U_0 be a tubular neighbourhood of \mathbb{R}^2 and define

$$(1.6) \quad B(U_0, \mathbf{m}) := \{u \in \operatorname{Hol}(U_0) \mid \exists C > 0, |u| \leq C \mathbf{m}\}.$$

The reader who is used to symbols classes in the smooth setting will notice the following.

Remark 1.2. Thanks to the Cauchy estimates, letting $U \subset U_0$ (another tubular neighbourhood of \mathbb{R}^2), satisfying $\overline{U} \subset U_0$, the set $B(U_0, \mathbf{m})$ is contained in $S(U, \mathbf{m})$ defined by

$$(1.7) \quad S(U, \mathbf{m}) = \{u \in \mathcal{C}^\infty(U) \mid \forall \alpha \in \mathbb{N}, |\partial^\alpha u| \leq C_\alpha \mathbf{m}\}.$$

Let us consider an interval of energies (E_1, E_2) , we suppose that for some $\delta > 0$, that the following assumptions *H.1*, *H.2*, *H.3* hold.

H.1 Localisation condition: $p^{-1}((-\infty, E_2 + \delta])$ is compact and non empty,

H.2 Description of the energy sets: $dp \neq 0$ on $p^{-1}((E_1 - \delta, E_2 + \delta))$ and the energy curves $E_\lambda := \{p = \lambda\}$ are connected for $\lambda \in (E_1 - \delta, E_2 + \delta)$.

H.3 Ellipticity: There exists $C, C_0 > 0$ such that $\frac{1}{C_0} \mathbf{m} \leq C + p \leq C_0 \mathbf{m}$ on U_0 .

Remark 1.3. In the case where $\liminf_{|(x, \xi)| \rightarrow +\infty} \mathbf{m} = +\infty$ Assumption *H.1* is a consequence of *H.3* if we assume $p^{-1}((-\infty, E_2 + \delta])$ is non empty.

By construction the operator P is symmetric and defined on the domain $\mathcal{S}(\mathbb{R})$. Thanks to the usual results on pseudo-differential operators, since on \mathbb{R}^2 , $|p - i| \gtrsim 1$ and $p - i \in S(\mathbb{R}^2, \mathbf{m})$, $P - i$ is invertible for \hbar small enough. This proves that $(P, \mathcal{S}(\mathbb{R}))$ admits a unique self-adjoint extension that we will denote $(P, \mathcal{D}(P))$. By Fredholm theory (for instance see [DZ19,

²That we will sometimes denote $\operatorname{Neightube}(S)$.

Appendix C]), since p is elliptic in $S(\mathbb{R}^2, \mathfrak{m})$ and $p^{-1}((-\infty, E_2 + \delta])$ is compact, this implies that P has discrete spectrum in $(-\infty, E_2 + \delta)$ for some smaller $\delta > 0$.

1.1.2. *Statement of the Main Theorem.* We will see that in this setting, the action integral $A(\lambda) = \int_{\{p=\lambda\} \cap \mathbb{R}^2} \xi dx$ is well defined for $\lambda \in (E_1 - \delta, E_2 + \delta)$ and invertible as a function of λ .

Theorem 1.4. *There exist a classical analytic symbol $\lambda(\hbar, I) = A^{-1}(I - \pi\hbar) + \mathcal{O}(\hbar^2)$ defined for I in a small neighbourhood of $A([E_1, E_2])$ and $\mathcal{C} > 0$ such that, for \hbar small enough*

$$(1.8) \quad \begin{aligned} \forall \lambda_{\hbar} \in \sigma_{\hbar}(P) \cap (E_1, E_2), \quad \exists! k \in \mathbb{Z}, \quad |\lambda_{\hbar} - \lambda(\hbar, 2\pi k\hbar)| \leq \frac{1}{\mathcal{C}} e^{-\mathcal{C}/\hbar}, \\ \forall k \in \mathbb{Z}, \quad \text{s.t. } \lambda(\hbar, 2\pi k\hbar) \in (E_1, E_2), \quad \exists! \lambda_{\hbar} \in \sigma_{\hbar}(P), \quad |\lambda_{\hbar} - \lambda(\hbar, 2\pi k\hbar)| \leq \frac{1}{\mathcal{C}} e^{-\mathcal{C}/\hbar}, \end{aligned}$$

and the eigenvalues of P in (E_1, E_2) are simple.

Paying the price of a small imprecision, it is more natural to remember that we want to prove that $\sigma_{\hbar}(P) \cap (E_1, E_2)$ is

$$(1.9) \quad \left\{ \lambda(\hbar, 2\pi k\hbar) + \mathcal{O}(e^{-\mathcal{C}/\hbar}) \mid k \in \mathbb{Z} \right\} \cap (E_1, E_2).$$

In the following sections, we show that for each $\lambda_0 \in \text{Neigh}_{\mathbb{R}}([E_1, E_2])$ Theorem 1.4 stands replacing (E_1, E_2) with some small open neighbourhood of λ_0 . By the patching Lemma A.7 the result will extend on the whole segment $[E_1, E_2]$.

1.2. Scheme of the Proof.

In the first section we follow [Sjö] or [Zwo12, Chapter 13] and conjugate $p^w(x, \hbar D_x)$ with the Bargmann transform into a new pseudo-differential operator $p_{\Phi_0}(z, \hbar D_z)$ acting on the Bargmann space. We then use Sjöstrand's result obtained in [Sjö82, Chapter 4] to build analytic WKB expansions around points of the complex energy curves. In the end, we show that the condition on λ for which we can merge these local WKB expansions (using a $\bar{\partial}$ lemma) into a full quasimode yields the Bohr-Sommerfeld quantization condition with exponential sharpness.

2. BARGMANN TRANSFORM AND COMPLEX PSEUDO-DIFFERENTIAL OPERATORS

In this section we introduce some aspects of the theory of the Bargmann space and complex analytic pseudo-differential operators. After conjugating P with the Bargmann transform, we write the operator as an analytic pseudo-differential operator as defined by Sjöstrand in [Sjö82, Chapter 3] as integrals on “good contours”.

2.1. The Bargmann Transform.

We define the spaces of weighted holomorphic functions, see for instance Zworski's book [Zwo12, Chapter 13]. Here we will only use the weight associated with the Bargmann transform

$$(2.1) \quad \Phi_0(z) = \frac{1}{2}|z|^2.$$

We define the global spaces with cursive notations, denoting $dL(z) = \frac{2}{i}d\bar{z} \wedge dz$ the Lebesgue measure,

$$(2.2) \quad \begin{aligned} L_{\Phi_0}^2(\mathbb{C}) &:= \left\{ f : \mathbb{C} \rightarrow \mathbb{C} \text{ measurable} \mid \int_{\mathbb{C}} |f(z)|^2 e^{-2\Phi_0(z)/\hbar} dL(z) < \infty \right\}, \\ L_{\Phi_0}^2(\mathbb{C}, \mathbf{m}) &:= \left\{ f : \mathbb{C} \rightarrow \mathbb{C} \text{ measurable} \mid \int_{\mathbb{C}} |f(z)|^2 \mathbf{m}(z)^2 e^{-2\Phi_0(z)/\hbar} dL(z) < \infty \right\}, \\ H_{\Phi_0}(\mathbb{C}) &:= L_{\Phi_0}^2(\mathbb{C}) \cap \text{Hol}(\mathbb{C}), \quad H_{\Phi_0}(\mathbb{C}, \mathbf{m}) := L_{\Phi_0}^2(\mathbb{C}, \mathbf{m}) \cap \text{Hol}(\mathbb{C}). \end{aligned}$$

The Bargmann transform $\mathcal{T}_0 : L^2(\mathbb{R}) \rightarrow H_{\Phi_0}(\mathbb{C})$ defined by

$$(2.3) \quad \forall v \in L^2(\mathbb{R}), \quad \mathcal{T}_0 v(z) := \frac{1}{(\pi\hbar)^{3/4}} \int_{\mathbb{R}} e^{\frac{i}{\hbar}\varphi_0(z,y)} v(y) dy, \quad \varphi_0(z,y) = i \left(\frac{z^2}{2} - \sqrt{2}yz + \frac{y^2}{2} \right),$$

is a unitary transform (see [Zwo12, Theorem 13.7]) and an isometry. It is associated with the symplectomorphism

$$(2.4) \quad \kappa_0 : \begin{cases} \mathbb{R}^2 & \longrightarrow \Lambda_{\Phi_0} \subset \mathbb{C}^2 \\ (x, \xi) & \longmapsto \frac{1}{\sqrt{2}}(x - i\xi, \xi - ix) \end{cases} \quad \text{and} \quad \kappa_0^{-1} : \begin{cases} \Lambda_{\Phi_0} & \longrightarrow \mathbb{R}^2 \\ (z, \zeta) & \longmapsto \frac{1}{\sqrt{2}}(z + i\zeta, \zeta + iz) \end{cases},$$

where

$$\Lambda_{\Phi_0} := \{(x - i\xi, \xi - ix) \mid (x, \xi) \in \mathbb{R}^2\} = \{(z, -i\bar{z}) \mid z \in \mathbb{C}\} = \left\{ \left(z, \frac{2}{i}\partial_z\Phi_0(z) \right) \mid z \in \mathbb{C} \right\},$$

plays the role of ‘‘phase space’’ in our new representation of $L^2(\mathbb{R})$ functions. We will denote (with a slight abuse of notations) $\mathbf{m} \circ \kappa_0^{-1}$ simply by \mathbf{m}

Let us recall without proof a result of Sjöstrand (see the lecture notes [Sjö, Section 12.2] or Zworski’s book [Zwo12, Theorem 13.9]) that is the starting point of our analysis. In the following theorem,

$$(2.5) \quad S(\Lambda_{\Phi_0}, \mathbf{m}) := \kappa_{0,*}(S(\mathbb{R}^2, \mathbf{m})) = \{v \in \mathcal{C}^\infty(\Lambda_{\Phi_0}) \mid |\partial^\alpha u| < C_\alpha \mathbf{m}, \forall \alpha \in \mathbb{N}^2\}.$$

Theorem 2.1 (Real Weyl Quantization on $H_{\Phi_0}(\mathbb{C})$).

Let $a \in S(\mathbb{R}^2, \mathbf{m})$ then $a^w(x, \hbar D_x)$ with domain $\mathcal{S}(\mathbb{R})$ is unitarily equivalent via \mathcal{T}_0 , to the operator $a_{\Phi_0}^w(z, \hbar D_z) = \mathcal{T}_0 A \mathcal{T}_0^* : e^{\Phi_0/\hbar} \mathcal{S}(\mathbb{C}) \cap \text{Hol}(\mathbb{C}) \rightarrow H_{\Phi_0}(\mathbb{C})$ given by

$$(2.6) \quad a_{\Phi_0}^w(z, \hbar D_z)u = \frac{1}{2\pi\hbar} \iint_{\Lambda_0(z)} a_{\Phi_0} \left(\frac{z+w}{2}, \zeta \right) e^{\frac{i}{\hbar}\zeta(z-w)} u(w) d\zeta \wedge dw,$$

where $\Lambda_0(z)$ is the contour defined with $\zeta = \frac{2}{i}\partial_z\Phi_0 \left(\frac{z+w}{2} \right)$ and $a_{\Phi_0} = a \circ \kappa_0^{-1}$.

We will often denote P_{Φ_0} for $\mathcal{T}_0 P \mathcal{T}_0^*$ instead of $p_{\Phi_0}(z, \hbar D_z)$ to lighten the writings. Since $p_{\Phi_0}(z, \zeta) = p\left(\frac{1}{\sqrt{2}}(z + i\zeta, \zeta + iz)\right)$, p_{Φ_0} can be extended by holomorphy to an element of $\mathcal{B}(\mathcal{U}_0, \mathbf{m})$ where $\mathcal{U}_0 = \kappa_0(U_0)$ is a tubular neighbourhood of Λ_{Φ_0} and the definition of $\mathcal{B}(\mathcal{U}_0, \mathbf{m})$ is

$$(2.7) \quad \mathcal{B}(\mathcal{U}_0, \mathbf{m}) = \{u \in \text{Hol}(\mathcal{U}_0) \mid |u(z, \zeta)| < C \mathbf{m}(z)\}$$

Since \mathcal{T}_0 is unitary, P_{Φ_0} with domain $e^{\Phi_0/\hbar} \mathcal{S}(\mathbb{C}) \cap \text{Hol}(\mathbb{C})$ has a unique self-adjoint extension that we still denote P_{Φ_0} . This extension is uniquely defined on the domain $\mathcal{D}(P_{\Phi_0}) = \mathcal{T}_0 \mathcal{D}(P)$ containing $H_{\Phi_0}(\mathbb{C}, \mathbf{m})$ (in fact because of the ellipticity assumption H.3 they are equal) and induces a continuous operator $H_{\Phi_0}(\mathbb{C}, \mathbf{m}) \rightarrow L_{\Phi_0}^2(\mathbb{C})$.

Let us now show that we can bound the L^∞ norms of the elements of $H_{\Phi_0}(\mathbb{C})$ by their $L_{\Phi_0}^2(\mathbb{C})$ norm. This can be interpreted as an obstacle for approaching Dirac distributions

(physically representing classical particles) with functions in $H_{\Phi_0}(\mathbb{C})$ (representing quantum states at fixed \hbar).

Lemma 2.2 (Uncertainty Principle).

$$(2.8) \quad \forall u_h \in H_{\Phi_0}(\mathbb{C}), \quad \forall z \in \mathbb{C}, \quad |u_h(z)| \leq \frac{2^{\frac{1}{4}}}{\sqrt{\pi\hbar}} e^{\Phi_0(z)/\hbar} \|u_h\|_{L^2_{\Phi_0}(\mathbb{C})}.$$

Proof. We write $z = x + i\xi$ we have $\operatorname{Re}(i\varphi_0(z, y)) = \frac{|z|^2}{2} - \frac{1}{2}(y - \sqrt{2}x)^2$, then denoting $v_h = \mathcal{T}_0^* u_h$ we have for all $z \in \mathbb{C}$,

$$\begin{aligned} |u_h(z)| &= \left| c_h \int_{\mathbb{R}} e^{i\varphi_0(z, \cdot)/\hbar} v_h(y) dy \right| \leq e^{\Phi_0(z)/\hbar} \int_{\mathbb{R}} c_h e^{-\frac{1}{2}(y - \sqrt{2}x)^2/\hbar} |v_h(y)| dy \\ &\leq e^{\Phi_0(z)/\hbar} \|c_h e^{-\frac{1}{2}(y - \sqrt{2}x)^2/\hbar}\|_{L^2(\mathbb{R})} \|v_h\|_{L^2(\mathbb{R})} \\ &\leq e^{\Phi_0(z)/\hbar} \|c_h e^{-\frac{1}{2}(y - \sqrt{2}x)^2/\hbar}\|_{L^2(\mathbb{R})} \|u_h\|_{L^2_{\Phi_0}(\mathbb{C})} \end{aligned}$$

$$\text{and } \|c_h e^{-\frac{1}{2}(y - \sqrt{2}x)^2/\hbar}\|_{L^2(\mathbb{R})} = \frac{2^{\frac{1}{4}}}{\sqrt{\pi\hbar}}. \quad \square$$

2.2. Complex pseudo-differential Operators. In the following lemma, we obtain an integral formula on a different contour (than that of (2.6)) for P_{Φ_0} . The benefit of such change of contour is that it can be shrank to a local contour making only ‘‘an exponentially small error’’.

Lemma 2.3. *Let $z \in \mathbb{C}$, for $C \in \mathbb{R}_+^*$, define the contour*

$$(2.9) \quad \Lambda_C(z) = \left\{ (w, \zeta) \in \mathbb{C}^2 \mid \zeta = \frac{2}{i} \partial_z \Phi_0 \left(\frac{z+w}{2} \right) + iC \frac{\overline{z-w}}{\langle z-w \rangle} \right\},$$

and choose $C \in \mathbb{R}_+^*$ small enough so that $\Lambda_C \subset \mathcal{U}_0$. We have the **exact** formula, for all $z \in \mathbb{C}$

$$(2.10) \quad \forall u_h \in H_{\Phi_0}(\mathbb{C}, \mathfrak{m}), \quad P_{\Phi_0} u_h(z) = \frac{1}{2\pi\hbar} \iint_{\Lambda_C(z)} p_{\Phi_0} \left(\frac{w+z}{2}, \zeta \right) e^{i\zeta(z-w)/\hbar} u_h(w) d\zeta \wedge dw.$$

Notice that in particular, the contour Λ_C will satisfy the estimate 2.12 and that the integral is converging. This lemma is proved in [HS15, Formula 1.4.17], here we briefly provide some elements of the proof which will be useful in the rest of the article. In the proof, for all $z \in \mathbb{C}$, we will denote $\mathfrak{m}(z, \frac{2}{i} \partial_z \Phi_0(z))$ simply by $\mathfrak{m}(z)$.

Proof. The distance between the contours Λ_C and Λ_{Φ_0} must be bounded to remain in the domain where p_{Φ_0} is holomorphic, that is why we cannot consider the more natural contour $\zeta = \frac{2}{i} \partial_z \Phi_0 \left(\frac{z+w}{2} \right) + iC \overline{(z-w)}$. We recall that for all $u_h \in e^{\Phi_0/\hbar} \mathcal{S}(\mathbb{C}) \cap H_{\Phi_0}(\mathbb{C})$, for all $N \in \mathbb{N}$,

$$|\mathfrak{m}(w) e^{-\Phi_0(w)/\hbar} u_h(w)| \leq \frac{C_N}{\langle w \rangle^N}, \text{ for some } C_N > 0. \text{ Using Assumption 1.5 on } \mathfrak{m}, \text{ by holomorphic}$$

change of contour we obtain

(2.11)

$$\forall u_h \in e^{\Phi_0/\hbar} \mathcal{S}(\mathbb{C}) \cap H_{\Phi_0}(\mathbb{C}), \quad P_{\Phi_0} u_h(z) = \frac{1}{2\pi\hbar} \iint_{\Lambda_C(z)} p \left(\frac{w+z}{2}, \zeta \right) e^{i\zeta(z-w)/\hbar} u_h(w) dw \wedge d\zeta.$$

The contour $\Lambda_C(z)$ verifies

$$\forall(w, \zeta) \in \Lambda_C(z), \quad -\text{Im}(\zeta(z-w)) + \Phi_0(w) = \Phi_0(z) - C \frac{|z-w|^2}{\langle z-w \rangle},$$

so that defining $\psi_z(w, \zeta) = -\text{Im}(\zeta(z-w)) + \Phi_0(w)$ we have

$$(2.12) \quad \forall(w, \zeta) \in \Lambda_C, \quad \psi_z(w, \zeta) - \psi_z\left(z, \frac{2}{i}\partial_z\Phi_0(z)\right) = -C \frac{|z-w|^2}{\langle z-w \rangle}.$$

We acknowledge that for all $u \in L^2_{\Phi_0}(\mathbb{C})$, $e^{-\Phi_0/\hbar}\mathbf{m}u \in L^2(\mathbb{C})$. Therefore the new formula (2.11) induces an operator $L^2(\mathbb{C}) \rightarrow L^2(\mathbb{C})$ with kernel given by

$$k_h(z, w) = \frac{1}{2\pi\hbar} \left(p_{\Phi_0}\left(\frac{z+w}{2}, \zeta(z, w)\right) e^{-\frac{C}{\hbar} \frac{|z-w|^2}{\langle z-w \rangle}} \right) \mathbf{m}^{-1}(w).$$

Using assumption 1.5 \mathbf{m} and the fact that $p \in S(U, \mathbf{m})$ ensures us that, choosing C small enough in (2.11),

$$\exists C_0 > 0, \exists N \in \mathbb{N}, \forall(z, w) \in \mathbb{C}^2, \quad |p_{\Phi_0}\left(\frac{z+w}{2}, \zeta(z, w)\right)| \leq C \mathbf{m}\left(\frac{z+w}{2}\right) \leq C_0 \langle z-w \rangle^N \mathbf{m}(w).$$

This implies the bound, for some other $C > 0$

$$\forall(z, w) \in \mathbb{C}^2, \quad |k_h(z, w)| \leq \frac{C}{2\pi\hbar} \langle z-w \rangle^N e^{-\frac{C}{\hbar} \frac{|z-w|^2}{\langle z-w \rangle}},$$

and we deduce that

$$C_h := \sup_z \iint_{w \in \mathbb{C}} |k_h(z, w)| dL(w) = \mathcal{O}(1) \text{ and } \tilde{C}_h := \sup_w \iint_{z \in \mathbb{C}} |k_h(z, w)| dL(z) = \mathcal{O}(1).$$

Schur's inequality implies that the formula (2.11) defines a bounded operator $L^2_{\Phi_0}(\mathbb{C}, \mathbf{m}) \rightarrow L^2_{\Phi_0}(\mathbb{C})$. Let us denote it momentarily \tilde{P}_{Φ_0} and notice that $P_{\Phi_0}, \tilde{P}_{\Phi_0}$ are both equal on $e^{\Phi_0/\hbar} \mathcal{S}(\mathbb{C})$ which is dense in $H_{\Phi_0}(\mathbb{C}, \mathbf{m})$ for the $L^2_{\Phi_0}(\mathbb{C}, \mathbf{m})$ norm. We therefore deduce that \tilde{P}_{Φ_0} and P_{Φ_0} coincide on $H_{\Phi_0}(\mathbb{C}, \mathbf{m})$. \square

Taking³ $\delta > 0$, we can truncate the contour and consider

$$(2.13) \quad \Lambda_{c,\delta}(z) = \left\{ (w, \zeta) \mid w \in \mathbb{C}, \zeta = \frac{2}{i}\partial_z\Phi_0\left(\frac{z+w}{2}\right) + ic \frac{\overline{z-w}}{\langle z-w \rangle} \text{ and } |z-w| < \delta \right\},$$

then the following lemma proves that the operator

$$(2.14) \quad \forall u \in H_{\Phi_0}(\mathbb{C}, \mathbf{m}), \quad P_{c,\delta}u := \frac{1}{2\pi\hbar} \iint_{\Lambda_{c,\delta}(z)} p_{\Phi_0}\left(\frac{w+z}{2}, \zeta\right) e^{i\zeta(z-w)/\hbar} u_h(w) dw \wedge d\zeta$$

approximates P_{Φ_0} with exponential sharpness.

Lemma 2.4. $P_{c,\delta}$ is a bounded linear operator $H_{\Phi_0}(\mathbb{C}, \mathbf{m}) \rightarrow L^2_{\Phi_0}(\mathbb{C})$ satisfying

$$(2.15) \quad \forall u \in H_{\Phi_0}(\mathbb{C}, \mathbf{m}), \quad \|(P_{\Phi_0} - P_{c,\delta})u\|_{L^2_{\Phi_0}(\mathbb{C})} \leq \mathcal{O}(e^{-\frac{c}{\hbar}}) \|u\|_{L^2_{\Phi_0}(\mathbb{C}, \mathbf{m})}.$$

³Even though there is a conflict of notation with the $\delta > 0$ defining $(E_1 - \delta, E_2 + \delta)$, it will be always clear to which δ we refer.

In this article, we will use the indicator function of a set $A \subset \mathbb{C}$ defined by $\mathbf{1}_A(z) := \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{if } z \notin A \end{cases}$, and in particular we will also denote $\mathbf{1}_{\mathbb{B}_\delta(z)}(w)$ the indicator function associated with the ball $\{w \mid |z - w| > \delta\}$.

Proof. Denoting $R = P_{\Phi_0} - P_{c,\delta}$ and using the estimate (1.5), for all $u \in H_{\Phi_0}(\mathbb{C}, \mathbf{m})$,

$$|Ru(z)| \leq \frac{C}{2\pi\hbar} e^{\frac{\Phi_0(z)}{\hbar}} \iint_{w \in \mathbb{C}} \mathbf{1}_{\mathbb{B}_\delta(z)}(w) \langle z - w \rangle^N e^{-\frac{C}{\hbar} \frac{|z-w|^2}{|z-w|}} |\mathbf{m}(w) e^{-\Phi_0(w)/\hbar} u(w)| dw \wedge d\bar{w}.$$

Similarly to the proof of the previous theorem, the approach with Schur's inequality allows us to conclude. In particular, we obtain that $P_{\Phi_0} - P_{c,\delta}$ is a bounded linear operator $H_{\Phi_0}(\mathbb{C}, \mathbf{m}) \rightarrow L^2_{\Phi_0}(\mathbb{C})$ and so is $P_{c,\delta}$. \square

Let us finish with a technical lemma that will be useful in the next sections and the next chapters. The idea is that the action of pseudo-differential operators on a set only depend on a small neighbourhood of this set.

Lemma 2.5 (Pseudolocality of pseudo-differential operators). *Let us consider two sets Ω, U such that Ω is open and $\bar{U} \subset \Omega$ then there exists $\mathcal{C}, \varrho > 0$ such that*

$$(2.16) \quad \forall u \in L^2_{\Phi_0}(\mathbb{C}, \mathbf{m}), \quad \|P_{\Phi_0} u\|_{L^2_{\Phi_0}(U, \mathbf{m})} \leq \varrho \|u\|_{L^2_{\Phi_0}(\Omega, \mathbf{m})} + \mathcal{C} e^{-\frac{1}{\mathcal{C}\hbar}} \|u\|_{L^2_{\Phi_0}(\mathbb{C}, \mathbf{m})}.$$

Proof. We have $P_{\Phi_0} = P_{c,\delta} + R$ and taking $\delta > 0$ small enough in (2.13) we make sure that the w -projection of $\Lambda_{C,\delta}(z)$ is included in Ω for all $z \in K$.

We have $\|P_{c,\delta} u\|_{L^2_{\Phi_0}(U)} = \|\mathbf{1}_U P_{c,\delta} \mathbf{1}_\Omega u\|_{L^2_{\Phi_0}(\mathbb{C})}$ and the integral kernel of $e^{-\Phi_0/\hbar} \mathbf{1}_U P_{c,\delta} e^{\Phi_0/\hbar} \mathbf{1}_\Omega$ (seen as a kernel operator of $L^2(\mathbb{C})$) is

$$(2.17) \quad k_h(z, w) = \frac{1}{2\pi\hbar} \mathbf{1}_U(z) \left(p_{\Phi_0} \left(\frac{z+w}{2}, \zeta(z, w) \right) e^{-\frac{C}{\hbar} \frac{|z-w|^2}{|z-w|}} \right) \mathbf{1}(|z-w| < \delta) \mathbf{1}_\Omega(w)$$

with $\zeta(z, w)$ given by (2.9). We notice that

$$C_h := \sup_z \iint_{w \in \mathbb{C}} |k_h(z, w)| \mathbf{m}^{-1}(w) dL(w) = \mathcal{O}(1), \quad \hbar \rightarrow 0^+$$

and similarly for the integral in w so that there exists $C(\delta) > 0$ such that $(C_h \tilde{C}_h)^{\frac{1}{2}} < C(\delta)$ for all $\hbar \in (0, \hbar_0)$. Schur's inequality for integral kernels yields

$$(2.18) \quad \|\mathbf{1}_U P_{c,\delta} u\|_{L^2_{\Phi_0}(\mathbb{C})} \leq (C_h \tilde{C}_h)^{\frac{1}{2}} \|\mathbf{1}_\Omega u\|_{L^2_{\Phi_0}(\mathbb{C}, \mathbf{m})} \leq C(\delta) \|u\|_{L^2_{\Phi_0}(\Omega, \mathbf{m})}.$$

Lemma 2.4 yields $\mathcal{C} > 0$ such that

$$\|P_{\Phi_0} u\|_{L^2_{\Phi_0}(U)} \leq C(\delta) \|u\|_{L^2_{\Phi_0}(\Omega, \mathbf{m})} + \mathcal{C} e^{-\frac{1}{\mathcal{C}\hbar}} \|u\|_{L^2_{\Phi_0}(\mathbb{C}, \mathbf{m})}.$$

The conclusion follows. \square

Using the FBI transform, we have taken advantage of the holomorphy of the elements in $H_{\Phi_0}(\mathbb{C}, \mathfrak{m})$ to reduce, modulo an exponentially small remainder, the definition of pseudo-differential operators to a local formula. This is key in order to prove that the WKB quasi-modes we build in Section 4 are sharp. In the next section we provide the construction of the phase function as well as the geometric results behind the Maslov index.

3. ANALYSIS OF THE GEOMETRY

In this subsection, we introduce the geometric results that we will need in the following section.

3.1. Action-Angle Lemma in the $L^2(\mathbb{R})$ setting. Similarly to the previous chapter, the energy sets E_λ we are interested in are within the scope of the classification of 1 dimensional compact connected submanifolds embedded in \mathbb{R}^2 . Since we ask $dp \neq 0$ along these submanifolds it is standard, using curvilinear coordinates, that they are diffeomorphic to the 1 dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In fact we even have the action-angle theorem illustrating that the geometry of this situation is not far from that of the excited harmonic oscillator and that we can expect similar results in the semiclassical regime. To define the action $A(\lambda)$, we parametrize E_λ by a simple loop in \mathbb{R}^2 oriented by the Hamiltonian flow. This allows us to set

$$(3.1) \quad \forall \lambda \in (E_1 - \delta, E_2 + \delta), \quad A(\lambda) = \int_{E_\lambda} \xi dx.$$

The usual results of Hamiltonian mechanics yield that, in that setting, the Hamiltonian flow is periodic of period $T(\lambda)$ given by

$$(3.2) \quad A'(\lambda) = T(\lambda).$$

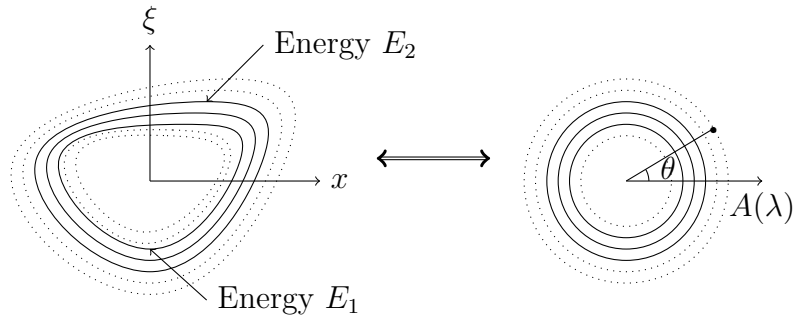


FIGURE 1. Illustration of the action-angle change of variables.

In the following lemma, we provide \mathbb{R}^2 with its natural symplectic structure $d\xi \wedge dx$ as well as $\mathbb{T} \times \mathbb{R}$ with $dA \wedge d\theta$ (letting A be the second coordinate).

Lemma 3.1 (Action-Angle). *There exists a map $\theta : p^{-1}((E_1 - \delta, E_2 + \delta)) \cap \mathbb{R}^2 \longrightarrow \mathbb{T}$ such that the map defined as follows*

$$(3.3) \quad \begin{aligned} p^{-1}((E_1 - \delta, E_2 + \delta)) \cap \mathbb{R}^2 &\longrightarrow \mathbb{T} \times \text{Neigh}_{\mathbb{R}}(A((E_1, E_2))) \\ (x, \xi) &\longmapsto (\theta, A(p(x, \xi))), \end{aligned}$$

is a smooth symplectomorphism. It satisfies $\nu_*(\mathcal{X}_p) = \frac{1}{T(\lambda)} \frac{\partial}{\partial \theta}$ where \mathcal{X}_p is the Hamiltonian vector field given by

$$(3.4) \quad \mathcal{X}_p = \partial_{\xi} p \frac{\partial}{\partial x} - \partial_x p \frac{\partial}{\partial \xi}.$$

Remark 3.2. In our setting, p being holomorphic it is even possible to build a holomorphic symplectomorphism, for instance see Ito's article [Ito91]. In the case of one degree of freedom one needs to show that the action A is holomorphic (see Lemma (3.5)) and then find an holomorphic solution $\theta(x, \xi)$ of the Poisson bracket equation $\{\theta, A \circ p\} = -1$.

3.2. Holomorphy of the action and eikonal equations. In what follows, we prove Lemma 3.3 which is the main preliminary technicality allowing us to locally solve the eikonal equation

$$(3.5) \quad p_{\Phi_0}(z, \varphi'_\lambda(z)) = \lambda.$$

and prove that the action A is holomorphic.

In the Bargmann setting, it is noteworthy that the natural projection

$$(3.6) \quad \pi_z : \begin{cases} (z, \zeta) & \longmapsto z \\ \Lambda_{\Phi_0} & \longrightarrow \mathbb{C} \end{cases},$$

is an isomorphism. In that regard, we will often identify the sets

$$(3.7) \quad \mathcal{E}_\lambda := \kappa_0(E_\lambda) = \{(z, \zeta) \in \Lambda_{\Phi_0} \mid p_{\Phi_0}(z, \zeta) = \lambda\}.$$

with the corresponding complex curve in \mathbb{C} .

Lemma 3.3. *Let $\lambda_1 \in (E_1 - \delta, E_2 + \delta)$, we can chose $\text{Neigh}(\mathcal{E}_{\lambda_1}, \mathbb{C}^2)$, a small enough neighbourhood of \mathcal{E}_{λ_1} in \mathbb{C}^2 on which the map $\rho : (z, \zeta) \longmapsto (z, p_{\Phi_0}(z, \zeta))$ is a biholomorphism onto its image.*

Proof. The Jacobian of ρ , $\text{Jac}(\rho) = \partial_\zeta p_{\Phi_0}$, is nonzero along \mathcal{E}_{λ_1} because we assume $dp \neq 0$ along $\{(x, \xi) \in \mathbb{R}^2 \mid p(x, \xi) = \lambda_1\}$. The local inverse function theorem shows that ρ is locally a biholomorphism near any point of \mathcal{E}_{λ_1} . We can cover a small neighbourhood of \mathcal{E}_{λ_1} with m sets of the form

$$\mathcal{U}_j := \{(z, \zeta) \in \mathbb{C}^2 \mid |z - z_j| < \epsilon_j \ \& \ |\zeta - \frac{2}{i} \partial_z \Phi_0(z)| < \epsilon_j\}, \quad j \in \llbracket 1, m \rrbracket,$$

where $(z_j, \frac{2}{i} \partial_z \Phi_0(z))$ is some point of \mathcal{E}_{λ_0} and $\epsilon_j > 0$ is small enough. We can further assume that on each \mathcal{U}_j we can apply the local inverse function theorem providing a unique inverse $\zeta_j(\lambda, z)$ satisfying

$$\forall (z, \zeta) \in \mathcal{E}_{\lambda_1} \cap \mathcal{U}_j, \quad \zeta_j(\lambda_1, z) := \frac{2}{i} \partial_z \Phi_0(z).$$

To conclude it suffices to notice that ρ is injective in the union of \mathcal{U}_j . Suppose there exists $(z_1, \zeta_1), (z_2, \zeta_2) \in \bigcup_{j=0}^m \mathcal{U}_j$ such that $z_1 = z_2$ and $p_{\Phi_0}(z_1, \zeta_1) = p_{\Phi_0}(z_2, \zeta_2)$. Up to re-indexing, we can suppose that

— either $(z_1, \zeta_1), (z_2, \zeta_2) \in \mathcal{U}_1$ in which case the local inverse function theorem yields the uniqueness,

— or $(z_1, \zeta_1) \in \mathcal{U}_1$ and $(z_2, \zeta_2) \in \mathcal{U}_2$. We have $z_1 = z_2$ and setting $\epsilon = \min(\epsilon_1, \epsilon_2)$ we have

$$\{(z_1, \zeta) \mid |\zeta - \frac{2}{i} \partial_z \Phi_0(z)| < \epsilon\} \subset \mathcal{U}_1 \cap \mathcal{U}_2$$

and this proves that one of the two points $(z_1, \zeta_1), (z_2, \zeta_2)$ belongs to $\mathcal{U}_1 \cap \mathcal{U}_2$. Therefore the uniqueness given by the local inverse function theorem yields $(z_1, \zeta_1) = (z_2, \zeta_2)$ and this seals the injectivity of ρ . \square

The relevance of such a lemma appears when acknowledging that it provides a biholomorphism, for all $\lambda \in \mathbb{C}$ near λ_1 ,

$$(3.8) \quad \begin{array}{ccc} z & \longmapsto & (z, \zeta(\lambda, z)) \\ \text{Neigh}_{\mathbb{C}}(\pi_z \mathcal{E}_{\lambda_1}) & \longrightarrow & \{p_{\Phi_0} = \lambda\} \cap \text{Neigh}_{\mathbb{C}^2}(\mathcal{E}_{\lambda_1}). \end{array}$$

This allows to solve easily the eikonal equation (3.5).

Lemma 3.4 (Local eikonal resolution). *Let $\lambda_1 \in (E_1 - \delta, E_2 + \delta)$ and $\omega \in \mathcal{E}_{\lambda_1}$. The solutions of the eikonal equation (3.5) near ω satisfying $\lambda \in \text{Neigh}_{\mathbb{C}}(\lambda_1)$ and $\varphi'_\lambda(\omega) = \zeta(\lambda, \omega) \in \text{Neigh}_{\mathbb{C}}(\frac{2}{i}\partial_z \Phi_0(\omega))$ are given by,*

$$(3.9) \quad \forall z \in \text{Neigh}_{\mathbb{C}}(\omega), \quad \varphi_\lambda(z) := \int_\omega^z \zeta(\lambda, w) dw + \text{constant}.$$

Proof. Let us suppose that φ_λ solves the eikonal equation and for all z near ω , λ near λ_0 , and $\varphi'_\lambda(z) = \zeta(\lambda, z)$. Lemma 3.3 ensures that there is a unique solution to the equation $p(z, \zeta) = \lambda$ near $(\omega, \frac{2}{i}\partial_z \Phi_0(\omega))$ given by $\zeta = \zeta(\lambda, z)$. Since $p_{\Phi_0}(z, \varphi'_\lambda(z)) = \lambda$ this yields $\varphi'_\lambda(z) = \zeta(\lambda, z)$ and the lemma follows. \square

Another useful consequence of Lemma 3.3 is that the action $A(\lambda)$ is in fact holomorphic in a small complex neighbourhood of (E_1, E_2) . In the following lemma, the orientation on \mathcal{E}_λ will be given by the Hamiltonian flow in the sense that \mathcal{E}_λ is identified with a simple loop in Λ_{Φ_0} given by the Hamiltonian flow of a point in \mathcal{E}_λ .

Let $\lambda \in (E_1 - \delta, E_2 + \delta)$, we have on Λ_{Φ_0} ,

$$\kappa_0^*(\zeta dz) = \frac{1}{2}(\xi - ix)d(x - i\xi) = \frac{1}{2}(\xi dx - x d\xi) - i\kappa_0^*(d\Phi_0) = \xi dx - \frac{1}{2}d(x\xi) - i\kappa_0^*(d\Phi_0),$$

so that

$$(3.10) \quad A(\lambda) = \int_{E_\lambda} \xi dx = \int_{E_\lambda} \xi dx - \frac{1}{2}d(x\xi) = \int_{\mathcal{E}_\lambda} (\zeta dz - id\Phi_0) = \int_{\mathcal{E}_\lambda} \zeta dz.$$

Lemma 3.5 (Holomorphic Quantum Action). *The action A extends by holomorphy to a complex neighbourhood of $[E_1, E_2]$ and we will denote by \mathcal{A} this extension.*

Proof. We will also consider $\lambda_1 \in (E_1 - \delta, E_2 + \delta)$. For all λ in a small real neighbourhood of λ_1 , by the change of contour $\pi_z(\mathcal{E}_\lambda) \rightarrow \pi_z(\mathcal{E}_{\lambda_1})$ and the holomorphy of $(\lambda, z) \mapsto \zeta(\lambda, z)$ we have

$$(3.11) \quad A(\lambda) = \int_{\mathcal{E}_\lambda} \zeta dz = \int_{\pi_z(\mathcal{E}_\lambda)} \zeta(\lambda, z) dz = \int_{\pi_z(\mathcal{E}_{\lambda_1})} \zeta(\lambda, z) dz.$$

We notice that $\mathcal{A}(\lambda) = \int_{\pi_z(\mathcal{E}_{\lambda_1})} \zeta(\lambda, z) dz$ defines a holomorphic map for λ near λ_1 . Since this is true for any $\lambda_1 \in (E_1 - \delta, E_2 + \delta)$ and by uniqueness of $A(\lambda)$ for $\lambda \in (E_1 - \delta, E_2 + \delta)$, the conclusion stems from the standard patching results for holomorphic functions. \square

Remark 3.6. Letting $\lambda \in (E_1 - \delta, E_2 + \delta)$ and using the “real expression” giving $A(\lambda)$, we notice that it corresponds to the area in \mathbb{R}^2 delimited by E_λ . This proves that the action doesn’t cancel thus the phases φ_λ which are primitives of $\zeta(\lambda, z)$ suffer from multivaluedness issues on a neighbourhood of \mathcal{E}_λ . In other words it φ_λ is only well defined on the universal cover (or the corresponding Riemann surface).

3.3. Index of the energy sets. Now we finish this section with the geometric result underpinning the Maslov correction that we will determine in Section 4.4.

In the usual functional setting $L^2(\mathbb{R})$, the energy sets E_λ cannot be projected on the variable x and this is an obstacle to building WKB quasimodes of the form $a_h(x)e^{\frac{i}{h}\phi(x)}$. Indeed at the critical points x where the projection

$$E_\lambda \ni (x, \xi) \longmapsto x$$

is not locally an isomorphism, the phase ϕ is not well defined. A way of getting around this issue is by looking at generalised WKB quasimodes defined either with special functions or as the inverse Fourier transform of functions of WKB type. The Maslov correction then appears as the condition on the eigenvalues to be able to glue the “usual” WKB quasimodes with the generalized ones defined at the “turning points”.

In the Bargmann’s setting $H_{\Phi_0}(\mathbb{C})$, (3.6) being an isomorphism, the energy sets \mathcal{E}_λ are projectable on the variable z , thus the lack of projectability is not of concern. As we will see in Section 4.4, the Maslov condition will still stem from a gluing condition. In particular it is due to the particular form of the principal symbol $\partial_\zeta p_{\Phi_0}(z, \zeta(\lambda, z))^{-\frac{1}{2}}$ of the WKB expansion that we will find in Section 4. We will see in section 4.4 that it amounts to the following result (Lemma 3.7) on the index of the energy sets in \mathbb{C} .

We let $\lambda \in (E_1 - \delta, E_2 + \delta)$ and define the loop $\gamma_\lambda : [0, T(\lambda)] \ni t \longmapsto z(t)$ with the Hamiltonian flow $(z(t), \zeta(t))$ of a point $(z, \zeta) \in \mathcal{E}_\lambda$. We recall that with $(x(t), \xi(t)) = \kappa_0^{-1}(z(t), \zeta(t))$

$$\frac{dz}{dt} = \frac{dx(t)}{dt} - i \frac{d\xi(t)}{dt} = \partial_\xi p(x(t), \xi(t)) + i \partial_x p(x(t), \xi(t)) = \partial_\zeta p_{\Phi_0}(z(t), \zeta(t)).$$

By Assumption H.2, ∇p does not vanish on E_λ thus $\gamma'_\lambda(t) = \partial_\zeta p_{\Phi_0}(z(t), \zeta(t)) \neq 0$. The goal is to prove the following lemma that will be useful for computing the subprincipal term of the Bohr-Sommerfeld expansion in Section 4.4. We are mainly interested in computing the index of the derivative of γ_λ . The idea is that Gauss-Bonnet allows to link it with that of γ_λ .

Lemma 3.7. *Let us consider the complex loop $\sigma : [0, T(\lambda)] \ni t \longmapsto \partial_\zeta p_{\Phi_0}(z(t), \zeta(t))$. It admits two square roots which are well defined on $[0, T(\lambda)]$. If we pick any square root that we denote by $\sqrt{\sigma}$ then*

$$(3.12) \quad \sqrt{\sigma}(0) = -\sqrt{\sigma}(T(\lambda))$$

In particular, this results prevents the existence of a squareroot of $\partial_\zeta p_{\Phi_0}(z, \zeta(\lambda, z))$ near the z -projection of \mathcal{E}_λ .

Proof. Let us first recall the definition of the index of a smooth closed curve $\gamma : [0, t_0] \longmapsto \mathbb{C}$ in the complex plane revolving around a point $a \in \mathbb{C}$. The universal cover of $\mathbb{C} \setminus \{a\}$ is \mathbb{C}

(thanks to the exponential function). By uniqueness of the lifted path, there exists a unique lifting $\theta : [0, t_0] \ni t \mapsto \theta(t) \in \mathbb{C}$ of γ satisfying $\gamma(t) = a + (\gamma(0) - a)e^{2i\pi\theta(t)}$ and $\theta(0) = 0$. The index of γ around a is then unambiguously given by $\text{Ind}(\gamma) := \theta(t_0) - \theta(0) \in \mathbb{Z}$.

The index enjoys, in particular, the property of being *constant on the homotopy classes of loops* as well as the following property that will be useful. In the case when γ' does not vanish on $[0, t_0]$, if we consider the *tangential Gauss map* defined by $[0, t_0] \ni t \mapsto \frac{\gamma'(t)}{|\gamma'(t)|} \in \mathbb{S}^1$, we have the following standard result as consequence of Gauss-Bonnet's Theorem.

Lemma 3.8. *The index of the smooth loop γ (such that γ' does not vanish) around a is also the index of its tangential Gauss map around 0.*

Proof. Let us proceed to a first a priori simplification: we can suppose for each $t \in [0, t_0]$, $|\gamma'(t)| = 1$. To see that we define $\beta(t) = \int_0^t |\gamma'(s)| ds$ and notice that it is a diffeomorphism $[0, t_0] \mapsto [0, \beta(t_0)]$ since by assumption γ' does not vanish. If we denote α its reciprocal we obtain that

$$\frac{d}{dt}\gamma(\alpha(t)) = \alpha'(t)\gamma'(\alpha(t)) = \frac{1}{|\gamma'(\alpha(t))|}\gamma'(\alpha(t))$$

is unitary. The parametrization $[0, \beta(t_0)] \ni t \mapsto \gamma(\alpha(t))$ then satisfies the desired properties.

In complex notation, the geodesic curvature at $t \in [0, t_0]$ of the curve γ is then simply $k_g(t) = \text{Im}(\overline{\gamma'(t)}\gamma''(t))$. We then assume that γ is a simple loop around a . Thanks to Gauss-Bonnet's theorem

$$\int_0^{t_0} k_g(t) dt = \pm 2\pi,$$

where the sign is a $+$ if the loop γ is homotopic to a circle (with usual trigonometric orientation) and $-$ if not. We deduce, since $\text{Ind}_a(\gamma)$ is either 1 or -1 ,

$$\text{Im}\left(\int_0^{t_0} \overline{\gamma'(t)}\gamma''(t) dt\right) = 2\pi \text{Ind}_a(\gamma).$$

Notice then that the previous formula extends by concatenation of loops to the case where γ is not a simple loop. The famous residue theorem for meromorphic functions allows to link the winding number with complex analysis

$$\text{Ind}_0(\gamma') = \frac{1}{2\pi} \text{Im}\left(\int_{\gamma'} \frac{dz}{z}\right) = \frac{1}{2\pi} \text{Im}\left(\int_0^{t_0} \overline{\gamma'(t)}\gamma''(t) dt\right) = \text{Ind}_a(\gamma).$$

□

To conclude, the loop $\gamma_\lambda : [0, 1] \ni t \mapsto z(t)$ is a simple, closed and regular loop whose index is ± 1 . By homotopy, the tangential Gauss map of γ_λ has the same index as γ'_λ . We thus obtain that the loop $\sigma : [0, T(\lambda)] \ni t \mapsto \gamma'_\lambda(t) = \partial_\zeta p_{\Phi_0}(z(t), \zeta(t))$ is a loop in \mathbb{C} of index 1 or -1 around 0. It then suffices to recall the following natural result.

Lemma 3.9. *Let \mathfrak{s} a loop in \mathbb{C}^* around 0, it admits two smooth squareroots paths that we denote arbitrarily $-\sqrt{\mathfrak{s}}, \sqrt{\mathfrak{s}}$, we have*

- (1) $\sqrt{\mathfrak{s}}(T(\lambda)) = -\sqrt{\mathfrak{s}}(0)$ if $\text{Ind}(\mathfrak{s}) \in 2\mathbb{Z} + 1$,
- (2) $\sqrt{\mathfrak{s}}(T(\lambda)) = \sqrt{\mathfrak{s}}(0)$ if $\text{Ind}(\mathfrak{s}) \in 2\mathbb{Z}$.

The loop $[0, T(\lambda)] \ni t \mapsto \partial_{\zeta} p_{\Phi_0}(z(t), \zeta(t))$ having index ± 1 around 0, we can then apply Lemma 3.9 and finish the proof of Lemma 3.7. \square

Remark 3.10. In [Arn78] V. Arnold developed the framework allowing to associate an index: *Maslov's index* with a *loop of Lagrangian spaces* in the Lagrangian Grassmanian $\Lambda(n)$ of \mathbb{R}^{2n} . In our situation Lemma 3.7 is linked to the computation of the Maslov's index of the path $[0, T_0] \ni t \mapsto \mathbb{R}\gamma'(t) \in \Lambda(1)$ and the relation between the indices comes from the diffeomorphism $\Lambda(1) \simeq \mathbb{R}\mathbf{P}^1 \simeq \mathbb{S}^1$.

4. CONSTRUCTION OF GLOBAL WKB QUASIMODES

In this section we start recalling a classical $\mathcal{O}(\hbar^2)$ result and then prove, extending the ideas of Fujie-Zerzeri [FZ04], the existence of WKB quasimodes which are exponentially sharp.

4.1. Normal form in the smooth setting. In this chapter, we will need a weak $\mathcal{O}(\hbar^2)$ version of a result that is stated with precision $\mathcal{O}(\hbar^\infty)$ in Vũ Ngọc's [Vũ 06, Proposition 5.1.3] or [Vũ 00, Section 5]. The idea of the following lemma is that the eigenvalues of P_{Φ_0} are contained in the reunion of the small intervals $[\mathcal{A}^{-1}(2\pi\hbar(k + \frac{1}{2})) - \mathcal{C}\hbar^2, \mathcal{A}^{-1}(2\pi\hbar(k + \frac{1}{2})) + \mathcal{C}\hbar^2]$.

Lemma 4.1 (Bohr-Sommerfeld $\mathcal{O}(\hbar^2)$). *There exists a constant $\mathcal{C} > 0$ such that for \hbar small enough*

$$(4.1) \quad \forall \lambda_{\hbar} \in \sigma_{\hbar}(P) \cap \text{Neigh}_{\mathbb{R}}(\lambda_0), \exists ! k \in \mathbb{Z}, \quad |\mathcal{A}(\lambda_{\hbar}) - 2\pi\hbar(k + \frac{1}{2})| \leq \mathcal{C}\hbar^2.$$

Moreover the eigenvalues of P are of multiplicity 1 and there is at most one eigenvalue satisfying (4.1).

In the lemma, we use the notation $\text{Neigh}_{\mathbb{R}}(\lambda_0)$ to denote a small \hbar -independent neighbourhood of λ_0 in \mathbb{R} .

Proof. We simply provide the philosophy of the proof. Considering $\lambda \in (E_1 - \delta, E_2 + \delta)$ the standard $\mathcal{O}(\hbar^2)$ analysis with Fourier Integral Operators guarantees the existence and uniqueness of microlocal solutions (defined with Fourier Integral Operators) of the equation $(P - \lambda)u = 0$ near any point of the energy curve. Since the eigenfunctions are microlocalised near E_{λ} the existence of an eigenfunction of P is equivalent to the existence of such semiglobal microlocal solution. This semiglobal microlocal solution exists if and only if it is possible to patch the fully microlocal solutions along the energy curve. The condition on the transition between these microlocal solutions stemming from Čech cohomology yields that, in that case, λ satisfies (4.1). \square

We now have a good idea of the shape of the spectrum. It remains to prove the reciprocal of the previous lemma, namely that we can build quasimodes giving an exponentially sharp approximation of the eigenvalues.

4.2. Microlocal WKB Expansions.

4.2.1. *Construction of analytic WKB expansions.* Let us start with finding microlocal WKB expansions. We let $z_0 \in \mathcal{E}_{\lambda_0}$ and denote $\phi(\lambda, z)$, for λ near λ_0 , the phase given by

$$(4.2) \quad \phi(\lambda, z) = \int_{z_0}^z \zeta(\lambda, z), \text{ satisfying } p_{\Phi_0}(z, \partial_z \phi(\lambda, z)) = \lambda.$$

Let us mention that we will sometimes use ϕ_λ to refer to $\phi(\lambda, \cdot)$ and $\phi'_\lambda = \partial_z \phi_\lambda$. We also define a pluriharmonic gauge $\Phi(\lambda, z) = -\text{Im}\phi(\lambda, z)$. In order to build microlocal WKB quasimodes for $P_{c,\delta} - \lambda$, we will first build quasimodes $a(\hbar, \lambda, z)e^{\frac{i}{\hbar}\phi(\lambda, z)}$ in the germ spaces $H_{\Phi,(\lambda_0, z_0)}$ (see Appendix A.3.2). The link with the operator P_{Φ_0} will then appear in Proposition 4.6. The reason for which we start with looking for quasimodes in $H_{\Phi,(\lambda_0, z_0)}$ are the following:

- it is easier to solve WKB equations $(P - \lambda)u \sim 0$ in spaces H_Φ with holomorphic gauges Φ ,
- seeing λ as a variable allows to link the microlocal solutions.

In Section A.3.2 of the appendix, we recalled the construction of pseudodifferential operators on germ spaces. This allows to define a pseudodifferential operator $\tilde{P} : H_{\Phi,(\lambda_0, z_0)} \rightarrow H_{\Phi,(\lambda_0, z_0)}$ formally given by the integral

$$(4.3) \quad \tilde{P}u_\hbar = \frac{1}{2\pi\hbar} \iint_{\mathcal{G}_{C,\delta}(\lambda, z)} e^{i\zeta(z-w)/\hbar} p_{\Phi_0}\left(\frac{z+w}{2}, \zeta\right) u_\hbar(w) d\zeta \wedge dw,$$

where

$$\mathcal{G}_{C,\delta}(\lambda, z) := \left\{ (w, \zeta) \mid |z - w| \leq \delta, \zeta = \frac{2}{i} \partial_z \Phi(\lambda, \frac{z+w}{2}) + iC\overline{z-w} \right\},$$

is a smooth family of good contours for $(w, \zeta) \mapsto -\text{Im}(\zeta(z-w)) + \Phi(\lambda, w)$.

Proposition 4.2 (Microlocal WKB expansions). *There exists $a(\hbar, \lambda, z)$ a classical analytic symbol in (λ, z) defined near (λ_0, z_0) such that the following holds.*

- (1) *There exists $\mathcal{C} > 0$ such that, in a small neighbourhood of (λ_0, z_0) ,*

$$(4.4) \quad |e^{-\frac{i}{\hbar}\phi}(\tilde{P} - \lambda)(e^{\frac{i}{\hbar}\phi}a(\hbar, \lambda, \cdot))| \leq \mathcal{O}(e^{-\mathcal{C}/\hbar}),$$

where \tilde{P} is defined in (4.3) for δ small enough.

- (2) *The principal symbol $a_0(\lambda, z)$ of $a(\hbar, \lambda, z)$ is a square root of $w \mapsto \partial_\zeta p_{\Phi_0}(w, \zeta(\lambda, w))^{-1}$.*
(3) *If $b(\hbar, \lambda, z)$ is a classical analytic symbol such that the WKB expansion $b(\hbar, \lambda, z)e^{\frac{i}{\hbar}\phi_\lambda(z)}$ satisfies equation (4.4), then there exists a classical analytic symbol $g(\hbar, \lambda)$ satisfying*

$$\forall \lambda \in \text{Neigh}_{\mathbb{C}}(\lambda_0), \forall z \in \text{Neigh}_{\mathbb{C}}(z_0), \quad |b(\hbar, \lambda, z) - g(\hbar, \lambda)a(\hbar, \lambda, z)| \leq \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}).$$

Proof. (1) The main result for finding WKB expansions $a(\hbar, \cdot, \cdot)e^{\frac{i}{\hbar}\phi}$ in the analytic setting is Sjöstrand [HS15, Theorem 2.8.1] that we recall here. The minimal ingredients for understanding the following result (among others the definitions of the space H_{0, z_0}) are stated in Appendix A. We have kept Sjöstrand notations and warn the reader that in the following lemma (x, ξ) denotes a point in \mathbb{C}^{2d} (with $d \in \mathbb{N}^*$) not a point in \mathbb{R}^2 as in the beginning of the chapter.

Theorem 4.3 (Theorem 2.8.1 [HS15]). *Let $f \in \text{Hol}(\text{Neigh}(x_0, \mathbb{C}^d))$ and $Q(x, \hbar D_x, \hbar) : H_{-\text{Im}f, x_0} \rightarrow H_{-\text{Im}f, x_0}$, be a complex pseudo-differential operator of order 0 such that the leading symbol satisfies*

$$(4.5) \quad q(x_0, \xi_0) = 0, \quad \partial_{\xi_d} q(x_0, \xi_0) \neq 0.$$

Assume that f solves the eikonal equation

$$(4.6) \quad \begin{cases} q(x, f'(x)) = 0, \\ f'(0) = \xi_0. \end{cases}$$

Let H be the hypersurface $\{x_n = x_{0,d}\}$ and denote $x = (x', x_d) \in \mathbb{C}^d$. Let $v(x, \hbar), \underline{a}(x', \hbar)$ be classical analytic symbols of order 0 defined near x_0 and x'_0 respectively. There exists a unique (modulo equivalence in H_{0,x_0}) classical analytic symbol $a(\hbar, \cdot)$ defined near x_0 such that

$$(4.7) \quad e^{-if/\hbar} Q e^{if/\hbar} a \sim \hbar v \text{ in } H_{0,x_0}, \quad a|_H = \underline{a}.$$

We apply this theorem with $x = (\lambda, w) \in \text{Neigh}_{\mathbb{C}^2}(\lambda_0, z_0)$, $Q = \tilde{P} - \lambda$, $v = 0$, $f(\lambda, w) = \phi_\lambda(w)$ and $H = \{w = z_0\}$ by asking $\underline{a}(\hbar, \lambda) = \partial_\zeta p_{\Phi_0}(z_0, \zeta(\lambda, z_0))^{-\frac{1}{2}}$ (choosing a local square-root) for all λ near λ_0 . This gives a classical analytic symbol denoted $a(\hbar, \lambda, w)$ defined near z_0 satisfying

$$(4.8) \quad \exists \mathcal{C} \in \mathbb{R}_+^*, \forall \lambda \in \text{Neigh}_{\mathbb{C}}(\lambda_0), \quad |e^{-\frac{i}{\hbar}\phi_\lambda}(\tilde{P} - \lambda) e^{\frac{i}{\hbar}\phi_\lambda} a(\hbar, \lambda, \cdot)| \leq \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}).$$

(2) Let us compute the principal symbol a_0 . For that, we define $P_\phi := e^{-\frac{i}{\hbar}\phi} \tilde{P} e^{\frac{i}{\hbar}\phi}$ and notice that for all $u \in \text{Hol}(\text{Neigh}_{\mathbb{C}}(z_0))$,

$$(4.9) \quad P_\phi u(z) = \frac{1}{2\pi\hbar} \iint_{\mathcal{G}_{C,\delta}(z)} e^{\frac{i}{\hbar}\zeta(z-w)} p_{\Phi_0}\left(\frac{z+w}{2}, \zeta\right) e^{\frac{i}{\hbar}(\phi(w)-\phi(z))} u(w) d\zeta \wedge dw.$$

Let us mention that, by Lemma A.14, $P_\phi : H_{0,(\lambda_0, z_0)} \rightarrow H_{0,(\lambda_0, z_0)}$ is a well defined complex pseudodifferential operator. Indeed (denoting $\phi_\lambda(z) = \phi(\lambda, z)$), the function $\theta(\lambda, z, w) = \frac{\phi_\lambda(w) - \phi_\lambda(z)}{z-w}$ extended by $-\phi'_\lambda(z)$ at $w = z$, is holomorphic therefore $\Lambda_\phi(z) := \{(w, \zeta + \theta(\lambda, z, w)) \mid (w, \zeta) \in \mathcal{G}_{C,\delta}(z)\}$ defines a good contour for the phase $(w, \eta) \mapsto -\text{Im}(\eta \cdot (z-w))$. Thanks to the equivalence of good contours we can suppose, making only an error $\mathcal{O}(e^{-\mathcal{C}/\hbar})$ that the new contours are

$$\mathcal{G}_{C,\delta}^0(z) := \{(w, \zeta) \mid \zeta = iC\overline{z-w}, |z-w| < \delta\},$$

for a different $\delta > 0$. Neglecting this exponentially small remainder, Formula (4.9) becomes

$$(4.10) \quad P_\phi u(z) = \frac{1}{2\pi\hbar} \iint_{\mathcal{G}_{C,\delta}^0(z)} e^{\frac{i}{\hbar}\zeta(z-w)} p_{\Phi_0}\left(\frac{z+w}{2}, \zeta - \theta(z, w)\right) u(w) d\zeta \wedge dw.$$

Therefore the classical analytic symbol $a(\hbar, \lambda, z)$ satisfies (4.4) if and only if it solves $(P_\phi - \lambda)a(\hbar, \lambda, \cdot) = 0$ in H_{0,z_0} . Formula (4.10) defines a complex pseudo-differential operator (with classical analytic symbol) on the space H_{0,z_0} . See for instance in [RSV20, Section 2.3] for a comprehensive presentation of complex pseudo-differential operators and their properties. Now thanks to [RSV20, Section 3.2] it is possible to find a classical analytic symbol b_\hbar such that $P_\phi = \text{Op}_\hbar^w(b_\hbar)$ where the formal expansion of b_\hbar is given by

$$(4.11) \quad b_\hbar(z, \zeta) = \exp\left[-\frac{i}{2\hbar}\hbar D_\zeta(\hbar D_z - \hbar D_w)\right] p_{\Phi_0}\left(\frac{z+w}{2}, \zeta - \theta(z, w)\right)\Big|_{w=z}.$$

Let us provide a quick formal explanation of the result to the reader in order to make it feel more natural. In order to shorten the notations, we will write ϕ_λ instead of ϕ and ϕ'_λ

instead of $\partial_z \phi$. The first step is to write (at least formally) the infinite Taylor expansion of $p_{\Phi_0}(\frac{z+w}{2}, \zeta - \theta(z, w))$ centered at $\frac{z+w}{2}$. Namely one finds p_1, \dots, p_k, \dots holomorphic such that

$$(4.12) \quad p_{\Phi_0}(\frac{z+w}{2}, \zeta - \theta(z, w)) = p_{\Phi_0}(\frac{z+w}{2}, \zeta + \phi'_\lambda(\frac{z+w}{2})) + (z-w)^2 p_1(\frac{z+w}{2}, \zeta) \\ + (z-w)^4 p_2(\frac{z+w}{2}, \zeta) + \dots + (z-w)^{2k} p_k(\frac{z+w}{2}, \zeta) + \dots$$

The idea of the transformation (4.11) is justified by applying the formal integration by parts, with $\mathcal{G} := \{i\bar{z} - \bar{w} \mid |z-w| \leq \epsilon\}$, for all $k \in \mathbb{N}$,

$$(4.13) \quad \iint_{\mathcal{G}} e^{i\hbar\zeta(z-w)} \frac{(z-w)^k}{k!} q(\frac{z+w}{2}, \zeta) u(w) d\zeta \wedge dw = \frac{(i\hbar)^k}{k!} \iint_{\mathcal{G}} e^{i\hbar\zeta(z-w)} \partial_\zeta^k q(\frac{z+w}{2}, \zeta) u(w) d\zeta \wedge dw,$$

to the Taylor expansion (4.12). The proof provided in [RSV20, Section 3.2] consists in proving that (4.11) is in fact a complex Fourier Integral Operator sending classical analytic symbols towards classical analytic symbols.

Finally, thanks to the (analytic) stationary phase, we find that in a small neighbourhood of z_0

$$(4.14) \quad (P_\phi - \lambda)a_0 = \text{Op}_\hbar^w(p_{\Phi_0}(\frac{z+w}{2}, \zeta + \phi'_\lambda(\frac{z+w}{2})))a_0 + \mathcal{O}_{L^\infty}(\hbar^2) \\ = (p_{\Phi_0}(z, \phi'_\lambda(z)) - \lambda)a_0 + \text{Op}_\hbar^w(\zeta \partial_\zeta p_{\Phi_0}(\frac{z+w}{2}, \phi'_\lambda(z)))a_0 + \mathcal{O}_{L^\infty}(\hbar^2) \\ = \frac{\hbar}{i} \left(\frac{1}{2} \partial_z (\partial_\zeta p_{\Phi_0}(z, \phi'_\lambda(z))) + \partial_\zeta p_{\Phi_0}(z, \phi'_\lambda(z)) \partial_z \right) a_0 + \mathcal{O}_{L^\infty}(\hbar^2).$$

This justifies that a_0 satisfies the transport equation

$$(4.15) \quad \left(\frac{1}{2} \partial_z (\partial_\zeta p_{\Phi_0}(z, \phi'_\lambda(z))) + \partial_\zeta p_{\Phi_0}(z, \phi'_\lambda(z)) \partial_z \right) a_0 = 0.$$

The way to solve this equation is to remember that a_0 should be linked to a half form (see [Zwo12, Theorem 10.6] for a more comprehensive discussion) and hence that it could be fruitful to deduce an equation on a_0^2 . Multiplying the previous equation with a_0 we simply get

$$(4.16) \quad \partial_z \left(a_0^2 \partial_\zeta p_{\Phi_0}(w, \phi'_\lambda(w)) \right) = 0.$$

This proves that $a_0^2(\lambda, w) = \partial_\zeta p_{\Phi_0}(w, \phi'_\lambda(w))^{-1}$ and if we set the initial condition, by choosing a local branch of the squareroot, $a_0^2(\lambda_0, z_0) = \partial_\zeta p_{\Phi_0}(z_0, \phi'_\lambda(z_0))^{-\frac{1}{2}}$ we deduce that for all (λ, w) near (λ_0, z_0)

$$(4.17) \quad a_0(\lambda, w) = [\partial_\zeta p_{\Phi_0}(w, \phi'_\lambda(w))]^{-\frac{1}{2}}.$$

(3) We now prove the last statement as follows. We consider $u(\hbar, \lambda, \cdot) = \frac{a(\hbar, \lambda, z)}{a(\hbar, \lambda, z_0)}$ and notice that it is a classical analytic symbol as quotient of classical analytic symbols (see A in which we prove that the class of classical analytic symbols with non vanishing principal symbol is stable by multiplication and inverse). It satisfies in particular:

- i) $(P_{\phi_\lambda} - \lambda)u(\hbar, \lambda, z) = 0$ in $H_{0,(\lambda_0, z_0)}$,
- ii) the formal analytic symbol $\sum_{n \geq 0} u_n(\lambda, z) \hbar^n$ of $u(\hbar, \lambda, z)$ satisfies $u_0(\lambda, z_0) = 1$ and $u_n(\lambda, z_0) = 0$ for $n \geq 1$.

It is then easy to check that both classical analytic symbols $b(\hbar, \lambda, z)$ and $b(\hbar, \lambda, z_0)u(\hbar, \lambda, z)$ satisfy for some $\mathcal{C} > 0$

$$(P_\phi - \lambda)b(\hbar, \lambda, \cdot) = \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}) \text{ and } (P_\phi - \lambda)b(\hbar, \lambda, z_0)u(\hbar, \lambda, z) = \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}).$$

At z_0 : $b(\hbar, \lambda, z_0) = b(\hbar, \lambda, z_0)u(\hbar, \lambda, z_0) + \mathcal{O}(e^{-\mathcal{C}/\hbar})$. This proves that the formal symbol of $b(\hbar, \lambda, z) - b(\hbar, \lambda, z_0)u(\hbar, \lambda, z)$ is 0 at z_0 . Therefore by the uniqueness in Theorem 4.3 we conclude that the formal symbols of $b(\hbar, \lambda, \cdot)$ and $b(\hbar, \lambda, z_0)u(\hbar, \lambda, \cdot)$ coincide for λ, z near λ_0, z_0 . This proves the result. \square

The previous proposition allows us to build WKB quasimodes when λ is $\mathcal{O}(\hbar)$ close to \mathbb{R} . Let us also choose some $M > 0$ and define the set $\Sigma_\hbar := \text{Neigh}(\lambda_0, \mathbb{R}) + i\hbar(-M, M)$ in the following section. The main idea in that in 4.10 we will find than under the condition that λ solves the equation $\mu(\hbar, \lambda)e^{\frac{i}{\hbar}\mathcal{A}(\lambda)}$ with $\mu(\hbar, \lambda) = 1 + \mathcal{O}(\hbar)$ it is possible to prove the existence of WKB quasimodes. However, since a priori we only know the solutions of this equations are real modulo $\mathcal{O}(\hbar^2)$, we prefer to restrict to a small neighbourhood of λ_0 where we expect to find such solutions. Note also that when we will say ‘‘by reducing Σ_\hbar ’’ we will mean shortening $\text{Neigh}_{\mathbb{R}}(\lambda_0, \mathbb{R})$ and considering smaller $M > 0$.

The small difficulty is that in the previous proposition, the pseudodifferential operator \tilde{P} acts on H_Φ with $\Phi(\lambda, z) = -\text{Im}\phi(\lambda, z)$ whereas we wish to build quasimodes in H_{Φ_0} . The aim of the following construction is to remedy this issue.

Note that we expect that these quasimodes are localized near the energy sets $\mathcal{E}_{\text{Re}(\lambda)}$ when $\lambda \in \Sigma_\hbar$. It will be useful to choose a smooth transversal $\text{Neigh}(\lambda_0, \mathbb{R}) \ni \lambda \mapsto z(\lambda)$ such that $(z(\lambda), \frac{2}{i}\partial_z\Phi_0(z(\lambda))) \in \mathcal{E}_\lambda$.

$$(4.18) \quad \varphi(\lambda, z) = \int_{z(\text{Re}(\lambda))}^z \zeta(\lambda, z)dz - i\Phi_0(z(\text{Re}(\lambda))),$$

especially so that $\varphi(\lambda, z(\text{Re}(\lambda))) = \Phi_0(z(\text{Re}(\lambda)))$ corresponding to a function localized at $\mathcal{E}_{\text{Re}(\lambda)}$. This will be clear thanks to Lemma 4.4 noticing that the $L_{\Phi_0}^2$ norm of $a(\hbar, \lambda, z)e^{\frac{i}{\hbar}\varphi(\lambda, z)}$ will be given by

$$\int_{\text{Neigh}(z_0, \mathbb{C})} e^{-2(\text{Im}\varphi(\lambda, z) + \Phi_0(z))/\hbar} |a(\hbar, \lambda, z)|^2 dL(z),$$

therefore it will be useful to estimate $\min_{z \in \text{Neigh}(z_0, \mathbb{C})} (\text{Im}\varphi(\lambda, z) + \Phi_0(z))$.

4.2.2. Refined analysis of the phase functions.

Lemma 4.4. *There exists $C > 0$ such that, for all $\lambda \in \text{Neigh}(\lambda_0, \mathbb{R})$, for all $z \in \text{Neigh}(z_0, \mathbb{C})$*

$$(4.19) \quad \frac{1}{C}d(z, \mathcal{E}_\lambda)^2 \leq \text{Im}\varphi(\lambda, z) + \Phi_0(z) \leq Cd(z, \mathcal{E}_\lambda)^2.$$

Proof. In what follows we will lighten the notations by denoting φ_λ instead of $\varphi(\lambda, \cdot)$ and define

$$\forall z \in \text{Neigh}(z_0, \mathbb{C}), \quad F(\lambda, \cdot) := \text{Im}\varphi(\lambda, \cdot) + \Phi_0.$$

In a nutshell, for $\lambda \in \text{Neigh}(\lambda_0, \mathbb{R})$, F_λ is strictly subharmonic and vanishes to second order on \mathcal{E}_λ . The strict subharmonicity implies that F_λ must grow quadratically in the normal direction to \mathcal{E}_λ which proves the result.

- We have on $\mathcal{E}_\lambda \subset \Lambda_{\Phi_0}$

$$d\text{Im}\varphi_\lambda = \frac{i}{2} \left(\overline{\varphi'_\lambda(z)} d\bar{z} - \varphi'_\lambda(z) dz \right) = -\frac{i}{2} (\zeta dz - \bar{\zeta} d\bar{z}),$$

as well as, on Λ_{Φ_0}

$$d\Phi_0 = \frac{1}{2} (\bar{z} dz + z d\bar{z}) = \frac{1}{2} (i\zeta dz - i\bar{\zeta} d\bar{z}) = \frac{i}{2} (\zeta dz - \bar{\zeta} d\bar{z}).$$

This proves that

$$(4.20) \quad d(\text{Im}\varphi_\lambda + \Phi_0)|_{\mathcal{E}_\lambda} = 0,$$

and achieves this upper bound because $\text{Im}\varphi_\lambda + \Phi_0|_{z(\lambda)} = 0$.

- Let us now give the quadratic estimate of F_λ . It is subharmonic as the sum of a harmonic map $\text{Im}\varphi_\lambda$ and a subharmonic map Φ_0 . Since F_λ takes real values, its critical points are exactly given by the $z \in \mathcal{E}_\lambda$. Since \mathcal{E}_λ is a one dimensional curve, we infer that the real Hessian of F_λ has, in a convenient basis, the form $\begin{pmatrix} 0 & 0 \\ 0 & \partial_{z,\bar{z}} F_\lambda \end{pmatrix}$. The eigenvectors associated with the eigenvalue 0 correspond to the tangential direction to \mathcal{E}_λ and that associated with $\partial_{z,\bar{z}} F_\lambda = \frac{1}{2}$ correspond to the tangential direction. The continuity of the Hessian in λ imply that

$$\forall \lambda \in \text{Neigh}_{\mathbb{R}}(\lambda_0), \forall z \in \text{Neigh}_{\mathbb{C}}(\mathcal{E}_{\lambda_0}), \quad F_\lambda(z) \asymp d(z, \mathcal{E}_\lambda)^2,$$

where⁴ the constants defining \asymp can be chosen uniform in λ . □

The previous estimate extends to Σ_{\hbar} .

Corollary 4.5. *We find $C_1, C_2 > 0$ and a function $\mathcal{O}(\hbar)$ such that all $\lambda \in \Sigma_{\hbar}$, letting $\mu := \text{Re}(\lambda)$ and for all $z \in \text{Neigh}(z_0, \mathbb{C})$*

$$(4.21) \quad C_1 d(z, \mathcal{E}_\mu)^2 - \mathcal{O}(\hbar) \leq (\text{Im}\varphi(\lambda, z) + \Phi_0(z)) \leq C_2 d(z, \mathcal{E}_\mu)^2 + \mathcal{O}(\hbar),$$

4.2.3. *Changing the gauge.* In the following proposition, we prove that our constructions yields quasimodes in $H_{\Phi_0}(\text{Neigh}(z_0, \mathbb{C}))$.

To do so, let us choose small neighbourhoods $\text{Neigh}(z_0, \mathbb{C})$, $\text{Neigh}(\lambda_0, \mathbb{C})$ on which the construction of the previous proposition holds.

Proposition 4.6. *Let $\lambda \in \Sigma_{\hbar}$ and $\text{Neigh}(z_0, \mathbb{C}) \ni w \mapsto a(\hbar, \lambda, w) e^{\frac{i}{\hbar}\varphi_\lambda(w)}$ be the quasimode constructed in the Proposition 4.2. Then, uniformly in λ ,*

$$(4.22) \quad (P_{c,\delta} - \lambda)(a(\hbar, \lambda, \cdot) e^{\frac{i}{\hbar}\varphi_\lambda}) = \mathcal{O}(e^{\Phi_0 - \mathcal{C}/\hbar}).$$

⁴ $A \lesssim_{\text{parameters}} B$ means that there exists a constant $K \in \mathbb{R}_+$ possibly depending on the parameters such that $A \leq KB$ and $A \asymp B$ means $A \lesssim B \lesssim A$.

Proof. The conclusion is a consequence of (4.4) and of the following result. There exists $\mathcal{C} > 0$ such that, up to choosing $\delta > 0$ small enough, for all $\lambda \in \Sigma_{\hbar}$,

$$(4.23) \quad (P_{c,\delta} - \lambda)(a(\hbar, \lambda, \cdot)e^{\frac{i}{\hbar}\varphi\lambda}) = \mathcal{O}(e^{(\Phi_0 - \mathcal{C} + \mathcal{O}(\hbar))/\hbar}) + (\tilde{P} - \lambda)(a(\hbar, \lambda, \cdot)e^{\frac{i}{\hbar}\varphi\lambda}),$$

in a small neighbourhood of (λ_0, z_0) .

The term $P_{c,\delta}(a(\hbar, \cdot, \cdot)e^{\frac{i}{\hbar}\varphi\lambda})$ is defined as an integral of a holomorphic 2-form on the good contour

$$\Lambda_{C,\delta} := \{(w, \zeta) \mid \zeta = \frac{2}{i}\partial_z\Phi_0\left(\frac{z+w}{2}\right) + \overline{iz-w}\},$$

for $-\text{Im}(\zeta(z-w)) + \Phi_0(w)$. We notice that by Corollary 4.5 $-\text{Im}\varphi(\lambda, z) \leq \Phi_0 + \mathcal{O}(\hbar)$ as long as $\lambda \in \Sigma_{\hbar}$. Note also that \tilde{P} is defined as the integral of the same 2-form than that of P_{Φ_0} but on the other contour

$$\mathcal{G}_{C,\delta}(\lambda, z) := \{(w, \zeta) \mid \zeta = \partial_z\varphi(\lambda, w) + \overline{iz-w}, |z-w| \leq \delta\}.$$

We can therefore apply Stokes lemma using the intermediary contours,

$$\mathcal{G}_{t,C,\delta}(\lambda, z) := \{(w, \zeta) \mid \zeta = (1-t)\partial_z\varphi(\lambda, w) + t\frac{2}{i}\partial_z\Phi_0(w) + \overline{icz-w}, |z-w| \leq \delta\}.$$

which are good contours for the phase

$$(w, \zeta) \longmapsto -\text{Im}(\zeta(z-w)) + \Phi_t(\lambda, w), \text{ with } \Phi_t(\lambda, w) = -(1-t)\text{Im}\varphi(\lambda, w) + t\Phi_0(w).$$

We infer, by continuity in (t, λ, z) that there exists $\mathcal{C} > 0$ such that for all $\lambda \in \Sigma_{\hbar}$ and $t \in [0, 1]$

$$-\text{Im}(\zeta(z-w)) + \varphi(\lambda, w) + \gamma(\lambda) \leq \Phi_0 - \mathcal{C} + \mathcal{O}(\hbar) \text{ on } \partial\mathcal{G}_{t,C,\delta}(\lambda, z),$$

and this proves (4.23). To conclude the proof, using (4.4) and (4.23), we infer that there exists $\mathcal{C} > 0$ such that

$$(P_{c,\delta} - \lambda)(a(\hbar, \cdot, \cdot)e^{\frac{i}{\hbar}\varphi\lambda}) = \mathcal{O}(e^{\Phi_0 - \mathcal{C} + \mathcal{O}(\hbar)/\hbar}) + \mathcal{O}(e^{(-\text{Im}(\varphi\lambda) - \mathcal{C})/\hbar}) = \mathcal{O}(e^{\Phi_0 - \mathcal{C} + \mathcal{O}(\hbar)/\hbar}).$$

Up to considering \hbar sufficiently small this proves the result. \square

We infer the existence of microlocal WKB quasimodes when $\lambda \in \Sigma_{\hbar}$.

Proposition 4.7. *There exists $\mathcal{C} > 0$ and a function $\mathcal{O}(e^{-\mathcal{C}/\hbar})$ such that for all $\lambda \in \Sigma_{\hbar}$,*

$$\|(P_{c,\delta} - \lambda)a(\hbar, \lambda, \cdot)e^{\frac{i}{\hbar}\varphi(\lambda, \cdot)}\|_{L^2_{\Phi_0}(\text{Neigh}(z_0, \mathbb{C}))} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar})\|a(\hbar, \lambda, \cdot)e^{\frac{i}{\hbar}\varphi(\lambda, \cdot)}\|_{L^2_{\Phi_0}(\text{Neigh}(z_0, \mathbb{C}))}.$$

Proof. There exists $C > 0$ such that for all $\lambda \in \Sigma_{\hbar}$, we have denoting $\mu = \text{Re}(\lambda)$

$$(4.24) \quad |a(\hbar, \lambda, w)e^{\frac{i}{\hbar}\varphi(\lambda, w)}|e^{-\Phi_0(w)/\hbar} \geq e^{\frac{-\text{Im}\varphi(\lambda, w) + \Phi_0(w)}{\hbar}} (|\partial_{\zeta}p_{\Phi_0}(w, -\varphi'_{\lambda}(w))|^{-1} + \mathcal{O}(\hbar)) \\ \geq Ce^{-\frac{Cd(w, \mathcal{E}_{\mu})^2}{\hbar}} (|\partial_{\zeta}p_{\Phi_0}(w, -\varphi'_{\lambda}(w))|^{-1} + \mathcal{O}(\hbar)).$$

We infer

$$(4.25) \quad |a(\hbar, \lambda, w)e^{\frac{i}{\hbar}\varphi(\lambda, w)}e^{-\Phi_0(w)/\hbar}| \geq Ce^{-\frac{Cd(w, \mathcal{E}_{\mu})^2}{\hbar}} (|\partial_{\zeta}p_{\Phi_0}(w, -\varphi'_{\lambda}(w))|^{-1} + \mathcal{O}(\hbar))$$

Let us consider the sets

$$\mathcal{U}_{\hbar}(\lambda) := \{z \in \mathbb{C} \mid d(z, \mathcal{E}_{\mu}) \leq \sqrt{\hbar}\} \cap \text{Neigh}(z_0, \mathbb{C})$$

and notice that there exists $C_0 > 0$ such that $\text{Vol}(\mathcal{W}_h(\lambda)) \geq C_0 \hbar^{\frac{1}{2}}$. We infer that there exists $C, K > 0$ such that

$$\begin{aligned} \| |a(\hbar, \lambda, \cdot) e^{\frac{i}{\hbar} \varphi(\lambda, \cdot)} \|_{L^2_{\Phi_0}(\text{Neigh}(z_0, \mathbb{C}))}^2 &\geq C \int_{\mathcal{W}_h(\lambda)} e^{-\frac{C d(w, \mathcal{E}_\mu)^2}{\hbar}} (|\partial_\zeta p_{\Phi_0}(w, -\varphi'_\lambda(w))|^{-1} + \mathcal{O}(\hbar)) dL(w) \\ &\geq C \int_{\mathcal{W}_h(\lambda)} e^{-C} (\min_{\Omega} |\partial_\zeta p_{\Phi_0}(z, \zeta(\lambda, z))|^{-1} + \mathcal{O}(\hbar)) dL(w) \\ &\geq e^{-C} \left(\min_{\Omega} |\partial_\zeta p_{\Phi_0}(z, \zeta(\lambda, z))|^{-1} + \mathcal{O}(\hbar) \right) \text{Vol}(\mathcal{W}_h(\lambda)) \\ &\geq C_0 e^{-C} \left(\min_{\text{Neigh}(\mathcal{E}_{\lambda_0}, \mathbb{C}^2)} |\partial_\zeta p_{\Phi_0}(z, \zeta(\lambda, z))|^{-1} + \mathcal{O}(\hbar) \right) \hbar^{\frac{1}{2}}. \end{aligned}$$

Since $|\partial_\zeta p_{\Phi_0}|^2 = \frac{1}{2} |\nabla p|_2^2$ does not vanish near \mathcal{E}_{λ_0} by Assumption [H.2](#). We infer that $\| |a_h(\lambda, \cdot) e^{\frac{i}{\hbar} \varphi_\lambda} \|_{L^2_{\Phi_0}(\mathbb{C})}$ uniformly in $\lambda \in \Sigma_h$. We then conclude thanks to Proposition [4.6](#). \square

4.3. Patching WKB expansions with a $\bar{\partial}$ Lemma. The goal of this section is to patch the microlocal WKB quasimodes constructed above and prove Proposition [4.10](#).

4.3.1. $\bar{\partial}$ Lemma adapted to WKB expansions. In order to patch the WKB expansions on a small neighbourhood of \mathcal{E}_{λ_0} we will use Hörmander's $\bar{\partial}$ lemma that we recall in what follows.

Lemma 4.8 (Hörmander $\bar{\partial}$ Lemma). *Let $f \in L^2_{\Phi_0}(\mathbb{C}, \mathfrak{m})$. Then there exists $u \in L^2_{\Phi_0}(\mathbb{C}, \mathfrak{m})$ such that for \hbar small enough*

$$(4.26) \quad \bar{\partial}_z u = f \quad \text{and} \quad \|u\|_{L^2_{\Phi_0}(\mathbb{C}, \mathfrak{m})} \leq 2\hbar \|f\|_{L^2_{\Phi_0}(\mathbb{C}, \mathfrak{m})}.$$

Proof. This is a straightforward consequence of Hörmander's Lemma [[Hör83](#), p. 15.1.1] with weight $\frac{\Phi_0}{\hbar} + \log(\mathfrak{m})$. \square

We will adapt the $\bar{\partial}$ lemma to glue WKB quasimodes in the following setting. Let us consider a family of complex balls $(\mathbb{B}_j)_{j \in \llbracket 0, m \rrbracket}$ that we suppose satisfy the following.

- A.1 For all $j \in \llbracket 0, m \rrbracket$, the ball \mathbb{B}_j is centered at some point $z_j \in \mathcal{E}_{\lambda_0}$.
- A.2 The intersection $\mathbb{B}_j \cap \mathbb{B}_k$ is empty if and only if $k \notin \llbracket j-1, j+1 \rrbracket$ (denoting $\mathbb{B}_{m+1} = \mathbb{B}_0$).
- A.3 The Hamiltonian flow of z_0 crosses \mathbb{B}_0 , then $\mathbb{B}_1, \mathbb{B}_2$ all the way back to \mathbb{B}_0 in the increasing order of indices.

Let $\Omega \Subset \bigcup_{j=0}^m \mathbb{B}_j$ be a connected open set containing \mathcal{E}_{λ_0} . The standard results on partition of unity gives us a family of smooth functions $(\chi_j)_{j \in \llbracket 0, m \rrbracket}$ satisfying that for all $j \in \llbracket 0, m \rrbracket$, $\chi_j \in \mathcal{C}_c^\infty(\mathbb{C})$, $\text{supp} \chi_j \subset \mathbb{B}_j$ and $\sum_{j=0}^m \chi_j = 1$ on Ω . This allows us to state the following Lemma in the spirit of Čech cohomology.

Lemma 4.9 (Patching Lemma in $L^2_{\Phi_0}(\mathbb{C}, \mathfrak{m})$). *For each $j \in \llbracket 0, m \rrbracket$, we consider $f_j, r_j \in \text{Hol}(\mathbb{B}_j)$ satisfying*

$$(4.27) \quad \forall j \in \llbracket 0, m \rrbracket, \exists c_{j,j+1} \in \mathbb{C}^*, \quad c_{j,j+1} f_{j+1} = f_j + r_j \text{ on } \mathbb{B}_j \cap \mathbb{B}_{j+1},$$

where we denote $\mathbb{B}_{m+1} = \mathbb{B}_0$ and $f_{m+1} = f_0$.

There exists a constant $C > 0$ (depending on the partition of unity) such that the following holds. If the condition $\prod_{j=0}^m c_{j,j+1} = 1$ is satisfied then there exists $u \in L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})$ such that, denoting $d_j = c_{0,1} \cdots c_{j-1,j}$, we have

$$(4.28) \quad \sum_{j=0}^m d_j \chi_j f_j - u \in H_{\Phi_0}(\mathbb{C}, \mathbf{m}) \text{ and } \|u\|_{L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})} \leq C \hbar \sum_{j=0}^m \left(\|d_j r_j\|_{L_{\Phi_0}^2(\mathbb{B}_j)} + \|d_j f_j\|_{L_{\Phi_0}^2(\mathbb{B}_j \setminus \Omega)} \right).$$

Proof. We consider $g := \sum_{j=0}^m d_j \chi_j f_j$ and notice that

$$\begin{aligned} & \text{— on } \mathbb{B}_j \cap \mathbb{B}_{j+1} \cap \Omega, \bar{\partial} g = \bar{\partial} \chi_j (d_j f_j - d_{j+1} f_{j+1}) = -d_j \bar{\partial} \chi_j r_j, \\ & \text{— on } \mathbb{C}, \bar{\partial} g = \sum_{j=0}^m d_j \bar{\partial} \chi_j f_j. \end{aligned}$$

This gives some $c > 0$ only depending on the partition of unity

$$(4.29) \quad \|\bar{\partial} g\|_{L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})} \leq c \sum_{j=0}^m \left(\|d_j r_j\|_{L_{\Phi_0}^2(\mathbb{B}_j)} + \|d_j f_j\|_{L_{\Phi_0}^2(\mathbb{B}_j \setminus \Omega)} \right).$$

We apply Hörmander $\bar{\partial}$ Lemma 4.8 giving $u \in L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})$ such that

$$(4.30) \quad g - u \in H_{\Phi_0}(\mathbb{C}, \mathbf{m}) \text{ and } \|u\|_{L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})} \leq 2c \hbar \sum_{j=0}^m \left(\|d_j r_j\|_{L_{\Phi_0}^2(\mathbb{B}_j)} + \|d_j f_j\|_{L_{\Phi_0}^2(\mathbb{B}_j \setminus \Omega)} \right).$$

□

From now on the energy sets \mathcal{E}_λ will be identified with their projections on the z variable thanks to the fact that (3.6) is an isomorphism.

4.3.2. *Construction of the quasimodes.* In what follows, we apply the $\bar{\partial}$ lemma and prove the following proposition. The undeterminacy of the sign of the principal symbol of $\mu(\hbar, \lambda)$ will be removed in the next section.

Proposition 4.10. *There exist $\mathcal{C} > 0$, a function $\mathcal{O}(e^{-\mathcal{C}/\hbar})$ and a classical analytic symbol $\mu(\hbar, \lambda) = \pm 1 + \mathcal{O}(\hbar)$ such that the following holds. For all λ in Σ_\hbar and satisfying*

$$(BS) \quad \mu(\hbar, \lambda) e^{\frac{i}{\hbar} \mathcal{A}(\lambda)} = 1,$$

there exists $\Psi_\lambda \in H_{\Phi_0}(\mathbb{C})$ such that

$$(4.31) \quad \|(P_{\Phi_0} - \lambda) \Psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar}) \|\Psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})}.$$

By compactness of \mathcal{E}_{λ_0} and thanks to Proposition 4.2, we find $m \in \mathbb{N}^*$ and a family of m balls, $(\mathbb{B}_j)_{j \in [0, m]}$ such that the following stands.

- i) The assumptions A.1, A.2, A.3 are verified.
- ii) There exists a smooth transversal $\lambda \mapsto z_j(\lambda)$ satisfying $z_j(\lambda) \in \mathcal{E}_{\text{Re}(\lambda)}$.
- iii) For all $j \in [0, m]$ the phase $\varphi_j(\lambda, \cdot)$ is well defined on \mathbb{B}_j by the construction of Section 4.2.2. It satisfies $i\varphi_j(\lambda, z_j(\lambda)) = \Phi_0(z_j(\lambda))$ and also Lemma 4.4.
- iv) There exists a classical analytic symbol $a_j(\hbar, \lambda, z)$ well defined for λ near λ_0 , z in \mathbb{B}_j and satisfying (4.4).
- v) The principal symbol of $a_j(\hbar, \lambda, \cdot)$ is non zero and satisfies $a_j^0(\lambda, z)^2 = \partial_\zeta p_{\Phi_0}(z, \zeta(\lambda, z))^{-1}$.

The last statement of Proposition 4.2 guarantees that for all $j \in \llbracket 0, m \rrbracket$ there exists a classical analytic symbol $\alpha_{j,j+1}(\hbar, \lambda)$ with principal symbol ± 1 satisfying for all λ close enough to λ_0 (denoting $\mathbb{B}_{m+1} = \mathbb{B}_0$)

$$(4.32) \quad \alpha_{j,j+1}(\hbar, \lambda) a_{j+1}(\hbar, \lambda, \cdot) = a_j(\hbar, \lambda, \cdot) + \varepsilon_j(\hbar, \lambda, \cdot) \text{ on } \mathbb{B}_j \cap \mathbb{B}_{j+1}.$$

where there is $\mathcal{C} > 0$ such that $|\varepsilon_j| \leq \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}})$.

Let us start with a brief look at the phase functions.

Lemma 4.11. *For all $j \in \llbracket 0, m \rrbracket$, there exists $\mathcal{A}_{j,j+1}(\lambda) \in \mathbb{C}$ such that for all $\lambda \in \Sigma_{\hbar}$,*

$$(4.33) \quad \varphi_j(\lambda, \cdot) - \varphi_{j+1}(\lambda, \cdot) = \mathcal{A}_{j,j+1}(\lambda) \text{ on } \mathbb{B}_j \cap \mathbb{B}_{j+1}, \quad \sum_{j=1}^m \mathcal{A}_{j,j+1}(\lambda) = \mathcal{A}(\lambda) \text{ and } \text{Im}(\mathcal{A}_{j,j+1}(\lambda)) = \mathcal{O}(\hbar).$$

Proof. The fact that $\mathbb{B}_j \cap \mathbb{B}_{j+1}$ is connected and that the maps $\varphi_{j+1}(\lambda, \cdot)$, $\varphi_j(\lambda, \cdot)$ have same derivative guarantees the existence of $\mathcal{A}_{j,j+1}(\lambda)$ satisfying (4.33). In particular $\mathcal{A}_{j,j+1}(\lambda)$ is a partial action in the sense that defining $\gamma_{j,j+1}$ as the simple path from $z_j(\lambda)$ to $z_{j+1}(\lambda)$ in $\pi_z \mathcal{E}_{\lambda}$, we have from the identity $\partial_z \varphi_j(\hbar, \lambda, z) = \zeta(\lambda, z)$,

$$(4.34) \quad \mathcal{A}_{j,j+1}(\lambda) = \int_{\gamma_{j,j+1}} \zeta(\lambda, z) dz + i \text{cobord} \quad \text{where } \text{cobord} = \Phi(z_j(\lambda)) - \Phi(z_{j+1}(\lambda)).$$

The concatenation of all the paths $\gamma_{j,j+1}$ is a simple loop going through \mathcal{E}_{λ} . Since the orientation of the loops are, by the assumptions at the beginning of this section, given by the Hamiltonian flow, we conclude that $\sum_{j=0}^m \mathcal{A}_{j,j+1}(\lambda) = \int_{\mathcal{E}_{\text{Re}(\lambda)}} \zeta(\lambda, z) dz = \mathcal{A}(\lambda)$. Eventually, from corollary 4.5, we infer that $\text{Im} \varphi_j(\lambda, \cdot)$ and $\text{Im} \varphi_{j+1}(\lambda, \cdot)$ coincide modulo $\mathcal{O}(\hbar)$ on $\mathcal{E}_{\text{Re}(\lambda)}$, this proves that $\varphi_j(\lambda, \cdot) - \varphi_{j+1}(\lambda, \cdot) = \mathcal{O}(\hbar)$. \square

We now apply the patching Lemma 4.9 to get the following Lemma, denoting $\mathcal{A}_j(\lambda) = \mathcal{A}_{0,1}(\lambda) + \dots + \mathcal{A}_{j-1,j}(\lambda)$,

$$\alpha_j(\hbar, \lambda) = \alpha_{0,1}(\hbar, \lambda) \cdots \alpha_{j-1,j}(\hbar, \lambda) \text{ and } \psi_{\lambda} := \sum_{j=0}^m \chi_j(\cdot) \alpha_j(\hbar, \lambda) a_j(\hbar, \lambda, \cdot) e^{\frac{i}{\hbar}(\varphi_j(\lambda, \cdot) + \mathcal{A}_j(\lambda))}.$$

Lemma 4.12. *There exists a classical analytic symbol $\mu(\hbar, \lambda)$ whose principal symbol is ± 1 such that the following holds. There exists $\mathcal{C} > 0$ and a function $\mathcal{O}(e^{-\mathcal{C}/\hbar})$ such that if $\mu(\hbar, \lambda) e^{\frac{i}{\hbar} \mathcal{A}(\lambda)} = 1$ for $\lambda \in \Sigma_{\hbar}$ we find $u_{\lambda} \in L^2_{\Phi_0}(\mathbb{C}, \mathfrak{m})$ satisfying*

$$(4.35) \quad \psi_{\lambda} - u_{\lambda} \in H_{\Phi_0}(\mathbb{C}, \mathfrak{m}) \text{ and } \|u_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C}, \mathfrak{m})} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar}).$$

Proof. We apply the patching lemma 4.9 (considering λ as a parameter) with

$$f_j = a_j(\hbar, \lambda, \cdot) e^{\frac{i}{\hbar} \varphi_j(\lambda, \cdot)}, \quad c_{j,j+1}(\hbar, \lambda) = e^{\frac{i}{\hbar} \mathcal{A}_{j,j+1}(\lambda)} \alpha_{j,j+1}(\hbar, \lambda), \quad d_j(\hbar, \lambda) = \alpha_j(\hbar, \lambda) e^{\frac{i}{\hbar} \mathcal{A}_j(\lambda)},$$

and $r_j(\hbar, \lambda, z) := \varepsilon_j(\hbar, \lambda, z) e^{\frac{i}{\hbar} \varphi_j(\lambda, z)}$, with ε_j defined in (4.32). Setting

$$\mu(\hbar, \lambda) := \alpha_{0,1}(\hbar, \lambda) \alpha_{1,2}(\hbar, \lambda) \cdots \alpha_{m,0}(\hbar, \lambda),$$

thanks to Lemma 4.11 the condition $\prod_{j=0}^m c_{j,j+1} = 1$ in Lemma 4.9 is equivalent to $\mu(\hbar, \lambda)e^{\frac{i}{\hbar}A(\lambda)} = 1$. As soon as this condition is satisfied we obtain $u_\lambda \in H_{\Phi_0}(\mathbb{C}, \mathbf{m})$ satisfying

$$(4.36) \quad \psi_\lambda - u_\lambda \in H_{\Phi_0}(\mathbb{C}, \mathbf{m}) \text{ and } \|u_\lambda\|_{L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})} \leq C \sum_{j=0}^m (\|d_j(\hbar, \lambda)r_j\|_{L_{\Phi_0}^2(\mathbb{B}_j)} + \|d_j(\hbar, \lambda)f_j\|_{L_{\Phi_0}^2(\mathbb{B}_j \setminus \Omega)}).$$

The assumption on the principal symbol of $a_{j+1}(\hbar, \lambda, \cdot)$ guarantees that $|\alpha_{j,j+1}(\hbar, \lambda)| = 1 + \mathcal{O}(\hbar)$ (uniformly in λ near λ_0) thus $\mu(\hbar, \lambda) = \pm 1 + \mathcal{O}(\hbar)$. Using Lemma 4.11 we recall that $\text{Im}(\mathcal{A}_j(\lambda)) = \mathcal{O}(\hbar)$ when $\lambda \in \Sigma_\hbar$ and $d_j(\hbar, \lambda) = \mathcal{O}(1)$. Therefore, there exists $\mathcal{C} > 0$ such that $\|d_j(\hbar, \lambda)r_j\|_{L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})} \leq \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}})$, uniformly in $\lambda \in \Sigma_\hbar$. Possibly restricting Σ_\hbar , we infer also by Corollary 4.5 that $\|d_j(\hbar, \lambda)f_j\|_{L_{\Phi_0}^2(\mathbb{B}_j \setminus \Omega)} \leq \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}})$. Eventually from (4.32) we infer that there exists (another) $\mathcal{C} > 0$ such that for all $\lambda \in \Sigma_\hbar$ satisfying $\mu(\hbar, \lambda)e^{\frac{i}{\hbar}A(\lambda)} = 1$, $\|u_\lambda\|_{L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})} \leq \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}})$. \square

The map $\Psi_\lambda := \psi_\lambda - u_\lambda \in H_{\Phi_0}(\mathbb{C}, \mathbf{m})$ defined in the previous lemma will be our candidate in the quest for quasimodes. Before providing the proof of Proposition 4.10 we let $K \Subset \Omega$ a compact neighbourhood of \mathcal{E}_{λ_0} and state the following estimates.

Lemma 4.13 (Necessary estimates). *Possibly restricting Σ_\hbar we assume that for all $\lambda \in \Sigma_\hbar$, $\mathcal{E}_\lambda \subset K$. There exists $\mathcal{C}, C > 0$ and a function $\mathcal{O}(e^{-\mathcal{C}/\hbar})$ such that for all $\lambda \in \Sigma_\hbar$ satisfying (BS):*

- i) $\|(P_{c,\delta} - \lambda)\psi_\lambda\|_{L_{\Phi_0}^2(K)} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar})\|\psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})}$ and $\|\psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})} \geq C\hbar^{\frac{1}{4}}$.
- ii) $\|\psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C}K, \mathbf{m})} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar})$
- iii) $\|\Psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})} \asymp \|\Psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C}, \mathbf{m})}$ and $\|\Psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})} \asymp \|\psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})}$ (with constants uniform in λ).

Proof. The estimates are elementary consequences of the previous lemmas. However for the sake of completeness we choose to provide some details to the reader.

i) By construction we have $K \subset \bigcup_{j \in \llbracket 0, m \rrbracket} \mathbb{B}_j$ and by Proposition 4.7, on each \mathbb{B}_j ,

$$\|(P_{c,\delta} - \lambda)a_j(\hbar, \lambda, \cdot)e^{\frac{i}{\hbar}\varphi_j(\lambda, \cdot)}\|_{L_{\Phi_0}^2(\mathbb{B}_j)} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar})\|a_j(\hbar, \lambda, \cdot)e^{\frac{i}{\hbar}\varphi_j(\lambda, \cdot)}\|_{L_{\Phi_0}^2(\mathbb{B}_j)}.$$

The estimate on $\|\psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})}$ is a straightforward consequence of the proof of Proposition 4.7.

ii) To prove $\|\chi_j(\cdot)d_j a_j e^{\frac{i}{\hbar}\varphi_j}\|_{L_{\Phi_0}^2(\mathbb{C}K, \mathbf{m})} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar})$ when $\lambda \in \Sigma_\hbar$, we recall that $\mathcal{E}_\lambda \subset K$, $d_j(\hbar, \lambda) = \mathcal{O}(1)$ and we apply the Corollary 4.5. This proves ii) by the triangular inequality.

iii) To prove the last statement, we notice that $\|\Psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})} \asymp \|\psi_\lambda\|_{L_{\Phi_0}^2(\mathbb{C})}$ by ii). The map ψ_λ has compact support in $\bigcup_{j=0}^m \mathbb{B}_j$ and we can conclude thanks to the equivalence $\|\cdot\|_{L_{\Phi_0}^2(\bigcup_{j=0}^m \mathbb{B}_j)} \asymp \|\cdot\|_{L_{\Phi_0}^2(\bigcup_{j=0}^m \mathbb{B}_j, \mathbf{m})}$. \square

Proof of Proposition 4.10. Let us then gather all the elements and prove Proposition 4.10. The idea is to apply the technical Lemma 2.5 by considering $K_1 \subset \overset{\circ}{K}$ another compact neighbourhood of \mathcal{E}_{λ_0} . *Norm estimate on $\mathbb{C}K$:* By the estimates of Lemma (4.13) we know

that there exists $\mathcal{C} > 0$ and a function $\mathcal{O}(e^{-\mathcal{C}/\hbar})$, possibly restricting Σ_{\hbar} , for all $\lambda \in \Sigma_{\hbar}$ satisfying (BS),

$$\|\psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C}K_{1,m})} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar}) \|\psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C},m)}.$$

Lemma (2.5) then yields, changing the value of $\mathcal{C} > 0$, $\mathcal{O}(e^{-\mathcal{C}/\hbar})$ at each line if need be,

$$(4.37) \quad \begin{aligned} \|(P_{\Phi_0} - \lambda)\Psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C}K)} &\leq \rho \|\Psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C}K_{1,m})} + \mathcal{O}(e^{-\mathcal{C}/\hbar}) \|\Psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C},m)} \\ &\leq \mathcal{O}(e^{-\mathcal{C}/\hbar}) \|\Psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C},m)}. \end{aligned}$$

Norm estimate on K : We take $P_{c,\delta}$ defined in (2.14) with δ smaller than $\text{dist}(K, \text{supp } \partial\Omega)$. The goal is to control the terms of the right hand side of

$$(4.38) \quad \begin{aligned} \|(P_{\Phi_0} - \lambda)\Psi_{\lambda}\|_{L^2_{\Phi_0}(K)} \\ \leq \|(P_{c,\delta} - \lambda)\psi_{\lambda}\|_{L^2_{\Phi_0}(K)} + \|P_{\Phi_0} - P_{c,\delta}\|_{L^2_{\Phi_0}(\mathbb{C},m) \rightarrow L^2_{\Phi_0}(\mathbb{C})} \|\Psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C},m)} + C \|u_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C},m)}. \end{aligned}$$

The term $\|u_{\lambda}\|$ is easily controlled thanks to the norm estimate of the $\bar{\partial}$ lemma. We also have by Lemma 2.4 that

$$\|P_{\Phi_0} - P_{c,\delta}\|_{L^2_{\Phi_0}(\mathbb{C},m) \rightarrow L^2_{\Phi_0}(\mathbb{C})} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar}),$$

for some $\mathcal{C} > 0$ and by the first estimate of Lemma (4.13) for another $\mathcal{C} > 0$,

$$(4.39) \quad \|(P_{c,\delta} - \lambda)\psi_{\lambda}\|_{L^2_{\Phi_0}(K)} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar}) \|\psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C})} \leq \mathcal{O}(e^{-\mathcal{C}/\hbar}) \|\Psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C})}.$$

We can conclude this estimate using the point *iii*) of the previous lemma and that fact that $\|\psi_{\lambda}\|_{L^2_{\Phi_0}(\mathbb{C})}$ is bounded by below by $c\hbar^{\frac{1}{4}}$ for some $c > 0$. \square

4.4. Maslov's correction.

Proposition 4.14. *We can compute the first order of $\mu(\hbar, \lambda)$ which is related to Maslov's correction. It is given by*

$$(4.40) \quad \mu(\hbar, \lambda) = -1 + \mathcal{O}(\hbar).$$

Proof. First, letting $j \in \llbracket 0, m \rrbracket$, as defined in Section 4.3.2 the principal symbol $a_j^0(\lambda, z)$ of $a_j(\hbar, \lambda, \cdot)$ is a local squareroot of $\partial_{\zeta} p_{\Phi_0}(z, \zeta(\lambda, z))^{-1}$. Possibly recursively replacing $a_j(\hbar, \lambda, z)$ by its opposite we can assume that for all $j \in \llbracket 0, m-1 \rrbracket$, $a_j^0(\lambda, z) = a_{j+1}^0(\lambda, z)$ on $\mathbb{B}_j \cap \mathbb{B}_{j+1}$ and for all λ near λ_0 . This yields that with $\alpha_{j,j+1}(\hbar, \lambda)$ defined in (4.32), $\alpha_{j,j+1}(\hbar, \lambda) = 1 + \mathcal{O}(\hbar)$ for $j = 1, 2, \dots, m-1$. Now we prove that necessarily $\alpha_{m,0}(\hbar, \lambda) = -1 + \mathcal{O}(\hbar)$. Indeed assume that this is not the case, that is to say $\alpha_{m,0}(\hbar, \lambda) = 1 + \mathcal{O}(\hbar)$. In other words we obtain

$$\forall j \in \llbracket 0, m \rrbracket, \quad a_j^0(\lambda, z) = a_{j+1}^0(\lambda, z) \text{ on } \mathbb{B}_j \cap \mathbb{B}_{j+1}$$

with the convention $a_{m+1}^0(\lambda, z) = a_0^0(\lambda, z)$ and $\mathbb{B}_0 = \mathbb{B}_{m+1}$. We can then apply the usual patching result of Čech cohomology for holomorphic functions providing a holomorphic function $b(\lambda, z)$ fitting with all the $a_j^0(\lambda, z)$ on the union of \mathbb{B}_j . In particular $b(\lambda, z)$ satisfies $b(\lambda, z)^2 = \partial_{\zeta} p_{\Phi_0}(z, \zeta(\lambda, z))^{-1}$ and we infer that $\partial_{\zeta} p_{\Phi_0}(z, \zeta(\lambda, z))$ has a holomorphic squareroot defined on a neighbourhood of \mathcal{E}_{λ_0} . This contradicts the result of Lemma 3.7. We conclude that $\alpha_{m,0}(\hbar, \lambda) = -1 + \mathcal{O}(\hbar)$ and

$$(4.41) \quad \mu(\hbar, \lambda) = \alpha_{0,1}(\hbar, \lambda) \alpha_{1,2}(\hbar, \lambda) \cdots \alpha_{m-1,m}(\hbar, \lambda) \alpha_{m,0}(\hbar, \lambda) = -1 + \mathcal{O}(\hbar).$$

\square

At order 2 it is then easy to see that (with a small abuse of notations) the equation (BS) becomes

$$(4.42) \quad \mathcal{A}(\lambda) + \pi\hbar + \mathcal{O}(\hbar^2) \in 2\pi\hbar\mathbb{Z}.$$

We obtain that the solutions of the equation (BS) near λ_0 belong to Σ_{\hbar} . This allows the existence of exponentially sharp quasimodes by Proposition 4.10. The term $\pi\hbar$ in the left hand side of (4.42) is deduced from Proposition 4.14 and usually known as Maslov's correction.

Remark 4.15 (Interpretation of Maslov's correction). From another perspective, by the previous construction, we can choose a phase φ_{λ} on the universal cover of Ω as a primitive of $\zeta(\lambda, z)$ and notice that the ‘‘principal symbol’’ $\partial_{\zeta} p_{\Phi_0}(z, \zeta(\lambda, z))^{-\frac{1}{2}}$ is also well defined on the universal cover (as it is simply connected). If we truncate at order 2 in \hbar , the condition (BS) becomes $\mathcal{A}(\lambda) + \pi\hbar \in 2\pi\hbar\mathbb{Z}$ and is equivalent to the condition at which $\partial_{\zeta} p_{\Phi_0}(z, \zeta(\lambda, z))^{-\frac{1}{2}} e^{\frac{i}{\hbar}\varphi_{\lambda}(z)}$ defines a holomorphic function on Ω .

It is interesting to notice that $\partial_{\zeta} p_{\Phi_0}(z, \zeta(\lambda, z))^{-\frac{1}{2}} e^{\frac{i}{\hbar}\varphi_{\lambda}(z)}$ is well defined ‘‘semi-globally’’ near \mathcal{E}_{λ_0} even though individually the symbol and the phase are not.

4.5. Inversion in the Class of Analytic Symbols and Conclusion. *The goal of this section is to invert (BS) in the class of analytic symbols to finish the proof of Theorem 1.4 in a small neighbourhood of λ_0 .*

We look for a classical analytic symbol $\lambda(\hbar, I) = \mathcal{A}^{-1}(I) + \mathcal{O}(\hbar)$ defined for $I \in \text{Neigh}_{\mathbb{C}}(\mathcal{A}(\lambda_0))$, taking values near λ_0 and solving the equation

$$(4.43) \quad \mu(\hbar, \lambda) e^{\frac{i}{\hbar}\mathcal{A}(\lambda)} = e^{iI/\hbar}.$$

For that, we will need a classical analytic symbol $\mathcal{F}(\lambda, \hbar) = \mathcal{A}(\lambda) + \pi\hbar + \mathcal{O}(\hbar^2)$ such that

$$\mu(\hbar, \lambda) e^{\frac{i}{\hbar}\mathcal{A}(\lambda)} = e^{\frac{i}{\hbar}\mathcal{F}(\lambda, \hbar)},$$

and then invert the equation $\mathcal{F}(\hbar, \lambda) = I$. This is the subject of the two following lemmas.

Lemma 4.16. *We find a classical analytic symbol $\gamma(\hbar, \lambda)$ of formal symbol $\sum_{k \geq 0} \hbar^k \gamma_k$ such that $\mu(\hbar, \lambda) = e^{i\gamma(\hbar, \lambda)}$ and $\gamma(\hbar, \lambda) = \pi + \mathcal{O}(\hbar)$.*

Proof. We apply Lemma A.10 with $\gamma(\hbar, \lambda) = i \log(\mu(\hbar, \lambda))$, where we choose the branch defined in a small neighbourhood of -1 satisfying $\log(-1) = -i\pi$. \square

Lemma 4.17. *Suppose that $(\hbar, \lambda) \mapsto f(\hbar, \lambda)$ is a classical analytic symbol defined in a neighbourhood of λ_0 with formal symbol $\sum_{k \geq 0} \hbar^k f_k(\lambda)$ satisfying $\partial_{\lambda} f_0(\lambda_0) \neq 0$. Then there exists a classical analytic symbol $g(\hbar, I)$ of formal symbol $\sum_{k \geq 0} \hbar^k g_k$ satisfying*

$$(4.44) \quad \exists \mathcal{C} > 0, \quad \forall I \in \text{Neigh}_{\mathbb{C}}(f_0(\lambda_0)), \quad f(\hbar, g(\hbar, I)) = I + \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}),$$

and

$$(4.45) \quad \exists \mathcal{C} > 0, \quad \forall \lambda \in \text{Neigh}_{\mathbb{C}}(\lambda_0), \quad g(\hbar, f(\hbar, \lambda)) = \lambda + \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}).$$

Proof. In the following proof, we denote formal symbols with majuscules and classical analytic symbols with minuscules. According to this remark, $F(\hbar, \lambda)$ will refer to the formal symbol associated with $f(\hbar, \lambda)$.

The asymptotic expansion is obtained by the usual iterative method giving a formal symbol $G = \sum_{k \geq 0} \hbar^k g_k$, which is the formal inverse of the formal symbol of $f(\hbar, \lambda)$. Of course the g_k are well defined and holomorphic in a small neighbourhood of λ_0 (independent of k). With no loss of generality, in order to simplify the notations, we will suppose that $\lambda_0 = 0$.

1) *Estimate on the formal symbol*

In their recent article [HZ24, Proposition 3.5] Hitrik and Zworski check by arm and hammer that the formal symbol G is a formal classical analytic symbol. Let us notice that a variant of their proof can be obtained after the reading of Komatsu [Kom79] (which proves similar results for ultradifferentiable mappings). In our situation, denoting $f_{k,n} = \frac{f_n^{(k)}(0)}{k!}$, the datum of a (1-variable) formal classical analytic symbol is equivalent to that of a formal series

$$\sum_{k,n \in \mathbb{N}^2} f_{k,n} \hbar^n \lambda^k, \quad \exists C, K > 0, \quad |f_{k,n}| \leq C^k K^n n!.$$

The key of the proof of Komatsu is that he had the idea of using Lagrange's inversion formula. Let us follow his insight and use (without proof) the following result.

Lemma 4.18 (Lagrange Inversion Formula). *Let $S(\hbar, \lambda) = \sum_{n \geq 1} \widehat{s}_n(\hbar) \lambda^n$ a formal power series with each $\widehat{s}_n := \sum_{k \geq 0} s_{k,n} \hbar^k \in \mathbb{C}[[\hbar]]$ is a formal symbol. Let us assume that $s_{1,0} \neq 0$ and denote $L(\hbar, \lambda) = \lambda/S(\hbar, \lambda)$. The formal series $S(\hbar, \lambda)$ then admits a formal inverse denoted $S^{-1}(\hbar, t) = \sum_{n \geq 1} \widehat{\alpha}_n(\hbar) t^n$, it is given by*

$$(4.46) \quad \widehat{\alpha}_j = \frac{1}{j} [L^j]_{j-1} \quad \text{where} \quad \left[\sum_{k \geq 0} \widehat{\beta}_k(\hbar) X^k \right]_j := \widehat{\beta}_j(\hbar).$$

We wish to apply the previous lemma with $S(\hbar, \lambda) = F(\hbar, \lambda) - F(\hbar, 0)$. We notice that by Lemmas A.10 and A.9, $L(\hbar, \lambda) = \lambda/S(\hbar, \lambda)$ is a formal classical analytic symbol. It can be seen as the datum of a formal series $\sum_{(k,n) \in \mathbb{N}^2} \ell_{n,k} \lambda^k \hbar^n$ with $|\ell_{n,k}| \leq C^{k+n} n!$ for some $C > 0$. Moreover $L^j(\hbar, \lambda)$ is a formal classical analytic symbol and by Lemma A.11 we infer that there exists another $C > 0$ such that uniformly in $j \in \mathbb{N}$, $[L^j]_{n,k} \leq 3^{j-1} C^{k+n} n!$, where $[\cdot]_{n,k}$ denotes naturally the coefficient in front of $\lambda^k \hbar^n$. From Formula 4.46 we obtain a formal classical analytic symbol $S^{-1}(\hbar, I)$ which is the formal inverse of $F(\hbar, \lambda) - F(\hbar, 0)$. In order to conclude, let us recall again that by Proposition A.8 the composition of formal classical analytic symbols is still a formal classical analytic symbol (in the sense of Definition A.4). We deduce, using this result, that $G(\hbar, \lambda) := S^{-1}(\hbar, I - F(\hbar, \lambda))$ is the formal inverse of $F(\hbar, \lambda)$ and that it is a formal classical analytic symbol.

2) *Conclusion*

Let us recall again that by Proposition A.8 the composition of classical analytic symbols is still a classical analytic symbol. Let us then define g as the resummation (in the sense of Proposition A.5) of the formal symbol constructed above. Therefore $f(\hbar, g(\hbar, I))$, respectively $g(\hbar, f(\hbar, \lambda))$ are classical analytic symbols whose formal symbols are I , respectively λ . Eventually Proposition A.5 implies that $f(\hbar, g(\hbar, I))$, respectively $g(\hbar, f(\hbar, \lambda))$ are exponentially close to the identity mapping. \square

In the end, for λ near λ_0 , with $f(\hbar, \lambda) = \mathcal{A}(\lambda) + \hbar\gamma(\hbar, \lambda)$ we have

$$\begin{aligned} \mu(\hbar, \lambda)e^{i\mathcal{A}(\lambda)/\hbar} = 1 &\iff e^{\frac{i}{\hbar}(\hbar\gamma(\hbar, \lambda) + \mathcal{A}(\lambda))} = 1 \iff e^{\frac{i}{\hbar}\mathcal{F}(\hbar, \lambda)} = 1, \\ &\iff \mathcal{F}(\hbar, \lambda) \in 2\pi\hbar\mathbb{Z} \cap \text{Neigh}_{\mathbb{R}}(\mathcal{A}(\lambda_0)). \end{aligned}$$

As $\partial_\lambda \mathcal{A}(\lambda_0) = T(\lambda_0) \neq 0$ therefore applying the local inverse function lemma for analytic symbols yields a classical analytic symbol

$$(4.47) \quad \lambda(\hbar, I) = \mathcal{A}^{-1}(I - \pi\hbar) + \mathcal{O}(\hbar^2),$$

solving $\lambda(\hbar, f(\hbar, \lambda)) = \lambda + \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}})$ for I close to $\mathcal{A}(\lambda_0)$. This yields, denoting with a slight abuse $\lambda(\hbar, 2\pi\hbar\mathbb{Z}) := \{\lambda(\hbar, 2\pi k\hbar) \mid k \in \mathbb{Z}, 2\pi\hbar k \in \text{Neigh}_{\mathbb{R}}(\mathcal{A}(\lambda_0))\}$, for λ near λ_0

$$(4.48) \quad \begin{aligned} \mu(\hbar, \lambda)e^{\frac{i}{\hbar}\mathcal{A}(\lambda)/\hbar} = 1 &\iff \mathcal{F}(\hbar, \lambda) \in 2\pi\hbar\mathbb{Z} \cap \text{Neigh}_{\mathbb{R}}(\mathcal{A}(\lambda_0)) \\ &\iff \lambda + \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}) \in \lambda(\hbar, 2\pi\mathbb{Z}\hbar). \end{aligned}$$

This proves thanks to Lemma 4.10 that, there exist another $\mathcal{C} > 0$ and a function $\mathcal{O}(e^{-\mathcal{C}/\hbar})$ such that for all $\lambda \in \text{Neigh}_{\mathbb{R}}(\lambda_0) \cap \lambda(\hbar, 2\pi\mathbb{Z}\hbar)$ there exists an eigenvalue λ_{\hbar} of P_{Φ_0} satisfying $|\lambda - \lambda_{\hbar}| \leq \mathcal{O}(e^{-\mathcal{C}/\hbar})$.

In conclusion, Theorem 1.4 is achieved in a small neighbourhood of λ_0 . The patching Lemma A.7 allows to extend this result in a small neighbourhood of (E_1, E_2) .

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APPENDIX A. ANALYTIC SYMBOLS AND ANALYTIC CONSTRUCTIONS

A.1. Standard Definitions. Let $\Omega \subset \mathbb{C}^d$ an open set.

Definition A.1 (Formal Classical Analytic Symbols). A *formal classical analytic symbol* \hat{a}_{\hbar} on Ω is a formal series $\hat{a}_{\hbar} = \sum_{n \geq 0} a_n \hbar^n$ where each $a_n \in \text{Hol}(\Omega)$ satisfies

$$(A.1) \quad \forall K \Subset \Omega, \exists C_K > 0, \forall n \in \mathbb{N}, \sup_K |a_n| \leq C_K^{n+1} n^n.$$

Remark A.2 (Stirling Equivalent). The reader should notice that the estimate (A.1) in the previous definition is, thanks to the Stirling equivalent of $n!$, equivalent to

$$(A.2) \quad \forall K \Subset \Omega, \exists C_K > 0, \forall n \in \mathbb{N}, \sup_K |a_n| \leq C_K^{n+1} n!.$$

The most natural class of symbols that we can define is the class of *classical symbols* which will be supposed to be holomorphic throughout this dissertation.

Definition A.3 (Classical Symbols). The map $a : (0, \hbar_0) \times \Omega \rightarrow \mathbb{C}$ is a classical symbol if for every $\hbar \in (0, \hbar_0)$, $a(\hbar, \cdot) \in \text{Hol}(\Omega)$ and admits a formal classical analytic symbol $\hat{a}_\hbar = \sum_{n \geq 0} a_n \hbar^n$ as asymptotic expansion in the sense that

$$(A.3) \quad \forall K \Subset \Omega, \quad \forall N \in \mathbb{N}, \quad \left| a(\hbar, \cdot) - \sum_{n=0}^{N-1} \hbar^n a_n \right|_{L^\infty(K)} = \mathcal{O}(\hbar^N).$$

Refining the constraint (A.3) yields the definition of classical analytic symbols.

Definition A.4 (Classical Analytic Symbols). The map $a : (0, \hbar_0) \times \Omega \rightarrow \mathbb{C}$ is a classical analytic symbol if for every $\hbar \in (0, \hbar_0)$, $a(\hbar, \cdot) \in \text{Hol}(\Omega)$ and a admits a formal classical analytic symbol $\hat{a}_\hbar = \sum_{n \geq 0} a_n \hbar^n$ as asymptotic expansion in the sense that

$$(A.4) \quad \forall K \Subset \Omega, \quad \exists C_K > 0, \quad \forall N \in \mathbb{N}, \quad \left| a(\hbar, \cdot) - \sum_{n=0}^{N-1} \hbar^n a_n \right|_{L^\infty(K)} \leq \hbar^N C_K^{N+1} N^N.$$

In fact (A.4) is crucial as it guarantees that classical analytic symbols come with a resummation process which is exponentially sharp. This is key to replace $\mathcal{O}(\hbar^\infty)$ remainders with exponentially sharp ones. The idea is to notice that when $n \leq \frac{1}{\hbar}$, $\hbar^n \leq n^{-n}$, cancelling the exponential growth of $|a_n|$.

Sometimes for reasons of convenience of the notation, we denote a_\hbar instead of $a(\cdot, \cdot)$ the classical analytic symbols. We will also say that a_\hbar is associated with the formal analytic symbol $\sum_{k \geq 0} a_k \hbar^k$.

Proposition A.5 (Resummation). *Let a_\hbar a classical analytic symbol on Ω of formal symbol $\sum_{n \geq 0} a_n \hbar^n$, $K \Subset \Omega$ and C_K given by Definition (A.1) then setting*

$$(A.5) \quad \forall K \Subset \Omega, \quad b_{\hbar, K} := \sum_{n=0}^{\lfloor 1/\mathcal{C}\hbar \rfloor} a_n \hbar^n, \quad \text{where } \mathcal{C} \geq C_K e,$$

we have $\sup_K |a_\hbar - b_{\hbar, K}| \leq C_K e^{-1/\mathcal{C}\hbar}$.

Proof. Replacing N with $\lfloor \frac{1}{\mathcal{C}\hbar} \rfloor$ in (A.4), we have on $K \Subset \Omega$

$$|a_\hbar - b_{\hbar, K}| = \left| a_\hbar - \sum_{n=0}^{\lfloor 1/\mathcal{C}\hbar \rfloor} a_n \hbar^n \right| \leq C_K e^{\frac{1}{\mathcal{C}\hbar} \ln(C_K \hbar / \mathcal{C}\hbar)} \leq C_K e^{-\frac{1}{\mathcal{C}\hbar}}.$$

□

Notice that this procedure allows for instance to distinguish a classical analytic symbol $a(\hbar, z)$ from for instance $a(\hbar, z) + e^{-\sqrt{-\ln(\hbar)}}$ which has the same classical formal symbol but doesn't satisfy (A.4). From the proof of the previous proposition, it is natural to notice the following.

Lemma A.6. *A classical symbol on Ω , holomorphic in z , satisfies the estimate of Definition A.4, if and only if it is exponentially close, on each compact of Ω , to a classical analytic symbol with same formal symbol.*

Proof. The direct sense is a straightforward consequence of Proposition A.5. Let us prove the reciprocal *i.e.* that classical symbols, holomorphic in z , of the form $a(\hbar, \cdot) + \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}})$ with $a(\hbar, \cdot)$ a classical analytic symbol and $\mathcal{C} > 0$, satisfy the inequality (A.4). In other words, we want, given $C > 0$, to prove that there exists $C' > 0$ such that

$$\forall \hbar \in (0, \hbar_0), \forall N \in \mathbb{N}, \quad (C\hbar N)^N + \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}) \leq (C'\hbar N)^N.$$

We have some $r > 0$ such that

$$(C\hbar N)^N + \mathcal{O}(e^{-\frac{\mathcal{C}}{\hbar}}) \leq (C'\hbar N)^N \left(\left(\frac{C}{C'} \right)^N + \frac{re^{-\frac{\mathcal{C}}{\hbar}}}{(C'\hbar N)^N} \right).$$

We want the term $\left(\frac{C}{C'} \right)^N + \frac{re^{-\frac{\mathcal{C}}{\hbar}}}{(C'\hbar N)^N}$ to be bounded uniformly in $N \in \mathbb{N}$ and $\hbar \in (0, \hbar_0)$. The non-divergence of $\left(\frac{C}{C'} \right)^N$ implies $C' > C$. Let us then estimate

$$(A.6) \quad \frac{e^{-\frac{\mathcal{C}}{\hbar}}}{(C'\hbar N)^N} = e^{-\frac{1}{\hbar}(\mathcal{C} + N\hbar \ln(C'\hbar N))}.$$

to obtain another condition on C' . The map $\hbar \mapsto N\hbar \ln(C'\hbar N)$ attains its minimum at $\hbar = \frac{1}{C'Ne}$ and its value is $-\frac{1}{C'e}$. Therefore choosing $C' \geq \frac{1}{\mathcal{C}e}$ is sufficient to guarantee the boundedness of (A.6). Eventually choosing $C' = \max(C, \frac{1}{\mathcal{C}e})$ seals the proof. \square

Let us now present a useful patching result which is independent of the other upcoming results. The set \mathcal{F} defined on the following lemma can be viewed for instance as the disjoint union $\bigsqcup_{\hbar \in (0, \hbar_0)} \sigma_{\hbar}(P_{\Phi_0})$.

Lemma A.7 (Patching Classical Analytic Symbols on a Grid). *Let $a_1(\hbar, \lambda), a_2(\hbar, \lambda)$ two classical analytic symbols (of the variable λ) with formal symbol $\sum_{n \geq 0} a_1^n(\lambda) \hbar^n$ and $\sum_{n \geq 0} a_2^n(\lambda) \hbar^n$ and defined on some connected open sets Ω_1, Ω_2 with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Suppose that there exists a set $\mathcal{F} \subset (0, \hbar_0) \times \Omega_1 \cap \Omega_2$ satisfying the following.*

— *There exists an infinite subset $\mathcal{S} \subset \Omega_1 \cap \Omega_2$ such that for every $\lambda \in \mathcal{S}$, there exists a sequence $(\hbar_j, \lambda_j)_{j \in \mathbb{N}}$ in \mathcal{F} converging towards $(0, \lambda)$.*

— *There exists $r, \epsilon_0 > 0$*

$$\forall (\hbar, \lambda) \in \mathcal{F}, \quad |a_1(\hbar, \lambda) - a_2(\hbar, \lambda)| \leq re^{-\epsilon_0/\hbar}.$$

Then for all $K \Subset \Omega_1 \cup \Omega_2$, we find a glued classical analytic symbol $a_K(\hbar, \cdot)$ and $C, \epsilon > 0$ such that for $i = 1, 2$,

$$|a_i(\hbar, \cdot) - a_K(\hbar, \cdot)| \leq Ce^{-\epsilon/\hbar} \text{ on } \Omega_i \cap K.$$

Proof. For all $\lambda \in \mathcal{S}$, we have a sequence $(\hbar_j, \lambda_j)_{j \in \mathbb{N}}$ in \mathcal{F} converging towards $(0, \lambda)$. This gives the recursion formula

$$a_1^n(\lambda_j) - a_2^n(\lambda_j) = \mathcal{O}(\hbar_j) \xrightarrow{\hbar_j \rightarrow 0} 0 \text{ i.e. } a_1^n = a_2^n \text{ on } \mathcal{S}.$$

This proves that for all $n \in \mathbb{N}$ and $\lambda \in \mathcal{S}$, $a_1^n(\lambda) = a_2^n(\lambda)$ so that by holomorphy they coincide on $\Omega_1 \cap \Omega_2$. This provides a formal analytic symbol on $\Omega_1 \cup \Omega_2$ coinciding with the formal analytic symbols of $a_1(\hbar, \cdot)$ and $a_2(\hbar, \cdot)$ on their domain. Applying the resummation Lemma A.5 seals the proof. \square

We now present some standard and useful results about classical analytic symbols.

A.2. Composition of classical analytic symbol.

A.2.1. *Statement of the result.* In all that follows, it will be implied that the formal symbols associated with classical analytic symbols $f(\hbar, \lambda)$, $g(\hbar, I)$ are $\sum_{n \geq 0} f_n(\lambda) \hbar^n$ and $\sum_{n \geq 0} g_n(I) \hbar^n$. The goal of this section is to prove that we can compose (in the sense of functions) classical analytic symbols.

Proposition A.8. *Let $f(\hbar, \lambda)$ (defined near λ_0) and $g(\hbar, I)$ (defined near $f_0(\lambda_0)$) be two classical analytic symbols, then $g(\hbar, f(\hbar, \lambda))$ is a classical analytic symbol near λ_0 .*

A.2.2. *Proof of the result.* To prove this result we must go through some useful tools and results. The first useful tools appear when trying to prove the following lemma.

Lemma A.9. *Let $a(\hbar, \lambda)$, $b(\hbar, \lambda)$ be two classical analytic symbols defined near λ_0 , then $a(\hbar, \lambda)b(\hbar, \lambda)$ is a classical analytic symbol.*

Proof. In what follows, we will consider a compact domain $\bar{\Omega}$ where both symbols are well defined and denote $|a_k| = |a_k|_{L^\infty}$ and similarly for b_k . To prove the result, we check that (A.1) and (A.4) are satisfied.

1) Estimate on the formal symbol

The idea is to use powerful results on converging power series. Let us notice that for $\rho > 0$ small enough, given a formal classical analytic symbol $D(\hbar, \lambda) = \sum_{k \geq 0} d_k(\lambda) \hbar^k$ the series given by the Borel transform $\sum_{n \in \mathbb{N}} \frac{|d_k|}{k!} \rho^k$ converges for ρ small enough. This is interpreted as a *pseudonorm* on the classical analytic symbol in the sense that a formal classical symbol is analytic if and only if there exists $\rho > 0$ such that this sum is converging. It is often denoted $\|D\|_\rho$ in the litterature. A great advantage is that this pseudonorm is multiplicative and additive,

$$\forall \rho > 0, \quad \|AB\|_\rho \leq \|A\|_\rho \|B\|_\rho \quad \|A + B\|_\rho \leq \|A\|_\rho + \|B\|_\rho.$$

This proves that the formal symbol AB of $a(\hbar, \lambda)b(\hbar, \lambda)$ is a classical analytic formal symbol.

2) Comparison between the symbol and the formal symbol

This is slightly more subtle. Let us write $d(\hbar, \lambda) = \sum_{m \in \mathbb{N}} d_m \hbar^m$ with $d_m = \sum_{k=0}^m a_k b_{m-k}$ the formal symbol of $a(\hbar, \lambda)b(\hbar, \lambda)$. We notice denoting $S_n(a) = \sum_{k=0}^n \hbar^k a_k$, $S_n(b) = \sum_{k=0}^n \hbar^k b_k$

$$\sum_{m=0}^n d_m \hbar^m = \sum_{k=0}^n \hbar^k a_k S_{n-k}(b).$$

Let us also denote⁵ $\delta_k(a) = a - S_k(a)$ and similarly for $\delta_k(b)$. We obtain

$$\sum_{m=0}^n d_m \hbar^m = \sum_{k=0}^n \hbar^k a_k (b - \delta_{n-k}(b)) = (a - \delta_n(a))b - \sum_{k=0}^n \hbar^k a_k \delta_{n-k}(b).$$

This proves the formula

$$(A.7) \quad \delta_n(ab) = \delta_n(a)b + \sum_{k=0}^n \hbar^k a_k \delta_{n-k}(b).$$

⁵Notice that $\delta_n(a), \delta_n(b)$ depend on \hbar .

From (A.4) we know that

$$\ell_n(a) := \sup_{\hbar \in (0, \hbar_0)} \frac{\delta_n(a)}{\hbar^n} < +\infty$$

which allows to define

$$\alpha_0(a) = \max\left(|a_0|, \sup_{\hbar \in (0, \hbar_0)} |a|\right) \text{ and } \alpha_n(a) = \max(|a_n|, \ell_{n-1}(a))$$

for $n \geq 1$, and similarly for b . This allows to define a pseudonorm ⁶ on classical analytic symbols using once again the Borel transform

$$\|d(\hbar, \cdot)\|_\rho = \sum_{k=0}^n \frac{\alpha_k(d)}{k!} \rho^k.$$

The finiteness of this new norm is therefore equivalent to the fact that the symbol satisfies simultaneously (A.1) (which is already known) and (A.4). Thanks to (A.7) this new pseudonorm is also multiplicative *i.e.* letting $a(\hbar, \lambda), b(\hbar, \lambda)$ two classical analytic symbols

$$\|a(\hbar, \cdot)b(\hbar, \cdot)\|_\rho \leq \|a(\hbar, \cdot)\|_\rho \|b(\hbar, \cdot)\|_\rho.$$

□

The next step is then to prove that the composition of classical analytic symbols with an holomorphic function is still a classical analytic symbol.

Lemma A.10. *Let $f(\hbar, \lambda)$ a classical analytic symbol near λ_0 and $g \in \text{Hol}(\text{Neigh}_{\mathbb{C}}(f_0(\lambda_0)))$ then the classical symbol $g(f(\hbar, \lambda))$ is a classical analytic symbol.*

Proof. We will use the norm $\|\cdot\|_\rho$ defined in the proof of the previous lemma. Let us notice, by elementary results of summable families that if $a(\hbar, \lambda) = \sum_{k \geq 0} a_k(\hbar, \lambda)$ is an infinite sum of classical symbols (whose formal symbol converges formally)

$$\|a(\hbar, \cdot)\|_\rho \leq \sum_{k=0}^{+\infty} \|a_k(\hbar, \cdot)\|_\rho,$$

where by convention $\|a(\hbar, \lambda)\|_\rho = +\infty$ if the series does not converge. We apply this to the estimate, letting $\rho > 0$ small enough and supposing $f_0(\lambda_0) = 0$ (up to replacing g with $g(\cdot - f_0(\lambda_0))$),

$$\|g(f(\hbar, \lambda))\|_\rho \leq \sum_{n=0}^{+\infty} \frac{|g^{(n)}(0)|}{n!} \|f(\hbar, \lambda)\|_\rho^n.$$

Let us now precise some points. We can suppose that g is defined on a disc $\mathbb{D}(0, r)$ of radius $r > 0$ small enough. Similarly, we can choose the domain of $f(\hbar, \lambda)$ to be another disc $\mathbb{D}(\lambda_0, \tilde{r})$ so that $|f_0|_{\mathbb{D}(\lambda_0, \tilde{r})} \leq r$.

Cauchy's estimates yields

$$\frac{|g^{(n)}(0)|}{n!} \leq r^{-n} |g|_{L^\infty(\mathbb{D}(r))}.$$

⁶Which is distinct from the pseudonorm on formal classical analytic symbols.

We infer

$$(A.8) \quad \|g(f(\hbar, \lambda))\|_\rho \leq |g|_{L^\infty(\mathbb{D}(\lambda_0, r))} \sum_{n=0}^{+\infty} \left(\frac{\|f(\hbar, \lambda)\|_\rho}{r} \right)^n \leq \frac{|g|_{L^\infty(\mathbb{D}(\lambda_0, r))}}{1 - (\|f(\hbar, \lambda)\|_\rho/r)}.$$

From the fact that $\|f(\hbar, \lambda)\|_\rho \xrightarrow{\rho \rightarrow 0^+} |f_0|_{L^\infty(\mathbb{D}(\lambda_0, \tilde{r}))}$, we infer that for $r > 0$ small enough $\|g(f(\hbar, \lambda))\|_\rho$ is finite proving that $g(f(\hbar, \lambda))$ is a classical analytic symbol. \square

To conclude the proof of Proposition A.8, we write

$$\|g(\hbar, f(\hbar, \lambda))\|_\rho \leq \sum_{n=0}^{+\infty} \|g_k(f(\hbar, \lambda))\|_\rho \frac{\rho^k}{k!} \leq \frac{1}{1 - \|f(\hbar, \lambda)\|_\rho/r} \sum_{k=0}^{+\infty} \frac{\|g_k\|_{L^\infty(\mathbb{D}(\lambda_0, r))}}{k!}.$$

We infer that choosing a small enough neighbourhood of λ , $\|g(\hbar, f(\hbar, \lambda))\|_\rho < +\infty$ for $\rho > 0$ small enough which seals the proof.

A.2.3. Other useful estimates. Let $a(\hbar, \lambda)$ be a classical analytic symbol. In this section we provide some estimates on the formal symbol of $a(\hbar, \lambda)^j$. The proof of Proposition A.8 gives that for all $j \in \mathbb{N}$, the power series $\sum_{k \geq 0} \frac{\rho^k}{k!} |a_k^j|_\infty$ have the same radius of convergence R_a than $\sum_{k \geq 0} \frac{\rho^k}{k!} |a_k|_\infty$. This allows to infer easily that $a_k^j = o_{k \rightarrow +\infty}(k!(R_a + \epsilon)^k)$ for all $\epsilon > 0$. In the following lemma, we provide a slightly more explicit estimate which will be useful for instance in Lemma 4.17.

Lemma A.11. *Let $A(\hbar, \lambda)$ a formal classical analytic symbol, then denoting $A^j(\hbar, \lambda) = \sum_{n \geq 0} a_n^j(\lambda) \hbar^n$,*

$$(A.9) \quad \exists C > 0, \forall (j, n) \in \mathbb{N}^* \times \mathbb{N}, |a_n^j| \leq 3^{j-1} C^n n!.$$

Proof. Let $a(\hbar, \lambda), b(\hbar, \lambda)$ be classical analytic symbols such that their formal symbols satisfy the estimate

$$\exists C, \alpha, \beta > 0, |a_k| \leq \alpha C^k k! \text{ and } |b_k| \leq \beta C^k k!.$$

If we denote $\sum_{m \geq 0} d_m \hbar^m$ the formal symbol of $a(\hbar, \lambda)b(\hbar, \lambda)$ let us prove that we have

$$(A.10) \quad |d_m| \leq 3\alpha\beta C^m m!.$$

In order not to overload the notations, the dependency in λ will be implied. For all $m \in \mathbb{N}$, from the formula of the product of power series, we infer

$$|d_m| \leq \sum_{k=0}^m \alpha C^k k! \beta C^{m-k} (m-k)! = \alpha\beta C^m m! \sum_{k=0}^m \binom{m}{k}^{-1}.$$

Let us define $S_m = \sum_{k=0}^m \frac{1}{\binom{m}{k}}$. Letting $k \in \llbracket 2, m-2 \rrbracket$ we have $\binom{m}{k} \geq \binom{m}{2}$ thus $|S_m - 2| \leq \frac{2}{m}$ (for $m \geq 4$) and computing $S_0 = 1, S_1 = 2, S_2 = 5/2, S_3 = 8/3$ we deduce that $S_m \leq 3$ for all $m \in \mathbb{N}$ and $|d_m| \leq 3\alpha\beta C^m m!$.

Therefore by a straightforward recursion the n -th term of $a(\hbar, \lambda)^j$ is bounded by $3^{j-1} C^n n!$. \square

A.3. Pseudodifferential operators with classical analytic symbols. Analytic symbols are useful for computing WKB expansions, which are functions of the form $a_h e^{i\varphi/h}$ where φ solves some eikonal equation and a_h is a classical analytic symbol. In [Sjö82], Sjöstrand has developed a functional framework of useful spaces of weighted holomorphic functions for finding and manipulating WKB expansions with exponential sharpness.

A.3.1. *Useful function spaces.*

Definition A.12 (Space $H_{\Phi}^{\text{loc}}(\Omega)$). Let Ω an open subset of \mathbb{C}^d . Let $\Phi \in \mathcal{C}^{\infty}(\Omega, \mathbb{R})$ and $u = (u_h)_{h < h_0}$ a family of functions indexed by h , then we say that $u \in H_{\Phi}^{\text{loc}}(\Omega)$ if

- (1) $\forall h \in (0, h_0), u_h \in \text{Hol}(\Omega)$;
- (2) $\forall K \Subset \Omega, \forall \epsilon > 0, \exists C_K > 0$, such that $|u_h| \leq C_K e^{(\Phi + \epsilon)/h}$ on K for all $h \in (0, h_0)$.

This allows to define the space of germs H_{Φ, x_0} at a given point $x_0 \in \mathbb{C}^d$ by the inductive limit

$$(A.11) \quad H_{\Phi, x_0} := \varinjlim_{\Omega \ni x_0} H_{\Phi}^{\text{loc}}(\Omega),$$

where Ω varies in the set of open neighbourhoods of x_0 . We can quotient these spaces by considering exponentially small germs as negligible.

Definition A.13. An element $u_h \in H_{\Phi, x_0}$ is called *negligible* if it belongs to the space

$$(A.12) \quad \mathcal{N} := \{u_h \in H_{\Phi, x_0} \mid \exists c > 0, \exists \Omega \text{ open neighbourhood of } x_0, u_h \in H_{\Phi - c, x_0}^{\text{loc}}(\Omega)\},$$

allowing to define a quotient space $H_{\Phi, x_0}/\mathcal{N}$.

We will still denote by H_{Φ, x_0} the quotient space not to overload the notation (and when the point of reference is clear) we simply write H_{Φ} and we prefer to denote by \sim the equality in H_{Φ} spaces. If $u_h \in H_{0, x_0}$, we say that u_h is an analytic symbol and by a simple estimate one can see that classical analytic symbols are indeed analytic symbols.

A.3.2. Defining complex Pseudodifferential operators. We then define complex pseudo-differential operators on the spaces of germs H_{Φ} and recall results which can be found in [RSV20, Chapter 3]. We will rather choose to write the operators in the Weyl's quantization, however the interested reader could consult [RSV20, Section 2] providing a good description of change of quantization in this ‘‘analytic’’ context.

Let $\Phi \in \mathcal{C}^{\infty}(\text{Neigh}_{\mathbb{C}^d}(x_0))$ subharmonic and $q_h(z, \zeta)$ a classical analytic symbol defined near $(x_0, \xi_0) := (x_0, \frac{2}{i} \partial_x \Phi(x_0))$. Using the good contour

$$\mathcal{G}_{C, \delta}(z) := \left\{ (w, \zeta) \mid |x - y| < \delta, \zeta = \frac{2}{i} \partial_x \Phi\left(\frac{x+y}{2}\right) + iC\overline{x - y} \right\}$$

one can define the Weyl's quantization in the H_{Φ, z_0} space formally by

$$(A.13) \quad \forall u_h \in H_{\Phi, x_0}, \quad \text{Op}_h^w(q_h)u_h = \frac{1}{2\pi\hbar} \iint_{\mathcal{G}_{C, \delta}(x)} e^{i\zeta(x-y)/\hbar} q_h\left(\frac{x+y}{2}, \xi\right) u_h(y) d\xi \wedge dy.$$

Similarly we can define the left quantization by

$$(A.14) \quad \forall u_h \in H_{\Phi, x_0}, \quad \text{Op}_h^{\ell}(q_h)u_h = \frac{1}{2\pi\hbar} \iint_{\mathcal{G}_{C, \delta}(x)} e^{i\xi(x-y)/\hbar} q_h(x, \xi) u_h(y) d\xi \wedge dy.$$

with similar contours. Indeed, let us consider $\text{Neigh}(x_0, \mathbb{C}^d)$ small neighbourhoods of x_0 , and $u_h \in H_\Phi(\text{Neigh}(x_0, \mathbb{C}))$. Then up to choosing δ small enough, for different δ , the integrals

$$(A.15) \quad \frac{1}{2\pi\hbar} \iint_{\mathcal{G}_{\mathbb{C},\delta}(x)} e^{i\xi(x-y)/\hbar} q_h(x, \xi) u_h(y) d\xi \wedge dy$$

define the same function near x_0 modulo an exponentially small remainder. This proves that formulas (A.13) and (A.14) define operators on $H_{\Phi, x_0} \rightarrow H_{\Phi, x_0}$. Notice also that it is possible to integrate on “fewer variables” and the same result holds (replace $(x, y, \xi) \in \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d$ in A.15 with $(x, y', \xi') \in \mathbb{C}^d \times \mathbb{C}^m \times \mathbb{C}^m$ for some $m \leq d$). The following remark, in the case when the gauge Φ is harmonic, is a well known consequence of [Sjö82, Chapters 1, 2] and will be useful.

Lemma A.14. *Let $\Phi = -\text{Im}(\varphi)$ be harmonic and $Q : H_{\Phi, x_0} \rightarrow H_{\Phi, x_0}$ be a pseudodifferential operator, then $Q_\varphi := e^{-\frac{i}{\hbar}\varphi} Q e^{\frac{i}{\hbar}\varphi}$ defines a complex pseudodifferential operator on H_{0, x_0} .*

Proof. Let us simply give the idea of the proof. This stems from the Kuranishi trick and equivalence of good contours. \square

Remark A.15. In the case where $\Phi = 0$, the operators $\text{Op}_\hbar^w(q_h), \text{Op}_\hbar^\ell(q_h) : H_0 \rightarrow H_0$ are well defined and sends classical analytic symbols toward classical analytic symbols. The action of $\text{Op}_\hbar^w(q_h), \text{Op}_\hbar^\ell(q_h)$ at the level of formal symbols is in fact given by that of a *formal pseudodifferential operator* which is an infinite sum of monomial differential operators. In the case of the left quantization the formal pseudodifferential operator is

$$q_h(z, \hbar D_z) = \sum_{k \geq 0} \frac{\partial_\zeta^{(k)} q_h(z, 0)}{k!} (\hbar D_z)^k,$$

while in the case of Weyl’s quantization it is $q_{\hbar, \frac{1}{2}}(z, \hbar D_z)$ with

$$q_{\hbar, \frac{1}{2}}(z, \zeta) := \exp\left(-\frac{i}{2\hbar}(\hbar D_\zeta \cdot (\hbar D_z - \hbar D_w))\right) q_h|_{w=z}.$$

Proof. Let us provide some elements of proof, we start with the following useful remark.

1) *The operators send classical analytic symbols towards classical analytic symbols.*

We let the reader convince himself that thanks to the holomorphic Morse lemma and by separation of variables in the integral, it is sufficient to prove the following. For all a classical analytic symbol $a(\hbar, z, w)$ defined near some (z_0, z_0) , the function

$$b(\hbar, w) = \frac{1}{\sqrt{2\pi\hbar}} \int_{\text{Neigh}_{\mathbb{R}}(0)} e^{-\frac{z^2}{2\hbar}} a(\hbar, z, w) dz,$$

is a classical analytic symbol.

— We check that the formal symbol of $b(\hbar, z)$ is a formal classical analytic symbol. Let us denote the formal symbols with majuscules. We obtain from [Sjö82, Theorem 2.1]

$$B(\hbar, w) = \sum_{k \geq 0} \hbar^k \sum_{j \geq 0} \hbar^j \frac{\Delta_z^k}{2^k k!} a_j(z, w)|_{z=0} = \sum_{m \geq 0} \hbar^m \left(\sum_{\nu=0}^m \frac{\Delta_z^\nu}{2^\nu \nu!} a_{m-\nu}(z, w)|_{z=0} \right).$$

In view of Lemma A.9, we wish to estimate $\|B(\hbar, w)\|_\rho$ and we do as follows. Formally, B is obtained by applying the formal sum of differential operators $\mathcal{I}(z, \hbar D_z) := \sum_{k \geq 0} \hbar^k \frac{\Delta_z^k}{2^k k!}$, to

a formal classical analytic symbol. This suggests to define a converging series of differential operators by the Borel transform

$$(A.16) \quad [\mathcal{I}(z, \hbar D_z)]_\rho = \sum_{k=0}^{+\infty} \frac{\Delta_z^k \rho^k}{2^k k! k!}.$$

Letting $\mathbb{D}(s) \Subset \mathbb{D}(t) \subset \mathbb{D}(t_0) \subset \mathbb{C}$ be small and non empty discs on which $a(\hbar, z, w)$ is well defined for w close enough to 0. We then infer that the operator $\frac{\Delta_z^\nu}{2^\nu \nu!}$ is a bounded operator $\text{Hol}(\mathbb{D}(t)) \rightarrow \text{Hol}(\mathbb{D}(s))$ with norm $\|\frac{\Delta_z^\nu}{2^\nu \nu!}\|_{s,t}$ lower than $C^\nu \nu! (t-s)^{-2\nu}$ for some $C > 0$ (independent of ν). This allows to define

$$\mathcal{N}_k = \sup_{0 < s < t \leq t_0} \left\| \frac{\Delta_z^\nu}{2^\nu \nu!} \right\|_{s,t} (t-s)^{2\nu}, \quad \|\mathcal{I}\|_\rho = \sum_{n=0}^{+\infty} \frac{\mathcal{N}_k}{k!} \rho^k$$

and notice that $\|\mathcal{I}\|_\rho$ converges for ρ small enough. We then infer that formally

$$\|\mathcal{I}(z, \hbar D_z) A(\hbar, \cdot, \cdot)\|_\rho \leq \|\mathcal{I}\|_\rho \|A(\hbar, \cdot, \cdot)\|_\rho.$$

This yields $\|B(\hbar, w)\|_\rho \leq \|\mathcal{I}\|_\rho \|A(\hbar, \cdot, \cdot)\|_\rho < +\infty$ if $\rho > 0$ is small enough. We conclude that $B(\hbar, w)$ is a formal classical analytic symbol.

— It remains to check that the new symbol satisfies the estimate (A.4). Let $N \in \mathbb{N}$ and write $a(\hbar, z, w) = a_{N-1}(\hbar, z, w) + \delta_N(\hbar, z, w)$, with a_{N-1} the partial sum to the order \hbar^{N-1} . We obtain

$$\int_{\text{Neigh}_{\mathbb{C}}(z_0)} e^{-\frac{z^2}{2\hbar}} a(\hbar, z, w) dz = \int_{\text{Neigh}_{\mathbb{C}}(z_0)} e^{-\frac{z^2}{2\hbar}} a_{N-1}(\hbar, z, w) dz + \int_{\text{Neigh}_{\mathbb{C}}(z_0)} e^{-\frac{z^2}{2\hbar}} \delta_N(\hbar, z, w) dz$$

and the second term that we denote $\tilde{\delta}_N$ satisfies $|\tilde{\delta}_N| \leq C_0 C^N N! \hbar^N$. Let us also denote $\mathcal{I}_m = \sum_{k=0}^m \hbar^k \frac{\Delta_z^k}{2^k k!}$ and recall that by the analytic stationary phase [Sjö82, Theorem 2.1]

$$\forall n \in \mathbb{N}, \quad \int_{\text{Neigh}_{\mathbb{C}}(z_0)} e^{-\frac{z^2}{2\hbar}} a_n(z, w) dz = \mathcal{I}_m a_n(z, w)|_{z=w} + R_{n,m}(\hbar, w)$$

where $|R_{m,n}(\hbar, w)| \leq (m+1)! C_1^{m+1} \hbar^{m+1} |a_n|_\infty$ and $C_1 > 0$ doesn't depend on n . We obtain

$$(A.17) \quad \begin{aligned} \int_{\text{Neigh}_{\mathbb{C}}(z_0)} e^{-\frac{z^2}{2\hbar}} a(\hbar, z, w) dz &= \sum_{n=0}^{N-1} \hbar^k (\mathcal{I}_{N-1-k} a_k|_{z=w} + R_{N-1-k,k}(\hbar, w)) + \tilde{\delta}_N \\ &= \sum_{n=0}^{N-1} b_k(w) \hbar^k + \sum_{n=0}^{N-1} \hbar^k R_{N-1-k,k}(\hbar, w) + \tilde{\delta}_N. \end{aligned}$$

It is then easy to see that A.4 is satisfied.

2) *The action of $\text{Op}_\hbar^w(q_\hbar), \text{Op}_\hbar^\ell(q_\hbar)$ on formal symbols is given by that of the formal pseudodifferential operators.*

The formal formula for $\text{Op}_\hbar^\ell(q_\hbar)$ is a direct consequence of the analytic stationary phase. The formal formula for $\text{Op}_\hbar^w(q_\hbar)$ can be also inferred from the analytic stationary phase but it is more easily obtained thanks to the formula for changing between quantization that is for instance detailed in [RSV20, Chapter 3]. \square

Let us then finish this presentation with recalling that the pseudonorms $\|\cdot\|_\rho$ defined on infinite series of differential operators is the main tool that allowed in [HS15, Theorem 2.8.1] to prove the existence of the analytic WKB expansions we use in Section 4.2. These

pseudonorms are also crucial to prove the existence of parametrix of elliptic pseudodifferential operators with classical analytic symbols (see [Sjö82, Theorem 1.5] or [HS15, Theorem 2.2.3]).

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