Conditional appearance of decay for the non-cutoff Boltzmann equation in a bounded domain

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CONDITIONAL APPEARANCE OF DECAY FOR THE NON-CUTOFF BOLTZMANN EQUATION IN A BOUNDED DOMAIN

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ABSTRACT. This work is concerned with the appearance of decay bounds in the velocity variable for solutions of the space-inhomogeneous Boltzmann equation without cutoff posed in a domain in the case of hard and moderately soft potentials. Such bounds are derived for general non-negative suitable weak subsolutions. These estimates hold true as long as mass, energy and entropy density functions are under control. The following boundary conditions are treated: in-flow, bounce-back, specular reflection, diffuse reflection and Maxwell reflection. The proof relies on a family of Truncated Convex Inequalities that is inspired by the one recently introduced by F. Golse, L. Silvestre and the first author (2023). To the best of our knowledge, the generation of arbitrary polynomial decay in the velocity variable for the Boltzmann equation without cutoff is new in the case of soft potentials, even for classical solutions.

CONTENTS

1. INTRODUCTION

1.1. **The Boltzmann equation.** We consider the Boltzmann equation posed in bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^d$ (with $d \geq 2$),

$$(\partial_t + v \cdot \nabla_x) f = Q(f,f), \qquad t \in (0,T), x \in \Omega, v \in \mathbb{R}^d \tag{1.1}$$

where the unknown function $f = f(t,x,v)$ is non-negative; $Q(f,f)$ denotes the Boltzmann collision operator,

$$Q(f,f)(v) = \iint_{\mathbb{S}^{d-1}\times\mathbb{R}^d} \big(f(v'_*)f(v') - f(v_*)f(v)\big) B(|v-v_*|,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_* \tag{1.2}$$

where v' and v'_* are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

and the collision kernel B satisfies

$$B\big(|v-v_*|,\sigma\big) = |v-v_*|^\gamma\, b(\cos\theta) \quad \text{ with } \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma,$$

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for $\gamma \in (-d, 1]$ and

$$b(\cos\theta) \approx |\theta|^{-d+1-2s}$$

for $s \in (0,1)$. This means that there exists constants $C_\pm$ such that $0 < C_- |\theta|^{-d+1-2s} \leq b(\cos\theta) \leq C_+ |\theta|^{-d+1-2s}$. More precisely, we can reformulate this as

$$b(\cos\theta) = |v - v'|^{-(d-1)-2s} |v - v_*|^{d-2-\gamma} |v - v'_*|^{\gamma+2s+1} \tilde{b}(\cos\theta),$$

where $\tilde{b}(\cos\theta)$ is such that $0 < \tilde{b}_- < \tilde{b}(\cos\theta) < \tilde{b}_+$ for constants $\tilde{b}_\pm > 0$. We are concerned with the case when $\gamma + 2s > 0$ corresponding to so-called moderately soft potentials.

The equation is supplemented with an initial condition $f(0,x,v) = f_{\text{in}}(x,v)$ for $(x,v) \in \Omega \times \mathbb{R}^d$ and a boundary condition. In order to describe the latter, we let n denote the outward unit normal vector on the boundary and Γ denote the domain that considers boundary points in space, that is $\Gamma := \partial\Omega \times \mathbb{R}^d$, by Γ_- the incoming part of the boundary, and by Γ_+ the outcoming part of the boundary, that is $\Gamma_- := \{(x,v) \in \Gamma : v \cdot n(x) < 0\}$, and $\Gamma_+ := \{(x,v) \in \Gamma : v \cdot n(x) > 0\}$.

The Boltzmann equation (1.1) is supplemented with the following boundary conditions that are commonly considered in the literature,

(i). In-flow: $f(t,x,v)|_{\Gamma_-} = f_b(t,x,v)$ for a given function f_b.
(ii). Bounce-back: $f(t,x,v) = f(t,x,-v)$ on Γ.
(iii). Specular reflection: $f(t,x,v) = f(t,x,\mathcal{R}_x v)$ with $\mathcal{R}_x v = v - 2(v \cdot n)n$.
(iv). Diffuse reflection: $f(t,x,v)|_{\Gamma_-} = c_\mu \mu(v) \int_{\Gamma_+} f(t,x,v')(v' \cdot n)\,\mathrm{d}v'$, where $\mu(v) = e^{-|v|^2}$ is the Maxwellian, and c_μ is a normalisation constant, such that $c_\mu \int_{\Gamma_+} \mu(v') v' \cdot n\,\mathrm{d}v' = 1$.
(v). Maxwell reflection: $f(t,x,v)|_{\Gamma_-} = (1-\iota) f(t,x,\mathcal{R}_x v) + \iota c_\mu \mu(v) \int_{\Gamma_+} f(t,x,v')(v' \cdot n)\,\mathrm{d}v'$ where $\iota : \partial\Omega \to [0,1]$ is the accomodation coefficient.

1.2. **Hydrodynamical quantities.** L. Silvestre [39] showed that when some hydrodynamical quantities are under control, the collision operator enjoys elliptic properties. It was further investigated by L. Silvestre and the second author in [29]. Throughout this work, we assume that there exist positive constants m_0, M_0, E_0, H_0 such that for almost every $(t,x) \in (0,T) \times \Omega$, the function $f(v) = f(t,x,v)$ satisfies,

$$m_0 \leq \int_{\mathbb{R}^d} f(v)\,\mathrm{d}v \leq M_0, \quad \int_{\mathbb{R}^d} f(v)|v|^2\,\mathrm{d}v \leq E_0, \quad \int_{\mathbb{R}^d} f(v) \ln f(v)\,\mathrm{d}v \leq H_0. \tag{1.3}$$

In particular, the entropy production estimate yields some integrability in (t,x,v) with a negative weight in velocity [11, Theorem 0.1].

1.3. **The collision operator.** In order to present the Truncated Convex Inequalities satisfied by our weak solutions, it is necessary to recall some facts about the collision operator and its kernel representation.

Using Carleman coordinates, the collision operator can be written as follows [41] – see also [39],

$$Q(f,f)(v) = \int_{\mathbb{R}^d} \left[f(v') K_f(v,v') - f(v) K_f(v',v) \right] \mathrm{d}v'$$

where the kernel $K_f(v,v')$ is given by

$$\begin{aligned} K_f(v,v') &= 2^{d-1} |v' - v|^{-1} \int_{w \perp v'-v} f(v+w) B(r, \cos\theta) r^{-d+2}\,\mathrm{d}w \\ &= 2^{d-1} |v' - v|^{-(d+2s)} \int_{w \perp v'-v} f(v+w) |w|^{\gamma+2s+1} \tilde{b}(\cos\theta)\,\mathrm{d}w, \end{aligned}$$

for $r^2 = |v'-v|^2 + |w|^2$ and $\cos\theta = \frac{|w|^2 - |v-v'|^2}{|w|^2 + |v-v'|^2}$. The function $\tilde{b}$ is bounded from above and below by positive constants.

1.4. **Truncated Convex Inequalities.** We derive decay estimates by studying the evolution along time of some L^{q_0}-Lebesgue norm of the function $(f-A)_+$. It is convenient to consider a general convex function $\varphi_0(r)$ instead of r^{q_0} and more generally, a general function $\varphi = \varphi(t,x,v,r)$ that is convex in r instead of $\varphi_0((f-A)_+)$. With such a general function φ, a formal computation yields,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(t,v,f)\,\mathrm{d}v\,\mathrm{d}x &= \iint_{\mathbb{R}^d\times\Omega}\partial_r\varphi(t,x,v,f)Q(f,f)\,\mathrm{d}v\,\mathrm{d}x + \iint_{\mathbb{R}^d\times\Omega}\partial_t\varphi(t,x,v,f)\,\mathrm{d}v\,\mathrm{d}x\\ &= \iint_{\mathbb{R}^d\times\Omega}\partial_r\varphi(t,x,v,f)\left\{\int_{\mathbb{R}^d}\left[f(v')K_f(v,v') - f(v)K_f(v',v)\right]\,\mathrm{d}v'\right\}\,\mathrm{d}v\,\mathrm{d}x\\ &\quad + \iint_{\mathbb{R}^d\times\Omega}\partial_t\varphi(t,x,v,f)\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

We denote in the sequel $f' = f(v')$ and $f = f(v)$. We can then add and subtract

$$\mathcal{D}_\varphi(f,f') := \varphi(t,x,v',f') - \varphi(t,x,v,f) - \partial_r\varphi(t,x,v,f)\,(f'-f)\,,$$

so that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(t,x,v,f)\,\mathrm{d}v\,\mathrm{d}x\\ &\quad= -\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\mathcal{D}_\varphi(f,f')K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &\qquad+\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\left[\partial_r\varphi(t,x,v,f)f - \varphi(t,x,v,f) - \partial_r\varphi(v',f')f' + \varphi(v',f')\right]K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &\qquad+\iint_{\mathbb{R}^d\times\Omega}\partial_t\varphi(t,x,v,f)\,\mathrm{d}v\,\mathrm{d}x - \iint_\Gamma\varphi(f)(v\cdot n)\,\mathrm{d}v\,\mathrm{d}S(x).\end{aligned}$$

We now consider

$$\Phi(f) := \partial_r\varphi(t,v,f)f - \varphi(t,x,v,f). \tag{1.4}$$

and we get,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(t,x,v,f)\,\mathrm{d}v\,\mathrm{d}x\\ &\quad= -\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\mathcal{D}_\varphi(f,f')K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &\qquad+\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\left[\Phi(f) - \Phi(f')\right]K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &\qquad+\iint_{\mathbb{R}^d\times\Omega}\partial_t\varphi(t,x,v,f)\,\mathrm{d}v\,\mathrm{d}x - \iint_\Gamma\varphi(f)(v\cdot n)\,\mathrm{d}v\,\mathrm{d}S(x).\end{aligned}$$

Due to the cancellation lemma, see (2.5), we are thus lead to,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi(t,x,v,f)\,\mathrm{d}v\,\mathrm{d}x &= -\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\mathcal{D}_\varphi(f,f')K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\ &\quad+ c_b\iint_{\mathbb{R}^d\times\Omega}\Phi(f)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x\\ &\quad+\iint_{\mathbb{R}^d\times\Omega}[\partial_t\varphi](t,x,v,f)\,\mathrm{d}v\,\mathrm{d}x - \iint_\Gamma\varphi(f)(v\cdot n)\,\mathrm{d}v\,\mathrm{d}S(x).\end{aligned}$$

We can now use the special structure of the function φ that we are going to use. More precisely,

$$\varphi(t,v,r) = \bar\varphi((r-A(t,v))_+)$$

for some convex function $\bar\varphi$ vanishing at 0 together with its derivative $\dot{\bar\varphi}$. Then we compute,

$$\Phi(f) = \bar\Phi((f-A)_+)$$

and

$$\mathcal{D}_\varphi(f,f') = d_{\bar\varphi}((f-A)_+,(f'-A')_+) + \dot{\bar\varphi}((f-A)_+)(A'-f')_+ - \dot{\bar\varphi}((f-A)_+)(A'-A), \tag{1.5}$$

where

$$d_{\bar\varphi}((f-A)_+,(f'-A')_+) = \bar\varphi((f'-A')_+) - \bar\varphi((f-A)_+) - \dot{\bar\varphi}((f-A)_+)\big((f'-A')_+ - (f-A)_+\big). \tag{1.6}$$

We remark that the first two terms for $\mathcal{D}_\varphi$ are non-negative, while the third one is an error that we will have to handle.

1.5. **Suitable weak subsolutions.** In order to define suitable weak solutions, we consider elementary functions $\bar\varphi$. Precisely, we consider the family of convex functions $\varphi_a(r) = (r-a)_+$ associated with $a>0$. These functions are commonly used in the theory of entropy solutions for scalar conservation laws and are known as *Kruzhkov's semi-entropies*, see for instance [4, 25]. In this case [19],

$$d_{\varphi_a}(r,s) = \begin{cases} (s-a)_+ & \text{if } r \le a \\ (a-s)_+ & \text{if } r > a \end{cases} \tag{1.7}$$

and $\Phi_a(r) = a\mathbf{1}_{\{f>a\}}$.

As far as integrability of solutions is concerned, we remark that the upper bound on the mass density – see condition (1.3) – and the bound on the time-integrated entropy production (see Theorem 2.3) suggest to consider solutions that are L^{q_0} in all variables with $q_0 = 1+\frac{2s}{d}$ – see Lemma 2.2.

In accordance with the formal computation that we performed above, we introduce the following notion of weak subsolutions for the inhomogeneous Boltzmann equation.

Definition 1.1 (Suitable weak subsolutions)**.** Let $T\in(0,+\infty]$ and let Ω be a C^1 domain of $\mathbb{R}^d$. A non-negative function $f\in L^1((0,T)\times\Omega\times\mathbb{R}^d)$ is a *suitable weak subsolution* of (1.1) if

(i). IN-FLOW: given $g\colon (0,T)\times\Gamma_-\to[0,+\infty)$, for any real number $a>0$ and any function $A\colon (0,T)\times\mathbb{R}^d\to(0,+\infty)$ such that $\partial_t A$ and $\partial^2_{v_i,v_j}A$ exist and are bounded continuous in $(0,T)\times\mathbb{R}^d$, there holds in $\mathcal{D}'((0,T))$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\iint_{\mathbb{R}^d\times\Omega}\varphi_a(f-A)\,\mathrm{d}v\,\mathrm{d}x \\ \qquad + \displaystyle\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega} d_{\varphi_a}((f-A)_+,(f'-A')_+)K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v'\,\mathrm{d}x \\ \qquad + \displaystyle\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\dot\varphi_a(f-A)(A'-f')_+K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v'\,\mathrm{d}x \\ \qquad \le \displaystyle\iint_{\mathbb{R}^d\times\Omega}\Big\{c_b\Big(\Phi_a(f-A)+A\dot\varphi_a(f-A)\Big)(f*_v|\cdot|^\gamma) - \dot\varphi_a(f-A)\partial_t A\Big\}\,\mathrm{d}v\,\mathrm{d}x \\ \qquad + \displaystyle\iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\Omega}\dot\varphi_a(f-A)(A'-A)K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v'\,\mathrm{d}x - \iint_{\Gamma_-}\varphi_a(f_b-A)(v\cdot n)\,\mathrm{d}v\,\mathrm{d}S(x) \end{cases} \tag{1.8}$$

with $\varphi_a(r)=(r-a)_+$ and $\Phi_a(r)=a\mathbf{1}_{\{f>a\}}$ and d_{φ_a} is given by (1.7).

(ii). BOUNCE-BACK: it satisfies the (1.8) with $f_b=0$.

(iii). SPECULAR REFLECTION: it satisfies (1.8) with $f_b=0$.

(iv). DIFFUSE REFLECTION: it satisfies (1.8) with $f_b(t,x,v) := c_\mu\mu(v)\int_{\Gamma_+}f(t,x,v')(v'\cdot n)\,\mathrm{d}v'$.

(v). MAXWELL REFLECTION : it satisfies (1.8) with $f_b(t,x,v) := \iota c_\mu\mu(v)\int_{\Gamma_+}f(t,x,v')(v'\cdot n)\,\mathrm{d}v'$.

Remark 1.2 (General convex functions). The family of inequalities (1.8) is only imposed for elementary (non-decreasing non-negative) convex functions $\varphi_a(r)=(r-a)_+$. But this implies that such inequalities hold true for general Lipschitz convex functions φ such that $\varphi(0)=\dot\varphi(0)=0$ by simply integrating in the parameter a. Indeed, as already mentioned in [19], a general C^2 convex function φ satisfies,

$$\varphi(r) = \varphi(0)+\dot\varphi(0)r + \int_0^{+\infty}\ddot\varphi(a)(r-a)_+\,\mathrm{d}a$$

and we can easily check that

$$\Phi(r) = \int_0^{+\infty} \ddot{\varphi}(a)\Phi_a(r)\,\mathrm{d}a \quad \text{and} \quad d_\varphi(r,\rho) = \int_0^{+\infty} \ddot{\varphi}(a) d_{\varphi_a}(r,\rho)\,\mathrm{d}a.$$

We will follow such an idea through a truncation procedure (see Section 3).

Remark 1.3 (Link with renormalized solutions). R. J. DiPerna and P.-L. Lions [15] constructed weak solutions of the Boltzmann equation in the cutoff case by considering the (Lipschitz) concave non-linear change of variables $\varphi(f)$ with $\varphi(r) = \ln(1+r)$. We consider here (Lipschitz) convex functions φ.

Remark 1.4 (Positive terms). The first positive term in the left hand side of (1.8) corresponds to the classical entropy production term. The second one is reminiscent of the "good extra term" first exhibited by L. Caffarelli, C. H. Chan and A. F. Vasseur in [6]. It also plays a crucial role in the recent work by Z. Ouyang and L. Silvestre [36].

Remark 1.5 (Error terms). We remark that right-hand sides of time differential inequalities are well-defined and integrable in time. Indeed, $\dot{\varphi}_a$ and Φ_a are bounded and $f *_v |\cdot|^\gamma$ is integrable in (t,x,v). The term involving the difference $A' - A$ comes from the dependance of the barrier function A on the velocity variable. It did not appear in the work of Q. Ouyang and L. Silvestre [36]. It is reminiscent of (and treated as) the "bad terms" that were treated by C. Mouhot, L. Silvestre and the first author in [26].

1.6. **Main result.**

Theorem 1.6 (Conditional decay estimates)**.** *Let the parameters $\gamma \in (-d, 1]$ and $s \in (0,1)$ of the non-cutoff collision kernel satisfy $\gamma + 2s \in (0,1)$, Let f_{in} (and f_b) denote the initial data (and the boundary data in the in-flow case).*

Let $T \in (0, +\infty]$ and f be a suitable weak subsolution of the Boltzmann equation (1.1) in $(0,T)\times\Omega\times\mathbb{R}^d$ with either in-flow, bounce-back, specular / diffuse / Maxwell reflection boundary conditions satisfying in $\mathcal{D}'((0,T))$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}^d} f\ln f\,\mathrm{d}x\,\mathrm{d}v + \int_\Omega D(f)\,\mathrm{d}x \le 0,$$

where $D(f) \ge 0$ denotes the entropy production. We also assume that f satisfies Condition (1.3) about mass, energy and entropy density functions.

Then for all $q \ge 0$, there exist $C_q > 0$ and $\beta_q > 1$, depending on $m_0, M_0, E_0, s, d, \gamma$, C_b and q such that, if

$$f_b(t,x,v) \le C_b\left(1 + t^{-\beta_q}\right)\langle v\rangle^{-q} \quad \text{a.e. in } (0,T)\times\Gamma_-$$

(only in case of in-flow or diffuse / Maxwell reflection), then the solution f satisfies,

$$f(t,x,v) \le C_q\left(1 + t^{-\beta_q}\right)\langle v\rangle^{-q} \quad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d.$$

Remark 1.7 (Entropy production). The definition of the entropy production $D(f)$ is recalled in the preliminary section, see (2.7).

Remark 1.8 (Integrability of solutions). A suitable weak subsolution of the Boltzmann equation is merely integrable in all variables. In the proof of our main theorem, we need more integrability. The entropy production estimate from Lemma 2.2 implies that $f \in L^1((0,T)\times\Omega; L^{p_0}_{k_0}(\mathbb{R}^d))$ for $\frac{1}{p_0} = 1 - \frac{2s}{d}$ and $k_0 = \gamma + 2s - \frac{2s}{d}$. Consequently, when combined with the hydrodynamical bounds (1.3), we obtain that the functions f we work with are such that $f \in L^{q_0}_{k_0^*}(0,T)\times\Omega\times\mathbb{R}^d)$ with $q_0 = \frac{d+2s}{d}$ and some $k_0^* \in (0,2)$.

Remark 1.9 (Algebraic time decay in v). For $q \in [0, q_{\mathrm{nsl}}]$, with q_{nsl} given in (vii) below, the time decay exponent is given by the formula $\beta_q = \frac{q}{2s}$. For large q's, it is proportional to a power of q. Precisely, $\beta_q \propto \left(1 + \frac{1}{2s}\right)^{\frac{q}{2s}}$.

Remark 1.10 (Large times). We emphasize that the estimates hold true uniformly in time, even in the case $T = +\infty$. This is important in the conditional regularity program. Obtaining bounds that are uniform in time will imply that the large time behaviour can be studied in the class of regular solutions, see in particular the work by L. Desvillettes and C. Villani [13].

Remark 1.11 (Generation of moments). We emphasise that the generation of any pointwise moments was unknown in the case of moderately soft potentials, even for classical solutions. In [26], the first author together with C. Mouhot and L. Silvestre established pointwise generation of moments for $q \in [0, q_{\rm nsl}]$ for classical solutions on the torus with rapid decay.

1.7. **Comments.** This work makes part of the large body of literature dealing with velocity decay of solutions of kinetic equations. Before reviewing the literature, we make several comments about our main theorem and its proof.

Conditional decay estimates. The conditional decay estimates that we obtain for the space-inhomogeneous Boltzmann equation in the non-cutoff case are to be compared with the ones obtained by C. Mouhot, L. Silvestre and the first author (2020). It extends them in two directions. First and most importantly, the Boltzmann equation is posed in a domain and is supplemented with boundary conditions. Second, we consider some weak (sub)solutions instead of classical ones. It is not only a technical challenge to be able to deal with solutions that are more likely to be constructed. The statements are also significantly improved since we generate any polynomial decay, even for moderately soft potentials.

We already mentioned that deriving decay estimates, such as pointwise or moment bounds, is a classical theme in kinetic theory, it is key if the conditional regularity program by L. Silvestre and the first author (see [30] and also [28]) is to be extended to the case of bounded domains.

Barriers and energy methods. The methods of proofs are developed after the article by Z. Ouyang and L. Silvestre [36] about conditional pointwise bounds of weak solutions: in order to prove that the solution f decays at a certain algebraic rate at infinity in velocity, a barrier method was used in [26] and a maximum principle argument was set up. More precisely, letting $A(t, v)$ denote the barrier, the method amounts to consider the first time of contact between f and A, as continuous functions. The technique used in [36] is closer to De Giorgi's truncation argument (revisited for instance by B. Perthame and A. F. Vasseur for evolution equations [37]): prove that the L^2-norm of the positive part of $(f - A)$, denoted by $(f - A)_+$, vanishes for all times. In order to do so, we study the time evolution of this L^2-norm along the flow of the equation.

Truncated Convex Inequalities. In this work, we choose to consider solutions with the mere integrability given by the control of the mass density and the entropy production (see Remark 1.8). We also choose the framework of weak solutions satisfying a family of inequalities associated with convex functions. Such weak solutions were recently introduced by F. Golse, L. Silvestre and the first author in the homogeneous case for very soft potentials [19]. It turns out that classical subsolutions of the Boltzmann equation satisfy them. We thus call them suitable weak subsolutions. We emphasize that we do not use the equation but only this family of inequalities. It is reminiscent of the notion of De Giorgi classes from classical elliptic regularity. This notion was introduced in [31] and the interested reader is referred to [32] for a modern presentation.

Coercive terms. In these inequalities, there are coercive terms, that is to say positive terms in the left hand side, and error terms, corresponding to all terms appearing on the right hand side (some of them are positive, others have *a priori* no sign). There are two types of coercive terms. Some are clear counterparts of the $\dot{H}^1$-norm in De Giorgi's original article. They ensure that the subsolution enjoys better integrability. These terms are typically the ones that are exploited in [19], thanks to the techniques developed in [11, 29]. They are non-local in nature, but also linear. Indeed, they all rely on the property of the kernel K that are derived from the hydrodynamical bounds – see Subsection 2.1. The second type of coercive term is what we call the "good extra term", in reference to [6], where an additional "coercive" term exhibited in the non-local setting. This term was exploited for kinetic equations for the first time in [33]. It turns out that this additional term is the most important in the proof.

Error terms. Error terms are of two types too. First, there is the error term coming from the dependance of the barrier A in the velocity variable. In order to control it, we rely on ideas from [26]. They have to be adapted since we use a different type of argument (pointwise contact vs. truncation) and a different way of splitting the collision operator. The other terms come from the cancellation lemma. In particular, they are non-linear in nature: they make appear a product with the convolution of f with $|\cdot|^\gamma$.

Generation of pointwise decay. Our main result asserts that, as long as mass, energy and entropy densities are under control, suitable subsolutions satisfy a polynomial decay in the velocity variable of any order q. The result is first proved for some exponent q_{nsl} (for not-so-large), by using the estimates from below for coercive terms and from above for error terms. We then use this first decay to improve iteratively the exponent q measuring the polynomial decay rate.

1.8. **Review of literature.** The study of velocity moments of solutions to kinetic equations plays a central role since Boltzmann's (and Landau's) collision operator integrates the velocity variable in the open space $\mathbb{R}^d$. It has a long history, mostly in the space-homogeneous setting, that goes back to the article by T. Carleman [9]. It is precisely described in the introduction of [26]. Let us give references, summarize the review of literature from [26] and review the literature written since then. The reader is also referred to references contained in the articles that are quoted in this subsection.

Velocity moments. We first review contributions to the study of moments of solutions in the velocity variable. Maxwell potentials are treated in [24, 40]. In the case of hard potentials and angular cutoff, Povzner's inequalities [38] are commonly used: see works by T. Elmroth [16], L. Desvillettes [12], S. Mischler and B. Wennberg [35], X. Lu [34] and B. Wennberg (in the non-cutoff case) [42]. A. Bobylev considered exponential moments in [5], see also [17], and in particular the work [2] in this direction, where they establish the creation and the propagation of exponential moments to the spatially homogeneous Boltzmann equation for hard potentials. The case of moderately soft potentials and angular cutoff is addressed by L. Desvillettes [12], see also [14, 41] in this direction. We finally mention the work by M. Gualdani, S. Mischler and C. Mouhot [20] that focusses on hard spheres and makes assumptions on hydrodynamical quantities that are similar to what is assumed in this work. More recently, C. Cao, L. B. He and J. Ji [8] studied the propagation of exponential moments in L^2 for (very) soft potentials in a perturbative regime.

Pointwise decay. There are fewer results about pointwise decay. It starts with works by T. Carleman [9,10] and extended by L. Arkeryd [3]. Exponential pointwise upper bounds were obtained in [17], see also [18].

More recently, the work by C. Mouhot, L. Silvestre and the first author [26] addressed the question of appearance and propagation of polynomial decay in the velocity variable under condition (1.3) on hydrodynamical quantities. In this vein, S. Cameron and S. Snelson [7] established similar results in the case $\gamma + 2s > 2$. The study of polynomial decay is also central in [22] in which the authors are able to deal with very soft potentials ($\gamma + 2s < 0$). See also [21] for results dealing with the Landau equation. In [23], C. Henderson and W. Wang are interested in very soft potentials and short time existence. They work in the class of solutions with polynomial decay.

1.9. **Open questions.** Solutions of the inhomogeneous Boltzmann equation converge to Maxwellians for large times [13]. We recall that under the condition (1.3), C. Mouhot, L. Silvestre and the first author [27] proved that solutions stay above a Maxwellian. It is thus natural to ask ourselves if they can be bounded from above by another Maxwellian. Unfortunately, our proof does not yield neither such an (optimal) upper bound nor any exponential decay (in v). The first open question is thus to show the propagation of pointwise Gaussian bounds. It is unnatural to expect to be able to propagate the Maxwellian with the right temperature.

Another natural open question is the possibility of bridging the gap between generating *any* polynomial pointwise bound to the generation of pointwise Gaussian bounds. Is it possible to show that solutions to the non-cutoff Boltzmann equation with moderately soft potentials generate Gaussian bounds of the form $e^{-a(t)\langle v\rangle^{\gamma+2s}}$ for some function $a = a(t)$?

The critical cases $\gamma + 2s = 0$ and $\gamma + 2s = 1$ are left open. Finally, the case of very soft potentials is also another natural open question.

1.10. **Notation.** For $p \in [1, \infty]$ and $k \in \mathbb{R}$, the weighted Lebesgue space L^p_k is given by

$$L^p_k(\mathbb{R}^d) := \left\{ f \colon \mathbb{R}^d \to \mathbb{R} \ \text{ s.t. } \int_{\mathbb{R}^d} f^p(v)\langle v\rangle^{kp}\,\mathrm{d}v < +\infty \right\},$$

where $\langle v\rangle := (1 + |v|^2)^{\frac{1}{2}}$. When $k = 0$, we simply write L^p.

For $a \in \mathbb{R}$ we define $a_+ := \max(a, 0)$.

The volume of the unit sphere of $\mathbb{R}^m$ is denoted by ω_{m-1}.

1.11. **Constants.** We gather here parameters and constants that are used repeatedly in statements and proofs.

(i). The dimension of the x and v variables is denoted by d. It is always larger than or equal to 2.

(ii). The parameters γ and s from the kernel satisfy: $\gamma \in (-d, 1]$ and $s \in (0,1)$ and $0 < \gamma + 2s < 1$.

(iii). The Lebesgue exponent $p_0 > 1$,
$$\frac{1}{p_0} = 1 - \frac{2s}{d}$$
comes from the entropy production estimate.

(iv). The Lebesgue exponent $q_0 > 1$,
$$q_0 = 1 + \frac{2s}{d}$$
is related to integrability of solutions.

(v). The moment exponent
$$k_0 = (\gamma + 2s) - \frac{2s}{d}$$
appears in the entropy production estimate.

(vi). To any decay exponent $q \geq 1$ is associated a moment exponent
$$l_q = (\gamma + 2s) + \frac{2s}{d} - q\left(1 + \frac{2s}{d}\right).$$

(vii). The decay exponent q_{nsl} (for not-so-large) is given by
$$q_{\mathrm{nsl}} = \begin{cases} d + 1 + \frac{d}{2s}\min(\gamma, 0), & \text{if } -\frac{2sd}{d+2s} \leq \gamma \leq 1, \\ d + 1 - \frac{d}{d+2s}(2-\gamma), & \text{if } -2s < \gamma < -\frac{2sd}{d+2s}. \end{cases}$$
It corresponds to the first decay that is generated by coercivity and error estimates.

2. Preliminaries

We gather in this section known results and technical lemmas that will be used in next sections.

2.1. **The collision operator.** We recall that the use of Carleman coordinates allows us to write the collision operator $Q(f,f)$ and the kernel K_f as follows,
$$Q(f,f)(v) = \int_{\mathbb{R}^d} \left[f(v')K_f(v,v') - f(v)K_f(v',v)\right] \mathrm{d}v'$$
and
$$K_f(v,v') = 2^{d-1}\left|v'-v\right|^{-1} \int_{w \perp v'-v} f(v+w)B(r,\cos\theta)r^{-d+2}\,\mathrm{d}w,$$
where $r^2 = |v'-v|^2 + |w|^2$ and $\cos\theta = \frac{|w|^2 - |v-v'|^2}{|w|^2 + |v-v'|^2}$.

2.1.1. *Coercivity.* Under the condition (1.3), we know that for any fixed $(t,x) \in [0,T]\times\Omega$ there exists a set $\Xi(v) \subset \mathbb{R}^d$ for every $v \in \mathbb{R}^d$ such that $\Xi(v)$ is a symmetric cone, and such that for $v' - v \in \Xi(v)$ there holds
$$K_f(v,v') \geq \lambda_0 \langle v\rangle^{\gamma+2s+1} \left|v-v'\right|^{-d-2s}, \tag{2.1}$$
with $\lambda_0 = \lambda_0(d, m_0, M_0, E_0, H_0)$. Moreover, there holds
$$\left|\Xi(v) \cap \mathbb{S}^{d-1}\right| \geq c_0 \langle v\rangle^{-1}, \tag{2.2}$$
for some constant $c_0 = c_0(d, m_0, M_0, E_0, H_0)$. The set $\Xi(v)$ is the *cone of non-degeneracy* of $K_f(v,v')$. We can also ensure that the following estimate holds,
$$|\Xi(v) \cap B_3 \setminus B_2| \geq c_0\langle v\rangle^{-1}. \tag{2.3}$$

2.1.2. *Upper Bound.* For $(t,x) \in [0,T]\times\Omega$ there exists a postive constant Λ_0 depending on m_0, M_0, E_0, H_0, d such that
$$\forall v \in \mathbb{R}^d,\ \forall r > 0, \qquad \int_{B_r(v)} K_f(v,v')\left|v'-v\right|^2 \mathrm{d}v' \leq \Lambda_0 \langle v\rangle^{\gamma+2s} r^{2-2s}. \tag{2.4}$$

2.1.3. *Cancellation.* For $(t,x) \in [0,T] \times \Omega$ there holds

$$\forall v \in \mathbb{R}^d : \int_{\mathbb{R}^d} (K_f(v,v') - K_f(v',v)) \, \mathrm{d}v' = c_b \int_{\mathbb{R}^d} f(v_*) \, |v - v_*|^\gamma \, \mathrm{d}v_*, \tag{2.5}$$

with

$$c_b = \int_{\mathbb{S}^{d-1}} \left\{ \frac{2^{\frac{d+\gamma}{2}}}{(1+\sigma\cdot\mathbf{e})^{\frac{d+\gamma}{2}}} - 1 \right\} b(\sigma\cdot\mathbf{e}) \, \mathrm{d}\sigma > 0,$$

for any $\mathbf{e} \in \mathbb{S}^{d-1}$. See [1].

2.1.4. *Symmetry.* For any $(t,x) \in [0,T] \times \Omega$ and $v, w \in \mathbb{R}^d$ there holds

$$K_f(v, v+w) = K_f(v, v-w). \tag{2.6}$$

2.2. **Integrability of solutions.** The following lemma is a consequence of Hölder's inequality and a proof is given in [19].

Lemma 2.1 (Hölder's inequality with weights)**.** *Let $p,q,r \in [1,+\infty]$ and $p \le r \le q$. Then,*

$$\|f\|_{L^r_{k_r}} \le \|f\|^{\alpha}_{L^p_{k_p}} \, \|f\|^{1-\alpha}_{L^q_{k_q}}$$

with $\alpha \in (0,1)$ such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ and $k_r = \alpha k_p + (1-\alpha) k_q$.

We can use the previous lemma to derive the following one.

Lemma 2.2 (Integrability of solutions)**.** *If $f \in L^\infty((0,T)\times\bar\Omega; L^1_2(\mathbb{R}^d))$ and $f \in L^1((0,T)\times\bar\Omega; L^{p_0}_{k_0}(\mathbb{R}^d))$ with k_0 given by (v), then $f \in L^{q_0}_{k_0^*}((0,T)\times\bar\Omega\times\mathbb{R}^d)$, where q_0 and k_0^* are given by*

$$q_0 = 2 - \frac{1}{p_0} = 1 + \frac{2s}{d} \in (1,2), \qquad k_0^* = \frac{2s + d(\gamma+2s)}{d+2s} > 0.$$

2.2.1. *Entropy production estimate.* The entropy production of a function f is given by the following formula,

$$\begin{aligned} D(f) &= -\langle Q(f,f), \ln f \rangle \\ &= \frac{1}{2} \iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} [f(v_*)f(v) - f(v'_*)f(v')] \ln \frac{f(v_*)f(v)}{f(v'_*)f(v')} B(|v-v_*|, \cos\theta) \, \mathrm{d}\sigma \, \mathrm{d}v \, \mathrm{d}v'. \end{aligned} \tag{2.7}$$

Theorem 2.3 (Entropy production estimate – [11, Theorem 0.1])**.** *Let $\gamma \in (-d, 1]$ and $s \in (0,1)$. Then, for any non-negative solution f of (1.1) verifying the bounds (1.3), the entropy production $D(f)$ satisfies*

$$\|f\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} \le C_0 D(f),$$

where p_0 is such that $\frac{1}{p_0} = 1 - \frac{2s}{d}$ and $k_0 = \gamma + 2s - \frac{2s}{d}$. The constant C_0 only depends on mass, energy and entropy of f.

In particular, as a consequence of the entropy production estimate and Lemma 2.2, we note that for any solution of (1.1) satisfying the hydrodynamical condition (1.3) there holds $f \in L^{q_0}_{k_0^*}((0,T)\times\bar\Omega\times\mathbb{R}^d)$.

2.3. **Interpolation estimates.** This subsection is devoted to the proofs of interpolation estimates that will be used in the proof of the main theorem.

Hölder's inequality with weights (see Lemma 2.1) applied to $p = r = q = 1$ yields the following result.

Lemma 2.4 (First weighted L^1 estimate)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$ have a finite 2-moment. Then there holds,*

$$\forall k \in \mathbb{R}, \ \forall \alpha_0 \in (0,1), \qquad \int_{\mathbb{R}^d} f(v)\langle v\rangle^k \, \mathrm{d}v \le E_0^{\alpha_0} \left(\int_{\mathbb{R}^d} f(v) \langle v\rangle^{\frac{k-2\alpha_0}{1-\alpha_0}} \, \mathrm{d}v \right)^{1-\alpha_0}.$$

Second, we use that $q_0 \in (1,2)$ to get $1 \in [q_0 - 1, q_0]$ and interpolate $f \in L^1$ between $f \in L^{q_0}$ and $f \in L^{q_0-1}$ (with weights). Since $q_0 - 1 < 1$, the interpolation is applied to $f^{(q_0-1)^{-1}}$.

Lemma 2.5 (Second weighted L^1 estimate)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$ have a finite 2-moment. Let $q_0 = \frac{d+2s}{d} \in (1,2)$, $\frac{1}{p_0} = 1 - \frac{2s}{d}$, $k_0 = \gamma + 2s - \frac{2s}{d}$. Then there holds,*

$$\forall k_2 \in \mathbb{R}, \qquad \int_{\mathbb{R}^d} f(v)\langle v\rangle^{k_2}\,\mathrm{d}v \le \|f^{q_0}\|_{L^{p_0}_{k_0}}^{\frac{d(d-2s)}{4s^2+d^2}} \left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_2}\,\mathrm{d}v\right)^{\frac{4sd}{4s^2+d^2}},$$

with

$$m_2 = \frac{4s^2+d^2}{4sd}\left[k_2 - \frac{d(d-2s)}{4s^2+d^2}k_0\right].$$

Proof. We interpolate

$$\int_{\mathbb{R}^d} f(v)\langle v\rangle^{k_2}\,\mathrm{d}v \le \left(\int_{\mathbb{R}^d} f^{q_0p_0}(v)\langle v\rangle^{k_0p_0}\,\mathrm{d}v\right)^{\frac{\alpha_3}{p_0}} \left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_2}\,\mathrm{d}v\right)^{\alpha_4},$$

where we need α_3, α_4 and m_2 to satisfy,

$$\frac{\alpha_3}{p_0} + \alpha_4 = 1, \qquad \alpha_3 q_0 + \alpha_4(q_0-1) = 1, \qquad \alpha_3 k_0 + \alpha_4 m_2 = k_2.$$

This yields

$$\alpha_3 = \frac{p(2-q_0)}{q_0p_0+1-q_0} = \frac{d(d-2s)}{4s^2+d^2}, \qquad \alpha_4 = \frac{q_0p_0-1}{q_0p_0+1-q_0} = \frac{4sd}{4s^2+d^2},$$
$$m_2 = \frac{1}{\alpha_4}(k_2 - \alpha_3 k_0) = \frac{4s^2+d^2}{4sd}\left[k_2 - \frac{d(d-2s)}{4s^2+d^2}\left(\gamma + 2s - \frac{2s}{d}\right)\right]. \qquad \square$$

We can then apply Lemma 2.1 with $p = 1$ and $q = p_0q_0$.

Lemma 2.6 (General weighted L^{p_1} estimate)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$. Then for all $p_1 \in [1, p_0q_0]$ and all $k_1 \in \mathbb{R}$,*

$$\|f\|_{L^{p_1}_{k_1/p_1}} \le \|f^{q_0}\|_{L^{p_0}_{k_0}}^{\frac{d(p_1-1)}{p_1 4s}} \left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{m_1}\,\mathrm{d}v\right)^{\frac{d+2s-p_1(d-2s)}{p_1 4s}},$$

with

$$m_1 = \frac{4s}{d+2s-p_1(d-2s)}\left[k_1 - \frac{d(p_1-1)}{4s}k_0\right].$$

Proof. We interpolate

$$\int_{\mathbb{R}^d} f^{p_1}(v)\langle v\rangle^{k_1}\,\mathrm{d}v \le \left(\int_{\mathbb{R}^d} f^{q_0p_0}(v)\langle v\rangle^{k_0p_0}\,\mathrm{d}v\right)^{\frac{\alpha_1}{p_0}} \left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{m_1}\,\mathrm{d}v\right)^{\alpha_2},$$

where we need α_1, α_2 and m_1 to satisfy,

$$\frac{\alpha_1}{p_0} + \alpha_2 = 1, \qquad \alpha_1 q_0 + \alpha_2 = p_1, \qquad \alpha_1 k_0 + \alpha_2 m_1 = k_1.$$

This yields

$$\alpha_1 = \frac{(p_1-1)p_0}{q_0p_0-1} = \frac{d(p_1-1)}{4s}, \qquad \alpha_2 = \frac{q_0p_0-p_1}{q_0p_0-1} = \frac{d+2s-p_1(d-2s)}{4s},$$
$$m_1 = \frac{1}{\alpha_2}(k_1 - \alpha_1 k_0) = \frac{4s}{d+2s-p_1(d-2s)}\left[k_1 - \frac{d(p_1-1)}{4s}\left(\gamma + 2s - \frac{2s}{d}\right)\right]. \qquad \square$$

Finally, as a consequence of Lemmas 2.4, 2.5 and 2.6, there holds the following estimate.

Lemma 2.7 (Weighted L^{q_0} estimate)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$ have a finite 2-moment. Then there holds for any $k_3 \in \mathbb{R}$ and any $\varepsilon \in (0,1)$*

$$\int_{\mathbb{R}^d} f^{q_0}(v)\langle v\rangle^{k_3}\,\mathrm{d}v \le \varepsilon\,\|f^{q_0}\|_{L^{p_0}_{k_0}} + C(\varepsilon, E_0)\left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_3}\,\mathrm{d}v\right),$$

with

$$m_3 = \left(1 + \frac{d}{2s}\right)k_3 - \frac{d}{2s}k_0 - 2,$$

and $C(\varepsilon, E_0)$ only depends on E_0, ε, s and d.

Proof. We first interpolate with Lemma 2.6 for $p_1 = q_0$ and $k_1 = k_3$,

$$\int_{\mathbb{R}^d} f^{q_0}(v)\langle v\rangle^{k_3}\,\mathrm{d}v \le \|f^{q_0}\|_{L^{p_0}_{k_0}}^{\alpha_1}\left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{m_1}\,\mathrm{d}v\right)^{\alpha_2},$$

where α_1, α_2, m_1 are given by

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = \frac{d+2s}{2d}, \qquad m_1 = \frac{2d}{d+2s}\left[k_3 - \frac{1}{2}\left(\gamma + 2s - \frac{2s}{d}\right)\right].$$

Then apply Lemma 2.4 for any $\alpha_0 \in (0,1)$ and with $k = m_1$.

$$\int_{\mathbb{R}^d} f(v)\langle v\rangle^{m_1}\,\mathrm{d}v \le E_0^{\alpha_0}\left(\int_{\mathbb{R}^d} f(v)\langle v\rangle^{\frac{m_1-2\alpha_0}{1-\alpha_0}}\,\mathrm{d}v\right)^{1-\alpha_0}.$$

Next we interpolate with Lemma 2.5 for $k_2 = \frac{m_1-2\alpha_0}{1-\alpha_0}$,

$$\int_{\mathbb{R}^d} f(v)\langle v\rangle^{\frac{m_1-2\alpha_0}{1-\alpha_0}}\,\mathrm{d}v \le \|f^{q_0}\|_{L^{p_0}_{k_0}}^{\alpha_3}\left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_2}\,\mathrm{d}v\right)^{\alpha_4},$$

where

$$\alpha_3 = \frac{d(d-2s)}{4s^2+d^2}, \qquad \alpha_4 = \frac{4sd}{4s^2+d^2}, \qquad m_2 = \frac{4s^2+d^2}{4sd}\left[\frac{m_1-2\alpha_0}{1-\alpha_0} - \frac{d(d-2s)}{4s^2+d^2}\left(\gamma+2s-\frac{2s}{d}\right)\right].$$

Finally we combine the three previous inequalities

$$\int f^{q_0}(v)\langle v\rangle^{k_3}\,\mathrm{d}v \le E_0^{\alpha_2\alpha_0}\,\|f^{q_0}\|_{L^{p_0}_{k_0}}^{\alpha_1+(1-\alpha_0)\alpha_2\alpha_3}\left(\int f^{q_0-1}\langle v\rangle^{m_2}\,\mathrm{d}v\right)^{(1-\alpha_0)\alpha_2\alpha_4}$$

we use Young's inequality for some $\theta > 1$ and for any $\varepsilon \in (0,1)$,

$$\le \varepsilon\,\|f^{q_0}\|_{L^{p_0}_{k_0}}^{\theta(\alpha_1+(1-\alpha_0)\alpha_2\alpha_3)} + C(\varepsilon, E_0)\left(\int_{\mathbb{R}^d} f^{q_0-1}(v)\langle v\rangle^{m_2}\,\mathrm{d}v\right)^{\frac{\theta(1-\alpha_0)\alpha_2\alpha_4}{\theta-1}}$$

with $C(\varepsilon, E_0) = E_0^{\frac{\alpha_2\alpha_0\theta}{\theta-1}}\,\varepsilon^{-\frac{1}{\theta-1}}$. We choose θ and α_0 such that

$$\theta(\alpha_1 + (1-\alpha_0)\alpha_2\alpha_3) = 1, \qquad \frac{\theta}{\theta-1}(1-\alpha_0)\alpha_2\alpha_4 = 1.$$

Then we showed

$$\int_{\mathbb{R}^d} f^{q_0}(v)\langle v\rangle^{k_3}\,\mathrm{d}v \le \varepsilon\,\|f^{q_0}\|_{L^{p_0}_{k_0}} + C(\varepsilon, E_0)\left(\int f^{q_0-1}\langle v\rangle^{m_2}\,\mathrm{d}v\right).$$

Note that we find

$$\theta = \frac{\alpha_3+\alpha_4}{\alpha_3+\alpha_1\alpha_4} = \frac{d+2s}{d}, \qquad \alpha_0 = \frac{\alpha_1 - 1 + \alpha_2(\alpha_3+\alpha_4)}{\alpha_2(\alpha_3+\alpha_4)} = \frac{4sd}{(d+2s)^2},$$

(in particular $\theta > 1$), so that

$$m_3 = m_2 = \frac{(d+2s)k_0 - d\left(\gamma+2s-\frac{2s}{d}\right)}{2s} - 2.$$

This yields the result for $\iota = 1$. To get it for any ι, apply the result to ε/ι and recall the definition of $C(\varepsilon, E_0)$. We get $\iota^{1+\frac{1}{\theta-1}}$ (with $\theta - 1 = \frac{2s}{d}$) in front of the second term. □

3. Truncated Lebesgue norms

In order to get the decay estimates from Theorem 1.6, we aim at proving that the following quantity equals zero,

$$\iint_{\mathbb{R}^d\times\Omega} (f(t,x,v)-A(t,v))_+^{q_0}\,\mathrm{d}v\,\mathrm{d}x$$

with $A(t,v)=\mathfrak{a}(t)\langle v\rangle^{-q}$ for some well chosen C^1 function $\mathfrak{a}>0$ and some exponent $q>0$. We recall that $q_0=1+\frac{2s}{d}$ (see Lemma 2.2) and that r_+ denotes $\max(0,r)$ for any real number r. In order to prove that this functional vanishes when applied to a solution of the space-inhomogeneous Boltzmann equation, we investigate how it evolves with time.

3.1. **Truncated Convex Inequalities.** Keeping in mind that $q_0>1$, we thus consider the convex function $\varphi_0(r)=r_+^{q_0}$ and we would like to apply the definition of suitable weak subsolutions – see (1.8) – and integrate against $\ddot{\varphi}_0$ (see Remark 1.2). Unfortunately, we are not sure that the right hand side of (1.8) is integrable with respect to a against $\ddot{\varphi}_0(a)$. For this reason, we follow [19] and approximate $\varphi_0(r)$ with $\varphi_{0,\kappa}(r)$ defined for $\kappa>0$ by

$$\varphi_{0,\kappa}(r)=\big(r_+\wedge\kappa\big)^{q_0}+q_0\kappa^{q_0-1}(r-\kappa)_+. \tag{3.1}$$

In particular $\varphi_{0,\kappa}(0)=(\dot{\varphi_{0,\kappa}})(0)=0$. We can now apply the definition of suitable weak subsolutions and integrate with respect to $\ddot{\varphi_{0,\kappa}}(a)$ for $a\in[0,\kappa]$. What we obtain is expressed in the following statement.

Lemma 3.1 (Truncated Convex Inequalities)**.** *Let f be a suitable weak subsolution of the Boltzmann equation with either in-flow, bounce-back, specular/ diffuse / Maxwell reflection boundary condition. Then,*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}^d}\varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x+\iiint_{\Omega\times\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}((f-A)_+,(f'-A')_+)K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\quad+\iiiint_{\Omega\times\mathbb{R}^{2d}}\dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\quad\le c_b\iint_{\Omega\times\mathbb{R}^d}\Phi\big(f\big)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x+\iiint_{\Omega\times\mathbb{R}^{2d}}\dot{\varphi_{0,\kappa}}(f-A)\,(A'-A)\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\quad-\iint_{\Omega\times\mathbb{R}^d}\dot{\varphi_{0,\kappa}}(f-A)\partial_t A\,\mathrm{d}v\,\mathrm{d}x-\iint_{\partial\Omega\times\mathbb{R}^d}\varphi_{0,\kappa}(f-A)(v\cdot n(x))\,\mathrm{d}v\,\mathrm{d}S(x),
\end{aligned} \tag{3.2}$$

with Φ given by,

$$\Phi(f):=\dot{\varphi_{0,\kappa}}(f-A)f-\varphi_{0,\kappa}(f-A)=\Phi_{0,\kappa}(f-A)+A\dot{\varphi_{0,\kappa}}(f-A). \tag{3.3}$$

3.2. **Useful properties associated with $\varphi_{0,\kappa}$.** The derivatives of the approximate convex function $\varphi_{0,\kappa}$ are given by the following formulas,

$$\dot{\varphi_{0,\kappa}}(r)=q_0\big(r_+\wedge\kappa\big)^{q_0-1}\quad\text{and}\quad\ddot{\varphi_{0,\kappa}}(r)=q_0(q_0-1)r^{q_0-2}\mathbf{1}_{\{0\le r\le\kappa\}}.$$

Moreover, we compute for $r\ge0$,

$$\Phi_{0,\kappa}(r)=(\dot{\varphi_{0,\kappa}})(r)r-\varphi_{0,\kappa}(r)=(q_0-1)\big(r\wedge\kappa\big)^{q_0},$$

and for $r,s\ge0$,

$$d_{\varphi_{0,\kappa}}(r,s)=\varphi_{0,\kappa}(s)-\varphi_{0,\kappa}(r)-\dot{\varphi_{0,\kappa}}(r)(s-r), \tag{3.4}$$

so that, if $0\le r,s\le\kappa$,

$$d_{\varphi_{0,\kappa}}(r,s)=s^{q_0}-r^{q_0}-q_0r^{q_0-1}(s-r).$$

Note that if $r\le k$, then for $q_0>1$

$$d_{\varphi_{0,\kappa}}\Big(r,\frac{r}{2}\Big)=c_{q_0}r^{q_0}\quad\text{with}\quad c_{q_0}=2^{-q_0}-1+\frac{q_0}{2}>0.$$

In particular,

$$d_{\varphi_0}(r,s)\ge c_{q_0}r^{q_0}\quad\text{in}\quad\{0\le s<r/2\} \tag{3.5}$$

with $c_{q_0} = d_{\varphi_0}(1, 1/2) = 2^{-q_0} - 1 + q_0/2 > 0$. We also have for $r \geq 0$,

$$d_{\varphi_{0,\kappa}}\Big(r, \frac{r \wedge \kappa}{2}\Big) = c_{q_0}\Big(r \wedge \kappa\Big)^{q_0}. \tag{3.6}$$

4. Coercivity estimates

This section is devoted to the study of positive terms appearing in the left hand side of the family of convex inequalities associated with the approximate L^{q_0}-norm, see (3.2). The term in which $(A' - f')_+$ appears is referred to as the "good extra term" (see the introduction).

4.1. First coercivity estimate.

In order to get our first coercivity estimate, we follow the method introduced in [11] and used in [36] and [19]. Precisely, the proof of the following proposition is adapted from the proof of [19, Lemma 3.1] in which $k_0 < 0$. We recall that $k_0 = (\gamma + 2s) - \frac{2s}{d}$.

Proposition 4.1 (Coercivity estimate for $k_0 \geq 0$)**.** *Assume f satisfies (1.3). If $k_0 \geq 0$, then there exists a constant $C_{\mathrm{cor}} > 0$, depending on d, γ, s and constants in (1.3), such that for all $f, g \colon \mathbb{R}^d \to [0, +\infty)$,*

$$\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g, g') K_f(v, v') \, \mathrm{d}v \, \mathrm{d}v' \geq C_{\mathrm{cor}} \left\| \big(g \wedge \kappa\big)^{q_0} \right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}. \tag{4.1}$$

Proof. For $R > 0$ (to be chosen), we define

$$T_R = \begin{cases} B_{R/2}(v) & \text{if } |v| \geq R \\ B_{3R}(v) \setminus B_{2R}(v) & \text{if } |v| \leq R. \end{cases}$$

In particular, for $v' - v \in T_R$, we have $|v| \leq 2|v'|$, which implies $\langle v \rangle \leq 2\langle v' \rangle$.

Then we use the non-degeneracy cone $\Xi(v)$ associated with the kernel K_f and get from (2.1) and (3.5)

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g, g') K_f(v, v') \, \mathrm{d}v \, \mathrm{d}v' \\
&\quad \geq c_{q_0} \int_{\mathbb{R}^d} \int_{\left\{g' < \frac{g \wedge \kappa}{2}\right\} \cap \{v' - v \in \Xi(v) \cap T_R\}} \big(g \wedge \kappa\big)^{q_0}(v) K_f(v, v') \, \mathrm{d}v' \, \mathrm{d}v \\
&\quad \geq c \int_{\mathbb{R}^d} \int_{\{g' < g/2\} \cap \{v' - v \in \Xi(v) \cap T_R\}} \big(g \wedge \kappa\big)^{q_0}(v) \langle v \rangle^{\gamma + 2s + 1} \, |v - v'|^{-d-2s} \, \mathrm{d}v' \, \mathrm{d}v \\
&\quad \geq c \int_{\mathbb{R}^d} \big(g \wedge \kappa\big)^{q_0}(v) R^{-d-2s} \langle v \rangle^{\gamma + 2s + 1} \left| \left\{ g' < \frac{g \wedge \kappa}{2} \right\} \cap \{v + \Xi(v) \cap T_R\} \right| \mathrm{d}v
\end{aligned}$$

where $c = c_{q_0} \lambda_0$. Moreover, for any $v \in \mathbb{R}^d$, we use either (2.3) or (2.2) and Chebyshev's inequality to estimate the Lebesgue norm of the sub-level set as follows,

$$\begin{aligned}
&\left| \left\{ g' < \frac{g \wedge \kappa}{2} \right\} \cap \{v + \Xi(v) \cap T_R\} \right| \\
&\quad = |\{v + \Xi(v) \cap T_R\}| - \left| \left\{ g' > \frac{g \wedge \kappa}{2} \right\} \cap \{v + \Xi(v) \cap T_R\} \right| \\
&\quad \geq c_0 R^d \langle v \rangle^{-1} - \frac{2^{p_0(q_0 - k_0)}}{\langle v \rangle^{k_0 p_0} \big(g \wedge \kappa\big)^{q_0 p_0}(v)} \int_{\mathbb{R}^d} (g' \wedge \kappa)^{q_0 p_0} \langle v' \rangle^{k_0 p_0} \, \mathrm{d}v'.
\end{aligned}$$

The last line uses that $k_0 \geq 0$. Then, we choose R such that

$$R^d = c_1 \, \|g^{q_0}\|^{p_0}_{L^{p_0}_{k_0}} \, \langle v \rangle^{-k_0 p_0 + 1} \big(g \wedge \kappa\big)^{-q_0 p_0}(v), \tag{4.2}$$

with $c_1 = \frac{2^{p_0(q_0 - k_0) + 1}}{c_0}$ and get

$$\left| \left\{ g' < \frac{g \wedge \kappa}{2} \right\} \cap \{v + \Xi(v) \cap T_R\} \right| \geq c_0 R^d \langle v \rangle^{-1} - \frac{c_0}{2} R^d \langle v \rangle^{-1} = \frac{c_0}{2} R^d \langle v \rangle^{-1}.$$

Therefore, with such a choice of R,

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g,g')K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \\
&\qquad \geq c\frac{c_0}{2}\int_{\mathbb{R}^d}\big(g\wedge\kappa\big)^{q_0}(v)R^{-2s}\langle v\rangle^{\gamma+2s}\,\mathrm{d}v \\
&\qquad = c\frac{c_0}{2}c_1^{-\frac{2s}{d}}\,\|g^{q_0}\|_{L^{p_0}_{k_0}}^{-\frac{2sp_0}{d}}\int_{\mathbb{R}^d}\langle v\rangle^{\gamma+2s+\frac{2s(k_0p_0-1)}{d}}\big(g\wedge\kappa\big)^{q_0\left(1+\frac{2sp_0}{d}\right)}(v)\,\mathrm{d}v,
\end{aligned}$$

and remarking that p_0 and k_0 are such that $p_0 = 1+\frac{2sp_0}{d}$ and $k_0p_0 = \gamma+2s+\frac{2s(k_0p_0-1)}{d}$,

$$\geq c\frac{c_0}{2}c_1^{-\frac{2s}{d}}\,\|g^{q_0}\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}^{1-p_0}\left\|\big(g\wedge\kappa\big)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}^{p_0}.$$

We reach the desired conclusion with $C_{cor} = c\frac{c_0}{2}c_1^{-\frac{2s}{d}}$. □

Proposition 4.2 (Coercivity estimate for $k_0<0$)**.** *Assume $k_0<0$. Then there exists $C_{\mathrm{cor}}>0$, depending on s,d,M_0,E_0, such that for all $f,g\colon\mathbb{R}^d\to[0,+\infty)$,*

$$\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g,g')K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \geq C_{\mathrm{cor}}\left\|\big(g\wedge\kappa\big)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} - C_{\mathrm{cor}}\int_{\mathbb{R}^d}(g\wedge\kappa)^{q_0}\langle v\rangle^{\gamma}\,\mathrm{d}v. \tag{4.3}$$

In particular,

$$\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g,g')K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \geq C_{\mathrm{cor}}\left\|\big(g\wedge\kappa\big)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} - C_{\mathrm{cor}}\left\|\big(g\wedge\kappa\big)^{q_0-1}\right\|_{L^1_{\gamma-d-1}(\mathbb{R}^d)}. \tag{4.4}$$

Proof. The estimate (4.3) follows the same lines as the proof of Proposition 4.1 and is proved in [19, Lemma 3.1].

We can obtain (4.4) by applying Lemma 2.7 to $g\wedge\kappa$ and $\varphi=1$ and $k_3=\gamma$ and get for all $\varepsilon\in(0,1)$,

$$\int_{\mathbb{R}^d}\big(g\wedge\kappa\big)^{q_0}\langle v\rangle^{\gamma}\,\mathrm{d}v \leq \varepsilon\left\|\big(g\wedge\kappa\big)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} + C(\varepsilon,E_0)\left\|\big(g\wedge\kappa\big)^{q_0-1}\right\|_{L^1_{m_3}}$$

with $m_3=\gamma-d-1$. □

We now prove another coercivity estimate for $\gamma<0$.

Proposition 4.3 (Second coercivity estimate)**.** *There exists two positive constants $C_{\mathrm{cor}'}$ and κ_0, only depending on s,d,γ and hydrodynamical bounds from (1.3), such that for all $\kappa\geq\kappa_0$ and all $f,g\colon\mathbb{R}^{2d}\to[0,+\infty)$,*

$$\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g,g')K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \geq C_{\mathrm{cor}'}\,\kappa^{\frac{2s}{d}q_0}N^{-\frac{2s}{d}}\int_{\mathbb{R}^d}(g(v)-\kappa)_+\langle v\rangle^{\gamma}\,\mathrm{d}v. \tag{4.5}$$

with $N=\int_{\mathbb{R}^d}(g\wedge\kappa)^{q_0-1}\langle v\rangle^{-(d-1)}\,\mathrm{d}v$.

Proof. We explain how to adapt the proof of [19, Lemma 3.2]. The only difference lies in the choice of R. In our case, we choose $R=R(v')$ such that

$$\frac{R^d}{\langle v'\rangle} = C_0\frac{N}{\kappa^{q_0-1}\langle v'\rangle^{-(d-1)}}.$$

We can argue as in [19] by remarking that $N\leq\kappa^{q_0-2}M_0$. We thus choose C_0 and κ_0 such that

$$C_0\frac{M_0}{\varepsilon_0^d\kappa_0}\leq 1$$

to ensure that $R\leq\varepsilon_0\langle v'\rangle$, which allows us to use [19, Lemma 2.6]. The idea is to consider the cone of non-degeneracy for fixed v'

$$S(v') = \left\{v\in\mathbb{R}^d : K(v,v')\geq\mu_0\langle v'\rangle^{\gamma+2s+1}\,|v-v'|^{-(d+2s)}\right\}.$$

Then Lemma 2.6 in [19] proves the existence of $\mu_0>0$ depending only on the hydrodynamical quantities such that

$$|S(v')\cap B_R(v')|\geq\mu_0R^d\langle v\rangle^{-1}.$$

The weight in N implies that the estimate of the sub-level set is modified as follows:

$$|\{g\wedge\kappa>\kappa/2\}\cap S(v')\cap B_R(v')| \le \frac{2^{q_0-1}}{\kappa^{q_0-1}}\int_{S(v')\cap B_R(v')}(g(v)\wedge\kappa)^{q_0-1}\frac{\langle v\rangle^{d-1}}{\langle v\rangle^{d-1}}\,\mathrm{d}v$$
$$\le \frac{2^{q_0+d-2}}{\kappa^{q_0-1}}\langle v'\rangle^{d-1}\int_{S(v')\cap B_R(v')}(g(v)\wedge\kappa)^{q_0-1}\langle v\rangle^{-(d-1)}\,\mathrm{d}v.$$

We used that for $v\in B_R(v')$ and R such that $R\le\varepsilon_0\langle v'\rangle$, we have $\langle v\rangle\le 2\langle v'\rangle$ (since $\varepsilon_0\le 1$).

Following the proof of [19, Lemma 3.2] and keeping in mind that $\varphi_{0,\kappa}(g')=q_0\kappa^{q_0-1}(g'-\kappa)$ in $\{g'>\kappa\}$, we get

$$\iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}(g,g')K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \ge C_{2,q_0}\frac{\mu_0^2}{2}\int_{\{g'>\kappa\}}\varphi_{0,\kappa}(g')\langle v'\rangle^{\gamma+2s}R^{-2s}\,\mathrm{d}v'$$
$$\ge C_{2,q_0}\frac{\mu_0^2}{2}C_0^{-\frac{2s}{d}}N^{-\frac{2s}{d}}q_0\kappa^{q_0-1}\kappa^{(q_0-1)\frac{2s}{d}}\int_{\mathbb{R}^d}(g'-\kappa)_+\langle v'\rangle^{\gamma}\,\mathrm{d}v'.$$

In view of the definition of k_0 and q_0, we obtain the announced lower bound for some constant $C_{\mathrm{cor}'}$ equal to $C_{2,q_0}\frac{\mu_0^2}{2}C_0^{-\frac{2s}{d}}q_0$. □

4.2. **The good extra term.** We exhibit a third coercivity term. It controls the weighted L^{q_0-1}-norm of $(f-A)_+$. This lemma is similar to [19, Lemma 5.3], but more general since A depends on v. In contrast with the previous coercivity estimates, it is nonlinear in nature, it cannot be stated for two general functions f and g.

Proposition 4.4 (The good extra term for not-too-large q's)**.** *Let $q\in[0,d+1]$. Then there exists a constant $C_{\mathrm{get}}>0$ depending on m_0,M_0,E_0,H_0,d, such that if $\mathfrak{a}$ satisfies*

$$\mathfrak{a}(t)\ge a_{\mathrm{get}} \quad \text{with} \quad a_{\mathrm{get}}:=2^{d+1}(M_0+E_0)c_0^{-1} \tag{4.6}$$

with c_0 from (2.2), then for any f satisfying (1.3), there holds,

$$\iint_{\mathbb{R}^{2d}}\dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \ge C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}}\int_{\mathbb{R}^d}\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\langle v\rangle^{l_q}\,\mathrm{d}v. \tag{4.7}$$

Proof. Let $R>0$ (to be chosen). We denote the "good" set of velocities by $G:=\{v:\langle v\rangle\ge 2R\}$. For $v'\in B_R(v)$ and $v\in G$, we have $\langle v'\rangle\le 2\langle v\rangle$ and $\langle v\rangle\le 2\langle v'\rangle$. In particular, $A(v')\ge 2^{-q}A(v)$ for such v's and v''s. We now use the non-degeneracy cone $\Xi(v)$, see in particular (2.1), in order to write,

$$\iint_{\mathbb{R}^{2d}}\dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v'$$
$$\ge \frac{q_0}{2}\int_G\int_{\left\{f'<\frac{A'}{2}\right\}\cap\{v+\Xi(v)\cap B_R\}}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)A'K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v$$
$$\ge \frac{q_0\lambda_0}{2^{q+1}}\int_G R^{-d-2s}\langle v\rangle^{\gamma+2s+1}\big((f-A)_+\wedge\kappa\big)^{q_0-1}A\left|\left\{f'<\frac{A'}{2}\right\}\cap\{v+\Xi(v)\cap B_R\}\right|\,\mathrm{d}v.$$

We then use (2.2) and Chebyshev's inequality to bound the sublevel set from below for $v\in G$ as follows,

$$\left|\left\{f'<\frac{A'}{2}\right\}\cap\{v+\Xi(v)\cap B_R\}\right| = |\{v+\Xi(v)\cap B_R\}| - \left|\left\{f'\ge\frac{A'}{2}\right\}\cap\{v+\Xi(v)\cap B_R\}\right|$$
$$\ge c_0R^d\langle v\rangle^{-1} - \int_{v+\Xi(v)\cap B_R}\frac{2}{A'}f'\,\mathrm{d}v'$$
$$\ge c_0R^d\langle v\rangle^{-1} - \frac{2}{\mathfrak{a}}\int_{v+\Xi(v)\cap B_R}\langle v'\rangle^{q-2}f(v')\langle v'\rangle^2\,\mathrm{d}v'$$

we use again that $\langle v'\rangle \le 2\langle v\rangle$ if $q \ge 2$ and $\langle v\rangle \le 2\langle v'\rangle$ if $q \le 2$ to get,

$$\begin{aligned}&\ge c_0 R^d \langle v\rangle^{-1} - \frac{2^{1+|q-2|}}{A(v)\langle v\rangle^2}\int_{\mathbb{R}^d} f(v')\langle v'\rangle^2 \,\mathrm{d}v' \\ &\ge c_0 R^d \langle v\rangle^{-1} - \frac{2^d}{A(v)\langle v\rangle^2}(M_0+E_0).\end{aligned}$$

We choose next $R = R(v)$ such that

$$R^d = \frac{2^{d+1}(M_0+E_0)}{c_0 A(v)\langle v\rangle}.$$

Then

$$\left|\left\{f' < \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}\right| \ge \frac{c_0}{2} R^d \langle v\rangle^{-1}.$$

With such an estimate of the sublevel set at hand, we can finish to justify the estimate from below of the good extra term,

$$\begin{aligned}&\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+ K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v \\ &\qquad\ge \frac{q_0\lambda_0 c_0}{2^{q+2}}\int_G R^{-2s}\langle v\rangle^{\gamma+2s}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)A(v)\,\mathrm{d}v \\ &\qquad= C_{\mathrm{get}}\int_G \langle v\rangle^{\gamma+2s+\frac{2s}{d}}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)A^{1+\frac{2s}{d}}(v)\,\mathrm{d}v\end{aligned}$$

with $C_{\mathrm{get}} = \frac{q_0\lambda_0 c_0}{2^{q+2}}\left(2^{d+1}c_0^{-1}(M_0+E_0)\right)^{-\frac{2s}{d}}$.

Finally, since $A(v) = \mathfrak{a}\langle v\rangle^{-q}$, we have

$$G = \{v : \langle v\rangle \ge 2R\} = \left\{v : \langle v\rangle^{d+1-q} \ge \frac{2^{d+1}(M_0+E_0)}{c_0\mathfrak{a}}\right\}.$$

If $q \le d+1$ and if $\mathfrak{a}$ is such that $\mathfrak{a} \ge 2^{d+1}(M_0+E_0)c_0^{-1}$ then $G = \mathbb{R}^d$. □

We conclude this section devoted to coercivity estimates by exhibiting an improved lower bound for the good extra term in the case of large decay exponents q (in contrast with assumptions of Proposition 4.4) by taking advantage of the fact that the function is already known to decay at some rate.

Proposition 4.5 (The very good extra term)**.** *Let $q \ge 0$. Let $f\colon \mathbb{R}^d \to [0,+\infty)$ and $A(v) := \mathfrak{a}\langle v\rangle^{-q}$ for some $\mathfrak{a} \ge 0$ and assume there is $\mathfrak{b}_* \ge 0$ depending on $m_0, M_0, E_0, s, d, \gamma$ (and time) such that*

$$f(v) \le \mathfrak{b}_*\langle v\rangle^{-q_*}.$$

for some $q_* \ge 0$. If $\mathfrak{a}$ satisfies

$$\mathfrak{a} \ge 2^{2+q-q_*}\mathfrak{b}_*,$$

then

$$\begin{aligned}&\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)(v)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \\ &\qquad\ge \frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}}\mathfrak{a}^{1+2s}\int_{\mathbb{R}^d}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\end{aligned} \tag{4.8}$$

for some a constant $C_{\mathrm{vget}} > 0$ depending on m_0, M_0, E_0, H_0, d, s.

Proof. The proof follows the same lines as for Proposition 4.4. We find

$$\begin{aligned}&\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v\,\mathrm{d}v' \\ &\qquad\ge \frac{q_0\lambda_0}{2^{q+1}}\int_G R^{-d-2s}\langle v\rangle^{\gamma+2s+1}\big((f-A)_+\wedge\kappa\big)^{q_0-1}A\left|\left\{f' < \frac{A'}{2}\right\}\cap\{v+\Xi(v)\cap B_R\}\right|\,\mathrm{d}v,\end{aligned}$$

where $G := \{v : \langle v\rangle \geq 2R\}$ for some $R \in (0,1)$ to be chosen. Then we use Chebyshev's inequality, so that for $v \in G$,

$$\begin{aligned}\left|\left\{f' < \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}\right| &= |\{v + \Xi(v) \cap B_R\}| - \left|\left\{f' \geq \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}\right| \\ &\geq |\{v + \Xi(v) \cap B_R\}| - \frac{2}{\mathfrak{a}} \int_{v+\Xi(v)\cap B_R} \langle v'\rangle^{q-q_*} f(v') \langle v'\rangle^{q_*} \,\mathrm{d}v' \\ &\geq |\{v + \Xi(v) \cap B_R\}| - \frac{2^{1+(q-q_*)}}{\mathfrak{a}} \mathfrak{b}_* \left|\{v + \Xi(v) \cap B_R\}\right|.\end{aligned}$$

We pick $\mathfrak{a}$ sufficiently large, so that

$$1 - \frac{2^{1+(q-q_*)}}{\mathfrak{a}} \mathfrak{b}_* \geq \frac{1}{2}.$$

Then, using (2.2), we bound

$$\left|\left\{f' < \frac{A'}{2}\right\} \cap \{v + \Xi(v) \cap B_R\}\right| \geq \frac{1}{2} c_0 R^d \langle v\rangle^{-1}.$$

Now pick R as follows,

$$R = \frac{2^{3+(q-q_*)}}{\mathfrak{a}} \mathfrak{b}_* \leq \frac{1}{2}.$$

The fact that $2R \leq 1$ implies in particular that the good set G is the whole space,

$$G = \{v : \langle v\rangle \geq 2R\} = \mathbb{R}^d.$$

We finally get

$$\begin{aligned}&\iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f - A)\,(A' - f')_+ \,K_f(v, v') \,\mathrm{d}v \,\mathrm{d}v' \\ &\quad\geq \frac{c_0 q_0 \lambda_0}{2^{q+2}} \int_G R^{-2s} \langle v\rangle^{\gamma+2s} \big((f - A)_+ \wedge \kappa\big)^{q_0 - 1} A \,\mathrm{d}v \\ &\quad= \frac{c_0 q_0 \lambda_0}{2^{q+2} 2^{2s(3+(q-q_*))}} \mathfrak{b}_*^{-2s} \mathfrak{a}^{1+2s} \int_{\mathbb{R}^d} \big((f - A)_+ \wedge \kappa\big)^{q_0 - 1} \langle v\rangle^{\gamma+2s-q} \,\mathrm{d}v. \qquad \square\end{aligned}$$

5. Error term due to the velocity dependance of the barrier

This section is devoted to the estimate of the first error term appearing in the right hand side of the Truncated Convex Inequalities, see Lemma 3.1. We recall that it has the following form,

$$\mathcal{E}_1 := \iint_{\mathbb{R}^{2d}} \big((f - A)_+ \wedge \kappa\big)^{q_0 - 1}(v)\,(A' - A)\,K_f(v, v') \,\mathrm{d}v' \,\mathrm{d}v.$$

In the next paragraph, we use ideas from [26].

5.1. Not-so-large decay exponents.

Lemma 5.1 (Error term in the velocity variable)**.** *We consider a function f satisfying (1.3). Let $q \in [0, d+1]$ and $A(v) := \mathfrak{a}\langle v\rangle^{-q}$ for some $\mathfrak{a} > 0$. There exists C_1 depending on d, m_0, M_0, E_0, H_0 such that*

$$\begin{aligned}&\iint_{\mathbb{R}^{2d}} \big((f - A)_+ \wedge \kappa\big)^{q_0 - 1}(v)\big(A(v') - A(v)\big) K_f(v, v') \,\mathrm{d}v' \,\mathrm{d}v \\ &\qquad\qquad \leq C_1 \mathfrak{a} \int_{\mathbb{R}^d} \big((f - A)_+ \wedge \kappa\big)^{q_0 - 1}(v) \langle v\rangle^{l_q} \,\mathrm{d}v,\end{aligned} \tag{5.1}$$

with l_q given by (vi).

Recalling that the kernel K_f is given by,

$$K_f(v, v + z) = \frac{1}{|z|^{d+2s}} \int_{v'_* \in v + z^\perp} f(v'_*) \,|v - v'_*|^{\gamma+2s+1} \,\tilde{b}(\cos\theta) \,\mathrm{d}v'_*,$$

we aim at estimating

$$\mathcal{E}_1 = \int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa\right)^{q_0-1}(v)\mathcal{I}(v)\,\mathrm{d}v$$

with

$$\begin{aligned}\mathcal{I}(v) &:= \int_{\mathbb{R}^d} (A(v')-A(v))\,K_f(v,v')\,\mathrm{d}v' \\ &= \int_{\mathbb{R}^d} \frac{(A(v+z)-A(v))}{|z|^{d+2s}} \left\{ \int_{v'_*\in v+z^\perp} f(v'_*)\,|v-v'_*|^{\gamma+2s+1}\,\tilde{b}(\cos\theta)\,\mathrm{d}v'_* \right\} \mathrm{d}z \\ &= \int_{\mathbb{R}^d} f(v'_*)\,|v-v'_*|^{\gamma+2s} \left\{ \int_{z\in(v-v'_*)^\perp} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z \right\} \mathrm{d}v'_*,\end{aligned}$$

where we changed the order of integration in the last equality.

We distinguish the singular part from the non-singular one,

$$\mathcal{I}(v) = \mathcal{I}_1(v) + \mathcal{I}_2(v)$$

with

$$\begin{aligned}\mathcal{I}_1(v) &= \int_{\mathbb{R}^d} f(v'_*)\,|v-v'_*|^{\gamma+2s} \left\{ \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z \right\} \mathrm{d}v'_* \\ \mathcal{I}_2(v) &= \int_{\mathbb{R}^d} f(v'_*)\,|v-v'_*|^{\gamma+2s} \left\{ \int_{z\in(v-v'_*)^\perp} \chi_{|z|<\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z \right\} \mathrm{d}v'_*.\end{aligned} \tag{5.2}$$

We now estimate each part separately.

Lemma 5.2 (Estimate of the non-singular part)**.**

$$\mathcal{I}_1(v) \le \begin{cases} CE_0\mathfrak{a}\langle v\rangle^{\gamma-2-\min(q,d-1)}, & \text{if } q\neq d-1, \\ CE_0\mathfrak{a}\langle v\rangle^{\gamma-d}, & \text{if } q = d-1. \end{cases} \tag{5.3}$$

In particular, if $q \le d+1$, then

$$\mathcal{I}_1(v) \le C\mathfrak{a}\langle v\rangle^{l_q}, \tag{5.4}$$

where l_q is given by (vi).

Proof. For the non-singular part we further distinguish two cases: $|v'_*| < \frac{1}{8}|v|$ and $\frac{1}{8}|v| < |v'_*|$,

$$\mathcal{I}_1(v) = \mathcal{I}_{1,1}(v) + \mathcal{I}_{1,2}(v)$$

with

$$\begin{cases} \mathcal{I}_{1,1}(v) = & \int_{|v'_*|<\frac{1}{8}|v|} f(v'_*)\,|v-v'_*|^{\gamma+2s} \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v'_* \\ \mathcal{I}_{1,2}(v) = & \int_{\frac{1}{8}|v|<|v'_*|} f(v'_*)\,|v-v'_*|^{\gamma+2s} \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v'_*. \end{cases}$$

The idea for $\mathcal{I}_{1,1}$ is to show that $|v+z| > |v|$ by exploiting the orthogonality $v+z \perp v'_*$ and the smallness $|v'_*| < \frac{1}{8}|v|$. In particular, $\mathcal{I}_{1,1}$ has a good sign. For the second term $\mathcal{I}_{1,2}$, we exploit the 2-moment of f.

We start with $\mathcal{I}_{1,1}$. Since and $z \perp (v-v'_*)$ and $|v'_*| < \frac{1}{8}|v|$ and $|z| > \frac{1}{2}\langle v\rangle > \frac{1}{2}|v|$, we find

$$\begin{aligned}|v+z|^2 &= |v|^2 + |z|^2 + 2v\cdot z = |v|^2 + |z|^2 + 2v'_*\cdot z \\ &\ge |v|^2 + |z|\,(|z| - 2|v'_*|) > |v|^2 + |z|\left(|z| - \frac{1}{4}|v|\right) \\ &> |v|^2 + \frac{1}{4}|z|\,|v| > \left(1+\frac{1}{8}\right)|v|^2.\end{aligned} \tag{5.5}$$

Since $q \ge 0$, $A(v+z) - A(v) = \mathfrak{a}\left[\langle v+z\rangle^{-q} - \langle v\rangle^{-q}\right] \le 0$, we conclude that $\mathcal{I}_{1,1}(v) < 0$.

For $\mathcal{I}_{1,2}$ we find

$$\mathcal{I}_{1,2}(v) = \int_{\frac{1}{8}|v|<|v'_*|} f(v'_*)\,|v-v'_*|^{\gamma+2s} \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{(A(v+z)-A(v))}{|z|^{d+2s-1}} \tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v'_*$$

we use that $\tilde{b}(\cos\theta) \le \tilde{b}_+$ and $A(v) \ge 0$,

$$\begin{aligned}
&\le \tilde{b}_+ \mathfrak{a} \int_{\frac{1}{8}|v|<|v'_*|} f(v'_*)\,|v-v'_*|^{\gamma+2s} \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z\,\mathrm{d}v'_* \\
&\le \tilde{b}_+ \mathfrak{a} \int_{\frac{1}{8}|v|<|v'_*|} f(v'_*) \frac{(9\,|v'_*|)^{\gamma+2s}\langle v'_*\rangle^2}{\langle v'_*\rangle^2} \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z\,\mathrm{d}v'_* \\
&\le \left(9^{\gamma+2s}8^{2-\gamma-2s}\tilde{b}_+\right)\mathfrak{a}\langle v\rangle^{\gamma+2s-2} \int_{\frac{1}{8}|v|<|v'_*|} f(v'_*)\langle v'_*\rangle^2 \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z\,\mathrm{d}v'_*
\end{aligned}$$

since $2 \ge \gamma + 2s$.

Finally, we compute,

$$\begin{aligned}
&\int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \\
&\quad= \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle}\chi_{|v+z|>|v|} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z + \int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle}\chi_{|v+z|<|v|} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z
\end{aligned}$$

we let ω_{d-2} denote the volume of the unit sphere of dimension $d-2$,

$$\begin{aligned}
&\le \omega_{d-2}\langle v\rangle^{-q} \int_{\frac{1}{2}\langle v\rangle}^\infty r^{-d+1-2s+d-2}\,\mathrm{d}r + (\langle v\rangle/2)^{-d-2s+1} \int_{z\in(v-v'_*)^\perp} \chi_{|v+z|<|v|}\langle v+z\rangle^{-q}\,\mathrm{d}z \\
&\le C\langle v\rangle^{-q-2s} + C\langle v\rangle^{-(d-1)-2s} \int_0^{|v|} \frac{r^{d-2}}{(1+r^2)^{q/2}}\,\mathrm{d}r
\end{aligned}$$

for some positive constant C only depending on d and s only.

We now distinguish cases. If $q > d-1$, then $\frac{r^{d-2}}{(1+r^2)^{q/2}} \in L^1(\mathbb{R})$ and

$$\int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \le C_{q,1}\langle v\rangle^{-(d-1)-2s}$$

with

$$C_{q,1} = C + C\int_0^{+\infty} \frac{r^{d-2}}{(1+r^2)^{q/2}}\,\mathrm{d}r \le C + C(1+2^{-q/2}(q-(d-1))^{-1})$$

(split the domain integration between $r \le 1$ and $r \ge 1$).

If $q < d-1$, then we simply write,

$$\int_0^{|v|} \frac{r^{d-2}}{(1+r^2)^{q/2}}\,\mathrm{d}r \le \int_0^{|v|} r^{d-2-q}\,\mathrm{d}r = \frac{|v|^{d-1-q}}{d-1-q} \le \frac{\langle v\rangle^{d-1-q}}{d-1-q}.$$

This implies that

$$\int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \le C\langle v\rangle^{-q-2s}.$$

If $q = d-1$, then

$$\int_0^{|v|} \frac{r^{d-2}}{(1+r^2)^{(d-1)/2}}\,\mathrm{d}r \le \int_0^{|v|} r^{-1}\,\mathrm{d}r = \ln|v|.$$

This implies in this case that

$$\int_{z\in(v-v'_*)^\perp} \chi_{|z|>\frac{1}{2}\langle v\rangle} \frac{\langle v+z\rangle^{-q}}{|z|^{d+2s-1}}\,\mathrm{d}z \le C\langle v\rangle^{-(d-1)-2s}\ln\langle v\rangle.$$

We thus proved that,

$$\mathcal{I}_{1,2}(v) \le \begin{cases} CE_0\mathfrak{a}\langle v\rangle^{\gamma-2-\min(q,d-1)}, & \text{if } q \neq d-1, \\ CE_0\mathfrak{a}\langle v\rangle^{\gamma-d}, & \text{if } q = d-1. \end{cases} \tag{5.6}$$

Using $\mathcal{I}_{1,1} < 0$ and (5.6) we obtain (5.3) and (5.4).

We can check that

$$\gamma - 2 - \min(q, d-1) \le (\gamma + 2s) + \frac{2s}{d} - q\frac{d+2s}{d} = l_q,$$

for $q \neq d-1$, and

$$\gamma - d \le (\gamma + 2s) + \frac{2s}{d} - (d-1)\frac{d+2s}{d} = l_{d-1}.$$

The proof is now complete. □

Lemma 5.3 (Estimate of the singular part)**.**

$$\mathcal{I}_2(v) \le C_{q,2}\mathfrak{a}\langle v\rangle^{\gamma-q} \tag{5.7}$$

with $C_{q,2} = \omega_{d-2}\frac{q(q+4)}{2}2^{\frac{3}{2}(q+2)}$.

In particular, for $q \le d+1$ *there holds* $\gamma - q \le l_q$ *where* l_q *is given in* (vi).

Proof. For the singular part, we do a Taylor expansion: there exists $\theta \in (0,1)$ such that

$$A(v+z) - A(v) - \nabla_v A(v+\theta z)\cdot z = \frac{1}{2}\mathrm{D}_v^2 A(v+\theta z)z\cdot z$$

for some $\theta \in (0,1)$. Moreover, $A(w) = \mathfrak{a}\mathcal{A}(|w|)$ with $\mathcal{A}(r) = (1+r^2)^{-q/2}$. In particular,

$$\frac{1}{\mathfrak{a}}D_v^2 A(w) = \mathcal{A}''(|w|)\frac{w}{|w|}\otimes\frac{w}{|w|} + \frac{\mathcal{A}'(|w|)}{|w|}\left(I - \frac{w}{|w|}\otimes\frac{w}{|w|}\right)$$

where I denotes the identity matrix. As far as derivatives of $\mathcal{A}$ are concerned, we have

$$\frac{|\mathcal{A}'(r)|}{r} \le q(1+r^2)^{-\frac{q+2}{2}} \quad \text{and} \quad |\mathcal{A}''(r)| \le q(q+3)(1+r^2)^{-\frac{q+2}{2}}.$$

This implies that for all $w, \xi \in \mathbb{R}^d$, we have

$$D_v^2 A(w)\xi\cdot\xi \le q(q+4)\mathfrak{a}\langle w\rangle^{-q-2}|\xi|^2.$$

In particular,

$$|A(v+z) - A(v) - \nabla_v A(v+\theta z)\cdot z| \le \frac{q(q+4)}{2}\mathfrak{a}\langle v+\theta z\rangle^{-q-2}|z|^2.$$

We further note for $|z| \le \frac{1}{2}\langle v\rangle$,

$$\langle v+\theta z\rangle \ge \frac{1}{2\sqrt{2}}\langle v\rangle. \tag{5.8}$$

Indeed,

$$\langle v+\theta z\rangle \ge \frac{1}{\sqrt{2}}(1+|v+\theta z|) \ge \frac{1}{\sqrt{2}}(1+|v|-|z|) \ge \frac{1}{\sqrt{2}}(1+|v|-\frac{1}{2}\langle v\rangle) \ge \frac{1}{2\sqrt{2}}\langle v\rangle.$$

We thus have,

$$|A(v+z) - A(v) - \nabla_v A(v+\theta z)\cdot z| \le C_{q,2}\mathfrak{a}\langle v\rangle^{-q-2}|z|^2$$

with $C_{q,2} = \frac{q(q+4)}{2}2^{\frac{3}{2}(q+2)}$. Due to the symmetry (2.6) and the fact that $\gamma + 2s > 0$, we thus obtain

$$\begin{aligned}
\mathcal{I}_2(v) &= \int_{\mathbb{R}^d} f(v_*')\,|v-v_*'|^{\gamma+2s}\int_{z\in(v-v_*')^\perp}\chi_{|z|<\frac{1}{2}\langle v\rangle}\frac{(A(v+z)-A(v))}{|z|^{d+2s-1}}\tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v_* \\
&\le C_{q,2}\tilde{b}_+\mathfrak{a}\langle v\rangle^{-q-2}\int_{\mathbb{R}^d} f(v_*')\,|v-v_*'|^{\gamma+2s}\int_{z\in(v-v_*')^\perp}\chi_{|z|<\frac{1}{2}\langle v\rangle}\,|z|^{2-d-2s+1}\,\mathrm{d}z\,\mathrm{d}v_* \\
&\le \omega_{d-2}C_{q,2}\tilde{b}_+\mathfrak{a}\langle v\rangle^{-q-2}\left\{\int_{\mathbb{R}^d} f(v_*')\left(|v|^{\gamma+2s}+|v_*'|^{\gamma+2s}\right)\mathrm{d}v_*'\right\}\left\{\int_0^{\frac{1}{2}\langle v\rangle} r^{3-d-2s+d-2}\,\mathrm{d}r\right\} \\
&\le \omega_{d-2}C_{q,2}\mathfrak{a}\langle v\rangle^{-q-2}\langle v\rangle^{\gamma+2s}\langle v\rangle^{-2s+2}.
\end{aligned}$$

This implies (5.7). □

Proof of Lemma 5.1. Therefore, if we now denote by $\mathcal{E}_1$ the left hand side of (5.1), then (5.4) and (5.7) imply

$$\begin{aligned}\mathcal{E}_1 &= \int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa\right)^{q_0-1}(v)\mathcal{I}(v)\,\mathrm{d}v = \int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa\right)^{q_0-1}(v)\left[\mathcal{I}_1(v)+\mathcal{I}_2(v)\right]\,\mathrm{d}v \\ &\leq C\mathfrak{a}\int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa\right)^{q_0-1}(v)\langle v\rangle^{l_q}\,\mathrm{d}v.\end{aligned}$$

This concludes the proof. □

5.2. **Large decay exponents.** For large q's, we need a better estimate than the one obtained in Lemma 5.1 above in order to establish the propagation of decay.

Lemma 5.4 (Error term in the velocity variable for large q's)**.** *Let $q > d+1$ and $A := \mathfrak{a}\langle v\rangle^{-q}$ for some $\mathfrak{a} \geq 0$. We consider a function f satisfying (1.3). Assume there is an exponent $q_* > d+\gamma+2s$ and a function on time $\mathfrak{b}_* = \mathfrak{b}_*(t) > 0$ depending only on $m_0, M_0, E_0, s, d, \gamma$ and q_* such that*

$$f(v) \leq \mathfrak{b}_*\langle v\rangle^{-q_*}. \tag{5.9}$$

Then there exists C_{1,q_*} depending on $\gamma, s, d, q_*, m_0, M_0, E_0, H_0$ such that

$$\begin{aligned}&\iint_{\mathbb{R}^{2d}} \left((f-A)_+ \wedge \kappa\right)^{q_0-1}(v)\left(A(v')-A(v)\right)K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v \\ &\qquad \leq C_{1,q_*}\mathfrak{a}\left(\mathfrak{b}_*\int_{\mathbb{R}^d}\left((f-A)_+ \wedge \kappa\right)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v + \int_{\mathbb{R}^d}\left((f-A)_+ \wedge \kappa\right)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v\right).\end{aligned} \tag{5.10}$$

Remark 5.5 (Growth of C_{1,q_*} with q_*). We will see that the constant C_{1,q_*} grows like 8^{q_*} (up to a multiplicative constant).

The proof of this lemma follows the same steps as the one of Lemma 5.1 above. We bound the left hand side of (5.10) by splitting the inner integral in v' into the non-singular and the singular parts as we did above in (5.2). For the singular part, we can use Lemma 5.3. For the non-singular part, we need to improve Lemma 5.2. We recall from (5.2) that the non-singular part $\mathcal{I}_1$ is given by

$$\mathcal{I}_1(v) = \int_{\mathbb{R}^d} f(v'_*)\,|v-v'_*|^{\gamma+2s}\left\{\int_{z\in(v-v'_*)^\perp}\chi_{|z|>\frac{1}{2}\langle v\rangle}\frac{(A(v+z)-A(v))}{|z|^{d+2s-1}}\tilde{b}(\cos\theta)\,\mathrm{d}z\right\}\mathrm{d}v'_*,$$

for which the following estimate holds.

Lemma 5.6 (Estimate of the non-singular part for large q's)**.** *Let $q > d+1$ and $A(t,v) = \mathfrak{a}(t)\langle v\rangle^{-q}$. Assume there is an exponent $q_* > d+\gamma+2s$ and a function $\mathfrak{b}_* = \mathfrak{b}_*(t) > 0$ depending only on $m_0, M_0, E_0, s, d, \gamma$ and q_* such that*

$$f(v) \leq \mathfrak{b}_*\langle v\rangle^{-q_*}.$$

Then there is a constant C depending on γ, s, d, q_* such that

$$\mathcal{I}_1(v) \leq C\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{\gamma-q_*}. \tag{5.11}$$

Proof. To bound $\mathcal{I}_1$, we distinguish the cases $\frac{1}{8}|v| > |v'_*|$ and $\frac{1}{8}|v| < |v'_*|$, and we denote these parts by $\mathcal{I}_{1,1}$ and $\mathcal{I}_{1,2}$, respectively. We note as above that $\mathcal{I}_{1,1} < 0$, see (5.5). For $\mathcal{I}_{1,2}$ we find

$$\mathcal{I}_{1,2}(v) = \int_{\frac{1}{8}|v|<|v'_*|} f(v'_*)\,|v-v'_*|^{\gamma+2s}\int_{z\in(v-v'_*)^\perp}\chi_{|z|>\frac{1}{2}\langle v\rangle}\frac{(A(v+z)-A(v))}{|z|^{d+2s-1}}\tilde{b}(\cos\theta)\,\mathrm{d}z\,\mathrm{d}v'_*$$

we change the order of integration

$$= \int_{z\in\mathbb{R}^d}\chi_{|z|>\frac{1}{2}\langle v\rangle}\frac{(A(v+z)-A(v))}{|z|^{d+2s}}\left\{\int_{v'_*\in v+z^\perp}\chi_{\frac{1}{8}|v|<|v'_*|}f(v'_*)\,|v-v'_*|^{\gamma+2s+1}\,\tilde{b}(\cos\theta)\,\mathrm{d}v'_*\right\}\mathrm{d}z$$

we use that $\tilde{b}(\cos\theta) \le \tilde{b}_+$ and $A(v) \ge 0$, $|z| > \langle v\rangle/2$, $|v| \le 8\,|v'_*|$ and $f(v'_*) \le \mathfrak{b}_*\langle v'_*\rangle^{-q_*}$

$$\begin{aligned}
&\le 2^{d+2s}\tilde{b}_+\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{-d-2s}\int_{z\in\mathbb{R}^d}\chi_{|z|>\frac{1}{2}\langle v\rangle}\langle v+z\rangle^{-q}\int_{v'_*\in v+z^\perp}\chi_{\frac{1}{8}|v|<|v'_*|}\langle v'_*\rangle^{-q_*}(9\langle v'_*\rangle)^{\gamma+2s+1}\,\mathrm{d}v'_*\,\mathrm{d}z\\
&\le 2^{d+2s}\omega_{d-2}9^{\gamma+2s+1}\tilde{b}_+\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{-d-2s}\int_{z\in\mathbb{R}^d}\chi_{|z|>\frac{1}{2}\langle v\rangle}\langle v+z\rangle^{-q}\int_{\frac{1}{8}\langle v\rangle}^{\infty} r^{\gamma+2s+1-q_*+d-2}\,\mathrm{d}r\,\mathrm{d}z\\
&\le C\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{-d-2s}\langle v\rangle^{\gamma+2s+d-q_*}\int_{z\in\mathbb{R}^d}\langle v+z\rangle^{-q}\,\mathrm{d}z,
\end{aligned}$$

with $C = 2^{d+2s}\omega_{d-2}9^{\gamma+2s+1}\tilde{b}_+8^{q_*-\gamma-2s-d}$, provided that $q_* > d+\gamma+2s$. Note that $\langle\cdot\rangle^{-q} \in L^1(\mathbb{R}^d)$ for $q > d$. Thus

$$\mathcal{I}_{1,2}(v) \le C\mathfrak{a}\mathfrak{b}_*\langle v\rangle^{\gamma-q_*}. \qquad \square$$

We use this to prove Lemma 5.4.

Proof of Lemma 5.4. We use Lemma 5.6 for the non-singular part $\mathcal{I}_1$, and Lemma 5.3 for the singular part $\mathcal{I}_2$, and find

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\,(A(v')-A(v))\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\\
&\quad=\int_{\mathbb{R}^d}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\big[\mathcal{I}_1(v)+\mathcal{I}_2(v)\big]\,\mathrm{d}v\\
&\quad\le C\mathfrak{a}\mathfrak{b}_*\int_{\mathbb{R}^d}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v + C\mathfrak{a}\int_{\mathbb{R}^d}\left((f-A)_+\wedge\kappa\right)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v. \qquad \square
\end{aligned}$$

6. Remaining error terms for hard potentials

In the case of hard potentials, that is to say in the case where $\gamma \ge 0$, we can treat error terms appearing in the right hand side of (3.2) all together. More precisely, we can estimate,

$$\int\Phi(f)\,(f*|\cdot|^\gamma)\,\mathrm{d}v = \int\left((q_0-1)\left((f-A)_+\wedge\kappa\right)^{q_0}(v) + q_0A\Big(\left((f-A)_+\wedge\kappa\right)^{q_0-1}\Big)(v)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\right.$$

with Φ from (3.3). Note that the case $\gamma \ge 0$ is significantly simpler than $\gamma < 0$.

6.1. Generation of decay.

We start with an estimate that is used to prove the generation of moments in the case of hard potentials.

Lemma 6.1 (Error terms for hard potentials)**.** *Let $f : \mathbb{R}^d \to [0,\infty)$ have finite mass M_0 and energy E_0 and let $q \in [0, d+1]$. There exists a constant $C_{\mathrm{hard}} > 0$ depending on M_0, E_0, d such that for any $\varepsilon \in (0,1)$ there holds*

$$\begin{aligned}
(6.1)\qquad &\int_{\mathbb{R}^d}\Phi(f)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\\
&\qquad\le \varepsilon\left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)} + C_{\mathrm{hard}}\,(\mathfrak{a}+C(\varepsilon))\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{\gamma-q}(\mathbb{R}^d)}.
\end{aligned}$$

Proof. We first remark that

$$\int_{\mathbb{R}^d} f(w)\,|v-w|^\gamma\,\mathrm{d}w \le C_\gamma\int_{\mathbb{R}^d} f(w)(\langle v\rangle^\gamma + |w|^2)\,\mathrm{d}w$$

for some constant C_γ only depending on γ. We used that for $\gamma \in [0,1]$, $(a+b)^\gamma \le C_\gamma((a^2+1)^{\frac{\gamma}{2}} + b^2)$. Let $C_{\gamma,0}$ denote $C_\gamma(M_0+E_0)$. We have,

$$\begin{aligned}
\int_{\mathbb{R}^d}\Phi(f)\,(f*|\cdot|^\gamma)\,\mathrm{d}v &\le C_{\gamma,0}\int_{\mathbb{R}^d}\Phi(f)\langle v\rangle^\gamma\,\mathrm{d}v\\
&\le q_0C_{\gamma,0}\int_{\mathbb{R}^d}\Big(\mathfrak{a}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q} + \big((f-A)_+\wedge\kappa\big)^{q_0}\langle v\rangle^\gamma\Big)\,\mathrm{d}v.
\end{aligned}$$

Using the interpolation from Lemma 2.7 with $k_3 = \gamma$ and $\iota = 1$, there holds for $m_3 = \gamma - (d+1) \le \gamma - q$, (recall $q \le d+1$) and any $\varepsilon \in (0,1)$

$$\int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0} \langle v\rangle^{\gamma} \,\mathrm{d}v \le \varepsilon \left\| \big((f-A)_+ \wedge \kappa\big)^{q_0} \right\|_{L^{p_0}_{k_0}} + C(\varepsilon, E_0) \left(\int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1} (v) \langle v\rangle^{\gamma-q} \,\mathrm{d}v \right).$$

This yields the claim (6.1). □

6.2. **Error terms for large exponents.** For large q's, the following estimate is used instead of Lemma 6.1 to establish the decay estimate.

Lemma 6.2 (Error terms for hard potentials and large q's)**.** *Let $\gamma > 0$. Let $f : \mathbb{R}^d \to [0,\infty)$ satisfy (1.3). Assume*

$$f \le \mathfrak{b}_* \langle v\rangle^{-q_*}$$

for some $q_* \ge 0$ and some $\mathfrak{b}_* = \mathfrak{b}_*(t) \ge 0$. We also suppose that $\mathfrak{a} \ge \left(\frac{2q_0 C_{\gamma,0}}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}} \mathfrak{b}_*$. Then, for any $q \ge 0$, there holds

$$\begin{aligned} \int_{\mathbb{R}^d} \Phi(f) \left(f * |\cdot|^{\gamma}\right) \,\mathrm{d}v \le & \frac{1}{2} \frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}} \mathfrak{a}^{1+2s} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \langle v\rangle^{\gamma+2s-q} \,\mathrm{d}v \\ & + q_0 C_{\gamma,0} \mathfrak{b}_* \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \langle v\rangle^{\gamma-q_*} \,\mathrm{d}v. \end{aligned} \tag{6.2}$$

Proof. As in the proof of Lemma 6.1, we have

$$\int_{\mathbb{R}^d} \Phi(f) \left(f * |\cdot|^{\gamma}\right) \,\mathrm{d}v \le q_0 C_{\gamma,0} \int_{\mathbb{R}^d} \Big(\mathfrak{a} \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \langle v\rangle^{\gamma-q} + \big((f-A)_+ \wedge \kappa\big)^{q_0} \langle v\rangle^{\gamma} \Big) \,\mathrm{d}v.$$

Then we remark first that $\gamma - q < \gamma + 2s - q$, and we thus bound for $\mathfrak{a} \ge \left(\frac{2q_0 C_{\gamma,0}}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}} \mathfrak{b}_*$,

$$q_0 C_{\gamma,0} \mathfrak{a} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \langle v\rangle^{\gamma-q} \,\mathrm{d}v \le \frac{C_{\mathrm{vget}}}{2} \frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \langle v\rangle^{\gamma+2s-q} \,\mathrm{d}v.$$

For the second term we use the decay assumption (5.9) and we find

$$q_0 C_{\gamma,0} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0} \langle v\rangle^{\gamma} \,\mathrm{d}v \le q_0 C_{\gamma,0} \mathfrak{b}_* \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1} \langle v\rangle^{\gamma-q_*} \,\mathrm{d}v.$$ □

7. Main error terms for soft potentials

We next explain how to estimate the following term in the case of soft potentials, *i.e.* when $\gamma < 0$,

$$\begin{aligned} & \iint_{\mathbb{R}^d \times \Omega} \Phi\big(f\big) \left(f * |\cdot|^{\gamma}\right) \,\mathrm{d}v \,\mathrm{d}x \\ & \qquad = \iint_{\mathbb{R}^d \times \Omega} \left((q_0 - 1) \Big((f-A)_+ \wedge \kappa \Big)^{q_0} (v) + q_0 A \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \right) (v) \left(f * |\cdot|^{\gamma}\right) \,\mathrm{d}v \,\mathrm{d}x. \end{aligned}$$

We will use the following auxiliary results.

7.1. Auxiliaries.

Lemma 7.1 (Convolution product)**.** *Let $\gamma < 0$ and $f : \mathbb{R}^d \to [0,\infty)$ have finite mass M_0. Let $A(t,v) = \mathfrak{a}\langle v\rangle^{-q}$ for some $q \ge 0$ and suppose, for all $t \ge 0$, $\mathfrak{a}(t) \ge 2^d \frac{M_0}{C_{q,3}}$ with $C_{q,3} = \frac{\omega_{d-1}(2\sqrt{2})^q}{(d+\gamma)}$. Then there exists $C_{\mathrm{conv}} > 0$ depending on d, γ, q and M_0, such that*

$$(f \wedge A) * |\cdot|^{\gamma} \le C_{\mathrm{conv}} \, \mathfrak{a}^{-\frac{\gamma}{d}} \langle v\rangle^{\gamma \min\left(1, \frac{q}{d}\right)}.$$

Proof. For $R > 0$ such that $R < \langle v\rangle/2$, we write

$$\begin{aligned} \int_{\mathbb{R}^d} (f \wedge A)(v-w) \, |w|^{\gamma} \,\mathrm{d}w &= \int_{B_R} (f \wedge A)(v-w) \, |w|^{\gamma} \,\mathrm{d}w + \int_{\mathbb{R}^d \setminus B_R} (f \wedge A)(v-w) \, |w|^{\gamma} \,\mathrm{d}w \\ &\le \mathfrak{a} \int_{B_R} \langle v-w\rangle^{-q} |w|^{\gamma} \,\mathrm{d}w + R^{\gamma} M_0 \end{aligned}$$

and for $R < \langle v\rangle/2$ and $w \in B_R$, we have $\langle v - w\rangle \geq \frac{1}{2\sqrt{2}}\langle v\rangle$ (argue as in (5.8)),

$$\leq \mathfrak{a}\,\omega_{d-1}(2\sqrt{2})^q\langle v\rangle^{-q}\frac{R^{d+\gamma}}{d+\gamma} + R^\gamma M_0 \leq C_{q,3}a\langle v\rangle^{-q}R^{d+\gamma} + M_0R^\gamma,$$

where

$$C_{q,3} = \frac{\omega_{d-1}(2\sqrt{2})^q}{(d+\gamma)}. \tag{7.1}$$

Choosing $R = \big(\frac{M_0}{C_{q,3}\mathfrak{a}}\big)^{\frac{1}{d}}\langle v\rangle^{\min(1,\frac{q}{d})} \leq \frac{1}{2}\langle v\rangle$ (since $\mathfrak{a} \geq 2^d\frac{M_0}{C_{q,3}}$) yields

$$\int_{\mathbb{R}^d}(f\wedge A)(v-w)\,|w|^\gamma\,\mathrm{d}w \leq (C_{q,3})^{-\frac{\gamma}{d}}M_0^{1+\frac{\gamma}{d}}\mathfrak{a}^{-\frac{\gamma}{d}}\langle v\rangle^{-q+(d+\gamma)\min(1,\frac{q}{d})} + (C_{q,3})^{-\frac{\gamma}{d}}M_0^{1+\frac{\gamma}{d}}\mathfrak{a}^{-\frac{\gamma}{d}}\langle v\rangle^{\gamma\min(1,\frac{q}{d})}$$
$$\leq 2(C_{q,3})^{-\frac{\gamma}{d}}M_0^{1+\frac{\gamma}{d}}\mathfrak{a}^{-\frac{\gamma}{d}}\langle v\rangle^{\gamma\min(1,\frac{q}{d})}. \qquad \square$$

We next collect auxiliary computations in the following lemma.

Lemma 7.2 (Auxiliary technical result)**.** *Let $\gamma \in (-2s, 0)$ and*

$$1 \leq q \leq 2 + \frac{d}{2s}k_0, \qquad l_1 = \min(2, q).$$

If we pick

$$\theta = \frac{d+2s}{d-2s}, \qquad \frac{1}{\beta} = \frac{d^2+4s^2}{(d+2s)^2}, \qquad \frac{1}{\beta'} = \frac{4sd}{(d+2s)^2}, \tag{7.2}$$

then

$$\frac{1}{\beta} + \frac{1}{\beta'} = 1, \qquad \frac{1}{\theta} + \frac{1}{\theta'} = 1, \qquad \gamma_2\theta = 1, \qquad \gamma_3\theta' = 1, \tag{7.3}$$

where

$$\gamma_1 := \frac{8s^2(d+\gamma)}{d(4sd+\gamma(d+2s))}, \qquad \gamma_2 := \frac{1}{\beta}\frac{(d-2s)(d+2s)}{d^2+4s^2}, \qquad \gamma_3 := \frac{1}{\beta}\frac{4s(d+2s)}{d^2+4s^2}. \tag{7.4}$$

Moreover,

$$\gamma_1\theta' < 1 + \frac{2s}{d}, \tag{7.5}$$

and there holds

$$m_1 \leq \frac{2}{\beta'} + \frac{m_3}{\beta}, \tag{7.6}$$

and

$$m_4 \leq \frac{2}{\beta'} + \frac{m_3}{\beta}, \tag{7.7}$$

with

$$m_1 = -\frac{4sd^2}{(d+2s)(4sd+\gamma(d+2s))}l_1 + \frac{\gamma d}{4sd+\gamma(d+2s)}k_0, \qquad m_3 = \frac{4sd}{d^2+4s^2}l_q + \frac{d(d-2s)}{d^2+4s^2}k_0,$$
$$m_4 = -2\frac{4sd}{(d+2s)(\gamma+2s)} - \frac{(2s-\gamma)d}{(d+2s)(\gamma+2s)}k_0.$$

Proof. The relations in (7.3) are verified by a straightforward computation. We then rewrite (7.5) as

$$\frac{2s(d+2s)(d+\gamma)}{d(4sd+\gamma(d+2s))} < 1 + \frac{2s}{d},$$

which holds for $\gamma + 2s > 0$.

It remains to check that (7.6) and (7.7) are satisfied. We rewrite (7.6) as

$$-\frac{4sd^2}{4sd+\gamma(d+2s)}l_1 + \frac{\gamma d(d+2s)}{4sd+\gamma(d+2s)}k_0 \leq 2\frac{4sd}{(d+2s)} + \frac{4sd}{(d+2s)}l_q + \frac{d(d-2s)}{(d+2s)}k_0.$$

Then using that $l_q = k_0 + 2\frac{2s}{d} - q\frac{d+2s}{d}$ we get

$$-\frac{4sd^2}{(4sd+\gamma(d+2s))}l_1 + \frac{\gamma d(d+2s)}{4sd+\gamma(d+2s)}k_0 \leq 8s - 4sq + dk_0,$$

or equivalently,

$$-d^2 l_1 \leq (2-q)(4sd+\gamma(d+2s)) + d^2 k_0.$$

If $2 \geq q$ and $k_0 \geq 0$ then (7.6) is satisfied since $l_1 \geq 0$.

If $2 \leq q$ and $k_0 \geq 0$ then $l_1 = 2$, and since $q \leq 2 + \frac{d}{2s}k_0$, (7.6) is satisfied if

$$-2d^2 \leq -\frac{d}{2s}k_0(4sd+\gamma(d+2s)) + d^2 k_0 = -d^2 k_0 - \frac{d}{2s}\gamma(d+2s)k_0,$$

which holds if

$$\left(\gamma + 2s - \frac{2s}{d}\right) = k_0 \leq 2,$$

which is true since $\gamma + 2s \leq 2$.

If $k_0 < 0$ then $q \leq 2 + \frac{d}{2s}k_0 < 2$ and in particular $l_1 = q$, so that (7.6) is satisfied if

$$0 \leq (2-q)(4sd+\gamma(d+2s)) + d^2 k_0 + d^2 q.$$

We remark that $4sd+\gamma(d+2s) = (\gamma+2s)d + 2s(\gamma+d) \geq 0$. Since $2 - q \geq \frac{d}{2s}(-k_0) \geq 0$ and $q \geq 1$, we have to check that

$$\left(\frac{\gamma}{d} + \frac{\gamma+2s}{2s}\right)k_0 \leq 1,$$

is satisfied. If $\frac{\gamma}{d} + \frac{\gamma+2s}{2s} \geq 0$ then (7.6) is satisfied. If $\frac{\gamma}{d} + \frac{\gamma+2s}{2s} \leq 0$, then we see that (7.6) holds as soon as

$$\gamma k_0 = \gamma\left(\gamma + 2s - \frac{2s}{d}\right) \leq d,$$

which is satisfied. Indeed, $0 < -\gamma < d$ and $s < 1$ ensures that $-\gamma\frac{2s}{d} \leq 2$. We conclude by recalling that $d \geq 2$. Thus in any case we see that (7.6) is satisfied for $1 \leq q \leq 2 + \frac{d}{2s}k_0$.

Finally, we rewrite (7.7) as

$$\begin{aligned}
&-2\frac{4sd}{(d+2s)(\gamma+2s)} - \frac{(2s-\gamma)d}{(d+2s)(\gamma+2s)}k_0 \leq 2\frac{4sd}{(d+2s)^2} + \frac{d^2+4s^2}{(d+2s)^2}\left(\frac{4sd}{d^2+4s^2}l_q + \frac{d(d-2s)}{d^2+4s^2}k_0\right),\\
&\Leftrightarrow -2\frac{4sd}{(\gamma+2s)} - \frac{(2s-\gamma)d}{(\gamma+2s)}k_0 \leq 2\frac{4sd}{(d+2s)} + \frac{1}{(d+2s)}\left(4sd\left[k_0 + 2\frac{2s}{d} - q\frac{d+2s}{d}\right] + d(d-2s)k_0\right),\\
&\Leftrightarrow 4sq \leq 8s + 2\frac{4sd}{(\gamma+2s)} + dk_0 + \frac{(2s-\gamma)d}{(\gamma+2s)}k_0.
\end{aligned}$$

We thus see, that for $q \leq 2 + \frac{d}{2s}k_0$, (7.7) is satisfied if

$$\gamma k_0 = \gamma\left(\gamma + 2s - \frac{2s}{d}\right) \leq 4s.$$

This is satisfied since $-\gamma \leq d$. □

7.2. **Splitting.** We split the error term involving $\Phi(f)$ by writing $f = (f \wedge A) + (f-A)_+ \wedge \kappa + (f - A - \kappa)_+$,

$$\int_{\mathbb{R}^d} \Phi\big(f\big)\,(f * |\cdot|^\gamma)\,\mathrm{d}v = \mathcal{E}_2 + \mathcal{E}_4 + \mathcal{E}_3 + \mathcal{E}_5$$

with

$$\text{(7.8)}\qquad \begin{cases}
\mathcal{E}_2 &= q_0 c_b \int A\Big((f-A)_+ \wedge \kappa\Big)^{q_0-1}\big((f\wedge A) * |\cdot|^\gamma\big)\,\mathrm{d}v\\
\mathcal{E}_3 &= q_0 c_b \int A\Big((f-A)_+ \wedge \kappa\Big)_+^{q_0-1}\big((f-A)_+ \wedge \kappa * |\cdot|^\gamma\big)\,\mathrm{d}v\\
\mathcal{E}_4 &= (q_0-1) c_b \int \Big((f-A)_+ \wedge \kappa\Big)^{q_0}\,(f * |\cdot|^\gamma)\,\mathrm{d}v\\
\mathcal{E}_5 &= q_0 c_b \int A\Big((f-A)_+ \wedge \kappa\Big)_+^{q_0-1}\big((f-A-\kappa)_+ * |\cdot|^\gamma\big)\,\mathrm{d}v.
\end{cases}$$

7.3. **The term $\mathcal{E}_2$.**

Lemma 7.3 (Estimate of $\mathcal{E}_2$). *Let $\gamma \in (-d, 0]$ and $\gamma + 2s > 0$ and $q \in [0, d+1]$. Then*

$$\mathcal{E}_2 \le \frac{1}{2} C_{\mathrm{get}}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \left((f-A)_+ \wedge \kappa\right)^{q_0-1} \right\|_{L^1_{l_q}}, \tag{7.9}$$

as soon as $\mathfrak{a}$ satisfies for all $t \ge 0$

$$\mathfrak{a}(t) \ge \underline{a}^{(2)} = \max\left(\frac{2^d M_0}{C_{q,3}}, \left(\frac{2}{C_{\mathrm{get}}}\right)^{\frac{d}{\gamma+2s}}, a_{get} \right) \tag{7.10}$$

(with $a_{get}, C_{\mathrm{get}}$ from Proposition 4.4).

Proof. By Lemma 7.1 (see the assumption on $\mathfrak{a}$) there holds

$$\mathcal{E}_2 \le C_{\mathrm{conv}}\, q_0 c_b \mathfrak{a}^{-\frac{\gamma}{d}} \int_{\mathbb{R}^d} A\Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma \min(1, \frac{q}{d})}\, \mathrm{d}v.$$

It suffices to check that $1 - \frac{\gamma}{d} < 1 + \frac{2s}{d}$, which is true since $\gamma + 2s > 0$, and that $\gamma \min(1, \frac{q}{d}) - q \le l_q$, which again is true since $q \le d+1$. Thus

$$\mathfrak{a}^{1-\frac{\gamma}{d}} \int \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{\gamma \min(1,\frac{q}{d}) - q}\, \mathrm{d}v \le \frac{1}{2} C_{\mathrm{get}}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \right\|_{L^1_{l_q}}$$

where C_{get} is the constant from Proposition 4.4. We used the assumption on $\mathfrak{a}$ from the statement. □

7.4. **The term $\mathcal{E}_3$.** We consider next $\mathcal{E}_3$ defined in (7.8).

Lemma 7.4 (Estimate of $\mathcal{E}_3$). *Let $\gamma < 0$ and $q \in [0, d+1+\frac{d}{2s}\gamma]$. Then there exists a constant $\underline{a}^{(3)} > 0$, only depending on the parameters $d, \gamma, s, 2 \wedge q$ and the constants c_b, a_{get}, C_{cor} and C_{get}, such that*

$$\mathcal{E}_3 \le \frac{1}{2} C_{\mathrm{cor}} \left\| \Big((f-A)_+ \wedge \kappa\Big)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{4} C_{\mathrm{get}}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \right\|_{L^1_{l_q}} \tag{7.11}$$

(with C_{cor}, a_{get} and C_{get} from Propositions 4.2 and 4.4) as soon as $a \ge \underline{a}^{(3)}$.

To prove this lemma, we show that we can split $\mathcal{E}_3$ as follows.

Lemma 7.5 (Splitting $\mathcal{E}_3$). *We have $\mathcal{E}_3 \le \mathcal{E}_3^{(i)} + \mathcal{E}_3^{(ii)}$ with*

$$\mathcal{E}_3^{(i)} = C_3^{(i)} \left\| (f \wedge A)\langle\cdot\rangle^{l_1} \right\|_{L^{p_1}} \left\| \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle\cdot\rangle^{-\frac{2sl_1}{d+2s}} \right\|_{L^{p_2}} \left\| \Big((f-A)_+ \wedge \kappa\Big)\langle\cdot\rangle^{-\frac{dl_1}{d+2s}} \right\|_{L^{q_3}},$$
$$\mathcal{E}_3^{(ii)} = q_0 c_b M_0 \mathfrak{a} \left\| \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \right\|_{L^1_{-q}}$$

where $l_1 = \min(q, 2)$, and $p_1, p_2, p_3, q_3 \in [1, +\infty]$ satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \qquad 1 + \frac{1}{p_3} = \frac{1}{q_3} + \frac{-\gamma}{d}. \tag{7.12}$$

The constant $C_3^{(i)}$ is given by $2^{\frac{dl_1}{2(d+2s)}} q_0 c_b \left\| |\cdot|^\gamma \right\|_{L^{\frac{-\gamma}{d},\infty}}$.

Proof. We use that for $w \in B_1$ and $v \in \mathbb{R}^d$, we have,

$$\langle v - w\rangle^2 \le 1 + |v|^2 + |w|^2 \le 2 + |v|^2 \le 2\langle v\rangle^2.$$

We now distinguish two domains of integration: $|w| \le 1$ and $|w| \ge 1$. We use the fact that $\gamma < 0$ to get $|w|^\gamma \le 1$ for $|w| \ge 1$ and we write for $l_1 = \min(q, 2)$,

$$\begin{aligned}
\mathcal{E}_3 &= q_0 c_b \int_{\mathbb{R}^d} (f \wedge A)\big((f-A)_+ \wedge \kappa\big)^{q_0-1} \left(\int_{B_1} \big((f-A)_+ \wedge \kappa\big)(v-w)\,|w|^\gamma \,\mathrm{d}w \right) \mathrm{d}v \\
&\quad + q_0 c_b \int_{\mathbb{R}^d} (f \wedge A)\big((f-A)_+ \wedge \kappa\big)^{q_0-1} \left(\int_{\mathbb{R}^d \setminus B_1} \big((f-A)_+ \wedge \kappa\big)(v-w)\,|w|^\gamma \,\mathrm{d}w \right) \mathrm{d}v \\
&\le q_0 c_b \int_{\mathbb{R}^d} (f \wedge A)\langle v\rangle^{l_1}\langle v\rangle^{-l_1}\big((f-A)_+ \wedge \kappa\big)^{q_0-1} \left(\int_{B_1} \big((f-A)_+ \wedge \kappa\big)(v-w)\,|w|^\gamma \,\mathrm{d}w \right) \mathrm{d}v \\
&\quad + q_0 c_b M_0 \mathfrak{a} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v \\
&\le 2^{\frac{dl_1}{2(d+2s)}} q_0 c_b \int_{\mathbb{R}^d} (f \wedge A)\langle v\rangle^{l_1}\big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-\frac{2sl_1}{d+2s}} \\
&\qquad\qquad \times \left(\int_{B_1} \langle v-w\rangle^{-\frac{dl_1}{d+2s}} \big((f-A)_+ \wedge \kappa\big)(v-w)\,|w|^\gamma \,\mathrm{d}w \right) \mathrm{d}v \\
&\quad + q_0 c_b M_0 \mathfrak{a} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v
\end{aligned}$$

we use weak Young's inequality,

$$\begin{aligned}
&\le 2^{\frac{dl_1}{2(d+2s)}} q_0 c_b \,\big\| |\cdot|^\gamma \big\|_{L^{\frac{-\gamma}{d},\infty}} \big\|(f \wedge A)\langle\cdot\rangle^{l_1}\big\|_{L^{p_1}} \Big\|\big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle\cdot\rangle^{-\frac{2sl_1}{d+2s}}\Big\|_{L^{p_2}} \\
&\qquad \times \Big\|\big((f-A)_+ \wedge \kappa\big)\langle\cdot\rangle^{-\frac{dl_1}{d+2s}}\Big\|_{L^{q_3}} \\
&\quad + q_0 c_b M_0 \mathfrak{a} \int_{\mathbb{R}^d} \big((f-A)_+ \wedge \kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v. \qquad \square
\end{aligned}$$

Then we estimate each piece separately.

Lemma 7.6 (Estimate of $\mathcal{E}_3^{(ii)}$)**.**

$$\mathcal{E}_3^{(ii)} \le \frac{1}{8} C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}} \Big\|\big((f-A)_+ \wedge \kappa\big)^{q_0-1}\Big\|_{L^1_{l_q}} \tag{7.13}$$

as soon as $\mathfrak{a} \ge \underline{a}^{(3,ii)} := \left(\frac{8q_0c_bM_0}{C_{\mathrm{get}}}\right)^{\frac{d}{2s}}$.

Proof. We note that $-q \le l_q$ for $q \le 2 + \frac{d}{2s}k_0$, and $1 < 1 + \frac{2s}{d}$, so that the conclusion holds true. $\square$

Lemma 7.7 (Estimate of $\mathcal{E}_3^{(i)}$)**.** *There exists* $\underline{a}^{(3,i)}$*, that only depends on* d, γ, s, E_0, l_1*,* c_b*,* C_{cor} *and* C_{get} *(see Propositions 4.1 and 4.4), such that*

$$\mathcal{E}_3^{(i)} \le \frac{1}{2} C_{\mathrm{cor}} \big\|\big((f-A)_+ \wedge \kappa\big)^{q_0}\big\|_{L^{p_0}_{k_0}} + \frac{1}{8} C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}} \Big\|\big((f-A)_+ \wedge \kappa\big)^{q_0-1}\Big\|_{L^1_{l_q}}$$

as soon as $\mathfrak{a} \ge \underline{a}^{(3,i)}$.

Proof. For $\mathcal{E}_3^{(i)}$, we first notice that,

$$\big\|(f \wedge A)\langle\cdot\rangle^{l_1}\big\|_{L^{p_1}} \le \big\|(f \wedge A)\langle\cdot\rangle^{l_1}\big\|_{L^1}^{\frac{1}{p_1}} \big\|(f \wedge A)\langle\cdot\rangle^{l_1}\big\|_{L^\infty}^{1-\frac{1}{p_1}} \le E_0^{\frac{1}{p_1}}\,\mathfrak{a}^{1-\frac{1}{p_1}}.$$

Next we choose $p_3 = \infty$ and $p_2 = \frac{q_3}{q_0-1} \ge 1$ in (7.12), which implies

$$\frac{1}{q_3} = \frac{d+\gamma}{d}, \qquad \frac{1}{p_2} = \frac{2s(d+\gamma)}{d^2}, \qquad 1 - \frac{1}{p_1} = \frac{2s(d+\gamma)}{d^2}.$$

Thus

$$\begin{aligned}\mathcal{E}_3^{(i)} &= C_3^{(i)} \left\|(f\wedge A)\langle\cdot\rangle^{l_1}\right\|_{L^{p_1}} \left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\langle\cdot\rangle^{-\frac{2sl_1}{d+2s}}\right\|_{L^{p_2}} \left\|\left((f-A)_+\wedge\kappa\right)\langle\cdot\rangle^{-\frac{dl_1}{d+2s}}\right\|_{L^{q_3}} \\ &\leq C_3^{(i)} E_0^{\frac{1}{p_1}} \mathfrak{a}^{\frac{2s(d+\gamma)}{d^2}} \left(\int \left((f-A)_+\wedge\kappa\right)^{\frac{d}{d+\gamma}} \langle v\rangle^{-\frac{d^2 l_1}{(d+2s)(d+\gamma)}} \,\mathrm{d}v\right)^{\frac{(d+\gamma)(d+2s)}{d^2}}.\end{aligned}$$

Step 1: First interpolation. Then we interpolate with Lemma 2.6 for $k_1 = -\frac{d^2 l_1}{(d+2s)(d+\gamma)}$

$$\int_{\mathbb{R}^d} \left((f-A)_+\wedge\kappa\right)^{\frac{d}{d+\gamma}} \langle v\rangle^{k_1} \,\mathrm{d}v \leq \left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\alpha_1} \left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}}^{\alpha_2},$$

where

$$\begin{aligned}&\alpha_1 = \frac{-\gamma d}{4s(d+\gamma)}, \qquad \alpha_2 = \frac{4sd+\gamma(d+2s)}{4s(d+\gamma)}, \\ &m_1 = -\frac{4sd^2}{(d+2s)(4sd+\gamma(d+2s))} l_1 + \frac{\gamma d}{4sd+\gamma(d+2s)} k_0.\end{aligned}$$

We can check that $\frac{-\gamma(d+2s)}{4sd}$ since $-\gamma \in [0,d]$ and $\gamma+2s\geq 0$. Together with Young's inequality, this implies that there exists $C = C(C_{\mathrm{cor}}, c_b, E_0, l_1, \gamma, d, s)$ such that,

$$\begin{aligned}\mathcal{E}_3^{(i)} &\leq C_3^{(i)} E_0^{\frac{1}{p_1}} \mathfrak{a}^{\frac{2s(d+\gamma)}{d^2}} \left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\frac{-\gamma(d+2s)}{4sd}} \left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}}^{\frac{(d+2s)(4sd+\gamma(d+2s))}{4sd^2}} \\ &\leq \frac{1}{4} C_{\mathrm{cor}} \left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}} + C\mathfrak{a}^{\gamma_1} \left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}}^{\frac{d+2s}{d}}\end{aligned}$$

where γ_1 is given in Lemma 7.2, see (7.4).

Step 2: Second interpolation. Now we pick $\beta, \beta' \geq 1$ as in (7.2), so that (7.6) implies together with Hölder's inequality

$$\begin{aligned}\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}} &\leq \left\|(f-A)_+\wedge\kappa\right\|_{L^1_{\frac{2}{\beta_1'}+\frac{m_3}{\beta_1}}} \leq \left\|(f-A)_+\wedge\kappa\right\|_{L^1_2}^{\frac{1}{\beta_1'}} \left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_3}}^{\frac{1}{\beta_1}} \\ &\leq E_0^{\frac{1}{\beta_1'}} \left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_3}}^{\frac{1}{\beta_1}}.\end{aligned}$$

To bound the right hand side of the previous inequality, we interpolate again, now with Lemma 2.5 and $k_2 = m_3$,

$$\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_3}} \leq \left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\alpha_3} \left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{m_2}}^{\alpha_4},$$

where

$$\alpha_3 = \frac{d(d-2s)}{d^2+4s^2}, \qquad \alpha_4 = \frac{4sd}{d^2+4s^2}, \qquad m_2 = \frac{4s^2+d^2}{4sd}\left[m_3 - \frac{d(d-2s)}{d^2+4s^2} k_0\right].$$

Note that the choice of m_3 in Lemma 7.2 is such that $m_2 = l_q$, that is $m_3 = \frac{4sd}{d^2+4s^2} l_q + \frac{d(d-2s)}{d^2+4s^2} k_0$. In particular, we find

$$\begin{aligned}\left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_1}}^{\frac{d+2s}{d}} &\leq E_0^{\frac{1}{\beta'}\frac{d+2s}{d}} \left\|(f-A)_+\wedge\kappa\right\|_{L^1_{m_3}}^{\frac{1}{\beta}\frac{d+2s}{d}} \\ &\leq E_0^{\frac{1}{\beta'}\frac{d+2s}{d}} \left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\gamma_2} \left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}}^{\gamma_3}\end{aligned}$$

where γ_2 and γ_3 also come from Lemma 7.2, see (7.4).

Step 3: Young's inequality. We combine Step 1 and Step 2 with Young's inequality, so that for $\theta \geq 1$ and any $\varepsilon \in (0,1)$

$$\begin{aligned}\mathcal{E}_3^{(i)} &\leq \frac{1}{4} C_{\rm cor} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + C\mathfrak{a}^{\gamma_1} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}}^{\gamma_2} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0-1} \right\|_{L^1_{l_q}}^{\gamma_3} \\ &\leq \frac{1}{4} C_{\rm cor} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{4} C_{\rm cor} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + \bar{C}\mathfrak{a}^{\gamma_1\theta'} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0-1} \right\|_{L^1_{l_q}}^{\gamma_3\theta'}\end{aligned}$$

with $\bar{C} = \bar{C}(C_{\rm cor}, c_b, E_0, l_1, \gamma, d, s)$ and $\theta' = \frac{\theta}{1-\theta}$, see Lemma 7.2 again. We used that $\gamma_2\theta = 1$, see (7.3). To complete the proof of Lemma 7.7, we use (7.5) from Lemma 7.2. □

Proof of Lemma 7.4. Combine Lemmas 7.5, 7.6 and 7.7. □

7.5. **The term $\mathcal{E}_4$.** The estimate of $\mathcal{E}_4$ defined in (7.8) follows with similar methods.

Lemma 7.8 (Estimate of $\mathcal{E}_4$)**.** *Let $f : [0,\infty) \to \mathbb{R}$ have finite mass M_0 and finite 2-moment E_0, and let $q \in [0, \frac{d}{2s} k_0]$.*

Then there exists a constant $\underline{a}^{(4)} > 0$, only depending on the parameters $d, \gamma, s, \min(2,q)$ and the constants c_b, $C_{\rm cor}$ and $C_{\rm get}$ (see Propositions 4.1 and 4.4), such that

$$\mathcal{E}_4 \leq \frac{1}{4} C_{\rm cor} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{8} C_{\rm get}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \right\|_{L^1_{l_q}} \tag{7.14}$$

(with $C_{\rm cor}$ and $C_{\rm get}$ from Propositions 4.2 and 4.4) as soon as $a \geq \underline{a}^{(4)}$.

To prove this lemma, we split $\mathcal{E}_4$ as follows.

Lemma 7.9 (Splitting $\mathcal{E}_4$)**.** *Let $p_4, p_5, q_5 \in [1, +\infty]$ satisfy*

$$\frac{1}{p_4} + \frac{1}{p_5} = 1, \qquad 1 + \frac{1}{p_5} = \frac{1}{q_5} + \frac{-\gamma}{d}. \tag{7.15}$$

We have $\mathcal{E}_4 \leq \mathcal{E}_4{}^{(i)} + \mathcal{E}_4{}^{(ii)}$ with

$$\begin{aligned}\mathcal{E}_4{}^{(i)} &= C_4^{(i)} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \langle\cdot\rangle^{-2} \right\|_{L^{p_4}} \left\| (f-A)_+ \langle\cdot\rangle^2 \right\|_{L^{q_5}}, \\ \mathcal{E}_4{}^{(ii)} &= M_0 (q_0 - 1) c_b \left\| (f-A)_+ \wedge \kappa \right\|_{L^{q_0}}^{q_0},\end{aligned}$$

with $C_4^{(i)} = 4(q_0-1)c_b \left\| |\cdot|^\gamma \right\|_{L^{\frac{-\gamma}{d},\infty}}$.

Proof. We use for $w \in B_1$ and $v \in \mathbb{R}^d$

$$|v| \leq |v-w| + |w| \leq |v-w| + 1,$$

which implies $\langle v \rangle \leq 2\langle v - w \rangle$, so that distinguishing two domains of integration, $|w| \leq 1$ and $|w| \geq 1$, we find

$$\begin{aligned}\mathcal{E}_4 &= (q_0-1)c_b \int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right)^{q_0} \langle v \rangle^{-2} \langle v \rangle^2 \left(\int_{B_1} f(v-w) \, |w|^\gamma \, \mathrm{d}w \right) \mathrm{d}v \\ &\quad + (q_0-1)c_b \int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right)^{q_0} \left(\int_{\mathbb{R}^d \setminus B_1} f(v-w) \, |w|^\gamma \, \mathrm{d}w \right) \mathrm{d}v \\ &\leq 4(q_0-1)c_b \int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right)^{q_0} \langle v \rangle^{-2} \left(\int_{B_1} \langle v-w \rangle^2 f(v-w) \, |w|^\gamma \, \mathrm{d}w \right) \mathrm{d}v \\ &\quad + M_0 (q_0-1) c_b \int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right)^{q_0} \mathrm{d}v \\ &\leq 4(q_0-1)c_b \left\| |\cdot|^\gamma \right\|_{L^{\frac{-\gamma}{d},\infty}} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \langle\cdot\rangle^{-2} \right\|_{L^{p_4}} \left\| f \langle\cdot\rangle^2 \right\|_{L^{q_5}} \\ &\quad + M_0 (q_0-1) c_b \left\| (f-A)_+ \wedge \kappa \right\|_{L^{q_0}}^{q_0}.\end{aligned}$$

□

Again we estimate each piece separately.

Lemma 7.10 (Estimate of $\mathcal{E}_4^{(ii)}$). *There holds*

$$\mathcal{E}_4^{(ii)} \le \frac{1}{8} C_{\rm cor} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{16} C_{\rm get}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0-1} \right\|_{L^1_{l_q}}$$

(with $C_{\rm cor}$ and $C_{\rm get}$ from Propositions 4.1 and 4.4) as soon as $\mathfrak{a} \ge \underline{a}^{(4,ii)}$ with $\underline{a}^{(4,ii)} > 0$ only depending on $C_{\rm cor}, c_b, q_0, C_{\rm get}, d, \gamma, s, M_0$.

Proof. We use Lemma 2.7 with $k_3 = 0$ to get for some constant $C = C(C_{\rm cor}, q_0, M_0, c_b, d, \gamma, s)$,

$$\mathcal{E}_4^{(ii)} \le \frac{1}{8} C_{\rm cor} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + C \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0-1} \right\|_{L^1_{m_3}},$$

with $m_3 = -\frac{d}{2s} k_0 - 2$. For $q \le 2 + \frac{d}{2s} k_0$, the exponent m_3 satisfies $m_3 \le l_q$. □

Lemma 7.11 (Estimate of $\mathcal{E}_4^{(i)}$). *There holds*

$$\mathcal{E}_4^{(i)} \le \frac{1}{8} C_{\rm cor} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + \frac{1}{16} C_{\rm get}\, \mathfrak{a}^{1+\frac{2s}{d}} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0-1} \right\|_{L^1_{l_q}}$$

as soon as $\mathfrak{a} \ge \underline{a}^{(4,i)}$ with $\underline{a}^{(4,i)} > 0$ only depending on $C_{\rm cor}, c_b, C_{\rm get}, d, \gamma, s, M_0$ (with $C_{\rm cor}$ and $C_{\rm get}$ from Propositions 4.1 and 4.4).

Proof. We choose $q_5 = 1$ in (7.15). Then

$$\frac{1}{p_5} = \frac{-\gamma}{d} \in (0,1), \qquad \frac{1}{p_4} = 1 + \frac{\gamma}{d} = \frac{d+\gamma}{d} \in (0,1),$$

and thus

$$\begin{aligned} \mathcal{E}_4^{(i)} &\le 4(q_0-1) c_b \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \langle \cdot \rangle^{-2} \right\|_{L^{p_4}} \left\| f \langle \cdot \rangle^2 \right\|_{L^{q_5}} \\ &\le 4(q_0-1) c_b E_0 \left(\int \left((f-A)_+ \wedge \kappa \right)^{\frac{d+2s}{d+\gamma}} \langle v \rangle^{-2\frac{d}{d+\gamma}} \, \mathrm{d}v \right)^{\frac{d+\gamma}{d}}. \end{aligned}$$

Step 1: First interpolation. We then interpolate thanks to Lemma 2.6 with $p_1 = \frac{d+2s}{d+\gamma} \in [1, p_0 q_0]$,

$$\begin{aligned} &\left(\int \left((f-A)_+ \wedge \kappa \right)^{\frac{d+2s}{d+\gamma}} \langle v \rangle^{-2\frac{d}{d+\gamma}} \, \mathrm{d}v \right)^{\frac{d+\gamma}{d}} \\ &\qquad \le \left(\int \left((f-A)_+ \wedge \kappa \right)^{q_0 p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{\alpha_5}{p_0}} \left(\int \left((f-A)_+ \wedge \kappa \right) \langle v \rangle^{m_4} \, \mathrm{d}v \right)^{\alpha_6}, \end{aligned}$$

where α_5, α_6, m_4 are given by

$$\alpha_5 = \frac{2s-\gamma}{4s}, \qquad \alpha_6 = \frac{(d+2s)(\gamma+2s)}{4sd}, \qquad m_4 = -2\frac{4sd}{(d+2s)(\gamma+2s)} - \frac{(2s-\gamma)d}{(d+2s)(\gamma+2s)} k_0.$$

We remark that $\alpha_5 \in (0,1)$. Then Young's inequality yields for any $\varepsilon_1 \in (0,1)$

$$\mathcal{E}_4^{(i)} \le \frac{1}{16} C_{\rm cor} \left\| \left((f-A)_+ \wedge \kappa \right)^{q_0} \right\|_{L^{p_0}_{k_0}} + C(C_{\rm cor}, M_0, E_0, c_b) \left\| (f-A)_+ \wedge \kappa \right\|^{\frac{d+2s}{d}}_{L^1_{m_4}}.$$

Step 2: Second interpolation. We pick $\beta, \beta' \ge 1$ as in (7.2). In particular, (7.7) allows us to use Hölder's inequality in order to get,

$$\left\| (f-A)_+ \wedge \kappa \right\|_{L^1_{m_4}} \le \left\| (f-A)_+ \wedge \kappa \right\|^{\frac{1}{\beta'}}_{L^1_2} \left\| (f-A)_+ \wedge \kappa \right\|^{\frac{1}{\beta}}_{L^1_{m_3}} \le E_0^{\frac{1}{\beta'}} \left\| (f-A)_+ \wedge \kappa \right\|^{\frac{1}{\beta}}_{L^1_{m_3}}.$$

If we now apply the interpolation Lemma 2.5 for $k_2 = m_3$, we find

$$\int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right) \langle v \rangle^{m_3} \, \mathrm{d}v \le \left(\int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right)^{q_0 p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{\alpha_3}{p_0}} \left(\int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa \right)^{q_0 - 1} \langle v \rangle^{m_2} \, \mathrm{d}v \right)^{\alpha_4},$$

where

$$\alpha_3 = \frac{d(d-2s)}{4s^s + d^2}, \qquad \alpha_4 = \frac{4sd}{4s^2+d^2}, \qquad m_2 = \frac{4s^2+d^2}{4sd} \left[m_3 - \frac{d(d-2s)}{4s^2+d^2} k_0 \right].$$

Again the choice of m_3 in Lemma 7.2 is such that $m_2 = l_q$. In particular, there holds

$$\|(f-A)_+ \wedge \kappa\|_{L^1_{m_4}}^{\frac{d+2s}{d}} \leq E_0^{\frac{1}{\beta'}\frac{d+2s}{d}} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\gamma_2} \|(f-A)_+ \wedge \kappa\|_{L^1_{l_q}}^{\gamma_3}$$

with γ_2, γ_3 given by (7.4).

Step 3: Young's inequality. We combine Step 1 and Step 2 with Young's inequality, so that for $\theta \geq 1$,

$$\begin{aligned}
\mathcal{E}_4{}^{(i)} &\leq \frac{1}{16} C_{\rm cor} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}} + C(C_{\rm cor}) \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}}^{\gamma_2} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0-1}\right\|_{L^1_{l_q}}^{\gamma_3} \\
&\leq \frac{1}{16} C_{\rm cor} \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}} + \frac{1}{16} C_{\rm cor} \left\|(f-A)_+^{q_0}\right\|_{L^{p_0}_{k_0}}^{\gamma_2\theta} + \bar{C}(C_{\rm cor}) \left\|\left((f-A)_+ \wedge \kappa\right)^{q_0-1}\right\|_{L^1_{l_q}}^{\gamma_3\theta'}.
\end{aligned}$$

If we pick θ as in (7.2), then Lemma 7.2 implies (7.3) and we conclude by choosing $\underline{a}^{(4,i)} = \underline{a}$ such that $\frac{1}{16} C_{\rm get}\, \underline{a}^{1+\frac{2s}{d}} \geq \bar{C}(C_{\rm cor})$. □

Proof of Lemma 7.8. The result follows as a consequence of Lemmas 7.9, 7.10 and 7.11. □

7.6. **The error term due to truncation for soft potentials.** Finally it remains to estimate $\mathcal{E}_5$ in (7.8).

Lemma 7.12 (Estimate of $\mathcal{E}_5$)**.** *Let $\gamma < 0$ and $q \leq q_{\rm nsl}$ where $q_{\rm nsl}$ is given in (vii). We also assume that $q \geq d - 1 - 2s\frac{d-2}{d+2s}$ if $\gamma < -\frac{2sd}{d+2s}$. There holds*

$$\begin{aligned}
\mathcal{E}_5 \leq & \frac{1}{16} C_{\rm get}\, \mathfrak{a}^{\frac{d+2s}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{l_q} \,\mathrm{d}v \\
& + \frac{1}{2} C_{\rm cor'}\, \kappa^{\frac{2s}{d}q_0} N^{-\frac{2s}{d}} \int_{\mathbb{R}^d} (f - A - \kappa)_+ \langle v\rangle^{\gamma} \,\mathrm{d}v \\
& + C_5 \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v\rangle^{k_0 p_0} \,\mathrm{d}v \right)^{\frac{1}{p_0}},
\end{aligned}$$

with $N = \int_{\mathbb{R}^d} ((f-A)_+ \wedge \kappa)^{q_0-1} \langle v\rangle^{-(d-1)} \,\mathrm{d}v$. The positive constant C_5 only depends on parameters d, γ, s and constants $c_b, E_0, M_0, C_{\rm get}$ and $C_{\rm cor'}$ (see Propositions 4.3 and 4.4).

Proof. To estimate $\mathcal{E}_5$, we first recall its definition,

$$\mathcal{E}_5 = q_0 c_b \int_{\mathbb{R}^d} A(v) \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \big((f-A-\kappa)_+ * |\cdot|^{\gamma}\big) \,\mathrm{d}v.$$

Step 1: Splitting. We split the integration domain as before and we find for a constant m to be chosen later,

$$\begin{aligned}
\mathcal{E}_5 &= q_0 c_b \int_{\mathbb{R}^d} A(v) \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{m} \langle v\rangle^{-m} \int_{B_1} (f-A-\kappa)_+(v-w)\,|w|^{\gamma} \,\mathrm{d}w\,\mathrm{d}v \\
&\quad + q_0 c_b \int_{\mathbb{R}^d} A(v) \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \int_{\mathbb{R}^d \setminus B_1} (f-A-\kappa)_+(v-w)\,|w|^{\gamma} \,\mathrm{d}w\,\mathrm{d}v \\
&\leq 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} \int_{\mathbb{R}^d} \langle v\rangle^{-q-m} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \int_{B_1} \langle v-w\rangle^{m} (f-A-\kappa)_+(v-w)\,|w|^{\gamma} \,\mathrm{d}w\,\mathrm{d}v \\
&\quad + q_0 c_b M_0 \mathfrak{a} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa\Big)^{q_0-1} \langle v\rangle^{-q} \,\mathrm{d}v \\
&=: \mathcal{E}_5^{(i)} + \mathcal{E}_5^{(ii)}.
\end{aligned}$$

Note that in case that $m \geq 0$, we used $\langle v\rangle \leq \langle v-w\rangle$ for $w \in B_1$ and $v \in \mathbb{R}^d$. In case that $m < 0$, we used $\langle v-w\rangle \leq \sqrt{2}\langle v\rangle$ for $w \in B_1, v \in \mathbb{R}^d$.

The term $\mathcal{E}_5^{(ii)}$ is treated by checking that $-q \leq l_q$ holds true for any $q \leq d + 1 + \frac{d\gamma}{2s}$.

We use again Hölder's and Young's inequalities to write,

$$\begin{aligned}\mathcal{E}_5^{(i)} &\le 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} \left\| \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle \cdot \rangle^{-(q+m)} \right\|_{L^{\frac{d}{d+\gamma}}} \|(f-A-\kappa)_+\|_{L^1_m} \\ &\le 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} \kappa^{-\gamma \frac{2s}{d^2}} \left(\int_{\mathbb{R}^d} ((f-A)_+ \wedge \kappa)^{q_0-1} \langle v \rangle^{-(q+m)\frac{d}{d+\gamma}} \, \mathrm{d}v \right)^{\frac{d+\gamma}{d}} \|(f-A-\kappa)_+\|_{L^1_m} .\end{aligned}$$

It is now convenient to introduce some notation. We consider for any $\ell \in \mathbb{R}$,

$$N_\ell = \int_{\mathbb{R}^d} ((f-A)_+ \wedge \kappa)^{q_0-1} \langle v \rangle^\ell \, \mathrm{d}v \quad \text{and} \quad M_\ell = \int_{\mathbb{R}^d} (f-A-\kappa)_+ \langle v \rangle^\ell \, \mathrm{d}v.$$

We also let k_m denote $-(q+m)\frac{d}{d+\gamma}$ and we write the previous inequality under the following compact form,

$$\mathcal{E}_5^{(i)} \le 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} \kappa^{-\gamma \frac{2s}{d^2}} N_{k_m}^{\frac{d+\gamma}{d}} M_m. \tag{7.16}$$

We distinguish two cases: $\frac{-2sd}{d+2s} \le \gamma < 0$ and $-2s < \gamma < \frac{-2sd}{d+2s}$.

Step 2: estimate of $\mathcal{E}_5^{(i)}$ for not too negative γ's. We start by assuming $\frac{-2sd}{d+2s} \le \gamma < 0$. We use Young's inequality, so that

$$\mathcal{E}_5^{(i)} \le \frac{1}{16} C_{\mathrm{get}} \, \mathfrak{a}^{\frac{d}{d+\gamma}} N_{k_m} + \left(\frac{1}{16} C_{\mathrm{get}} \right)^{\frac{d+\gamma}{\gamma}} \left(2^{\frac{m}{2}} q_0 c_b \right)^{-\frac{d}{\gamma}} \kappa^{\frac{2s}{d}} M_m^{\frac{d}{-\gamma}}.$$

We pick $m = k_0 = \gamma + 2s - \frac{2s}{d}$. We then observe that for any $\gamma \ge -\frac{2sd}{d+2s}$, there holds $\frac{d}{d+\gamma} \le 1 + \frac{2s}{d}$, and moreover $k_m \le l_q$, provided that $q \le q_{\mathrm{nsl}} = d + 1 + \frac{\gamma d}{2s}$, $\gamma \ge -\frac{2sd}{d+2s}$ and $d \ge 2$, see Lemma 7.13. This shows

$$\frac{1}{16} C_{\mathrm{get}} \, \mathfrak{a}^{\frac{d}{d+\gamma}} N_{k_m} \le \frac{1}{16} C_{\mathrm{get}} \, \mathfrak{a}^{1+\frac{2s}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{l_q} \, \mathrm{d}v.$$

Moreover, we use Hölder's and Chebyshev's inequalities to bound

$$\begin{aligned}\kappa^{\frac{2s}{d}} M_{k_0}^{\frac{d}{-\gamma}} &= \kappa^{\frac{2s}{d}} M_{k_0}^{\frac{d}{-\gamma}-1} M_{k_0} \le \kappa^{\frac{2s}{d}} M_{k_0}^{\frac{d+\gamma}{-\gamma}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}} |\{f-A > \kappa\}|^{\frac{2s}{d}} \\ &\le \kappa^{\frac{2s}{d}} \kappa^{-\frac{2s}{d}} E_0^{\frac{d+\gamma}{-\gamma}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}} \left(\int_{\mathbb{R}^d} (f-A)_+ \, \mathrm{d}v \right)^{\frac{2s}{d}} \\ &= E_0^{\frac{d+\gamma}{-\gamma}} M_0^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}}.\end{aligned}$$

We used that $k_0 < 2$ to bound M_{k_0} from above by E_0.

Gathering the estimates, we get in this first case,

$$\begin{aligned}\mathcal{E}_5^{(i)} \le & \frac{1}{16} C_{\mathrm{get}} \, \mathfrak{a}^{1+\frac{2s}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{l_q} \, \mathrm{d}v \\ & + \left(\frac{1}{16} C_{\mathrm{get}} \right)^{\frac{d+\gamma}{\gamma}} \left(2^{\frac{m}{2}} q_0 c_b \right)^{-\frac{d}{\gamma}} E_0^{\frac{d+\gamma}{-\gamma}} M_0^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v \rangle^{k_0 p_0} \, \mathrm{d}v \right)^{\frac{1}{p_0}}.\end{aligned} \tag{7.17}$$

Step 3: estimate of $\mathcal{E}_5^{(i)}$ for very negative γ's. We now assume $-2s < \gamma < \frac{-2sd}{d+2s}$. It remains to bound $\mathcal{E}_5^{(i)}$ for this range of γ. We start from (7.16) and we recall $k_m = -(q+m)\frac{d}{d+\gamma}$ for some $m \in \mathbb{R}$ to be determined,

$$\mathcal{E}_5^{(i)} \le 2^{\frac{m}{2}} q_0 c_b \mathfrak{a} N_{k_m}^{\frac{d+\gamma}{d}} M_m = 2^{\frac{m}{2}} q_0 c_b \kappa^{\frac{-\gamma 2s}{d^2}} \mathfrak{a} N_{k_m}^{\frac{d}{d+2s}} N_{k_m}^{\frac{d+\gamma}{d} - \frac{d}{d+2s}} M_m.$$

Note that

$$\frac{d+\gamma}{d} - \frac{d}{d+2s} < 0$$

for $\gamma < -\frac{2sd}{d+2s}$.

We use Young's inequality

$$\mathcal{E}_5^{(i)} \leq \frac{1}{16} C_{\mathrm{get}}\, \mathfrak{a}^{\frac{d+2s}{d}} N_{k_m} + C_5^{(i)} \kappa^{-\frac{\gamma(d+2s)}{d^2}} N_{k_m}^{\left(\frac{d+\gamma}{d} - \frac{d}{d+2s}\right)\frac{d+2s}{2s}} M_m^{\frac{d+2s}{2s}},$$

where $C_5^{(i)} = \left(\frac{1}{16} C_{\mathrm{get}}\right)^{-\frac{d}{2s}} \left(2^{\frac{m}{2}} q_0 c_b\right)^{\frac{d+2s}{2s}}$. We further bound for any $\alpha_0 \in [0,1)$

$$M_m \leq E_0^{\alpha_0} M_{m_{\alpha_0}}^{1-\alpha_0} \quad \text{with} \quad m_{\alpha_0} := \frac{m - 2\alpha_0}{1-\alpha_0}$$

thanks to Lemma 2.4. We found,

$$\mathcal{E}_5^{(i)} \leq \frac{1}{16} C_{\mathrm{get}}\, \mathfrak{a}^{\frac{d+2s}{d}} N_{k_m} + C_5^{(i)} E_0^{\alpha_0 \frac{d+2s}{2s}} \kappa^{-\frac{\gamma(d+2s)}{d^2}} N_{k_m}^{\left(\frac{d+\gamma}{d} - \frac{d}{d+2s}\right)\frac{d+2s}{2s}} M_{m_{\alpha_0}}^{(1-\alpha_0)\frac{d+2s}{2s}}. \tag{7.18}$$

Then, in order to absorb the first term in the good extra term (Proposition 4.4), we need

$$k_m := -(q+m)\frac{d}{d+\gamma} \leq l_q = k_0 + 2\frac{2s}{d} - q\frac{d+2s}{d}.$$

For the second term, we want to use the second coercivity estimate from Proposition 4.3. To this end, we require

$$k_m := -(q+m)\frac{d}{d+\gamma} \geq -(d-1),$$

and

$$m_{\alpha_0} := \frac{m - 2\alpha_0}{1-\alpha_0} \leq k_0 + \frac{2s}{d} k_m.$$

In order to satisfy the first two constraints, q has to satisfy,

$$-(d-1) \leq l_q,$$

which holds true for $q \leq q_{\mathrm{nsl}}$.

We pick m such that

$$k_m = -(q+m)\frac{d}{d+\gamma} = -(d-1),$$

and we use Proposition 4.3 with $k_m = -(d-1)$ for which we have

$$N := \int_{\mathbb{R}^d} \left((f-A)_+ \wedge \kappa\right)^{q_0 - 1} \langle v \rangle^{-(d-1)} \,\mathrm{d}v = N_{-(d-1)}$$

Thus we consider

$$\bar{\theta} = -\frac{\left(\frac{d+\gamma}{d} - \frac{d}{d+2s}\right)\frac{d+2s}{2s}}{\frac{2s}{d}} = \frac{-\gamma(d+2s) - 2sd}{4s^2}$$

and we find

$$\begin{aligned}
\kappa^{-\frac{\gamma(d+2s)}{d^2}} N_{k_m}^{\left(\frac{d+\gamma}{d} - \frac{d}{d+2s}\right)\frac{d+2s}{2s}} M_{m_{\alpha_0}}^{\frac{d+2s}{2s}(1-\alpha_0)} &= \kappa^{\frac{2s}{d} q_0 \bar{\theta}} N^{-\frac{2s}{d}\bar{\theta}} M_{m_{\alpha_0}}^{\bar{\theta}} \kappa^{-\frac{\gamma(d+2s)}{d^2} - \frac{2s}{d} q_0 \bar{\theta}} M_{m_{\alpha_0}}^{\frac{(d+2s)(1-\alpha_0)}{2s} - \bar{\theta}} \\
&= \kappa^{\frac{2s}{d} q_0 \bar{\theta}} N^{-\frac{2s}{d}\bar{\theta}} M_{m_{\alpha_0}}^{\bar{\theta}} \kappa^{\frac{(\gamma+2s)(d+2s)}{2sd}} M_{m_{\alpha_0}}^{\frac{(d+2s)(\gamma+2s)}{4s^2} + \frac{d-\alpha_0(d+2s)}{2s}}.
\end{aligned}$$

We plug this into (7.18) and use Young's inequality, to get

$$\mathcal{E}_5^{(i)} \leq \frac{1}{16} C_{\mathrm{get}}\, \mathfrak{a}^{\frac{d+2s}{d}} N_{k_m} + \frac{1}{8} C_{cor'} \kappa^{\frac{2s}{d} q_0} N^{-\frac{2s}{d}} M_{m_{\alpha_0}} + C_5 \kappa^{\frac{2s}{d}} M_{m_{\alpha_0}}^{1 + \frac{2sd}{(\gamma+2s)(d+2s)} - \alpha_0 \frac{2s}{\gamma+2s}}. \tag{7.19}$$

where $C_5 := \left(\frac{1}{8} C_{cor'}\right)^{\frac{\gamma(d+2s)+2sd}{(\gamma+2s)(d+2s)}} \left(C_5^{(i)} E_0^{\alpha_0 \frac{d+2s}{2s}}\right)^{\frac{4s^2}{(\gamma+2s)(d+2s)}}$. We used that

$$\frac{1}{1-\theta} = \frac{4s^2}{(\gamma+2s)(d+2s)}.$$

Note that the estimate (7.19) holds for any $0 \leq \alpha_0 < 1$, but we want to pick α_0, such that

$$1 + \frac{2sd}{(\gamma+2s)(d+2s)} - \alpha_0 \frac{2s}{\gamma+2s} = 1, \tag{7.20}$$

but at the same time we need to make sure that

$$m_{\alpha_0} = \frac{m-2\alpha_0}{1-\alpha_0} \leq k_0 + \frac{2s}{d}k_m = \gamma, \tag{7.21}$$

where we recall that we picked m and k_m as

$$m = -q + \frac{(d-1)(d+\gamma)}{d}, \qquad k_m = -(d-1),$$

We verify (7.21) in Lemma 7.14 below.

Finally, we conclude the proof just as in the case for larger γ above: we use that $\frac{m-2\alpha_0}{1-\alpha_0} \leq k_0$ and find with Chebyshev's inequality

$$\begin{aligned}
\kappa^{\frac{2s}{d}} M_{m_{\alpha_0}} &\leq \kappa^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v\rangle^{k_0 p_0} \,\mathrm{d}v \right)^{\frac{1}{p_0}} |\{f-A>\kappa\}|^{\frac{2s}{d}} \\
&\leq \kappa^{\frac{2s}{d}} \kappa^{-\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v\rangle^{k_0 p_0} \,\mathrm{d}v \right)^{\frac{1}{p_0}} \left(\int_{\mathbb{R}^d} (f-A)_+ \,\mathrm{d}v \right)^{\frac{2s}{d}} \\
&\leq M_0^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v\rangle^{k_0 p_0} \,\mathrm{d}v \right)^{\frac{1}{p_0}}.
\end{aligned}$$

We plug this into (7.19) and use (7.20) to find

$$\begin{aligned}
\mathcal{E}_5^{(i)} &\leq \frac{1}{16} C_{\mathrm{get}} \mathfrak{a}^{\frac{d+2s}{d}} N + \frac{1}{8} C_{cor'} \kappa^{\frac{2s}{d} q_0} N^{-\frac{2s}{d}} M_\gamma + C_5 \kappa^{\frac{2s}{d}} M_{m_{\alpha_0}} \\
&\leq \frac{1}{16} C_{\mathrm{get}} \mathfrak{a}^{\frac{d+2s}{d}} N + \frac{1}{8} C_{cor'} \kappa^{\frac{2s}{d} q_0} N^{-\frac{2s}{d}} M_\gamma + C_5 M_0^{\frac{2s}{d}} \left(\int_{\mathbb{R}^d} (f-A-\kappa)_+^{p_0} \langle v\rangle^{k_0 p_0} \,\mathrm{d}v \right)^{\frac{1}{p_0}}.
\end{aligned} \tag{7.22}$$

We conclude the proof for $\gamma \in \left[-\frac{2sd}{d+2s}, 0 \right)$ using (7.17), and for $\gamma \in \left(-2s, -\frac{2sd}{d+2s} \right)$ with (7.22). □

Lemma 7.13. *Let $d \geq 2$, $s \in (0,1)$, $\gamma < 0$ and $q \leq d+1+\frac{d\gamma}{2s}$. Then*

$$-(q+k_0)\frac{d}{d+\gamma} \leq l_q = k_0 + 2\frac{2s}{d} - q\frac{d+2s}{d}. \tag{7.23}$$

Proof. We rewrite (7.23) as

$$\left(\frac{d+2s}{d} - \frac{d}{d+\gamma} \right) q \leq \left(1 + \frac{d}{d+\gamma} \right) k_0 + 2\frac{2s}{d}.$$

or also

$$\left(\gamma(d+2s) + 2sd\right) q \leq d\left(2d+\gamma\right) k_0 + 4s(d+\gamma).$$

If we now use $q \leq \frac{d}{2s}k_0 + 2$, we find

$$\left(\gamma(d+2s) + 2sd\right) \left(\frac{d}{2s}k_0 + 2 \right) \leq d\left(2d+\gamma\right) k_0 + 4s(d+\gamma).$$

or equivalently,

$$\left(\gamma\frac{d^2}{2s} + d^2 \right) k_0 \leq 2d^2 k_0 - 2\gamma d,$$

which is satisfied if

$$0 \leq \left(d - \gamma\frac{d}{2s} \right) k_0 - 2\gamma.$$

Now we use that $k_0 = \gamma + 2s - \frac{2s}{d}$. We find

$$0 \leq \left(d - \gamma\frac{d}{2s} \right) \left(\gamma + 2s - \frac{2s}{d} \right) - 2\gamma = d\left(\frac{2s-\gamma}{2s} \right)(\gamma+2s) - (2s+\gamma).$$

We divide by $\gamma + 2s > 0$ to find

$$0 \le \frac{d}{2s}(2s-\gamma) - 1,$$

which is true for any $d \ge 1$ and $\gamma < 0$. □

Lemma 7.14. *Let $d \ge 2$, $s \in (0,1)$, $-2s < \gamma < -\frac{2sd}{d+2s}$ and $q \ge d - 1 - 2s\frac{d-2}{d+2s}$. There exists $\alpha_0 \in (0,1)$ such that (7.20) and(7.21) hold true.*

Proof. In view of (7.20), we pick

$$\alpha_0 = \frac{d}{d+2s} \in (0,1).$$

We rewrite the constraint (7.21) as

$$\frac{d^2 - d - \gamma - dq}{d(2-\gamma)} \le \alpha_0$$

and check that (7.21) is satisfied. It is equivalent to,

$$(d+2s)(d^2-\gamma) - d(d+2s)(q+1) \le d^2(2-\gamma).$$

We now remark that for $q \ge d - 1 - 2s\frac{d-2}{d+2s}$, we have

$$(d+2s)(q+1) \ge d^2 + 4s.$$

We are thus left with checking that

$$(d+2s)(d^2-\gamma) - d(d^2+4s) \le d^2(2-\gamma).$$

We rearrange terms and get,

$$(-\gamma)(d+2s-d^2) - 4sd - (2-2s)d^2 \le 0.$$

We conclude by remarking that for $d \ge 2$ and $s \in (0,1)$, we have $d + 2s - d^2 \le 0$. □

7.7. **Final Estimate.** Combining Lemmas 7.3, 7.4, 7.8 and 7.12 implies the following statement.

Lemma 7.15 (Error terms for soft potentials)**.** *Let $\gamma < 0$. Let $0 \le q \le q_{\rm nsl}$ where $q_{\rm nsl}$ is given by (vii) and $q \ge d - 1 - 2s\frac{d-2}{d+2s}$ if $\gamma < -\frac{2sd}{d+2s}$. There exists a positive constant $\underline{a}$ only depending on d, γ, s, M_0, E_0, c_b, $C_{\rm get}$, $C_{\rm cor}$ (see Propositions 4.1 and 4.4) and $C_{q,3}$ (see formula (7.1)) such that*

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\Omega} \Phi\big(f\big)\,(f * |\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad \le \frac{3}{4} C_{\rm cor}\,\big\|\big((f-A)_+ \wedge \kappa\big)^{q_0}\big\|_{L^{p_0}_{k_0}} + \frac{15}{16} C_{\rm get}\,\mathfrak{a}^{1+\frac{2s}{d}}\,\Big\|\big((f-A)_+\wedge\kappa\big)^{q_0-1}\Big\|_{L^1_{l_q}} \\
&\qquad + C_5 \int_\Omega \|(f-A-\kappa)_+\|_{L^{p_0}_{k_0}}\,\mathrm{d}x \\
&\qquad + \frac{1}{2} C_{\rm cor'}\,\kappa^{\frac{2s}{d}q_0} \int_\Omega \left[\Big\|\big((f-A)_+\wedge\kappa\big)^{q_0-1}\Big\|^{-\frac{2s}{d}}_{L^1_{-(d-1)}} \int_{\mathbb{R}^d} (f-A-\kappa)_+ \langle v\rangle^\gamma\,\mathrm{d}v\right]\mathrm{d}x
\end{aligned} \tag{7.24}$$

as soon as $\mathfrak{a} \ge \underline{a}$.

Remark 7.16. The constants $C_{\rm cor}$, $C_{\rm cor'}$, $C_{\rm get}$ and C_5 appear in Propositions 4.2, 4.3, 4.4 and Lemma 7.12 respectively.

Proof. Estimate (7.24) is a consequence of (7.8) and Lemmas 7.3, 7.4, 7.8, and 7.12. □

8. Estimates for large exponents

In this section, we establish estimates that we will use to generate more decay.

8.1. **Convolution product.**

Lemma 8.1 (Convolution product for decaying functions and soft potentials)**.** *Let $\gamma < 0$ and $f : [0,T]\times\mathbb{R}^d \to [0,\infty)$ have finite mass M_0. Assume that*

$$f(t,v) \le \mathfrak{b}_*\langle v\rangle^{-q_*}, \qquad \forall (t,v) \in [0,T]\times\mathbb{R}^d,$$

for some $q_* > 0$ and some $\mathfrak{b}_* = \mathfrak{b}_*(t)$ such that $\mathfrak{b}_* \ge 2^d \frac{M_0}{c_{q_*}}$ with $c_{q_*} = \frac{|\partial B_1|(2\sqrt{2})^{q_*}}{(d+\gamma)}$. Then

$$f * |\cdot|^\gamma \le C_{\mathrm{conv}'}\, \mathfrak{b}_*^{-\frac{\gamma}{d}} \langle v\rangle^{\gamma \min(1,\frac{q_*}{d})}$$

with $C_{\mathrm{conv}'} = 2c_{q_*}^{-\frac{\gamma}{d}} M_0^{1+\frac{\gamma}{d}}$.

Proof. We proceed as in the proof of Lemma 7.1, exploiting the assumed pointwise decay. For $R > 0$ such that $R < \langle v\rangle/2$, we write

$$\begin{aligned}\int_{\mathbb{R}^d} f(v-w)\,|w|^\gamma\,\mathrm{d}w &= \int_{B_R} f(v-w)\,|w|^\gamma\,\mathrm{d}w + \int_{\mathbb{R}^d\setminus B_R} f(v-w)\,|w|^\gamma\,\mathrm{d}w\\ &\le \mathfrak{b}_*\int_{B_R}\langle v-w\rangle^{-q_*}|w|^\gamma\,\mathrm{d}w + R^\gamma M_0 \le \mathfrak{b}_*|\partial B_1|(2\sqrt{2})^{q_*}\langle v\rangle^{-q_*}\frac{R^{d+\gamma}}{d+\gamma} + R^\gamma M_0\\ &\le c_{q_*}\mathfrak{b}_*\langle v\rangle^{-q_*}R^{d+\gamma} + M_0R^\gamma\end{aligned}$$

with $c_{q_*} = \frac{|\partial B_1|(2\sqrt{2})^{q_*}}{(d+\gamma)}$. Then we pick $R = \big(\frac{M_0}{c_{q_*}\mathfrak{b}_*}\big)^{\frac{1}{d}}\langle v\rangle^{\min(1,\frac{q_*}{d})} \le \frac{1}{2}\langle v\rangle$ (since $\mathfrak{b}_* \ge 2^d\frac{M_0}{c_{q_*}}$) so that

$$\begin{aligned}\int_{\mathbb{R}^d} f(v-w)\,|w|^\gamma\,\mathrm{d}w &\le c_{q_*}^{-\frac{\gamma}{d}}\mathfrak{b}_*^{-\frac{\gamma}{d}}M_0^{1+\frac{\gamma}{d}}\Big(\langle v\rangle^{-q_*+(d+\gamma)\min(1,\frac{q_*}{d})} + \langle v\rangle^{\gamma\min(1,\frac{q_*}{d})}\Big)\\ &\le 2c_{q_*}^{-\frac{\gamma}{d}}\mathfrak{b}_*^{-\frac{\gamma}{d}}M_0^{1+\frac{\gamma}{d}}\langle v\rangle^{\gamma\min(1,\frac{q_*}{d})}.\end{aligned}$$

We thus get the desired estimate. □

8.2. **Estimate of the remaining error term.**

Lemma 8.2 (Error terms for soft potentials and decaying functions)**.** *Let $\gamma < 0$. Assume $f : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}_+$ has finite mass M_0. If for a.e. $(t,x,v) \in [0,T]\times\Omega\times\mathbb{R}^d$,*

$$f(t,x,v) \le \mathfrak{b}_*(t)\langle v\rangle^{-q_*}$$

for some $q_* \ge 0$ and some $\mathfrak{b}_* \ge 2^d\frac{M_0}{c_{q_*}}$ with $c_{q_*} = \frac{\omega_{d-1}(2\sqrt{2})^{q_*}}{(d+\gamma)}$, then

$$\begin{aligned}\iint_{\mathbb{R}^d\times\Omega}\Phi\big(f\big)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x \le{}& \frac{1}{2}C_{\mathrm{vget}}\,\frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}}\iint_{\mathbb{R}^d\times\Omega}\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x\\ &+ \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}}\iint_{\mathbb{R}^d\times\Omega}\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\langle v\rangle^{\gamma\min(1,\frac{q_*}{d})-q_*}\,\mathrm{d}v\,\mathrm{d}x,\end{aligned}\tag{8.1}$$

as soon as $\mathfrak{a}$ satisfies $\mathfrak{a} \ge \bar{C}\mathfrak{b}_*^{1-\frac{\gamma}{2sd}}$ with $\bar{C} = 16q_0c_b\frac{C_{\mathrm{conv}'}}{C_{\mathrm{vget}}}$ and $\underline{C} = C_{\mathrm{conv}'}\,(q_0-1)c_b$.

Proof. In order to prove the lemma, we split the error term involving $\Phi(f)$ naturally into two pieces,

$$\int_{\mathbb{R}^d}\Phi\big(f\big)\,(f*|\cdot|^\gamma)\,\mathrm{d}v = \bar{\mathcal{E}} + \underline{\mathcal{E}}$$

with

$$\begin{cases}\bar{\mathcal{E}} &= q_0c_b\int A\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\big(f*|\cdot|^\gamma\big)\,\mathrm{d}v,\\ \underline{\mathcal{E}} &= (q_0-1)c_b\int\Big((f-A)_+\wedge\kappa\Big)^{q_0}\,(f*|\cdot|^\gamma)\,\mathrm{d}v.\end{cases}\tag{8.2}$$

We now estimate each part.

Estimate of $\bar{\mathcal{E}}$. We first claim that

$$\bar{\mathcal{E}} \le \frac{1}{2} C_{\mathrm{vget}} \frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{\gamma \min(1, \frac{q_*}{d}) - q} \, \mathrm{d}v, \tag{8.3}$$

as soon as the constant $\mathfrak{a}$ satisfies

$$\mathfrak{a} \ge \bar{C} \mathfrak{b}_*^{1 - \frac{\gamma}{2sd}}$$

for some constant $\bar{C} = 2q_0 c_b \frac{C_{\mathrm{conv}'}}{C_{\mathrm{vget}}}$ with C_{vget} from Proposition 4.5 and $C_{\mathrm{conv}'}$ from Lemma 8.1. Indeed, by Lemma 8.1 there holds

$$\begin{aligned} \bar{\mathcal{E}} &\le C_{\mathrm{conv}'} \, q_0 c_b \mathfrak{b}_*^{-\frac{\gamma}{d}} \int_{\mathbb{R}^d} A \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{\gamma \min(1, \frac{q_*}{d})} \, \mathrm{d}v \\ &\le \frac{1}{2} C_{\mathrm{vget}} \frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{\gamma + 2s - q} \, \mathrm{d}v. \end{aligned}$$

The last inequality follows from the condition imposed on a.

Estimate of $\underline{\mathcal{E}}$. We next claim there exists a constant $\underline{C} > 0$ depending on s, d, γ, c_b, q_* such that

$$\underline{\mathcal{E}} \le \underline{C} \mathfrak{b}_*^{1-\frac{\gamma}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{\gamma \min(1, \frac{q_*}{d}) - q_*} \, \mathrm{d}v. \tag{8.4}$$

Indeed, by Lemma 8.1 there holds

$$\begin{aligned} \underline{\mathcal{E}} &\le C_{\mathrm{conv}'} \, (q_0 - 1) c_b \mathfrak{b}_*^{-\frac{\gamma}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0} \langle v \rangle^{\gamma \min(1, \frac{q_*}{d})} \, \mathrm{d}v \\ &\le C_{\mathrm{conv}'} \, (q_0 - 1) c_b \mathfrak{b}_*^{1-\frac{\gamma}{d}} \int_{\mathbb{R}^d} \Big((f-A)_+ \wedge \kappa \Big)^{q_0-1} \langle v \rangle^{\gamma \min(1, \frac{q_*}{d}) - q_*} \, \mathrm{d}v. \end{aligned}$$

We used that $(f-A)_+ \le f \le \mathfrak{b}_* \langle v \rangle^{-q_*}$ to get the second line.

Conclusion. Estimate (8.1) is derived from the Truncated Convex Inequalities (3.2) after using (8.2) together with 8.3 and 8.4. □

9. Proof of the decay estimate

9.1. **Monotonicity for generation.** The next lemma is obtained by combining Lemma 3.1 with (coercivity and error) estimates from previous sections.

Lemma 9.1 (Monotonicity)**.** *Let f be a suitable weak subsolution of the Boltzmann equation with either in-flow, bounce-back, specular / diffuse / Maxwell reflection boundary condition. Let*

$$A(t,v) := \mathfrak{a}(t) \langle v \rangle^{-q_{\mathrm{nsl}}} \qquad \text{with} \qquad \mathfrak{a}(t) := a_* \left(1 + t^{-\frac{d}{2s}} \right)$$

with q_{nsl} from (vii) and for some constant $a_* > 0$ large enough depending on $d, \gamma, s, c_b, M_0, E_0$ and C_b. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^d \times \Omega} \varphi_{0,\kappa} (f - A) \, \mathrm{d}v \, \mathrm{d}x \le \mathcal{I}_\kappa(t) \qquad \text{in } \mathcal{D}'((0,T)), \tag{9.1}$$

with $\mathcal{I}_\kappa(t)$ such that for any $T > 0$, we have

$$\int_0^T \mathcal{I}_\kappa(t) \, \mathrm{d}t \xrightarrow[\kappa \to \infty]{} 0.$$

Remark 9.2. The constant C_b comes from the assumption on the boundary data f_b in the statement of the main result, see Theorem 1.6.

Proof. We write q instead of q_{nsl} for clarity. We assume that a_* satisfies

$$a_* \ge a_{\mathrm{get}} \qquad \text{and} \qquad a_* \ge C_b \quad (\text{and } a_* \ge \underline{a} \text{ if } \gamma < 0) \tag{9.2}$$

with a_{get} given in (4.6) and C_b in the statement of Theorem 1.6 (it is related to boundary values) and $\underline{a}$ comes from Lemma 7.15. The first condition ensures that we can use Proposition 4.4 about the good extra term while the second one ensures that boundary terms will not appear in the Truncated Convex Inequalities, see Lemma 3.1.

We distinguish the cases of hard and soft potentials.

THE CASE OF HARD POTENTIALS. In this case, we apply Lemma 3.1 and collect estimates (4.4) with $g = (f-A)_+$, (4.7), (5.1), (6.1) to get that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad\le -C_{\mathrm{cor}}\int_\Omega \left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x + C_{\mathrm{cor}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{\gamma-d-1}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad - C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + (C_1+C_{\mathrm{hard}})\mathfrak{a}(t)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + \varepsilon\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x + C_{\mathrm{hard}}\,C(\varepsilon)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad - \dot{\mathfrak{a}}(t)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{-q}(\mathbb{R}^d)}\,\mathrm{d}x
\end{aligned}$$

we then pick $\varepsilon = C_{\mathrm{cor}}/2$ and get,

$$\begin{aligned}
&\le -\frac{C_{\mathrm{cor}}}{2}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\quad + \bar{C}\left(-\bar{c}\mathfrak{a}^{1+\frac{2s}{d}}+1+\mathfrak{a}(t)-\dot{\mathfrak{a}}(t)\right)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x
\end{aligned}$$

for some positive constants $\bar{C}$ and $\bar{c}$ depending on C_{cor}, C_{get}, C_{hard}, $C(C_{\mathrm{cor}})$, C_1, d and s. The last inequality follows from the fact that $\gamma - d - 1 \le l_q$ and $-q \le l_q$ (recall that $\dot{\mathfrak{a}} < 0$ and l_q is given in (vi)) for $q \le d+1$ and $\gamma \ge 0$. Finally, we choose a_* large enough (depending on $\bar{c}$, d and s) such that

$$-\bar{c}\,\mathfrak{a}^{1+\frac{2s}{d}}(t) + 1 + \mathfrak{a}(t) - \dot{\mathfrak{a}}(t) \le 0. \tag{9.3}$$

In particular, this yields, $\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}\big((f-A)_+\big)\,\mathrm{d}v\,\mathrm{d}x \le 0$ and concludes the proof for hard potentials ($\gamma \ge 0$).

THE CASE OF SOFT POTENTIALS. In this case, we apply Lemma 3.1 and collect estimates (4.4) with $g = (f-A)_+$, (4.7), (5.1), (7.24) and use Lemma 3.1 to deduce,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad\le -C_{\mathrm{cor}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x + C_{\mathrm{cor}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad - C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x + C_1\mathfrak{a}(t)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + \frac{3}{4}C_{\mathrm{cor}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0}\right\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x + \frac{15}{16}C_{\mathrm{get}}\,\mathfrak{a}^{1+\frac{2s}{d}}\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\qquad + C_5\int_\Omega\|(f-A-\kappa)_+\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x - \dot{\mathfrak{a}}(t)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{-q}(\mathbb{R}^d)}\,\mathrm{d}x.
\end{aligned}$$

Using that $-q \le l_q$ for $q \le d+1+\frac{\gamma d}{2s}$ (and in particular for $q = q_{\mathrm{nsl}}$), we get,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x &\le \bar{C}\left(-\bar{c}\mathfrak{a}^{1+\frac{2s}{d}}(t)+\mathfrak{a}(t)-\dot{\mathfrak{a}}(t)\right)\int_\Omega\left\|\left((f-A)_+\wedge\kappa\right)^{q_0-1}\right\|_{L^1_{l_q}(\mathbb{R}^d)}\,\mathrm{d}x\\
&\quad + C_5\int_\Omega\|(f-A-\kappa)_+\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x.
\end{aligned}$$

Finally, we choose a_* such that both (9.2) and (9.3) (for a new $\bar{c}$) hold true and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}\big((f-A)_+\big)\,\mathrm{d}v\,\mathrm{d}x\le\mathcal{I}_\kappa(t),$$

with

$$\mathcal{I}_\kappa(t)=C_5\int_\Omega\|(f-A-\kappa)_+\|_{L^{p_0}_{k_0}(\mathbb{R}^d)}\,\mathrm{d}x.$$

Finally we notice that $f\in L^1([0,T]\times\Omega;L^{p_0}_{k_0}(\mathbb{R}^d))$ ensures that,

$$\int_0^T\mathcal{I}_\kappa(t)\,\mathrm{d}t\to 0\qquad\text{as }\kappa\to\infty.$$

This concludes the proof in the case of soft potentials. □

9.2. **Proof of generation for not so large exponents.** We now state and prove a result that will be used several times.

Lemma 9.3 (Monotonicity implies decrease)**.** *Let f be a suitable weak subsolution of the Boltzmann equation with either in-flow, bounce-back, specular / diffuse / Maxwell reflection boundary condition. Let*

$$A(t,v):=\mathfrak{a}(t)\langle v\rangle^{-q}$$

for some smooth bounded function $\mathfrak{a}\colon(0,T)\to(0,+\infty)$ and for some $q\ge 0$. Assume that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x\le\mathcal{I}_\kappa(t)\qquad\text{in }\mathcal{D}'((0,T)),\tag{9.4}$$

for some $\mathcal{I}_\kappa(t)$ such that for any $T>0$, we have

$$\int_0^T\mathcal{I}_\kappa(t)\,\mathrm{d}t\xrightarrow[\kappa\to\infty]{}0.$$

Then $f(t,x,v)\le A(t,v)$ almost everywhere in $(0,T)\times\Omega\times\mathbb{R}^d$.

Proof. For all $t\in(0,T)$, we define

$$m_\kappa(t):=\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)(t,x,v)\,\mathrm{d}v\,\mathrm{d}x.\tag{9.5}$$

Since $\varphi_{0,\kappa}(f-A)\in L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, there holds $m_\kappa\in L^1((0,T))$.

Step 1. Lebesgue points. We now take two Lebesgue points $t_1,t_2\in(0,T)$ such that $t_1<t_2$, and we consider for any $\varepsilon>0$ the following cutoff $\eta_\varepsilon\in C_c^\infty(\mathbb{R})$ in time given by

$$\eta_\varepsilon(t)=\begin{cases}\frac{1}{\varepsilon}(t_2-t), & \text{if } t\in[t_2-\varepsilon,t_2),\\ 1, & \text{if } t\in(t_1+\varepsilon,t_2-\varepsilon),\\ \frac{1}{\varepsilon}(t-t_1), & \text{if } t\in(t_1,t_1+\varepsilon],\\ 0, & \text{else.}\end{cases}$$

Then we test (9.1) with η_ε and integrate over $(0,T)$, so that

$$\int_0^T m_\kappa'(t)\eta_\varepsilon(t)\,\mathrm{d}t\le\int_0^T\mathcal{I}_\kappa(t)\eta_\varepsilon(t)\,\mathrm{d}t.$$

The left hand side yields

$$\int_0^T m_\kappa'(t)\eta_\varepsilon(t)\,\mathrm{d}t=-\int_0^T m(t)\eta_\varepsilon'(t)\,\mathrm{d}t=\frac{1}{\varepsilon}\int_{t_2-\varepsilon}^{t_2}m(t)\,\mathrm{d}t-\frac{1}{\varepsilon}\int_{t_1}^{t_1+\varepsilon}m(t)\,\mathrm{d}t,$$

so that we deduce after taking $\varepsilon\to 0$, we deduce that for almost every $t_1,t_2\in(0,T)$ with $t_1<t_2$,

$$m_\kappa(t_2)-m_\kappa(t_1)\le\int_{t_1}^{t_2}\mathcal{I}_\kappa(t)\,\mathrm{d}t.\tag{9.6}$$

Note that the right hand side converges to 0 as $\kappa \to \infty$. It also converges to 0 as $t_2 \to t_1$ or as $t_1 \to t_2$, so that $m_\kappa(t)$ is (coincides a.e. with a) càdlàg (function) and the values of $m_\kappa(t)$ are well-defined for all $t \in (0,T)$, see [36, Corollary 4.9.1]. We deduce from (9.6) that m_∞ is non-increasing in $(0,T)$.

Step 2. Shifting the barrier in time. To finish the proof of Theorem 1.6, it is sufficient to show that $m_\infty(t) = 0$ for a.e. $t \in (0,T)$. Indeed this in turn implies $f(t,x,v) \le \mathfrak{a}(t)\langle v\rangle^{-q}$ for almost every $(t,x,v) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Even if m_∞ is monotonically decreasing, the difficulty is that $m_\infty(t)$ is not defined at $t = 0$. The following argument is taken from [36, Proof of Theorem 1.1]. We define a shifted version of the function m defined in (9.5),

$$m_{\kappa,t_1}(t) := \iint_{\mathbb{R}^d\times\Omega} \varphi_{0,\kappa}\big(f(t,x,v) - A(t-t_1,v)\big)\,\mathrm{d}v\,\mathrm{d}x.$$

After updating $\mathfrak{a}$, we know from Step 1 that $m_{\infty,t_1}(t)$ is monotonically decreasing for $t \in (t_1,T)$. Since $\mathfrak{a}(t) \to \infty$ as $t \to 0$, we note

$$\lim_{t_1\to t_2} m_{\infty,t_1}(t_2) = 0.$$

Thus for any $\varepsilon \in (0,1)$ there exists $t_1 \in (0,t_2)$ such that $m_{\infty,t_1}(t_2) < \varepsilon$.

Moreover, there holds for a.e. $t \in (t_2,T)$

$$m_{\infty,t_2}(t) \le m_{\infty,t_1}(t) \le m_{\infty,t_1}(t_2) < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we deduce $m_{\infty,t_2}(t) = 0$ for a.e. $t \in (t_2,T)$. This implies that for a.e. $t_2 \in (0,T)$ and $t \in (t_2,T)$, we have $(x,v) \in \Omega \times \mathbb{R}^d$, $f(t,x,v) \le A(t-t_2,v)$. We can now fix $t_0 > 0$ and consider $t_2 < t_0 < t$ and outside the sets of null measure. By letting $t_2 \to 0$, this implies $f(t,x,v) \le A(t,v)$ for $t > t_0$. Since t_0 is arbitrary, the proof is complete. □

Proof of Theorem 1.6 for not so large q's. It is enough to deal with the case $q = q_{\mathrm{nsl}}$. In this case, the conclusion of the theorem is a consequence of Lemmas 9.1 and 9.3. □

9.3. **Improving decay.** In order to prove the appearance of decay for large values of the exponent q, we improve it iteratively. We start from Lemma 3.1 and write,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_{\mathbb{R}^d} \varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x + \int_\Omega \iint_{\mathbb{R}^{2d}} d_{\varphi_{0,\kappa}}((f-A)_+,(f'-A')_+)K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\quad + \int_\Omega \iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)\,(A'-f')_+\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\quad \le c_b\int_\Omega\int_{\mathbb{R}^d} \Phi\big(f\big)\,(f*|\cdot|^\gamma)\,\mathrm{d}v\,\mathrm{d}x + \int_\Omega \iint_{\mathbb{R}^{2d}} \dot{\varphi_{0,\kappa}}(f-A)\,(A'-A)\,K_f(v,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\\
&\quad - \int_\Omega\int_{\mathbb{R}^d} \dot{\varphi_{0,\kappa}}(f-A)\partial_t A\,\mathrm{d}v\,\mathrm{d}x - \int_{\partial\Omega}\int_{\mathbb{R}^d} \varphi_{0,\kappa}(f-A)(v\cdot n(x))\,\mathrm{d}v\,\mathrm{d}S(x).
\end{aligned}$$

We recall that we assume that, depending on the boundary condition that we impose, $f \le f_b \le \mathfrak{a}(t)\langle v\rangle^{-q}$ on Γ_-, so that the boundary term vanishes.

When q is lower or equal to $d+1$, we can use the lower bound from the good extra term from Proposition 4.4 and estimate the error term due to the v-dependance of A thanks to Lemma 5.1 and the convolution term thanks to Lemma 6.1.

When q is greater than $d+1$, for the term on the second line we use the coercivity estimate from Proposition 4.5. Finally, for the error terms and q large, we use Lemma 5.4 to estimate $\mathcal{E}_1$ and either Lemma 6.2 (in case of hard potentials) or Lemma 8.2 (in case of soft potentials) to estimate the term involving the convolution.

9.3.1. *Hard potentials.*

Lemma 9.4 (Improving decay – hard potentials)**.** *Let $\gamma \ge 0$ be such that $0 < \gamma + 2s < 1$ and $q_* \ge d+1$ and $q = q_* + 2s$. If*

$$f \le C_{q_*}(1 + t^{-\beta_{q_*}})\langle v\rangle^{-q_*}, \qquad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d,$$

then

$$f \le C_q(1 + t^{-\beta_q})\langle v\rangle^{-q}, \qquad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d$$

with $\beta_q = \frac{1+2s}{2s}\beta_{q_*}$ *and* $C_q = a_* C_{q_*}^{1+\frac{1}{2s}}$ *with* a_* *only depending on* $d, \gamma, s, C_{\mathrm{vget}}, C_{\gamma,0}, C_b$, β_{q_*} *and* C_{1,q_*} *from Lemma 5.4.*

Proof. We consider the barrier $A(t,v) = \mathfrak{a}(t)\langle v\rangle^{-q}$ and the function $\mathfrak{a}(t) = C_q(1+t^{-\beta_q})$. It is convenient to write

$$\mathfrak{b}_*(t) = C_{q_*}(1+t^{-\beta_{q_*}}). \tag{9.7}$$

Truncated Convex Inequalities. Without loss of generality, we can assume that $C_{q_*} \geq 1$, so that $\mathfrak{b}_*(t) \geq 1$. We impose that

$$\mathfrak{a} \geq \underline{c}\, \mathfrak{b}_* \quad \text{with} \quad \underline{c} = \max\left(2^{2+2s}, \left(\frac{2q_0 C_{\gamma,0}}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}}, C_b\right) \tag{9.8}$$

so that we can use Proposition 4.5, Lemma 5.4 (since $q > d+1$ and $q_* > d+\gamma+2s$), Lemma 6.2 (since $q_* \geq 0$) in order to write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x \leq\ & C_{1,q_*}\mathfrak{a}(t)\mathfrak{b}_*(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\
&+ C_{1,q_*}\mathfrak{a}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v\,\mathrm{d}x\\
&+ q_0 C_{\gamma,0}\mathfrak{b}_*(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\
&- \dot{\mathfrak{a}}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v\,\mathrm{d}x\\
&- \frac{1}{2}\frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Using the fact that $q = q_* + 2s$ and $-2s \leq \gamma$, we get,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}\big(f-A\big)\,\mathrm{d}v\,\mathrm{d}x \leq\ & C_{1,q_*}\left(\mathfrak{b}_*(t)+1\right)\mathfrak{a}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\
&+ q_0 C_{\gamma,0}\mathfrak{b}_*(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\
&- \dot{\mathfrak{a}}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\
&- \frac{1}{2}\frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We also impose to $\mathfrak{a}$ to satisfy,

$$-\dot{\mathfrak{a}}(t) + C_{1,q_*}\big(\mathfrak{b}_*(t)+1\big)\mathfrak{a}(t) + q_0 C_{\gamma,0}\mathfrak{b}_*(t) \leq \frac{C_{\mathrm{vget}}}{2\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t). \tag{9.9}$$

Construction of the time barrier. We now consider

$$\mathfrak{a}(t) = a_*\mathfrak{b}_*^{1+\frac{1}{2s}}$$

and we choose $a_* \geq 1$ large enough so that $\mathfrak{a}$ satisfies both (9.8) and (9.9). In order to satisfy (9.8), we remark that $\mathfrak{b}_* \geq C_{q_*}$ and in particular $\mathfrak{a} \geq a_* C_{q_*}^{\frac{1}{2s}}\mathfrak{b}_*$, so that (9.8) holds true if we impose

$$a_* \geq C_{q_*}^{-\frac{1}{2s}}\underline{c} = C_{q_*}^{-\frac{1}{2s}}\max\left(2^{2+2s}, \left(\frac{2q_0 C_{\gamma,0}}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}}, C_b\right). \tag{9.10}$$

As far as (9.9) is concerned, we plug $\mathfrak{a} = a_*\mathfrak{b}_*^{1+\frac{1}{2s}}$ into the ODE and get (after dividing by $a_*\mathfrak{b}_*^{\frac{1}{2s}}$),

$$\frac{1+2s}{2s}(-\dot{\mathfrak{b}_*})(t) + C_{1,q_*}\mathfrak{b}_*^2(t) + (C_{1,q_*} + \underline{c}^{-1}q_0 C_{\gamma,0})\mathfrak{b}_*(t) \leq \frac{C_{\mathrm{vget}}}{2}a_*^{2s}\mathfrak{b}_*^2(t)$$

(we used that $a_* \geq C_{q_*}^{-\frac{1}{2s}} \underline{c} \geq \mathfrak{b}_*^{-\frac{1}{2s}} \underline{c}$). Since we have $\mathfrak{b}_*(t) \geq 1$, this amounts to check that

$$\frac{1+2s}{2s}(-\dot{\mathfrak{b}_*})(t) + \left(2C_{1,q_*} + \underline{c}^{-1} q_0 C_{\gamma,0}\right)\mathfrak{b}_*^2(t) \leq \frac{C_{\mathrm{vget}}}{2} a_*^{2s}\mathfrak{b}_*^2(t).$$

We now pick a_* such that

$$a_* \geq \left(2C_{1,q_*} + \underline{c}^{-1} q_0 C_{\gamma,0}\right)^{\frac{1}{2s}} \left(\frac{4}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}} \tag{9.11}$$

and we are left with verifying that

$$(-\dot{\mathfrak{b}_*})(t) \leq \frac{2sC_{\mathrm{vget}}}{4(1+2s)} a_*^{2s}\mathfrak{b}_*^2(t).$$

Recalling that $\mathfrak{b}_*$ is given by (9.7), we have to check that for all $t > 0$,

$$C_{q_*}^{-1}\beta_{q_*} t^{-\beta_{q_*}-1} \leq \frac{2sC_{\mathrm{vget}}}{4(1+2s)} a_*^{2s}(1+t^{-\beta_{q_*}})^2.$$

It is enough to pick a_* such that

$$a_* \geq C_{q_*}^{-\frac{1}{2s}} \left(\frac{4(1+2s)\beta_{q_*}}{2sC_{\mathrm{vget}}}\right)^{\frac{1}{2s}} \tag{9.12}$$

(we used that $t^{-\beta_{q_*}-1} \leq (1+t^{-\beta_{q_*}})^2$ for all $t > 0$ since $\beta_{q_*} > 1$).

Conclusion. Thanks to Lemma 9.3, we thus conclude for $\beta_q = \left(1+\frac{1}{2s}\right)\beta_{q_*}$

$$f \leq a_* C_{q_*}^{1+\frac{1}{2s}} (1+t^{-\beta_{q_*}})^{1+\frac{1}{2s}} \langle v\rangle^{-q} \leq a_* C_{q_*}^{1+\frac{1}{2s}} 2^{\frac{1}{2s}} (1+t^{-\beta_q})\langle v\rangle^{-q}.$$

(we used the convexity of $r \mapsto r^{1+\frac{1}{2s}}$) as soon as $a_* \geq 1$ satisfies (9.10), (9.11) and (9.12).

Since we assume $C_{q_*} \geq 1$, it is enough to impose,

$$a_* \geq \max\left(\underline{c}, \left(2C_{1,q_*} + \underline{c}^{-1} q_0 C_{\gamma,0}\right)^{\frac{1}{2s}} \left(\frac{4}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}}, \left(\frac{4(1+2s)\beta_{q_*}}{2sC_{\mathrm{vget}}}\right)^{\frac{1}{2s}}\right)$$

with $\underline{c} = \max\left(2^{2+2s}, \left(\frac{2q_0 C_{\gamma,0}}{C_{\mathrm{vget}}}\right)^{\frac{1}{2s}}, C_b\right)$. □

9.3.2. *Soft potentials.*

Lemma 9.5 (Improving decay – soft potentials)**.** *Let $\gamma < 0$ be such that $0 < \gamma + 2s < 1$ and q_{nsl} be given by (vii). If for some $q_* \geq q_{\mathrm{nsl}}$,*

$$f \leq C_{q_*}(1+t^{-\beta_{q_*}})\langle v\rangle^{-q_*}, \qquad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d,$$

then for $q = q_* + \delta_*$, there holds

$$f \leq C_q(1+t^{-\beta_q})\langle v\rangle^{-q}, \qquad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d$$

with

$$\delta_* = \begin{cases} \min\left(\gamma+2s-\gamma\frac{q_*}{d}, d+1-q_*\right) & \text{if } q_* < d \\ \min(2s, d+1-q_*) & \text{if } d \leq q_* < d+1 \\ 2s & \text{if } q_* \geq d+1 \end{cases} \tag{9.13}$$

and $\beta_q = \frac{1+2s}{2s}\beta_{q_*}$ and $C_q = a_* C_{q_*}^{1+\frac{1}{2s}}$ with a_* only depending on $d, \gamma, s, C_{\mathrm{vget}}, C_{\gamma,0}, C_b$ and q_*, β_{q_*} and C_{q_*}.

Remark 9.6. See (9.15) and (9.17) in the proof of the lemma for conditions on a_*.

Proof. We consider again the barrier $A(t,v) = \mathfrak{a}(t)\langle v\rangle^{-q}$ with $q = q_* + \delta_*$ (with δ_* to be chosen) and the function $\mathfrak{a}$ such that for all $t \in [0,T)$, $\mathfrak{a}(t) \geq C_b$. We also write $\mathfrak{b}_*(t)$ for $C_{q_*}(1 + t^{-\beta_{q_*}})$ and we can assume without loss of generality that

$$C_{q_*} \geq 2^d \frac{M_0}{c_{q_*}} \quad \text{and} \quad C_{q_*} \geq 1$$

(see the statement of Lemma 8.2 for the definition of c_{q_*}).

Let $q_* < d+1$. We remark that in this case, δ_* is chosen so that $q = q_* + \delta_* \leq d+1$. We impose that

$$\mathfrak{a} \geq C_b \quad \text{and} \quad \mathfrak{a} \geq 2^{2+\delta_*}\mathfrak{b}_* \quad \text{and} \quad \mathfrak{a} \geq \overline{C}\mathfrak{b}_*^{1-\frac{\gamma}{2sd}} \tag{9.14}$$

so that we can apply Proposition 4.5, Lemma 5.1 (since $q \leq d+1$), and Lemma 8.2, and get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega} \varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x + \frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}}\mathfrak{a}^{1+2s}\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x \\
\leq \frac{1}{2}C_{\mathrm{vget}}\frac{\mathfrak{a}^{1+2s}}{\mathfrak{b}_*^{2s}}\iint_{\mathbb{R}^d\times\Omega}\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x \\
+ \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}}\iint_{\mathbb{R}^d\times\Omega}\Big((f-A)_+\wedge\kappa\Big)^{q_0-1}\langle v\rangle^{\gamma\min(1,\frac{q_*}{d})-q_*}\,\mathrm{d}v\,\mathrm{d}x \\
+ C_1\mathfrak{a}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{l_q}\,\mathrm{d}v\,\mathrm{d}x \\
- \dot{\mathfrak{a}}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We remark that $l_q \leq \gamma + 2s - q$ since $q > q_{\mathrm{nsl}} \geq 1$. We also have $\gamma\min(1,\frac{q_*}{d}) - q_* \leq \gamma + 2s - q$ by the choice of δ_* in (9.13). We look for $\mathfrak{a}$ such that

$$-\dot{\mathfrak{a}} + C_1\mathfrak{a} + \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}} \leq \frac{C_{\mathrm{vget}}}{2\mathfrak{b}_*^{2s}}\mathfrak{a}^{1+2s}.$$

We plug next $\mathfrak{a} = a_*\mathfrak{b}_*^{1+\frac{1}{2s}}$ and get (after dividing by $a_*\mathfrak{b}_*^{\frac{1}{2s}}$),

$$-\frac{1+2s}{2s}\dot{\mathfrak{b}_*} + C_1\mathfrak{b}_* + \underline{C}a_*^{-1}\mathfrak{b}_*^{1-\frac{\gamma}{d}-\frac{1}{2s}} \leq \frac{C_{\mathrm{vget}}}{2}a_*^{2s}\mathfrak{b}_*^2.$$

We remark next that since $\mathfrak{b}_* \geq 1$ and $a_* \geq 1$ and $1 - \frac{\gamma}{d} - \frac{1}{2s} \leq 2$, it is sufficient to check that $\mathfrak{b}_*$ and a_* satisfy

$$-\frac{1+2s}{2s}\dot{\mathfrak{b}_*} + (C_1 + \underline{C})\mathfrak{b}_*^2 \leq \frac{C_{\mathrm{vget}}}{2}a_*^{2s}\mathfrak{b}_*^2.$$

We thus pick

$$a_* \geq \max(C_b, 2^{2+\delta_*}, 1, \overline{C}) \quad \text{and} \quad a_*^{2s} \geq \frac{4(C_1+\underline{C})}{C_{\mathrm{vget}}} \quad \text{and } a_* \geq C_{q_*}^{-\frac{1}{2s}}\left(\frac{4(1+2s)\beta_{q_*}}{2sC_{\mathrm{vget}}}\right)^{\frac{1}{2s}} \tag{9.15}$$

(see (9.12) above for the last condition) so that (9.14) holds true together with the differential inequality. We can conclude as in the case of hard potentials (thanks to Lemma 9.3) that

$$f \leq a_*C_{q_*}^{1+\frac{1}{2s}}(1+t^{-\beta_{q_*}})^{1+\frac{1}{2s}}\langle v\rangle^{-q} \leq a_*C_{q_*}^{1+\frac{1}{2s}}2^{\frac{1}{2s}}(1+t^{-\beta_q})\langle v\rangle^{-q},$$

where $\beta_q = \beta_{q_*}\big(1+\frac{1}{2s}\big)$ and a_* satisfies (9.15).

Assume now that $q_* \geq d+1 > d+\gamma+2s$, we pick $\delta_* = 2s$ and $q = q_* + \delta_* \geq d+1+2s$. We impose that

$$\mathfrak{a} \geq C_b \quad \text{and} \quad \mathfrak{a} \geq 2^{2+2s}\mathfrak{b}_* \quad \text{and} \quad \mathfrak{a} \geq \overline{C}\mathfrak{b}_*^{1-\frac{\gamma}{2sd}} \tag{9.16}$$

so that we can use Proposition 4.5, Lemma 5.4 (since $q > d+1$ and $q_* > d+\gamma+2s$) and Lemma 8.2 and find,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\Omega}\varphi_{0,\kappa}(f-A)\,\mathrm{d}v\,\mathrm{d}x + \frac{C_{\mathrm{vget}}}{\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x,\\ \leq C_{1,q_*}\mathfrak{a}(t)\mathfrak{b}_*(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\ + C_{1,q_*}\mathfrak{a}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q}\,\mathrm{d}v\,\mathrm{d}x\\ + \frac{C_{\mathrm{vget}}}{2\mathfrak{b}_*^{2s}(t)}\mathfrak{a}^{1+2s}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma+2s-q}\,\mathrm{d}v\,\mathrm{d}x\\ + \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}(v)\langle v\rangle^{\gamma-q_*}\,\mathrm{d}v\,\mathrm{d}x\\ - \dot{\mathfrak{a}}(t)\iint_{\mathbb{R}^d\times\Omega}\big((f-A)_+\wedge\kappa\big)^{q_0-1}\langle v\rangle^{-q}\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

We remark that $\gamma - q_* = \gamma+2s-q$, that $\gamma - q \leq \gamma+2s-q$ and that $-q \leq \gamma+2s-q$. We thus look for $\mathfrak{a}$ such that

$$-\dot{\mathfrak{a}} + 2C_{1,q_*}\mathfrak{a}\mathfrak{b}_* + \underline{C}\mathfrak{b}_*^{1-\frac{\gamma}{d}} \leq \frac{C_{\mathrm{vget}}}{2\mathfrak{b}_*^{2s}}\mathfrak{a}^{1+2s}$$

(we used that $\mathfrak{b}_* \geq 1$). We plug once again $\mathfrak{a} = a_*\mathfrak{b}_*^{1+\frac{1}{2s}}$ and get (after dividing by $a_*\mathfrak{b}_*^{\frac{1}{2s}}$),

$$-\frac{1+2s}{2s}\dot{\mathfrak{b}_*} + (2C_{1,q_*}+\underline{C})\mathfrak{b}_*^2 \leq \frac{C_{\mathrm{vget}}}{2}a_*^{2s}\mathfrak{b}_*^2.$$

We thus pick

$$a_* \geq \max(C_b, 2^{2+2s}, 1, \overline{C}) \quad \text{and} \quad a_*^{2s} \geq \frac{4(2C_{1,q_*}+\underline{C})}{C_{\mathrm{vget}}} \quad \text{and } a_* \geq C_{q_*}^{-\frac{1}{2s}}\left(\frac{4(1+2s)\beta_{q_*}}{2sC_{\mathrm{vget}}}\right)^{\frac{1}{2s}} \tag{9.17}$$

(see (9.12) above for the last condition) so that (9.16) holds true together with the differential inequality. We can conclude as in the case of hard potentials (thanks to Lemma 9.3) that

$$f \leq a_*C_{q_*}^{1+\frac{1}{2s}}(1+t^{-\beta_{q_*}})^{1+\frac{1}{2s}}\langle v\rangle^{-q} \leq a_*C_{q_*}^{1+\frac{1}{2s}}2^{\frac{1}{2s}}(1+t^{-\beta_q})\langle v\rangle^{-q},$$

where $\beta_q = \beta_{q_*}\big(1+\frac{1}{2s}\big)$ and a_* satisfies (9.17). □

9.4. **Proof of generation for large exponents.**

Proof of Theorem 1.6 for large q's. The idea is to iterate the gain of decay from Lemma 9.4 and Lemma 9.5 for hard and soft potentials respectively. We have shown in Subsection 9.2 that

$$f \leq C_{q_*}(1+t^{-\beta_{q_*}})\langle v\rangle^{-q_*}, \quad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d,$$

with

$$q_* = q_{\mathrm{nsl}} =: q^{(0)}, \qquad \beta_{q_*} = \frac{d}{2s} =: \beta^{(0)}.$$

where we recall that q_{nsl} is given by (vii).

Hard potentials. Lemma 9.4 then implies that there is $C_q = a_0C_{q_*}$ for some $a_0 > 0$ such that

$$f \leq C_q(1+t^{-\beta_q})\langle v\rangle^{-q}, \quad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d$$

with

$$q = q_* + 2s =: q^{(1)}, \qquad \beta_q = \frac{1+2s}{2s}\beta_{q_*} =: \beta^{(1)}.$$

We apply Lemma 9.4 again, now with $q^{(1)}$ and with $\beta^{(1)}$. We find

$$f \leq C_q(1+t^{-\beta^{(2)}})\langle v\rangle^{-q^{(2)}}, \quad \text{a.e. in } (0,T)\times\Omega\times\mathbb{R}^d$$

with

$$q^{(2)} = q^{(1)} + 2s = q_* + 2 \cdot 2s = d + 1 + 4s, \qquad \beta^{(2)} = \left(\frac{1+2s}{2s}\right)^2 \beta_{q_*} = \left(\frac{1+2s}{2s}\right)^2 \frac{d}{2s}.$$

We iterate this process and conclude for any $n \geq 0$

$$f \leq C_q (1 + t^{-\beta^{(n)}}) \langle v \rangle^{-q^{(n)}}, \quad \text{a.e. in } (0,T) \times \Omega \times \mathbb{R}^d$$

with

$$q^{(n)} = d + 1 + n \cdot 2s, \qquad \beta^{(n)} = \left(\frac{1+2s}{2s}\right)^n \frac{d}{2s}.$$

Soft potentials. We follow the same reasoning in the case of soft potentials. We apply Lemma 9.5 with $q_* = q_{\rm nsl}$ and $\beta_{q_*} = \frac{d}{2s}$: then there is $C_q = a_0 C_{q_*}$ with $a_0 > 0$ such that

$$f \leq C_q (1 + t^{-\beta_q}) \langle v \rangle^{-q}, \quad \text{a.e. in } (0,T) \times \Omega \times \mathbb{R}^d,$$

with

$$q = q_* + \delta_* =: q^{(1)}, \qquad \beta_q = \frac{1+2s}{2s} \beta_{q_*} =: \beta^{(1)},$$

where $\delta_* > 0$ depends on q_*. We iterate this process and we conclude for any $n \geq 0$

$$f \leq C_q \left(1 + t^{-\beta_q^{(n)}}\right) \langle v \rangle^{-q^{(n)}}, \quad \text{a.e. in } (0,T) \times \Omega \times \mathbb{R}^d,$$

with

$$q^{(n)} = q^{(n-1)} + \delta_*^{(n-1)}, \qquad \beta^{(n)} = \left(\frac{1+2s}{2s}\right)^n \frac{d}{2s}.$$

Writing $q^{(0)} = q_{\rm nsl}$, we remark that there exists $n_0 \in \mathbb{N}$ such that

$$q^{(n_0+1)} = d + 1.$$

More precisely, we remark that

$$q^{(n+1)} = \begin{cases} \min\left(q^{(n)} + (\gamma + 2s) + (-\gamma)\frac{q^{(n)}}{d}, d+1\right) & \text{if } q^{(n)} < d \\ \min\left(q^{(n)} + 2s, d+1\right) & \text{if } d \leq q^{(n)} < d+1 \\ q^{(n)} + 2s & \text{if } q^{(n)} \geq d+1. \end{cases}$$

In particular, as long as $q^{(n)} < d+1$, we have $q^{(n+1)} \geq q^{(n)} + (\gamma + 2s)$ and since $\gamma + 2s > 0$, this cannot hold for all $n \geq 1$.

We thus have for $n \geq n_0 + 1$,

$$q^{(n)} = d + 1 + (n - n_0 - 1)2s,$$

ensuring in particular $q^{(n)} \to \infty$ as $n \to \infty$. □

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