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Approximating the probabilistic p -Center problem under pressure

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Abstract

The Probabilistic p -Center problem under Pressure ($\text{Min } PpCP$) is a variant of the usual $\text{Min } p$ -Center problem we recently introduced in the context of wildfire management. The problem is to locate p shelters minimizing the maximum distance people will have to cover in case of fire in order to reach the closest accessible shelter. The landscape is divided into zones and is modeled as an edge-weighted graph with vertices corresponding to zones and edges corresponding to direct connections between two adjacent zones. The risk associated with fire outbreaks is modeled using a finite set of fire scenarios. Each scenario corresponds to a fire outbreak on a single zone (i.e., on a vertex) with the main consequence of modifying evacuation paths in two ways. First, an evacuation path cannot pass through the vertex on fire. Second, the fact that someone close to the fire may not take rational decisions when selecting a direction to escape is modeled using new kinds of evacuation paths. In this paper, we characterize the set of feasible solutions of $\text{Min } PpCP$ -instance. Then, we propose some approximation results for $\text{Min } PpCP$. These results require approximation results for two variants of the (deterministic) $\text{Min } p$ -Center problem called $\text{Min } MAC$ p -Center and $\text{Min } \text{Partial } p$ -Center.

Keywords Variants of the p -Center problem · Shelter location under indeterminacy · Under pressure decision model · Probabilistic combinatorial optimization · Approximation algorithms

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1 Introduction

The problem $\text{Min } p\text{PCP}$ was introduced in Demange et al. (2018) as a variant of the usual $\text{Min } p\text{-Center}$ problem with indeterminacy on vertices. In the same paper, we presented our motivation in the context of wildfire management and discussed it further in Demange et al. (2020). In our model, the landscape is represented by an adjacency graph $G = (V, E)$. Each vertex corresponds to a zone and two vertices i and j are linked by an edge if and only if it is possible to go directly from one to the other without passing through another area. We assume that it is a symmetric relation, which makes this graph non-directed. Each edge (i, j) is weighted with a positive number ℓ_{ij} that can be seen as a distance or a traveling time; we will call it *the length* of the edge (i, j) . For every two vertices i, j , $d(i, j)$ will denote the shortest path distance between i and j in G and for any set of vertices $C \subset V$, we denote $d(v, C) = \min_{c \in C} d(v, c)$ the distance from v to C . By convention, we will set $d(v, \emptyset) = +\infty$.

For a given integer p , the objective is to select a set C (called p -center) of at most p vertices, i.e. zones, where to locate fire-proof shelters so as to minimize the maximum traveling time from a zone to a shelter. In a deterministic setup, this problem is the classical $\text{Min } p\text{-Center}$ problem that aims to locate facilities on vertices of a network modeled by a graph. Given our motivating context, centers will just be called shelters and, when no ambiguity occurs, we will just use the term shelter to refer to a vertex where to install a shelter. For a set C of shelters and a vertex j , $d(j, C)$ will be called *distance to shelters* of j and the (deterministic) *radius* of C , denoted $r(C)$, corresponds to the longest distance to shelters of vertices: $r(C) = \max_{v \in V} d(v, C)$. $\text{Min } p\text{-Center}$ is to find, for any p , a set C , $|C| \leq p$ of minimum radius.

Since adding a shelter to C cannot increase its radius, it is straightforward that, if $p \leq |C|$, then there is an optimal solution with exactly p shelters; however, this is not a necessary condition for optimal solutions. Consider indeed the graph of Fig. 1 with all edge lengths equal to 1; if $p = 2$, then the minimum possible radius is 2 but $r(\{x\}) = 2$.

$\text{Min } p\text{-Center}$ and numerous versions have been extensively studied both from a graph theory perspective and for various applications [see, for instance (Calik et al. 2015)]. It is a well known NP-hard problem, even in the class of planar graphs with degrees less than 3 (Kariv and Hakimi 1979) that is particularly relevant in our motivating context. $\text{Min } p\text{-Center}$ is known to be 2-approximable (Hochbaum and

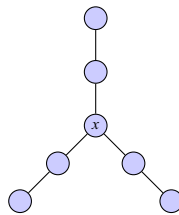


Fig. 1 An example where, for $p = 2$, a singleton minimizes the radius

Table 1 Main hardness and approximation results in the paper appear in bold characters

Instance class	Complexity	Approximation Lower bound	Upper bound
<i>Min PpCP</i>			
Any graph G with lengths in $[\ell, 2\ell]$	NP-hard Demange et al. (2018)	$\frac{20}{19}$	$\overline{4\deg(G)} + 2$ Theorem 2
Tree and all edge lengths 1	?		3 Corollary 5
<i>Min MAC p-Center</i>			
Any	NP-hard Prop. 5	2	2 Theorem 1
<i>Min Partial p-Center</i>			
Any	NP-hard Hsu and Nemhauser (1979)	2	2 Proposition 6

Shmoys 1985) and is not approximated with a constant ratio strictly smaller than 2, unless P=NP (Hsu and Nemhauser 1979). Similar results can be obtained for variants of *Min p-Center*. For instance, in Chaudhuri et al. (1998), the generalization of *Min p-Center* is investigated, where, given a number k , we have to place p centers so as to minimize the maximum distance of any non-center node to its k th closest center. A 2-approximation algorithm is proposed for this problem, and it is shown as the best possible. In this paper, to establish approximation results for the problem we deal with, we will need to establish similar results for two variants of *Min p-Center*: *Min MAC p-Center* defined in Sect. 3.2 and *Min Partial p-Center* introduced in Daskin and Owen (1999). The former problem aims at finding a p -center that is a feasible solution for the problem *Min PpCP* (see Sect. 3.1) and of minimum radius. The latter problem is to find a p -center of minimum partial radius, where only some vertices are taken into account to compute the partial radius (only these vertices are required to be close to a center).

Min PpCP is a version of *Min p-Center* with indeterminacy on vertices: with some probability, a vertex may become unavailable due to a fire outbreak. We present this problem in details in Sect. 2 after giving required related definitions and we discuss the difference with other versions of *Min p-Center* under indeterminacy. In Sect. 3, we characterize the feasible solutions (Sect. 3.1) and define *Min MAC p-Center*. We investigate some approximation results in Sects. 4 and 5. These results are based on an approximation preserving reduction (Sect. 4) between *Min MAC p-Center* and *Min PpCP*. In Sect. 5, we devise 2-approximation polynomial time algorithms for *Min Partial p-Center* and *Min MAC p-Center* and we show that these results are the best possible. These results then lead us to constant approximation algorithms for *Min PpCP*.

All definitions of problems used in the paper are recalled in appendix at the end of the paper. Main hardness and approximation results are reported in Table 1.

This paper lies in the theoretical branch of combinatorial optimization that interfaces computational complexity and polynomial approximation theories. Polynomial

approximation theory (or just approximation theory) can be seen as an extension of complexity theory and aims at characterizing the complexity of finding a feasible solution for a combinatorial optimization problem that guarantees bounds on the ratio between the value of the computed solution and the optimal value. For a minimization problem as we consider in this work, the ratio is not smaller than 1 and the smaller the better. An approximation ratio of 1 corresponds to an exact algorithm and cannot be guaranteed by a polynomial algorithm for an NP-hard problem, unless $P = NP$. Upper bounds in Table 1 correspond to approximation results (existence of a polynomial algorithm guaranteeing this bound) while lower bounds correspond to hardness in approximation (NP-hardness of finding a solution within this guarantee). For details about this domain and the related terminology, the reader is referred to Ausiello et al. (1999) and Garey and Johnson (1979). Considering such results is relevant for problems that are identified as NP-hard and then, exactly as NP-hardness results, they are critical milestones towards understanding the hardness of the problem and for the design of efficient heuristics in practice.

2 The probabilistic p -Center problem under pressure

2.1 Definition of the problem

Without any specification, graphs will be considered as non-directed; else we will specify directed graph (or digraph) or mixed graph as defined later. For all graph theory terms not defined here, the reader is referred to Diestel (2018). Graphs are also supposed simple, i.e., without loop or parallel edges with same extremities. Let G be an edge-weighted graph; we will denote it $G = (V, E, L)$ with $L = (\ell_{ij})_{i,j \in V}$ the matrix of lengths. If \mathbb{Q} denotes the set of rational numbers, L has entries in $\mathbb{Q} \cup \{\infty\}$ such that $\ell_{ij} < \infty \Leftrightarrow (i, j) \in E$. We will denote ℓ_m and ℓ_M , respectively the smallest and the largest edge lengths (i.e., ℓ_M is the largest finite entry in L). We will refer as *uniform* the case where all edge lengths are equal to 1.

A *mixed graph* is a graph with both directed and non-directed edges. When no ambiguity occurs, we will use similar notations for graphs and mixed graphs. In the mixed case, we will just identify directed edges and denote them with an arrow in the related drawing. All non-directed notions in graphs also apply to mixed graphs by considering the *non-directed version* of the mixed graph obtained by replacing directed edges by non-directed ones. Similarly, all directed notions apply to mixed graphs since a mixed graph can be seen as a digraph with non-directed edges replaced by two directed edges in opposite directions. For instance, when speaking about distances in a mixed graph, paths are meant to respect the edge orientations and thus, the matrix of distances is not symmetric anymore. In an edge-weighted graph $G = (V, E, L)$, two vertices i, j are in different connected components if and only if $d(i, j) = +\infty$. In a mixed graph however, we may have $d(i, j) = +\infty$ with i and j in the same connected component. It just means that there is no path from i to j respecting the orientation of directed edges. For example, in the mixed graph represented in Fig. 2, $d(2, 6) = 5$ while $d(6, 2) = \infty$.

In our motivating application, fire hazards (or any hazard occurring on vertices) is modeled using scenarios. The landscape is represented by an edge-weighted graph $G = (V, E, L)$. A scenario is associated with each specific fire outbreak. We restrict ourselves to single fire outbreak and consequently, each scenario s corresponds to a single vertex s on fire. This restriction is motivated by our primary focus on a relatively short time period after outbreak which assumes an efficient early warning system. In this case, everybody can escape to a shelter before the fire spreads to adjacent zones. The *operational graph* associated with the scenario s , denoted by G^s , is a mixed graph $G^s = (V, E^s, L^s)$ obtained from $G = (V, E, L)$ by replacing the edges (s, v) incident to s by directed edges (s, v) . All weights are preserved. Consequently, in G^s , vertex s is no longer accessible from another vertex.

For every two vertices i, j , the distance from i to j in G^s is denoted by $d^s(i, j)$. Note that for all $j \in V \setminus \{s\}$, we have $d^s(j, s) = +\infty$.

In this paper, we consider a uniform distribution of probabilities over all scenarios: each scenario $s_i, i \in V$ has probability $\frac{1}{|V|}$ and these events are all independent.

In most p -Center problems under indeterminacy, given a solution C with p vertices or less, and given a scenario s , the evacuation distance of a vertex j is usually the shortest distance between j and its nearest shelter, $d(j, C)$. This strategy is not adapted to our context and we consider a different evacuation strategy introduced and explained in our previous paper (Demange et al. 2020). This evacuation strategy induces new evacuation distances to shelters. If s is on fire, we have:

1. for people on s , two cases have to be considered. If a shelter is located on s , then people present on vertex s are considered as safely sheltered in it, otherwise we assume that they first run away from the fire in any direction and after they reach a neighbor j , they evacuate to the shortest shelter from j in G^s .
2. for people who are not on s , say on $j \neq s$, the evacuation distance from j to shelter k corresponds to $d^s(j, k)$ in graph G^s , i.e., avoiding vertex s .

These evacuation distances make our problem specific compared to the literature and induce some additional complexity. The justification of this measure for people escaping from s is twofold. First, since the area s may be relatively large, a single scenario may correspond to many possible fire configurations, each prohibiting some paths in the zone. The second motivation is to represent decision under stress, a very important characteristic in emergency management: somebody close to the fire may not take rational decisions when selecting a direction while people in another zone can be assumed to behave more rationally.

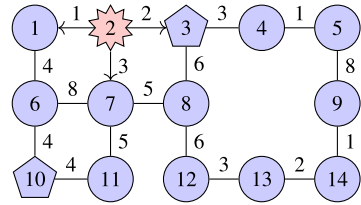
For a given set $C \subset V$ seen as a set shelter's locations and a given scenario s , the evacuation distance of a zone j is denoted by $r^s(C, j)$. If a shelter is located on j , $r^s(C, j) = 0$ otherwise we have:

$$r^s(C, j) = \begin{cases} d^s(j, C) & \text{if } j \neq s \\ \max_{v \in N_{G^s}^+(s)} \{\ell_{sv} + d^s(v, C)\} & \text{otherwise} \end{cases} \tag{1}$$

where $N_{G^s}^+(s)$ is the set of all vertices v such that $(s, v) \in E^s$.

Notice that $r^s(C, j)$ is equal to $+\infty$ if j can't reach any shelter in G^s .

Fig. 2 The operational graph associated with scenario 2 with $C = \{3, 10\}$



The *evacuation radius* associated with scenario s is defined as $r^s(C) = \max_{j \in V} r^s(C, j)$. Note that $r^s(C)$ is not equal to the usual radius computed in G^s : $r^s(C) \geq \max_{v \in V} d^s(v, C)$ and for any scenario s and vertex v , $d^s(v, C) \geq d(v, C)$ since evacuation paths in G may pass through s . Consequently, we have $r^s(C) \geq r(C)$.

Example 1 This example is adapted from Demange et al. (2020) and allows to better understand the evacuation radius $r^s(C)$ and the operational graph.

Let us consider $p = 2$ and the non-directed version $G = (V, E, L)$ of the graph in Fig. 2. We consider the scenario $s = 2$. The related operational mixed graph is given in Fig. 2. Vertices 3 and 10 represented by pentagons correspond to shelters' locations ($C = \{3, 10\}$). In case of fire on vertex 2 (scenario 2), the modification of the graph and the evacuation strategy induce:

- The shortest path length from 1 to 3 is no longer 3 but 23, using the shortest path 1, 6, 7, 8, 3. Consequently, the nearest shelter from vertex 1 is 10 at a distance of 8. Thus the evacuation distance of 1 in scenario 2, is equal to 8 and vertex 1 is evacuated to vertex 10.
- To compute the evacuation distance of vertex 2 in scenario 2, we have to consider three neighbors:
 - for neighbor 1, the distance to the nearest shelter 10 is 8;
 - for neighbor 7, the distance to the nearest shelter 10 is 9;
 - for neighbor 3 with a shelter, the distance is 0.

Consequently, $r^2(C, 2) = \max\{1 + 8, 3 + 9, 2 + 0\} = 12$.

- The evacuation radius of the scenario 2 is given by $r^2(C) = \max_{j=1, \dots, 14} r^2(C, j) = r^2(C, 13) = 15$.

□

We are now ready to define the problem $\text{Min } p\text{PCP}$. A $\text{Min } p\text{PCP}$ -instance will be an edge-weighted graph G and an integer p and a solution C will correspond to a set of at most p vertices where to locate shelters. A given solution C corresponds to $n = |V|$ evacuation radius $r^1(C), \dots, r^n(C)$ for n different scenarios. We associate to C the expected value $\mathbb{E}(C)$ of these evacuation radius over all scenarios; since all scenarios are supposed equi-probable, it is just the average value:

$$\mathbb{E}(C) = \frac{1}{|V|} \sum_{s \in V} r^s(C) = \frac{1}{|V|} \sum_{s \in V} \max_{j \in V} r^s(C, j) \tag{2}$$

$\mathbb{E}(C)$ is called probabilistic radius. For any set C of centers, it can be computed in polynomial time: for each scenario it requires to compute the matrix of shortest path values in the related operational graph, which requires $O(|V|^3)$ operations. So, $\mathbb{E}(C)$ can be computed in $O(|V|^3)$. The $\text{Min } p\text{CP}$ problem is then to determine a solution C^* minimizing \mathbb{E} .

We synthesize below the formal definition of the problem:

$\text{Min } p\text{CP}$

Instance: An edge-weighted graph $G = (V, E, L)$ and an integer p ; the instance is denoted (G, p)

Feasible solutions: Any p -center $C \subset V$, $|C| \leq p$ satisfying $\mathbb{E}(C) < \infty$ (see Relation 2)

Objective: Minimizes $\mathbb{E}(C)$.

In a more general setting we could add a probability distribution on vertices but in this work we only consider the uniform probability distribution. To avoid any confusion, we recall that, in this work, the term uniform refers to the case where L is the matrix (ℓ_{ij}) with $\ell_{ij} = 1 \Leftrightarrow (i, j) \in E$ and $\ell_{ij} = \infty$ otherwise.

Note that in our definition, p is part of the instance. We can define natural sub-problems by restricting the possible values for p . If p is a fixed value, then the related sub-problem is polynomial since all possible p -centers can be enumerated in polynomial time and the probabilistic radius (objective value) of each one can be determined in polynomial time.

2.2 Related work

A variant of the $\text{Min } p\text{-Center}$ problem for large-scale emergencies is proposed in Huang et al. (2010), where the disaster affects a single vertex s , including any facility on this vertex. This model incorporates both indeterminacy in the facility availability and in the demand: any facility on an affected vertex is no longer available and only the population on this vertex requires evacuation. Our context is really different since we consider that all zones must be evacuated in each scenario s and that a shelter always secures at least the people from the corresponding area.

Numerous models for $\text{Min } p\text{-Center}$ under indeterminacy have already been developed. In a non-deterministic environment, problems are generally described in two stages: first, before the indeterminacy is resolved (i.e., when the instance is still subject to indeterminacy), we need to choose locations. Then, once the effective instance is known, we can react, for example by assigning vertices to centers. This description matches real life situations like facility breakdown or natural disaster cutting off communication, and it has been addressed using various approaches [see for example the reviews (Caunhye et al. 2012; Laporte et al. 2015; Snyder 2006; Correia and da Gama 2015)]. We briefly present some $\text{Min } p\text{-Center}$ variants under indeterminacy, as well as some other models relevant for our context.

In some models, non determinate parameters may vary independently one from each other, for example in Averbakh and Berman (1997), Averbakh (2003), Lu (2013) and Taghavi and Shavandi (2012) lengths in the graph are described as intervals. In our

context, this independence hypothesis is not relevant since, if a fire ignites on a vertex, then all lengths of the edges incident to this vertex are modified in a same way. So, we focus on the $\text{Min } p\text{-Center}$ variants where indeterminacy is represented by a set of discrete scenarios.

In such a decision-making environment, we usually distinguish two contexts: risk and uncertainty. Risk refers to situations where the values of some parameters are governed by given probability distributions. In uncertainty on the contrary, no probabilistic information is used, either because it is not available or because the decision maker prefers not to resort to it. The first context is referred as *stochastic*, or *probabilistic* models and the second one corresponds to *robust* models. In this second context, a measure of robustness is usually considered for evaluating the performance of a solution. For example, in Du et al. (2020), a robust variant of $\text{Min } p\text{-Center}$ with scenarios is studied. In Demange et al. (2020), we investigated a robust variant of $\text{Min } PpCP$ where the objective is to minimize the maximum (worst) evacuation radius over all scenarios instead of minimizing their expected value. For this version, we proposed NP-hardness results in various classes of graphs that include subgrids. Our application motivates this class. We also proposed exact algorithms based on Integer Linear Programming formulation.

Here, we adopt a stochastic/probabilistic approach and for this reason we assume some probability distribution over scenarios. We focus on a probabilistic variant of $\text{Min } p\text{-Center}$ with a fixed set of discrete scenarios. In stochastic/probabilistic optimization, one generally optimizes the expected value of a given objective function or maximizes the probability that the solution is “good”. Such problems can be solved using a specific algorithm, like in Martínez-Merino et al. (2017), or using general stochastic programming techniques, as in Bayram and Yaman (2018). The problem studied in Martínez-Merino et al. (2017) is different from ours since the indeterminacy is associated to the demand and it is not possible to re-affect the vertices to a center. Note that, despite $\text{Min } PpCP$ falls into the paradigm of stochastic optimization, we cannot easily reduce it to classical $p\text{-center}$ variants. In particular, $\text{Min } PpCP$ cannot be seen as a variant in which new distances are associated to each scenario. Indeed, due to our specific evacuation process, in particular for people in the zone on fire, the new evacuation distances to a shelter do not systematically correspond to a shortest path in the new graph G^s .

$\text{Min } PpCP$ itself has been introduced in Demange et al. (2018). We have proposed an explicit solution for the uniform case (all edge lengths are 1) on paths and cycles. The decision version of $\text{Min } PpCP$ is in NP. Indeed, if we consider a set of centers C , then for each scenario s , the evacuation radius $r^s(C)$ can be computed in polynomial time using a shortest path algorithm. Then, $\mathbb{E}(C)$ can be computed in polynomial time using Relation 2. So, a non-deterministic algorithm will infer a $p\text{-center}$ and verify in polynomial time whether its value exceeds or not the target. As a consequence, any hardness in approximation result for $\text{Min } PpCP$ can be immediately turned into a NP-completeness result.

Regarding hardness results for $\text{Min } PpCP$, note that it does not count the classic deterministic $\text{Min } p\text{-Center}$ problem as one of its specific cases since we defined it with fixed uniform probabilities. So, the hardness of $\text{Min } PpCP$ cannot be directly deduced from the NP-hardness of $\text{Min } p\text{-Center}$ (Kariv and Hakimi 1979). In Demange

et al. (2018), we showed that $\text{Min } PpCP$ is not approximable on planar graphs of degree 2 or 3 within a ratio less than $\frac{20}{19}$, unless $P=NP$.

Hardness results for $\text{Min } PpCP$ motivate designing new approximation results for this problem, even in restricted classes of graphs. This is the aim of the remaining part of this paper. In the next section, we characterize the set of feasible solutions of $\text{Min } PpCP$ and define the $\text{Min } MAC$ p -Center problem, for which we will give approximation results.

3 Feasibility and the $\text{Min } MAC$ p -Center problem

3.1 Feasible solutions

In this subsection, we analyze necessary and sufficient conditions for a solution to be feasible for a given $\text{Min } PpCP$ -instance (G, p) . Without loss of generality we will consider that $G = (V, E, L)$ is a connected graph. A vertex $a \in G$ is an *articulation point* if and only if removing a disconnects the graph G . We denote by $\mathcal{A}(G)$ the set of articulation points of G .

For any articulation point a , a connected component of $G \setminus \{a\}$ is called *articulation component*. Then, every vertex $a \in \mathcal{A}(G)$ is associated to at least 2 articulation components, and every articulation component is associated to one articulation point. A graph is 2-connected if it has no articulation point; in this case there is no articulation component.

A *minimal articulation component*, or *MAC* for short, is an articulation component that does not strictly contain another articulation component. We denote $\Upsilon(G)$ the set of minimal articulation components. Note that an articulation component that is a singleton $\{v\}$ is necessarily minimal and this occurs if and only if v is a vertex of degree 1.

Lemma 1 *A is a minimal articulation component of G if and only if A is an articulation component which does not include an articulation point of G .*

Proof \Rightarrow By contrapositive we prove that if an articulation component A includes an articulation point, then A is not minimal. Let A be an articulation component induced by the articulation point $a \in V$. Suppose $b \in A$ is an articulation point of G . Then b induces at least two disjoint connected components in $G \setminus \{b\}$. Since $b \neq a$, a is in one connected component of $G \setminus \{b\}$, consequently $G \setminus A$ is a subset of this connected component. It follows that at least another component of $G \setminus \{b\}$ is contained in A , which means that A is not minimal.

\Leftarrow The proof is also by contrapositive. We prove that if A is a non-minimal articulation component, then A includes an articulation point. Let A an articulation component that is not minimal. Then there is an articulation component $B \subset A$ induced by the articulation point $b \in V$, such that $B \neq A$. Consider $x \in A \setminus B$ and $y \in B$. Since A is connected, x and y are connected in A by a path; this path necessarily crosses b and in particular $b \in A$. \square

Lemma 2 *All minimal articulation components of G are pairwise disjoint.*

Proof By contrapositive, we assume $A \in \Upsilon(G)$ and B an articulation component such that $B \neq A$ and $B \cap A \neq \emptyset$. We prove then that B is not minimal.

Let $x \in A \cap B$. Since A is a MAC, $B \not\subseteq A$. Then there is a vertex $y \in B \cap (V \setminus A)$. Every path between x and y in G crosses a . As B is a connected component, there is a path from x to y in B , thus $a \in B$, and B is not a MAC by Lemma 1. \square

Given an edge-weighted graph $G = (V, E, L)$ and p , we denote with $\mathcal{C}_p(G)$ the set of feasible solutions of the Min PpCP -instance (G, p) .

Proposition 1 *Let (G, p) be an instance of Min PpCP with $|V| \geq 2$. A solution $C \subset V$, $|C| \leq p$ is in $\mathcal{C}_p(G)$ if and only if $|C| \geq 2$ and C includes at least one vertex in each minimal articulation component of G .*

Proof Suppose C is a feasible solution for Min PpCP on G . We have seen that C is a feasible solution for Min PpCP if and only if $r^s(C, j) \in \mathbb{R}, \forall j, s \in V$, i.e. all the evacuation distances over all vertices and all scenarios are finite.

First suppose there is no articulation point, then G has no articulation component. Let $s \in C$, and $x \in V, x \neq s$. In scenario s , x is assigned to a center that is not s . Thus $|C| \geq 2$. Conversely, if $|C| \geq 2$, for any scenario s , $G \setminus \{s\}$ is connected and contains at least one center.

Second, suppose that G has at least one articulation point and consequently at least 2 disjoint articulation components. In addition, if A is an articulation component of G induced by the articulation point a , then $\forall j \in A, r^a(C, j) \in \mathbb{R}$ if and only if $C \cap A \neq \emptyset$. Then C intersects all articulation components. In particular $|C| \geq 2$ and C intersects all minimal ones. Conversely, if C intersects all MACs then $|C| \geq 2$ due to Lemma 2 and it intersects all articulation components since any articulation component contains a MAC. \square

Corollary 1 *If G has at least 2 vertices, $\mathcal{C}_1(G) = \emptyset$.*

As a consequence, from now we will consider only Min PpCP instances satisfying $p \geq 2$.

Corollary 2 *For a given p , we can verify in polynomial time whether $\mathcal{C}_p(G) \neq \emptyset$.*

Proof For $G = (V, E)$, we generate $\mathcal{A}(G)$ in $O(|V| + |E|)$ using Tarjan’s Algorithm (Tarjan 1972). The minimal connected components of G are the connected components of $G \setminus \mathcal{A}(G)$ adjacent to at most one articulation point in G , where a set V' of vertices is said adjacent to a vertex if this vertex has at least one neighbor in V' .

There is a feasible solution for Min PpCP on G if p is greater or equal to the number of MACs. \square

Corollary 3 *For all $C \in \mathcal{C}_p(G)$, C necessarily includes all vertices of degree 1.*

Proof Every vertex of degree 1 is a MAC of G . Then by Proposition 1, a feasible solution includes all vertices of degree 1. \square

3.2 The $\text{Min MAC } p\text{-Center}$ problem

Proposition 1 ensures that, given an edge-weighted graph $G = (V, E, L)$, a set $C \subset V$ is in $\mathcal{C}_p(G)$ (i.e., is feasible for $\text{Min } PpCP$) if and only if $|C| \geq 2$ and C intersects all MACs. For any $p \leq |V|$, we call *MAC p -Center* a p -center intersecting all MACs. For $p \geq 2$, $\mathcal{C}_p(G)$ is the set of MAC p -centers.

By analogy to the usual p -Center problem, this makes natural considering the problem of finding a MAC p -center of minimum radius. We call this problem $\text{Min MAC } p\text{-Center}$. Recall that the radius of C is $r(C) = \max_{v \in V} d(v, C)$; it is a common measure used to evaluate a p -center. Note that for any scenario $s \in V$, $r^s(C) \geq r(C)$. The $\text{Min MAC } p\text{-Center}$ problem has a feasible solution for a graph G if and only if p is at least the number of MACs in G , i.e., $p \geq |\Upsilon(G)|$. This problem is close to the usual $\text{Min } p\text{-Center}$ problem with the particularity that it has the same feasible solutions as $\text{Min } PpCP$ in the same graph.

$\text{Min MAC } p\text{-Center}$ will be used in the two following sections to devise approximation results for $\text{Min } PpCP$. In Sect. 4, we propose an approximation preserving reduction between the $\text{Min MAC } p\text{-Center}$ problem and $\text{Min } PpCP$ (Proposition 4): it allows to derive an approximation result for the latter from an approximation result for the former. Then, in Sect. 5.3, we propose an approximation algorithm for $\text{Min MAC } p\text{-Center}$ (Theorem 1) that leads to an approximation result for $\text{Min } PpCP$ (Theorem 2 in Sect. 5). The result is limited to the class of instances where edge lengths lie into $[l, 2l]$ for a positive l since this is a restriction in Proposition 4 for the reduction to be valid.

4 A polynomial approximation preserving reduction

In what follows, we describe an approximation preserving reduction between $\text{Min MAC } p\text{-Center}$ and $\text{Min } PpCP$. A polynomial approximation algorithm for the former leads to a polynomial approximation algorithm for the latter with a ratio that depends on the average degree $\overline{deg}(G) = \frac{2|E|}{|V|}$ of G . More precisely, the reduction is even the identity: we analyze in Proposition 4 how good for the problem $\text{Min } PpCP$ an approximated MAC p -center can be. Then, in Sect. 5, we show that $\text{Min MAC } p\text{-Center}$ can be approximated within the ratio 2 (Theorem 1), which leads to a $(4\overline{deg}(G) + 2)$ -approximation for the $\text{Min } PpCP$ where all edge-lengths lie in the interval $[l, 2l]$ for any positive l (Theorem 2).

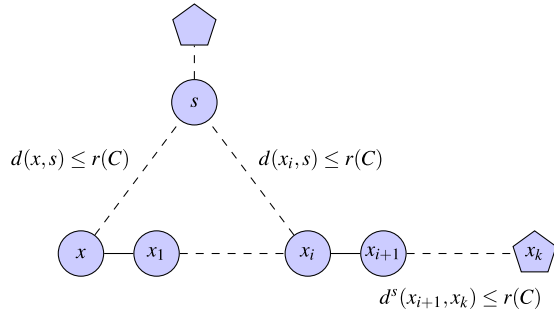
Proposition 4 first requires a technical result described in Proposition 2, an upper bound on $\mathbb{E}(C)$ for a graph G . Then, we improve the upper bound in the case of trees (Proposition 3) before describing the main result of this section (Proposition 4).

For a real number x , we denote $x^+ = \max(x, 0)$ its positive part.

Proposition 2 *On an edge weighted graph G with lengths in $[\ell_m, \ell_M]$, $\forall C \in \mathcal{C}_p(G)$, we have:*

$$\mathbb{E}(C) \leq (2\overline{deg}(G) + 1)r(C) + (\ell_M - 2\ell_m)^+ \overline{deg}(G)$$

Fig. 3 Distance relations between vertices x, s and x_k used for Proposition 2 if $x \neq s$



Proof Let us consider any scenario $s \in V$ of degree $deg(s)$ and number $1, 2, \dots, deg(s)$ the edges incident to s . We claim that:

$$r^s(C) \leq (2deg(s) + 1)r(C) + deg(s)(\ell_M - 2\ell_m)^+ \tag{3}$$

and we then will conclude taking the average value over all vertices.

The end of the proof aims to prove Relation 3. Consider $x \in V$ such that $r^s(C, x) = r^s(C)$. We already noticed in Sect. 2.1 that for any scenario s , $r^s(C) \geq r(C)$ and so, $r^s(C, x) \geq r(C)$. If $r^s(C, x) = r(C)$, then the claim is satisfied since $(2deg(s) + 1) \geq 1$ and $(\ell_M - 2\ell_m)^+ \geq 0$. So, let us assume from now that $r^s(C, x) > r(C)$.

We consider two cases whether $x \neq s$ or $x = s$.

Case 1 $x \neq s$. For this case, the reader is referred to Fig. 3. In the figure, no shelter is located on s , but the reasoning is the same if there is one.

Using Relation 1, we have $r^s(C, x) = d^s(x, C)$ and it is the length of a path $\mu = [x_0, x_1, \dots, x_k]$, where $x_0 = x, x_k \in C$ and μ is a minimum path between x and C in G^s . We also know that x_k is the only vertex of μ that is in C .

Since $d^s(x, x_k) = d^s(x_0, x_k) > r(C)$, we can define $i = \max\{j \in \{0, \dots, k - 1\}, d^s(x_j, x_k) > r(C)\}$; as a consequence, $d^s(x_{i+1}, x_k) \leq r(C)$ and $x_i \notin C$.

Let $j \in \{0, \dots, i\}$, the path $[x_j, \dots, x_k]$ is a minimum path between x_j and C in G^s and is of length greater than $r(C)$. This implies that the distance $d(x_j, C)$ between x_j and C in G is less than $d^s(x_j, C)$ and consequently, any evacuation path of vertex x_j in C passes through s (else it would be an evacuation path in G^s). We deduce that, in G , x_j is at distance at most $r(C)$ from s : $d(x_j, s) \leq r(C)$ as illustrated in Fig. 3.

For every $j \in \{0, \dots, i\}$, we consider a minimum path in G from x_j to s , of value at most $r(C)$. We then assign to x_j a color in $N_1, \dots, N_{deg(s)}$ depending on the last edge of the minimum path we have fixed for x_j : x_j is assigned the color N_t if the related minimum path between x_j and s terminates with the t th edge incident to s .

The distance, in G^s , between two vertices $x_j, x_{j'}$ of the same color is at most $2r(C) - 2\ell_m$. Consider indeed two minimum paths in G from these vertices to s and sharing the last edge (t, s) ; as seen above, their length is at most $r(C)$. We deduce a walk avoiding s between them by concatenating the paths from x_j to t and from t to $x_{j'}$. Since the length of the edge (t, s) is at least ℓ_m , this walk is of length at most $2r(C) - 2\ell_m$. Such a walk includes a path in G^s of length at most $2r(C) - 2\ell_m$ between these two vertices.

This allows us to derive an upper bound of $d^s(x, x_i)$. Suppose x is of color N_{i_1} and consider the last vertex x_{j_1} of color N_{i_1} along the path μ ; we have $d^s(x, x_{j_1}) \leq 2r(C) - 2\ell_m$. If $j_1 < i$, then the vertex x_{j_1+1} is of color N_{i_2} and $d^s(x_{j_1}, x_{j_1+1}) \leq \ell_M$ (we denote such an edge a bridge). Using the same reasoning for all non-empty colors gives:

$$d^s(x, x_i) \leq deg(s)(2r(C) - 2\ell_m) + (deg(s) - 1)\ell_M \tag{4}$$

In Relation 4, $deg(s)$ is the maximum number of non-empty colors and $deg(s) - 1$ is the maximum number of bridges.

Taking into account the edge (x_i, x_{i+1}) and the fact that $d^s(x_{i+1}, x_k) \leq r(C)$ we deduce:

$$\begin{aligned} r^s(C) = r^s(C, x) = d^s(x, C) &\leq d^s(x, x_i) + l_{x_i x_{i+1}} + d^s(x_{i+1}, x_k) \\ &\leq deg(s)(2r(C) - 2\ell_m) + (deg(s) - 1)\ell_M + \ell_M + r(C) \\ &\leq (2deg(s) + 1)r(C) + deg(s)(\ell_M - 2\ell_m) \\ &\leq (2deg(s) + 1)r(C) + deg(s)(\ell_M - 2\ell_m)^+ \end{aligned} \tag{5}$$

So, Relation 3 holds in this case, which concludes the first case.

Case 2 $x = s$. In this case, since $r^s(C, x) > r(C) \geq 0$, we know that $s \notin C$. Then, Relation 1 indicates this time that $r^s(C, s)$ is the length of a path $\mu = [x_0, x_1, \dots, x_k]$, where $x_0 = s, x_k \in C$ and $[x_1, \dots, x_k]$ is a minimum path between x_1 and C in G^s . So, the length of μ is $l_{sx_1} + d^s(x_1, x_k)$. We use similar arguments as in the first case, but some details require our attention. First, if $d^s(x_1, x_k) \leq r(C)$, then the length of μ is at most

$$\begin{aligned} \ell_M + r(C) &\leq r(C) + 2r(C) + (\ell_M - 2\ell_m) \\ &\leq (2deg(s) + 1)r(C) + deg(s)(\ell_M - 2\ell_m) \\ &\leq (2deg(s) + 1)r(C) + deg(s)(\ell_M - 2\ell_m)^+ \end{aligned} \tag{6}$$

where the first inequality holds because $r(C) \geq \ell_m$, the second because $deg(s) \geq 1$ and the last because $(\ell_M - 2\ell_m) \leq (\ell_M - 2\ell_m)^+$. This is exactly the Relation 3.

If now $d^s(x_1, x_k) > r(C)$, we define $i = \max\{j \in \{1, \dots, k - 1\}, d^s(x_j, x_k) > r(C)\}$ by analogy with the Case 1. As already noticed in the Case 1, all evacuation paths of $x_j, j = 1, \dots, i$, in G , pass through s . We select one such evacuation path for each $x_j, j \in \{1, \dots, i\}$. Since $s \notin C$, we can assume that they all share a non-empty minimum path, in G , from s to C . The main change with Case 1 is that we now can define at most $deg(s) - 1$ colors for x_1, \dots, x_i ($x_0 = s$ does not have a color) depending on which edge incident to s contributes to the selected evacuation path of x_j and taking into account that one edge incident to s is used for the minimum path from s to C in G (second part of the evacuation path after s).

As in Case 1, the distance, in G^s , between two vertices of the same color is at most $2r(C) - 2\ell_m$.

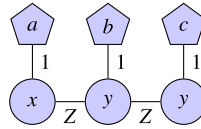


Fig. 4 A case where $\frac{\mathbb{E}(C)}{r(C)} = Z + 1$

Since we have at most $\text{deg}(s) - 1$ colors, the length of μ satisfies:

$$\begin{aligned} r^s(C) &\leq l_{xx_1} + (\text{deg}(s) - 1)(2r(C) - 2\ell_m) + (\text{deg}(s) - 2)\ell_M + \ell_M + r(C) \\ &\leq (2\text{deg}(s) - 1)r(C) + \text{deg}(s)(\ell_M - 2\ell_m) + 2\ell_m \tag{7} \\ &\leq (2\text{deg}(s) + 1)r(C) + \text{deg}(s)(\ell_M - 2\ell_m)^+ \end{aligned}$$

since $l_{xx_1} \leq \ell_M$ and $\ell_m \leq r(C)$.

So, Relation 4 holds in all cases as we claimed. By taking the average value, we deduce that

$$\mathbb{E}(C) = \frac{1}{|V|} \sum_{s \in V} r^s(C) \leq (2\overline{\text{deg}}(G) + 1)r(C) + (\ell_M - 2\ell_m)^+ \overline{\text{deg}}(G)$$

which concludes the proof. □

On a tree, the analysis can be improved:

Proposition 3 *On a tree with edge lengths in $[\ell_m, \ell_M]$, $\forall C \in \mathcal{C}_p(G)$, we have:*

$$\mathbb{E}(C) \leq 3r(C) + (\ell_M - 2\ell_m)^+$$

Proof Consider, for a scenario s , and a vertex x , $r^s(C, x) = r^s(C)$, the same analysis as in the proof of Proposition 2. Since there is no cycle, all vertices x, \dots, x_i are of the same color. Equation 5 becomes

$$r^s(C) \leq 3r(C) + (\ell_M - 2\ell_m)^+$$

which concludes the proof. □

Remark 1 In Demange et al. (2018), we have shown that, on paths with all edge-weight 1, there is an optimal solution C^* of Min MAC p -Center such that $\mathbb{E}(C^*) = r(C^*)$.

Propositions 2 and 3 can be seen as upper bounds of the ratio $\frac{\mathbb{E}(C)}{r(C)}$ that play a crucial role in the next proposition. In particular, the ratio is bounded if $(\ell_M - 2\ell_m)^+$ is bounded. We cannot expect a similar results in graphs with general lengths as outlined in the following example. Consider the caterpillar H (in particular it is a tree) of Fig. 4 with three internal vertices x, y, z and edges (x, y) and (y, z) of length Z and three pendent vertices a, b, c , respectively linked to x, y, z with edges of length 1.

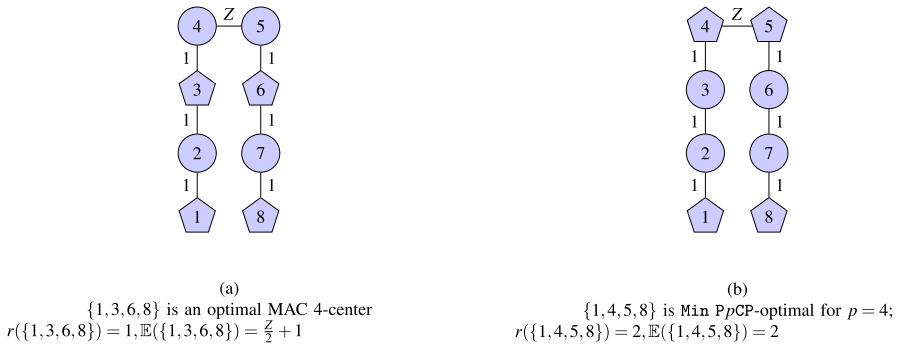


Fig. 5 With general weights, an optimal MAC p -center can be a very bad Min PpCP solution

$\{a, b, c\}$ is the unique feasible solution of the Min PpCP-instance $(H, 3)$. We have $r(\{a, b, c\}) = 1$. However, for any scenario s , $r^s(\{a, b, c\}) = Z + 1$, which implies $\mathbb{E}(\{a, b, c\}) = Z + 1$. It gives an example where the upper bound of Proposition 3 is achieved ($Z + 1 = 3 \times 1 + Z - 2 \times 1$) but also shows that the ratio $\frac{\mathbb{E}(C)}{r(C)}$ can infinitely grow when edge lengths are not bounded.

We are now ready to show the main result of this section, namely a polynomial approximation reduction that allows to approximate Min PpCP using an approximation for Min MAC p -Center.

Proposition 4 *Suppose a class of edge-weighted graphs $G = (V, E, L)$ with $\ell_M \leq 2\ell_m$ for which Min MAC p -Center can be approximated with $\rho(G)$. Then, Min PpCP can be approximated with $(2\overline{\text{deg}}(G) + 1)\rho(G)$ on the same class.*

Proof Given a graph G in the class, we build a p -center C in $\mathcal{C}_p(G)$, if it exists, of value at most $\rho(G)r^*(G)$, where $r^*(G)$ denotes the optimal radius of a MAC p -center in G . Using Proposition 2 and $\ell_M \leq 2\ell_m$, we have $\mathbb{E}(C) \leq (2\overline{\text{deg}}(G) + 1)r(C) \leq (2\overline{\text{deg}}(G) + 1)\rho(G)r^*(G)$.

Now if C^* is an optimum solution for Min PpCP, we have $\mathbb{E}(C^*) \geq r(C^*) \geq r^*(G)$. This concludes the proof. □

As noticed in the following example in Fig. 5, the situation with a general weight system may be totally different. In this example the graph is a path on 8 vertices with only one edge of weight $Z > 1$ and all other edges of weight 1 and $p = 4$. There is a unique optimal MAC 4-center (Fig. 5a) and, for large values of Z , its value is very bad compared to an optimal Min PpCP solution shown in Fig. 5b.

5 Constant approximation algorithms

The main objective of this section is to derive approximation results for Min PpCP using Proposition 4. This first requires a constant approximation result for Min MAC p -Center. To this aim, we propose Algorithm 1 and prove it is a 2-approximation polynomial-time algorithm for Min MAC p -Center.

To properly explain the ideas of Algorithm 1, we will need another p -center problem called *Min Partial p -Center* that was introduced in Daskin and Owen (1999). Given an edge-weighted graph $G = (V, E, L)$ and a set of vertices $U \subset V$, *Min Partial p -Center* is to minimize the *partial radius* $r(C, U)$ of a p -center C , where $r(C, U) = \max_{x \in U} d(x, C)$. The underlying logic is that only vertices in U need to be close to a center. However, centers can be any vertex in G and distances are computed in G (within our terminology, it means that the evacuation paths toward a shelter are not required to stay in U).

The idea of Algorithm 1 is to reduce *Min MAC p -Center* to *Min Partial p -Center* through a pre-processing that allocates some centers to MACs. Then, the solution is completed using *Min Partial p -Center*. As we will see, *Min Partial p -Center* can be approximated by generalizing the 2-approximation algorithm for *Min p -Center* in Hochbaum and Shmoys (1985) or using the general method described in Hochbaum and Shmoys (1986). However, since it cannot be directly deduced from existing results, we will give a direct proof through a few claims.

This section is organized as follows. In Sect. 5.1, we first show that our problems, *Min Partial p -Center* and *Min MAC p -Center* are not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$ in graphs with all edge lengths 1, unless $P=NP$. This gives a lower bound on their approximation ratios. Then, in Sect. 5.2, we establish (Proposition 6) that *Min Partial p -Center* is polynomially 2-approximable. In Sect. 5.3, we present Algorithm 1 and show (Theorem 1) that it guarantees a polynomial 2-approximation for *Min MAC p -Center*. Finally, we deduce in Sect. 5.4 the approximation results for *Min p CP*.

5.1 Lower bound on the approximation ratios

We recall, that a *dominating set* in a graph $G = (V, E)$ is a subset U of V such that every vertex not in U is adjacent to at least one member of U . The *Min Dominating Set* problem is to find a dominating set of minimum size.

For a graph $G = (V, E)$, $U \subseteq V$ is a dominating set, if and only if U is a $|U|$ -center of radius 1.

Lemma 3 *Min Partial p -Center is not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$ in planar bipartite graphs of degree 3 with all edge lengths 1, unless $P=NP$.*

Proof If $U = V$, then $r(C, V) = r(C)$ and *Min Partial p -Center* is just the usual *Min p -Center* problem. So, *Min p -Center* is a particular case of *Min Partial p -Center*. In particular, *Min Partial p -Center* is NP-hard and not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$, unless $P=NP$ by using the same hardness result for *Min p -Center* proved in Hsu and Nemhauser (1979). This result is directly obtained from the NP-hardness of *Min Dominating Set* and holds in the uniform case (all edges have the length 1). Since *Min Dominating Set* remains NP-hard in planar bipartite graphs of degree 3 (Clark et al. 1990), *Min p -Center*, and by consequence *Min Partial p -Center*, are not approximable within $2 - \varepsilon$

for any $\varepsilon > 0$ in planar bipartite graphs of degree 3 with all edge lengths 1, unless $P=NP$. \square

Note that the argument used for $\text{Min Partial } p\text{-Center}$ cannot be easily adapted to $\text{Min MAC } p\text{-Center}$ since this latter problem is not an immediate generalization of $\text{Min } p\text{-Center}$. However, we can show that:

Proposition 5 *Min MAC p -Center in graphs with edge lengths all equal to 1 is not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$, unless $P=NP$.*

Proof We first notice an immediate hardness result for the general weighted case (edge weights can be any non-negative number). For any edge weighted graph $G = (V, E, L)$, instance of $\text{Min } p\text{-Center}$, the instance is equivalent to the instance (K, \tilde{L}) , where K is the complete graph over V and \tilde{L} denotes the minimum path distance, i.e., $\forall i, j \in V, \tilde{L}_{ij} = d(i, j)$, where the distance d is the distance in G . Both instances G and K have the same feasible solutions with the same values and thus, the same optimal solutions. To guarantee finite edge lengths in K , we just consider that G is connected. Since K is 2-connected as soon as $|V| \geq 2$, $\text{Min MAC } p\text{-Center}$ is equivalent to $\text{Min } p\text{-Center}$ on K . Since the hardness result for $\text{Min } p\text{-Center}$ still holds in connected graphs, $\text{Min MAC } p\text{-Center}$ is not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$, unless $P=NP$.

To show that this hardness results already holds for the uniform case, i.e., where all edge lengths are 1, we revisit the classical reduction of $\text{Min Dominating Set}$ to $\text{Min } p\text{-Center}$ given in Hsu and Nemhauser (1979). Consider a graph $G = (V, E)$, instance of $\text{Min Dominating Set}$ and an integer k . As already noticed, there is in G a dominating set of cardinality k if and only if there is a k -center of radius 1. Any polynomial approximation algorithm guaranteeing the ratio $2 - \varepsilon$ for $\text{Min } p\text{-Center}$ allows to discriminate, for any k , between instances with a k -center of radius 1 to instances with all k -centers of radius at least 2. Using such an algorithm for $k \in \{1, \dots, |V|\}$ allows to compute in polynomial time a minimum dominating set.

On the other hand, $\text{Min MAC } p\text{-Center}$ and $\text{Min } p\text{-Center}$ are equivalent on the class of 2-connected graphs (i.e., without articulation point). As a consequence, to prove Proposition 5, it is enough to prove that $\text{Min Dominating Set}$ remains NP-hard in 2-connected graphs. Given a graph $G = (V, E)$ instance of $\text{Min Dominating Set}$, we construct G' from G as follows: for every articulation point a of G , create a twin vertex a' linked to a and to all neighbors of a . G' is 2-connected and the $\text{Min Dominating Set}$ problems in G and G' are equivalent: there is in G a dominating set of cardinality k if and only if there is in G' a dominating set of cardinality k since any minimal dominating set in G' never includes both a and a' for an articulation point a of G . Therefore, $\text{Min Dominating Set}$ is NP-hard in 2-connected graphs and the proof is complete. \square

In the next section, we show that $\text{Min Partial } p\text{-Center}$ is polynomially 2-approximable; Lemma 3 ensures that it is the best possible approximation for this problem.

5.2 Approximation results for Min Partial p -Center

Consider an instance (G, U) of Min Partial p -Center, where $G = (V, E, L)$ is a graph with positive lengths on edges and $U \subset V$. We denote $n = |V|$. We can compute the complete graph $K = (V, \tilde{E}, \tilde{L})$ in $O(n^3)$ as explained in the proof of Proposition 5. We denote $SL = \{d(x, y), x, y \in V\}$ the set of edge lengths in K (note that $|SL| \leq n^2$) and for any $d \in SL$, $K_d = (V, E_d)$ is the partial graph of K where E_d is the set of edges of length at most d . Note that for any p -center, its radius is in SL .

A p -center of partial radius d in (G, U) can be seen as a *partial dominating set* of (K_d, U) , where a partial dominating set X is a set of vertices such that every vertex in U has at least one neighbor in X . If A_d is the adjacency matrix of K_d with additional 1s on the diagonal (alternatively A_d is the adjacency matrix of K_d with additional loops on each vertex), we denote $A_{d,U}$ the sub-matrix of A_d corresponding to rows in U (it has $|U|$ rows and $|V|$ columns). The problem of finding a minimum partial dominating set can be formulated by the following mathematical program $PDS(G, U, d)$, where the 1s on the diagonal represent the fact that a vertex dominates itself and the notation $\mathbf{1}_d$, for an integer d , denotes the column vector of dimension d with only 1-entries:

$$PDS(G, U, d) : \begin{cases} \min & \langle \mathbf{1}_{|V|}, x \rangle \\ & A_{d,U} x \geq \mathbf{1}_{|U|} \\ & x \in \{0, 1\}^{|V|} \end{cases}$$

We then consider the mathematical program $SIS(G, U, d)$ that corresponds to finding a maximum *strong independent set* of K_d contained in U , where a strong independent set $S \subset V$ is an independent set (every two vertices in S are not adjacent) such that every vertex in $V \setminus S$ has at most one neighbor in S .

$$SIS(G, U, d) : \begin{cases} \max & \langle \mathbf{1}_{|U|}, y \rangle \\ & A_{d,U}^\top y \leq \mathbf{1}_{|V|} \\ & y \in \{0, 1\}^{|U|} \end{cases}$$

We first establish three claims that will be used in Proposition 6.

Claim 1 *The cardinality of any strong independent set of K_d contained in U is not more than the cardinality of any partial dominating set of (K_d, U) .*

Proof The relaxations of mathematical programs $PDS(G, U, d)$ and $SIS(G, U, d)$, replacing the binary conditions with non negative conditions, are dual linear programming problems. The result is an immediate consequence of the weak duality theorem. □

Let $d_{max} = \max(SL)$. We denote $K_{2d,U}$ the graph $K_{\min(2d, d_{max})}[U]$.

Claim 2 *For a given distance $d \in SL$, let S_d be a maximal independent set of $K_{2d,U}$. S_d is a partial $|S_d|$ -center in (G, U) of partial radius $r(S_d, U) \leq 2d$.*

Proof Consider any vertex $u \in U \setminus S_d$. Since S_d is maximal, $S_d \cup \{u\}$ is not independent in $K_{2d,U}$, which means $d(u, S_d) \leq 2d$ and the claim is proved. □

Claim 3 *Any independent set S of $K_{2d,U}$ is a strong independent set of K_d contained in U .*

Proof By definition, $S \subset U$. Since S is independent in $K_{2d,U}$, it is independent in $K_{d,U}$, a partial graph of $K_{2d,U}$. So, it is an independent set of K_d . The result then follows by contrapositive: if there is a vertex $u \in V \setminus S$ adjacent, in K_d , to two vertices of S , then these two vertices would be at distance at most $2d$, so would be adjacent in $K_{2d,U}$. \square

Claims 1, 2 and 3 immediately allow to derive an approximation algorithm for `Min Partial p -Center`. Even if this result is not strictly used for Theorem 1, it is worth to mention and it helps understand the main ideas of Algorithm 1.

Proposition 6 `Min Partial p -Center` is polynomially 2-approximable and this is the best possible constant ratio.

Proof We already noted that 2 is a lower bound for any constant approximation ratio of `Min Partial p -Center` (Lemma 3). So, we only need to prove that this bound can be guaranteed.

For a given instance (G, U) , we can compute SL (all distances $d(i, j), i, j \in V$) in $O(n^3)$. Then, for any $d \in SL$, we can compute a maximal independent set S_d of $K_{2d,U}$ and then select $S_{\tilde{d}}$, where $\tilde{d} \in \operatorname{argmin}_{d \in SL, |S_d| \leq p} (r(S_d))$. In other words, $S_{\tilde{d}}$ is of minimum value among all S_d s of cardinality at most p .

Denote r_U^* the minimum partial radius of a p -center in (G, U) . $r_U^* \in SL$, so we can consider the set $S_{r_U^*}$ corresponding to $d = r_U^*$. Since $S_{r_U^*}$ is an independent set of $K_{2r_U^*,U}$, Claim 3 ensures that it is a strong independent set of $K_{2r_U^*}$. On the other hand, by definition of $2r_U^*$, there is a p -center of partial radius $2r_U^*$ in (G, U) that can be seen as a partial dominating set of $(K_{2r_U^*}, U)$ and then, Claim 1 ensures that $|S_{r_U^*}| \leq p$. Consequently, \tilde{d} exists and $r(S_{\tilde{d}}) \leq r(S_{r_U^*})$. Using Claim 2, we deduce $r(S_{r_U^*}) \leq 2r_U^*$, which completes the proof. \square

Note that, using a binary search on the same model as the 2-approximation algorithm for `Min p -Center` proposed in Hochbaum and Shmoys (1985), we can design a 2-approximation algorithm of complexity $O(n^2 \log n)$ as soon as all distances between two vertices in G are computed.

In the next section, we use similar ideas and the same claims to derive a polynomial 2-approximation algorithm for `Min MAC p -Center` (Algorithm 1).

5.3 Approximation algorithm for `Min MAC p -Center`

To simplify the description of Algorithm 1, we introduce some notations used in the description of the algorithm. Given the instance $G = (V, E, L)$, we denote by k the number of MACs of G . These MACs are denoted by A_1, \dots, A_k and the related articulation points are called a_1, \dots, a_k (we may have $a_i = a_j, i \neq j$). As previously, we define $SL = \{d(i, j), i, j \in V, \}$; for any $d \in SL$, we partition $I = \{1, \dots, k\}$ into $I = I_d^- \sqcup I_d^+$ (\sqcup denotes the disjoint union), where $I_d^- = \{i \in I, \max_{x \in A_i} d(x, a_i) \leq d\}$ and $I_d^+ = \{i \in I, \max_{x \in A_i} d(x, a_i) > d\}$. MACs A_i for $i \in I_d^-$ are seen as small MACs relative to d , while MACs A_i for $i \in I_d^+$ are seen as large ones. “No-solution output” is any output we use to indicate that the problem has no feasible solution.

Algorithm 1 2-approximation for Min MAC p -Center and Min PpCP

Input: Edge weighted graph $G = (V, E, L)$ (lengths are non negative) and $p \geq 2$.

Output: C , a MAC p -center if it exists.

```

1: Compute  $A_1, \dots, A_k$ , and  $a_1, \dots, a_k$ ;
2: if  $k > p$  then No-solution output;
3: else
4:    $SL := \{d(i, j), i, j \in V\}$ ;
5:    $\widetilde{SL} := \emptyset$ ;
6:   for  $d \in SL$  do
7:      $I_d^- := \{i \in I, \max_{x \in A_i} d(x, a_i) \leq d\}$ ;
8:      $I_d^+ := \{i \in I, \max_{x \in A_i} d(x, a_i) > d\}$ ;
9:      $C_d := \emptyset$ ;
10:    for  $i \in I_d^-$  do
11:      Select  $x \in A_i$ ;
12:       $C_d := C_d \cup \{x\}$ ;
13:     $V'_d := \{v \in V, d(v, \{a_i, i \in I_d^-\}) > d\}$ ;
14:     $S_d := \emptyset$ ;
15:    for  $i \in I_d^+$  do
16:      Select  $y \in \operatorname{argmax}_{x \in A_i} d(x, a_i)$ ;
17:       $S_d := S_d \cup \{y\}$ ;
18:    while  $\exists v \in V'_d, d(v, S_d) > 2d$  do
19:       $S_d := S_d \cup \{v\}$ ;
20:    if  $|S_d| \leq p - |I_d^-|$  then
21:       $\widetilde{SL} := \widetilde{SL} \cup \{d\}$ ;
22:       $C_d := C_d \cup S_d$ ;
23:     $\tilde{d} := \operatorname{argmin}_{d \in \widetilde{SL}} (r(C_d))$ ;
24:     $C := C_{\tilde{d}}$ ;
25:  return  $C$ .

```

The idea of the Algorithm is as follows:

1. We first compute all MACs A_1, \dots, A_k and related articulation points a_1, \dots, a_k using Tarjan’s Algorithm (Tarjan 1972).
2. If the number k of MACs is more than p , then there is obviously no solution.
3. Else, for every distance $d \in SL$, Algorithm 1 tries to compute a MAC p -center C_d of radius at most $2d$; only feasible MAC p -centers obtained through this process will be kept and \widetilde{SL} is the set of distances d for which it will occur;
4. C_d is built as follows:
 - (a) The algorithm selects one center per small MAC $A_i, i \in I_d^-$;
 - (b) For each $i \in I_d^-$, all vertices at distance at most d from a_i are allocated to the related center (by definition of I_d^- , this includes in particular all vertices of A_i).

- (c) V'_d is the set of uncovered vertices. If possible, the algorithm completes C_d with a partial $(p - |I_d^-|)$ -center of $(G \setminus \bigcup_{i \in I_d^-} A_i, V'_d)$ of partial radius at most $2d$. In this process, it uses the same ideas as in Proposition 6: it constructs a maximal independent set S_d of K_{2d, V'_d} while ensuring that it intersects all A_i , $i \in I_d^+$. To this aim, it first selects one vertex in each of these components before completing S_d greedily. Claim 5, given below, ensures that S_d has the required property. If $|S_d| \leq p - |I_d^-|$, then d is added to \widetilde{SL} ;

5. The best solution $C_{\bar{d}}, d \in \widetilde{SL}$ is selected as an approximated solution for Min MAC p -Center.

Theorem 1 analyzes the approximation guarantee of Algorithm 1. Its proof will require three claims.

Claim 4 *If C_{MAC}^* is an optimal MAC p -center of radius d^* , then $d^* \in \widetilde{SL}$.*

Proof Since C_{MAC}^* has at least one center per MAC, C_{MAC}^* has at most $(p - |I_{d^*}^-|)$ centers in $V \setminus \bigcup_{i \in I_{d^*}^-} A_i$. In addition, vertices in V'_{d^*} cannot be associated with (i.e., evacuated to) centers in $\bigcup_{i \in I_{d^*}^-} A_i$ since these centers are at distance more than d^* . This means that $C_{MAC}^* \cap (V \setminus \bigcup_{i \in I_{d^*}^-} A_i)$ is a $(p - |I_{d^*}^-|)$ -center of partial radius at most d^* in $(G \setminus \bigcup_{i \in I_{d^*}^-} A_i, V'_{d^*})$.

As a consequence $C_{MAC}^* \cap (V \setminus \bigcup_{i \in I_{d^*}^-} A_i)$ is a partial dominating set in (K_{d^*}, V'_{d^*}) .

Using Claims 1 and 3, we get $|S_{d^*}| \leq |C_{MAC}^* \cap (V \setminus \bigcup_{i \in I_{d^*}^-} A_i)| \leq p - |I_{d^*}^-|$, which means $d^* \in \widetilde{SL}$. □

Claim 5 $\forall d \in SL, S_d$ is a maximal independent set in K_{2d, V'_d} that intersects all $A_i, i \in I_d^+$.

Proof The algorithm initializes S_d by selecting, in each MAC $A_i, i \in I_d^+$, a vertex at maximum distance from a_i . This ensures that, at Line 17, S_d includes one element per MAC $A_i, i \in I_d^+$ and is an independent set (possibly empty) in K_{2d, V'_d} . Indeed, if y_i, y_j are respectively selected at Line 16 for $i, j \in I_d^+, i \neq j$, then any path between them passes through a_i and a_j (we may have $a_i = a_j$) and is of length greater than $2d$. S_d is a maximal independent set in K_{2d, V'_d} since otherwise, a new vertex will be added to it at Line 18. □

Claim 6 *If C_{MAC}^* is an optimal MAC p -center of radius d^* , then $r(C_{d^*}) \leq 2d^*$.*

Proof Consider first a vertex $v \in V'_{d^*}$ and use the same argument as in the proof of Proposition 6. We have $d(v, C_{d^*}) \leq d(v, C_{d^*} \setminus \bigcup_{i \in I_{d^*}^-} A_i) \leq r(S_{d^*}, V'_{d^*})$. Using Claims 2 and 5, we have $r(S_{d^*}, V'_{d^*}) \leq 2d^*$ and thus:

$$\forall v \in V'_{d^*}, d(v, C_{d^*}) \leq 2d^*. \tag{8}$$

Consider now a vertex $v \in V \setminus V'_{d^*}$. By definition of V'_{d^*} , it means that $d(v, \{a_i, i \in I_{d^*}^-\}) \leq d^*$ and by definition of $I_{d^*}^-$, it ensures that $\exists i \in I_{d^*}^-, \forall u \in A_i, d(v, u) \leq 2d^*$. This ensures:

$$\forall v \in V'_{d^*}, d(v, C_{d^*}) \leq 2d^*. \tag{9}$$

Eqsuations 8 and 9 ensure $r(C_{d^*}) \leq 2d^*$. □

Theorem 1 *Algorithm 1 is a polynomial 2-approximation algorithm for Min MAC p -Center and this is the best possible constant ratio.*

Proof We already noted that 2 is a lower bound for constant approximation ratios (Proposition 5). So, we only need to prove that this bound can be guaranteed. Assume that $k \leq p$; then the instance of Min MAC p -Center has feasible solutions and thus, also an optimal solution.

Fix a distance $d \in SL$. Note first that, by definition of I_d^- and I_d^+ , V'_d computed at line 13 satisfies $V'_d \subset V \setminus \bigcup_{i \in I_d^-} A_i$ and $\forall i \in I_d^+, A_i \cap V'_d \neq \emptyset$. Then, the algorithm computes the set S_d from Lines 14 to Line 19.

\widetilde{SL} , computed by the algorithm (Lines 21), is the set of distances d such that S_d is of size at most $p - |I_d^-|$. Claim 4 ensures that $\widetilde{SL} \neq \emptyset$ and consequently \tilde{d} computed at Line 23 is well defined. Since d^* and \tilde{d} are both in \widetilde{SL} , the algorithm computes both sets C_{d^*} and $C_{\tilde{d}}$ by selecting one vertex per $A_i, i \in I_{d^*}^-$ and one vertex per $A_i, i \in I_{\tilde{d}}^-$, respectively (from Line 9 to Line 12) and completing with S_{d^*} and $S_{\tilde{d}}$, respectively. Using Claim 5, this ensures that both C_{d^*} and $C_{\tilde{d}}$ are MAC p -centers. Finally, $C_{\tilde{d}}$ is selected as approximated solution and Line 23 ensures that

$$r(C_{\tilde{d}}) \leq r(C_{d^*}) \tag{10}$$

Claim 6 and Eq. 10 imply $r(C_{\tilde{d}}) \leq 2d^*$, which concludes the proof of Theorem 1. □

We conclude this section with an easy remark on trees.

Proposition 7 *Min MAC p -Center is polynomial on trees with general lengths.*

Proof Given a tree \mathcal{T} , for any distance d we consider the tree \mathcal{T}_d obtained from \mathcal{T} by gluing to each pending vertex v a path of length d . Then, \mathcal{T} has a MAC p -center of radius d if and only if \mathcal{T}_d has a p -center of radius d . The result immediately follows from the fact that p -Center is polynomial on trees. □

5.4 Approximation results of Min pPCP

We immediately deduce from Theorem 1 and Proposition 4 the main result of this section:

Theorem 2 *For edge weighted graphs with lengths in $[\ell, 2\ell]$, Algorithm 1 is a polynomial time approximation algorithm for Min pPCP guaranteeing the ratio $4\overline{deg}(G) + 2$.*

In particular, on graphs with bounded degree, the ratio is constant:

Corollary 4 *Min PpCP is constant approximable for graphs of bounded degree and edge lengths in $[\ell, 2\ell]$.*

In the specific case of trees, using Proposition 3 and the analysis of Proposition 4, we get:

Corollary 5 *There is a polynomial algorithm for Min PpCP guaranteeing the ratio 3 on trees with all edge values 1.*

Remark 2 Note however that we leave open the problem of whether Min PpCP is NP-hard or polynomial on trees.

6 Conclusion

In this paper, we strengthen the analysis of Min PpCP initiated in Demange et al. (2018). Former hardness results for this problem motivate devising polynomial approximation algorithms, the main objective of this work. We propose a constant approximation for a large class of instances of Min PpCP using an approximation preserving reduction from Min MAC p -Center. This result holds for graphs of bounded degree and with edge lengths in $[\ell, 2\ell]$. To our knowledge, this is the first approximation result for this problem and in addition, it holds for a class of instances on which all known hardness results apply. This class also includes natural particular cases arising in the context of wild fire management application that motivated this study. These results provide a first gap between constant approximation ratios and the hardness in approximation. Narrowing this gap for intermediate classes of graphs is a natural open question for further researches. In Sect. 5, we even show a stronger approximation result on trees. This motivates the problem of whether Min PpCP is NP-hard or polynomial on trees, a question that we leave open.

Proposition 4 and Theorem 2 are valid for the case where edge lengths lie in $[\ell, 2\ell]$. Finding polynomial cases and approximation results for Min PpCP with general length system remains an important open question that would require new methods or tools.

Finally, when considering the feasibility conditions for Min PpCP, we have introduced the notion of minimal articulation components (MACs) and the related Min MAC p -Center Problem. We have shown that this problem is 2-approximable and that this is the best possible constant approximation ratio (Theorem 1). It is also polynomial on trees. Strengthening the study of this notion and the complexity and approximation results for this problem on specific classes of instances is another question raised by the paper.

6.1 List of problems

Min p -Center

Instance: An edge-weighted graph $G = (V, E, L)$ and an integer p
Feasible solutions: Any p -center $C \subset V, |C| \leq p$
Objective: Minimize $r(C) = \max_{v \in V} d(v, C)$.

Min $PpCP$

Instance: An edge-weighted graph $G = (V, E, L)$ and an integer p ; the instance is denoted (G, p)
Feasible solutions: Any p -center $C \subset V, |C| \leq p$ satisfying $\mathbb{E}(C) < \infty$
Objective: Minimize $\mathbb{E}(C)$.

Min MAC p -Center

Instance: An edge-weighted graph $G = (V, E, L)$ and an integer p
Feasible solutions: Any p -center $C \subset V, |C| \leq p$ that intersects all Minimal Articulation Components.
Objective: Minimize $r(C)$.

Min Partial p -Center

Instance: An edge-weighted graph $G = (V, E, L)$, a subset $U \subset V$ and an integer p
Feasible solutions: Any p -center $C \subset V, |C| \leq p$
Objective: Minimize $r(C, U)$.

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