



HAL
open science

Kinetic model for myxobacteria with directional diffusion

Laura Kanzler, Christian Schmeiser

► **To cite this version:**

Laura Kanzler, Christian Schmeiser. Kinetic model for myxobacteria with directional diffusion. Communications in Mathematical Sciences, 2023, 21 (1), pp.107-126. 10.4310/cms.2023.v21.n1.a5 . hal-04869751

HAL Id: hal-04869751

<https://hal.science/hal-04869751v1>

Submitted on 7 Jan 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Kinetic Model for Myxobacteria with Directional Diffusion

L. Kanzler*

C. Schmeiser†

Abstract

In this article a kinetic model for the dynamics of myxobacteria colonies on flat surfaces is investigated. The model is based on the kinetic equation for collective bacteria dynamics introduced in [S. Hittmeir, L. Kanzler, A. Manhart, C. Schmeiser, KRM, 14 (1), pp. 1–24, 2021], which is based on the assumption of hard binary collisions of two different types: alignment and reversal, but extended by additional Brownian forcing in the free flight phase of single bacteria. This results in a diffusion term in velocity direction at the level of the kinetic equation, which opposes the concentrating effect of the alignment operator. A global existence and uniqueness result as well as exponential decay to uniform equilibrium is proved in the case where the diffusion is large enough compared to the total bacteria mass. Further, the question whether in a small diffusion regime nonuniform stable equilibria exist is positively answered by performing a formal bifurcation analysis, which revealed the occurrence of a pitchfork bifurcation. These results are illustrated by numerical simulations.

Keywords: myxobacteria, inelastic Boltzmann equation, hypocoercivity, entropy, bifurcation, small diffusion parameter, fixed-point, decay to equilibrium

AMS subject classification: 35Q20, 35B40, 35B32

Acknowledgements: This work has been supported by the Austrian Science Fund, grants no. W1245 and F65.

1 Introduction

The aim of this work is to investigate a model for the dynamics of myxobacteria colonies moving on flat substrates. The equation of interest is the kinetic transport equation

$$\partial_t f + \omega(\varphi) \cdot \nabla_x f = \mu \partial_\varphi^2 f + Q(f, f). \quad (1)$$

for the distribution function $f(x, \varphi, t) \geq 0$, where $x \in \mathbb{T}^2$, $\varphi \in \mathbb{T}^1$ and $t \geq 0$ denote position, the directional angle, and time, respectively. We consider the collision operator Q introduced in [29] and extend the model by a diffusion term with diffusivity $\mu > 0$ in the angular direction. Under the assumption of constant speed (normalized to 1), the velocity is given by $\omega(\varphi) = (\cos \varphi, \sin \varphi)$. The notation \mathbb{T}^1 and \mathbb{T}^2 is used for the one- and, respectively, two-dimensional flat tori with 2π -periodicity. The collision operator is of the form

$$Q(f, g) = 2 \int_{\mathbb{T}_{\rightarrow\varphi}^{AL}} b(\tilde{\varphi}, \varphi_*) \tilde{f} g_* d\varphi_* + \int_{\mathbb{T}_{\varphi}^{REV}} b(\varphi^\downarrow, \varphi_*^\downarrow) f^\downarrow g_*^\downarrow d\varphi_* - \int_{\mathbb{T}^1} b(\varphi, \varphi_*) f g_* d\varphi_*, \quad (2)$$

*Sorbonne Université, Laboratoire Jacques-Louis Lions, F-75005 Paris, France.
laura.kanzler@sorbonne-universite.fr

†University of Vienna, Faculty for Mathematics, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria.
christian.schmeiser@univie.ac.at

where

$$\mathbb{T}_\varphi^{REV} := \left(\varphi + \frac{\pi}{2}, \varphi + \frac{3\pi}{2} \right), \quad \mathbb{T}_{\rightarrow\varphi}^{AL} = \left(\varphi - \frac{\pi}{4}, \varphi + \frac{\pi}{4} \right),$$

and

$$\tilde{\varphi} := 2\varphi - \varphi_*, \quad \varphi^\downarrow = \varphi + \pi, \quad \varphi_*^\downarrow = \varphi_* + \pi.$$

Super- and subscripts on f and g denote evaluation at φ with the same super- and subscripts. The model describes movement along trajectories governed by *Brownian motion* in velocity direction, interrupted by hard binary collisions with *collision cross-section* $b(\varphi, \varphi_*)$. Its dependence on the pre-collisional directions φ and φ_* is due to the shape of the bacteria. In this work we consider two possible choices: Rod shaped bacteria are described by $b(\varphi, \varphi_*) = |\omega_* \cdot \omega^\perp| = |\sin(\varphi - \varphi_*)|$. On the other hand, bacteria with circular shape yield a collision rate independent from the pre-collisional directions. By analogy to a similar simplification of the gas dynamics Boltzmann equation [18], we use the name '*Maxwellian myxos*' for this imagined species, modeled by $b(\varphi, \varphi_*) \equiv 1$. We may note at this point the reflection and rotation symmetries $b(\varphi, \varphi_*) = b(\varphi^\downarrow, \varphi_*^\downarrow)$ and, respectively, $b(\varphi + \alpha, \varphi_* + \alpha) = b(\varphi, \varphi_*)$, $\alpha \in \mathbb{T}^1$. The gain terms in (2) describe two different types of collisions:

- *Alignment*: $(\tilde{\varphi}, \varphi_*) \rightarrow (\varphi, \varphi)$ with $\varphi = (\tilde{\varphi} + \varphi_*)/2$, if two myxobacteria moving in directions $\tilde{\varphi}$ and φ_* meet at an angle smaller than $\pi/2$. The factor 2 is due to the fact that an alignment collision produces 2 myxobacteria with the same direction. The set $\mathbb{T}_{\rightarrow\varphi}^{AL}$ describes all angles φ_* , which can produce the angle φ upon collision.
- *Reversal*: $(\varphi, \varphi_*) \rightarrow (\varphi^\downarrow, \varphi_*^\downarrow)$, if two myxobacteria with directions φ and φ_* meet at an angle larger than $\pi/2$. The set \mathbb{T}_φ^{REV} describes all angles φ_* such that a collision between φ^\downarrow and φ_*^\downarrow can produce the angle φ .

The alignment collisions described above resemble a perfectly inelastic collision between particles in the framework of gas dynamics, where the involved particles undergo a complete loss of kinetic energy, hence they stick together. This results in the corresponding operator to be dissipative and closely connected to the inelastic Boltzmann equation, widely investigated over the last years, see [1, 26, 35, 40, 39] for instance. On the other hand, the invertible reversal collisions are even an involution, which results in the fact that the corresponding part of the collision operator is conservative. Properties of the model without directional diffusion

$$\partial_t f + \omega(\varphi) \cdot \nabla_x f = Q(f, f), \tag{3}$$

introduced and investigated in [29], will serve as motivation for the dynamics we expect in (1) in the small diffusion regime. In both (1) and (3) the total mass is conserved and denoted by

$$M := \int_{\mathbb{T}^1 \times \mathbb{T}^2} f(x, \varphi, t) \, d\varphi \, dx.$$

Throughout all of this paper, we denote the *uniform distribution* by

$$f_0 := \frac{M}{2\pi}, \tag{4}$$

which defines an equilibrium for both (1) and (3). Numerical experiments in the non-diffusive case [29] suggest instability of f_0 and convergence as $t \rightarrow \infty$ to an equilibrium measure of the form

$$f_\infty(\varphi) := \rho_+ \delta(\varphi - \varphi_+) + \rho_- \delta(\varphi - \varphi_+^\downarrow), \tag{5}$$

where $\rho_{\pm} := \int_{\mathbb{T}_{\pm}^1} f \, d\varphi$, with $\mathbb{T}_{+}^1 := [0, \pi]$, $\mathbb{T}_{-}^1 := [-\pi, 0]$ and $\varphi_{+} \in \mathbb{T}_{+}^1$. The convergence can be proved for the spatially homogeneous equation with special initial conditions. These observations give rise to the assumption that for small μ the uniform equilibrium f_0 will also be unstable for (1) and other equilibria will occur.

In [12] a similar model, also of Boltzmann-type but just describing alignment interactions, was introduced as binary collision counterpart of the Vicsek model for swarm dynamics, which on the other hand is based on nonlocal alignment interactions between agents [41]. It was investigated further in [13] as well as in [17], where additionally the case of Brownian forcing between binary interactions was considered. Before, such a diffusive kinetic equation modelling alignment between agents was already introduced and studied in [9].

Section 2 of this article is dedicated to establish existence and uniqueness of solutions of (1), as well as asymptotic stability of the uniform equilibrium. This can only be expected under the assumption of large enough diffusivity μ compared to the total mass M .

Theorem 1. *Let $f_I \in L^2(\mathbb{T}_{\varphi}^1; H^2(\mathbb{T}_x^2))$, $f_I \geq 0$, and let μ/M be large enough with $M = \int_{\mathbb{T}^2 \times \mathbb{T}^1} f_I \, d\varphi \, dx$. Let furthermore $\|f_I - f_0\|_{H^2(\mathbb{T}_x^2)L^2(\mathbb{T}_{\varphi}^1)}$ be small enough with $f_0 = M/(2\pi)$. Then equation (1) subject to the initial condition $f(t=0) = f_I$ has a unique global solution $f \in C([0, \infty), L^2(\mathbb{T}_{\varphi}^1; H^2(\mathbb{T}_x^2)))$, satisfying*

$$\|f(t) - f_0\|_{L^2(\mathbb{T}_{\varphi}^1; H^2(\mathbb{T}_x^2))} \leq C e^{-\lambda t} \|f_I - f_0\|_{L^2(\mathbb{T}_{\varphi}^1; H^2(\mathbb{T}_x^2))}, \quad C, \lambda > 0.$$

This result relies on a perturbative approach including the proof of spectral stability of the equilibrium before extending it to the nonlinear framework, close to equilibrium. We want to mention at this point that the theory for the dissipative Boltzmann equation is much less developed than the one of the conservative Boltzmann equation, which is due to the lack of a-priori estimates given by an entropy. Global existence results for the spatially inhomogeneous Cauchy problem in the inelastic case are only known for near vacuum data [1] (i.e. the collisions do not have much impact on the dynamics) inspired by the method using Kaniel & Shinbrot iterates [33]. More recently in [40] existence in the spatially inhomogeneous framework for inelastic collisions could be established without the closeness to vacuum restriction. Further, theory in the one-dimensional situation, where grazing collisions are almost elastic, can be found in [8]. Another important work in the one dimensional case has been done in [31], carrying out the rigorous macroscopic limit towards pressureless gas dynamics. Many more results have been established in an homogeneous framework, see e.g. [26] and [38] for investigations on the existence and uniqueness of solutions. Besides work on the Cauchy problem, a number of results on existence and further properties of self-similar profiles for diffusively excited inelastic hard sphere models have been obtained. Among them to mention [14], [26] and [35, 36, 37] for the case of a constant coefficient of restitution, while we refer the reader to [3] for considerations on the non-constant case.

In Section 3, restricting ourselves to the spatially homogeneous setting, we carry out an explicit analysis of the spectral stability of f_0 . Using bifurcation theory [19] via Fourier series expansion, we establish the occurrence of a supercritical pitchfork bifurcation for both the cases of rod-shaped and Maxwellian myxos. Although the aforesaid calculations remain formal, they do provide new insights into the behavior of the model, while being consistent with already existing results. Indeed, in [17] a rigorous proof of existence of a pitchfork bifurcation in the noisy version of a Boltzmann-type alignment model for swarming behavior is stated. It is interesting to note that our considerations revealed that the branch of nontrivial equilibria occurring in the pitchfork bifurcation is two-dimensional with two opposite peaks of equal height, as opposed to the non-diffusive case (5) where the set of equilibria with nonsymmetrical distributed masses is three-dimensional. In the second part of Section 3 we ask the

questions whether the manifold of non-trivial equilibria, obtained in the bifurcation analysis before, can be extended for arbitrarily small diffusion and eventually approaching the one for the system without diffusion (5). We were able to prove the existence of a formal approximation of spatially homogeneous equilibria in the Maxwellian case, which can be seen as regularizations of (5) in the case of equal masses $\rho_+ = \rho_-$. A rigorous proof that these converge to (5) for vanishing diffusivity remains open.

In Section 4 we present results of numerical simulations for the spatially homogeneous equation, providing evidence for the bifurcation results of Section 3.

2 Decay to the uniform equilibrium

This section is dedicated to the proof of Theorem 1, relying heavily on insights from the L^2 -hypo-coercivity theory [28, 42]. The first step will be a proof of spectral stability by an application of the L^2 -*hypo-coercivity method* of [23]. Then this result will be extended to an H^2 -setting in order to be able to control the quadratic nonlinearities of the collision operator.

2.1 Spectral stability by hypo-coercivity

Following the notation of [23], we write the linearization of (1) around $f_0 = M/(2\pi)$ in the abstract form

$$\partial_t f + Tf = Lf + Q_M f, \quad (6)$$

with the dissipative operator $L := \mu \partial_\varphi^2$, the conservative transport operator $T := \omega(\varphi) \cdot \nabla_x$, and the linearized collision operator $Q_M f := Q(f_0, f) + Q(f, f_0)$, treated as a perturbation. The linear operators T , L , and Q_M are closed on the Hilbert space

$$\mathcal{H} := \left\{ f \in L^2(\mathbb{T}^2 \times \mathbb{T}^1) : \int_{\mathbb{T}^2 \times \mathbb{T}^1} f \, d\varphi \, dx = 0 \right\},$$

and $L + Q_M - T$ generates the strongly continuous semigroup $e^{(L+Q_M-T)t}$ on \mathcal{H} . The scalar product and the norm on \mathcal{H} will be denoted by $\langle \cdot, \cdot \rangle$ and, respectively, $\| \cdot \|$. The orthogonal projection to the nullspace $\mathcal{N}(L)$ of L is given by the average with respect to the angle:

$$\Pi f := \frac{1}{2\pi} \int_{\mathbb{T}^1} f \, d\varphi.$$

The decay to equilibrium relies on two coercivity properties:

Microscopic coercivity:

$$-\langle Lf, f \rangle = \mu \int_{\mathbb{T}^2 \times \mathbb{T}^1} (\partial_\varphi f)^2 \, d\varphi \, dx \geq \mu \|f - \Pi f\|^2, \quad (7)$$

where the last inequality is the Poincaré inequality on \mathbb{T}^1 with optimal Poincaré constant 1.

Macroscopic coercivity:

$$\|T\Pi f\|^2 = \pi \int_{\mathbb{T}^2} |\nabla_x \Pi f|^2 \, dx \geq 8\pi^3 \int_{\mathbb{T}^2} (\Pi f)^2 \, dx = 4\pi^2 \|\Pi f\|^2, \quad (8)$$

where now the Poincaré inequality on \mathbb{T}^2 with optimal Poincaré constant $8\pi^2$ has been used. The macroscopic coercivity constant $4\pi^2$ can be seen as a lower bound for the spectrum of the symmetric operator $(T\Pi)^* T\Pi$ on $\mathcal{N}(L)$.

Diffusive macroscopic limit: The method of [23] relies on an algebraic property, which guarantees that the macroscopic limit, when the dissipative operator L dominates the transport operator T , is diffusive:

$$\Pi T \Pi = 0. \quad (9)$$

It is easily verified in our situation. The macroscopic limit of (6) without the perturbation ($Q_M = 0$) is the heat equation on \mathbb{T}^2 .

The modified entropy: A natural entropy for the unperturbed version of (6) is given by the square of the norm:

$$\frac{d}{dt} \frac{\|f\|^2}{2} = \langle Lf, f \rangle + \langle Q_M f, f \rangle.$$

The semidefiniteness of the dissipation $\langle Lf, f \rangle$, which vanishes on $\mathcal{N}(L)$, can be remedied by introducing the modified entropy (see [23])

$$H[f] := \frac{1}{2} \|f\|^2 + \varepsilon \langle Af, f \rangle, \quad (10)$$

with an appropriately chosen small parameter $\varepsilon > 0$, with the operator

$$A = (1 + (T\Pi)^* T \Pi)^{-1} (T\Pi)^*. \quad (11)$$

It has been shown in [23, Lemma 1] that under the assumption (9), A and TA are bounded operators with

$$\|Af\| \leq \frac{1}{2} \|(1 - \Pi)f\|, \quad \|TAf\| \leq \|(1 - \Pi)f\|. \quad (12)$$

For $\varepsilon < 1$, the bound on A implies the equivalence inequalities

$$\frac{1 - \varepsilon}{2} \|f\|^2 \leq H[f] \leq \frac{1 + \varepsilon}{2} \|f\|^2. \quad (13)$$

The time derivative of the modified entropy is written as

$$\frac{d}{dt} H[f] = -D[f], \quad (14)$$

where the dissipation is given by

$$\begin{aligned} D[f] := & -\langle Lf, f \rangle + \varepsilon \langle AT\Pi f, f \rangle + \varepsilon \langle AT(1 - \Pi)f, f \rangle - \varepsilon \langle ALf, f \rangle - \varepsilon \langle TAf, f \rangle \\ & - \langle Q_M f, f \rangle - \varepsilon \langle AQ_M f, f \rangle. \end{aligned} \quad (15)$$

We want to note here that the terms $-\varepsilon \langle Af, Lf \rangle$ and $-\varepsilon \langle Af, Q_M f \rangle$ are not represented in the formulation of $D[f]$, since they vanish due to the easily checked properties $A = \Pi A$ as well as

$$Q_M = (1 - \Pi)Q_M(1 - \Pi), \quad (16)$$

which the linearized collision operator

$$\begin{aligned} Q_M f = & 2f_0 \int_{\mathbb{T}^{AL} \rightarrow \varphi} b(\tilde{\varphi}, \varphi_*) (\tilde{f} + f_*) d\varphi_* + f_0 \int_{\mathbb{T}^{REV}} b(\varphi^\downarrow, \varphi_*^\downarrow) (f^\downarrow + f_*^\downarrow) d\varphi_* \\ & - f_0 \int_{\mathbb{T}^1} b(\varphi, \varphi_*) (f + f_*) d\varphi_* \end{aligned}$$

inherits from Q due to mass conservation. Coercivity is provided by the first two terms as a combination of microscopic and macroscopic coercivity and of the observation that $AT\Pi$ can be interpreted as the application of the map $z \mapsto z/(1+z)$ to the operator $(T\Pi)^*T\Pi$:

$$-\langle Lf, f \rangle + \varepsilon \langle AT\Pi f, f \rangle \geq \mu \|(1-\Pi)f\|^2 + \varepsilon \frac{4\pi^2}{1+4\pi^2} \|\Pi f\|^2. \quad (17)$$

It remains to show that the last five terms in (15) can be controlled by the first two. We start with the last term of the first line. The property $A = \Pi A$ and (9) imply $TA = (1-\Pi)TA$ and therefore, with (12),

$$|\langle T Af, f \rangle| = |\langle T Af, (1-\Pi)f \rangle| \leq \|(1-\Pi)f\|^2. \quad (18)$$

The operator AT is bounded if and only if its adjoint is bounded which, after using the self-adjointness of Π and the skew-symmetry of T , can be written as

$$(AT)^* = -T^2\Pi[1 + (T\Pi)^*(T\Pi)]^{-1}.$$

Let us define $g := [1 + (T\Pi)^*(T\Pi)]^{-1}f$, giving

$$(AT)^*f = -T^2\Pi g.$$

Furthermore, the definition of g is equivalent to $g - \Pi(\omega(\varphi) \cdot \nabla_x(\omega(\varphi) \cdot \nabla_x \Pi g)) = f$. After applying Π on both sides and using the notation $\rho_g := \Pi g$ and $\rho_f := \Pi f$, the equation reads

$$\rho_g - \frac{1}{2}\Delta_x \rho_g = \rho_f.$$

Testing against $\Delta_x \rho_g$ implies $\|\Delta_x \rho_g\|_{L_x^2} \leq 2\|\rho_f\|_{L_x^2}$. Therefore

$$\|(AT)^*f\|^2 = \|T^2\rho_g\|^2 \leq \pi \|\nabla_x^2 \rho_g\|_{L_x^2}^2 = \pi \|\Delta_x \rho_g\|_{L_x^2}^2 \leq 4\pi \|\rho_f\|_{L_x^2}^2 = 2\|\Pi f\|^2,$$

implying

$$|\langle AT(1-\Pi)f, f \rangle| = |\langle (1-\Pi)f, (AT)^*f \rangle| \leq \sqrt{2} \|\Pi f\| \|(1-\Pi)f\|. \quad (19)$$

Since, by a straightforward computation, $\Pi T L = -\mu \Pi T$ we have $AL = -\mu A$ and, thus,

$$|\langle ALf, f \rangle| = \mu |\langle Af, \Pi f \rangle| \leq \frac{\mu}{2} \|\Pi f\| \|(1-\Pi)f\|. \quad (20)$$

Finally, we deal with the perturbation terms. Using $0 \leq b \leq 1$ we easily conclude

$$(Q_M f) f \leq f_0 |f| \left(6 \int_{\mathbb{T}^1} |f_*| d\varphi_* + \pi |f^\downarrow| \right),$$

and therefore, with (16) and with the Cauchy-Schwarz inequality,

$$\langle Q_M f, f \rangle \leq 13\pi f_0 \|(1-\Pi)f\|^2 = \frac{13}{2} M \|(1-\Pi)f\|^2. \quad (21)$$

Similarly,

$$|Q_M f| \leq f_0 \left(6 \int_{\mathbb{T}^1} |f_*| d\varphi_* + \pi |f^\downarrow| + 2\pi |f| \right),$$

implying

$$\|Q_M f\| \leq f_0 \left(6\sqrt{2\pi} \|f\| + 3\pi \|f\| \right) = 3M \left(2\sqrt{\frac{2}{\pi}} + 1 \right) \|f\|.$$

Combining this with (12), $A = \Pi A$, and with (16) gives

$$|\langle A Q_M f, f \rangle| \leq 3M \left(\sqrt{\frac{2}{\pi}} + \frac{1}{2} \right) \|\Pi f\| \|(1-\Pi)f\|. \quad (22)$$

Hypoocoercivity: Using our results (17), (18), (19), (20), (21), (22) in (14), (15) gives

$$\begin{aligned} \frac{d}{dt}H[f] &\leq -\left(\mu - \frac{13}{2}M - \varepsilon\right) \|(1 - \Pi)f\|^2 - \varepsilon \frac{8\pi^3}{1 + 8\pi^3} \|\Pi f\|^2 \\ &\quad + \varepsilon \left(\sqrt{2} + \frac{\mu}{2} + 3M \left(\sqrt{\frac{2}{\pi}} + \frac{1}{2}\right)\right) \|\Pi f\| \|(1 - \Pi)f\|. \end{aligned}$$

Obviously for $\mu > 13M/2$ (as requested in Theorem 1) and for ε small enough, the right hand side is negative definite and controls $\|f\|^2 = \|\Pi f\|^2 + \|(1 - \Pi)f\|^2$. With (13) we obtain the existence of $\lambda > 0$, such that

$$\frac{d}{dt}H[f] \leq -2\lambda H[f],$$

and therefore exponential decay of the modified entropy and also of $\|f\|$ by another application of (13). This proves spectral stability of the uniform equilibrium in L^2 .

Theorem 2. *Let $\mu/M > 13/2$. Then there exist positive constants λ and C , such that for any initial datum $f_I \in \mathcal{H}$, we have*

$$\|e^{t(L+Q_M-T)}f_I\| \leq Ce^{-\lambda t}\|f_I\|, \quad t \geq 0. \quad (23)$$

This result can easily be extended to the Sobolev space $L^2\left(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2)\right) \cap \mathcal{H}$. Indeed, by noting that $f \in L^2\left(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2)\right) \cap \mathcal{H}$ implies that the partial derivatives of f w.r.t. x lie in \mathcal{H} . Therefore Theorem 2 immediately carries over to the pure x -derivatives, since the coefficients in (6) are x -independent and the x -derivatives thus solve the same equation. Hence, we have the following Corollary of Theorem 2.

Corollary 3. *For μ/M large enough there exist positive constants λ and C , such that for any initial datum $f_I \in L^2\left(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2)\right) \cap \mathcal{H}$, we have*

$$\|e^{t(L+Q_M-T)}f_I\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))} \leq Ce^{-\lambda t}\|f_I\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))}, \quad t \geq 0. \quad (24)$$

2.2 Nonlinear stability of the uniform equilibrium

This section is devoted to the proof of Theorem 1. We introduce the perturbation

$$h := f - f_0 \in L^2\left(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2)\right) \cap \mathcal{H},$$

satisfying, with the notation introduced above,

$$\partial_t h + Th = Lh + Q_M h + Q(h, h), \quad h(t=0) = f_I - f_0, \quad (25)$$

and consider the mild formulation

$$h(t) = e^{t(L+Q_M-T)}(f_I - f_0) + \int_0^t e^{(t-s)(L+Q_M-T)}Q(h(s), h(s)) ds.$$

For the estimation of the semigroup, Corollary 3 will be used, and apart from that we need estimates of the quadratic collision operator.

Lemma 4. *Let $h_1, h_2 \in L^2\left(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2)\right) \cap \mathcal{H}$. Then $Q(h_1, h_2) \in L^2\left(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2)\right) \cap \mathcal{H}$ and there exists a constant \bar{Q} such that*

$$\|Q(h_1, h_2)\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))} \leq \bar{Q} \|h_1\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))} \|h_2\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))}.$$

Proof. Because of the Sobolev inequality

$$\|h\|_{L^\infty(\mathbb{T}_x^2)} + \|\nabla_x h\|_{L^4(\mathbb{T}_x^2)} \leq c_S \|h\|_{H^2(\mathbb{T}_x^2)}, \quad (26)$$

it will be sufficient to find estimates in terms of the L^∞ -norms of h_1 and h_2 or of the L^4 -norms of the first order derivatives with respect to x . We start with the observation due to the Young's inequality for convolutions

$$|Q(h_1, h_2)(\varphi)| \leq 3 \|h_1(\cdot, \varphi)\|_{L^\infty(\mathbb{T}_x^2)} \int_{\mathbb{T}^1} |h_2| \, d\varphi,$$

implying

$$\|Q(h_1, h_2)\| \leq \sqrt{3} \|h_1\|_{L^2(\mathbb{T}_\varphi^2; L^\infty(\mathbb{T}_x^2))} \|h_2\|, \quad (27)$$

and similarly,

$$\|Q(h_1, h_2)\| \leq \sqrt{3} \|h_2\|_{L^2(\mathbb{T}_\varphi^2; L^\infty(\mathbb{T}_x^2))} \|h_1\|, \quad (28)$$

Alternatively it is, by the convolution structure of the collision terms, straightforward to show

$$\int_{\mathbb{T}^1} Q(h_1, h_2)^2 \, d\varphi \leq 6\pi \int_{\mathbb{T}^1} h_1^2 \, d\varphi \int_{\mathbb{T}^1} h_2^2 \, d\varphi,$$

with the consequence

$$\|Q(h_1, h_2)\| \leq \sqrt{6\pi} \left(\int_{\mathbb{T}^1} \|h_1\|_{L^4(\mathbb{T}_x^2)}^2 \, d\varphi \int_{\mathbb{T}^1} \|h_2\|_{L^4(\mathbb{T}_x^2)}^2 \, d\varphi \right)^{\frac{1}{2}}. \quad (29)$$

By elliptic regularity, we may use the equivalent norm $\|h\|_* = \|h\| + \|\Delta_x h\|$ on $H^2(\mathbb{T}_x^2)$. We have

$$\begin{aligned} \|\Delta_x Q(h_1, h_2)\| &\leq \|Q(\Delta_x h_1, h_2)\| + \|Q(h_1, \Delta_x h_2)\| + 2\|Q(\nabla_x h_1, \nabla_x h_2)\| \\ &\leq \sqrt{3} \|h_1\|_{L^2(\mathbb{T}_\varphi^2; L^\infty(\mathbb{T}_x^2))} \|h_2\| + \sqrt{3} \|h_2\|_{L^2(\mathbb{T}_\varphi^2; L^\infty(\mathbb{T}_x^2))} \|h_1\| \\ &\quad + \sqrt{6\pi} \left(\int_{\mathbb{T}^1} \|h_1\|_{L^4(\mathbb{T}_x^2)}^2 \, d\varphi \int_{\mathbb{T}^1} \|h_2\|_{L^4(\mathbb{T}_x^2)}^2 \, d\varphi \right)^{\frac{1}{2}} \\ &\leq c_S (2\sqrt{3} + \sqrt{6\pi}) \|h_1\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))} \|h_2\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))}, \end{aligned}$$

where we have used (27)–(29) as well as (26), which completes the proof. \square

Lemma 4 implies local Lipschitz continuity of Q , considered as a map on $L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2)) \cap \mathcal{H}$ and therefore local existence and uniqueness of a mild solution. Corollary 3 and Lemma 4 imply

$$\|h(t)\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))} \leq C e^{-\lambda t} \|f_I - f_0\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))} + C \bar{Q} \int_0^t e^{\lambda(s-t)} \|h(s)\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))}^2 \, ds.$$

It is easily checked that for

$$\|f_I - f_0\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))} \leq \frac{\lambda}{4C^2 \bar{Q}},$$

Picard iteration preserves the inequality

$$\|h(t)\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))} \leq 2C e^{-\lambda t} \|f_I - f_0\|_{L^2(\mathbb{T}_\varphi^1; H^2(\mathbb{T}_x^2))},$$

completing the proof of Theorem 1.

3 Spatially Homogeneous Non-Uniform Equilibria

This section focuses on finding nonuniform, *spatially homogeneous* equilibria of (1), i.e. stationary solutions of the equation

$$\begin{aligned}\partial_t f &= \mu \partial_\varphi^2 f + Q(f, f), \quad \varphi \in \mathbb{T}^1, t > 0, \\ f(\varphi, 0) &= f_I(\varphi), \quad \varphi \in \mathbb{T}^1,\end{aligned}\tag{30}$$

and further investigate their stability. We expect the uniform equilibrium

$$f_0 = \frac{M}{2\pi}, \quad \text{with } M = \int_{\mathbb{T}^1} f_I(\varphi) d\varphi,$$

to be stable for sufficiently large diffusion and intend to find other equilibria in the collision dominated regime.

We approach this problem in two ways. On the one hand, in Section 3.1 a formal bifurcation analysis with bifurcation parameter μ shows a supercritical pitchfork bifurcation away from the uniform equilibrium, producing a branch of nontrivial equilibria for μ less than a critical value μ^* . The nontrivial equilibria have two reflection symmetries with two opposite maxima and two opposite minima. We restrict ourselves to the formal computations, noting that they can be made rigorous in a straightforward way, following the theory of bifurcations from a simple eigenvalue (see, e.g., [19]).

On the other hand, we investigate the case $\mu \ll 1$. The picture we have in mind is that, as $\mu \rightarrow 0+$, the nontrivial equilibrium converges to

$$f_\infty(\varphi) = \frac{M}{2}(\delta(\varphi) - \delta(\varphi - \pi)),$$

or to a rotated version. This is motivated by the fact that f_∞ is an equilibrium for $\mu = 0$ [29]. In Section 3.2 we construct a formal approximation for a nontrivial equilibrium, which is a smoothed version of f_∞ .

3.1 Bifurcation from the Uniform Equilibrium

Stability of the uniform equilibrium: We start by analyzing the spectral stability of f_0 by linearization of (30):

$$\partial_t f^* = (L + Q_M) f^*, \quad \int_{\mathbb{T}^1} f^* d\varphi = 0,\tag{31}$$

where f^* is the perturbation, and we recall $L = \mu \partial_\varphi^2$ and the linearization $Q_M f^* = Q(f_0, f^*) + Q(f^*, f_0)$ of Q around f_0 . For the collision kernel, the model $b(\varphi, \varphi_*) = |\sin(\varphi - \varphi_*)|$ for rod-shaped bacteria will be used in this section (see, however, the remark at the end of the section). The Fourier series expansion

$$f^*(\varphi, t) = \sum_{n=1}^{\infty} a_n(t) \cos(n\varphi) + \sum_{n=1}^{\infty} b_n(t) \sin(n\varphi)\tag{32}$$

diagonalizes the problem and leads to

$$\dot{a}_n = \lambda_n a_n, \quad \dot{b}_n = \lambda_n b_n, \quad n \geq 1,$$

with the eigenvalues

$$\lambda_1 = -\mu - \frac{f_0}{3} (4\sqrt{2} - 1),$$

$$\lambda_2 = -4\mu + \frac{2f_0}{3},$$

$$\lambda_n = -n^2\mu + 2f_0 \left(\frac{4n \sin(n\pi/4) - 8}{n^2 - 4} + \frac{n \sin(n\pi/2) + (-1)^n}{n^2 - 1} + (-1)^n - 2 \right), \quad n > 2.$$

It is easily checked that $\lambda_n < 0$ for $n \neq 2$ and for all $\mu > 0$. Thus, by the sign of λ_2 , the uniform equilibrium is spectrally stable for

$$\mu \geq \mu^* := \frac{f_0}{6}, \quad (33)$$

and spectrally unstable for $\mu < \mu^*$.

Pitchfork bifurcation: For understanding the nature of the steady-state bifurcation at $\mu = \mu^*$ we observe that the problem has both a rotation symmetry ($\varphi \leftrightarrow \varphi + \varphi_0$ with arbitrary φ_0) and a flip symmetry ($\varphi \leftrightarrow -\varphi$). The rotation symmetry is the reason for the double eigenvalues in the previous section. It can be eliminated by the additional auxiliary condition

$$f(\varphi = 0) = f_0.$$

This makes the eigenvalues (in particular λ_2) simple, and we can expect a branch of bifurcating solutions [19]. Because of the flip symmetry the generic bifurcation to be expected is a pitchfork. We shall construct a supercritical pitchfork bifurcation with a bifurcating branch for $\mu \leq \mu^*$, and therefore make the ansatz

$$\mu = \mu^* - \delta^2, \quad f(\varphi) = f_0 + \delta f_1(\varphi) + \delta^2 f_2(\varphi) + \delta^3 f_3(\varphi) + \mathcal{O}(\delta^4), \quad (34)$$

with $0 < \delta \ll 1$. The corrections have to satisfy the additional auxiliary condition and mass conservation:

$$f_k(0) = 0, \quad \int_{\mathbb{T}^1} f_k \, d\varphi = 0, \quad k \geq 1. \quad (35)$$

Substitution of (34) in the stationary version of (30) yields

$$0 = (L^* + Q_M)f_1 + \delta((L^* + Q_M)f_2 + Q(f_1, f_1)) + \delta^2((L^* + Q_M)f_3 - \partial_\varphi^2 f_1 + Q(f_1, f_2) + Q(f_2, f_1)) + \mathcal{O}(\delta^3), \quad (36)$$

with $L^* := \mu^* \partial_\varphi^2$.

Lemma 5. *The null space of $L^* + Q_M$ subject to (35) is one-dimensional and spanned by $\sin(2\varphi)$. The solvability condition for the equation $(L^* + Q_M)f = g$ is*

$$\int_{\mathbb{T}^1} g \sin(2\varphi) \, d\varphi = 0.$$

Proof. The result on the null space is a consequence of the computations in the previous section and of the observation that the $\cos(2\varphi)$ -contribution is eliminated by the additional auxiliary condition. A straightforward computation shows that Q_M is symmetric with respect to the L^2 scalar product, and so is of course L^* , completing the proof. \square

Equation (36) with $\delta = 0$ implies

$$f_1(\varphi) = b \sin(2\varphi),$$

with $b \in \mathbb{R}$ still to be determined. By a straightforward computation the inhomogeneity in the $\mathcal{O}(\delta)$ -equation is given by

$$Q(f_1, f_1) = -\frac{4}{3}b^2 \cos(4\varphi),$$

which satisfies the solvability condition for

$$(L^* + Q_M)f_2 + Q(f_1, f_1) = 0.$$

The computations in the previous section show that $(L^* + Q_M) \cos(4\varphi) = \lambda_4^* \cos(4\varphi)$ with

$$\lambda_4^* = \lambda_4 \big|_{\mu=\mu^*} = -\frac{88}{15}f_0.$$

Therefore, considering (35), we obtain

$$f_2(\varphi) = \frac{5b^2}{22f_0}(\cos(2\varphi) - \cos(4\varphi)).$$

The final computation is the evaluation of the solvability condition

$$\int_{\mathbb{T}^1} \left(-\partial_\varphi^2 f_1 + Q(f_1, f_2) + Q(f_2, f_1) \right) \sin(2\varphi) \, d\varphi = 0$$

for the $\mathcal{O}(\delta^2)$ -equation, which gives

$$b \left(\frac{440f_0}{15} - b^2 \right) = 0.$$

The nontrivial solutions determine the bifurcating branch

$$f(\varphi) = f_0 \pm \sqrt{\frac{440f_0(\mu^* - \mu)}{15}} \sin(2\varphi) + \mathcal{O}(\mu^* - \mu), \quad \mu \leq \mu^*,$$

of nontrivial equilibria. Supercriticality implies stability of the bifurcating branch (see the bifurcation diagram in Figure 1). Recalling mass conservation and rotational symmetry we obtain for each $0 < \mu^* - \mu \ll 1$ a two-dimensional set of nontrivial equilibria of the form

$$f(\varphi) = \frac{M}{2\pi} + \sqrt{\frac{220M(\mu^* - \mu)}{15\pi}} \sin(2(\varphi - \varphi_0)) + \mathcal{O}(\mu^* - \mu),$$

parametrized by the mass $M \geq 0$ and the rotation angle $\varphi_0 \in [0, \pi)$. Note that the different signs in the second term can be realized by rotation by $\pi/2$.

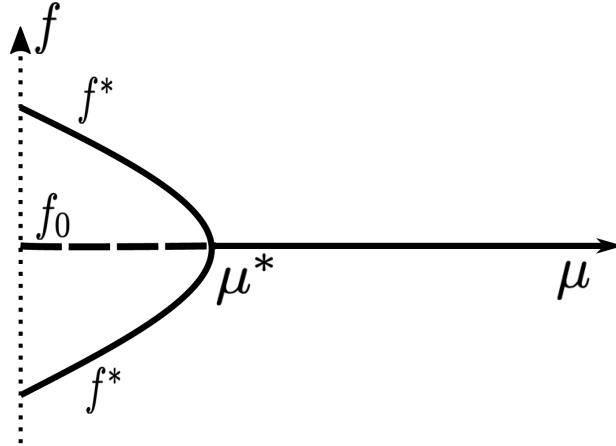


Figure 1: Bifurcation diagram of the supercritical pitchfork bifurcation. Solid lines represent stable branches, while the dashed line represents the unstable region of the uniform equilibrium.

Remark 1. The above calculations have been carried out for $b(\varphi, \varphi_*) = |\sin(\varphi - \varphi_*)|$, modelling the case of rod-shaped myxobacteria. It is easily checked that the important property that the linearized operator is diagonalized by Fourier decomposition holds for every model of the physically reasonable form $b(\varphi, \varphi_*) = \hat{b}(|\varphi - \varphi_*|)$. In particular, the same type of bifurcation result holds for Maxwellian myxos, i.e. $b \equiv 1$, with the bifurcation value

$$\mu_{Maxwell}^* = f_0 \left(1 - \frac{\pi}{4}\right) > \mu^*.$$

3.2 Equilibria for Small Diffusivity

In [29] it has been shown that the set of nontrivial equilibria of Q is three-dimensional and of the form

$$f_\infty(\varphi) := \rho_+ \delta(\varphi - \varphi_0) + \rho_- \delta(\varphi - \varphi_0^\downarrow), \quad (37)$$

with arbitrary $\varphi_0 \in \mathbb{T}^1$ and $\rho_+, \rho_- \geq 0$, satisfying $\rho_+ + \rho_- = M$. On the other hand, in the previous section we have found a manifold of equilibria of $\mu \partial_\varphi^2 + Q$, which is two-dimensional for each μ smaller than and close to μ^* . The question is: Can these results be connected by the limit $\mu \rightarrow 0+$? Motivated by the fact that the bifurcating equilibria of the previous section have two symmetric maxima, our conjecture is the following: *The manifold of nontrivial equilibria starting at the bifurcation at $\mu = \mu^*$ can be extended to arbitrarily small $\mu > 0$. Its limit as $\mu \rightarrow 0+$ is the family*

$$f_\infty(\varphi) := \frac{M}{2} \delta(\varphi - \varphi_0) + \frac{M}{2} \delta(\varphi - \varphi_0^\downarrow), \quad M \geq 0, \quad \varphi_0 \in \mathbb{T}^1. \quad (38)$$

For small μ , $\mu \partial_\varphi^2 + Q$ possesses a three-dimensional family of metastable states close to (37).

So far we cannot prove any of this, but in the remainder of this section we shall present a first small step: We shall prove the existence of a formal approximation for equilibria of $\mu \partial_\varphi^2 + Q$ close to (38) for the case of Maxwellian myxos. The corresponding considerations for the kernel $b(\varphi, \varphi_*) = |\sin(\varphi - \varphi_*)|$ are more involved due to the convolution structure of the nonlocal ODE, which hindered us to come to the desired conclusion. Some more evidence will be provided by the numerical simulations presented in the following section.

The stationary equation with $b(\varphi, \varphi_*) \equiv 1$ can be written as

$$0 = \mu \partial_\varphi^2 f(\varphi) + 2 \int_{\varphi - \frac{\pi}{4}}^{\varphi + \frac{\pi}{4}} f(2\varphi - \varphi_*) f(\varphi_*) \, d\varphi_* + f(\varphi + \pi) \int_{\varphi - \frac{\pi}{2}}^{\varphi + \frac{\pi}{2}} f(\varphi_*) \, d\varphi_* - M f(\varphi). \quad (39)$$

We look for a reflection symmetric solution ($f(\varphi) = f(\varphi + \pi)$) with mass concentrated around $\varphi = 0$ and $\varphi = \pi$ (close to (38) with $\varphi_0 = 0$). Concentrating on the peak at $\varphi = 0$, we introduce the new unknown $F(\xi)$ by the scaling

$$\varphi = \sqrt{\frac{2\mu}{M}} \xi, \quad f = \frac{M}{2} \sqrt{\frac{M}{2\mu}} F,$$

and rewrite (39) as

$$0 = \partial_\xi^2 F(\xi) + 2 \int_{\xi - \frac{\pi}{4} \sqrt{\frac{M}{2\mu}}}^{\xi + \frac{\pi}{4} \sqrt{\frac{M}{2\mu}}} F(2\xi - \xi_*) F(\xi_*) \, d\xi_* + F(\xi) \left(\frac{2}{M} \int_{\xi \sqrt{\frac{2\mu}{M}} - \frac{\pi}{2}}^{\xi \sqrt{\frac{2\mu}{M}} + \frac{\pi}{2}} f(\varphi_*) \, d\varphi_* - 2 \right)$$

Since, by the symmetry assumption,

$$\int_{-\frac{\pi}{2} \sqrt{\frac{M}{2\mu}}}^{\frac{\pi}{2} \sqrt{\frac{M}{2\mu}}} F(\xi_*) \, d\xi_* = \frac{2}{M} \int_{-\pi/2}^{\pi/2} f(\varphi_*) \, d\varphi_* = 1,$$

holds, the limit $\mu \rightarrow 0$ gives

$$0 = \partial_\xi^2 F(\xi) + 2 \int_{-\infty}^{\infty} F(2\xi - \xi_*) F(\xi_*) \, d\xi_* - F(\xi), \quad \int_{-\infty}^{\infty} F(\xi) \, d\xi = 1. \quad (40)$$

With the Green's function of ∂_ξ^2 -id we can rewrite (40) as the fixed point problem

$$F = \mathcal{S}(F) \quad \text{with} \quad \mathcal{S}(F)(\xi) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-|\xi - \tilde{\xi}|} F(2\tilde{\xi} - \xi_*) F(\xi_*) \, d\xi_* \, d\tilde{\xi}.$$

We claim that \mathcal{S} maps the set

$$\mathcal{B} := \left\{ F \in L_+^1(\mathbb{R}) \cap C_B(\mathbb{R}) : F(\xi) = F(-\xi), \int_{-\infty}^{\infty} F(\xi) \, d\xi = 1, \int_{-\infty}^{\infty} \xi^2 F(\xi) \, d\xi = 4 \right\}.$$

into itself. For $F \in \mathcal{B}$ we obviously have $\mathcal{S}(F) \geq 0$, $\mathcal{S}(F)(\xi) = \mathcal{S}(F)(-\xi)$, and

$$\int_{-\infty}^{\infty} \mathcal{S}(F)(\xi) \, d\xi = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(2\tilde{\xi} - \xi_*) F(\xi_*) \, d\xi_* \, d\tilde{\xi} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\hat{\xi}) F(\xi_*) \, d\xi_* \, d\hat{\xi} = 1.$$

A preliminary computation for the evaluation of the variance is

$$\int_{-\infty}^{\infty} \xi^2 e^{-|\xi - \tilde{\xi}|} \, d\xi = 4 + 2\tilde{\xi}^2,$$

implying

$$\begin{aligned} \int_{-\infty}^{\infty} \xi^2 \mathcal{S}(F)(\xi) \, d\xi &= 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(2\tilde{\xi} - \xi_*) F(\xi_*) \, d\xi_* \, d\tilde{\xi} + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\xi}^2 F(2\tilde{\xi} - \xi_*) F(\xi_*) \, d\xi_* \, d\tilde{\xi} \\ &= 2 + \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\xi_* + \hat{\xi})^2 F(\hat{\xi}) F(\xi_*) \, d\xi_* \, d\hat{\xi} \\ &= 2 + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\xi_*^2 + \xi_* \hat{\xi}) F(\hat{\xi}) F(\xi_*) \, d\xi_* \, d\hat{\xi} = 4, \end{aligned}$$

where the evenness of F has been used in the last equality. Finally, for any $\xi \in \mathbb{R}$,

$$\mathcal{S}(F)(\xi) \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(2\tilde{\xi} - \xi_*) F(\xi_*) d\xi_* d\tilde{\xi} = \frac{1}{2},$$

whence $\mathcal{S} : \mathcal{B} \rightarrow \mathcal{B}$, since the uniform continuity of $\mathcal{S}(F)$ is obvious.

Lemma 6. *With the above definitions, the set $\mathcal{S}(\mathcal{B})$ is relatively compact in $C_B(\mathbb{R})$.*

Proof. By the estimate

$$|\mathcal{S}(F)(\xi_1) - \mathcal{S}(F)(\xi_2)| \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\xi_1 - \xi_2| F(2\tilde{\xi} - \xi_*) F(\xi_*) d\xi_* d\tilde{\xi} = \frac{1}{2} |\xi_1 - \xi_2|,$$

$\mathcal{S}(\mathcal{B})$ is equi-Lipschitz-continuous. By the Arzelá-Ascoli theorem, a sequence $\{F_n\} \subset \mathcal{S}(\mathcal{B})$ possesses for every compact set $K \subset \mathbb{R}$ a subsequence, which converges uniformly on K . The standard diagonal procedure produces one subsequence $\{G_n\} \subset \{F_n\}$, such that $G_n \rightarrow G$ pointwise in \mathbb{R} and uniformly on each compact set. It remains to prove that the convergence is uniform on \mathbb{R} .

Let $\xi_0 \geq X > 0$. Then, as a consequence of the Lipschitz continuity,

$$F(\xi) \geq F(\xi_0) - \frac{1}{2}(\xi - \xi_0), \quad \text{for } \xi_0 \leq \xi \leq \xi_0 + 2F(\xi_0).$$

Thus,

$$F(\xi_0)^2 = \int_{\xi_0}^{\xi_0 + 2F(\xi_0)} \left(F(\xi_0) - \frac{1}{2}(\xi - \xi_0) \right) d\xi \leq \int_X^{\infty} F(\xi) d\xi \leq \int_0^{\infty} \frac{\xi^2}{X^2} F(\xi) d\xi = \frac{2}{X^2}$$

With the analogous estimate for $\xi_0 \leq -X$ we have

$$F(\xi) \leq \frac{\sqrt{2}}{X}, \quad \text{for } F \in \mathcal{S}(\mathcal{B}), \quad |\xi| \geq X.$$

The same is true for the pointwise limit G of $\{G_n\}$ and, thus,

$$\sup_{\mathbb{R}} |G_n - G| = \max \left\{ \frac{2\sqrt{2}}{X}, \sup_{(-X, X)} |G_n - G| \right\},$$

which can be made arbitrarily small by choosing first X and then n sufficiently large. \square

Theorem 7. *Problem (40) has a nonnegative smooth solution satisfying*

$$\int_{-\infty}^{\infty} \xi^2 F(\xi) d\xi \leq 4.$$

Proof. For an application of the Schauder fixed point theorem it remains to prove continuity of \mathcal{S} with respect to the supremum norm: For $F_1, F_2 \in \mathcal{B}$,

$$\begin{aligned} |\mathcal{S}(F_1)(\xi) - \mathcal{S}(F_2)(\xi)| &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-|\xi - \tilde{\xi}|} \left(|F_1(2\tilde{\xi} - \xi_*) - F_2(2\tilde{\xi} - \xi_*)| F_1(\xi_*) \right. \\ &\quad \left. + |F_1(\xi_*) - F_2(\xi_*)| F_2(2\tilde{\xi} - \xi_*) \right) d\xi_* d\tilde{\xi} \\ &\leq \sup_{\mathbb{R}} |F_1 - F_2| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-|\xi - \tilde{\xi}|} \left(F_1(\xi_*) + F_2(2\tilde{\xi} - \xi_*) \right) d\xi_* d\tilde{\xi} \\ &= 4 \sup_{\mathbb{R}} |F_1 - F_2|. \end{aligned}$$

An application of the Schauder theorem shows that \mathcal{S} has a Lipschitz continuous fixed point. The boundedness of the variance implies tightness and therefore the fixed point is a probability density, satisfying the upper bound of the variance. It is easily seen that the map $\xi \mapsto \int_{-\infty}^{\infty} F(2\xi - \xi_*)F(\xi_*) d\xi_*$ is Lipschitz continuous and therefore the differential equation (40) implies $F \in C^2(\mathbb{R})$. Bootstrapping gives higher regularity. \square

A formal approximation as $\mu \rightarrow 0+$ for an equilibrium can now be given as

$$f(\varphi) \approx \begin{cases} \frac{M}{2} \sqrt{\frac{M}{2\mu}} F\left(\varphi \sqrt{\frac{M}{2\mu}}\right), & |\varphi| \leq \frac{\pi}{2}, \\ \frac{M}{2} \sqrt{\frac{M}{2\mu}} F\left((\varphi - \pi) \sqrt{\frac{M}{2\mu}}\right), & |\varphi - \pi| \leq \frac{\pi}{2}. \end{cases}$$

The rigorous justification remains open.

Remark 2. For the case of the rod-shaped myxos the steady-state equation is of the form

$$\begin{aligned} 0 = & \mu \partial_{\varphi}^2 f(\varphi) + 2 \int_{\varphi - \frac{\pi}{4}}^{\varphi + \frac{\pi}{4}} b(2\varphi - \varphi_*, \varphi_*) f(2\varphi - \varphi_*) f(\varphi_*) d\varphi_* \\ & + f(\varphi) \int_{\varphi - \frac{\pi}{2}}^{\varphi + \frac{\pi}{2}} b(\varphi, \varphi_*) f(\varphi_*) d\varphi_* - f(\varphi) \int_{\mathbb{T}^1} b(\varphi, \varphi_*) f(\varphi_*) d\varphi_*. \end{aligned} \quad (41)$$

In this case, we introduce the scaling

$$\varphi = \mu^{1/3} \xi, \quad f = \mu^{-1/3} F,$$

through which the steady state equation (41) can be written as

$$\begin{aligned} 0 = & \partial_{\xi}^2 F(\xi) + 2\mu^{-\frac{1}{3}} \int_{\xi - \mu^{-\frac{1}{3}} \frac{\pi}{2}}^{\xi + \mu^{-\frac{1}{3}} \frac{\pi}{2}} b\left(\mu^{\frac{1}{3}}(2\xi - \xi_*), \mu^{\frac{1}{3}} \xi_*\right) F(2\xi - \xi_*) F(\xi_*) d\xi_* \\ & - \mu^{-\frac{1}{3}} F(\xi) \int_{\xi - \mu^{-\frac{1}{3}} \frac{\pi}{2}}^{\xi + \mu^{-\frac{1}{3}} \frac{\pi}{2}} b\left(\mu^{\frac{1}{3}} \xi, \mu^{\frac{1}{3}} \xi_*\right) F(\xi_*) d\xi_*. \end{aligned}$$

Remembering $b(\varphi, \varphi_*) = |\sin(\varphi - \varphi_*)|$ the above equation becomes

$$0 = \partial_{\xi}^2 F(\xi) + 4 \int_{\mathbb{R}} |\xi - \xi_*| F(2\xi - \xi_*) F(\xi_*) d\xi_* - F(\xi) \int_{\mathbb{R}} |\xi - \xi_*| F(\xi_*) d\xi_*. \quad (42)$$

after considering the formal limit $\mu \rightarrow 0$. An existence analysis similar to that of (40) might be possible. However, because of the ξ -dependent coefficient in the last term, an appropriate Green function would not be explicit and dependent on F .

4 Numerical Simulations with the Spatially Homogeneous Model

Discretization: The results of the preceding section will be illustrated by numerical simulations with the spatially homogeneous model (30) with the collision kernel $b(\varphi, \varphi_*) = |\sin(\varphi - \varphi_*)|$ for rod-shaped myxos. Discretization in the angular direction is based on an equidistant grid

$$\varphi_k = \frac{k\pi}{n}, \quad k \in \mathbb{Z}_{2n},$$

with an even number of grid points, representing a discrete torus. The collision operator is approximated by quadrature, chosen such that mass is conserved and post-collisional states are on the grid (see [29, Section 5] for details). Diffusion is discretized by the standard three-point scheme, and the explicit Euler scheme is used for the time discretization, such that mass conservation is guaranteed. The scheme has been implemented in MATLAB.

All simulations have been carried out with $n = 51$ and with time steps satisfying a parabolic CFL condition. This has not been too restrictive since only rather small values for the diffusivity μ have been used. The mass has been normalized, i.e. $M = 1$, leading to the bifurcation value (see (33))

$$\mu^* = \frac{1}{12\pi} \approx 0.0265.$$

The plots in the figures below show distributions initially (red dotted lines), at an intermediate time (blue dashed lines), and at the end of the simulation time (black solid lines), the latter typically close to an equilibrium state.

Simulations in the bifurcation regime: First we show simulations with values of the diffusivity μ just below and just above the bifurcation value μ^* .

In Figure 2 the initial data have been chosen as random perturbations of the constant equilibrium, which is stable for $\mu > \mu^*$ (right), and unstable for $\mu < \mu^*$ (left). In the latter case, the solution converges to a nonuniform steady state with peaks centered around two unpredictable, but always opposite points.

In Figure 3 the constant equilibrium is initially perturbed only at one grid point. The results are as above.

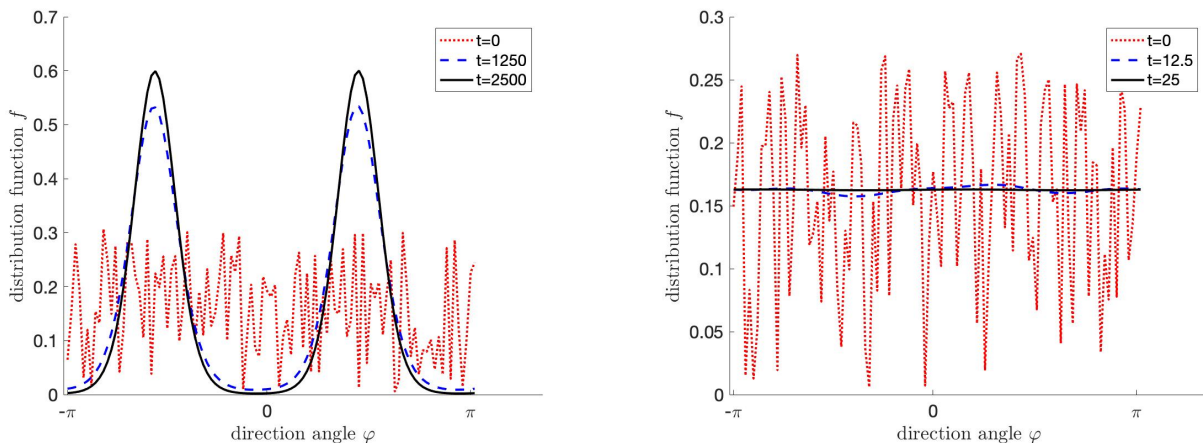


Figure 2: Random perturbation of the constant equilibrium as initial conditions. *Left:* With diffusivity smaller than the bifurcation value ($\mu = 0.02$) the solution converges to a nonuniform equilibrium with peaks at unpredictable positions. *Right:* For $\mu = 0.03 > \mu^*$ convergence to the constant steady state $f_0 = 1/(2\pi)$ is observed.

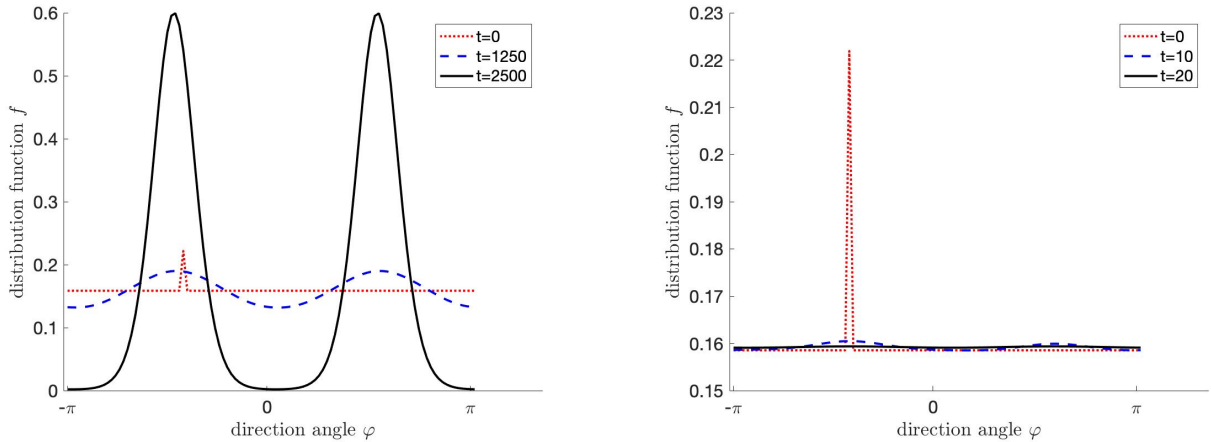


Figure 3: Initial perturbation of the constant equilibrium at one grid point. *Left:* $\mu = 0.02 < \mu^*$. *Right:* $\mu = 0.03 > \mu^*$.

Simulations in the small diffusion regime: The remaining simulation results support the conjecture formulated in Section 3.2. For the value $\mu = 0.001$ of the diffusivity we always observe convergence to a nonuniform equilibrium with opposite peaks of equal mass. As expected the dynamics passes through metastable states with two peaks of different masses, where the convergence to the final equilibrium becomes slower with decreasing values of μ . This is the reason why a rather moderate value has been chosen, where the concentration effect is not too strong.

The simulations shown in Figure 4 start with initial datum nonzero only at the two opposite points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, weighted differently. We observe two time scales. First the peaks are smoothed and then mass is transferred to make them of equal size. In Figure 5 the initial datum again is nonzero only at two points with different weights, but placed at non-opposite angles. In this case not only mass has to be transferred, but the peaks also move to produce the distance π between them.

Both figures also show the masses in opposite half intervals, which stay approximately constant at the beginning of the simulation. In a second phase they converge towards each other. As soon as the reversal operator comes to play, they start converging towards each other. For $\mu > 0$ we never observe non-symmetric equilibria in simulations, which exist and are stable for $\mu = 0$ [29]. (For further insights we refer to [29], Section 3.) Symmetrization requires the interplay between diffusion and reversal. Figure 6 shows simulations with initial data as in Figure 5, but without reversal. In this case the smaller peak is absorbed by the larger one as a consequence of the alignment interaction. The role of the reversal operator seems to be rather subtle and is the subject of ongoing research.

Nonetheless, further simulations in Figure 6, which were performed in absence of the reversal dynamics, encourage the conjecture that the reversal and the diffusion operator are conjointly responsible for the symmetrization. We again start with an initial datum only nonzero at two points with different weights, placed at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Also here, the smoothening of the peaks can be observed first, followed by redistribution of the mass from the small peak to the bigger peak leading eventually to the disappearance of the smaller peak.

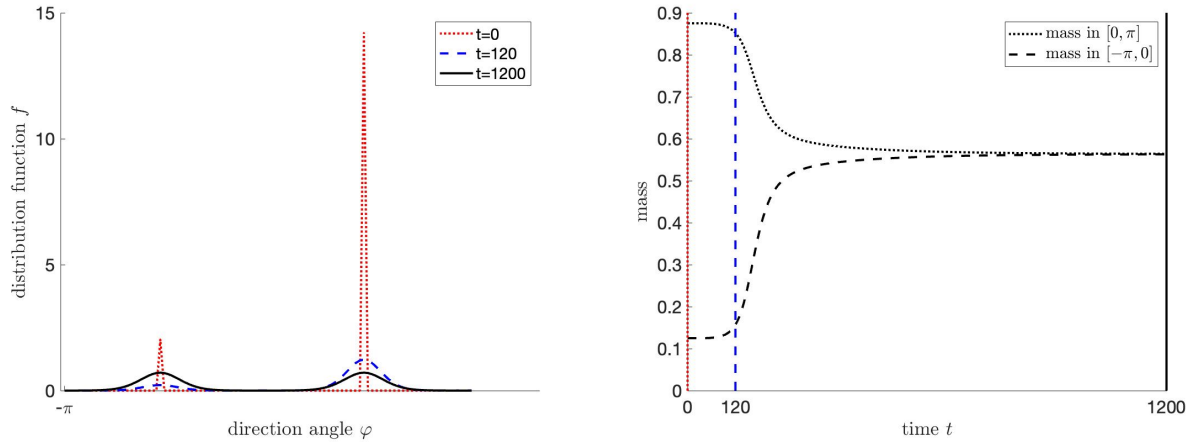


Figure 4: Initial condition with masses concentrated at $-\pi/2$ and $\pi/2$, weighted differently. Diffusion constant $\mu = 0.001$. *Left*: Time evolution with smoothing of the peaks, followed by redistribution of mass. *Right*: Time evolution of the mass in $[-\pi, 0]$ and in $[0, \pi]$.

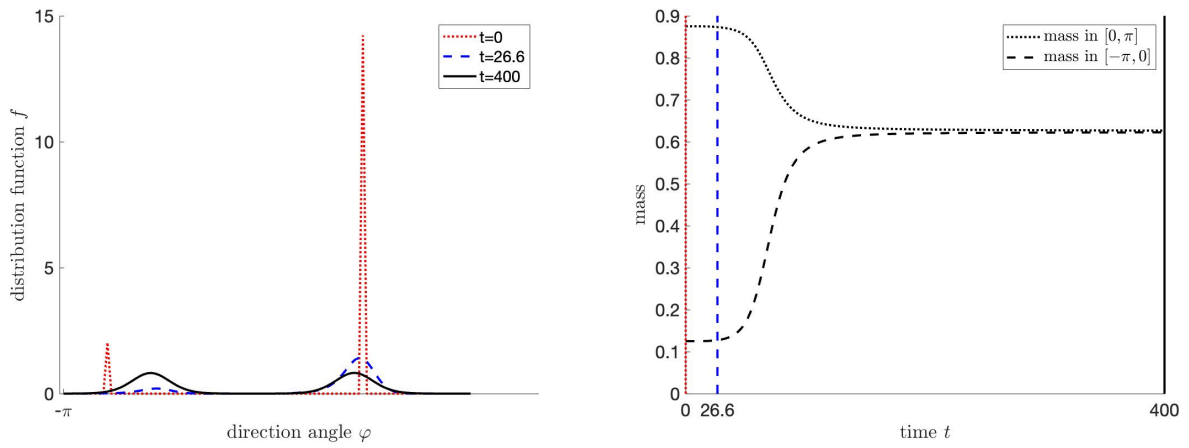


Figure 5: Different masses initially concentrated at $-3\pi/4$ and at $\pi/2$, diffusion constant $\mu = 0.001$. *Left*: Time evolution with smoothing and relocation of the peaks, followed by redistribution of mass. *Right*: Time evolution of the mass in $[-\pi, 0]$ and in $[0, \pi]$.

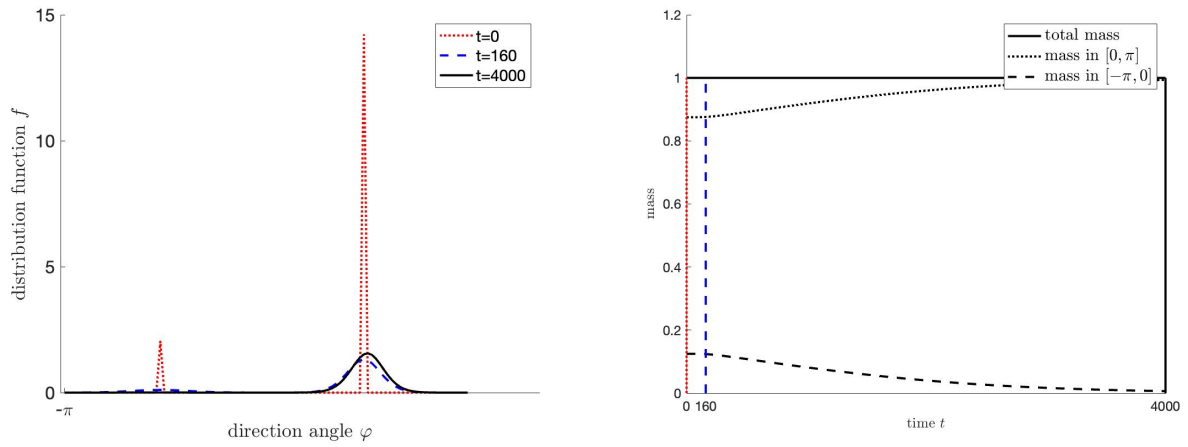


Figure 6: Different masses initially concentrated at $-\pi/2$ and at $\pi/2$, diffusion constant $\mu = 0.001$. These simulations were produced just with the alignment and the diffusion operator. *Left:* Time evolution with smoothing of the peaks and redistribution of the mass to the bigger of the two peaks, while the smaller one vanishes completely. *Right:* Time evolution of the mass in $[-\pi, 0]$ and in $[0, \pi]$, where it becomes even clearer that the peak in the negative part of the torus vanishes.

References

- [1] R.J. Alonso, *Existence of global solutions to the Cauchy problem for the inelastic Boltzmann equation with near-vacuum data*, JSTOR 58 (2009), pp. 999–1022.
- [2] R.J. Alonso, V. Bagland, Y. Cheng, B. Lods, *One-Dimensional Dissipative Boltzmann Equation: Measure Solutions, Cooling Rate, and Self-Similar Profile*, SIAM J. Math. Anal. (2018), 50, pp. 1278–1321.
- [3] R. J. Alonso, and B. Lods, *Uniqueness and regularity of steady states of the Boltzmann equation for viscoelastic hard-spheres driven by a thermal bath*. Commun. Math. Sci. 11, 4 (2013), 851–906.
- [4] R. J. Alonso, B. Lods, I. Tristani, *Fluid dynamic limit of Boltzmann equation for granular hard-spheres in a nearly elastic regime*, arXiv:2008.05173v4
- [5] K. Asano, S. Ukai, *On the Cauchy problem of the Boltzmann equation with a soft potential*. Publ. RIMS, Kyoto Univ. 18, 57–99 (1982).
- [6] A. Baskaran, M.C. Marchetti, *Nonequilibrium statistical mechanics of self propelled hard rods*, J. Stat. Mech. 2010 (2010), P04019.
- [7] E. Bertin, M. Droz, G. Gregoire, *Hydrodynamic equations for self-propelled particles: microscopic derivation and stability analysis*, J. Phys. A: Math. Theor. 42 (2006), 445001.
- [8] D. Benedetto, M. Pulvirenti, *On the one-dimensional Boltzmann equation for granular flows*, M2AN 35 (2001), pp. 899–905.
- [9] E. Ben-Naim, P.L. Krapivsky, *Alignment of rods and partition of integers*, Phys. Rev. E Stat. Nonlin. Soft Matter Phys. 73 (2006), 031109.
- [10] A.V. Bobylev, C. Cercignani, *Self-Similar Asymptotics for the Boltzmann Equation with Inelastic and Elastic Interactions*, J. Stat. Phys. 110 (2003), pp. 333–375.
- [11] A. V. Bobylev, J. A. Carrillo, I. Gamba, *On Some Properties of Kinetic and Hydrodynamic Equations for Inelastic Interactions*, J. Statist. Phys (2000), 98, pp. 743–773.
- [12] E. Bertin, M. Droz, G. Gregoire, *Boltzmann and hydrodynamic description for self-propelled particles*, (2006) Phys. Rev. E 74 022101
- [13] E. Bertin, M. Droz, G. Gregoire, *Hydrodynamic equations for self-propelled particles: microscopic derivation and stability analysis*, J. Phys. A: Math. Theor. 42 (2006), 445001.
- [14] A. V. Bobylev, I. M. Gamba, V. A. Panferov, *Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions*. J. Statist. Phys. 116, 5-6 (2004), 1651–1682.
- [15] L. Boltzmann, *Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen*, Sitzungsberichte Akad. Wiss., Vienna, part II, 66 (1872), pp. 275–370.
- [16] J. Carrillo, G. Toscani, *Contractive probability metrics and asymptotic behavior of dissipative kinetic equations*, Riv. Mat. Univ. Parma (7) 6 (2007), pp. 75–198.
- [17] E. Carlen, M. C. Carvalho, P. Degond, B. Wennberg, *A Boltzmann model for rod alignment and schooling fish*, Nonlinearity 28 (2015), pp. 1783–1804.

- [18] C. Cercignani, R. Illner, M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Springer-Verlag, New York, 1994.
- [19] M. G. Crandall, P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, Journal of Functional Analysis, 8, Issue 2, October 1971, pp 321–340.
- [20] P. Degond, A. Frouvelle, G. Raoul, *Local stability of perfect alignment for a spatially homogeneous kinetic model*, J. Stat. Phys. 157 (2014), pp. 84–112.
- [21] P. Degond, A. Manhart, H. Yu, *A continuum model of nematic alignment of self-propelled particles*, DCDS-B 22 (2017), pp. 1295–1327.
- [22] P. Degond, A. Manhart, H. Yu, *An age-structured continuum model for myxobacteria*, M3AS 28 (2018), pp. 1737–1770.
- [23] J. Dolbeault, C. Mouhot, C. Schmeiser, *Hypocoercivity for linear kinetic equations conserving mass*, Trans. AMS 367 (2015), pp. 3807–3828.
- [24] I.M. Gamba, V. Panferov & C. Villani, *On the Boltzmann Equation for Diffusively Excited Granular Media*, Communications in Mathematical Physics volume 246, pages503–541(2004)
- [25] M. P. Gualdani, S. Mischler, and C. Mouhot, *Factorization for non-symmetric operators and exponential H-theorem*. <http://hal.archives-ouvertes.fr/ccsd-00495786> (2013).
- [26] I.M. Gamba, V. Panferov, C. Villani, *On the Boltzmann equation for diffusively excited granular media*. Commun. Math. Phys. 246, 503-541 (2004).
- [27] P. Haff, *Grain flow as a fluid-mechanical phenomenon*, J. Fluid Mech. 134 (1983), pp. 401–30.
- [28] F. Herau, *Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation*. Asymptot. Anal. 46, 3-4 (2006), 349359
- [29] S. Hittmeir, L. Kanzler, A. Manhart, C. Schmeiser, *Kinetic modelling of colonies of myxobacteria*, Kinetic and Related Models 14, Number 1, (2021), pp. 1–24, doi:10.3934/krm.2020046
- [30] J. Hodgkin, D. Kaiser, *Genetics of gliding motility in Myxococcus xanthus (Myxobacterales): two gene systems control movement*, Mol. Gen. Genet. 171 (1979), pp. 177–191.
- [31] P.-E. Jabin, T. Rey, *Hydrodynamic limit of granular gases to pressureless Euler in dimension 1*, Quart. Appl. Math. 75 (2017), 155–179.
- [32] Y. Jiang, O. Sozinova, M. Alber, *On modeling complex collective behavior in myxobacteria*, Adv. in Complex Syst. 9 (2006), pp. 353–367.
- [33] S. Kaniel, M. Shinbrot, *The Boltzmann equation, uniqueness and local existence*. Commun. Math. Phys. 58, 65-84 (1978).
- [34] O. E. Lanford, *Time evolution of large classical systems*, Lect. Notes Phys. 38 (1975), pp. 1–111.
- [35] S. Mischler, C. Mouhot, *Cooling Process for Inelastic Boltzmann Equations for Hard Spheres, Part II: Self-Similar Solutions and Tail Behavior*, J. Stat. Phys. 124 (2006), pp. 703–746.
- [36] S. Mischler, C. Mouhot, *Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres*. Comm. Math. Phys. 288, 2 (2009), 431–502.

- [37] S. Mischler, C. Mouhot, *Stability, convergence to the steady state and elastic limit for the Boltzmann equation for diffusively excited granular media*. Discrete Contin. Dyn. Syst. 24, 1 (2009), 159–185.
- [38] S. Mischler, C. Mouhot & M. Rodriguez Ricard, *Cooling Process for Inelastic Boltzmann Equations for Hard Spheres, Part I: The Cauchy Problem*, Journal of Statistical Physics volume 124, pages655–702(2006)
- [39] G. Toscani, *Hydrodynamics from the dissipative Boltzmann equation*, in: G. Capriz, P.M. Mariano, P. Giovine (eds), Mathematical Models of Granular Matter, Lect. Notes in Math. 1937, Springer, Berlin–Heidelberg, 2008.
- [40] I. Tristani, *Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting*, Journal of Functional Analysis, Volume 270, Issue 5, 2016, Pages 1922-1970, SSN 0022-1236.
- [41] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, and O. Shochet. *Novel type of phase transition in a system of self-driven particles*. Phys. Rev. Lett., 75(6):12261229, 1995.
- [42] C. Villani, *Hypocoercivity*, Mem. Amer. Math. Soc., 202(950): iv+141, 2009